PERGAMON

Transportation Research Part E 38 (2002) 281–303

TRANSPORTATION RESEARCH PART E

www.elsevier.com/locate/tre

A supply chain network equilibrium model

Anna Nagurney [a,*], June Dong [b], Ding Zhang [b]

[a] *Department of Finance and Operations Management, Isenberg School of Management, University of Massachusetts, Amherst, MA 01003, USA*

[b] *Department of Marketing and Management, School of Business, State University of New York at Oswego, Oswego, NY 13126, USA*

Received 14 March 2001; received in revised form 5 July 2001; accepted 15 October 2001

Abstract

In this paper, an equilibrium model of a competitive supply chain network is developed. Such a model is sufficiently general to handle many decision-makers and their independent behaviors. The network structure of the supply chain is identified and equilibrium conditions are derived. A finite-dimensional variational inequality formulation is established. Qualitative properties of the equilibrium model and numerical examples are given. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Supply chains; Networks; Equilibrium; Variational inequalities; Decentralized decision-making

1. Introduction

The topic of supply chain analysis is interdisciplinary by nature since it involves manufacturing, transportation and logistics, as well as retailing/marketing. It has been the subject of a growing body of literature (cf. Stadtler and Kilger, 2000 and the references therein) with the associated research being both conceptual in nature (see, e.g., Poirier, 1996, 1999; Mentzer, 2000; Bovet, 2000), due to the complexity of the problem and the numerous agents such as manufacturers, retailers, and consumers involved in the transactions, as well as analytical (cf. Federgruen and Zipkin, 1986; Federgruen, 1993; Slats et al., 1995; Bramel and Simchi-Levi, 1997; Miller, 2001;

* Corresponding author. Tel.: +1-413-545-5635; fax: +1-413-545-3858.
E-mail address: nagurney@gbfin.umass.edu (A. Nagurney).

1366-5545/02/$ - see front matter
PII: S1366-5545(01)00020-5

Hensher et al., 2001 and the references therein). See Erengüç et al. (1999) for a recent survey on supply chains.

Lee and Billington (1993) expressed the need for decentralized models that allow for a generalized network structure and simplicity in the study of supply chains. Anupindi and Bassok (1996), in turn, addressed the challenges of formulating systems consisting of decentralized retailers with information sharing. Lederer and Li (1997), on the other hand, studied competition among firms that produce goods or services for customers who are sensitive to delay time. Corbett and Karmarkar (2001) were concerned with the equilibrium number of firms in oligopolistic competition in a supply chain. In order to allow for the closed form determination of the equilibrium number of firms they assumed that the firms in the same tier were characterized by identical linear production cost functions. Equilibrium models have a long tradition in transportation modeling (cf. Florian and Hearn, 1995) as well as in economics (cf. Arrow and Intrilligator, 1982) and in finance (see Nagurney and Siokos, 1997).

Many researchers, in addition to, practitioners, have described the various networks that underlie supply chain analysis and management with the goal being primarily that of *optimization*. In this paper, in contrast, we develop an equilibrium model of competitive supply chain networks. Such a model at our level of generality has not appeared heretofore in the literature. It provides a benchmark against which one can evaluate both price and product flows. The equilibrium model captures both the independent behavior of the various decision-makers as well as the effect of their interactions. Finally, it provides the foundation for developing dynamic models for the study of the evolution of supply chains.

The equilibrium model is drawn from economics and, in particular, from network economics (cf. Nagurney, 1999). Manufacturers are assumed to be involved in the production of a homogeneous product which is then shipped to the retailers. Manufacturers obtain a price for the product (which is endogenous) and seek to determine their optimal production and shipment quantities, given the production costs as well as the *transaction* costs associated with conducting business with the different retailers.

Retailers, in turn, must agree with the manufacturers as to the volume of shipments since they are faced with the handling cost associated with having the product in their retail outlet. In addition, they seek to maximize their profits with the price that the consumers are willing to pay for the product being endogenous. Consumers determine their optimal consumption levels from the various retailers subject both to the prices charged for the product as well as the cost of conducting the transaction (which, of course, may include the cost of transportation associated with obtaining the product from the retailer).

The paper is organized as follows. In Section 2, we present the competitive supply chain network model, derive optimality conditions for its decision-makers, and then present the governing equilibrium conditions. We also derive the finite-dimensional variational inequality formulation of the problem. In Section 3, we provide qualitative properties of the equilibrium pattern and establish the properties needed for proving convergence of the algorithm. In Section 4, we describe the computational procedure, along with convergence results. The algorithm resolves the network problem into subproblems, each of which can be solved exactly and in closed form. In Section 5, we apply the algorithm to numerical examples to determine the equilibrium product flows and prices and also provide a discussion of the model and results. We conclude the paper with Section 6.

2. The supply chain network model with decentralized decision-makers

In this section, we develop the supply chain network model with manufacturers, retailers, and consumers. The supply chain network structure, at equilibrium, which we establish in this section, is as depicted in Fig. 1. Specifically, we consider m manufacturers who are involved in the production of a product, which can then be purchased by n retailers, who, in turn, make the product available to consumers located at o demand markets. We denote a typical manufacturer by i, a typical retailer by j, and a typical demand market by k. Note that the manufacturing firms are located at the top tier of nodes in the network in Fig. 1; the retailers are located at the middle tier, whereas the demand markets are located at the third or bottom tier. The links in the supply chain network denote transportation/transaction links.

We first focus on the manufacturers. We then turn to the retailers and, subsequently, to the consumers. The complete equilibrium model is then constructed along with the variational inequality formulation of the governing equilibrium conditions.

2.1. The behavior of the manufacturers and their optimality conditions

Let q_i denote the nonnegative production output of the product by manufacturer i. We group the production outputs of all manufacturers into the column vector $q \in R^m_+$. We assume that each manufacturer i is faced with a production cost function f_i, which can depend, in general, on the entire vector of production outputs, that is,

$$f_i = f_i(q) \quad \forall i. \tag{1}$$

A manufacturer may ship the product to the retailers, with the amount of the product shipped (or transacted) between manufacturer i and retailer j denoted by q_{ij}. We associate with each manufacturer and retailer pair (i, j) a transaction cost denoted by c_{ij}. The transaction cost includes the cost of shipping the product. We group the product shipments between the manufacturers and

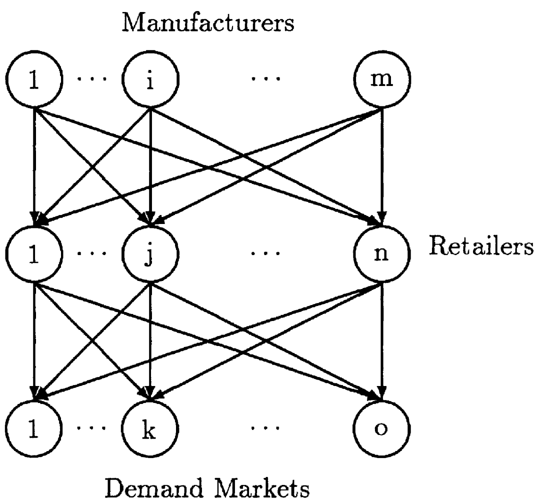

Fig. 1. The network structure of the supply chain at equilibrium.

the retailers into the mn-dimensional column vector Q^1. We consider the situation in which the transaction cost between a manufacturer and retail pair is given by

$$c_{ij} = c_{ij}(q_{ij}) \quad \forall i, j. \tag{2}$$

To help fix ideas (cf. Fig. 2), and in order to facilitate the ultimate construction of the supply chain network in equilibrium, we depict the manufacturers and retailers as nodes and the transactions between a manufacturer i and the retailers j, $j = 1, \ldots, n$, as links.

The quantity produced by manufacturer i must satisfy the following conservation of flow equation:

$$q_i = \sum_{j=1}^{n} q_{ij}, \tag{3}$$

which states that the quantity produced by manufacturer i is equal to the sum of the quantities shipped from the manufacturer to all retailers.

The total costs incurred by a manufacturer i, thus, are equal to the sum of his production cost plus the total transaction costs. His revenue, in turn, is equal to the price that the manufacturer charges for the product (and the retailers are willing to pay) times the total quantity obtained/purchased of the product from the manufacturer by all the retail outlets. If we let ρ^*_{1ij} denote the price charged for the product by manufacturer i to retailer j (i.e., the supply price), and note the conservation of flow equations (3), we can express the criterion of profit maximization for manufacturer i as

$$\text{Maximize} \quad \sum_{j=1}^{n} \rho^*_{1ij} q_{ij} - f_i(Q^1) - \sum_{j=1}^{n} c_{ij}(q_{ij}), \tag{4}$$

subject to $q_{ij} \geqslant 0$ for all j.

