Strictly as per Revised Syllabus of

ANNA UNIVERSITY

Choice Based Credit System(CBCS)-2017

Semester - V (EEE)

DIGITAL SIGNAL PROCESSING

Dr. J. S. Chitode

M.E. (Electronics), Ph.D.
Formerly Professor & Head,
Department of Electronics Engineering
Bharati Vidyapeeth Deemed University
College of Engineering, Pune

Price : ₹ 495/-

ISBN 978-93-332-0210-7

TECHNICAL PUBLICATIONS

An Up-Thrust for Knowledge

DIGITAL SIGNAL PROCESSING

Semester - V (Electrical and Electronics Engineering)

First Edition : June 2019

Published by :

TECHNICAL PUBLICATIONS

Amit Residency, Office No.1, 412, Shaniwar Peth, Pune - 411030, M.S. INDIA
Ph.: +91-020-24495496/97, Telefax : +91-020-24495497
Email : sales@technicalpubilcations.org Website : www.technicalpubilcations.org

Printer :

Yogiraj Printers & Binders
Sr.No. 10\1A,
Ghule Industrial Estate, Nanded Village Road,
Tal-Haveli, Dist-Pune - 411041.

Price : ₹ 495/-

ISBN 978-93-332-0210-7

9 789333 202107

AU 17

PREFACE

The importance of **Digital Signal Processing** is well known in various engineering fields. Overwhelming response to my books on various subjects inspired me to write this book. The book is structured to cover the key aspects of the subject **Digital Signal Processing.**

The book uses plain, lucid language to explain fundamentals of this subject. The book provides logical method of explaining various complicated concepts and stepwise methods to explain the important topics. Each chapter is well supported with necessary illustrations, practical examples and solved problems. All chapters in this book are arranged in a proper sequence that permits each topic to build upon earlier studies. All care has been taken to make students comfortable in understanding the basic concepts of this subject.

Representative questions have been added at the end of each section to help the students in picking important points from that section.

The book not only covers the entire scope of the subject but explains the philosophy of the subject. This makes the understanding of this subject more clear and makes it more interesting. The book will be very useful not only to the students but also to the subject teachers. The students have to omit nothing and possibly have to cover nothing more.

I wish to express my profound thanks to all those who helped in making this book a reality. Much needed moral support and encouragement is provided on numerous occasions by my whole family. I wish to thank the **Publisher** and the entire team of **Technical Publications** who have taken immense pain to get this book in time with quality printing.

Any suggestion for the improvement of the book will be acknowledged and well appreciated.

Author
Dr. J. S. Chitode

Dedicated at the Lotus Feet of Lord Krishna

SYLLABUS

Digital Signal Processing [EE8591]

Unit - I Introduction (Chapter - 1)

Classification of systems : Continuous, Discrete, Linear, Causal, Stable, Dynamic, Recursive, Time variance; Classification of signals : Continuous and discrete, Energy and power; Mathematical representation of signals; Spectral density; Sampling techniques, Quantization, Quantization error, Nyquist rate, Aliasing effect.

Unit - II Discrete Time System Analysis (Chapter - 2)

z-transform and its properties, Inverse z-transforms, Difference equation - Solution by z-transform, Application to discrete systems - Stability analysis, Frequency response - Convolution - Discrete time Fourier transform, Magnitude and phase representation.

Unit - III Discrete Fourier Transform and Computation (Chapter - 3)

Discrete Fourier Transform - properties, Magnitude and phase representation - Computation of DFT using FFT algorithm - DIT and DIF using radix 2 FFT - Butterfly structure.

Unit - IV Design of Digital Filters (Chapter - 4)

FIR and IIR filter realization - Parallel and cascade forms. FIR design : Windowing techniques - Need and choice of windows - Linear phase characteristics. Analog filter design - Butterworth and Chebyshev approximations; IIR filters, Digital design using impulse invariant and bilinear transformation - Warping, Prewarping.

Unit - V Digital Signal Processors (Chapter - 5)

Introduction - Architecture - Features - Addressing formats - Functional modes - Introduction to commercial DS Processors.

Table of Contents

Chapter - 1 Introduction (1 - 1) to (1 - 70)

1.1 Introduction to Digital Signal Processing .... 1 - 2

1.1.1 Basic Elements of Digital Signal Processing .... 1 - 2

1.1.2 Advantages of DSP over Analog Signal Processing .... 1 - 2

1.1.3 Disadvantages of Digital Signal Processing Systems .... 1 - 4

1.1.4 DSP Applications .... 1 - 4

1.2 Classification of Signals .... 1 - 5

1.2.1 CT and DT Signals .... 1 - 6

1.2.2 Periodic and Non-Periodic Signals .... 1 - 7

1.2.3 Even and Odd Signals .... 1 - 9

1.2.4 Energy and Power Signals .... 1 - 11

1.2.5 Deterministic and Random Signals .... 1 - 13

1.2.6 Multichannel and Multidimensional Signals .... 1 - 14

1.2.7 Sesmic Signals .... 1 - 14

1.3 Elementary Signals and Operations on them .... 1 - 21

1.3.1 Unit Step Function .... 1 - 21

1.3.2 Unit Impulse or Delta Function .... 1 - 22

1.3.3 Unit Ramp Function .... 1 - 23

1.3.4 Complex Exponential and Sinusoidal Signals .... 1 - 24

1.3.5 Time Delay / Advancing of Signals .... 1 - 25

1.3.6 Time Folding of Signals .... 1 - 25

1.4 Systems .... 1 - 26

1.4.1 Static and Dynamic Systems (Systems with Memory or without Memory) . . . 1 - 26
1.4.2 Time Invariant and Time Variant Systems . . . 1 - 27
1.4.3 Linear and Non-linear Systems . . . 1 - 30
1.4.4 Causal and Non-causal Systems . . . 1 - 33
1.4.5 Stable and Unstable Systems . . . 1 - 34
1.5 Nyquist-Shannon Sampling Theorem . . . 1 - 52
1.5.1 Sampling, Quantization and Quantization Error . . . 1 - 52
1.5.2 Sampling Theorem for Low Pass (LP) Signals . . . 1 - 53
1.5.3 Effects of Undersampling (Aliasing) . . . 1 - 58
1.5.4 Nyquist Rate and Nyquist Interval . . . 1 - 60
1.5.5 Reconstruction Filter (Interpolation Filter) . . . 1 - 61
1.5.6 Antialiasing Filter . . . 1 - 62
1.6 Short Answered Questions [2 Marks Each] . . . 1 - 64

Chapter - 2 Discrete Time System Analysis (2 - 1) to (2 - 132)

2.1 Introduction . . . 2 - 2
2.2 z-Transform . . . 2 - 2
2.2.1 Definition of z-Transform . . . 2 - 2
2.2.2 Types of z-Transform : Unilateral and Bilateral . . . 2 - 2
2.2.3 Region of Convergence (ROC) . . . 2 - 2
2.3 Properties of the ROC . . . 2 - 9
2.4 Properties of z-Transform . . . 2 - 10
2.4.1 Linearity . . . 2 - 10
2.4.2 Time Shifting or Translation . . . 2 - 11
2.4.3 Scaling in z-Domain or Multiplication by Exponential . . . 2 - 11
2.4.4 Time Reversal . . . 2 - 12
2.4.5 Differentiation in z-Domain or Multiplication by a Ramp . . . 2 - 12
2.4.6 Convolution in Time Domain . . . 2 - 13

2.4.7 Correlation of Two Sequences . . . . 2 - 13
2.4.8 Multiplication of Two Sequences or Convolution in z-Domain . . . . 2 - 14
2.4.9 Conjugation of a Complex Sequence . . . . 2 - 15
2.4.10 z-Transform of Real Part of a Sequence . . . . 2 - 15
2.4.11 z-Transform of Imaginary Part of Sequence . . . . 2 - 15
2.4.12 Parseval's Relation . . . . 2 - 16
2.4.13 Initial Value Theorem . . . . 2 - 17
2.5 Inverse z-Transform .... 2 - 26
2.5.1 Inverse z-Transform using Power Series Expansion . . . . 2 - 26
2.5.2 Inverse z-Transform using Partial Fraction Expansion . . . . 2 - 27
2.6 Unilateral z-transform .... 2 - 39
2.6.1 Properties of Unilateral z-Transform . . . . 2 - 39
2.6.1.1 Time Shift . . . . 2 - 39
2.6.1.2 Final Value Theorem . . . . 2 - 40
2.7 Convolution .... 2 - 42
2.7.1 Discrete Time Signal as Weighted Impulses . . . . 2 - 42
2.7.2 Convolution Formula . . . . 2 - 46
2.8 Properties of Impulse Response Representations for LTI Systems .... 2 - 65
2.8.1 Commutative Property . . . . 2 - 65
2.8.2 Cascade Connection of Systems . . . . 2 - 66
2.8.3 Parallel Connection of Systems . . . . 2 - 67
2.8.4 Causality of LTI Systems . . . . 2 - 71
2.8.5 Stability of LTI Systems . . . . 2 - 73
2.8.6 Memoryless and with Memory Systems . . . . 2 - 74
2.8.7 Step Response . . . . 2 - 75
2.9 Transform Analysis of LTI Systems .... 2 - 75
2.9.1 Pole-Zero Plots . . . . 2 - 75

2.9.2 Transfer Function of the LTI System . . . . 2 - 77

2.9.3 Causality and Stability interms of z-Transform . . . . 2 - 80

2.9.4 Inverse Systems . . . . 2 - 91

2.9.5 Solution of Difference Equations using z-Transform . . . . 2 - 93

2.10 Fourier Transform of Discrete Time Signals . . . . 2 - 98

2.10.1 Definition . . . . 2 - 98

2.10.2 Properties of DTFT . . . . 2 - 99

2.10.2.1 Periodicity . . . . 2 - 99

2.10.2.2 Linearity . . . . 2 - 99

2.10.2.3 Time Shift . . . . 2 - 100

2.10.2.4 Frequency Shift . . . . 2 - 101

2.10.2.5 Scaling . . . . 2 - 101

2.10.2.6 Differentiation in Frequency Domain . . . . 2 - 102

2.10.2.7 Time Reversal . . . . 2 - 102

2.10.2.8 Convolution . . . . 2 - 103

2.10.2.9 Multiplication in Time Domain (Modulation) . . . . 2 - 104

2.10.2.10 Parseval's Theorem . . . . 2 - 105

2.10.3 Inverse Fourier Transform . . . . 2 - 110

2.10.4 Magnitude / Phase Transfer Functions using Fourier Transform . . . . 2 - 111

2.10.5 Relationship between Fourier Transform and z-Transform . . . . 2 - 117

2.11 Short Answered Questions [2 Marks Each] . . . . 2 - 123

Chapter - 3 Discrete Fourier Transform and Computation (3 - 1) to (3 - 142)

3.1 Introduction to DFT . . . . 3 - 2

3.1.1 Definition of DFT and IDFT . . . . 3 - 2

3.1.2 DFT as a Linear Transformation . . . . 3 - 3

3.1.3 Periodicity Property of W_N . . . . 3 - 4

3.1.4 Comparison between DTFT and DFT . . . . 3 - 15

3.2 Relationship of DFT with Other Transforms ....3 - 16

3.2.1 Relationship between DFT and z-transform ....3 - 16

3.2.2 Relationship between DFT and DFS ....3 - 16

3.2.3 Relationship between DFT and DTFT ....3 - 17

3.3 Properties of DFT ....3 - 18

3.3.1 Periodicity....3 - 18

3.3.2 Linearity ....3 - 19

3.3.3 Circular Symmetries of a Sequence ....3 - 20

3.3.4 Symmetry Properties....3 - 25

3.3.5 Circular Convolution ....3 - 28

3.3.6 Time Reversal of a Sequence ....3 - 47

3.3.7 Circular Time Shift of a Sequence....3 - 49

3.3.8 Circular Frequency Shift ....3 - 52

3.3.9 Complex Conjugate Properties....3 - 52

3.3.10 Circular Correlation ....3 - 53

3.3.11 Multiplication of Two Sequences....3 - 55

3.3.12 Parseval's Theorem ....3 - 55

3.4 Applications of DFT....3 - 64

3.4.1 Linear Filtering using DFT (Linear Convolution using DFT)....3 - 64

3.4.2 DFT for Linear Filtering of Long Duration Sequences ....3 - 76

3.4.3 Overlap Save Method for Linear Filtering ....3 - 76

3.4.4 Overlap Add Method for Linear Filtering....3 - 79

3.4.5 Spectrum Analysis using DFT ....3 - 81

3.5 Computation of DFT using FFT Algorithms ....3 - 83

3.5.1 Direct Computation of DFT ....3 - 83

3.5.2 Properties of W_N ....3 - 89

3.5.3 Classification of FFT Algorithms ....3 - 91

3.6 Radix-2 FFT Algorithms ....3 - 91
3.6.1 Radix-2 DIT-FFT Algorithm ....3 - 92
3.6.1.1 Butterfly Computation ....3 - 102
3.6.1.2 Computational Complexity Compared to Direct Computation....................3 - 102
3.6.1.3 Memory Requirement and Inplace Computations....................3 - 104
3.6.1.4 Bit Reversal ....3 - 105
3.6.2 Radix-2 DIF FFT Algorithm....................3 - 118
3.6.2.1 Butterfly Operation in DIF-FFT ....3 - 121
3.6.2.2 Computational Complexity ....3 - 124
3.6.2.3 Memory Requirement and Inplace Computation ....3 - 124
3.6.2.4 Bit Reversal....................3 - 125
3.6.3 Comparison between Radix-2 DIT and DIF FFT Algorithms ....3 - 131
3.7 Inverse DFT using FFT Algorithms....................3 - 132
3.8 Short Answered Questions [2 Marks Each] ....3 - 138

Chapter - 4 Design of Digital Filters (4 - 1) to (4 - 168)

4.1 FIR and IIR Filter Realization ....4 - 2
4.1.1 Describing Equation....................4 - 2
4.1.2 Elementary Blocks for Filter Realization ....4 - 2
4.2 Basic FIR Filter Structures ....4 - 3
4.2.1 Direct Form Structure of FIR System ....4 - 4
4.2.2 Linear Phase FIR Structures....................4 - 7
4.2.3 Cascade Form Structure for FIR System....................4 - 13
4.3 Basic IIR Filter Structures ....4 - 18
4.3.1 Direct Form Structures for IIR Systems ....4 - 18
4.3.1.1 Direct Form-I Structure of IIR System....................4 - 18
4.3.1.2 Direct Form-II Structure for IIR System ....4 - 22
4.3.2 Cascade Form Structure for IIR Systems ....4 - 30
4.3.3 Parallel Form Structure for IIR Systems ....4 - 33
4.4 Properties of FIR Digital Filters ....4 - 40

4.4.1 Inherent Stability of FIR Filters . . . . 4 - 40
4.4.2 Magnitude and Phase Response of FIR Filters and Linear Phase Property . . . . 4 - 40
4.4.3 Magnitude Characteristics and Order of FIR Filter . . . . 4 - 44
4.5 Different Types of Windows . . . . 4 - 45
4.5.1 Rectangular Window . . . . 4 - 45
4.5.2 Bartlett (Triangular) Window . . . . 4 - 47
4.5.3 Blackmann Window . . . . 4 - 48
4.5.4 Hamming Window . . . . 4 - 49
4.5.5 Hanning Window . . . . 4 - 50
4.5.6 Attenuation and Transition Width of Windows . . . . 4 - 51
4.6 Design of Linear Phase FIR Filters using Windows . . . . 4 - 52
4.6.1 Rectangular Window for FIR Filter Design . . . . 4 - 52
4.6.2 Gibbs Phenomenon . . . . 4 - 53
4.6.3 Design using Other Window Functions . . . . 4 - 54
4.7 Analog Filter Design . . . . 4 - 73
4.7.1 Analog Filter Design using Butterworth Approximation . . . . 4 - 73
4.7.2 Analog Filter Design using Chebyshev Approximation . . . . 4 - 80
4.7.3 Analog Frequency Transformations . . . . 4 - 96
4.7.4 Digital Frequency Transformations . . . . 4 - 98
4.7.5 Comparison between Butterworth and Chebyshev Approximations . . . . 4 - 100
4.8 Design of IIR Filters from Analog Filters . . . . 4 - 101
4.8.1 IIR Filter Design by Solution of Differential Equations
(Approximation of Derivatives) . . . . 4 - 102
4.8.1.1 $s \rightarrow z$ Transformation . . . . 4 - 102
4.8.1.2 s-plane to z-plane Mapping . . . . 4 - 103
4.8.2 IIR Filter Design by Impulse Invariance . . . . 4 - 105
4.8.2.1 $s \rightarrow z$ Transformation . . . . 4 - 105
4.8.2.2 s-plane to z-plane Mapping . . . . 4 - 106
4.8.2.3 Aliasing in Frequency Domain . . . . 4 - 108

4.8.2.4 Frequency Response of Digital Filter Designed using Impulse Invariance . . . 4 - 108

4.8.3 IIR Filter Design by Bilinear Transformation . . . . 4 - 112

4.8.3.1 $s \rightarrow z$ Transformation . . . . 4 - 112

4.8.3.2 s-plane to z-plane Mapping . . . . 4 - 112

4.8.3.3 Frequency Warping. . . . . 4 - 113

4.8.4 Comparison between Impulse Invariance Method with Bilinear Transform Method . . . . 4 - 115

4.9 Design of IIR Filters using Butterworth and Chebyshev Approximations ......4 - 121

4.9.1 Design Steps . . . . 4 - 121

4.9.2 Solved Examples on Butterworth Approximation . . . . 4 - 123

4.9.3 Solved Examples on Chebyshev Approximation . . . . 4 - 150

4.10 Comparison of FIR and IIR Filters.......................4 - 157

4.11 Short Answered Questions [2 Marks Each] ....4 - 159

Chapter - 5 Digital Signal Processors (5 - 1) to (5 - 28)

5.1 Introduction.......................5 - 2

5.1.1 Desirable Features of DSP Processors . . . . 5 - 2

5.2 Computer Architectures for Signal Processing ....5 - 3

5.2.1 Von-Neumann Architecture . . . . 5 - 3

5.2.2 Harvard Architecture . . . . 5 - 3

5.2.3 Modified Harvard Architecture. . . . . 5 - 4

5.2.4 Hardware Multiplier-Accumulator (MAC) Unit . . . . 5 - 5

5.2.5 Multiple Access Memory. . . . . 5 - 6

5.2.6 Multiported Memory. . . . . 5 - 7

5.2.7 Pipelining . . . . 5 - 7

5.2.8 Extended Parallelism . . . . 5 - 8

5.2.8.1 VLIW Architecture and Multiple ALUs . . . . 5 - 8

5.2.8.2 Single Instruction Multiple Data (SIMD) . . . . 5 - 9

5.2.8.3 Superscalar Processing . . . . 5 - 10

5.2.9 Special Instructions . . . . . 5 - 11
5.2.10 Replication . . . . . 5 - 11
5.2.11 On-chip Memory/Cache . . . . . 5 - 11
5.3 Addressing Formats . . . . . 5 - 12
5.3.1 Short Immediate Addressing . . . . . 5 - 12
5.3.2 Short Direct Addressing . . . . . 5 - 12
5.3.3 Memory-Mapped Addressing . . . . . 5 - 12
5.3.4 Indirect Addressing . . . . . 5 - 13
5.3.5 Bit Reversed Addressing Mode . . . . . 5 - 13
5.3.6 Circular Addressing . . . . . 5 - 13
5.4 On-Chip Peripherals . . . . . 5 - 13
5.4.1 On-Chip Timer . . . . . 5 - 13
5.4.2 Serial Port . . . . . 5 - 14
5.4.3 TDM Serial Port . . . . . 5 - 14
5.4.4 Parallel Port . . . . . 5 - 15
5.4.5 Bit I/O Ports . . . . . 5 - 15
5.4.6 Host Port . . . . . 5 - 15
5.4.7 Comm Ports . . . . . 5 - 16
5.4.8 A/D and D/A Converter . . . . . 5 - 16
5.5 Introduction to Commercial Digital Signal Processors . . . . . 5 - 16
5.5.1 TMS 320C5x Processors . . . . . 5 - 16
5.5.2 TMS 302C54X Processors . . . . . 5 - 18
5.5.2.1 Features of 'C54X . . . . . 5 - 18
5.5.2.2 Architecture of 'C54X . . . . . 5 - 19
5.6 Selecting Digital Signal Processors . . . . . 5 - 23
5.6.1 Comparison between DSP Processors and General Purpose Processors . . . . . 5 - 24
5.7 Short Answered Questions [2 Marks Each] . . . . . 5 - 26

Solved Question Paper of Anna University (S - 1) to (S - 4)

May - 2015 . . . . . (S - 1) to (S - 4)

1 Introduction

Syllabus

Classification of systems : Continuous, Discrete, Linear, Causal, Stable, Dynamic, Recursive, Time variance; Classification of signals : Continuous and discrete, Energy and power; Mathematical representation of signals; Spectral density; Sampling techniques, Quantization, Quantization error, Nyquist rate, Aliasing effect.

Contents

1.1 Introduction to Digital Signal Processing **May-03, 04, 05, 14,**
. . . . **Dec.-06, 16,** · · · · · · · · · · · Marks 16
1.2 Classification of Signals **May-04, 05, 06, 08, 11, 16,**
. . . . **Dec.-05, 06, 08, 09,**
. . . . **Dec.-10, 11, 12, 15, 16,** · · · Marks 10
1.3 Elementary Signals and Operations on them
1.4 Systems **May-07, 08, 09, 11, 12, 14, 15, 17,**
. . . . **Dec.-07, 09, 10, 11,**
. . . . **Dec.-12, 13, 15, 16,** · · · · · · Marks 16
1.5 Nyquist-Shannon Sampling Theorem. **May-06, 11, 15, 16, 17,**
. . . . **Dec.-05, 06, 08, 10,**
. . . . **Dec.-11, 12, 15,** · · · · · · · · · Marks 10
1.6 Short Answered Questions [2 Marks Each]

1.1 Introduction to Digital Signal Processing

AU : May-03, 04, 05, 14, Dec.-06, 16

1.1.1 Basic Elements of Digital Signal Processing

Fig. 1.1.1 shows the basic elements of digital signal processing system. Most of the signals generated are analog in nature. For example sound, video, temperature, pressure, flow, seismic signals, biomedical signals etc. If such signals are processed by a digital signal processing system, then the signals must be digitized. Hence input is given through analog-to-digital converter and output is obtained through digital-to-analog converter.

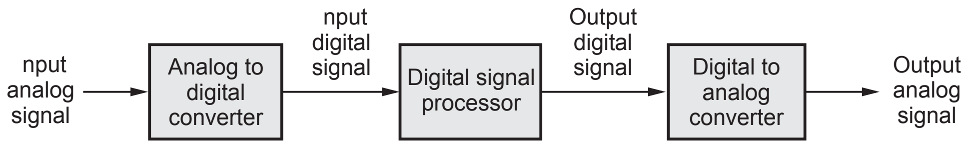


Fig. 1.1.1 Basic elements of digital signal processing

Analog to digital converter :

The A/D converter converts analog input to digital input. This signal is processed by a DSP system. The A/D converter determines sampling rate and quantization error in digitizing operation.

Digital signal processor : It is also called DSP processor. It performs amplification, attenuation, filtering, spectral analysis, feature extraction etc operations on digital data. The digital signal processor consists of ALU, shifter, serial ports, interrupts, address generators etc. for its functioning. The DSP processor has special architectural features due to which DSP operations are implemented fast on it compared to general purpose microprocessors.

Digital to analog converter : Some of the processed signals are required back in their analog form. For example sound, image video signals are required in analog form. Hence the DSP processor output is given to digital to analog converter. The D/A converter converts digital output of DSP processor to its analog equivalent. Such analog output is processed signal.

Most of the DSP systems use the three basic elements discussed above since majority of applications have analog signals.

1.1.2 Advantages of DSP over Analog Signal Processing

The digital signal processing offers many advantages over analog signal processing. These advantages are discussed next.

1. Flexibility : Digital signal processing systems are flexible. The system can be reconfigured for some other operation by simply changing the software program. For example, the high pass digital filter can be changed to low pass digital filter by simply changing the software. For this, no changes in the hardware are required. Thus digital signal processing systems are highly flexible. But this type of change is not easily possible in analog system. An analog system which is performing as high pass filter, is to be totally replaced to get lowpass filter operation.

