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Some Problems on Determined Circles

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Abstract

In this report we deal with the following problem: Given a set P consisting of n points, not all on a line or a circle. What is the minimal possible number of circles determined by P? What is the lower bound on the number of circles determined under certain constraints on the arrangement? We develop the solution with the help of the results on ordinary lines by Kelley-Moser and Purdy-Smith's result on the number of determined planes. We give a lower bound for the number of circles determined by P under the constraints of the form that exactly $n-k$ points are co-circular in our arrangement. We also suggest an extension of our method to give an alternate proof for Elliott's results on the number of determined circles. Apart from this we also suggest a different lower bound for the number of planes determined by exactly three points using recent result of Green-Tao on ordinary lines.

Contents

1 Determined Lines

1.1 Number of determined lines by n points.

Theorem 1.1.1 (De Bruijn-Erdős. [1]). Let P be a configuration of n points in the real plane, not all on a line. Let t be the number of lines determined by P. Then

1. $t \geq n$, and
2. if $t = n$, any two lines have exactly one point of P in common. In this case exactly $n-1$ of the points are collinear.

Using the Sylvester-Gallai theorem. The statement is true for $n = 3$.
Let P be a set of n (≥ 4) points not all collinear. Let us assume the theorem holds true for $n-1$ points. By Sylvester-Gallai theorem [2] there exist a line containing exactly two points of P, say a and b. Consider the set $P \backslash \{a\}$.

- Case 1: If $P \backslash \{a\}$ is a set of $n-1$ collinear points then P generates $n-1$ ordinary lines through a plus the line containing the $n-1$ collinear points.
- Case 2: If $P \backslash \{a\}$ is not a set of $n-1$ collinear points then by induction hypothesis $P \backslash \{a\}$ determines at least $n-1$ lines. The ordinary line determined by a and b is not among these, so P determines at least n lines.

□

Using Linear Algebra. Let B be the incidence matrix of P defined as following : the rows are indexed by the points and the columns by the lines $l_1, \dots, l_t$ determined by P, and

$$B_{xl_i} = \left\{ \begin{array}{ll} 1 & \text{if } x \in l_i \\ 0 & \text{otherwise} \end{array} \right\}$$

Consider the product BB^T. For $x_1 \neq x_2$, $(BB^T)_{x_1x_2} = 1$ since two lines share precisely one point. Therefore

$$BB^T = \begin{bmatrix} r_{x_1}-1 & 0 & 0 & \cdots & 0 \\ 0 & r_{x_2}-1 & 0 & \cdots & 0 \\ 0 & 0 & r_{x_3}-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{x_n}-1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

where r_{x_i} is the number of lines containing ${x_i}^{th}$ point. First matrix is positive definite because it has only positive eigenvalues and the second matrix is positive semi-definite with eigenvalues n and 0 therefore BB^T is positive definite and has rank $= n$. From Linear Algebra we know that $\text{rank}(BB^T) \leq \text{rank}(B)$. Therefore the rank of the $(n \times t)$ matrix B is at least n and therefore $n \leq t$.

□

Using combinatorial argument. Let r_x be the number of lines containing a fixed point x and $|l_i|$ be the number of points on ${l_i}^{th}$ line.
If $x \notin l_i$ then $r_x \geq |l_i|$ because each point on l_i has an incident line on x. Suppose $t < n$ then $t|l_i| < nr_x$ and $m(n - |l_i|) > n(m - r_x)$ for $x \notin A_i$. So,

$$1 = \sum_{x \in X} \frac{1}{n} = \sum_{x \in X} \sum_{l_i : x \notin l_i} \frac{1}{n(m - r_x)} > \sum_{l_i} \sum_{x : x \notin l_i} \frac{1}{m(n - |l_i|)} = \sum_{l_i} \frac{1}{m} = 1,$$

which is a contradiction. Therefore our assumption $t < n$ is wrong and hence $t \geq n$ □

2 Ordinary lines in $\mathbb{R}^2$

In 1893 J. J. Sylvester posed the following problem [2]: Given a collection of points in the plane, not all lying on a line, prove that there exists a line which passes through precisely two of the points. Sylvesters problem was reposed by Erdős in 1944 [3] and which was solved the same year by T. Gallai [4]. This result is referred to as Sylvester-Gallai theorem which we used in the last chapter to prove *Theorem 1.1.1.* After this result there was considerable work done on proving a lower bound on the number of ordinary lines determined by n points in a plane. Dirac [5] and Motzkin [6] separately conjectured that, given n points as in the statement of Sylvesters Theorem, there must be at least $n/2$ ordinary lines.

In 1958 Kelly and Moser [7] showed that a set of n not all collinear points must admit at least $3n/7$ ordinary lines which is the subject of study for this chapter. Quite recently in 2013 Green and Tao [8] proved the Dirac-Motzkin's conjecture for all n larger than some fixed constant. The proof of this result is beyond the scope of this report. We mention the result again in Chapter 4 and use it to improve the bound on the number of planes determined by exactly three points.