We assume that the manufacturers compete in a noncooperative fashion. Also, we assume that the production cost functions and the transaction cost functions for each manufacturer are continuous and convex. Given that the governing optimization/equilibrium concept underlying noncooperative behavior is that of Cournot (1838), Nash (1950, 1951), which states that each manufacturer will determine his optimal production quantity and shipments, given the optimal ones of the competitors, the optimality conditions for all manufacturers *simultaneously* can be

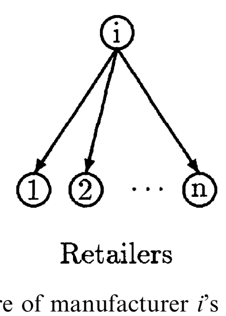

Fig. 2. Network structure of manufacturer i's transactions with retailers.

expressed as the following variational inequality (cf. Bazaraa et al., 1993, Gabay and Moulin, 1980; see also Dafermos and Nagurney, 1987; Nagurney, 1999): Determine $Q^{1^*} \in R_+^{mn}$ satisfying

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial f_i(Q^{1^*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} - \rho_{1ij}^* \right] \times \left[q_{ij} - q_{ij}^* \right] \geqslant 0 \quad \forall Q^1 \in R_+^{mn}. \tag{5}$$

The optimality conditions as expressed by (5) have a nice economic interpretation, which is that a manufacturer will ship a positive amount of the product to a retailer (and the flow on the corresponding link will be positive) if the price that the retailer is willing to pay for the product is precisely equal to the manufacturer's marginal production and transaction costs associated with that retailer. If the manufacturer's marginal production and transaction costs exceed what the retailer is willing to pay for the product, then the flow on the link will be zero.

2.2. The behavior of the retailers and their optimality conditions

The retailers, in turn, are involved in transactions both with the manufacturers since they wish to obtain the product for their retail outlets, as well as with the consumers, who are the ultimate purchasers of the product. Hence, the network structure of retailer *j*'s transactions is as depicted in Fig. 3. Thus, a retailer conducts transactions both with the manufacturers as well as with the consumers at the demand markets. Note that Fig. 3, as did Fig. 2, only depicts the network structure of the transactions involved. Later, we will also associate flows with the links as well as prices with the nodes.

A retailer j is faced with what we term a *handling* cost, which may include, for example, the display and storage cost associated with the product. We denote this cost by c_j and, in the simplest case, we would have that c_j is a function of $\sum_{i=1}^{m} q_{ij}$, that is, the holding cost of a retailer is a function of how much of the product he has obtained from the various manufacturers. However, for the sake of generality, and to enhance the modeling of competition, we allow the function to, in general, depend also on the amounts of the product held by other retailers and, therefore, we may write

$$c_j = c_j(Q^1) \quad \forall j. \tag{6}$$

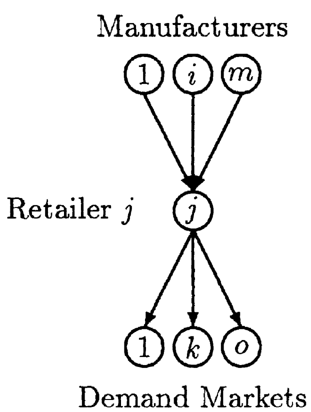

Fig. 3. Network structure of retailer *j*'s transactions.

The retailers associate a price with the product at their retail outlet, which is denoted by ρ_{2j}^*, for retailer j. This price, as we will show, will also be endogenously determined in the model. Assuming, as mentioned in Section 1, that the retailers are also profit-maximizers, the optimization problem of a retailer j is given by

$$\text{Maximize} \quad \rho_{2j}^* \sum_{k=1}^{o} q_{jk} - c_j(Q^1) - \sum_{i=1}^{m} \rho_{1ij}^* q_{ij} \tag{7}$$

subject to:

$$\sum_{k=1}^{o} q_{jk} \leqslant \sum_{i=1}^{m} q_{ij}, \tag{8}$$

and the nonnegativity constraints: $q_{ij} \geqslant 0$ and $q_{jk} \geqslant 0$ for all i and k. Objective function (7) expresses that the difference between the revenues minus the handling cost and the payout to the manufacturers should be maximized. Constraint (8) simply expresses that consumers cannot purchase more from a retailer than is held in stock.

We now consider the optimality conditions of the retailers assuming that each retailer is faced with the optimization problem (7) subject to (8), and the nonnegativity assumption on the variables. Here, we also assume that the retailers compete in a noncooperative manner so that each maximizes his profits, given the actions of the other retailers. Note that, at this point, we consider that retailers seek to determine not only the optimal amounts purchased by the consumers from their specific retail outlet but, also, the amount that they wish to obtain from the manufacturers. In equilibrium, all the shipments between the tiers of network agents will have to coincide.

Assuming that the handling cost for each retailer is continuous and convex, the optimality conditions for all the retailers coincide with the solution of the variational inequality: Determine $(Q^{1^*}, Q^{2^*}, \gamma^*) \in R_+^{mn+no+n}$ satisfying

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial c_j(Q^{1^*})}{\partial q_{ij}} + \rho_{1ij}^* - \gamma_j^* \right] \times \left[q_{ij} - q_{ij}^* \right] + \sum_{j=1}^{n} \sum_{k=1}^{o} \left[- \rho_{2j}^* + \gamma_j^* \right] \times \left[q_{jk} - q_{jk}^* \right]$$
$$+ \sum_{j=1}^{n} \left[\sum_{i=1}^{m} q_{ij}^* - \sum_{k=1}^{o} q_{jk}^* \right] \times \left[\gamma_j - \gamma_j^* \right] \geqslant 0 \quad \forall (Q^1, Q^2, \gamma) \in R_+^{mn+no+n}, \tag{9}$$

where the term γ_j is the Lagrange multiplier associated with constraint (8) for retailer j, γ the n-dimensional column vector of all the multipliers, and Q^2 denotes the no-dimensional vector of product flows between the retailers and the demand markets. For further background on such a derivation, see Bertsekas and Tsitsiklis (1989). In this derivation, as in the derivation of inequality (5), we have not had the prices charged be variables. They become endogenous variables in the complete equilibrium model.

We now highlight the economic interpretation of the retailers' optimality conditions. From the second term in inequality (9), we have that, if consumers at demand market k purchase the product from a particular retailer j, that is, if the q_{jk}^* is positive, then the price charged by retailer j, ρ_{2j}^*, is precisely equal to γ_j^*, which, from the third term in the inequality, serves as the price to clear the market from retailer j. Also, note that, from the second term, we see that if no product is sold by a particular retailer, then the price associated with holding the product can exceed the price charged to the consumers. Furthermore, from the first term in inequality (9), we can infer that, if a manufacturer transacts with a retailer resulting in a positive flow of the product between the two, then the price γ_j^* is precisely equal to the retailer j's payment to the manufacturer, ρ_{1ij}^*, plus its marginal cost of handling the product from the retailer.

2.3. The consumers at the demand markets and the equilibrium conditions

We now describe the consumers located at the demand markets. The consumers take into account in making their consumption decisions not only the price charged for the product by the retailers but also the transaction cost to obtain the product. We let c_{jk} denote the transaction cost associated with obtaining the product by consumers at demand market k from retailer j and recall that q_{jk} denotes the amount of the product purchased (or flowing) between retailer j and consumers at demand market k. We assume that the transaction cost is continuous, positive, and of the general form

$$c_{jk} = c_{jk}(Q^2) \quad \forall j, k, \tag{10}$$

where recall that Q^2 is the no-dimensional column vector of product flows between the retailers and the demand markets.

In Fig. 4, the network of transactions between the retailers and the consumers at demand market k is depicted. Each demand market is represented by a node and the transactions, as previously, by links.

Let now ρ_{3k} denote the price of the product at demand market k. Further, denote the demand for the product at demand market k by d_k and assume, as given, the continuous demand functions

$$d_k = d_k(\rho_3) \quad \forall k, \tag{11}$$

where ρ_3 is the o-dimensional column vector of demand market prices. Hence, according to (11), the demand for consumers for the product at a demand market depends, in general, not only on the price of the product at that demand market but also on the prices of the product at the other demand markets. Thus, consumers at a demand market, in a sense, also compete with consumers at other demand markets.

The consumers take the price charged by the retailers for the product, whose, recall was denoted by ρ_{2j}^* for retailer j, plus the transaction cost associated with obtaining the product, in making their consumption decisions.

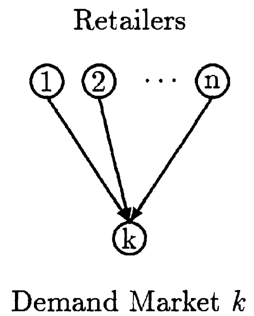

Fig. 4. Network structure of consumers' transactions at demand market k.