2. Accuracy : Accuracy of digital signal processing systems is much higher than analog systems. The analog systems suffer from component tolerances, their breakdown etc. problems. Hence it is difficult to attain high accuracy in analog systems. But in digital signal processing systems, these problems are absent. The accuracy of digital signal processing systems is decided by resolution of A/D converter, number of bits to represent digital data, floating/fixed point arithmetic etc. But these factors are possible to control in digital signal processing systems to get high accuracy.

3. Easy storage : The digital signals can be easily stored on the storage media such as magnetic tapes, disks etc. Whereas the analog signals suffer from the storage problems like noise, distortion etc. Hence digital signals are easily transportable compared to analog signals. Thus remote processing of digital signals is possible compared to analog signals.

4. Mathematical processing : Mathematical operations can be accurately performed on digital signals compared to analog signals. Hence mathematical signal processing algorithms can be routinely implemented on digital signal processing systems. Whereas such algorithms are difficult to implement on analog systems.

5. Cost : When there is large complexity in the application, then digital signal processing systems are cheaper compared to analog systems. The software control algorithm can be complex, but it can be implemented accurately with less efforts.

6. Repeatability : The processing of the signals is completely digital in digital signal processing systems. Hence the performance of these systems is exactly repeatable. For example the lowpass filtering operation performed by digital filter today, will be exactly same even after ten years. But the performance may deteriorate in analog systems because of noise effects and life of components etc.

7. Adaptability : The digital signal processing systems are easily upgradable since they are software controlled. But such easy upgradation is not possible in analog systems.

8. Universal compatibility : The digital signal processing systems use digital computers or standard digital signal processors as their hardware. Almost all the applications use this as standard hardware with minor modifications. The operation of the digital signal processing is decided mainly by software program. Hence universal compatibility is possible in digital signal processing systems. Whereas it is not possible in analog systems. Since a simple analog low pass filter can be implemented by large number of ways.

9. Size and Reliability : The digital signal processing systems are small in size, more reliable and less expensive compared to the analog systems.

1.1.3 Disadvantages of Digital Signal Processing Systems

Even though the digital signal processing systems have all the above advantages, they have few drawbacks as follows :

1. When the analog signals have wide bandwidth, then high speed A/D converters are required. Such high speeds of A/D conversion are difficult to achieve for same signals. For such applications, analog systems must be used.
2. The digital signal processing systems are expensive for small applications. Hence the selection is done on the basis of cost complexity and performance.

The advantages of digital communication systems outweigh the above drawbacks.

1.1.4 DSP Applications

In the last section we discussed the advantages of DSP. Now let us see what is the range of applications of DSP. The summary of important DSP applications is presented below.

1. DSP for Voice and Speech : Speech recognition, voice mail, speech vocoding, speaker verification, speech enhancement, speech synthesis, text to speech etc.

2. DSP for Telecommunications : FAX, cellular phone, speaker phones, digital speech interpolation, video conferencing, spread spectrum communications, packet switching, echo cancellation, digital EPABXs, ADPCM transponders, channel multiplexing, modems adaptive equalizers, data encryption and line repeaters etc.

3. DSP for Consumer Applications : Digital audio / video / Television / Music systems, Music synthesizer, Toys etc.

4. DSP for Graphics and Imaging : 3-D and 2-D visualization, animation, pattern recognition, image transmission and compression, image enhancement, robot vision, satellite imaging for multipurpose applications etc.

5. DSP for Military/Defence : Radar processing, Sonar processing, Navigation, missile guidance, RF modems, secure communications.

6. DSP for Biomedical Engineering : X-ray storage and enhancement, ultrasound equipment, CT scanning equipments, ECG analysis, EEG brain mappers, hearing aids, patient monitoring systems, diagnostic tools etc.

7. DSP for Industrial Applications : Robotics, CNC, security access and power line monitors etc.

8. DSP for Instrumentation : Spectrum analysis, function generation, transient analysis, digital filtering, phase locked loops, seismic processing, pattern matching etc.

9. DSP for Control Applications : Servo control, robot control, laser printer control, disk control, engine control and motor control etc.

10. DSP for Automotive Applications : Vibration analysis, voice commands, digital radio, engine control, navigation, antiskid brakes, cellular telephones, noise cancellation, adaptive ride control etc.

More applications are given in chapters as per the topics.

Review Questions

1. *Compare DSP with ASP.*
2. *Discuss the advantages and disadvantages of digital processing of signals.*

AU : May-04, Marks 16, Dec.-06, Marks 8

3. *Explain the DSP system with necessary sketches and give its merits and demerits.*

AU : May-14, Marks 16, Dec.-16, Marks 8

1.2 Classification of Signals

AU : May-04, 05, 06, 08, 11, 16, Dec.-05, 06, 08, 09, 10, 11, 12, 15, 16

A function of one or more independent variables which contain some information is called signal.

The signals can be classified into two parts depending upon independent variable (time).

A) Continuous Time (CT) signals.

B) Discrete Time (DT) signals.

Both the CT and DT signals can be classified into following parts :

i) Periodic and non-periodic signals.

ii) Even and odd signals.

iii) Energy and power signals.

iv) Deterministic and random signals.

v) Multichannel and multidimensional signals.

1.2.1 CT and DT Signals

> ***Definition : A CT signal is defined continuously with respect to time. A DT signal is defined only at specific or regular time instants.***

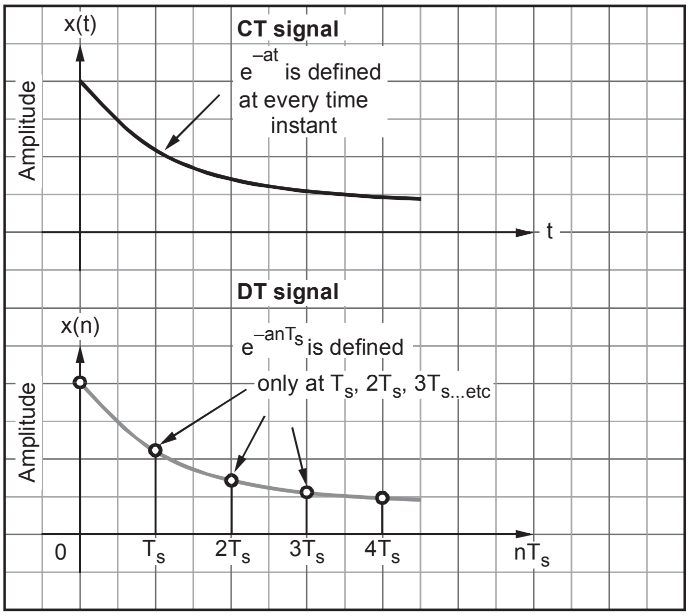


Fig. 1.2.1 CT & DT signals

Examples :

- Fig. 1.2.1 shows an example of CT signal, which is $x(t) = e^{-at}$. Note that this signal is continuous function of time.
- Fig. 1.2.1 also shows a DT version of exponential signal. It is defined as
  $x(n) = e^{-anT_s} = e^0, e^{-aT_s}, e^{-a2T_s}, e^{-a3T_s}, ..$
- Thus the DT signal has values only at $0, T_s, 2T_s, 3T_s, ...$ It is not defined over continuous time.

Significance :

i) Analog circuits process CT signals. Such circuits are op-amps, filters, amplifiers etc.

ii) Digital circuits process DT signals. Such circuits are microprocessors, counters, flip-flops etc.

Analog and digital signals :

i) When amplitude of CT signal varies continuously, it is called analog signal. In other words amplitude and time both are continuous for analog signal.

ii) When amplitude of DT signal takes only finite values, it is called digital signal. In other words amplitude and time both are discrete for digital signal.

Fig. 1.2.2 below shows the summary of CT/DT, analog/digital signals.

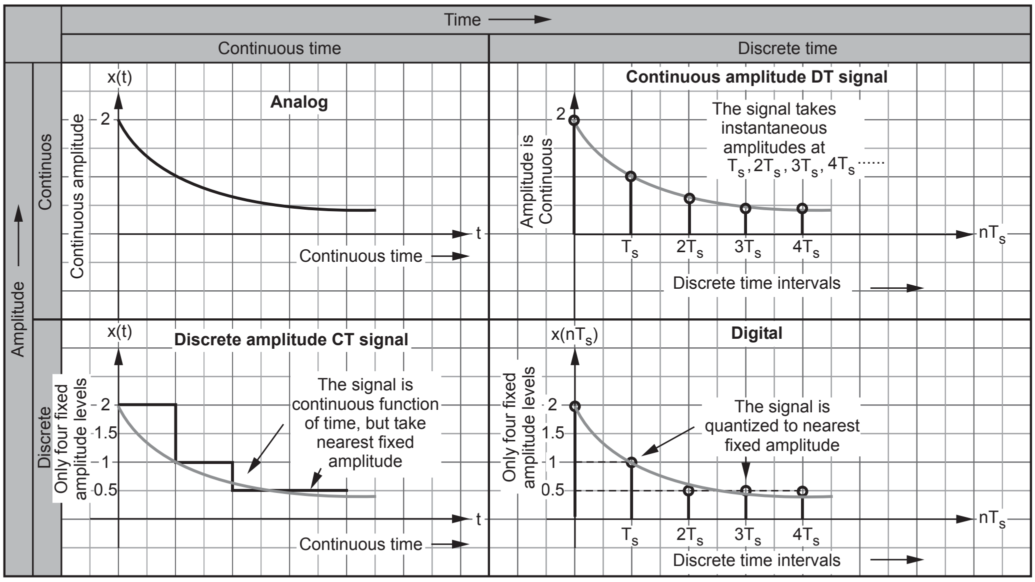


Fig. 1.2.2 Exponential signal as CT/DT, analog/digital

1.2.2 Periodic and Non-Periodic Signals

> *Definition : A signal is said to be periodic if it repeats at regular intervals. Non-periodic signals do not repeat at regular intervals.*

- **Examples of CT and DT periodic/non-periodic signals**

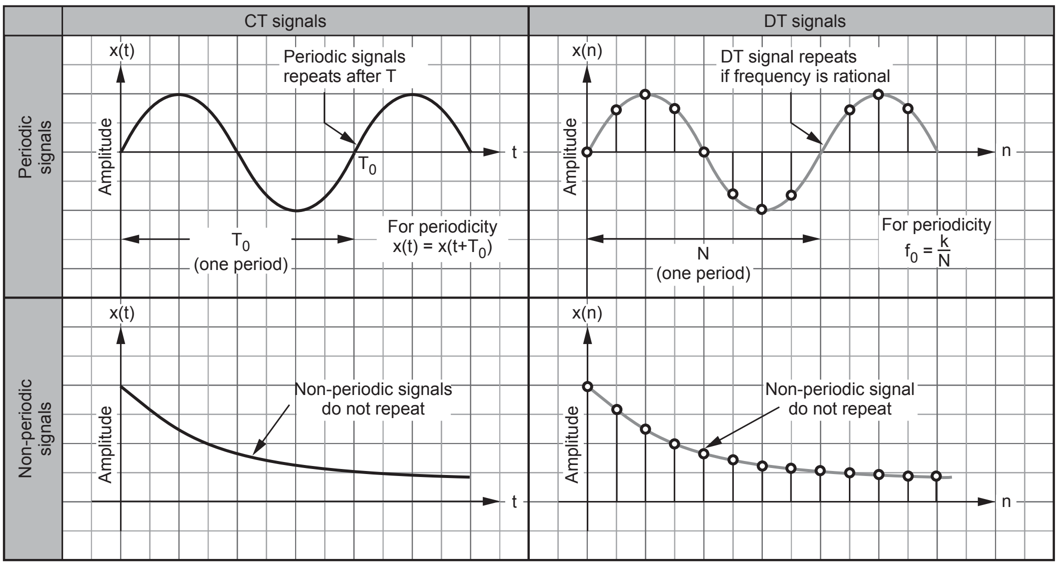


Fig. 1.2.3 Examples of periodic and non-periodic signals

- **Condition for periodicity of CT signal**

The CT signal repeat after certain period T_0 i.e.,

$$x(t) = x(t+T_0) \quad \ldots(1.2.1)$$

- **Condition for periodicity of DT signal**

Consider DT cosine wave, $x(n) = \cos(2\pi f_0 n)$

$$\therefore \quad x(n+N) = \cos[2\pi f_0 (n+N)]$$

$$= \cos(2\pi f_0 n + 2\pi f_0 N)$$

For periodicity, $x(n) = x(n+N)$

$$\therefore \quad \cos(2\pi f_0 n) = \cos(2\pi f_0 n + 2\pi f_0 N)$$

Above equation is satisfied only if $2\pi f_0 N$ is integer multiple of 2π . i.e.,

$$2\pi f_0 N = 2\pi k, \text{ where } k \text{ is integer}$$

$$\therefore \quad \boxed{f_0 = \frac{k}{N}} \quad \ldots(1.2.2)$$

The above condition shows that DT signal is periodic only if its frequency is rational (function of two integers).

- **Periodicity of signal $x_1(t)+x_2(t)$**

Let us consider that the signal $x(t) = x_1(t) + x_2(t)$

Then $x_1(t)$ will be periodic if,

$$x_1(t) = x_1(t+T_1) = x_1(t+2T_1) = \ldots$$

$$\text{or} \quad x_1(t) = x_1(t+mT_1), \quad \text{Here '}m\text{' is an integer.}$$

Similarly $x_2(t)$ will be periodic if,

$$x_2(t) = x_2(t+T_2) = x_2(t+2T_2) = \ldots$$

$$\text{or} \quad x_2(t) = x_2(t+nT_2), \quad \text{Here '}n\text{' is an integer}$$

Then $x(t)$ will be periodic if,

$$mT_1 = nT_2 = T_0, \quad \text{Here } T_0 \text{ is period of } x(t)$$

This means 'T_0' is integer multiple of periods of $x_1(t)$ and $x_2(t)$. From above equation we have,

$$\boxed{\frac{T_1}{T_2} = \frac{n}{m}, \text{ i.e. ratio of two integers}}$$

This is the condition for periodicity.

The period of $x(t)$ will be least common multiple of T_1 and T_2.

- **Periodicity of $x_1(n) + x_2(n)$**

Here $x(n) = x_1(n) + x_2(n)$ is periodic if,

$$\frac{N_1}{N_2} = \frac{n}{m} \quad \text{i.e. ratio of two integers}$$

The period of $x(n)$ will be least common multiple of N_1 and N_2.

1.2.3 Even and Odd Signals

Definition of even signal : A signal is said to be even signal if inversion of time axis does not change the amplitude. i.e.,

Condition for signal to be even : $\begin{cases} x(t) = x(-t) & \dots (1.2.3) \\ x(n) = x(-n) & \dots (1.2.4) \end{cases}$

- Even signals are also called symmetric signals.

Definition of odd signal : A signal is said to be odd signal if inversion of time axis also inverts amplitude of the signal i.e.,

Condition for signal to be odd : $x(t) = -x(-t)$ ... (1.2.5)

$x(n) = -x(-n)$ ... (1.2.6)

- Odd signals are also called anti-symmetric signals.

Examples of even and odd signals

- Cosine wave is an example of even signal. Since $\cos\theta = \cos(-\theta)$.
- Sine wave is an example of odd signal. Since $\sin(\theta) = -\sin(-\theta)$

Significance of even and odd signals

- Even or odd symmetry of the signal have specific harmonic or frequency content.
- Even or odd symmetry property is used in filter design.

Representation of signal in even and odd parts

i) Let the signal be represented into its even and odd parts as,

$$x(t) = x_e(t) + x_o(t) \quad \dots(1.2.7)$$

Here $x_e(t)$ is even part of $x(t)$ and

$x_o(t)$ is odd part of $x(t)$

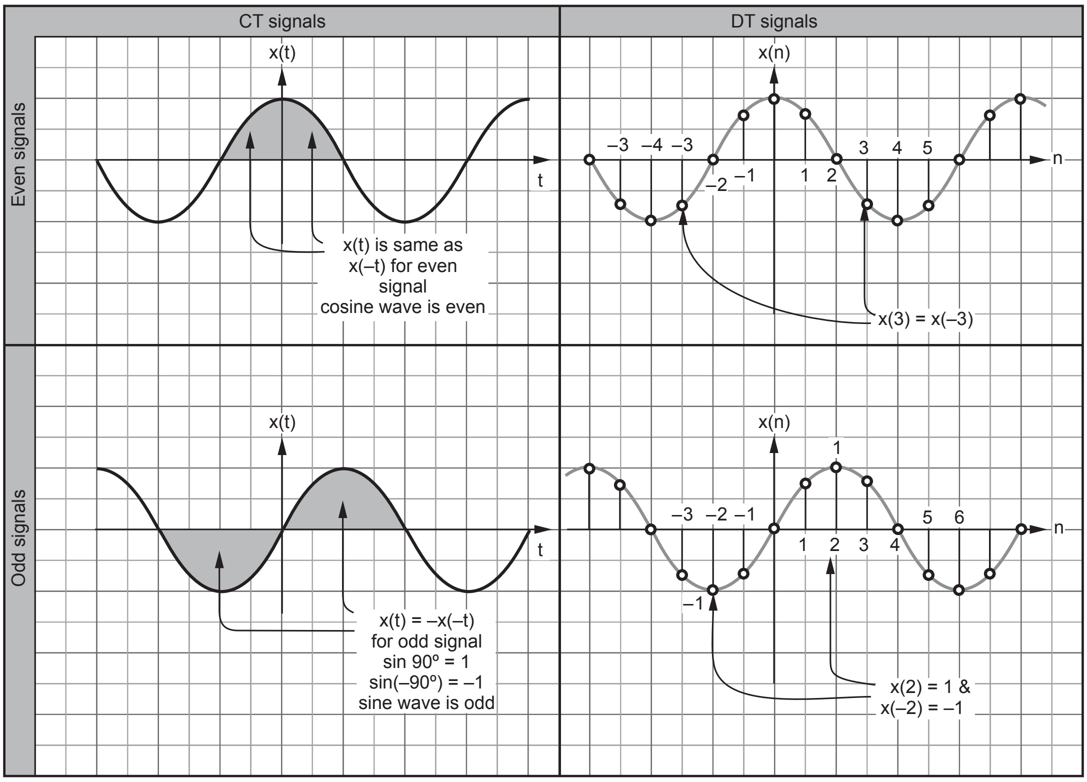


Fig. 1.2.4 Examples of even and odd signals

ii) Substitute –t for t in above equation,

$$x(-t) = x_e(-t) + x_o(-t)$$

Now by definition of even signal, $x_e(-t) = x_e(t)$ and by definition of odd signal $x_o(-t) = -x_o(t)$. Hence above equation will be,

$$x(-t) = x_e(t) - x_o(t) \qquad \dots(1.2.8)$$

iii) Adding equation (1.2.7) and equation (1.2.8),

$$x(t) + x(-t) = 2x_e(t) \Rightarrow x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$$

Subtracting equation (1.2.8) from equation (1.2.7).

$$x(t) - x(-t) = 2x_o(t) \Rightarrow x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$$

$$\textbf{Even part :}\ x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$$
$$\textbf{Odd part :}\ x_o(t) = \frac{1}{2}\{x(t) - x(-t)\} \qquad \ldots(1.2.9)$$

Similarly for DT signals we can write,

$$\textbf{Even part :}\ x_e(n) = \frac{1}{2}\{x(n) + x(-n)\}$$
$$\textbf{Odd part :}\ x_o(n) = \frac{1}{2}\{x(n) - x(-n)\} \qquad \ldots(1.2.10)$$

1.2.4 Energy and Power Signals

- **Instantaneous power dissipation**

For circuit of Fig. 1.2.5, the instantaneous power dissipated in load resistance 'R' will be given as,

$$p(t) = \frac{v^2(t)}{R} = i^2(t)R \qquad \ldots(1.2.11)$$

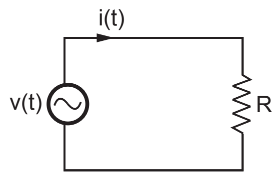


Fig. 1.2.5

- **Normalized power :** It is the power dissipated in R = 1 Ω load. Hence from equation (1.2.11) we can write,

Normalized power, $p(t) = v^2(t) = i^2(t)$

- **Significance of using normalized power**

Let $v(t)$ or $i(t)$ be denoted by $x(t)$. Then normalized power will be,

$$p(t) = x^2(t) \qquad \ldots(1.2.12)$$

Thus for current as well as voltage the equation for normalized power is same.

- **Definition of energy and power**

1. Energy of CT and DT signals : It is given by following equations,

$$\text{Energy,} \quad E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \quad \text{for CT signal} \qquad \ldots(1.2.13)$$

$$\text{and,} \quad E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad \text{for DT signal} \qquad \ldots(1.2.14)$$

2. Power of CT and DT signals : It is given by following equations,

$$\text{Power,} \quad P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \qquad \text{for CT signal} \qquad \ldots(1.2.15)$$

$$\text{and,} \quad P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \qquad \text{for DT signal} \qquad \ldots(1.2.16)$$

If the signal is periodic, then the period (T or N) have finite value. Hence there is no need to take limits.

- **Definition of power signal**

A signal is said to be power signal if its normalized power is non zero and finite. i.e.,

For power signal, $0 < P < \infty$

- **Definition of energy signal**

A signal is said to be energy signal if its total energy is finite and non-zero. i.e.,

For energy signal, $0 < E < \infty$

Examples of energy signal and power signal

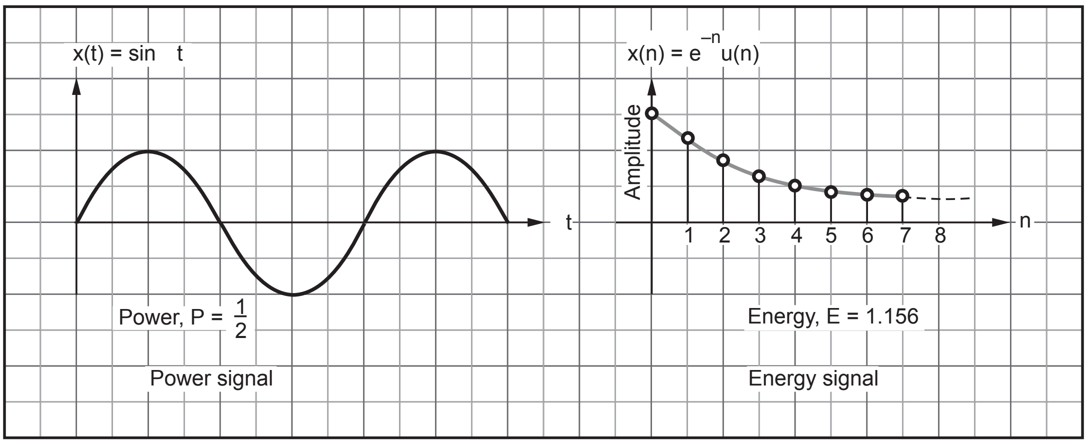


Fig. 1.2.6 Examples of energy and power signals

- **Comparison between power signal and energy signal**

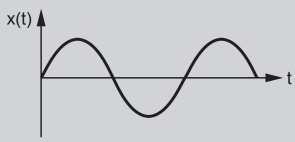
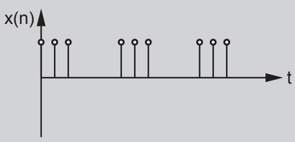
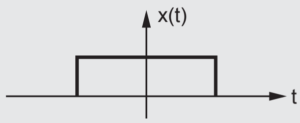
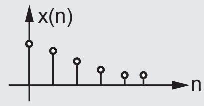
Sr. No.	Parameter	Power signal	Energy signal
1.	Definition	$0 < P < \infty$	$0 < E < \infty$
2.	Equation	$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt$ $= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \lvert x(n) \rvert^2$	$E = \int_{-\infty}^{\infty} \lvert x(t) \rvert^2\, dt$ $= \sum_{n=-\infty}^{\infty} \lvert x(n) \rvert^2$
3.	Periodicity	Most of periodic signals are power signals	Most of the non-periodic signals are energy signals.
4.	Energy and power	Energy of the power signal is infinite.	Power of the energy signal is zero.
5.	Examples	 	 

Table 1.2.1

1.2.5 Deterministic and Random Signals

Definition of deterministic signal :

> *A deterministic signal can be completely represented by mathematical equation at any time.*

Example :

sine wave, $x(t) = \cos \omega t$

$$x(n) = \cos 2\pi f n$$

Exponential pulse, triangular wave, square pulse etc.