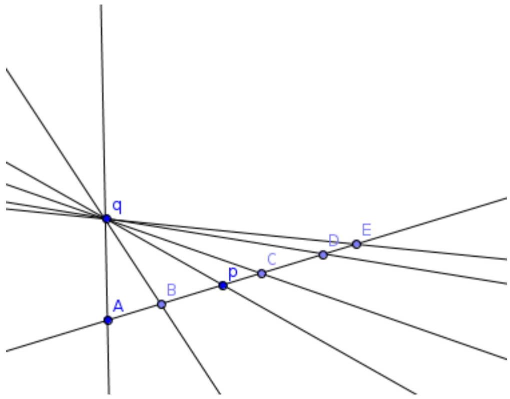
2.1 Kelly-Moser's theorem

Let P be a set of points in $\mathbb{R}^2$ and L a set of connecting lines. Let p and l be any point and line respectively. The set of lines of L which do not go through p subdivide the plane into polygonal regions. p is contained in one of these polygonal regions, which is called its *residence*.

Some preliminary definitions which are required to understand the following proofs are :

- *Neighbor of p* : a line of S containing an edge of the residence of p.
- *Order of p* : the number of ordinary lines passing through p.
- *Rank of p* : the number of neighbors of p which are ordinary lines.
- *Index of p* : the sum of its order and rank.
- *Pencil* : the set of all lines through a point
- *Near-Pencil* : an arrangement of $n \geq 3$ points such that $n - 1$ of them are collinear.

Lemma 2.1.1. If P does not have $n-1$ collinear points then each point of P has at least three neighbors.



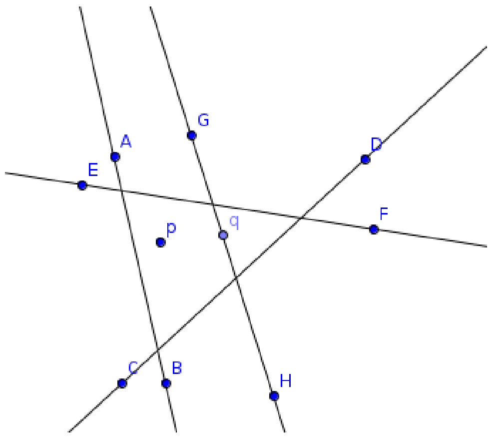
Proof. Suppose a point q has exactly one neighbor then all other lines pass through q. This is possible only in the above arrangement.

Now suppose a point p has exactly two neighbors. The lines of L that do not pass through p form a pencil, or else p would have at least three neighbors. This is also possible only in the above arrangement.

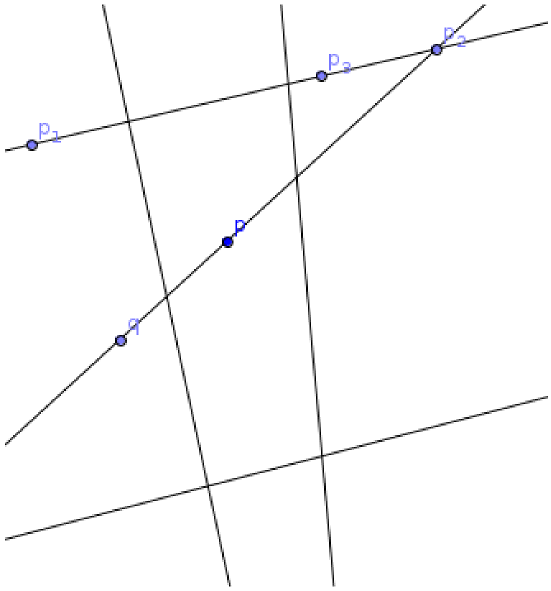
Therefore if P does not have $n-1$ collinear points then each point of P has at least three neighbors. □

Lemma 2.1.2. If the order of p is zero, every neighbor of p is an ordinary line.

Proof. Observe that no point will lie on the edges of the cell in which p is located otherwise there will be a line passing through that point and dividing C into a smaller region. See the figure below:



Now suppose that there is a line l which is the neighbor of the point P but passes through three points p_1, p_2 and p_3. We present only one case of arrangement of points and refer reader to [7] for complete discussion. Assume that p_1 is separated from p_2 and p_3. Consider the line p_2p. It also has a third point (say q) on it since the order of p is zero. Now either p_1q or p_3q will intersect the cell C which is a contradiction.