The equilibrium conditions for consumers at demand market k, hence, take the form: For all retailers j, $j = 1, \ldots, n$,

$$\rho_{2j}^* + c_{jk}(Q^{2^*}) \begin{cases} = \rho_{3k}^* & \text{if } q_{jk}^* > 0, \\ \geqslant \rho_{3k}^* & \text{if } q_{jk}^* = 0 \end{cases} \tag{12}$$

and

$$d_k(\rho_3^*) \begin{cases} = \sum_{j=1}^n q_{jk}^* & \text{if } \rho_{3k}^* > 0, \\ \leqslant \sum_{j=1}^n q_{jk}^* & \text{if } \rho_{3k}^* = 0. \end{cases} \tag{13}$$

Conditions (12) state that, in equilibrium, if the consumers at demand market k purchase the product from retailer j, then the price charged by the retailer for the product plus the transaction cost does not exceed the price that the consumers are willing to pay for the product. Conditions (13) state, in turn, that if the equilibrium price the consumers are willing to pay for the product at the demand market is positive, then the quantities purchased of the product from the retailers will be precisely equal to the demand for that product at the demand market. These conditions correspond to the well-known spatial price equilibrium conditions (cf. Samuelson, 1952; Takayama and Judge, 1971; Nagurney, 1999 and the references therein).

In equilibrium, conditions (12) and (13) will have to hold for all demand markets k, and these, in turn, can also be expressed as a variational inequality problem (see, e.g., Nagurney, 1999), akin to (5) and (9), and given by: Determine $(Q^{2^*}, \rho_3^*) \in R_+^{no+n}$ such that

$$\sum_{j=1}^n \sum_{k=1}^o \left[\rho_{2j}^* + c_{jk}(Q^{2^*}) - \rho_{3k}^*\right] \times \left[q_{jk} - q_{jk}^*\right] + \sum_{k=1}^o \left[\sum_{j=1}^n q_{jk}^* - d_k(\rho_3^*)\right] \times \left[\rho_{3k} - \rho_{3k}^*\right] \geqslant 0$$
$$\forall (Q^{2^*}, \rho_3^*) \in R_+^{no+o}. \tag{14}$$

Observe that, in the context of the consumption decisions, we have utilized demand functions, rather than utility functions, as was the case for the manufacturers and the retailers, who were assumed to be faced with profit functions, which correspond to utility functions. Of course, demand functions can be derived from utility functions (cf. Arrow and Intrilligator, 1982). Since we expect the number of consumers to be much greater than that of the manufacturers and the retailers we believe that the above formulation is the more natural and tractable one.

2.4. The equilibrium conditions of the supply chain

In equilibrium, the shipments of the product that the manufacturers ship to the retailers must be equal to the shipments that the retailers accept from the manufacturers. In addition, the amounts of the product purchased by the consumers at the demand markets must be equal to the amounts sold by the retailers. Furthermore, the equilibrium shipment and price pattern in the supply chain must satisfy the sum of inequalities (5), (9), and (14), in order to formalize the agreements between the tiers. We now state this explicitly in the following definition:

Definition 1 (*Supply chain network equilibrium*). The equilibrium state of the supply chain is one where the product flows between the distinct tiers of the decision-makers coincide and the product flows and prices satisfy the sum of the optimality conditions (5) and (9) and conditions (14).

We now establish the following:

Theorem 1 (Variational inequality formulation). *The equilibrium conditions governing the supply chain model with competition are equivalent to the solution of the variational inequality problem given by: Determine* $(Q^{1^*}, Q^{2^*}, \gamma^*, \rho_3^*) \in \mathscr{K}$ *satisfying*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1^*})}{\partial q_{ij}}+\frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}}+\frac{\partial c_j(Q^{1^*})}{\partial q_{ij}}-\gamma_j^*\right]\times\left[q_{ij}-q_{ij}^*\right]$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[c_{jk}(Q^{2^*})+\gamma_j^*-\rho_{3k}^*\right]\times\left[q_{jk}-q_{jk}^*\right]+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}q_{ij}^*-\sum_{k=1}^{o}q_{jk}^*\right]\times\left[\gamma_j-\gamma_j^*\right]$$
$$+\sum_{k=1}^{o}\left[\sum_{j=1}^{n}q_{jk}^*-d_k(\rho_3^*)\right]\times\left[\rho_{3k}-\rho_{3k}^*\right]\geqslant 0 \quad \forall (Q^1,Q^2,\gamma,\rho_3)\in\mathscr{K}, \tag{15}$$

where $\mathscr{K} \equiv \{(Q^1, Q^2, \gamma, \rho_3) \,|\, (Q^1, Q^2, \gamma, \rho_3) \in R_+^{mn+no+n+o}\}$.

Proof. We first establish that the equilibrium conditions imply variational inequality (15). Indeed, the summation of (5), (9), and (14), yields, after algebraic simplification, inequality (15).

We now establish the converse, that is, that a solution to variational inequality (15) satisfies the sum of inequalities (5), (9), and (14), and is, hence, an equilibrium according to Definition 1. To inequality (15) add the term $-\rho_{1ij}^* + \rho_{1ij}^*$ to the term in the first set of brackets preceding the multiplication sign and add the term $-\rho_{2j}^* + \rho_{2j}^*$ to the term preceding the second multiplication sign. Such "terms" do not change the value of the inequality since they are identically equal to zero, with the resulting inequality of the form

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1^*})}{\partial q_{ij}}+\frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}}+\frac{\partial c_j(Q^{1^*})}{\partial q_{ij}}-\gamma_j^*-\rho_{1ij}^*+\rho_{1ij}^*\right]\times\left[q_{ij}-q_{ij}^*\right]$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[c_{jk}(Q^{2^*})+\gamma_j^*-\rho_{3k}^*-\rho_{2j}^*+\rho_{2j}^*\right]\times\left[q_{jk}-q_{jk}^*\right]+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}q_{ij}^*-\sum_{k=1}^{o}q_{jk}^*\right]\times\left[\gamma_j-\gamma_j^*\right]$$
$$+\sum_{k=1}^{o}\left[\sum_{j=1}^{n}q_{jk}^*-d_k(\rho_3^*)\right]\times\left[\rho_{3k}-\rho_{3k}^*\right]\geqslant 0 \quad \forall (Q^1,Q^2,\gamma,\rho_3)\in\mathscr{K}, \tag{16}$$

which, in turn, can be rewritten as

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1^*})}{\partial q_{ij}}+\frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}}-\rho_{1ij}^*\right]\times\left[q_{ij}-q_{ij}^*\right]\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial c_j(Q^{1^*})}{\partial q_{ij}}+\rho_{1ij}^*-\gamma_j^*\right]\times\left[q_{ij}-q_{ij}^*\right]$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[-\rho_{2j}^*+\gamma_j^*\right]\times\left[q_{jk}-q_{jk}^*\right]+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}q_{ij}^*-\sum_{k=1}^{o}q_{jk}^*\right]\times\left[\gamma_j-\gamma_j^*\right]$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\rho_{2j}^*+c_{jk}(Q^{2^*})-\rho_{3k}^*\right]\times\left[q_{jk}-q_{jk}^*\right]\sum_{k=1}^{o}\left[\sum_{j=1}^{n}q_{jk}^*-d_k(\rho_3^*)\right]\times\left[\rho_{3k}-\rho_{3k}^*\right]\geqslant 0$$
$$\forall (Q^1,Q^2,\gamma,\rho_3)\in\mathscr{K}. \tag{17}$$

But inequality (17) is equivalent to the price and shipment pattern satisfying the sum of (5), (9), and (14). The proof is completed. □

For easy reference in the subsequent sections, variational inequality problem (15) can be rewritten in standard variational inequality form (cf. Nagurney, 1999) as follows: Determine $X^* \in \mathscr{K}$ satisfying

$$\langle F(X^*), X - X^* \rangle \geqslant 0 \quad \forall X \in \mathscr{K}, \tag{18}$$

where $X \equiv (Q^1, Q^2, \gamma, \rho_3)$, $F(X) \equiv (F_{ij}, F_{jk}, F_j, F_k)_{i=1,\ldots,m,j=1,\ldots,n,k=1,\ldots,o}$, and the specific components of F are given by the functional terms preceding the multiplication signs in (15). The term $\langle \cdot, \cdot \rangle$ denotes the inner product in N-dimensional Euclidean space.