Definition of random signal :

> *A signal which cannot be represented by any mathematical equation is called random signal.*

1.2.6 Multichannel and Multidimensional Signals

Multichannel signal : When the signal is generated by multiple sources or multiple sensors, it is called multichannel signal. These signals are represented by vectors. For example 3-lead and 12-lead ECG are multichannel signals.

Multidimensional signal

If the signal is the function of 'M' independent variables, then it is called multi-dimensional signal. The TV signal is three dimensional signal since brightness of the pixel is represented as $I(x, y, t)$. Where x and y indicate position of the pixel and 't' indicate time of the signal.

1.2.7 Sesmic Signals

Sesmic signals are generated because of an earthquake. Types of sesmic waves are (i) Primary (P) (ii) Secondary (S) and (iii) Surface wave.

Example : Noise generated in electronic components, transmission channels, cables etc.

Examples for Understanding

Example 1.2.1 *Determine whether the following DT signals are periodic or not ? If periodic, determine fundamental period.*

i) $\cos(0.01\pi n)$ *ii)* $\cos(3\pi n)$ *iii)* $\sin 3n$

iv) $\cos\frac{2\pi n}{5}+\cos\frac{2\pi n}{7}$ *v)* $\cos\left(\frac{n}{8}\right)\cos\frac{n\pi}{8}$ *vi)* $\sin(\pi+0.2n)$ *vii)* $e^{\left(j\frac{\pi}{4}\right)n}$

AU : Dec.-09, Marks 4, May-11, Marks 5

Solution : (i) $x(n) = \cos 0.01\,\pi\, n$

Compare with, $x(n) = \cos 2\pi f n$

$$\therefore \quad 2\pi f n = 0.01\,\pi\, n \;\Rightarrow\; f = \frac{0.01}{2} = \frac{1}{200} = \frac{k}{N}$$

Here f is expressed as ratio of two integers with $k = 1$ and $N = 200$. Hence the signal is, $\boxed{\text{periodic with } N = 200.}$

ii) $x(n) = \cos\ 3\pi n$

Compare with $x(n) = \cos 2\pi fn$

$$\therefore \quad 2\pi f n = 3\pi n \;\Rightarrow\; f = \frac{k}{N} = \frac{3}{2}$$ i.e. ratio of two integers.

Hence this signal is, $\boxed{\text{periodic with } N = 2}$

iii) $x(n) = \sin 3n$

Compare with $x(n) = \cos\ 2\pi f n$

$\therefore \quad 2\pi f n = 3n \Rightarrow f = \frac{k}{N} = \frac{3}{2\pi}$ which is not ratio of two integers.

Hence this signal is, $\boxed{\text{non-periodic}}$

iv) $x(n) = \cos\frac{2\pi n}{5} + \cos\frac{2\pi n}{7}$

Compare with, $x(n) = \cos 2\pi f_1 n + \cos 2\pi f_2 n$

$$\therefore \quad 2\pi f_1 n = \frac{2\pi n}{5} \Rightarrow f_1 = \frac{1}{5} = \frac{k_1}{N_1}, \quad \therefore N_1 = 5$$

$$\text{and} \quad 2\pi f_2 n = \frac{2\pi n}{7} \Rightarrow f_2 = \frac{1}{7} = \frac{k_2}{N_2}, \quad \therefore N_2 = 7$$

Here since $\frac{N_1}{N_2} = \frac{5}{7}$ is the ratio of two integers, the sequence is periodic. The period of $x(n)$ is least common multiple of N_1 and N_2. Here least common multiple of $N_1 = 5$ and $N_2 = 7$ is 35. Therefore this sequence is, $\boxed{\text{periodic with N = 35.}}$

v) $x(n) = \cos\left(\frac{n}{8}\right)\cos\frac{n\pi}{8}$

$$\text{Here} \quad 2\pi f_1 n = \frac{n}{8} \Rightarrow f_1 = \frac{1}{16\pi}, \quad \text{which is not rational}$$

$$\text{and} \quad 2\pi f_2 n = \frac{n\pi}{8} \Rightarrow f_1 = \frac{1}{16}, \quad \text{which is rational}$$

Thus $\cos\left(\frac{n}{8}\right)$ is non-periodic and $\cos\left(\frac{n\pi}{8}\right)$ is periodic. $x(n)$ is non-periodic since it is the product of periodic and non-periodic signal.

vi) $x(n) = \sin\left(\pi + 0.2n\right)$

Compare with, $x(n) = \sin(2\pi f n + \theta)$

$\therefore \quad \theta = \pi$ i.e. phase shift

and $\quad 2\pi f n = 0.2n \Rightarrow f = \frac{0.2}{2\pi} = \frac{1}{10\pi}$ which is not rational.

Hence this signal is, $\boxed{\textit{non-periodic.}}$

vii) $x(n) = e^{\left(j\frac{\pi}{4}\right)n}$

$$= \cos\frac{\pi}{4}n + j\sin\frac{\pi}{4}n$$

Compare with, $x(n) = \cos 2\pi f n + j \sin 2\pi f n$

Here $\quad 2\pi f n = \frac{\pi}{4} n \Rightarrow f = \frac{1}{8} = \frac{k}{N},$ which is rational.

Hence this signal is, $\boxed{\text{periodic with } N = 8}$

Example 1.2.2 *Find and sketch the even and odd components of the following :*
i) $x(n) = e^{-(n/4)}\ u(n)$ *ii)* $x(n) = \text{Im}[e^{jn\pi/4}]$

Solution : i) $x(n) = e^{-(n/4)}\ u(n)$

Even and odd parts of the sequence $x(n)$ are given by equation (1.2.10) as,

Even part, $x_e(n) = \frac{1}{2}\{x(n) + x(-n)\}$ and

Odd part, $x_o(n) = \frac{1}{2}\{x(n) - x(-n)\}$

Following figure shows the steps to obtain even and odd parts as per above equations.

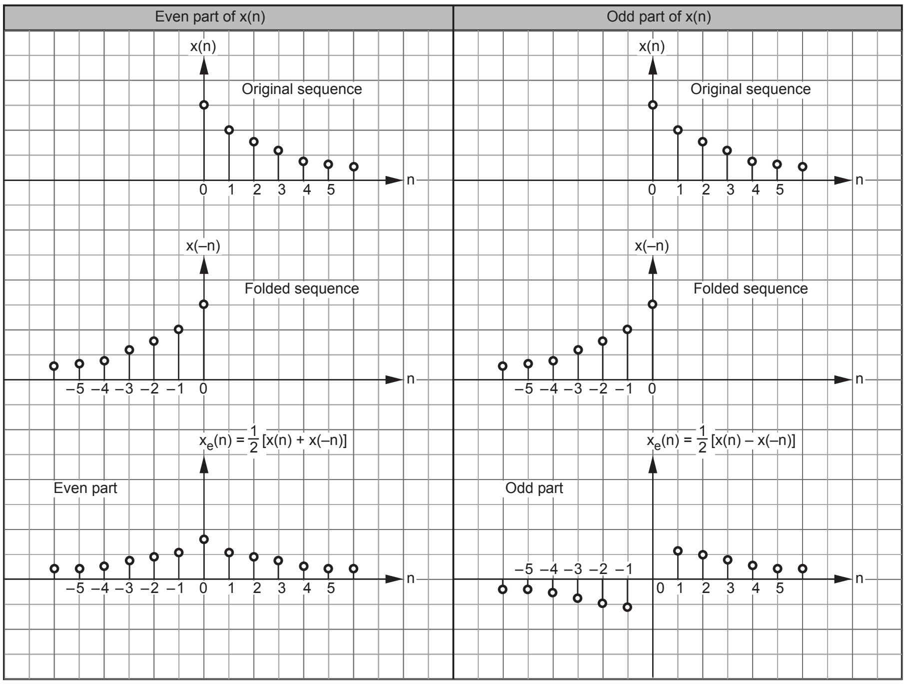


Fig. 1.2.7 Odd and even parts of x(n)

ii) $x(n) = \mathrm{Im}[e^{jn\pi/4}]$

$$= \mathrm{Im}\left[\cos\frac{n\pi}{4} + j\sin\frac{n\pi}{4}\right], \quad \sin e^{j\theta} = \cos\theta + j\sin\theta$$

$$= \sin\frac{n\pi}{4}$$

Compare this equation with $x(n) = \sin 2\pi f n$, hence $2\pi f n = \frac{n\pi}{4} \Rightarrow f = \frac{1}{8}$ cycles/sample. Since $f = \frac{k}{N} = \frac{1}{8}$. There will be 8 samples in one period of DT sine wave.

Fig. 1.2.8 shows the waveform of $x(n) = \sin\frac{n\pi}{4}$ and its even and odd parts are also shown. Even and odd parts are given as,

$$\text{Even part, } x_e(n) = \frac{1}{2}\{x(n) + x(-n)\} \text{ and}$$

$$\text{Odd part, } x_o(n) = \frac{1}{2}\{x(n) - x(-n)\}$$

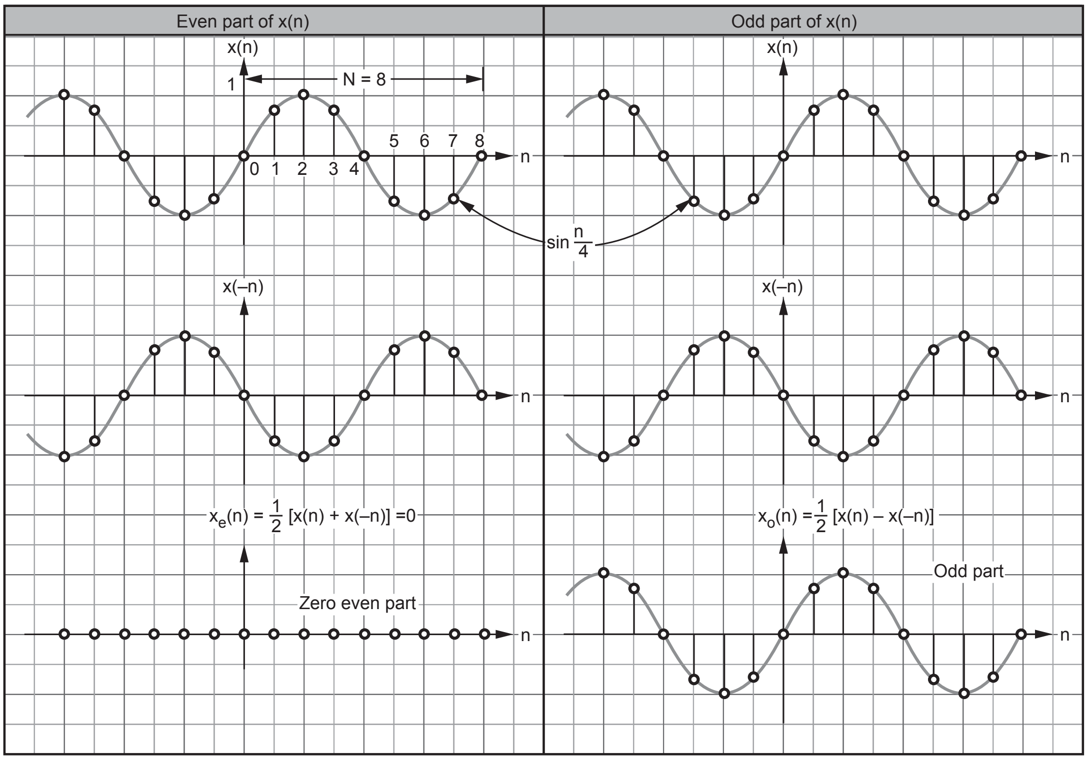


Fig. 1.2.8 Even and odd parts of $x(n)$

Example 1.2.3 *Determine whether the following signals are energy signals or power signals and calculate their energy or power.*

i) $x(n) = \left(\frac{1}{2}\right)^n u(n)$

AU : Dec.-15, May-16, Marks 4

ii) $x(n) = u(n)$

AU : Dec.-15, Marks 2

iii) $x(n) = \sin\left(\frac{\pi n}{6}\right)$

AU : Dec.-15, Marks 4

iv) $x(n) = e^{j\left(\frac{\pi n}{3}+\frac{\pi}{6}\right)}$

v) $x(n) = e^{2n}u(n)$

AU : May-16, Marks 4

AU : May-04, Marks 2; May-08, Marks 6; Dec.-08, Marks 8 May-11, Dec.-11, 12, Marks 4

Solution : When solving such examples we don't know whether the signal have finite power or finite energy. Hence follow the steps as given below :

Important tips :

Step 1 :	Observe the signal carefully. If it is periodic and infinite duration then it can be power signal. Hence calculate its power directly.
Step 2 :	If the signal is periodic but of finite duration, then it can be energy signal. Hence calculate its energy directly.
Step 3 :	If the signal is not periodic, then it can be energy signal. Hence calculate its energy directly.

i) $\boldsymbol{x(n) = \left(\frac{1}{2}\right)^n u(n)}$

This signal is not periodic. Hence as per step 3, calculate its energy directly.

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \qquad \text{By definition}$$

$$= \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^n\right]^2 = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \qquad \text{Since } u(n) = 1 \text{ for n = 0 to } \infty$$

Here use, $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ for $|a| < 1$. The above equation will be

$$E = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

Since energy is finite and non-zero, it is **energy signal with $E = \frac{4}{3}$**.

ii) *x(n) = u(n)*

This signal is periodic (since $u(n)$ repeats after every sample) and of inifinite duration. Hence it may be power signal. Therefore let us calculate power directly,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} (1)^2 \qquad \text{Since } u(n) = 1 \text{ for } 0 \le n \le \infty$$

Here $\sum_{n=0}^{N} (1)^2$ means 1 + 1 + 1 + 1...... for $n = 0$ to N. In other words,

1 + 1 + 1 + 1 ...... (N + 1) times = (N + 1). Therefore above equation will be,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \cdot (N+1)$$

$$= \lim_{N \to \infty} \frac{N+1}{2N+1}$$

$$= \lim_{N \to \infty} \frac{1+\frac{1}{N}}{2+\frac{1}{N}} = \frac{1}{2},$$ Power is finite, this is **power signal**.

iii) *x(n) = sin* $\left(\frac{\pi n}{6}\right)$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_2(n)|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left(\sin\frac{\pi}{6}n\right)^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1-\cos\frac{2\pi n}{6}}{2}$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1}{2} - \underbrace{\lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1}{2} \cos \frac{2\pi n}{6}}_{\text{Summation of cosine wave over complete cycle. Hence it will be zero}}$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \frac{1}{2} \cdot (2N+1) = \frac{1}{2}$$, Since power is finite, this is **power signal.**

iv) $x(n) = e^{j\left(\frac{\pi n}{3} + \frac{\pi}{6}\right)}$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_3(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| e^{j\left(\frac{\pi n}{3} + \frac{\pi}{6}\right)} \right|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1 \quad , \quad \text{since } \left| e^{j\theta} \right| = 1$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \cdot (2N+1) = 1$$, Since power is finite, this is **power signal.**

v) $x(n) = e^{2n} u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x_4(n)|^2 = \sum_{n=-\infty}^{\infty} \left| e^{2n} u(n) \right|^2$$

$$= \sum_{n=0}^{\infty} (e^4)^n = \sum_{n=0}^{\infty} (54.6)^n = \infty$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_4(n)|^2 = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e^{4n} u(n)$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} e^{4n} = \lim_{N \to \infty} \frac{1}{2N+1} \frac{e^{u(N+1)} - 1}{e - 1} = \infty$$

This signal is neither power nor energy signal.

Example 1.2.4 *Determine whether or not each of the following signals is periodic. If the signal is periodic, specify its fundamental period.*

1) $x(n) = e^{j6\pi n}$

2) $x(n) = \cos\frac{\pi}{3}n + \cos\frac{3\pi}{4}n$

AU : May-16, Marks 10

Solution :

1) $$x(n) = e^{j6\pi n}$$

Compare this phasor with $x(n) = e^{j2\pi fn}$.

Hence, $2\pi fn = 6\pi n \Rightarrow f = 3$, which is rational. Hence this signal is *periodic*.

$$f = 3 = \frac{3}{1} = \frac{k}{N}.$$ Hence fundamental period $N = 1$ sample.

2) $$x(n) = \cos\left(\frac{\pi n}{3}\right) + \cos\left(\frac{3\pi n}{4}\right)$$

Compare this equation with $x(n) = \cos(2\pi f_1 n) + \cos(2\pi f_2 n)$

Hence $f_1 = \frac{1}{6} = \frac{k_1}{N_1} \Rightarrow N_1 = 6$

and $f_2 = \frac{3}{8} = \frac{k_2}{N_2} \Rightarrow N_2 = 8$

Here $\frac{N_1}{N_2} = \frac{6}{8} = \frac{3}{4}$, which is rational. Hence $x(n)$ is *periodic*. Fundamental period is LCM of $N_1 = 6$ and $N_2 = 8$. That is N = 24 samples.

Examples for Practice

Example 1.2.5 *Check whether the following are energy or power signals.*
$x(n) = Ae^{j\omega_0 n}$

AU : Dec.-11, Marks 5, May-11, Marks 8

[Hint and Ans. : P = $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |Ae^{j\omega_0 n}|^2 = A^2$**, Power signal]**

Example 1.2.6 *Let x (n) = {1, 2, 3, 2, 1}. Find* $x_e(n) = \frac{x(n)+x(-n)}{2}$ *and* $x_o(n) = \frac{x(n)-x(n)}{2}$. *Obtain x(n) interms of* $x_e(n)$ *and* $x_o(n)$.

AU : Dec.-06, Marks 8, May-06, Marks 4

[**Ans. :** $x_e(n) = \left\{\frac{1}{2}, 1, \frac{3}{2}, 1, \underset{\uparrow}{1}, 1, \frac{3}{2}, 1, \frac{1}{2}\right\}$, $x_o(n) = \left\{-\frac{1}{2}, -1, -\frac{3}{2}, -1, \underset{\uparrow}{0}, 1, \frac{3}{2}, 1, \frac{1}{2}\right\}$]

Review Questions

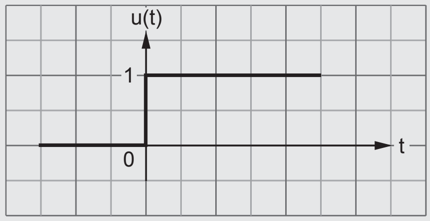
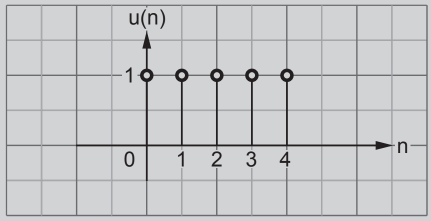
1. *Draw analog discrete, quantized and digital signal with an example.* **AU : May-05, Marks 8**
2. *Derive the condition for periodicity of DT signal.*
3. *Explain the classification of DT signals.* **AU : Dec.-10, Marks 10, May-06, Marks 8**
4. *Define energy and power. Hence define energy signal and power signal.*
5. *Compare energy signal and power signal.* **AU : Dec.-16, Marks 4**
6. *Explain different types of signal representation.* **AU : Dec.-08, Marks 6**

1.3 Elementary Signals and Operations on them

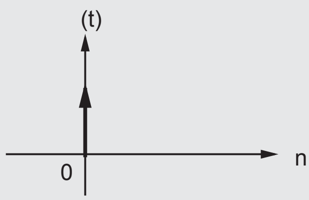
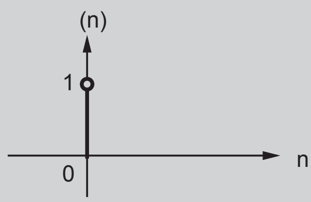
- Standard signals are used for the analysis of systems. These standard signals are,
  i) Unit step function.
  ii) Unit impulse function.
  iii) Unit ramp function.
  iv) Complex exponential function.
  v) Sinusoidal function.

1.3.1 Unit Step Function

Parameter	CT unit step signal u(t)	DT unit step signal u(n)
Definition	*The unit step signal has amplitude of '1' for positive values of independent variable. And it has amplitude of '0' for negative values of independent variable.*	
Mathematical representation	$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$	$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$ or $u(n) = \{0, 0, \underset{\uparrow}{1}, 1, 1, 1, 1, \ldots\}$

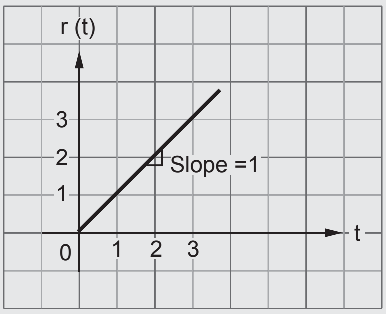
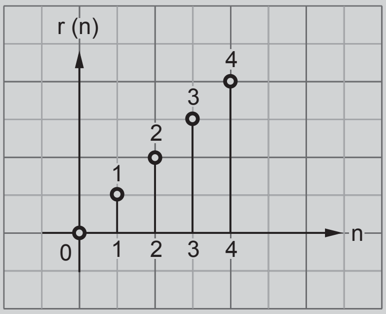
Waveform		
Significance	• DT unit step signal is sampled version of CT unit step. • Unit step signal is generated when a DC supply is applied to the circuit. Unit step is generated when a switch is closed at t = 0.	

1.3.2 Unit Impulse or Delta Function

Parameter	Unit impulse signal $\delta(t)$	Unit sample signal δ (n)
Definition	Area under unit impulse approaches '1' as its width approaches zero. Thus it has zero value everywhere except t = 0.	Amplitude of unit sample is '1' at n = 0 and it has zero value at all other values of n.
Mathematical representation	$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$ and $t \to 0$ $\delta(t) = 0$ for $t \neq 0$	$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$ or $\delta(n) = \{0, 0, 0, \underset{\uparrow}{1}, 0, 0, 0\}$
Waveform		
Significance	• $\delta(n)$ is not the sampled version of $\delta(t)$. The main difference is : Area under $\delta(t) = 1$, Amplitude of $\delta(n) = 1$. • Unit impulse or unit sample functions are used to determine impulse response of the system. • Unit impulse or unit sample functions contain all the frequencies from $-\infty$ to ∞.	

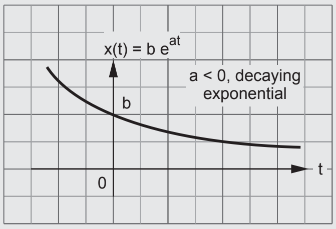
Properties	i) **Sifting property :** $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$ … (1.3.1) Here $\delta\,(t-t_0)$ is time shifted delta function. ii) **Replication property :** $\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$ … (1.3.2) or $x(t) * \delta(t) = x(t)$ Thus convolution of any function with delta function leaves that function unchanged.

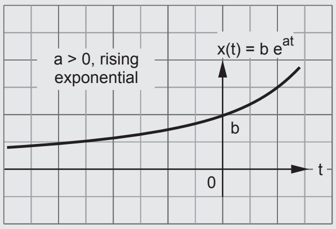
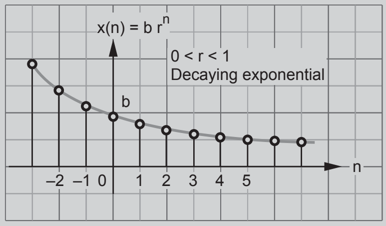
1.3.3 Unit Ramp Function

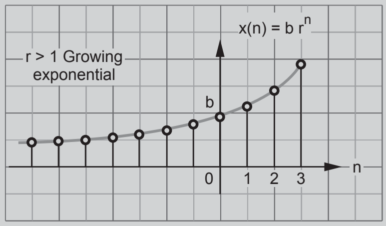
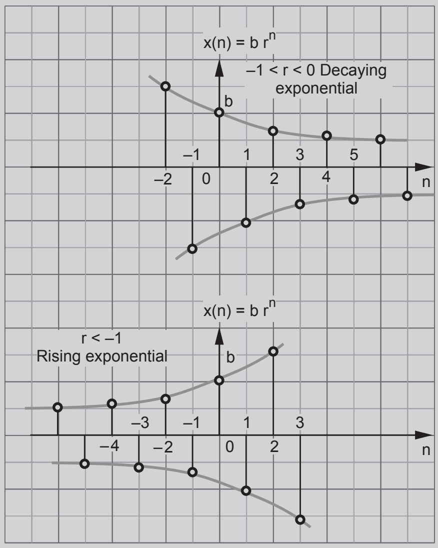
Parameter	CT unit impulse signal $r(t)$	DT unit ramp signal r (n)
Definition	It is linearly growing function for positive values of independent variable.	The amplitude of every sample increases linearly with its number(n) for positive values of 'n'
Mathematical representation	$r(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$ $= t\,u(t)$ since $u(t) = 1$ for $t \geq 0$ and $u(t) = 0$ for $t < 0$	$r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$ $= n\,u(n)$ since $u(n) = 1$ for $n \geq 0$ and $u(n) = 0$ for $n < 0$ $r(n) = \{\underset{\uparrow}{0}, 1, 2, 3, 4, 5, \ldots\ldots\ldots\}$
Waveform		
Significance	• The ramp function indicates linear relationship. • It indicates constant current charging of the capacitor.	