□

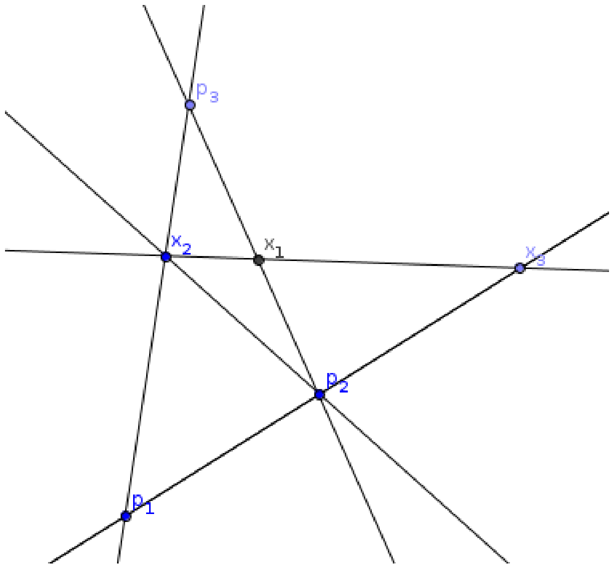
Lemma 2.1.3. Any point of P not of order two has index at least three.

Proof. If a point $p \in P$ has order 0 then it has at least three ordinary neighbors therefore it has rank at least 3 which means that it has index at least 3.
If a point $p \in P$ has order 3 then it has index at least 3.
If a point $p \in P$ has order 1 then it has exactly one ordinary line passing through it. Suppose the other point on this ordinary line is p_1. Let us also assume that at most one neighbor of p is ordinary other wise the point p_1 will have index three. By proof of *Lemma 2.1.2* any neighbor of point p which is not ordinary has to pass through p_1. Since not more than two neighbors of p can pass through p_1 therefore we can conclude that p has exactly three neighbors, two of which, say l_1 and l_2, pass through p_1 and the third one is an ordinary line which does not pass through p_1. The lines l_1 and l_2 have at least three points including p_1. It can be deduced that if p_1, p_2, p_3 lie on l_1 and p_1, p_4, p_5 lie on l_2 then either p_2p_4 and p_3p_5 pass through p or p_2p_5 and p_3p_4 pass through p. But then p must have two neighbors beside l_1 and l_2, which is in contradiction with the fact that p has three neighbors and no more. The above reasoning was due to Dirac's review [9] of Kelly-Moser's paper. □

Theorem 2.1.4. If a line l of L is a neighbor of three points p_1, p_2 and p_3 then the points of P which lie on s are on the connecting lines determined by p_1, p_2 and p_3.

Proof. Three points which share a neighbor can not be collinear. Suppose p_1, p_2 and p_3 share a common neighbor l. Without loss of generality assume that p_1 and p_2 lie on the same side of l and p_3 lies on the opposite side. All three can not lie on the same side of l because in that case one of the line p_ip_j will shield the other p_k from the line l and they can not share it as a neighbor. Here i, j, k is a permutation of $1, 2, 3$.

Let the intersections of p_1p_2 , p_2p_3 , p_3p_1 with l be x_3 , x_1 , x_2 respectively. If p is a point of P on l such that x_ix_j separate x_kp, then pp_i and pp_j separate l from p_k. Here again i, j, k is a permutation of $1, 2, 3$.



□

Theorem 2.1.5. A line l of L is a neighbor of at most four points.

Proof. On the contrary assume that l is a neighbor of 5 points $p_1, p_2 \ldots, p_5$. Consider the figure shown above. At least 2 of the x_1, x_2 and x_3 are the elements of P. Assume x_2, x_3 are elements of P. But neither of x_2 nor x_3 can be on the lines p_1p_4 or p_1p_5 because no three points which share a neighbor can be collinear. This means that one of the points of P on l is not on the connecting lines of the set p_1, p_4 and p_5

□

Theorem 2.1.6. Let m be the number of ordinary lines in L and n be the number of points in P. If I_i is the index of the point p_i, then

$$6m \geq \sum_{i=1}^{n} I_i$$

Proof. Each ordinary line can be counted at most 6 times — four times as a neighbor and twice as being incident to each of its points — so the sum of the index over all the points is at most six times the number of ordinary lines. □

Theorem 2.1.7 (Kelly-Moser's Theorem [7]). Let m be the number of ordinary lines in L and n be the number of points in P. Then

$$m \geq \frac{3n}{7}$$

Proof. Suppose k is the number of points of order 2. Then using *Theorem 2.1.6* and counting indices for such points explicitly we get

$$6m \geq 3(n-k) + 2k = 3n - k$$

This gives $6m \geq 3n - k \geq 3n - m$ because number of ordinary lines is at least as many as number of points with order 2. Therefore $m \geq 3n/7$. □

3 Determined Circles in $\mathbb{R}^2$

Kelly-Moser in their paper also suggested that for n points in P such that all did not lie on a circle or a line then for some suitable constant $c > 0$, P determines at least cn^2 distinct circles. This was proved by Elliott [10] in 1967. Using similar methods Bálintová and Bálint[11] further extended the results of Elliott. Elliott's 1967 reulst was slightly wrong which was pointed out by Purdy and Smith in [12]. We present important theorems from Elliott's paper in this chapter and give Bálintová and Bálint's theorem without proof.