The variables in the variational inequality problem are: the product shipments from the manufacturers to the retailers, Q^1 (from which one can then recover also the production outputs through (3)), the product flows from the retailers to the demand markets, Q^2, the prices associated with handling the product by the retailers, γ, and the demand market prices, ρ_3. The solution of the variational inequality problem (15), in turn, is given by $(Q^{1^*}, Q^{2^*}, \gamma^*, \rho_3^*)$.

We now discuss how to recover the equilibrium manufacturers' prices, ρ_{1ij}^*, for all i, j, and the retailers' equilibrium prices, ρ_{2j}^*, for all j, from the solution of variational inequality (15). (In Section 4 we describe an algorithm for computing the solution.) Recall that, in the preceding discussions, we have noted that if $q_{jk}^* > 0$ for some k and j, then ρ_{2j}^* is precisely equal to γ_j^*, which can be obtained from the solution of (15). The prices ρ_{1ij}^*, in turn (cf. also (5)), can be obtained by finding a $q_{ij}^* > 0$, and then setting

$$\rho_{1ij}^* = \left[\frac{\partial f(Q^{1^*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} \right].$$

We now construct the supply chain network in equilibrium (cf. Fig. 1), using, as building blocks, the previously drawn networks in Figs. 2–4 corresponding, respectively, to the transactions of a typical manufacturer, a typical retailer, and the consumers at a typical demand market. First, however, we need to establish the result that, in equilibrium, the sum of the product shipments to each retailer is equal to the sum of the product shipments out. This means that each retailer, assuming profit-maximization, only purchases from the manufacturers the amount of the product that is actually consumed at the demand markets. In order to establish this result, we utilize variational inequality (15). Clearly, we know that, if $\gamma_j^* > 0$, then the "market clears" for that retailer, that is, $\sum_{i=1}^m q_{ij}^* = \sum_{k=1}^o q_{jk}^*$. Let us now consider the case where $\gamma_j^* = 0$ for some retailer j. From the first term in inequality (15), since the production cost functions, and the transaction cost functions and handling cost functions have been assumed to be convex and, assuming, which is not unreasonable, that either the marginal production cost or the marginal transaction cost or the marginal handling cost for each manufacturer/retailer pair is strictly positive at equilibrium, then we know that

$$\frac{\partial f_i(Q^{1^*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} + \frac{\partial c_j(Q^{1^*})}{\partial q_{ij}} > 0,$$

which implies that $q_{ij}^* = 0$, and this holds for all i, j. (Note that we could set $Q^2 = Q^{2^*}$, $\gamma = \gamma^*$, and $\rho_3 = \rho_3^*$ in (15) to observe this.) It follows then from the third term in (15) that $\sum_{k=1}^o q_{jk}^* = 0$, and, hence, the market clears also in this case since the flow into a retailer is equal to the flow out and equal to zero. We have thus established the following:

Corollary 1. *The market for the product clears for each retailer in the supply chain network equilibrium.*

In Fig. 1, we depict the structure of the supply chain network in equilibrium consisting of all the manufacturers, all the retailers, and all the demand markets. Hence, we replicate Fig. 2 for all manufacturers, Fig. 3, for all retailers, and Fig. 4 for all demand markets. These resulting networks represent the possible transactions of all the economic agents. In addition, since there must be agreement between/among the transactors at equilibrium, the analogous links (and equilibrium flows on them) must coincide, yielding the network structure given in Fig. 1.

The equilibrium product shipments between the manufacturers and the retailers are given by the components of the vector Q^{1^*} and flow on the links connecting the top tier of nodes with the middle tier of nodes in Fig. 1. The equilibrium product shipments between the retailers and the demand markets are given by the components of the vector Q^{2^*} and flow on the links connecting the middle tier of nodes with the bottom tier of nodes in Fig. 1. The equilibrium prices associated with the demand markets are associated with the bottom tier nodes in Fig. 1 and are given by the components of the vector ρ_3^*. The equilibrium prices associated with the middle tier of nodes in Fig. 1 corresponding to the retailers are given by the ρ_{2j}^*'s and γ_j^*'s. Finally, the equilibrium prices associated with the manufacturers at the top tier of nodes in Fig. 1 are given by the ρ_{1ij}^*'s for all i, j.

3. Qualitative properties

In this section, we provide some qualitative properties of the solution to variational inequality (15). In particular, we derive the existence and uniqueness results. We also investigate properties of the function F (cf. (18)).

Since the feasible set underlying the variational inequality problem (15) is not compact we cannot derive existence of a solution simply from the assumption of continuity of the functions. Nevertheless, we can impose a rather weak condition to guarantee the existence of a solution pattern. Let

$$\mathscr{K}_b = \{(Q^1, Q^2, \gamma, \rho_3) \,|\, 0 \leqslant Q^1 \leqslant b_1;\ 0 \leqslant Q^2 \leqslant b_2;\ 0 \leqslant \gamma \leqslant b_3;\ 0 \leqslant \rho_3 \leqslant b_4\}, \tag{19}$$

where $b = (b_1, b_2, b_3, b_4) \geqslant 0$ and $Q^1 \leqslant b_1; Q^2 \leqslant b_2; \gamma \leqslant b_3; \rho_3 \leqslant b_4$ means that $q_{ij} \leqslant b_1$, $q_{jk} \leqslant b_2$, $\gamma_j \leqslant b_3$, and $\rho_{3k} \leqslant b_4$ for all i, j, k. Then $\mathscr{K}_b$ is a bounded, closed convex subset of $R^{mn+no+n+o}$. Thus, the following variational inequality:

$$\langle F(X^b), X - X^b \rangle \geqslant 0 \quad \forall X^b \in \mathscr{K}_b \tag{20}$$

admits at least one solution $X^b \in \mathscr{K}_b$, from the standard theory of variational inequalities, since $\mathscr{K}_b$ is compact and F is continuous. Following Kinderlehrer and Stampacchia (1980) (see also Theorem 1.5 in Nagurney, 1999), we then have:

Lemma 1. *Variational inequality* (18) *admits a solution if and only if there exists a* $b > 0$ *such that variational inequality* (20) *admits a solution in* $\mathscr{K}_b$ *with*

$$Q^{1b} < b_1, \quad Q^{2b} < b_2, \quad \gamma^b < b_3, \quad \rho_3^b < b_4. \tag{21}$$

Under the conditions in Theorem 2 below it is possible to construct b_1, b_2, b_3, and b_4 large enough so that the restricted variational inequality (20) will satisfy the boundedness condition (21) and, thus, existence of a solution to the original variational inequality problem according to Lemma 1 will hold.

Theorem 2 (Existence). *Suppose that there exist positive constants* M, N, R *with* $R > 0$ *such that*:

$$\frac{\partial f_i(Q^1)}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij})}{\partial q_{ij}} + \frac{\partial c_j(Q^1)}{\partial q_{ij}} \geqslant M \quad \forall Q^1 \textit{ with } q_{ij} \geqslant N, \ \forall i, j, \tag{22}$$

$$\begin{aligned} &c_{jk}(Q^2) \geqslant M \quad \forall Q^2 \textit{ with } q_{jk} \geqslant N, \ \forall j, k, \\ &d_k(\rho_3) \leqslant N \quad \forall \rho_3 \textit{ with } \rho_{3k} > R, \ \forall j. \end{aligned} \tag{23}$$

Then variational inequality (15) *admits at least one solution.*

Proof. Follows from Lemma 1. See also the proof of existence for Proposition 1 in Nagurney and Zhao (1993) and the existence proof in Nagurney et al. (2001). □

Assumptions (22) and (23) are reasonable from an economics perspective, since when the product shipment between a manufacturer and retail pair is large, we can expect the marginal production cost plus the marginal transaction cost plus the marginal handling cost to exceed a positive lower bound. Also, when the product flow between a retail and demand market pair is high, we can expect the transaction cost associated with that pair to be nonnegative and to exceed a lower bound. Moreover, in the case where the demand market price at a demand market is high, we can expect that the demand for the product to be low at that demand market.

We now recall the definition of an additive production cost functions introduced in Zhang and Nagurney (1996) for establishing certain qualitative properties in dynamic network oligopoly problems. Such cost functions will be assumed here in order to obtain certain monotonicity properties of the function F in variational inequality (18).

Definition 2 (*Additive production cost*). Suppose that for each manufacturer i the production cost f_i is additive, that is,

$$f_i(q) = f_i^1(q_i) + f_i^2(\bar{q}_i), \tag{24}$$

where $f_i^1(q_i)$ is the internal production cost that depends solely on the manufacturer's own output level q_i, which may include the production operation and the facility maintenance, etc., and $f_i^2(\bar{q}_i)$ is the interdependent part of the production cost that is a function of all the other manufacturers' output levels $\bar{q}_i = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m)$ and reflects the impact of the other manufacturers' production patterns on manufacturer i's cost. This interdependent part of the production cost may describe the competition for the resources, consumption of the homogeneous raw materials, etc.