1.3.4 Complex Exponential and Sinusoidal Signals

CT and DT real exponential signals

<table>
<tr><th>Parameter</th><th>CT real exponential signal</th><th>DT real exponential signal</th></tr>
<tr><td>Definition</td><td colspan="2">It is exponentially growing or decaying signal.</td></tr>
<tr><td>Mathematical representation</td><td>$x(t) = b\, e^{at}$
Here b and a are real.</td><td>$x(n) = b\, r^n$
If $r = e^{\alpha}$, then,
$x(n) = b\, e^{\alpha n}$
Here b and α are real.</td></tr>
<tr><td>Waveform</td><td>


</td><td>


</td></tr>
<tr><td colspan="2">Significance/Uses of CT exponential signal
i) Charging and discharging of a capacitor.
ii) Current flow through an inductor.
iii) Radioactive decay.
Significance/Uses of DT exponential signal
i) Population growth as a function of generation.
ii) Return on investment as a function of day, month or quarter.</td><td>
</td></tr>
</table>

1.3.5 Time Delay / Advancing of Signals

Fig. 1.3.1 shows time delay/advance operations on DT unit step signal.

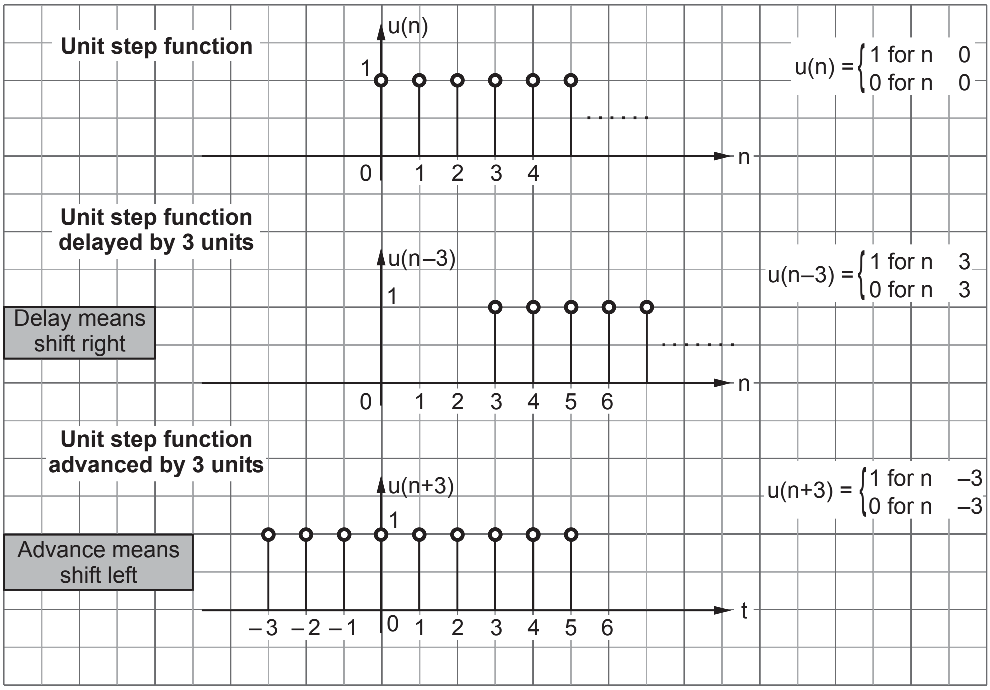


Fig. 1.3.1 Delayed/advanced discrete time signals

1.3.6 Time Folding of Signals

Let us consider the signal $x(n)$. Then its folded signal can be obtained by replacing n with $-n$. i.e.,

$$y(n) = x(-n)$$

Fig. 1.3.2 shows the sketch of *x (n)* and *x (–n)*. Here *x (n)* is considered as unit step signal *u(n)*.

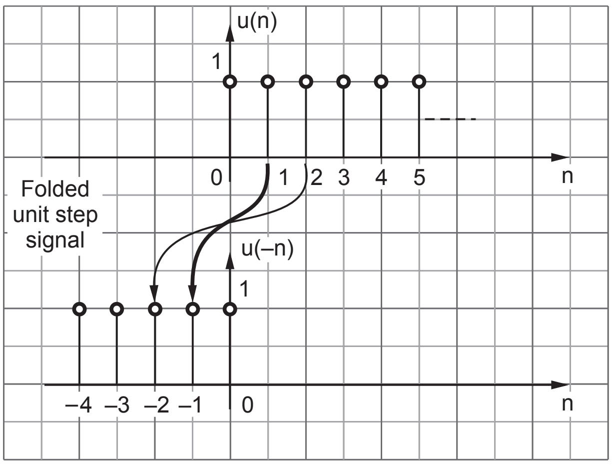


Fig. 1.3.2 Time folding operation on unit step signal

1.4 Systems AU : May-07, 08, 09, 11, 12, 14, 15, 17, Dec.-07, 09, 10, 11, 12, 13, 16

Definition :

> *A system is a set of elements or functional blocks that are connected together and produce an output in response to an input signal.*

Classification

- There are two types of systems : (i) continuous time and (ii) discrete time systems.
- Continuous time (CT) systems handle continuous time signals. Analog filters, amplifiers, attenuators, analog transmitters and receivers etc. are examples of continuous time systems.
- Discrete time (DT) systems handle discrete time signals. Computers, printers, microprocessors, memories, shift registers etc. are examples of discrete time systems. They operate only on discrete time signals.

Continuous as well as discrete time systems can be further classified based on their properties. These properties are as follows :

i) Dynamicity property : Static and dynamic systems.

ii) Shift invariance : Time invariant and time variant systems.

iii) Linearity property : Linear and non-linear systems.

iv) Causality property : Causal and non-causal systems.

v) Stability property : Stable and unstable systems.

vi) Invertibility property : Inversible and non-inversible systems.

In this section we will study all the above properties for continuous and discrete time systems.

1.4.1 Static and Dynamic Systems (Systems with Memory or without Memory)

> *Definition : The continuous time system is said to be static or (memoryless, instantaneous) if its output depends upon the present input only.*

If output of the discrete time system depends upon the present input sample only, then it is called static or memoryless or instantaneous system. For example,

$$y(n) = 10 \cdot x(n)$$

or $$y(n) = 15 \cdot x^2(n) + 10\,x(n)$$

are the static systems. Here the $y(n)$ depends only upon n^{th} input sample. Hence such systems do not need memory for its operation. A system is said to be dynamic if the output depends upon the past values of input also. For example,

$$y(n) = x(n)+x(n-1)$$

This is the dynamic system. In this system the n^{th} output sample value depends upon n^{th} input sample and just previous i.e. $(n-1)^{th}$ input sample. This systems need to store the previous sample value. Consider the following equation of a system.

$$y(n) = \sum_{k=0}^{4} x(n-k)$$

Expanding this equation,

$$y(n) = x(n)+x(n-1)+x(n-2)+x(n-3)+x(n-4)$$

This also represents a dynamic system. The output depends upon the present input and preceeding four input samples. Hence the preceeding four input samples i.e. $x(n-1), x(n-2), x(n-3)$ and $x(n-4)$ are stored in the memory.

1.4.2 Time Invariant and Time Variant Systems

Definition : ***A continuous time system is time invariant if the time shift in the input signal results in corresponding time shift in the output.***

If the input/output characteristics of the discrete time system do not change with shift of time origin, such systems are called shift invariant or time invariant systems. Let the system has input $x(n)$ and corresponding output $y(n)$, i.e. $y(n)=f[x(n)]$. Then the system is shift invariant or time invariant if and only if,

$$f[x(n-k)] = y(n-k) \quad \text{... (1.4.1)}$$

Here 'k' is constant. If the input is delayed by 'k' samples, then output is also delayed by the same number of samples for time invariant system. Above equation is not satisfied by time variant system.

Cooking a rice is the time invariant operation since every day it requires the same amount of time. It is independent of time/day/year of cooking. Ambient temperature is the time variant parameter since it depends upon the period of time. For example temperature is high in May-June whereas it is minimum in November-December. Like these, there are many physical examples of time invariant and time variant systems.

Steps to test for time invariance property :

Step 1 : For discrete time systems determine the output of the system for delayed input $x(n-k)$ i.e.,

$$y(n,k) = f\left[x(n-k)\right]$$

Step 2 : Then delay the output itself by 'k' samples, i.e. $y(n-k)$. Then

Step 3 : if
$$\left.\begin{array}{ll} y(n,k) \neq y(n-k) & \text{time variant} \\ y(n,k) = y(n-k) & \text{time invariant} \end{array}\right\} \quad \text{... (1.4.2)}$$

Example 1.4.1 *Determine whether the following discrete time systems are time invariant or not ?*

(i) $y(n) = x(n) - x(n-1)$ *(ii)* $y(n) = n\,x(n)$

(iii) $y(n) = x(-n)$ *(iv)* $y(n) = x(n) \cos \omega_0 n$

Solution : (i) Consider the system described by

$$y(n) = f\left[x(n)\right] = x(n) - x(n-1) \quad \text{... (1.4.3)}$$

Step 1 : Let us apply the input to this system which is delayed by 'k' samples. Then the output will be,

$$y(n,k) = f\left[x(n-k)\right]$$

$$= x(n-k) - x(n-k-1) \quad \text{... (1.4.4)}$$

Step 2 : Now let us delay the output $y(n)$ given by equation (1.4.3) by 'k' samples i.e.,

$$y(n-k) = x(n-k) - x(n-k-1) \quad \text{... (1.4.5)}$$

Step 3 : Here observe that

$$y(n,k) = y(n-k)$$

Hence the system is time invariant.

(ii) The given discrete time system equation is

$$y(n) = f\left[x(n)\right] = n\,x(n) \quad \text{... (1.4.6)}$$

Step 1 : When input $x(n)$ is delayed by 'k' samples, the response is,

$$y(n,k) = f\left[x(n-k)\right]$$

$$= n\,x(n-k) \quad \text{... (1.4.7)}$$

Here observe that only input $x(n)$ is delayed. The multiplier 'n' is not part of the input. Hence it cannot be written as $(n-k)$.

Step 2 : Now let us shift or delay the output $y(n)$ given by equation (1.4.6) by 'k' samples. i.e.,

$$y(n-k) = (n-k)\,x(n-k) \quad \text{... (1.4.8)}$$

Here both n and $x(n)$ in the equation $y(n) = n\,x(n)$ will be shifted by 'k' samples since they are part of the output sequence.

Step 3 : It is clear from equation (1.4.7) and above equation that,

$$y(n,k) \neq y(n-k)$$

Hence the system is time variant.

(iii) The given discrete time system equation is,

$$y(n) = f\left[x(n)\right]$$
$$= x(-n) \quad \text{... (1.4.9)}$$

Step 1 : Here let us delay the input $x(n)$ by 'k' samples. The output $y(n)$ is equal to folded input, i.e. $x(-n)$. Hence $x(-n)$ will also be delayed by 'k' samples. i.e.,

$$y(n,k) = f\left[x(n-k)\right]$$
$$= x\left[(-n)-k\right] \quad \text{... (1.4.10)}$$
$$= x(-n-k) \quad \text{... (1.4.11)}$$

Here observe that we cannot replace 'n' by simply $n-k$. Since we are delaying $x(n);\,x(-n)$ will also be delayed by same amount. The equation 1.4.10 is specifically written to indicate this operation.

Step 2 : Now let us delay the output $y(n)$ given by equation 1.4.9 by 'k' samples. i.e.,

$$y(n-k) = x\left[-(n-k)\right] \quad \text{... (1.4.12)}$$
$$= x(-n+k) \quad \text{... (1.4.13)}$$

Here observe that we are delaying the output $y(n)$. That is n is converted to $n-k$ in the equation for $y(n)$. This is specifically indicated in equation (1.4.12).

Step 3 : On comparing equation (1.4.12) and equation (1.4.11) we observe that,

$$y(n,k) \neq y(n-k),$$

Hence the system is time variant.

(iv) The given response equation is,

$$y(n) = f[x(n)]$$

$$= x(n) \cos \omega_0 n \qquad \text{... (1.4.14)}$$

Step 1 : The response to the delayed input $x(n-k)$ will be,

$$y(n,k) = x(n-k) \cos \omega_0 n \qquad \text{... (1.4.15)}$$

Here observe that 'n' in $\cos \omega_0 n$ is not written as $(n-k)$ since the term '$\cos \omega_0 n$' is not part of the input.

Step 2 : Now let us delay the output $y(n)$ by 'k' samples given by equation (1.4.14),

$$y(n-k) = x(n-k) \cos \omega_0 (n-k) \qquad \text{... (1.4.16)}$$

Here 'n' is converted to $(n-k)$ since both $x(n)$ and $\cos \omega_0 n$ in equation (1.4.14) are part of output $y(n)$.

Step 3 : On comparing above equation with equation (1.4.15) we find that,

$$y(n,k) \neq y(n-k)$$

Hence the system is time variant.

From this example it is clear that if the system alters the timing properties of the signal, then it is time variant system.

1.4.3 Linear and Non-linear Systems

Definition : *A system is said to be linear if it satisfies the superposition principle.*

The discrete time system is said to be linear if it satisfies superposition principle. Consider the two systems defined as follows :

$y_1(n) = f[x_1(n)]$ i.e. $x_1(n)$ is input and $y_1(n)$ is output.

$y_2(n) = f[x_2(n)]$ i.e. $x_2(n)$ is input and $y_2(n)$ is output.

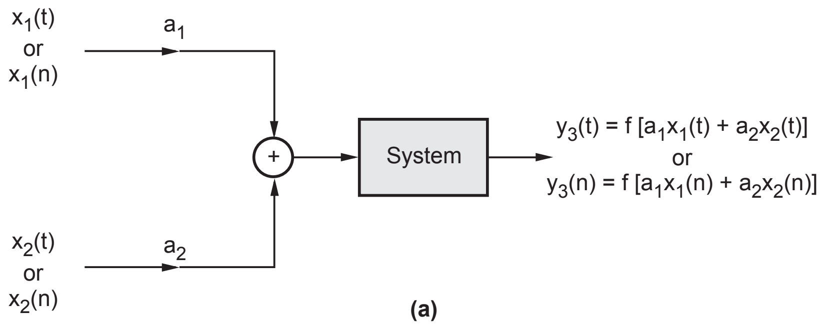
Then the discrete time system is linear if,

$$f[a_1 x_1(n) + a_2 x_2(n)] = a_1 y_1(n) + a_2 y_2(n) \qquad \text{... (1.4.17)}$$

Here a_1 and a_2 are arbitrary constants. The linearity principle is explained in Fig.1.4.1.

Fig. 1.4.1 (a) shows the output of the system due to linear combination of inputs. This figure represents L.H.S of equation (1.4.17). Fig. 1.4.1 (b) shows the linear combination of outputs due to individual inputs. This figure represents R.H.S. of equation (1.4.17). For the system to be linear,

$$y_3(n) = y'_3(n) \qquad \ldots (1.4.18)$$



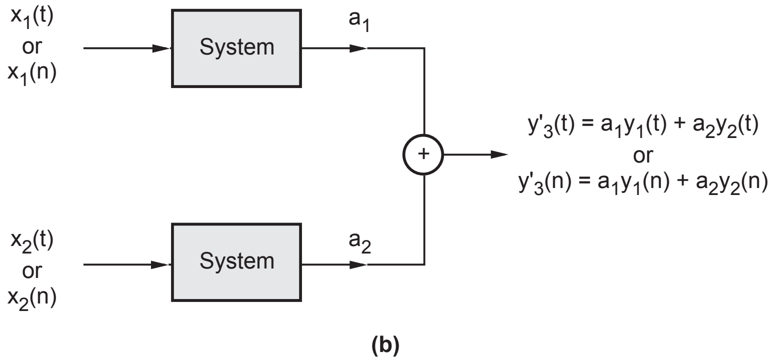


Fig. 1.4.1 (a) Response of the system to linear combination of inputs (b) Linear combination of outputs due to individual inputs

A linear system has the important property that, it produces zero output if the input is zero under the relaxed condition. If the system does not satisfy this condition, it is called non-linear system.

Example 1.4.2 *Determine whether the following discrete time systems are linear or non-linear.*
(i) $y(n) = x(n^2)$ *(ii)* $y(n) = x^2(n)$

Solution : (i) The given equation is,

$$y(n) = x(n^2)$$

For the two separate inputs $x_1(n)$ and $x_2(n)$ the system produces the response of,

$$\left.\begin{aligned} y_1(n) &= x_1(n^2) \\ \text{and} \quad y_2(n) &= x_2(n^2) \end{aligned}\right\} \quad ...(1.4.19)$$

The response of the system to the linear combination of $x_1(n)$ and $x_2(n)$ will be,

$$y_3(n) = f\left[a_1 x_1(n) + a_2 x_2(n)\right]$$

Since the linear systems satisfy *additive* property, the above equation can be written as,

$$y_3(n) = f\{a_1 x_1(n)\} + f\{a_2 x_2(n)\}$$

The linear systems also satisfy scaling property. Hence we can write above equation as,

$$y_3(n) = a_1 f\{x_1(n)\} + a_2 f\{x_2(n)\}$$

$$= a_1 x_1(n^2) + a_2 x_2(n^2) \quad \text{... (1.4.20)}$$

This is the response of the system to linear combination of two inputs. This type of operation is illustrated in Fig. 1.4.1 (a). Now the response of the system due to linear combination of two outputs will be,

$$y'_3(n) = a_1 y_1(n) + a_2 y_2(n)$$

Putting expressions from equation (1.4.19) in above equation we get,

$$y'_3(n) = a_1 x_1(n^2) + a_2 x_2(n^2) \quad \text{... (1.4.21)}$$

On comparing above equation with equation (1.4.20) we observe that,

$$y_3(n) = y'_3(n)$$

Hence the system is linear.

(ii) The given equation is,

$$y(n) = x^2(n)$$

When the inputs $x_1(n)$ and $x_2(n)$ are applied separately, the responses $y_1(n)$ and $y_2(n)$ will be,

$$\left.\begin{aligned} y_1(n) &= x_1^2(n) \\ \text{and} \quad y_2(n) &= x_2^2(n) \end{aligned}\right\} \quad \text{...(1.4.22)}$$

The response of the system to the linear combination of $x_1(n)$ and $x_2(n)$ will be,

$$y_3(n) = T\{a_1 x_1(n) + a_2 x_2(n)\}$$

$$= [a_1 x_1(n) + a_2 x_2(n)]^2 \quad \text{Since } y(n) = x^2(n)$$

$$= a_1^2 x_1^2(n) + 2a_1 a_2 x_1(n) x_2(n) + a_2^2 x_2^2(n) \quad \text{... (1.4.23)}$$

The linear combination of two outputs given by equation (1.4.22) will be,

$$y'_3(n) = a_1 x_1^2(n) + a_2 x_2^2(n)$$

On comparing above equation with equation (1.4.23) we find that,

$$y_3(n) \neq y'_3(n)$$

Hence the system is non-linear.

1.4.4 Causal and Non-causal Systems

Definition : ***The system is said to be causal if its output at any time depends upon present and past inputs only.***

A discrete time system is said to be causal if its output at any instant depends upon present and past input samples only. i.e.,

$$y(n) = f\left[x(k);\ k \leq n\right] \quad \text{... (1.4.24)}$$

Thus the output is the function of $x(n)$, $x(n-1), x(n-2), x(n-3)$ ... etc. For causal system. The system is non-causal if its output depends upon future inputs also, i.e. $x(n+1)$, $x(n+2)$, $x(n+3)$ ... etc.

Normally all causal systems are physically realizable. There is no system which can generate the output for inputs which will be available in future. Such systems are non-causal, and they are not physically realizable.

Example 1.4.3 *Check whether the discrete time systems described by following equations are causal or non-causal.*

(i) $y(n) = x(n) + x(n-1)$

(ii) $y(n) = x(n) + x(n+1)$

(iii) $y(n) = x(2n)$

Solution : (i) The given system equation is,

$$y(n) = x(n) + x(n-1)$$

Here $y(n)$ depends upon $x(n)$ and $x(n-1)$. $x(n)$ is the present input and $x(n-1)$ is the previous input. Hence the system is causal.

(ii) The given system equation is,

$$y(n) = x(n) + x(n+1)$$

Here $y(n)$ depends upon $x(n)$ and $x(n+1)$. $x(n)$ is the present input and $x(n+1)$ is the next input. Hence the system is non-causal.

(iii) The given system equation is,

$$y(n) = x(2n)$$

Here when $\quad n = 1 \Rightarrow y(1) = x(2)$

$$n = 2 \Rightarrow y(2) = x(4) \; ....$$

Thus the output $y(n)$ depends upon the future inputs. Hence the system is non-causal.

1.4.5 Stable and Unstable Systems

> **Definition :** *When every bounded input produces bounded output, then the system is called Bounded Input Bounded Output (BIBO) stable.*

This criteria is applicable for both the continuous time and discrete time systems. The input is said to be bounded if there exists some finite number M_x such that,

$$|x(n)| \leq M_x < \infty \qquad \text{... (1.4.25)}$$

Similarly the output is said to be bounded if there exists some finite number M_y such that,

$$|y(n)| \leq M_y < \infty \qquad \text{... (1.4.26)}$$

If the system produces unbounded output for bounded input, then it is unstable.

Example 1.4.4 *Determine whether the following discrete time systems are stable or not ?*

(i) $y(n) = x(n)+x(n-1)+y(n-1)$ *(ii)* $y(n) = r^n x(n)$, $r > 1$

Solution : (i) $\boldsymbol{y(n) = \quad (n) + \quad (n- \quad) + y(n- \quad)}$

In the above equation $x(n)$ and $x(n-1)$ are present and previous inputs. $y(n-1)$ is the previous output. As long as $x(n)$ is bounded $x(n-1)$ will also be bounded. Hence $y(n)$ will also be bounded. Hence the system is BIBO stable.

(ii) $\boldsymbol{y(n) = r^n x(n)}$ **,** $\boldsymbol{r > 1}$

Here $r > 1$. Hence as $n \to \infty$, $r^n \to \infty$. Therefore $y(n)$ will be unbounded even if $x(n)$ is bounded. Hence this system is unstable.

Example 1.4.5 *Few discrete time systems are given below :*

(i) $y(n) = \cos[x(n)]$ **AU : Dec.-10, May-12, Marks 3, Dec.-11, Marks 8, May-17, Marks 4**

(ii) $y(n) = \sum_{k=-\infty}^{n+1} x(k)$

(iii) $y(n) = x(n) \cos(\omega_0 n)$ **AU : Dec.-09, Marks 16, May-12, Marks 3**

(iv) $y(n) = x(-n+2)$ **AU : Dec.-10, May-12, Marks 3, May-09, Marks 8**

(v) $y(n) = |x(n)|$

(vi) $y(n) = x(n) u(n)$

(vii) $y(n) = x(n) + n\,x(n+1)$

AU : May-11, Dec.-11, Marks 8, May-12, Marks 3, Dec.-13, Marks 16, May-17, Marks 4

(viii) $y(n) = x(2n)$ **AU : May-17, Marks 4**

(ix) $y(n) = x(-n)$

(x) $y(n) = sgn\left[x(n)\right]$

Check whether these systems are :

1. Static or dynamic

2. Linear or non-linear

3. Shift invariant or shift variant

4. Causal or non-causal

5. Stable or unstable.

Solution : (i) $\boldsymbol{y(n) = \cos\left[x(n)\right]}$

1. A system is static if its output depends only upon the present input sample. Here since $y(n)$ depends upon cosine of $x(n)$, i.e. present input sample, the system is static.