3.1 Elliott's theorem on number of determined circles

Let C be a circle with radius r and center O. Let T be the map that takes a point P to a point P on the ray OP such that $OP \times OP' = r^2$. Then, T is an inversion in the circle C. In this report we denote a inverse of a set S with $\overline{S}$ which means $\overline{S}$ contains inverse images of all the points in S with a reference point specified.

Some properties of circular inversion useful for the proofs given below are stated without proof.

- A circle that passes through the center O of the reference circle inverts to a line not passing through O, but parallel to the tangent to the original circle at O, and vice versa; whereas a line passing through O is inverted into itself.
- A circle not passing through O inverts to a circle not passing through O. If the circle meets the reference circle, these invariant points of intersection are also on the inverse circle. A circle (or line) is unchanged by inversion if and only if it is orthogonal to the reference circle at the points of intersection.

Lemma 3.1.1. Let p_i be any point of a set P of n points in the Euclidean plane not all on a circle or a straight line. Then P determines at least $\frac{2(n-1)}{21}$ circles containing exactly three points p_j, one of which is p_i.

Proof. Invert the set $P\backslash\{p_i\}$ with respect to p_i. The new set $\overline{P\backslash\{p_i\}}$ becomes a set of n points which are not on a straight line. The desired result can be obtained if we can show that $\overline{P\backslash\{p_i\}}$ generates at least $\frac{2(n-1)}{21}$ ordinary lines not passing through p_i.

Case 1: Suppose that p_i lies on at most $\frac{n-1}{3}$ ordinary lines of the total $\frac{3(n-1)}{7}$ ordianry lines generated by $\overline{P \backslash \{p_i\}}$. Then the number of ordinary lines not passing through p_i is atleast $\frac{2(n-1)}{21}$.

Case 2: Suppose that p_i does not satisfy the condition of *Case 1*.Then p_i can not lie on more than $\frac{n-1}{3}$ of the ordinary lines determined by $\overline{P \backslash \{p_i\}} \cup p_i$. Using Kelly-Moser's result we can say that the number of such ordinary lines do not exceed $\frac{3(n-1)}{7}$, therefore the proof is complete. □

Theorem 3.1.2. Let P be a set of n points in the Euclidean plane not all on a circle or a straight line. Then if $n > 3$, S determines at least $2n(n-1)/63$ circles containing exactly three points.

Proof. The proof follows from the above lemma applied on all the points and noting that each circle is counted at most three times. □

Theorem 3.1.3 (Elliot's Theorem [10]). Let P be a set of n points in the Euclidean plane not all on a circle. If $n > 393$ then S determines at least $1 + \binom{n-1}{2} - \lfloor \frac{n-1}{2} \rfloor$ circles.

Theorem 3.1.4 (Bálintová - Bálint's Theorem[11]). Let P be a set of $n \geq 6$ points in $\mathbb{R}^2$ not all on a circle or a line. Then P determines at least $\frac{15n(n-1)+1678}{266}$ circles.

So Elliott's bound is slightly better than Bálintová - Bálint's bound but works only for $n > 393$. Interested reader is encouraged to go through the details of the proofs of both the results from [10] and [11] respectively.

4 Determined Planes in $\mathbb{R}^3$

In this chapter we explore the results by Purdy and Smith[13] on the counting the number of determined planes. We present their proof of the theorem which we use to present the alternate method of counting number of determined circles. In this chapter we also present a better lower bound for the number of planes determined by exactly three points for large values of n.

4.1 Planes determined by points in $\mathbb{R}^3$

Let P be a set of n points in $\mathbb{R}^3$, not all co-planar and no three collinear. We are interested in the number of planes determined by these n points. We count this by selecting a point p and then projecting the remaining $n-1$ points on a plane π which intersects all the lines determined by P. Let $p_i \in P$ be the point from which we project and let $l_{i,j}$ be the line connecting point p_i with another point $p_j \in P$. The projection of the point p_j on π is the point of intersection of $l_{i,j}$ with π. Since no three points are collinear, this projection of p_j is unique with respect to p_i

Some preliminary definitions which are required to understand the following proofs are:

- $t_k(p)$: Number of lines determined on π containing k points.
- $m_k(p)$: Number of planes determined which have k points including p.
- $t(p) = \sum_{k\geq 2} t_k(p)$
- $m(p) = \sum_{k\geq 3} m_k(p)$

Lemma 4.1.1. $t_k(p) = m_{k+1}(p)$.