We now establish additional qualitative properties both of the function F that enters the variational inequality problem, as well as uniqueness of the equilibrium pattern. Monotonicity and Lipschitz continuity of F will be utilized in the subsequent section for proving convergence of the algorithmic scheme.

Lemma 2 (Monotonicity). *Suppose that the production cost functions f_i, $i = 1, \ldots, m$, are additive, as defined in Definition* 2, *and f_i^1, $i = 1, \ldots, m$, are convex functions. If the c_{ij} and c_j functions are convex, the c_{jk} functions are monotone increasing, and the d_k functions are monotone decreasing functions of the generalized prices, for all i, j, k, then the vector function F that enters the variational inequality* (18) *is monotone, that is,*

$$\langle F(X') - F(X''), X' - X'' \rangle \geqslant 0 \quad \forall X', X'' \in \mathcal{K}. \tag{25}$$

Proof. Let $X' = (Q^{1\prime}, Q^{2\prime}, \gamma', \rho_3')$, $X'' = (Q^{1\prime\prime}, Q^{2\prime\prime}, \gamma'', \rho_3'')$ with $X' \in \mathcal{K}$ and $X'' \in \mathcal{K}$. Then, inequality (25) can been seen in the following deduction:

$$\begin{aligned}
\langle F(X^1) - F(X^2), X^1 - X^2 \rangle &= \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial f_i(Q^{1\prime})}{\partial q_{ij}} - \frac{\partial f_i(Q^{1\prime\prime})}{\partial q_{ij}} \right] \times \left[q_{ij}' - q_{ij}'' \right] \\
&+ \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial c_j(Q^{1\prime})}{\partial q_{ij}} - \frac{\partial c_j(Q^{1\prime\prime})}{\partial q_{ij}} \right] \times \left[q_{ij}' - q_{ij}'' \right] \\
&+ \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial c_{ij}(q_{ij}')}{\partial q_{ij}} - \frac{\partial c_{ij}(q_{ij}'')}{\partial q_{ij}} \right] \times \left[q_{ij}' - q_{ij}'' \right] \\
&+ \sum_{j=1}^{n} \sum_{k=1}^{o} \left[c_{jk}(Q^{2\prime}) - c_{jk}(Q^{2\prime\prime}) \right] \times \left[q_{jk}' - q_{jk}'' \right] \\
&+ \sum_{k=1}^{o} \left[- d_k(\rho_3') + d_k(\rho_3'') \right] \times \left[\rho_{3k}' - \rho_{3k}'' \right] \\
&= (I) + (II) + (III) + (IV) + (V).
\end{aligned} \tag{26}$$

Since f_i, $i = 1, \ldots, m$, are additive, and f_i^1, $i = 1, \ldots, m$, are convex functions, one has

$$(I) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial f_i^1(Q^{1\prime})}{\partial q_{ij}} - \frac{\partial f_i^1(Q^{1\prime\prime})}{\partial q_{ij}} \right] \times \left[q_{ij}' - q_{ij}'' \right] \geqslant 0. \tag{27}$$

The convexity of c_j, for all j, and c_{ij}, for all i, j, gives, respectively,

$$(II) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial c_j(Q^{1\prime})}{\partial q_{ij}} - \frac{\partial c_j(Q^{1\prime\prime})}{\partial q_{ij}} \right] \times \left[q_{ij}' - q_{ij}'' \right] \geqslant 0 \tag{28}$$

and

$$(III) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial c_{ij}(q_{ij}')}{\partial q_{ij}} - \frac{\partial c_{ij}(q_{ij}'')}{\partial q_{ij}} \right] \times \left[q_{ij}' - q_{ij}'' \right] \geqslant 0. \tag{29}$$

Since c_{jk}, for all j, k, are assumed to be monotone increasing, and d_k, for all k, are assumed to be monotone decreasing, we have

$$(IV) = \sum_{j=1}^{n} \sum_{k=1}^{o} \left[c_{jk}(Q^{2\prime}) - c_{jk}(Q^{2\prime\prime})\right] \times \left[q'_{jk} - q''_{jk}\right] \geqslant 0 \tag{30}$$

and

$$(V) = \sum_{k=1}^{o} \left[- d_k(\rho'_3) + d_k(\rho''_3)\right] \times \left[\rho'_{3k} - \rho''_{3k}\right] \geqslant 0. \tag{31}$$

Bringing (27)–(31) into the right-hand side of (26), we conclude that (26) is nonnegative. The proof is completed. □

Lemma 3 (Strict monotonicity). *Assume all the conditions of Lemma* 2. *In addition, suppose that one of the three families of convex functions* f_i^1, $i = 1, \ldots, m, c_{ij}$, $i = 1, \ldots, n, j = 1, \ldots, n$, *and* c_j, $j = 1, \ldots, n$, *is a family of strictly convex functions. Suppose that* c_{jk}, $j = 1, \ldots, n, k = 1, \ldots, o$, *and* d_k, $k = 1, \ldots, o$, *are strictly monotone. Then, the vector function F that enters the variational inequality* (18) *is strictly monotone, with respect to* (Q^1, Q^2, ρ_3), *that is, for any two* $X', X'' \in \mathscr{K}$ *with* $(Q^{1\prime}, Q^{2\prime}, \rho'_3) \neq (Q^{1\prime\prime}, Q^{2\prime\prime}, \rho''_3)$:

$$\langle F(X') - F(X''), X' - X''\rangle > 0. \tag{32}$$

Proof. For any two distinct $(Q^{1\prime}, Q^{2\prime}, \rho'_3)$, $(Q^{1\prime\prime}, Q^{2\prime\prime}, \rho''_3)$, we must have at least one of the following three cases:

(i) $Q^{1\prime} \neq Q^{1\prime\prime}$,
(ii) $Q^{2\prime} \neq Q^{2\prime\prime}$,
(iii) $\rho'_3 \neq \rho''_3$.

Under the condition of the theorem, if (i) holds true, then, at the right-hand side of (26), at least one of (I), (II) and (II) is positive. If (ii) is true, then (IV) is positive. In case of (iii), (V) is positive. Hence, we can conclude that the right-hand side of (26) is greater than zero. The proof is completed. □

Lemma 3 has an important implication for the uniqueness of product shipments, Q^1, the retailer shipments, Q^2, and the prices at the demand markets, ρ_3, at the equilibrium. We note also that no guarantee of a unique γ_j, $j = 1, \ldots, n$, can be generally expected at the equilibrium.

Theorem 3 (Uniqueness). *Under the conditions of Lemma* 3, *there is a unique product shipment pattern* Q^{1*}, *a unique retail shipment* (*consumption*) *pattern* Q^{2*}, *and a unique demand price vector* ρ_3^* *satisfying the equilibrium conditions of the supply chain. In other words, if the variational inequality* (18) *admits a solution, that should be the only solution in* Q^1, Q^2, *and* ρ_3.

Proof. Under the strict monotonicity result of Lemma 3, uniqueness follows from the standard variational inequality theory (cf. e.g., Kinderlehrer and Stampacchia, 1980). □

Lemma 4 (Lipschitz continuity). *The function that enters the variational inequality problem* (18) *is Lipschitz continuous, that is,*

$$\|F(X') - F(X'')\| \leqslant L\|X' - X''\| \quad \forall X', X'' \in \mathcal{K}, \tag{33}$$

under the following conditions:
(i) *Each* f_i, $i = 1, \ldots, m$, *is additive and has a bounded second-order derivative*;
(ii) c_{ij} *and* c_j *have bounded second-order derivatives for all* i, j;
(iii) c_{jk} *and* d_k *have bounded first-order derivatives for all* j, k.

Proof. The result is direct by applying a mid-value theorem from calculus to the vector function F that enters the variational inequality problem (18). □

4. The algorithm

In this section, an algorithm is presented which can be applied to solve any variational inequality problem in standard form (see (18)). The algorithm is the modified projection method of Korpelevich (1977) and is guaranteed to converge provided that the function F that enters the variational inequality is monotone and Lipschitz continuous (and that a solution exists). The realization of the algorithm (for further details see also Nagurney, 1999) for the supply chain network model is as follows, where $\mathcal{T}$ denotes an iteration counter:

Modified projection method for the solution of variational inequality (15)

Step 0. Initialization

Set $(Q^{10}, Q^{20}, \gamma^0, \rho_3^0) \in \mathcal{K}$. Let $\mathcal{T} = 1$ and set α such that $0 < \alpha \leqslant 1/L$, where L is the Lipschitz constant (cf. (33)) for the problem.