2. For two separate inputs the system produces the response of,

$$y_1(n) = T\left\{x_1(n)\right\} = \cos\left[x_1(n)\right]$$

and

$$y_2(n) = T\left\{x_2(n)\right\} = \cos\left[x_2(n)\right]$$

The response of the system to linear combination of two inputs will be,

$$y_3(n) = T\left\{a_1\, x_1(n) + a_2\, x_2(n)\right\} = \cos\left[a_1\, x_1(n) + a_2\, x_2(n)\right]$$

The linear combination of two outputs will be,

$$y'_3(n) = a_1\, y_1(n) + a_2\, y_2(n) = a_1 \cos\left[x_1(n)\right] + a_2 \cos\left[x_2(n)\right]$$

Clearly $y_3(n) \neq y'_3(n)$. Hence system is non-linear.

3. The system is said to be shift invariant or time invariant if its characteristics do not change with shift of time origin. The given system is,

$$y(n) = T\left\{x(n)\right\} = \cos\left[x(n)\right] \quad \text{... (1.4.27)}$$

Let us delay the input by k samples. Then output will be,

$$y(n, k) = T\left\{x(n-k)\right\} = \cos\left[x(n-k)\right] \quad \text{... (1.4.28)}$$

Now let us delay the output $y(n)$ given by equation (1.4.27) by 'k' samples, i.e. $y(n-k)$. This is equivalent to replacing n by $n-k$ in equation (1.4.27). i.e.,

$$y(n-k) = cos\left[x(n-k)\right]$$

Comparing above equation with equation 1.4.28 we observe that,

$$y(n,k) = y(n-k)$$

This shows that the system is shift invariant.

4. The system is said to be causal if output depends upon past and present inputs only. The output is given as,

$$y(n) = cos\left[x(n)\right]$$

Here observe that n^{th} sample of output depends upon n^{th} sample of input $x(n)$. Hence the system is a causal system.

5. For any bounded value of $x(n)$ the cosine function has bounded value. Hence $y(n)$ has bounded value. Therefore the system is said to be BIBO stable.

Thus the given system is,

static, non-linear, shift invariant, causal and stable.

(ii) $\boldsymbol{y(n) = \sum_{k=-\infty}^{n+1} x(k)}$

1. Here observe that the system's output for n^{th} sample is equal to summation of all past input samples, present input sample and one next input sample. Hence the system needs to store these samples. Therefore the system requires memory. Hence this system is dynamic system.

2. The output $y(n)$ can be written as,

$$y(n) = x(-\infty) + \ldots + x(-2) + x(-1) + x(0) + x(1)$$
$$+ x(2) + x(3) + \ldots + x(n) + x(n+1)$$

From this equation it is clear that the system is linear since it is the summation of individual inputs. We know that the summation operation is a linear operation. Hence the system is a linear system.

3. The given system produces output $y(n)$ as a linear summation of inputs. If we shift the inputs, then there will be corresponding shift in the output. Hence the system is shift invariant.

4. The output $y(n)$ can be expressed as,

$$y(n) = x(-\infty) + \ldots + x(-2) + x(-1) + x(0) + x(1)$$
$$+ x(2) + x(3) + \ldots + x(n) + x(n+1)$$

In the above equation the present output $y(n)$ depends upon past inputs like $x(-\infty), x(-2), x(-1), x(0) \ldots x(n-1)$. It also depends upon the present input $x(n)$ and future or next input $x(n+1)$. Hence the system is non-causal since its output depends upon the future inputs also.

5. Consider that the system has input $x(k)$ as unit sample sequence $u(k)$. We know that,

$$u(k) = \begin{cases} 1 & \text{for} \quad k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence the system equation becomes,

$$y(n) = \sum_{k=0}^{n+1} u(k)$$

$$= 1+1+1+1+\ldots(n+1) \text{ number of times ' 1 ' are added}$$

For example if $n=5$, then above equation becomes,

$$y(5) = 1+1+1+1+1+1+1=7$$

if $n=10$, $\quad y(10) = 1+1+1+1+1+1+1+1+1+1+1+1=12$

It is clear from above equation that as $n \to \infty$, $y(n) \to \infty$. Here the input $u(k)$ is the unit sample sequence and it is bounded. But output $y(n)$ is unbounded as $n \to \infty$. Hence the system is unstable. Thus the given system is,

dynamic, linear, shift invariant, non-causal and unstable.

(iii) $y(n) = x(n) \cos(\omega_0 n)$

1. This is a static system since, the output of the system depends only upon the present input sample. i.e., n^{th} output sample depends upon n^{th} input sample. Hence this is a static system.

2. We know that the given system is,

$$y(n) = T\{x(n)\} = x(n) \cos(\omega_0 n)$$

When the two inputs $x_1(n)$ and $x_2(n)$ are applied separately, the responses $y_1(n)$ and $y_2(n)$ will be,

$$\left.\begin{aligned} y_1(n) &= T\{x_1(n)\} = x_1(n) \cos(\omega_0 n) \\ y_2(n) &= T\{x_2(n)\} = x_2(n) \cos(\omega_0 n) \end{aligned}\right\} \quad \ldots (1.4.29)$$

The response of the system due to linear combination of inputs will be,

$$y_3(n) = T\{a_1 x_1(n) + a_2 x_2(n)\} = [a_1 x_1(n) + a_2 x_2(n)]\cos(\omega_0 n)$$

$$= a_1 x_1(n)\cos(\omega_0 n) + a_2 x_2(n)\cos(\omega_0 n) \quad \text{... (1.4.30)}$$

Now the linear combination of the two outputs will be,

$$y'_3(n) = a_1 y_1(n) + a_2 y_2(n)$$

$$= a_1 x_1(n)\cos(\omega_0 n) + a_2 x_2(n)\cos(\omega_0 n)$$

from equation (1.4.29)

From above equation and equation (1.4.30),

$y_3(n) = y'_3(n)$. Hence the system is linear.

3. The system equation is,

$$y(n) = x(n)\cos(\omega_0 n)$$

The response of the system to delayed input will be,

$$y(n,k) = T\{x(n-k)\}$$

$$= x(n-k)\cos(\omega_0 n) \quad \text{... (1.4.31)}$$

Now let us delay or shift the output $y(n)$ by 'k' samples. i.e.,

$$y(n-k) = x(n-k)\cos[\omega_0 (n-k)]$$

Here every 'n' is replaced by $n-k$. On comparing above equation with equation (1.4.31) we find that,

$y(n,k) \neq y(n-k)$. Hence the system is shift variant.

4. In the given system, $y(n)$ depends upon $x(n)$, i.e. present output depends upon present input. Hence the system is causal.
5. The given system equation is,

$$y(n) = x(n)\cos(\omega_0 n)$$

Here value of $\cos(\omega_0 n)$ is always bounded. Hence as long as $x(n)$ is bounded, $y(n)$ is also bounded. Hence the system is stable.

This system is,

static, linear, shift variant, causal and stable.

(iv) $y(n) = x(-n+2)$

1. It is clear from above equation that n^{th} sample of output is equal to $(-n+2)^{th}$ sample of input. Hence the system needs memory storage. Therefore the system is dynamic.
2. It is very easy to prove that this system is linear.
3. The output $y(n)$ for delayed input will be,

$$\begin{aligned} y(n, k) &= T\{x(n-k)\} \\ &= x(-n+2-k) \end{aligned} \quad \text{... (1.4.32)}$$

Now the delayed output will be obtained by replacing n by $n-k$ in the system equation i.e.,

$$\begin{aligned} y(n-k) &= x[-(n-k)+2] \\ &= x(-n+2+k) \end{aligned}$$

On comparing above equation with equation (1.4.32) we find that,

$y(n, k) \neq y(n-k)$. Hence the system is shift variant

4. In the given system equation, when we put $n = 0$ we get,

$$y(0) = x(2)$$

Thus the output depends upon future inputs. Hence the system is non-causal.

5. It is clear from the given system equation that, as long as input is bounded, the output is bounded. Hence the system is a stable system.

Thus the given system is,

dynamic, linear, shift variant, non-causal and stable.

(v) $y(n) = |x(n)|$

1. The output is equal to magnitude of present input sample. Hence the system does not need memory storage. Therefore the system is static.
2. The given system equation is,

$$y(n) = T\{x(n)\} = |x(n)|$$

For two separate inputs $x_1(n)$ and $x_2(n)$ the system has the response of,

$$\left.\begin{aligned} y_1(n) &= T\{x_1(n)\} = |x_1(n)| \\ y_2(n) &= T\{x_2(n)\} = |x_2(n)| \end{aligned}\right\} \quad \text{... (1.4.33)}$$

The response of the system to linear combination of two inputs $x_1(n)$ and $x_2(n)$ will be,

$$y_3(n) = T\{a_1 x_1(n) + a_2 x_2(n)\}$$
$$= |a_1 x_1(n) + a_2 x_2(n)| \qquad \text{... (1.4.34)}$$

Now the linear combination of two outputs will be,

$$y'_3(n) = a_1 y_1(n) + a_2 y_2(n)$$
$$= a_1 |x_1(n)| + a_2 |x_2(n)|$$

Here observe that $y_3(n) \neq y'_3(n)$. Hence the system is non-linear.

3. Delaying the input by 'k' samples, output will be,

$$y(n, k) = T\{x(n-k)\}$$
$$= |x(n-k)|$$

And the delayed output will be,

$$y(n-k) = |x(n-k)|$$

Since $y(n, k) = y(n-k)$, the system is shift invariant.

4. The system equation is $y(n) = |x(n)|$. The output depends upon present input. Hence the system is causal.
5. From the given equation it is clear that as long as $x(n)$ is bounded, $y(n)$ will be bounded. Hence the system is stable.

Thus the given system is,

static, non-linear, shift invariant, causal and stable.

(vi) $y(n) = x(n)u(n)$

1. The output depends upon present input only. Hence the system is static.
2. The given system equation is,

$$y(n) = T\{x(n)\} = x(n)u(n)$$

The response of this system to the two inputs $x_1(n)$ and $x_2(n)$ when applied separately will be,

$$\left.\begin{aligned} y_1(n) &= T\{x_1(n)\} = x_1(n)u(n) \\ y_2(n) &= T\{x_2(n)\} = x_2(n)u(n) \end{aligned}\right\} \qquad \text{... (1.4.35)}$$

The response of the system to linear combination of inputs $x_1(n)$ and $x_2(n)$ will be,

$$y_3(n) = T\{a_1 x_1(n) + a_2 x_2(n)\}$$
$$= [a_1 x_1(n) + a_2 x_2(n)] u(n)$$
$$= a_1 x_1(n) u(n) + a_2 x_2(n) u(n) \quad \text{... (1.4.36)}$$

The linear combination of two outputs of equation (1.4.35) will be,

$$y'_3(n) = a_1 y_1(n) + a_2 y_2(n)$$
$$= a_1 x_1(n) u(n) + a_2 x_2(n) u(n)$$

From above equation and equation (1.4.36) we find that,

$y_3(n) = y'_3(n)$, hence the system is linear.

3. The response of the system to delayed input will be,

$$y(n,k) = T\{x(n-k)\}$$
$$= x(n-k) u(n) \quad \text{... (1.4.37)}$$

The delayed output will be obtained by replacing 'n' by $n-k$. i.e.,

$$y(n-k) = x(n-k) u(n-k) \quad \text{... (1.4.38)}$$

Here, on comparing above equation and equation (1.4.37), we find that,

$y(n,k) \neq y(n-k)$. Hence the system is shift variant.

4. The system equation is, $y(n) = x(n) u(n)$. The output depends upon present input only. Hence the system is causal.

5. We know that $u(n) = 1$ for $n \geq 0$ and $u(n) = 0$ for $n < 0$. This means $u(n)$ is a bounded sequence. Hence as long as $x(n)$ is bounded, $y(n)$ is also bounded. Hence this system is stable.

Thus, the given system is,

static, linear, shift variant, causal and stable.

(vii) $y(n) = x(n) + n\, x(n+1)$

1. From the given equation it is clear that, the output depends upon the present input and next input. Hence system is dynamic.

2. The given system equation is,

$$y(n) = T\{x(n)\} = x(n) + n\, x(n+1) \quad \text{... (1.4.39)}$$

If we apply two inputs $x_1(n)$ and $x_2(n)$ separately, then the outputs become,

$$\left.\begin{aligned} y_1(n) &= T\{x_1(n)\} = x_1(n) + n\,x_1(n+1) \\ \text{and}\quad y_2(n) &= T\{x_2(n)\} = x_2(n) + n\,x_2(n+1) \end{aligned}\right\} \qquad \text{... (1.4.40)}$$

Response of the system to linear combination of inputs $x_1(n)$ and $x_2(n)$ will be,

$$\begin{aligned} y_3(n) &= T\{a_1\,x_1(n) + a_2\,x_2(n)\} \\ &= a_1\left[x_1(n) + n\,x_1(n+1)\right] + a_2\left[x_2(n) + n\,x_2(n+1)\right] \end{aligned} \qquad \text{... (1.4.41)}$$

The linear combination of two outputs given by equation (1.4.40) will be,

$$\begin{aligned} y_3'(n) &= a_1\,y_1(n) + a_2\,y_2(n) \\ &= a_1\left[x_1(n) + n\,x_1(n+1)\right] + a_2\left[x_2(n) + n\,x_2(n+1)\right] \end{aligned}$$

On comparing above equation with equation (1.4.41) we observe that,

$y_3(n) = y_3'(n)$. Hence the system is linear.

3. The given system equation is,

$$y(n) = T\{x(n)\} = x(n) + n\,x(n+1) \qquad \text{... (1.4.42)}$$

Response of the system to delayed input will be,

$$\begin{aligned} y(n,k) &= T\{x(n-k)\} \\ &= x(n-k) + n\,x(n-k+1) \end{aligned} \qquad \text{... (1.4.43)}$$

Now let us delay the output of equation (1.4.42) by 'k' samples. i.e.,

$$y(n-k) = x(n-k) + (n-k)\,x(n-k+1)$$

Here we have replaced 'n' by '$n-k$'. On comparing above equation with equation 1.4.43 we observe that,

$y(n,k) \neq y(n-k)$. Hence the system is shift variant.

4. The given system equation is,

$$y(n) = x(n) + n\,x(n+1)$$

Here observe that n^{th} output sample depends upon $(n+1)^{th}$ i.e. next input sample, that is the output depends upon future input. Hence the system is non-causal.

5. In the given system equation observe that as $n \to \infty, y(n) \to \infty$ even if $x(n)$ is bounded. Hence the system is unstable.

Thus the given system is,

dynamic, linear, shift variant, non-causal and unstable.

(viii) $y(n) = x(2n)$

1. By putting $n = 1$ in the given system equation,

$$y(1) = x(2) \text{ similarly,}$$
$$n = 2 \Rightarrow y(2) = x(4)$$
$$n = 3 \Rightarrow y(3) = x(6) \quad \text{and so on.}$$

Thus the system needs to store the future input samples. Hence it requires memory. Therefore the system is dynamic.

2. The given system equation is,

$$y(n) = T\{x(n)\} = x(2n) \qquad \text{... (1.4.44)}$$

For two separate inputs $x_1(n)$ and $x_2(n)$ the system produces the response of

$$\left.\begin{aligned} y_1(n) &= T\{x_1(n)\} = x_1(2n) \\ \text{and } y_2(n) &= T\{x_2(n)\} = x_2(2n) \end{aligned}\right\} \qquad \text{... (1.4.45)}$$

The response of the system to linear combination of $x_1(n)$ and $x_2(n)$ will be,

$$y_3(n) = T\{a_1 x_1(n) + a_2 x_2(n)\}$$
$$= a_1 x_1(2n) + a_2 x_2(2n) \qquad \text{... (1.4.46)}$$

Now the linear combination of two outputs given by equation (1.4.45) will be,

$$y'_3(n) = a_1 y_1(n) + a_2 y_2(n)$$
$$= a_1 x_1(2n) + a_2 x_2(2n)$$

On comparing above equation with equation (1.4.46) we find that,

$y_3(n) = y'_3(n)$, Hence the system is linear.

3. The given system equation is,

$$y(n) = T\{x(n)\} = x(2n) \qquad \text{... (1.4.47)}$$

The response of the system to delayed input $x(n-k)$ will be,

$$y(n,k) = T\{x(n-k)\} = x(2n-k) \qquad \text{... (1.4.48)}$$

Now let us delay the output $y(n)$ given by equation (1.4.47) by 'k' samples. This is obtained by replacing 'n' by $n-k$ in equation (1.4.47). i.e.,

$$y(n-k) = x[2(n-k)]$$

$$\therefore \quad y(n-k) = x(2n-2k)$$

On comparing above equation with equation (1.4.48) we find that,

$y(n, k) \neq y(n-k)$. Hence the system is shift variant.

4. The given system equation is,

$$y(n) = x(2n)$$

Here output depends upon future inputs. i.e. n^{th} sample of output depends upon $(2n)^{th}$ sample of input. Clearly the system is non-causal.

5. As long as $x(n)$ is bounded, then $x(2n)$ is also bounded. Hence output $y(n)$ is also bounded. Therefore the system is stable.

Thus the given system is,

dynamic, linear, shift variant, non-causal and stable.

(ix) $y(n) = x(-n)$

1. If the present input is $x(n)$, then output $y(n)$ is $x(-n)$. That is if input is $x(4)$ then output is $x(-4)$. Thus the system has to store input sequence in the memory. Hence the system is static.

2. The given system equation is,

$$y(n) = T\{x(n)\} = x(-n) \qquad \text{... (1.4.49)}$$

When the two inputs $x_1(n)$ and $x_2(n)$ are applied separately, then the responses $y_1(n)$ and $y_2(n)$ will be,

$$\left.\begin{aligned} y_1(n) &= T\{x_1(n)\} = x_1(-n) \\ y_2(n) &= T\{x_2(n)\} = x_2(-n) \end{aligned}\right\} \qquad \text{... (1.4.50)}$$

The response of the system to the linear combination of $x_1(n)$ and $x_2(n)$ will be,

$$y_3(n) = T\{a_1 x_1(n) + a_2 x_2(n)\}$$

$$= a_1 x_1(-n) + a_2 x_2(-n) \qquad \text{... (1.4.51)}$$

The linear combination of two outputs $y_1(n)$ and $y_2(n)$ given by equation (1.4.50) will be,

$$y'_3(n) = a_1 y_1(n) + a_2 y_2(n)$$

$$= a_1 x_1(-n) + a_2 x_2(-n) \qquad \text{... (1.4.52)}$$

On comparing above equation with equation (1.4.51) we find that,

$y_3(n) = y'_3(n)$. Hence the system is a linear system.

3. Let us apply the delayed input to the system. Then the response will be,

$$y(n,k) = T\{x(n-k)\}$$
$$= x(-n-k) \qquad \text{... (1.4.53)}$$

We know that the system equation is given as,

$$y(n) = x(-n)$$

Now let us delay the output $y(n)$ by 'k' samples. This can be obtained by replacing 'n' by $(n-k)$ in the above equation, i.e.,

$$y(n-k) = x[-(n-k)]$$
$$= x(-n+k) \qquad \text{... (1.4.54)}$$

On comparing above equation with equation (1.4.53) we observe that,

$y(n,k) \neq y(n-k)$, hence the system is shift variant.

4. The given system equation is,

$$y(n) = x(-n)$$

For $n=-2 \Rightarrow y(n) = x(2)$

For $n=-1 \Rightarrow y(n) = x(1)$ etc.

Thus the output depends upon future inputs. Hence the system is non-causal.

5. As long as $x(n)$ is bounded, $x(-n)$ will also be bounded. Hence the output $y(n)$ is also bounded. Therefore the system is stable.

Thus the given system is,

static, linear, shift variant, non-causal and stable.

(x) $y(n) = sgn[x(n)]$

1. Since output depends upon the present input only, then this system is static.
2. The given system equation is,

$$y(n) = T\{x(n)\} = sgn[x(n)]$$

When the two inputs $x_1(n)$ and $x_2(n)$ are applied separately, the responses $y_1(n)$ and $y_2(n)$ will be,

$$\left.\begin{aligned} y_1(n) &= T\{x_1(n)\} = sgn\left[x_1(n)\right] \\ y_2(n) &= T\{x_2(n)\} = sgn\left[x_2(n)\right] \end{aligned}\right\} \quad \text{... (1.4.55)}$$

The response of the system to linear combination of $x_1(n)$ and $x_2(n)$ will be,

$$\begin{aligned} y_3(n) &= T\{a_1 x_1(n) + a_2 x_2(n)\} \\ &= sgn\left[a_1 x_1(n) + a_2 x_2(n)\right] \quad \text{... (1.4.56)} \end{aligned}$$

Basically $sgn\left[x(n)\right] = 1$ for $n > 0$ and $sgn\left[x(n)\right] = -1$ for $n < 0$. In the above equation $y_3(n)$ will have a value of '1' for $n > 0$ and '−1' for $n < 0$.

The linear combination of two outputs given by equation (1.4.55) will be,

$$\begin{aligned} y'_3(n) &= a_1 y_1(n) + a_2 y_2(n) \\ &= a_1\, sgn\left[x_1(n)\right] + a_2\, sgn\left[x_2(n)\right] \quad \text{... (1.4.57)} \end{aligned}$$

In the above equation $sgn\left[x_1(n)\right] = 1$ for $n > 0$ and -1 for $n < 0$. Similarly $sgn\left[x_2(n)\right] = 1$ for $n > 0$ and -1 for $n < 0$. Hence the values of $y'_3(n)$ of above equation and $y_3(n)$ of equation (1.4.56) are not same.

i.e. $y_3(n) \neq y'_3(n)$. Hence the system is non-linear.

3. The given system equation is,

$$y(n) = T[x(n)] = sgn\left[x(n)\right] \quad \text{... (1.4.58)}$$

The response of the system to delayed input $x(n-k)$ will be,

$$y(n, k) = T\left\{x(n-k)\right\} = sgn\left[x(n-k)\right] \quad \text{... (1.4.59)}$$

$$\begin{aligned} \text{Here } sgn\left[x(n-k)\right] &= 1 \quad \text{for } n > 0 \text{ and} \\ &= -1 \quad \text{for } n < 0 \end{aligned}$$

The delayed output $y(n-k)$ can be obtained from equation (1.4.58) by replacing 'n' by $(n-k)$ i.e.,

$$y(n-k) = sgn\left[x(n-k)\right] \quad \text{... (1.4.60)}$$

On comparing above equation with equation (1.4.59) we find that,

$y(n, k) = y(n-k)$, hence the system is shift invariant.

4. Since $y(n)$ depends upon the present input. This is a causal system.
5. Since $sgn\left[x(n)\right]$ has a value of ± 1 depending upon 'n', this is a stable system.

Thus the given system is,

static, non-linear, shift invariant, causal and stable.

Examples with Solution

Example 1.4.6 *For the systems represented by following functions, determine whether every system is,*

(i) Stable (ii) Causal (iii) Linear (iv) Shift invariant.

1) $T[x(n)] = e^{x(n)}$ **AU : May-08, Marks 8**

2) $T[x(n)] = a\,x(n) + b$ **AU : Dec.-07, Marks 8**

Solution : (1) $y(n) = T[x(n)] = e^{x(n)}$

(i) The given system equation is,

$$y(n) = T\{x(n)\} = e^{x(n)} \qquad \text{... (1.4.61)}$$

For bounded $x(n)$, $e^{x(n)}$ is also bounded. [Note that value of 'e' is basically 2.7182818]. Hence the given system is a stable system.

(ii) Present value of output depends upon present value of input. Hence this is a causal system.

(iii) The response of the system to linear combination of two inputs $x_1(n)$ and $x_2(n)$ will be,

$$y_3(n) = T\{a_1 x_1(n) + a_2 x_2(n)\}$$

From equation (1.4.61) we can write,

$$y_3(n) = e^{a_1 x_1(n) + a_2 x_2(n)}$$

$$= e^{a_1 x_1(n)} \cdot e^{a_2 x_2(n)} \qquad \text{... (1.4.62)}$$

The response of the system to two inputs $x_1(n)$ and $x_2(n)$ when applied separately is given as,

$$\left.\begin{aligned} y_1(n) &= T\{x_1(n)\} = e^{x_1(n)} \\ y_2(n) &= T\{x_2(n)\} = e^{x_2(n)} \end{aligned}\right\} \qquad \text{... (1.4.63)}$$

The linear combination of two outputs given by above equation will be,

$$y'_3(n) = a_1 y_1(n) + a_2 y_2(n)$$

$$= a_1 e^{x_1(n)} + a_2 e^{x_2(n)} \quad \text{Putting values from equation (1.4.63)}$$

On comparing above equation with equation (1.4.62), we find that,

$y_3(n) \neq y'_3(n)$. Hence this system is non-linear.