Proof. Since each point in P - $\{p\}$ has a unique projection onto π, because no three are collinear, it follows that any two points on π which determine a line also determine a plane in $\mathbb{R}^3$ containing p. $\square$

Theorem 4.1.2 (Euler Formula). Let V, E and F denote the number of vertices, edges and faces respectively, then for a graph in $\mathbb{R}^2$ the following relation holds true:

$$V - E + F = 1.$$

Theorem 4.1.3 (Melchior's Inequality [14]). Let t_k be the number of intersection points through which k lines pass. The following holds true:

$$t_2 \geq 3 + \sum_{k\geq 3}(k-3)t_k$$

.

Proof. Let f_k be the total number of faces surrounded by k edges. If not all lines are concurrent then each face is bounded by at least three edges. We have $F = \sum_{k\geq 3} f_k$ and $V = \sum_{k\geq 2} t_k$ where F, V and E are the number of faces vertices and edges respectively. Also,

$$2\sum_{k\geq 2} kt_k = 2E = \sum_{k\geq 3} kf_k$$

$$2E = \sum_{k\geq 3} kf_k \geq 3F$$

$$0 \leq 2E - 3F = 3(V-1) - E$$

$$0 \leq 3V - E - 3 = 3\sum_{k\geq 2} t_k - \sum_{k\geq 2} kt_k - 3 = \sum_{k\geq 2}(3-k)t_k - 3$$

Therefore we get the result $\sum_{k\geq 2}(k-3)t_k \leq -3$. □

Theorem 4.1.4. Let P be a set of n points in $\mathbb{R}^3$ not all co-planar no three collinear. Let m_k be the number of planes determined by points in P containing exactly k points. Then

$$-3n \geq \sum_{k\geq 3} k(k-4)m_k$$

holds true.

Proof. We apply Melchior's inequality to $t_k(p_i)$ and get

$$-3 \geq \sum_{k\geq 2}(k-3)t_k(p_1) = \sum_{k\geq 3}(k-4)m_k(p_1) \tag{4.1}$$

Adding the above equation for each point in P we get

$$-3n \geq \sum_{i=1}^{n}\sum_{k\geq 3}(k-4)m_k(p_i) \tag{4.2}$$

Since each $k-$plane is counted exactly k times hence we get the result. □

Corollary 4.1.5. $m_3 \geq n + \sum_{k\geq 4} \frac{k(k-4)}{3} m_k$

Proof. Using *Theorem 4.1.4*, we get

$$-3n \geq \sum_{k\geq 3} k(k-4)m_k$$

$$-n \geq \sum_{k\geq 3} \frac{k(k-4)}{3} m_k$$

$$-n \geq -m_3 + \sum_{k\geq 4} \frac{k(k-4)}{3} m_k$$

$$m_3 \geq n + \sum_{k\geq 4} \frac{k(k-4)}{3} m_k$$

□

Let m be the total number of determined planes.

Corollary 4.1.6. $m_3 + m_4 \geq \frac{5m+3n}{8}$

Proof. Using Theorem 4.1.4

$$-3n \geq \sum_{k\geq 3} k(k-4)m_k$$

$$3n \leq 3m_3 + 0m_4 - 5m_5 - 12m_6 + \dots$$

Therefore by adding $5m$ to both the sides

$$5m + 3n \leq 8m_3 + 5m_4 + 0m_5 - 7m_6 + \dots$$

$$5m + 3n \leq 8(m_3 + m_4)$$

□

4.2 Planes determined by exactly three points

Theorem 4.2.1. (Purdy-Smith [13, Theorem 3.8]) Let P be a set of n points, n being sufficiently large, in $\mathbb{R}^3$, no three collinear and not all co-planar. Then there exist at least $\frac{4}{13}\binom{n}{2}$ planes determined by exactly three points.

Proof. In [15], Csima and Sawyer published their result that among any set of $n \neq 7$ points in a plane, not all collinear, there exist $\geq \frac{6}{13}$ lines incident to exactly two points. We can use this result to prove an analogous result for the number of planes determined by exactly three points $\mathbb{R}^3$. We pick a point p_1 and project $n-1$ points onto a plane π. Since all the points are not co-planar, not all the $n-1$ points are collinear. Using Csima and Sawyer's result for $n-1$ we get

$$m_3(p_1) = t_2(p_1) \geq \frac{6(n-1)}{13}$$

We can use this inequality for all the n points and note that we would count each three point plane three times.

$$m_3 = \frac{1}{3}(m_3(p_1) + m_3(p_2) + \ldots + m_3(p_n)) \geq \frac{4}{13}\binom{n}{2}$$

□

For large values of n we can use following recent result from Green and Tao and improve the previous bound.

Theorem 4.2.2. (Green-Tao [8, Theorem 1.2]) Suppose P be a finite set points in a plane not all being collinear. Suppose $n \geq n_0$ for a sufficiently large constant n_0. Then P spans at least $\frac{n}{2}$ ordinary lines.