Step 1. Computation

Compute $(\bar{Q}^{1^{\mathcal{T}}}, \bar{Q}^{2^{\mathcal{T}}}, \bar{\gamma}^{\mathcal{T}}, \bar{\rho}_3^{\mathcal{T}}) \in \mathcal{K}$ by solving the variational inequality

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\bar{q}_{ij}^{\mathcal{T}} + \alpha\left(\frac{\partial f_i(Q^{1^{\mathcal{T}-1}})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^{\mathcal{T}-1})}{\partial q_{ij}} + \frac{\partial c_j(Q^{1^{\mathcal{T}-1}})}{\partial q_{ij}} - \gamma_j^{\mathcal{T}-1}\right) - q_{ij}^{\mathcal{T}-1}\right] \times \left[q_{ij} - \bar{q}_{ij}^{\mathcal{T}}\right]$$
$$+ \sum_{j=1}^{n}\sum_{k=1}^{o}\left[\bar{q}_{jk}^{\mathcal{T}} + \alpha\left(c_{jk}(Q^{2^{\mathcal{T}-1}}) + \gamma_j^{\mathcal{T}-1} - \rho_{3k}^{\mathcal{T}-1}\right) - q_{jk}^{\mathcal{T}-1}\right] \times \left[q_{jk} - \bar{q}_{jk}^{\mathcal{T}}\right]$$
$$+ \sum_{j=1}^{n}\left[\bar{\gamma}_j^{\mathcal{T}} + \alpha\left(\sum_{i=1}^{m} q_{ij}^{\mathcal{T}-1} - \sum_{k=1}^{o} q_{jk}^{\mathcal{T}-1}\right) - \gamma_j^{\mathcal{T}-1}\right] \times \left[\gamma_j - \bar{\gamma}_j^{\mathcal{T}}\right]$$
$$+ \sum_{k=1}^{o}\left[\bar{\rho}_{3k}^{\mathcal{T}} + \alpha\left(\sum_{j=1}^{n} q_{jk}^{\mathcal{T}-1} - d_k\left(\rho_3^{\mathcal{T}-1}\right)\right) - \rho_{3k}^{\mathcal{T}-1}\right] \times \left[\rho_{3k} - \bar{\rho}_{3k}^{\mathcal{T}}\right] \geqslant 0 \quad \forall (Q^1, Q^2, \gamma, \rho_3) \in \mathcal{K}. \tag{34}$$

Step 2. Adaptation

Compute $(Q^{1^{\mathscr{T}}}, Q^{2^{\mathscr{T}}}, \gamma^{\mathscr{T}}, \rho_3^{\mathscr{T}}) \in \mathscr{K}$ by solving the variational inequality

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[q_{ij}^{\mathscr{T}} + \alpha\left(\frac{\partial f_i(\bar{Q}^{1^{\mathscr{T}}})}{\partial q_{ij}} + \frac{\partial c_{ij}(\bar{q}_{ij}^{\mathscr{T}})}{\partial q_{ij}} + \frac{\partial c_j(\bar{Q}^{1^{\mathscr{T}}})}{\partial q_{ij}} - \bar{\gamma}_j^{\mathscr{T}}\right) - q_{ij}^{\mathscr{T}-1}\right] \times \left[q_{ij} - q_{ij}^{\mathscr{T}}\right]$$
$$+ \sum_{j=1}^{n}\sum_{k=1}^{o}\left[q_{jk}^{\mathscr{T}} + \alpha\left(c_{jk}(\bar{Q}^{2^{\mathscr{T}}}) + \bar{\gamma}_j^{\mathscr{T}} - \bar{\rho}_{3k}^{\mathscr{T}}\right) - q_{jk}^{\mathscr{T}-1}\right] \times \left[q_{jk} - q_{jk}^{\mathscr{T}}\right]$$
$$+ \sum_{j=1}^{n}\left[\gamma_j^{\mathscr{T}} + \alpha\left(\sum_{i=1}^{m} q_{ij}^{\mathscr{T}-1} - \sum_{k=1}^{o} q_{jk}^{\mathscr{T}}\right) - \gamma_j^{\mathscr{T}-1}\right] \times \left[\gamma_j - \gamma_j^{\mathscr{T}}\right]$$
$$+ \sum_{k=1}^{o}\left[\rho_{3k}^{\mathscr{T}} + \alpha\left(\sum_{j=1}^{n} \bar{q}_{jk}^{\mathscr{T}} - d_k(\bar{\rho}_3^{\mathscr{T}})\right) - \rho_{3k}^{\mathscr{T}-1}\right] \times \left[\rho_{3k} - \rho_{3k}^{\mathscr{T}}\right] \geqslant 0 \quad \forall (Q^1, Q^2, \gamma, \rho_3) \in \mathscr{K}. \tag{35}$$

Step 3. Convergence verification

If $|q_{ij}^{\mathscr{T}} - q_{ij}^{\mathscr{T}-1}| \leqslant \epsilon$, $|q_{jk}^{\mathscr{T}} - q_{jk}^{\mathscr{T}-1}| \leqslant \epsilon$, $|\gamma_j^{\mathscr{T}} - \gamma_j^{\mathscr{T}-1}| \leqslant \epsilon$, $|\rho_{3k}^{\mathscr{T}} - \rho_{3k}^{\mathscr{T}-1}| \leqslant \epsilon$ for all $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, o$, with $\epsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $\mathscr{T} := \mathscr{T} + 1$, and go to Step 1.

Note that the variational inequality subproblems (34) and (35) can be solved explicitly and in closed form since the feasible set is that of the nonnegative orthant. Indeed, they yield subproblems in the q_{ij}, q_{jk}, γ_j and ρ_{3k} variables for all i, j, k.

We now state the convergence result for the modified projection method for this model.

Theorem 4 (Convergence). *Assume that the function that enters the variational inequality* (15) (*or* (18)) *satisfies the conditions in Theorem* 2 *and Lemmas* 2 *and* 4. *Then the modified projection method described above converges to the solution of the variational inequality* (15) *or* (18).

Proof. According to Korpelevich (1977), the modified projection method converges to the solution of the variational inequality problem of the form (18), provided that the function F that enters the variational inequality is monotone and Lipschitz continuous and that a solution exists. Existence of a solution follows from Theorem 2. Monotonicity follows Lemma 2. Lipschitz continuity, in turn, follows from Lemma 4. The proof is completed. □

5. Numerical examples and discussion

In this section, we apply the modified projection method to four numerical examples and also provide a discussion of the results. Section 5.1 describes the numerical examples and their solutions whereas Section 5.2 discusses more fully the model in the context of the examples solved.

5.1. Numerical examples

The algorithm was implemented in FORTRAN and the computer system used was a DEC Alpha system located at the University of Massachusetts at Amherst. The convergence criterion used was that the absolute value of the flows and prices between two successive iterations differed by no more than 10^{-4}.

Example 1. The first numerical example, depicted in Fig. 5, consisted of two manufacturers, two retailers, and two demand markets.

The data for this example were constructed for easy interpretation purposes. The production cost functions for the manufacturers were given by

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 2q_2.$$

The transaction cost functions faced by the manufacturers and associated with transacting with the retailers were given by

$$c_{11}(q_{11}) = .5q_{11}^2 + 3.5q_{11}, \quad c_{12}(q_{12}) = .5q_{12}^2 + 3.5q_{12}, \quad c_{21}(q_{21}) = .5q_{21}^2 + 3.5q_{21},$$
$$c_{22}(q_{22}) = .5q_{22}^2 + 3.5q_{22}.$$

The handling costs of the retailers, in turn, were given by

$$c_1(Q^1) = .5\left(\sum_{i=1}^{2} q_{i1}\right)^2, \quad c_2(Q^1) = .5\left(\sum_{i=1}^{2} q_{i2}\right)^2.$$

The demand functions at the demand markets were

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000, \quad d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000,$$

and the transaction costs between the retailers and the consumers at the demand markets were given by

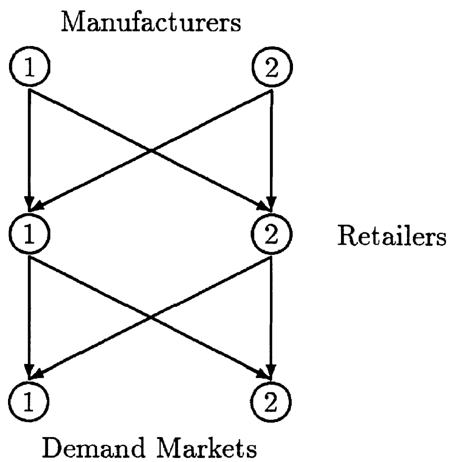

Fig. 5. Supply chain network for numerical examples 1 and 2.

$$c_{11}(Q^2) = q_{11} + 5, \quad c_{12}(Q^2) = q_{12} + 5, \quad c_{21}(Q^2) = q_{21} + 5, \quad c_{22}(Q^2) = q_{22} + 5.$$

The parameter α in the modified projection method was set to .05 for both Examples 1 and 2. The modified projection method converged in 257 iterations and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two retailers were Q^{1^*}: $q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 16.608$, the product shipments (consumption volumes) between the two retailers and the two demand markets were Q^{2^*}: $q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 16.608$, the vector γ^*, which was equal to the prices charged by the retailers ρ_2^*, had components $\gamma_1^* = \gamma_2^* = 254.617$, and the demand prices at the demand markets were $\rho_{31}^* = \rho_{32}^* = 276.224$.