(iv) The response of the system to delayed input will be,

$$y(n, k) = T\{x(n-k)\} = e^{k(n-k)} \qquad \text{... (1.4.64)}$$

The delayed output can be obtained by replacing 'n' by '$n-k$' in equation (1.4.61). i.e.,

$$y(n-k) = e^{x(n-k)}$$

On comparing above equation with equation (1.4.64) we find that,

$y(n, k) = y(n-k)$, Hence the system is shift invariant.

Thus this given system is,

stable, causal, non-linear and shift invariant.

(2) $y(n) = T\{x(n)\} = a\, x(n) + b$

(i) In the given system equation a and b are constants. As long as $x(n)$ is bounded, $y(n)$ is also bounded. Hence the system is stable.

(ii) Present output depends upon present input only. Hence this is a causal system.

(iii) The response of the system to linear combination of two inputs $x_1(n)$ and $x_2(n)$ will be,

$$y_3(n) = T\{a_1 x_1(n) + a_2 x_2(n)\}$$

$$= a[a_1 x_1(n) + a_2 x_2(n)] + b \qquad \text{... (1.4.65)}$$

Here above equation is written from given system equation with $(x(n) = a_1 x_1(n) + a_2 x_2(n).)$ The response of the system to two inputs $x_1(n)$ and $x_2(n)$ when applied separately is given as,

$$\left.\begin{aligned} y_1(n) &= T\{x_1(n)\} = a\, x_1(n) + b \\ \text{and } y_2(n) &= T\{x_2(n)\} = a\, x_2(n) + b \end{aligned}\right\} \qquad \text{... (1.4.66)}$$

The linear combination of two outputs given by above equation will be,

$$y'_3(n) = a_1 y_1(n) + a_2 y_2(n)$$

$$= a_1 [a\, x_1(n) + b] + a_2 [a\, x_2(n) + b]$$

Putting values from equation (1.4.66).

On comparing above equation with equation (1.4.65) we find that,

$y_3(n) \neq y'_3(n)$. Hence the system is non-linear.

(iv) The response of the system to the input delayed by 'k' samples will be,

$$y(n,k) = T\{x(n-k)\} = a\,x(n-k)+b \quad \text{... (1.4.67)}$$

And the delayed output by 'k' samples will be obtained by replacing 'n' by $(n-k)$ in given system equation. i.e.,

$$y(n-k) = a\,x(n-k)+b$$

On comparing above equation with equation (1.4.67) we find that,

$y(n,k) = y(n-k)$. Hence the system is shift invariant.

Thus the given system is,

stable, causal, non-linear and shift invariant.

Example 1.4.7 *(i) Check the causality and stability of the system,*

$y(n) = x(-n) + x(n-2) + x(2n-1)$

(ii) Check the linearity and time invariance of the system,

$y(n) = (n-1)\,x(n) + C$ **AU : May-14, Dec.-15, Marks 16**

Solution : (i) $y(n) = x(-n) + x(n-2) + x(2n-1)$

Causality : For $n = -1$, $y(-1) = x(1) + x(-3) + x(-3)$

Here $y(-1)$ is present output and $x(1)$ is future input. Since output depends upon future inputs, the system is *noncausal.*

Stability : As long as the inputs $x(-n)$, $x(n-2)$ and $x(2n-1)$ are bounded, the output is the sum of these inputs and hence it will be bounded. Hence the system is *BIBO stable.*

(ii) $y(n) = (n-1)\,x(n) + C$

Linearity : Response of the system to linear combination of inputs $x_1(n)$ and $x_2(n)$ will be,

$$y_3(n) = T\{a_1 x_1(n) + a_2 x_2(n)\} = (n-1)[a_1 x_1(n) + a_2 x_2(n)] + C$$

$$= a_1(n-1)x_1(n) + a_2(n-1)x_2(n) + C$$

Output of the system to inputs $x_1(n)$ and $x_2(n)$ will be,

$$y_1(n) = T\{x_1(n)\} = (n-1)x_1(n) + C$$

$$y_2(n) = T\{x_2(n)\} = (n-1)x_2(n) + C$$

Linear combination of two outputs $y_1(n)$ and $y_2(n)$ wil be,

$$y'_3(n) = a_1 y_1(n) + a_2 y_2(n)$$

$$= a_1(n-1)x_1(n) + a_1 C + a_2(n-1)x_2(n) + a_2 C$$

Since $y_3(n) \neq y'_3(n)$, the system is *nonlinear*.

Time invariance : Response of the system to delayed input will be,

$$y(n, k) = T\{x(n-1)\} = (n-1)\, x(n-k) + C$$

Delayed output by 'k' samples will be,

$$y(n-k) = (n-k-1)\, x(n-k) + C$$

Since $y(n, k) \neq y(n-k)$, the system is *time variant*.

Example 1.4.8 *For each impulse response listed below, determine if the corresponding system is (i) Causal (ii) Stable*

1) $h(n) = 2^n u(-n)$ 2) $h(n) = \sin \frac{n\pi}{2}$ 3) $h(n) = \delta(n) + \sin \pi n$ 4) $h(n) = \rho^{2n} u(n-1)$

AU : May-07, Marks 16

Solution : 1) $h(n) = 2^n u(-n)$

Causality

$$u(-n) = \begin{cases} 0 & \text{for } n > 0 \\ 1 & \text{for } n \leq 0 \end{cases}$$

Hence $h(n) = 2^n$ for $n \leq 0$. Since $h(n) \neq 0$ for $n < 0$, this system is noncausal.

Stability

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=-\infty}^{0} 2^k \quad \text{since n (–k) = 1 for k} \le 0$$

$$= \sum_{k=0}^{\infty} 2^{-k} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

$$= \frac{1}{1-\frac{1}{2}} = 2$$

Since $\sum_{k=-\infty}^{\infty} |h(k)| = 2 < \infty$, this system is stable.

2) $h(n) = \sin \frac{n\pi}{2}$

$$= \{\ldots 1, 0, -1, 0, 1, 0, -1, \underset{\uparrow}{0}, 1, 0, -1, 0, 1, 0, -1, \ldots\}$$

i) Since h(n) ≠ 0 for n < 0, this sequence is noncausal.

ii) This sequence is of infinite duration. Hence $\sum_{k=-\infty}^{\infty} |h(k)|$ will also be infinite. Hence this system is unstable.

3) $h(n) = \delta(n) + \sin \pi n$

sin π n is always zero for any integer value of n.

$$\therefore \quad h(n) = \delta(n)$$

This system is causal as well as stable.

4) $h(n) = \rho^{2n}\, u(n-1)$

i) Since $u(n-1) = 1$ for n ≥ 1, $h(n) = 0$ for n < 1. Therefore this system is causal.

ii) $\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=-\infty}^{\infty} \rho^{2k} u(k-1) = \sum_{k=1}^{\infty} (\rho^2)\, k$

Let us rearrange above equation as,

$$= \sum_{k=0}^{\infty} (\rho^2)^{\,k} - 1$$

$$= \frac{1}{1-\rho^2} - 1 = \frac{\rho^2}{1-\rho^2} \quad \text{if } \rho^2 < 1$$

Thus system is stable if $\rho^2 < 1$, otherwise it is unstable.

Example 1.4.9 *A discrete time system is represented by the following difference equation in which x(n) is input and y(n) is output.* $y(n) = 3y(n-1) - nx(n) + 4x(n-1) + 2x(n+1)$ *and* $n \geq 0$. *Is this system linear ? Shift invariant ? Causal ? In each case, justify your answer.* **AU : May-15, Marks 12**

Solution : Linearity : This is a constant coefficient difference equation without any constant values. Hence the system is *linear.*

Shift invariance : Observe that one of the input terms is multiplied by 'n' [n x(n) term in the given equation]. Since 'n' is time/shift dependent it makes the system *shift variant.*

Causality : The present output y(n), depends upon future inputs [x(n + 1) term in the given equation] also. Hence the system is non-causal.

Examples for Practice

Example 1.4.10 *State whether the following systems are (i) Static (ii) Linear (iii) Shift invariant (iv) Causal and (v) Stable.*

(a) $y(n) = n\,x(n)$ **[Ans. : Static, linear, shift variant, causal, unstable]**

(b) $y(n) = a\,x(n)$ **[Ans. : Static, linear, shift invariant, causal, stable]**

(c) $y(n) = x(n^2)$ **[Ans. : Dynamic, linear, shift invariant, non-causal, stable]**

(d) $y(n) = \sum_{k=-\infty}^{n} x(k)$ **[Ans. : Dynamic, linear, shift invariant, causal, unstable]**

(e) $y(n) = x(n) + 3u(n+1)$ **[Ans. : Static, non-linear, shift variant, non-causal, stable]**

(f) $y(n) = g(n)\,x(n)$. *Here* $g(n)$ *is another sequence,* **[Ans. : Static, linear, shift variant, causal, unstable]**

Example 1.4.11 *Determine the stability for each of the following linear system :*

(1) $y_1(n) = \sum_{k=0}^{\infty} (3/4)^k x(n-k)$ (2) $y_2(n) = \sum_{k=0}^{\infty} 2^k x(n-k)$.

AU : Dec.-12, Marks 8

[Ans. : Hint and Ans. : 1) As $k \to \infty, \left(\frac{3}{4}\right)^k \to 0$, stable
2) As $k \to \infty, 2^k \to \infty$, Unstable]

Example 1.4.12 *For each of the following systems, determine whether the system is stable, causal, linear and time invariant ?*

(i) $T\{x(n)\} = g(n)\, x(n)$

(ii) $T\{x(n)\} = \sum_{k=n_0}^{n} x(k)$

AU : Dec.-07, Marks 8

[Ans. : (i) stable, causal, linear and time variant
(ii) stable, causal, linear and time invariant]

Review Questions

1. *What is causality and stability of a system ?* **AU : Dec.-12, Marks 4**
2. *Explain how systems are classified.*
3. *What is BIBO stability for systems ?*
4. *Explain the properties of discrete time system.* **AU : May-15, Marks 4**
5. *Distinguish between time variant vs time invariant system.* **AU : Dec.-16, Marks 4**

1.5 Nyquist-Shannon Sampling Theorem

AU : May-06, 11, 15, 16, 17, Dec.-05, 06, 08, 10, 11, 12, 15

1.5.1 Sampling, Quantization and Quantization Error

Sampling : For digital processing, it is necessary to sample the continuous time signal. Fig. 1.5.1 (a) shows analog/continuous time signal and its sampled version is shown in Fig. 1.5.1 (b). The sampled signal takes instantaneous amplitudes of analog signal at the time of sampling.

Quantization : Analog signal is continuous in amplitude. However digital signal fed to the processor can accept only discrete amplitude levels. Hence analog amplitude of the sample is rounded off to nearest discrete amplitude level. This process is called *quantization*. Fig. 1.5.1 (c) shows such quantized signal. Observe that the samples x(1), x(2), x(3) and x(4) are quantized to nearest amplitude level of 3 V even though their analog amplitudes are different.

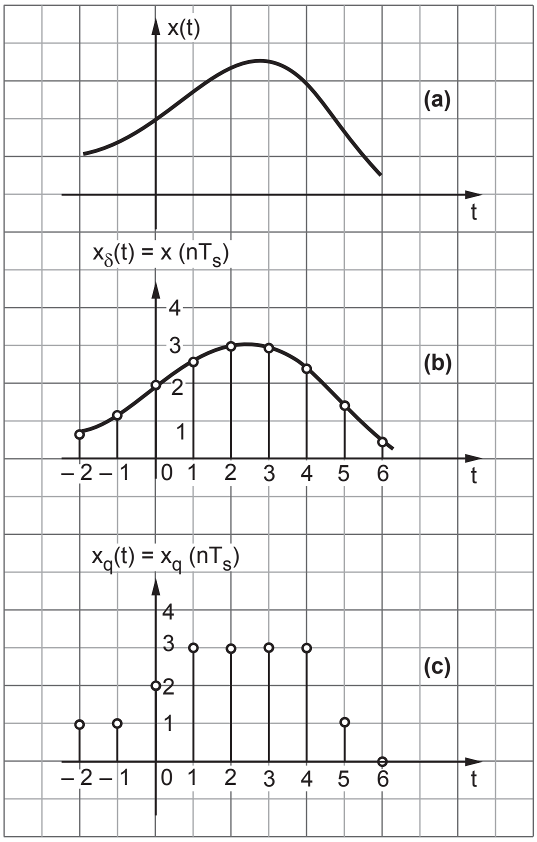


Fig. 1.5.1 Sampling and quantization

The quantized levels are digitally represented by binary bits. For example if there are four quantization levels, then they can be represented as 0(00), 1(01), 2(10), 3(11). Thus two digits are enough to represent four quantization levels.

Quantization error (ε) : It is the difference between quantized value of the sample and its actual value. i.e.,

$$\varepsilon = x_q(nT_s) - x(nT_s)$$

If the difference between the two quantization levels is 'δ', then maximum quantization error will be $\frac{\delta}{2}$. Thus,

$$-\frac{\delta}{2} \leq \varepsilon \leq \frac{\delta}{2}$$

Once the sample is quantized, the quantization error is permanently introduced in the digital signal. It affects the performance of digital filters, FFT algorithms and filter realization.

1.5.2 Sampling Theorem for Low Pass (LP) Signals

A low pass or LP signal contains frequencies from 1 Hz to some higher value.

Statement of sampling theorem

1) ***A band limited signal of finite energy, which has no frequency components higher than W Hertz, is completely described by specifying the values of the signal at instants of time separated by*** $\frac{1}{2W}$ ***seconds and***

2) ***A band limited signal of finite energy, which has no frequency components higher than W Hertz, may be completely recovered from the knowledge of its samples taken at the rate of 2W samples per second.***

The first part of above statement tells about sampling of the signal and second part tells about reconstruction of the signal. Above statement can be combined and stated alternately as follows :

A continuous time signal can be completely represented in its samples and recovered back if the sampling frequency is twice of the highest frequency content of the signal. i.e.,

$$f_s \geq 2W$$

Here f_s is the sampling frequency and

W is the higher frequency content

Proof of sampling theorem

There are two parts : (I) Representation of $x(t)$ in terms of its samples

(II) Reconstruction of $x(t)$ from its samples.

Part I : Representation of $x(t)$ in its samples $x(nT_s)$

Step 1 : Define $x_\delta(t)$

Step 2 : Fourier transform of $x_\delta(t)$ i.e. $X_\delta(f)$

Step 3 : Relation between $X(f)$ and $X_\delta(f)$

Step 4 : Relation between $x(t)$ and $x(nT_s)$

Step 1 : Define x_δ t)

Refer Fig. 1.5.1. The sampled signal $x_\delta(t)$ is given as,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s) \qquad \ldots (1.5.1)$$

Here observe that $x_\delta(t)$ is the product of x_δ and impulse train $\delta(t)$ as shown in Fig. 1.5.1. In the above equation $\delta(t-nT_s)$ indicates the samples placed at $\pm T_s$, $\pm 2T_s$, $\pm 3T_s$ ... and so on.

Step 2 : FT of x_δ t) i.e. X_δ f)

Taking FT of equation (1.5.1).

$$X_\delta(f) = \text{FT}\left\{\sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s)\right\}$$

$$= \text{FT}\ \{\text{ Product of } x(t) \text{ and impulse train }\}$$

We know that FT of product in time domain becomes convolution in frequency domain. i.e.,

$$X_\delta(f) = \text{FT}\ \{x(t)\} * FT\ \{\delta(t-nT_s)\} \qquad \ldots(1.5.2)$$

By definitions, $x(t) \xleftrightarrow{FT} X(f)$ and

$$\delta(t-nT_s) \xleftrightarrow{FT} f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

Hence equation (1.5.2) becomes,

$$X_\delta(f) = X(f) * f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

Since convolution is linear,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f - nf_s)$$

$$= f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \qquad \textit{By shifting property of impulse function}$$

$$= \cdots f_s\, X(f-2f_s) + f_s\, X(f-f_s) + f_s\, X(f) + f_s\, X(f-f_s) + f_s\, X(f-2f_s) + \cdots$$

Comments (i) The R.H.S. of above equation shows that *X(f)* is placed at $\pm f_s, \pm 2f_s, \pm 3f_s, \cdots$

(ii) This means *X(f)* is periodic in f_s.

(iii) If sampling frequency is $f_s = 2W$, then the spectrums $X(f)$ just touch each other.

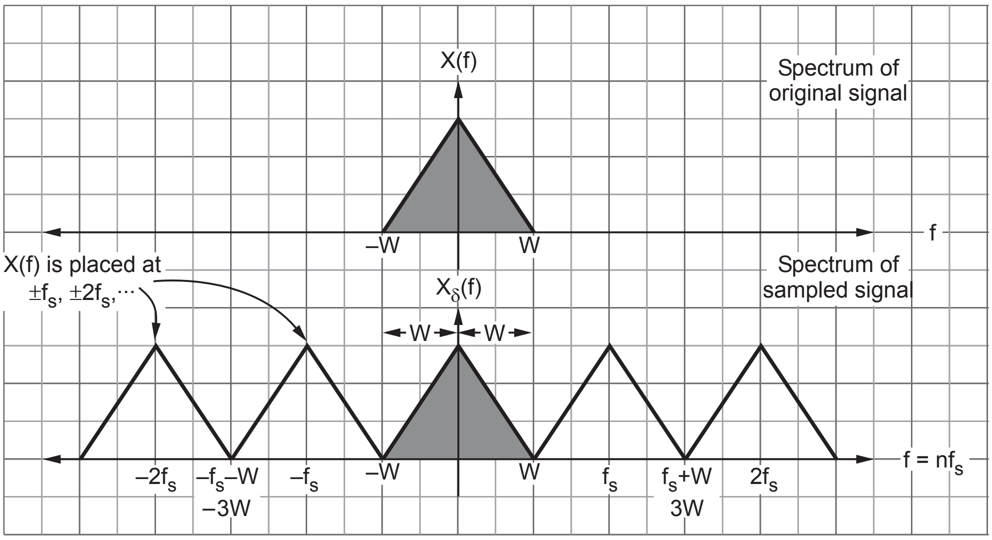


Fig. 1.5.2 Spectrum of original signal and sampled signal

Step 3 : Relation between $X\ f)$ and $X_\delta\ f)$

Important assumption : Let us assume that $f_s = 2W$, then as per above diagram.

$$X_\delta(f) = f_s\, X(f) \qquad \text{for } -W \le f \le W \text{ and } f_s = 2W$$

or

$$X(f) = \frac{1}{f_s} X_\delta(f) \qquad \ldots (1.5.3)$$

Step 4 : Relation between $x(t)$ and $x(nT_s)$

DTFT is, $$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$\therefore \quad X(f) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi f n} \qquad \dots (1.5.4)$$

In above equation 'f' is the frequency of DT signal. If we replace $X(f)$ by $X_\delta(f)$, then 'f' becomes frequency of CT signal. i.e.,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi \frac{f}{f_s} n}$$

In above equation 'f' is frequency of CT signal. And $\frac{f}{f_s}$ = Frequency of DT signal in equation (1.5.4). Since $x(n) = x(nT_s)$, i.e. samples of $x(t)$, then we have,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s} \quad \text{since } \frac{1}{f_s} = T_s$$

Putting above expression in equation (1.5.3),

$$X(f) = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s}$$

Inverse Fourier transform (IFT) of above equation gives $x(t)$ i.e.,

$$x(t) = IFT\left\{\frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s}\right\} \qquad \dots (1.5.5)$$

Comments :

i) Here $x(t)$ is represented completely in terms of $x(nT_s)$.

ii) Above equation holds for $f_s = 2W$. This means if the samples are taken at the rate of 2 W or higher, $x(t)$ is completely represented by its samples.

iii) First part of the sampling theorem is proved by above two comments.

Part II : Reconstruction of $x(t)$ from its samples

Step 1 : Take inverse fourier transform of $X(f)$ which is in terms of $X_\delta(f)$.

Step 2 : Show that $x(t)$ is obtained back with the help of interpolation function.

Step 1 : The IFT of equation (1.5.5) becomes,

$$x(t) = \int_{-\infty}^{\infty} \left\{ \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s} \right\} e^{j2\pi f t}\, df$$

Here the integration can be taken from $-W \le f \le W$. Since $X(f) = \frac{1}{f_s} X_\delta(f)$ for $-W \le f \le W$. (See Fig. 1.5.2).

$$\therefore \quad x(t) = \int_{-W}^{W} \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s} \cdot e^{j2\pi f t}\, df$$

Interchanging the order of summation and integration,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{1}{f_s} \int_{-W}^{W} e^{j2\pi f(t-nT_s)}\, df$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \cdot \left[\frac{e^{j2\pi f(t-nT_s)}}{j\,2\pi\,(t-nT_s)} \right]_{-W}^{W}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \left\{ \frac{e^{j2\pi W(t-nT_s)} - e^{-j2\pi W(t-nT_s)}}{j\,2\pi\,(t-nT_s)} \right\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \cdot \frac{\sin 2\pi W\,(t-nT_s)}{\pi\,(t-nT_s)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \pi\,(2Wt - 2W\,nT_s)}{\pi\,(f_s\,t - f_s\,nT_s)}$$

Here $f_s = 2W$, hence $T_s = \frac{1}{f_s} = \frac{1}{2W}$. Simplifying above equation,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \pi\,(2Wt-n)}{\pi\,(2Wt-n)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,\mathrm{sinc}(2Wt-n) \qquad \text{since} \quad \frac{\sin \pi\theta}{\pi\theta} = \mathrm{sinc}\,\theta \qquad \ldots(1.5.6)$$

Step 2 : Let us interpret the above equation. Expanding we get,

$$x(t) = \cdots + x(-2T_s)\,\mathrm{sinc}(2Wt+2) + x(-T_s)\,\mathrm{sinc}(2Wt+1) + x(0)\,\mathrm{sinc}(2Wt) + x(T_s)\,\mathrm{sinc}(2Wt-1) + \cdots$$

Comments :

(i) The samples $x(nT_s)$ are weighted by sinc functions.

(ii) The sinc function is the interpolating function. Fig. 1.5.3 shows, how $x(t)$ is interpolated. (See Fig. 1.5.3 on next page)

Step 3 : Reconstruction of x t) by lowpass filter

When the interpolated signal of equation (1.5.6) is passed through the lowpass filter of bandwidth $-W \leq f \leq W$, then the reconstructed waveform shown in above Fig. 1.5.3 (b) is obtained. The individual sinc functions are interpolated to get smooth $x(t)$.