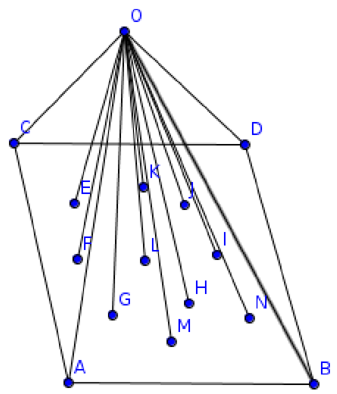
Using the above result and the same reasoning as in Theorem 4.2.1 we get

Theorem 4.2.3. Let P be a set of n points, n being sufficiently large, in $\mathbb{R}^3$, no three collinear and not all coplanar. Then there exist at least $\frac{1}{3}\binom{n}{2}$ planes determined by exactly three points.

Lemma 4.2.4. Let P be a set of n points in $\mathbb{R}^3$, not all coplanar and no three collinear. If m is the total number of planes determined by P, then if exactly $n-k$ of the points are co-planar we get :

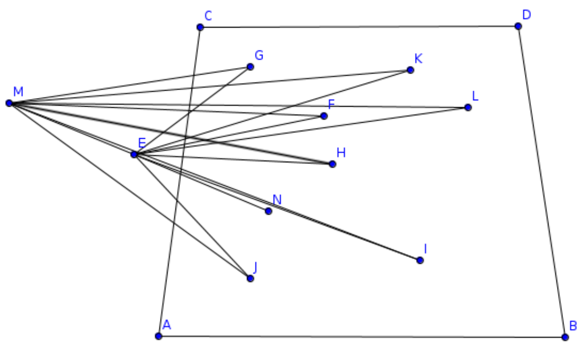
$$m \geq 1 + k\binom{n-k}{2} - \binom{k}{2}\frac{n-k}{2}$$

Proof. First we can see the result for $k=1$ and $k=2$ and generalize from there.
Case $k=1$*:* Here only one point (say O) of P is not co-planar with the rest $n-1$ points. This gives us the following figure



Each pair of points on the plane $ABCD$ will form a distinct three point plane with the point O. This gives us $\binom{n-1}{2}$ planes with the point O and a single plane $ABCD$. Therefore the total number of planes $m = 1 + \binom{n-1}{2}$

Case $k = 2$: Here we have two points (say M and E) of P which are not coplanar with the other $n-2$ points. Let N be the projection of the line ME on the plane $ABCD$. This gives us the following figure.



Each pair of points on the plane $ABCD$ will form a three-point plane with the point M and E. But some of these planes are double counted because a pair of points collinear with the projection of ME on the plane $ABCD$ (i.e with the point N) will give us the same plane with M and also with E. In other words the pair of points which are collinear with N will also be coplanar with M, E and N. Since no three-points are collinear in our starting situation therefore at most $\left\lfloor \frac{n-2}{2} \right\rfloor$ pairs can be collinear with the point N. Therefore

$$m \geq 1 + 2\binom{n-2}{2} - \binom{2}{2}\left\lfloor \frac{n-2}{2} \right\rfloor$$

Case $k \geq 3$: Here we have k points of P which are not coplanar with the other $n-k$ points. Now with each pair from $n-k$ coplanar points we can choose one point from the k points to get a plane. But here also we over-count the planes. Since for every pair of points from k points we can have at most $\left\lfloor \frac{n-k}{2} \right\rfloor$ points which are co-planar with the planes formed by the pair and its projection on the plane therefore we get

$$m \geq 1 + k\binom{n-k}{2} - \binom{k}{2}\left\lfloor \frac{n-k}{2} \right\rfloor$$

$\square$

Lemma 4.2.5. Let m_k be the number of planes incident to exactly k points. Let m be the total number of planes determined by P. Given a set of n points, no three collinear and not all co-planar,

$$6m \geq 3n + \sum_{k\geq 3} \binom{k}{2} m_k$$

Proof. Using *Theorem 4.1.4*, we get

$$-3n \geq \sum_{k\geq 3} k(k-4)m_k = \sum_{k\geq 3}(k^2-k)m_k - 3\sum_{k\geq 3} km_k \tag{1}$$

By negating the inequality we get

$$3m_3 + 0m_4 - 5m_5 - 12m_6 - 21m_7 - 32m_8 \ldots \geq 3n \tag{2}$$

Again using equation 1 we can get

$$9m_3 + 12m_4 + 15m_5 + 18m_6 + 21m_7 \ldots \geq 3n + \sum_{k\geq 3}(k^2-k)m_k \tag{3}$$

Adding (2) and (3) we get

$$12m_3 + 12m_4 + 10m_5 + 6m_6 \geq 6n + \sum_{k\geq 3}(k^2-k)m_k$$

Therefore,

$$12m \geq 12(m_3 + m_4 + m_5 + m_6) \geq 6n + \sum_{k\geq 3}(k^2-k)m_k$$

which gives us *Lemma 4.2.5* on dividing the inequality on both the sides.