It is easy to verify that the optimality/equilibrium conditions were satisfied with good accuracy.

Example 2. We then modified Example 1 as follows: The production cost function for manufacturer 1 was now given by

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 12q_1,$$

whereas the transaction costs for manufacturer 1 were now given by

$$c_{11}(q_{11}) = q_{11}^2 + 3.5q_{11}, \quad c_{12}(q_{12}) = q_{12}^2 + 3.5q_{12}.$$

The remainder of the data was as in Example 1. Hence, both the production costs and the transaction costs increased for manufacturer 1.

The modified projection method converged in 258 iterations and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two retailers were now Q^{1^*}: $q_{11}^* = q_{12}^* = 14.507$, $q_{21}^* = q_{22}^* = 17.230$, the product shipments (consumption amounts) between the two retailers and the two demand markets were now Q^{2^*}: $q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 15.869$, the vector γ^* was now equal to $\gamma_1^* = \gamma_2^* = 255.780$, and the demand prices at the demand markets were $\rho_{31}^* = \rho_{32}^* = 276.646$.

Hence, manufacturer 1 now produced less than it did in Example 1, whereas manufacturer 2 increased his production output. The price charged by the retailers to the consumers increased, as did the demand price at the demand markets, with a decrease in the incurred demand.

Example 3. The third supply chain network problem consisted of two manufacturers, three retailers, and two demand markets, as depicted in Fig. 6.

The data were constructed from Example 2, but we added data for the manufacturers' transaction costs associated with the third retailer; handling cost data for the third retailer, as well as the transaction cost data between the new retailer and the demand markets. Hence, the complete data for this example were given by:

The production cost functions for the manufacturers were given by

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 12q_2.$$

The transaction cost functions faced by the two manufacturers and associated with transacting with the three retailers were given by

$$c_{11}(q_{11}) = q_{11}^2 + 3.5q_{11}, \quad c_{12}(q_{12}) = q_{12}^2 + 3.5q_{12}, \quad c_{13}(q_{13}) = .5q_{13}^2 + 5q_{13},$$
$$c_{21}(q_{21}) = .5q_{21}^2 + 3.5q_{21}, \ c_{22}(q_{22}) = .5q_{22}^2 + 3.5q_{22}, \ c_{23}(q_{23}) = .5q_{23}^2 + 5q_{23}.$$

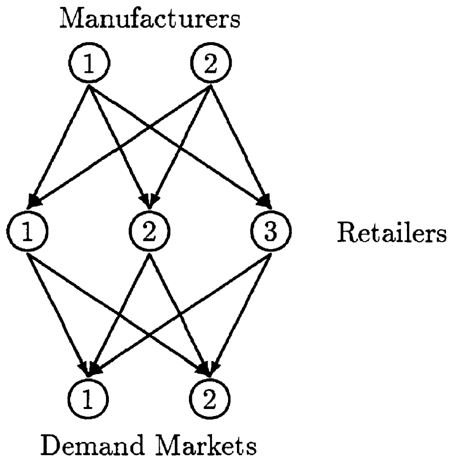

Fig. 6. Supply chain network for Example 3.

The handling costs of the retailers, in turn, were given by

$$c_1(Q^1) = .5\left(\sum_{i=1}^{2} q_{i1}\right)^2, \quad c_2(Q^1) = .5\left(\sum_{i=1}^{2} q_{i2}\right)^2, \quad c_3(Q^1) = .5\left(\sum_{i=1}^{2} q_{i3}\right)^2.$$

The demand functions at the demand markets, again, were

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000, \quad d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000,$$

and the transaction costs between the retailers and the consumers at the demand markets were given by

$$c_{11}(Q^2) = q_{11} + 5, \quad c_{12}(Q^2) = q_{12} + 5,$$
$$c_{21}(Q^2) = q_{21} + 5, \quad c_{22}(Q^2) = q_{22} + 5,$$
$$c_{31}(Q^2) = q_{31} + 5, \quad c_{32}(Q^2) = q_{32} + 5.$$

The α parameter in the modified projection method was now set to .03. The modified projection method converged in 361 iterations and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the three retailers were Q^{1^*}: $q^*_{11} = q^*_{12} = 9.243$, $q^*_{13} = 14.645$, $q^*_{21} = q^*_{22} = 13.567$, $q^*_{23} = 9.726$, the product shipments between the three retailers and the two demand markets were Q^{2^*}: $q^*_{11} = q^*_{12} = q^*_{21} = q^*_{22} = 11.404$, $q^*_{31} = q^*_{32} = 12.184$. The vector γ^* had components $\gamma^*_1 = \gamma^*_2 = 259.310$, $\gamma^*_3 = 258.530$, and the demand prices at the demand markets were $\rho^*_{31} = \rho^*_{32} = 275.717$.

Note that the demand prices at the demand markets were now lower than in Example 2, since there is now an additional retailer and, hence, increased competition. The incurred demand also increased at both demand markets, as did the production outputs of both manufacturers. Since the retailers now handled a greater volume of product flows, the prices charged for the product at the retail outlets, nevertheless, increased due to increased handling cost.

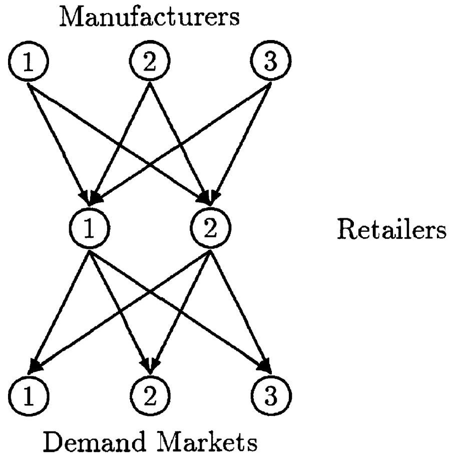

Fig. 7. Supply chain network for numerical example 4.

Example 4. The fourth numerical example consisted of three manufacturers, two retailers, and three demand markets. Hence, the supply chain network, in equilibrium, was as depicted in Fig. 7.

The data for this example were constructed from the data for Example 1, but we added the necessary functions for the third manufacturer and the third demand market resulting in the following functions:

The production cost functions for the manufacturers were given by

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 2q_2, \quad f_3(q) = .5q_3^2 + .5q_1q_3 + 2q_3.$$

The transaction cost functions faced by the manufacturers and associated with transacting with the retailers were given by

$$c_{11}(q_{11}) = .5q_{11}^2 + 3.5q_{11}, \quad c_{12}(q_{12}) = .5q_{12}^2 + 3.5q_{12}, \quad c_{21}(q_{21}) = .5q_{21}^2 + 3.5q_{21},$$
$$c_{22}(q_{22}) = .5q_{22}^2 + 3.5q_{22}, \quad c_{31}(q_{31}) = .5q_{31}^2 + 2q_{31}, \quad c_{32}(q_{32}) = .5q_{32}^2 + 2q_{32}.$$

The handling costs of the retailers, in turn, were given by

$$c_1(Q^1) = .5\left(\sum_{i=1}^{2} q_{i1}\right)^2, \quad c_2(Q^1) = .5\left(\sum_{i=1}^{2} q_{i2}\right)^2.$$

The demand functions at the demand markets were

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000, \quad d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000,$$
$$d_3(\rho_3) = -2\rho_{33} - 1.5\rho_{31} + 1000,$$

and the transaction costs between the retailers and the consumers at the demand markets were given by

$$c_{11}(Q^2) = q_{11} + 5, \quad c_{12}(Q^2) = q_{12} + 5, \quad c_{13}(Q^2) = q_{13} + 5,$$
$$c_{21}(Q^2) = q_{21} + 5, \quad c_{22}(Q^2) = q_{22} + 5, \quad c_{23}(Q^2) = q_{23} + 5.$$

The parameter α in the modified projection method was set to .05 for this example. The modified projection method converged in 230 iterations and yielded the following equilibrium pattern: the product shipments between the three manufacturers and the two retailers were Q^{1^*}: $q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 12.395$, $q_{31}^* = q_{32}^* = 50.078$. The product shipments (consumption levels) between the two retailers and the three demand markets were computed as Q^{2^*}: $q_{11}^* = q_{12}^* = q_{13}^* = q_{21}^* = q_{22}^* = q_{23}^* = 24.956$, whereas the vector γ^* was now equal to $\gamma_1^* = \gamma_2^* = 241.496$, and the demand prices at the three demand markets were $\rho_{31}^* = \rho_{32}^* = \rho_{33}^* = 271.454$.