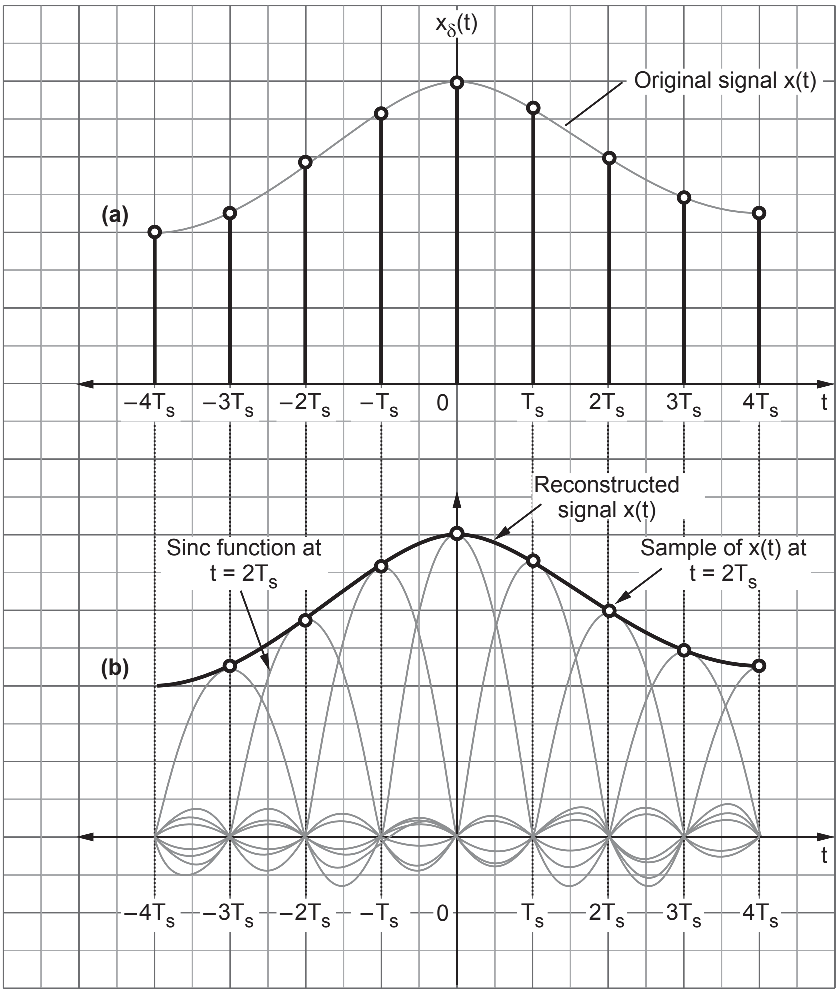


Fig. 1.5.3 (a) Sampled version of signal $x(t)$
(b) Reconstruction of $x(t)$ from its samples

1.5.3 Effects of Undersampling (Aliasing)

While proving sampling theorem we considered that $f_s = 2W$. Consider the case of $f_s < 2W$. Then the spectrum of $X_\delta(f)$ shown in Fig. 1.5.4 will be modified as follows :

Comments :

i) The spectrums located at $X(f), X(f-f_s), X(f-2f_s), \ldots$ overlap on each other.

ii) Consider the spectrums of $X(f)$ and $X(f-f_s)$ shown as magnified in Fig. 1.5.4. The frequencies from $(f_s - W)$ to W are overlapping in these spectrums.

iii) The high frequencies near 'ω' in $X(f-f_s)$ overlap with low frequencies $(f_s - W)$ in $X(f)$.

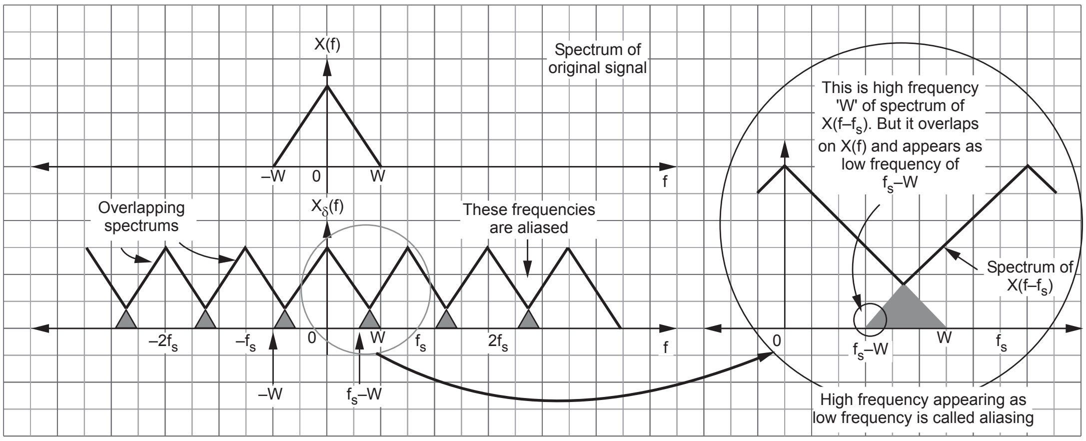


Fig. 1.5.4 Effects of undersampling or aliasing

Definition of aliasing : When the high frequency interferes with low frequency and appears as low frequency, then the phenomenon is called aliasing.

Effects of aliasing : i) Since high and low frequencies interfere with each other, distortion is generated.

ii) The data is lost and it cannot be recovered.

Different ways to avoid aliasing

Aliasing can be avoided by two methods :

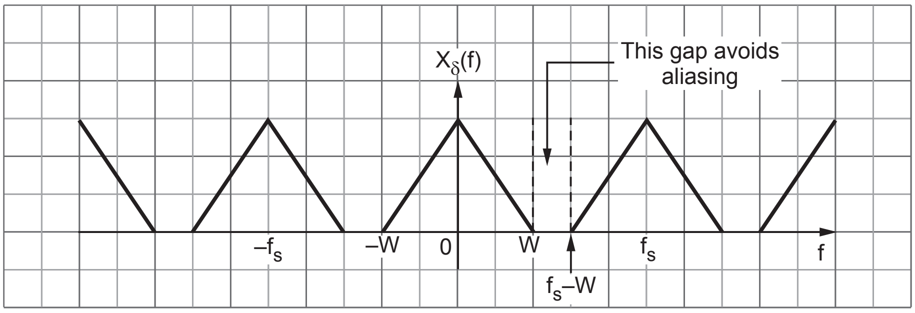


Fig. 1.5.5 $f_s \geq 2$ W avoids aliasing by creating a bandgap

i) Sampling rate $f_s \geq 2W$.

ii) Strictly bandlimit the signal to 'W'.

i) Sampling rate $f_s \geq 2W$

When the sampling rate is made higher than $2W$, then the spectrums will not overlap and there will be sufficient gap between the individual spectrums. This is shown in

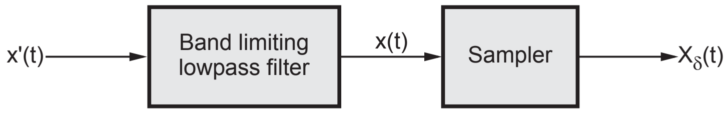


Fig. 1.5.6 Bandlimiting the signal. The bandlimiting LPF is called prealias filter

Fig. 1.5.5. (See Fig. 1.5.5 on next page).

ii) Bandlimiting the signal

The sampling rate is, f_s = 2W. The ideally speaking there should be no aliasing. But there can be few components higher than 2W. These components create aliasing. Hence a lowpass filter is used before sampling the signals as shown in Fig. 1.5.6. Thus the output of lowpass filter is strictly bandlimited and there are no frequency components higher than 'W'. Then there will be no aliasing.

1.5.4 Nyquist Rate and Nyquist Interval

Nyquist rate : When the sampling rate becomes exactly equal to '$2W$' samples/sec, for a given bandwidth of W Hertz, then it is called Nyquist rate.

Nyquist interval : It is the time interval between any two adjacent samples when sampling rate is Nyquist rate.

$$\text{Nyquist rate} = 2W \text{ Hz} \quad \dots (1.5.7)$$

$$\text{Nyquist interval} = \frac{1}{2W} \text{ seconds} \quad \dots (1.5.8)$$

1.5.5 Reconstruction Filter (Interpolation Filter)

Definition

In section 1.5.2 we have shown that the reconstructed signal is the succession of sinc pulses weighted by $x(nT_s)$. These pulses are interpolated with the help of a lowpass filter. It is also called *reconstruction* filter or *interpolation* filter.

Ideal filter

Fig. 1.5.7 shows the spectrum of sampled signal and frequency response of required filter. When the sampling frequency is exactly 2W, then the spectrums just touch each

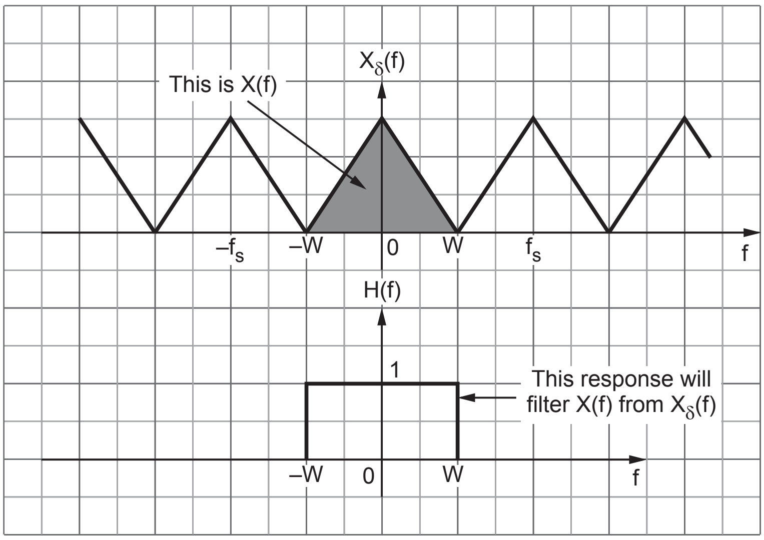


Fig. 1.5.7 Ideal reconstruction filter

other as shown in Fig. 1.5.7. The spectrum of original signal, $X(f)$ can be filtered by an ideal filter having passband from $-W \leq f \leq W$. (See Fig. 1.5.7 on next page)

Non-ideal filter

As discussed above, an ideal filter of bandwidth 'W' filters out an original signal. But practically ideal filter is not realizable. It requires some transition band. Hence f_s must be greater than $2W$. It creates the gap between adjacent spectrums of $X_\delta(f)$. This gap can be used for the transition band of the reconstruction filter. The spectrum $X(f)$ is then properly filtered out from $X_\delta(f)$. Hence the sampling frequency must be greater than '2W' to ensure sufficient gap for transition band.

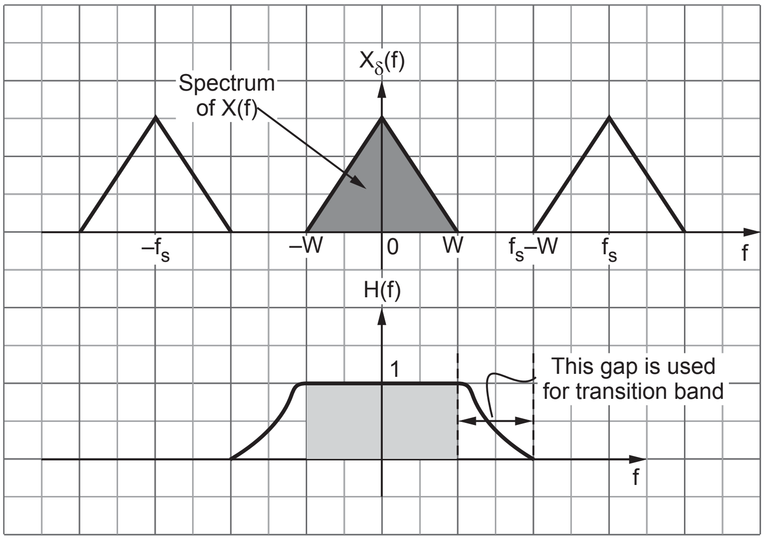


Fig. 1.5.8 Practical reconstruction filter

1.5.6 Antialiasing Filter

Sampling theorem states that the signal $x(t)$ can be completely recovered from its samples if

$$\Omega_{sF} \geq 2\,\Omega_m$$

Here Ω_{sF} is the sampling frequency

and Ω_m is the highest frequency in $x(t)$.

If above condition is not satisfied, then aliasing takes place. Hence the signal $x(t)$ is passed through an analog antialiasing lowpass filter before sampling. The antialiasing filter blocks all the frequency components higher than $\frac{\Omega_{sF}}{2}$. i.e.,

$$H_a(j\,\Omega) = \begin{cases} 1 & for\,|\Omega| < \dfrac{\Omega_{sF}}{2} \\ 0 & for\,|\Omega| > \dfrac{\Omega_{sF}}{2} \end{cases} \quad \ldots(1.5.9)$$

Above filter is an ideal filter, since it has no transition band. Hence it cannot be realized. Hence some filter approximation can be used to design anti-aliasing filter. For the practical antialiasing filter following condition must be satisfied,

$$\Omega_p < \frac{\Omega_{sF}}{2} \quad \ldots(1.5.10)$$

Here Ω_p is the passband edge frequency. It is normally the highest signal frequency Ω_m. Fig. 1.5.9 shows the magnitude response of antialiasing filter.

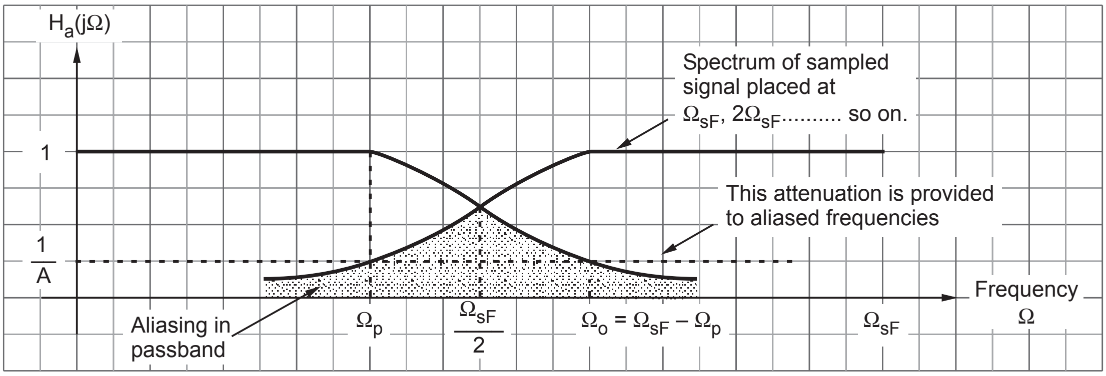


Fig. 1.5.9 Magnitude response of antialiasing filter

In above figure observe that the anti-aliasing filter has passband from 0 to Ω_p. There is no attenuation in passband. The maximum signal frequency $\Omega_m \leq \Omega_p$. Hence all the signal frequencies are passed without attenuation. There is transition band from Ω_p to Ω_o. Observe that aliasing frequencies are present at Ω_p. But its magnitude must be less than $\frac{1}{A}$. In the above figure observe that aliased component Ω_o is present at Ω_p. It is clear from spectrum of sampled signal and it is present at Ω_{sF}. All the frequencies higher than $\frac{\Omega_{sF}}{2}$ appear as aliasing frequencies. The filter provides an attenuation of at least $\frac{1}{A}$ to aliased frequencies less than Ω_p. This attenuation is within the tolerable limits.

Example for Understanding

Example 1.5.1 *Find the Nyquist rate and Nyquist interval for following signals.*

i) $m(t) = \frac{1}{2\pi}\cos(4000\pi t)\cos(1000\pi t)$ *ii)* $m(t) = \frac{\sin 500\pi t}{\pi t}$

Solution :

i)
$$m_1(t) = \frac{1}{2\pi}\cos(4000\pi t)\cos(1000\pi t)$$

$$= \frac{1}{2\pi}\left\{\frac{1}{2}\left[\cos(4000\pi t - 1000\pi t) + \cos(4000\pi t + 1000\pi t)\right]\right\}$$

$$= \frac{1}{4\pi}\left[\cos 3000\pi t + \cos 5000\pi t\right]$$

$$= \frac{1}{4\pi}\left[\cos 2\pi f_1 t + \cos 2\pi f_2 t\right]$$

Comparing, we get, $f_1 = 1500$ Hz and $f_2 = 2500$ Hz

Here highest frequency $W = f_2 = 2500$ Hz

$\therefore$ Nyquist rate $= 2W = 2 \times 2500 = 5000$ Hz

$$\text{Nyquist interval} = \frac{1}{2W} = \frac{1}{2 \times 2500} = 0.2 \text{ msec}$$

ii) $$m_2(t) = \frac{\sin 500\pi t}{\pi t}$$

$$= \frac{\sin 2\pi f t}{\pi t}$$

Comparing, we get, f = 250 Hz or W = 250 Hz

$\therefore$ Nyquist rate $= 2W = 2 \times 250 = 500$ Hz

$$\text{Nyquist interval} = \frac{1}{2W} = \frac{1}{2 \times 250} = 2 \text{ msec}$$

Example 1.5.2 *A signal x(t) =* sinc $(50\pi t)$ *is sampled at a rate of (1) 20 Hz (2) 50 Hz* and *(3) 75 Hz. For each of these cases, explain if you can recover the signal x(t) from the samples signal.* **AU : May-16, Marks 6**

Solution : We know that

$$\text{sinc}\,\theta = \frac{\sin \pi\theta}{\pi\theta}$$ is normalized sinc function and

$$\text{sinc}\,\theta = \frac{\sin\theta}{\theta}$$ is unnormalized sinc function. Here $x(t) = \text{sinc}\,(50\pi\,t) = \dfrac{\sin 50\pi t}{50\pi t}$

is unnormalized sinc function. Here,

$$2\pi f t = 50\pi t \Rightarrow f = 25 \text{ Hz}$$

Here observe that the frequency components are located at ± 25 Hz.

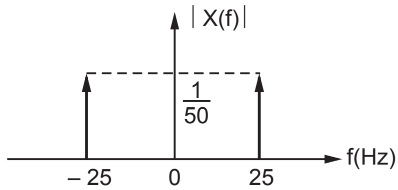


Fig. 1.5.10

Hence $W = f = 25$ Hz

1) f_s **= 20 Hz**

Nyquist rate = 2W = 2 × 25 = 50 Hz. Since $f_s < 2W$, original signal cannot be recovered from its samples due to aliasing.

2) f_s **= 50 Hz and** f_s **= 75 Hz**

Since $f_s \geq 50$ Hz in both the cases, there will be no aliasing and original signal can be recovered completely from its samples.

Example 1.5.3 *Complete the Nyquist sampling frequency of the signal $x(t) = 4\ \mathrm{sinc}(3t/\pi)$.*

AU : May-17, Marks 3

Solution :

$$x(t) = 4\ \mathrm{sinc}(3t/\pi) = \frac{4 \sin\left(\frac{3t\,\pi}{\pi}\right)}{\left(\frac{\pi\,3t}{\pi}\right)} = \frac{4 \sin 3t}{3t}$$

Here $\omega t = 3t$

$\therefore$ $2\pi f = 3$

$f = 3/2\pi$ Hz

Thus $W = f = \dfrac{3}{2\pi}$

$\therefore$ Nyquist sampling frequnecy $f_s = 2W = 2 \times \dfrac{3}{2\pi} = \dfrac{3}{\pi}$

Review Questions

1. *State and prove sampling theorem for lowpass signals.* **AU : Dec.-12, Marks 4, Dec.-15, Marks 8**
2. *Define Nyquist rate and Nyquist interval.* **AU : Dec.-08, 10, Marks 6, Dec.-05, Marks 8**
3. *Explain the concept of sampling, quantization and quantization error.* **AU : Dec.-06, Marks 10, Dec.-10, Marks 8, Dec.-11, Marks 6**
4. *State amd prove sampling theorem in frequency domain.* **AU : May-11, Marks 10**
5. *How sampling in time domain results its spectrum to be periodic ?* **AU : Dec.-06, Marks 10**
6. *Explain the concept of aliasing. How it can be avoided ?* **AU : May-06, Marks 8**
7. *Explain the concept of antialiasing filter.* **AU : Dec.-05, Marks 6**
8. *What is meant by quantization and quantization error ?* **AU : May-15, Marks 4, May-17, Marks 10**

1.6 Short Answered Questions [2 Marks Each]

Q.1 What is the need of antialiasing filter ? **AU : May-16**

Ans. : Antialiasing filter is a bandlimiting filter. It removes the higher frequency components higher than $\frac{\Omega_{SF}}{2}$ frequency. Here Ω_{SF} is the sampling frequency. Aliasing takes place if frequencies greater than $\frac{\Omega_{SF}}{2}$ are present. Thus anti-aliasing filter avoids aliasing.

Q.2 State sampling theorem. AU : May-04, 05, 11, 17, Dec.-05, 11

OR

State Shannon's sampling theorem. AU : May-14

Ans. : A continuous time signal can be completely represented in its samples and recovered back, if the sampling frequency $f_s \geq 2W$. Here f_s is the sampling frequency and ' W ' is the maximum frequency present in the signal.

Q.3 What is aliasing ? How it is corrected ?

AU : May-04, 06, 07, 08, 09, 11, Dec.-06, 12

OR

Define aliasing effect. AU : Dec.-10

Ans. : Aliasing effect takes place when sampling frequency $f_s < 2W$, i.e. less than Nyquist rate. Higher frequency components take the form of low frequencies in the spectrum. This effect is called aliasing. When the signal is sampled, its spectrums are placed periodically at $0, \pm f_s, \pm 2f_s, \ldots$ etc. If $f_s < 2W$, then the spectrums overlap and the frequency components mixup. This mixing effect is called aliasing. Aliasing can be avoided by two ways :

i) Use of antialiasing filter before A/D converter.

ii) Sampling rate must be greater than Nyquist rate.

Q.4 What is the criteria for designing reconstruction filter ?

Ans. : i) Reconstruction filter should pass spectrum of required signal without modification.

ii) It should attenuate the replicas of spectrum which are located at $\pm \Omega_{sF}$, $\pm \Omega_{sF}$, … etc.

iii) It should have transition band between the two spectrums where there are no frequency components.

Q.5 Define continuous-time, discrete-time, quantized time signal. AU : May-06

Ans. : CT signal : The CT signal is defined continuously with respect to time. For example, sin ωt, e^{at}, $u(t)$.

DT signal : The DT signal is defined only at specific or regular time instants. For example, sin ωnT, e^{anT}, $u(n)$

Quantized time signal : This signal assumes fixed quantization levels. It doesn't have continuous amplitude range.

Q.6 Differentiate among analog, discrete, quantized and digital signal.

AU : May-05, Dec.-16

Ans. : Analog signal : Continuous amplitude and time

Discrete signal : Continuous in amplitude but discrete in time.

Quantized signal : Discrete in amplitude but continuous time.

Digital signal : Amplitude as well as time is discrete and it has fixed amplitude levels.

Q.7 Draw the basic block diagram of a digital processing of analog signal.

AU : May-05

Ans. : Refer Fig. 1.1.1 in section 1.1.1.

Q.8 What are the basic operations on the sequence ? **AU : Dec.-05**

Ans. : Operations on independent variable (time)

- Time shift / advance
- Time scaling : Compression and expansion.
- Time folding

Operations on dependent variable (Amplitude)

- Amplitude scaling
- Addition / subtraction
- Multiplication / division
- Accumulation

Q.9 What do you understand by multichannel and multidimensional signals ?

AU : May-06

Ans. : Multichannel signal : When the signal is generated by multiple sources or multiple sensors, it is called multichannel signal. These signals are represented by rectors. For example, 3-lead and 12-lead ECG are multichannel signals.

Multidimensional signal : If the signal is the function of 'm' independent variables, then it is called multidimensional signal. The TV signal is three dimensional signal since brightness of the pixel is represented as $I(x, y, t)$. Where x and y indicate position of the pixel and 't' indicate time of the signal.

Q.10 What are sesmic signals and mention the types of basic sesmic waves ?

AU : May-04; Dec.-05

Ans. : Sesmic signals are generated because of an earthquake. Types of sesmic waves are (i) Primary (P) (ii) Secondary (S) and (iii) Surface wave.

Q.11 What are the disadvantages of analog signal processing ? **AU : Dec.-06**

Ans. : 1) For the signals having wide bandwidth, highspeed A/D converters are required, which are difficult to design.

2) For small applications, DSP systems are expensive.

Q.12 Define symmetric and anti symmetric signals. **AU : May-07**

OR

What are even and odd signals ? **AU : Dec.-10**

Ans. : Symmetric signal (Even signal) : A signal is said to be symmetric or even if inversion of the time axis does not change the amplitude i.e,

$$x(t) = x(-t) \text{ and } x(n) = x(-n)$$

For example, cos θ, rectangular pulse etc.

Anti symmetric signal (Odd signal) : A signal is said to be anti symmetric or odd if inversion of time axis also inverts amplitude of the signal i.e.,

$$x(t) = -x(-t) \text{ and } x(n) = -x(-n)$$

For example, sin θ.

Q.13 A system is characterised by y (n) = x (n) + nx (n–1). Is the system linear?

Madras Univ. : April-01

Ans. : Let $y_1(n) = x_1(n)+nx_1(n-1)$

and $y_2(n) = x_2(n)+nx_2(n-1)$

Linear combination of two outputs becomes,

$$y_3(n) = a_1 y_1(n)+a_2 y_2(n)$$
$$= a_1\left[x_1(n)+nx_1(n-1)\right]+a_2\left[x_2(n)+nx_2(n-1)\right]$$

Response to linear combination of two inputs will be,

$$y'_3(n) = T\left\{a_1x_1(n)+a_2x_2(n)\right\}$$
$$= a_1x_1(n)+a_2x_2(n)+n\left[a_1x_1(n-1)+a_2x_2(n-1)\right]$$
$$= a_1x_1(n)+a_2x_2(n)+na_1x_1(n-1)+na_2x_2(n-1)$$
$$= a_1\left[x_1(n)+nx_1(n-1)\right]+a_2\left[x_2(n)+nx_2(n-1)\right]$$

Thus $y_3(n) = y'_3(n)$, hence this is linear system.