□

Using the previous lemma we can derive an extension of Kelly-Moser's theorem (*Theorem 2.1.7*) to three dimensions.

Theorem 4.2.6. (Purdy-Smith [13, Theorem 3.11]) Let P be a set of n points, in $\mathbb{R}^3$, no three collinear and at most $n-k$ co-planar. If $n \geq g(k) := 54k^2 + \frac{9}{2}k$, the total number of planes determined by P is at least $1 + k\binom{n-k}{2} - \binom{k}{2}\frac{n-k}{2}$.

We define function $f(k) = 1 + k\binom{n-k}{2} - \binom{k}{2}\frac{n-k}{2}$. Let c_1 and c_2, where $c_1 < c_2$, be the function's two local extrema at $\frac{1}{9}(-1 + 5n \pm \sqrt{1-n+7n^2})$. For all $n \geq 4$, $f(c_1) > 0$ and $f(c_2) < 0$. Also $f''(c_1) < 0$ and $f''(c_2) > 0$.

Proof. P contains $\binom{n}{2}$ pair of points. Define the degree for a pair of points to be the number of determined planes to which the pair is incident.

- *Case 1:* More than $\frac{n}{2}$ pairs of points have a degree $< 6k$.
  These pairs can not form a matching, hence two pairs of low degree must share a point. Let these pairs be $\{p, q\}$ and $\{p, r\}$ and M be the plane determined by the points p, q and r. Let $a < 6k$ be the number of planes determined by P, excluding M, incident to the pair $\{p, q\}$. Similarly let $b < 6k$ be the number of planes incident to the pair of $\{p, r\}$. Since no three points are collinear, any plane passing through $\{p, q\}$ can share at most one point of P, other than p, with a plane though the pair $\{p, r\}$. Therefore, at most $a \times b < 36k^2$ points of P are not M. If M has exactly $n - x$ points on it, then $k \leq x < 36k^2$ The following two conditions hold true for $n \geq g(k)$:

  1. $36k^2 < c_2$, where c_2 is the second local extremum of $f(k)$.
     It is sufficient to see $(5 + \sqrt{6})n \geq 324k^2 + 1$ and this is true for all $n > 44k^2$.
  2. $f(36k^2) \geq f(k)$.
     To verify this consider $f_1(n, 36k^2) - f_1(n, k)$ where $f_1(n, k) := 1 + k\binom{n-k}{2} - \frac{n-k}{2}\binom{k}{2}$. $f_1(n, 36k^2) - f_1(n, k)$ is a convex quadratic function of n. By solving for the roots of this equation we can see that for all $n \geq g(k)$, $f_1(n, 36k^2) - f_1(n, k)$ is positive.

  From the above properties we can conclude that for all x such that $k \leq x < 36k^2$, $f(x) \geq f(k)$.

- *Case 2:* At most $\frac{n}{2}$ pairs of points have a degree $< 6k$.
  There are at least $\binom{n}{2}$ - $\frac{n}{2} = \frac{1}{2}n(n-2)$ pairs of degree $\geq 6k$. Let P_i be the number of pairs of the points incident to exactly i planes. By using Lemma 4.2.5,

  $$6m > \sum_{k\geq 3} \binom{k}{2} m_k = \sum_{i\geq 2} iP_i \geq \frac{1}{2}n(n-2)6k$$

  Thus for all $k \geq 1$ (and $n \geq 4$),

  $$m > \frac{1}{2}n(n-2)k \geq \frac{1}{2}(n-k)(n-k-1)k + 1 = 1 + k\binom{n-k}{2}$$

  Thus the inequality holds true. [13]

$\square$

5 Determined circles revisited

5.1 Alternate proof for determined circles

We saw in Elliott's proof for number of determined circles in $\mathbb{R}^2$ in Chapter 3. The proof uses circular inversion of the points with respect to each point of P and then counts number of ordinary lines using Kelly-Moser's result.

In the previous chapter we saw an extension of Kelly-Moser's theorem to three dimension (*Theorem 4.2.6*). That theorem estimates the number of planes determined by n points when at most $n-k$ of those are co-planar.

We suggest an alternate way to count the number of determined circles and show how it works for a base case when $n-1$ points are coplanar.