Note that, in comparison to the results in Example 1, with the addition of a new manufacturer, the price charged at the retailer outlets was now lower due to the competition, and the increased supply of the product. The consumers at the three demand markets benefited, as well, with a decrease in the demand market prices and an increased demand.

5.2. Discussion

The preceding examples demonstrate the type of supply chain network problems that can be solved using the modified projection method given in Section 4. We note that these examples had nonlinear production costs associated with the manufacturers, nonlinear handling costs associated with the retailers, and nonlinear transaction costs between the manufacturers and the retailers. Moreover, the demand functions at the demand markets were not separable as was the case in the oligopolistic supply chain problems studied in a game theoretic framework by Corbett and Karmarkar (2001). Furthermore, we established convergence of the modified projection method in Theorem 4. That theorem characterizes the types of functions in the supply chain network model which guarantee convergence of the computational method to the equilibrium price and product flow pattern. Of course, the algorithm may, nevertheless, converge for supply chain network problems in which the function F entering the variational inequality problem (18) is no longer monotone and Lipschitz continuous and, if it converges, it converges to the equilibrium pattern.

6. Conclusion

This paper has developed an equilibrium model of competitive supply chain networks. Prices associated with manufacturers, retailers, and consumers are endogenous, as are the product shipment and consumption flows. An equilibrium framework provides a benchmark against which existing product shipments and prices can be compared. Moreover, it provides the foundation for the development of dynamic supply chain network models and their evolution.

Qualitative properties of the equilibrium pattern were established, notably, the existence of a solution, as well as uniqueness, under reasonable assumptions on the underlying functions. The modified projection method was proposed for the computation of the equilibrium prices and product shipments. Several illustrative supply chain network examples were considered in the computations.

This work demonstrates both theoretically and empirically that solutions to supply chain network equilibrium problems with nonlinear and nonseparable functions can be computed using the modified projection method. The convergence of the method is guaranteed under reasonable assumptions underlying the various production, handling, and transaction costs, and demand functions, which were given in this paper.

For future research, the model should be adapted to include distribution centers and raw material suppliers. The authors plan to extend the model to include the disequilibrium dynamics associated with supply chains.

Acknowledgements

The research of the first and second authors was supported, in part, by NSF Grant No. IIS-0002647. The research of the first author was also supported, in part, by NSF Grant No. CMS-0085720 and by NSF Grant No.: INT-0000309. The support is gratefully acknowledged. The first author is grateful to Debra O'Connor for the helpful discussions and references and to Jon Loo for the stimulating discussions. The authors are grateful to the editor and to two anonymous reviewers for many helpful comments and detailed suggestions.

References

Anupindi, R., Bassok, Y., 1996. Distribution channels, information systems and virtual centralization. In: Proceedings of the Manufacturing and Service Operations Management Society Conference, pp. 87–92.

Arrow, K.J., Intrilligator, M.D. (Eds.), 1982. Handbook of Mathematical Economics. Elsevier, New York.

Bazaraa, M.S., Sherali, H.D., Shetty, C.M., 1993. Nonlinear Programming: Theory and Algorithms. Wiley, New York.

Bertsekas, D.P., Tsitsiklis, N., 1989. Parallel and Distributed Computation—Numerical Methods. Prentice-Hall, Englewood Cliffs, NJ.

Bovet, D., 2000. Value Nets: Breaking the Supply Chain to Unlock Hidden Profits. Wiley, New York.

Bramel, J., Simchi-Levi, D., 1997. The Logic of Logistics: Theory, Algorithms and Applications for Logistics Management. Springer, New York.

Corbett, C.J., Karmarkar, U.S., 2001. Competition and structure serial supply chains with deterministic demand. Management Science 47, 966–978.

Cournot, A.A., 1838. Researches into the Mathematical Principles of the Theory of Wealth. MacMillan, UK (English translation).

Dafermos, S., Nagurney, A., 1987. Oligopolistic and competitive behavior of spatially separated markets. Regional Science and Urban Economics 17, 245–254.

Erengüç, S.S., Simpson, N.C., Vakharia, A.J., 1999. Integrated production/distribution planning in supply chains: An invited review. European Journal of Operations Research 115, 219–236.

Federgruen, A., 1993. Centralized planning models for multi-echelon inventory systems under uncertainty. In: Graves, S.C., Rinooy Kan, A.H.G., Zipkin, P. (Eds.), Handbooks in Operations Research and Management Science: Volume on Logistics of Production and Inventory. Elsevier, Amsterdam, pp. 133–173.

Federgruen, A., Zipkin, P., 1986. An inventory model with limited production capacity and uncertain demands I: The average cost criterion. Mathematics of Operations Research 11, 193–207.

Florian, M., Hearn, D., 1995. Network equilibrium models and algorithms. In: Ball, M.O., Magnanti, T.L., Monma, C.L., Nemhauser, G.L (Eds.), Network Routing. Handbooks in Operations Research and Management Science, vol. 8. Elsevier, Amsterdam, pp. 485–550.

Gabay, D., Moulin, H., 1980. On the uniqueness and stability of Nash equilibria in noncooperative games. In: Bensoussan, A., Kleindorfer, P., Tapiero, C.S. (Eds.), Applied Stochastic Control of Econometrics and Management Science. North-Holland, Amsterdam.

Hensher, D., Button, K., Brewer, S. (Eds.), 2001. Handbook of Logistics and Supply Chain Management. Elsevier, Oxford, UK.
Kinderlehrer, D., Stampacchia, G., 1980. An Introduction to Variational Inequalities and their Application. Academic Press, New York.
Korpelevich, G.M., 1977. The extragradient method for finding saddle points and other problems. Matekon 13, 35–49.
Lederer, P.J., Li, L., 1997. Pricing, production, scheduling, and delivery-time competition. Operations Research 4, 407–420.
Lee, L., Billington, C., 1993. Material management in decentralized supply chains. Operations Research 41, 835–847.
Mentzer, J.T. (Ed.), 2000. Supply Chain Management. Sage, Thousand Oaks, CA.
Miller, T.C., 2001. Hierarchical Operations and Supply Chain Planning. Springer, London, UK.
Nagurney, A., 1999. Network Economics: A Variational Inequality Approach, second and revised ed. Kluwer Academic Publishers, Dordrecht.
Nagurney, A., Dong, J., Zhang, D., 2001. Multicriteria spatial price networks: Statics and dynamics. In: Daniele, P., Maugeri, A., Giannessi, F. (Eds.), Equilibrium Problems and Variational Models. Kluwer Academic Publishers, Dordrecht (to appear).
Nagurney, A., Siokos, S., 1997. Financial Networks: Statics and Dynamics. Springer, Heidelberg.
Nagurney, A., Zhao, L., 1993. Networks and variational inequalities in the formulation and computation of market disequilibria: The case of direct demand functions. Transportation Science 27, 4–15.
Nash, J.F., 1950. Equilibrium points in *n*-person games. In: Proceedings of the National Academy of Sciences, USA, vol. 36, pp. 48–49.
Nash, J.F., 1951. Noncooperative games. Annals of Mathematics 54, 286–298.
Poirier, C.C., 1996. Supply Chain Optimization: Building a Total Business Network. Berrett-Kochler Publishers, San Francisco, CA.
Poirier, C.C., 1999. Advanced Supply Chain Management: How to Build a Sustained Competitive Advantage. Berrett-Kochler Publishers, San Francisco, CA.
Samuelson, P., 1952. Spatial price equilibrium and linear programming. American Economic Review 42, 293–303.
Slats, P.A., Bhola, B., Evers, J.J., Dijkhuizen, G., 1995. Logistic chain modelling. European Journal of Operations Research 87, 1–20.
Stadtler, H., Kilger, C. (Eds.), 2000. Supply Chain Management and Advanced Planning. Springer, Berlin.
Takayama, T., Judge, G.G., 1971. Spatial and Temporal Price and Allocation Models. North-Holland, Amsterdam.
Zhang, D., Nagurney, A., 1996. Stability analysis of an adjustment process for oligopolistic market equilibrium modeled as a projected dynamical system. Optimization 36, 263–285.