5.1.1 A new map

Consider a set P of n points.
Define a mapping $(X,Y) \to (X,Y,X^2+Y^2)$
All the n points in $\mathbb{R}^2$ are mapped to $\mathbb{R}^3$.
Note that under the defined map, all the points incident to a particular determined plane in $\mathbb{R}^3$ are co-circular in $\mathbb{R}^2$ except those points which are incident to a vertical plane. They do not determine a circle together in $\mathbb{R}^2$. This can be seen more formally by considering algebraic equation for a circle. Any circle in $\mathbb{R}^2$ can be represented by following equation :

$$(x-a)^2 + (y-b)^2 = r^2$$

where (a,b) is the center of the circle and r is the radius of the circle. On expanding and rearranging the terms we get

$$x^2 + y^2 - 2ax - 2by - r^2 = 0$$

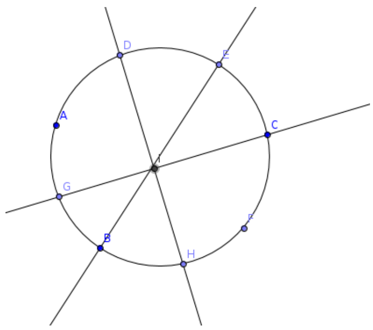
Using $z = x^2 + y^2$ from the definition of our map we get

$$z - 2ax - 2by - r^2 = 0$$

This is a general equation for all the planes in $\mathbb{R}^3$ excluding vertical planes because coefficient of z should be a nonzero. Therefore we should be able to count the number of determined circles in $\mathbb{R}^2$ if we count the number of determined non-vertical planes in $\mathbb{R}^3$ under the mapping defined above.

We will calculate the number of circles determined if exactly $n-k$ of the points are co-circular.

Consider a base case when $k = 1$ i.e $n-1$ points are on a fixed circle and one point is off the circle. Number of circles determined by this configuration is $1 + \binom{n-1}{2} - \lfloor \frac{n-1}{2} \rfloor$ because we have to subtract those choices of three points which are collinear.



We can obtain the same count by counting the number of determined planes in $\mathbb{R}^3$ when we translate the points under the above defined map.

Note that the co-circular points remain on a coplanar in $\mathbb{R}^3$ under our map. The one point off the circle is also off the plane determined by the co-circular points when mapped in $\mathbb{R}^3$.

Number of planes determined by this configuration in $\mathbb{R}^3 = 1 + \binom{n-1}{2}$. Not all planes correspond to the circles in $\mathbb{R}^2$. We have to exclude the vertical planes. Maximum number of vertical planes in the configuration are $\lfloor \frac{n-1}{2} \rfloor$ Therefore number of planes excluding vertical planes are $1 + \binom{n-1}{2} - \lfloor \frac{n-1}{2} \rfloor$ which is equal to the number of circles in $\mathbb{R}^2$.

Consider $k = 2$. Number of determined planes is at least $1 + 2\binom{n-2}{2} - \binom{2}{2} \lfloor \frac{n-2}{2} \rfloor$. Maximum number of vertical planes in the configuration is $2 \times \lfloor \frac{n-2}{2} \rfloor$. Notice that we are over counting the number of vertical planes because it is not possible for the two points which are off the plane to have a projection such that both of them have $\lfloor \frac{n-2}{2} \rfloor$ vertical planes passing through them. Therefore the number of determined circles in $\mathbb{R}^2$ when exactly $n-2$ points are co-circular is at least $1 + 2\binom{n-2}{2} - \binom{2}{2} \lfloor \frac{n-2}{2} \rfloor - 2 \times \lfloor \frac{n-2}{2} \rfloor$.

In general case also we can use similar argument. Number determined planes is at least $1 + k\binom{n-k}{2} - \binom{k}{2} \lfloor \frac{n-k}{2} \rfloor$ We can count the vertical planes by $k \times \lfloor \frac{n-k}{2} \rfloor$ like before. Therefore in this case the number of circles determined by n points of which exactly $n-k$ are co-circular is at least $1 + k\binom{n-k}{2} - \binom{k}{2}\left\lfloor \frac{n-k}{2} \right\rfloor - k \times \left\lfloor \frac{n-k}{2} \right\rfloor$ This gives us the following theorem.

Theorem 5.1.1. Let P be a set of n points, in $\mathbb{R}^2$, and exactly $n-k$ co-circular. Then total number of circles determined by P is at least $1 + k\binom{n-k}{2} - \binom{k}{2}\left\lfloor \frac{n-k}{2} \right\rfloor - k \times \left\lfloor \frac{n-k}{2} \right\rfloor$

5.2 Future direction of work

Having dealt with case when we knew that exactly how many points were co-circular in the configuration of n points. We wish to extend our method for a more general case where can can talk about configurations in which at most $n-k$ points are co-circular. Here we would like to use Theorem 4.2.6 due to Purdy and Smith and do a similar analysis on the number of planes determined and exclude vertical planes from the count. This can probably lead a lower bound on n for which we can give an estimate on minimum number of circles determined and extend Elliott's theorem for $n < 393$.

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