Q.14 What is causal system? **Madras Univ. : April-98, 2000, Oct.-97**

Ans. : A system is said to be causal if its output depends upon present input, past inputs or past outputs only. Response of such system begins only when input is applied. For example,

$$y(n) = x(n)+x(n-1)-y(n-1)$$

This is causal system. And,

$$y(n) = x(n)+x(n+1)$$

is non causal since output depends upon future input x(n+1).

Q.15 What is a shift invariant system? Give an example.

Madras Univ. : Oct.-98, April-96

Ans. : If the input is delayed, then output of the shift invariant system is also delayed by same amount. In other words, input/output relationship of shift invariant system is not affected by changing the time origin of input. For example,

$$y(n) = x(n)+x(n-1)$$

Response of the system to delayed input will be,

$$y(n,k) = T\{x(n-k)\} = x(n-k)+x(n-1-k)$$

And let us delay output itself by same number of samples. i.e.

$$y(n-k) = x(n-k)+x(n-k-1)$$

Thus $y(n,k) = y(n-k)$. This is shift invariant system.

Q.16 Check whether the system characterised by $y(n) = 2x(n)+\frac{1}{x(n-1)}$ is linear.

Madras Univ. : Oct.-97, 98

Ans. : Let

$$y_1(n) = 2x_1(n)+\frac{1}{x_1(n-1)}$$

$$y_2(n) = 2x_2(n)+\frac{1}{x_2(n-1)}$$

Linear combination of output will be,

$$\begin{aligned} y_3(n) &= a_1 y_1(n)+a_2 y_2(n) \\ &= 2a_1 x_1(n)+\frac{a_1}{x_1(n-1)}+2a_2 x_2(n)+\frac{a_2}{x_2(n-1)} \end{aligned}$$

Output due to linear combination of inputs will be,

$$\begin{aligned} y_3'(n) &= T\{a_1 x_1(n)+a_2 x_2(n)\} \\ &= 2[a_1 x_1(n)+a_2 x_2(n)]+\frac{1}{a_1 x_1(n-1)+a_2 x_2(n-1)} \end{aligned}$$

Here observe that $y_3(n) \neq y_3'(n)$. Hence the system is nonlinear.

Q.17 What is linear time invariant system ? **AU : May-08, Dec.-11, 12**

Ans. : A system that satisfies superposition principle are time shift in the input signal results in corresponding time shift in the output is called linear time invariant system.

Q.18 Distinguish between causal and noncausal system. **AU : Dec.-08**

Ans. :

Sr. No.	Causal system	Noncausal system
1.	Output depends upon past and present inputs and past outputs.	Output depends upon future inputs are present and future outputs.
2.	$y(n) = x(n)+x(n-1)$	$y(n) = x(n)+x(n+1)$

Q.19 What is the total energy of the discrete time signal x (n) which takes the value of unity at n = –1, 0, 1? AU : Dec.-09

Ans. :

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-1}^{1} |1|^2 = 1+1+1 = 3 \text{ J}$$

Q.20 Determine the system described by input-output equation is linear or non-linear?

y (n) = n x (n) AU : May-09

Ans. :

$$y_1(n) = n\, x_1(n)$$

$$y_2(n) = n\, x_2(n)$$

Linear combination of output will be,

$$\therefore \quad y'_3(n) = [a_1\, y_1(n) + a_2\, y_2(n)]$$

$$= a_1\, n x(n) + a_2\, n x_2(n)$$

$$= n[a_1\, x_1(n) + a_2\, x_2(n)]$$

Response of the system to linear combination of inputs will be,

$$y_3\ (\text{n}) = n[a_1 x_1(n) + a_2 x_2(n)]$$

Here $y_3\ (n) = y'_3(\text{n})$, hence the system is linear.

Q.21 Define sampling of analog signals. AU : May-10

Ans. : Analog signals cannot be digitally processed. It is necessary to define analog signals only at discrete time intervals. Amplitude of the analog signal at the time of sampling is taken as pulse amplitude. This process is called sampling of analog signals.

Q.22 Define Nyquist rate. AU : May-12

Ans. : Nyquist rate : Nyquist rate is the minimum sampling rate that avoids aliasing. For the signal of bandwidth 'W' Hz, the Nyquist rate will be be 2 W samples/sec.

Q.23 Determine the fundamental period of $\cos\left(\frac{\pi 30n}{105}\right)$. AU : Dec.-13

Ans. : Here $2\pi f_0 n = \frac{\pi 30n}{105}$, hence $f_0 = \frac{15}{105} = \frac{1}{7} = \frac{k}{N}$

Here N = 7 samples is period of given signal.

Q.24 What is the Nyquist rate for the signal, $x(t) = 3\cos(50\pi t) + 10\sin(300\pi t) - \cos(100\pi t)$. AU : Dec.-13, May-14

Ans. : Compare given signal with,

$$x(t) = A_1 \cos(2\pi f_1 t) + A_2 \sin(2\pi f_2 t) + A_3 \sin(2\pi f_3 t)$$

Here f_1 = 25 Hz, f_2 = 150 Hz and f_3 = 50 Hz

Here highest signal frequency is $W = f_2$ = 150 Hz

∴ Nyquist rate = 2 W = 2×150 = 300 Hz

Q.25 Differentiate between energy and power singals. AU : May-15

Ans. : Refer Table 1.2.1.

Q.26 State any two application of DSP. AU : May-15

Ans. : Refer section 1.1.4.

Q.27 Given a continuous signal *x(t)* = 2cos 300π*t*. What is the nyquist rate and fundamental frequency of the signal. AU : Dec.-15

Ans. : Here $A \cos 2\pi ft = 2\cos 300\pi t$. Hence highest frequency component is

f = W = 150 Hz. Hence,

Nyquist rate = 2W = 2 × 150 = 300 Hz.

Fundamental frequency = f = 150 Hz.

Q.28 Determine *x(n)* = *u(n)* is a power signal or an energy signal. AU : Dec.-15

Ans. : Refer Example 1.2.3 (ii).

Q.29 Determine if the system described by the equation $y(n) = x(n) + \frac{1}{x(n-1)}$ is causal or non causal. AU : May-16

Ans. : Here $x(n)$ is present input and $x(n-1)$ is past input. Since output $y(n)$ depends upon present and past inputs, this system is causal.

Q.30 If *y(n)* = x(n + 1) + x(n – 2), is the systm causal ? AU : Dec.-16

Ans. : Here output at n^{th} time instant depends upon future input at $(n+1)^{st}$ time instant become of the term $x(n+1)$. Hence this system is **not causal.**

Q.31 What are the energy and power of discrete signal ? AU : May-17

Ans. : Energy of the discrete time signal $x(n)$ is given as,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

And power of the discrete time signal is given as,

$$P = \underset{N \to \infty}{Lim} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

❑❑❑

2 Discrete Time System Analysis

Syllabus

z-transform and its properties, Inverse z-transforms, Difference equation - Solution by z-transform, Application to discrete systems - Stability analysis, Frequency response - Convolution - Discrete Time Fourier transform, Magnitude and phase representation.

Contents

2.1 Introduction

2.2 z-Transform . . . . **May-09, 11, 15, Dec.-16,** · · · Marks 8

2.3 Properties of the ROC

2.4 Properties of z-Transform **May-05, 07, 08, 10, 11, 14, 15, 17,**

. . . . **Dec.-06, 09, 10, 12, 13, 15** · Marks 16

2.5 Inverse z-Transform **May-11, 14, 15, 16, 17,**

. . . . **Dec.-05, 06, 13, 15, 16,** · · · Marks 13

2.6 Unilateral z-transform

2.7 Convolution **May-05, 10, 11,**

. . . . **Dec.-10, 11, 12** · · · · · · · · · Marks 10

2.8 Properties of Impulse Response Representations for LTI Systems

. . . . **Dec.-12** · · · · · · · · · · · · · · · · Marks 8

2.9 Transform Analysis of LTI Systems **May-05, 06, 07, 11, 12,15, 16**

. . . . **Dec.-06, 08, 10, 11, 15, 16,** Marks 16

2.10 Fourier Transform of Discrete Time Signals. . . **May-04, 06, 10, 14, 15, 16**

. . . . **Dec.-16** · · · · · · · · · · · · · · · Marks 16

2.11 Short Answered Questions [2 Marks Each]

2.1 Introduction

- Discrete time systems are analyzed with the help of z-transform and Fourier transform.
- Stability, causality, impulse response, step response, pole-zero plot etc. can be obtained using z-transform.
- Magnitude and phase response can be obtained using Fourier transform.
- The solution of linear difference equation becomes easy with the help of z-transform. The linear difference equation is converted to algebraic equation with the help of z-transform.

2.2 z-Transform

AU : May-09, 11, 15, Dec.-16

2.2.1 Definition of z-Transform

The z-transform of x(n) is denoted by X(z). It is defined as,

z-transform :

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \quad \text{... (2.2.1)}$$

Here z is complex variable. x(n) and X(z) is called z-transform pair. It is represented as,

z-transform pair : $x(n) \xleftrightarrow{z} X(z)$

2.2.2 Types of z-Transform : Unilateral and Bilateral

i) Bilateral z-transform : The z-transform defined above has both sided summation. It is called bilateral or both sided z-transform.

ii) Unilateral or one sided z-transform : It is defined as,

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} \quad \text{...(2.2.2)}$$

Here the summation is from n = 0 to ∞, i.e. one sided.

2.2.3 Region of Convergence (ROC)

Definition : ROC is the region where z-transform converges. From definition, it is clear that z-transform is an infinite power series. This series is not convergent for all values of z. Hence ROC is useful in mentioning z-transform.

Significance of ROC :

i) ROC gives an idea about values of z for which z-transform can be calculated.

ii) ROC can be used to determine causality of the system.

iii) ROC can be used to determine stability of the system.

Examples for Understanding

Example 2.2.1 *Determine z-transform of following sequences*

i) $x_1(n) = \{1, 2, 3, 4, 5, 0, 7\}$

ii) $x_2(n) = \{1, 2, 3, \underset{\uparrow}{4}, 5, 0, 7\}$

AU : Dec.-16, Marks 2

Solution : i) $x_1(n) = \{1, 2, 3, 4, 5, 0, 7\}$

i.e. $x_1(0) = 1$, $x_1(1) = 2$, $x_1(2) = 3$, $x_1(3) = 4$, $x_1(4) = 5$, $x_1(5) = 0$, $x_1(6) = 7$

By definition, $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$\therefore \quad X_1(z) = \sum_{n=0}^{6} x_1(n) z^{-n}$$

Putting for $x_1(n)$, $= 1+2z^{-1}+3z^{-2}+4z^{-3}+5z^{-4}+0z^{-5}+7z^{-6}$

$$\therefore \quad X_1(z) = 1+\frac{2}{z}+\frac{3}{z^2}+\frac{4}{z^3}+\frac{5}{z^4}+\frac{7}{z^6}$$

Result : i) $X_1(z)$ is as calculated above.

ii) $X_1(z) = \infty$ for z = 0, i.e. $X_1(z)$ is convergent for all values of z, except z = 0.

iii) Hence ROC : Entire z-plane except z = 0.

ii) $x_2(n) = \{1, 2, 3, \underset{\uparrow}{4}, 5, 0, 7\}$

i.e. $x_2(0) = 4$, $x_2(1) = 5$, $x_2(2) = 0$, $x_2(3) = 7$ and

$$x_2(-1) = 3,\ x_2(-2) = 2,\ x_2(-3) = 1$$

$$\therefore \qquad X_2(z) = \sum_{n=-3}^{3} x_2(n)\, z^{-n}$$

Putting for $x_2(n)$,

$$= 1\cdot z^3 + 2\cdot z^2 + 3z^1 + 4z^0 + 5z^{-1} + 0z^{-2} + 7z^{-3}$$

$$= z^3 + 2z^2 + 3z + 4 + \frac{5}{z} + \frac{7}{z^3}$$

Result : i) Above equation gives $X_2(z)$.

ii) $X_2(z) = \infty$ for z = 0 and z = ∞.

iii) Hence ROC : Entire z-plane except z = 0 and ∞.

Example 2.2.2 *z-transform of* $\delta(n)$.

Solution : We know that $\delta(n) = \begin{cases} 1 & for \quad n = 0 \\ 0 & for \quad n \neq 0 \end{cases}$

$$\therefore \qquad X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \delta(n)\, z^{-n}$$

$$= 1\cdot z^0 = 1$$

This is fixed value for any z. Hence ROC will be entire z-plane.

$$\boxed{\delta(n) \xleftrightarrow{z} 1, \qquad ROC : \text{Entire z-plane}} \qquad \ldots (2.2.3)$$

Example 2.2.3 *z-transform of unit step sequence, u(n).*

Solution : Unit step sequence, u(n) = $\begin{cases} 1 & for \quad n \geq 0 \\ 0 & for \quad n < 0 \end{cases}$

$$\therefore \qquad X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Putting u(n),

$$= \sum_{n=0}^{\infty} 1\cdot z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n$$

$$= 1 + (z^{-1}) + (z^{-1})^2 + (z^{-1})^3 + (z^{-1})^4 + \cdots$$

Here use, $1+A+A^2+A^3+A^4+\cdots = \frac{1}{1-A}$, $|A| < 1$. Then above equation will be,

$$X(z) = \frac{1}{1-z^{-1}}, |z^{-1}| < 1$$

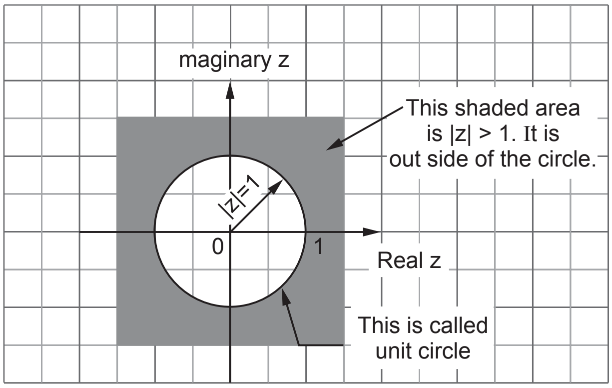


Fig. 2.2.1 ROC of u(n)

Here $|z^{-1}| < 1$ is equal to $|z| > 1$. This condition is nothing but ROC of $X(z)$. Here $|z|$ represents a circle in z-plane. $|z| = 1$ represents a circle of radius 1. And $|z| > 1$ indicates area outside of the circle. Fig. 2.2.1 shows the ROC by shaded area. Note that X(z) is convergent in this area. The circle of radius '1' i.e. $|z| = 1$ is called *unit circle* in the z-plane. The imaginary values are plotted with respect to real values in the z-plane.

$$u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \quad ROC : |z| > 1 \qquad \ldots (2.2.4)$$

Example 2.2.4 *z-transform of right hand sided sequence $x(n) = a^n u(n)$.*

AU : May-09, Marks 8, May-11, Marks 3, May-15, Marks 2

Solution : By definition of z-transform, $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \quad \text{since u(n) = 1 for n = 0 to } \infty$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= 1+(a z^{-1})+(a z^{-1})^2+(a z^{-1})^3+(a z^{-1})^4+\cdots$$

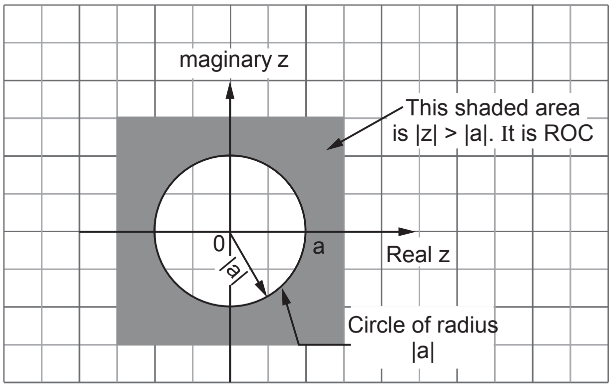


Fig. 2.2.2 ROC of $a^n u(n)$

Here use $1+A+A^2+A^3+\cdots = \frac{1}{1-A}$, $|A| < 1$. Then above equation will be,

$$X(z) = \frac{1}{1-a z^{-1}}, \quad |a z^{-1}| < 1$$

Here $|a z^{-1}| < 1$ is equal to $|z| > |a|$. This is the ROC. Fig. 2.2.2 shows this ROC. It is the shaded area outside the circle of radius $|a|$. Note that $|z|>|a|$ indicates the area outside the circle.

ROC of the right hand sided sequence (i.e. causal sequence) is outside the circle.

$$a^n u(n) \xleftrightarrow{z} \frac{1}{1-a z^{-1}}, \qquad ROC : |z| > |a| \qquad \ldots (2.2.5)$$

Examples with Solution

Example 2.2.5 *z-transform of left hand sided sequence.*

$x(n) = -a^n u(-n-1)$

Solution : Here $x(n) = \begin{Bmatrix} -a^n & for\ n \leq -1 \\ 0 & for\ n \geq 0 \end{Bmatrix} u(-n-1)=1$ for $n = -1$ to $-\infty$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

Let $\quad n = -l,\ X(z) = -\sum_{l=\infty}^{1} a^{-l} z^{l}$

$$= -\sum_{l=1}^{\infty} (a^{-1} z)^{l}$$

$$= -\left\{(a^{-1} z) + (a^{-1} z)^2 + (a^{-1} z)^3 + (a^{-1} z)^4 + \cdots\right\}$$

$$= -(a^{-1} z)\left\{1 + a^{-1} z + (a^{-1} z)^2 + (a^{-1} z)^3 + (a^{-1} z)^4 + \cdots\right\}$$

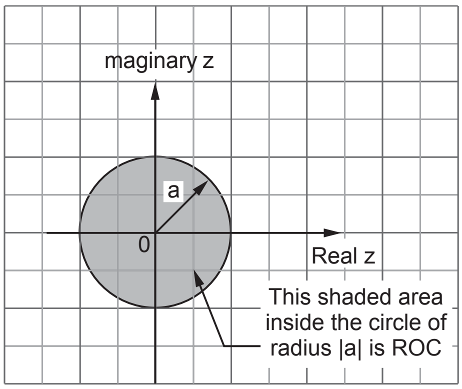


Fig. 2.2.3 ROC of $-a^n u(-n-1)$

For the term in bracket use,

$$1 + A + A^2 + A^3 + \cdots = \frac{1}{1-A}, \ |A| < 1. \text{ i.e.,}$$

$$X(z) = -(a^{-1} z) \cdot \frac{1}{1 - a^{-1} z}, \ |a^{-1} z| < 1$$

$$= \frac{1}{1 - a z^{-1}}, \ |a^{-1} z| < 1$$

Here $|a^{-1} z| < 1$ is equal to $|z| < |a|$. This ROC is the area that lies inside the circle of radius $|a|$. It is shown in Fig. 2.2.3.

ROC of left sided sequence (i.e. noncausal sequence) is an area inside the circle.

Important comment : z-transforms of $a^n u(n)$ and $-a^n u(-n-1)$ are same, i.e. $\frac{1}{1-az^{-1}}$, but their ROCs are different.

$$-a^n u(-n-1) \xleftrightarrow{z} \frac{1}{1-az^{-1}}, \quad ROC : |z| < |a| \qquad \ldots (2.2.6)$$

Example 2.2.6 *z-transform of both sided sequence.* $x(n) = a^n u(n) + b^n u(-n-1)$

Solution : Here let $x_1(n) = a^n u(n)$ and $x_2(n) = b^n u(-n-1)$

$$\therefore \qquad x(n) = x_1(n) + x_2(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} [x_1(n) + x_2(n)] z^{-n}$$

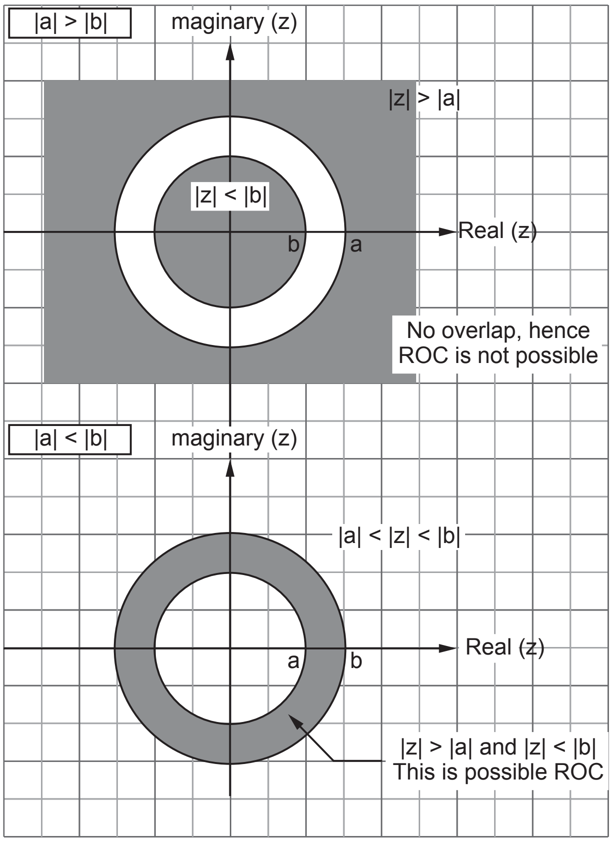


Fig. 2.2.4 ROC of both sided sequence

$$= \sum_{n=-\infty}^{\infty} x_1(n)\, z^{-n} + \sum_{n=-\infty}^{\infty} x_2(n)\, z^{-n}$$

From equation (2.2.5) and equation (2.2.6) above equation becomes,

$$X(z) = \frac{1}{1-a z^{-1}} + \frac{1}{1-b z^{-1}}, \text{ ROC : } |z|>|a| \text{ and } |z|<|b|$$

$$\text{i.e. } |a| < |z| < |b|$$

Comment on ROC : $|a| < |z| < |b|$

i) When $|a|>|b|$

As shown Fig. 2.2.4, there is no overlap between the shaded areas for $|z|>|a|$ and $|z|<|a|$. Hence both the terms of X(z) do not converge simultaneously. Therefore ROC is not possible.

ii) When $|a|<|b|$

For this case, as shown in Fig. 2.2.4, the shaded area shows the overlap of $|z|>|a|$ and $|z|<|b|$. This area is $|a|<|z|<|b|$. In this area both the terms of X(z) converge simultaneously. Hence the ring shown by $|a|<|z|<|b|$ is ROC of X(z).

2.3 Properties of the ROC

Property 1 : The ROC for a finite duration sequence includes entire z-plane, except $z=0$, and/or $|z| = \infty$.

Proof : Consider the finite duration sequence $x(n) = \{1\ 2\ \underset{\uparrow}{1}\ 2\}$

$$\therefore \quad X(z) = 1\cdot z^2 + 2\cdot z + 1 z^0 + 2 z^{-1} = z^2 + 2z + 1 + \frac{2}{z}$$

Here X(z) = ∞ for z = 0 and ∞. This proves first property.

Property 2 : ROC does not contain any poles.

Proof : The z-transform of $a^n u(n)$ is calculated as,

$$X(z) = \frac{1}{1-a z^{-1}}$$

$$= \frac{z}{z-a} \qquad ROC : |z|>|a|$$

This function has pole at z = a. Note that ROC is $|z|>|a|$. This means poles do not lie in ROC. Actually X(z) = ∞ at poles by definition of pole.

Property 3 : ROC is the ring in the z-plane centered about origin.

Proof : Consider $a^n\, u(n) \xleftrightarrow{z} \dfrac{1}{1-az^{-1}}$, $ROC : |z|>|a|$

or $$-a^n\, u(-n-1) \xleftrightarrow{z} \frac{1}{1-az^{-1}}\ ,\ ROC : |z|<|a|$$

Here observe that $|z|$ is always a circular region (ring) centered around origin.

Property 4 : ROC of causal sequence (right hand sided sequence) is of the form $|z|>$ r.

Proof : Consider right hand sided sequence $a^n u(n)$. It's ROC is $|z| > |a|$. Thus the ROC of right hand sided sequence is of the form of $|z|>r$ where 'r' is the radius of the circle.

Property 5 : ROC of left sided sequence is of the form $|z|<r$.

Proof : Consider left sided sequence $-a^n\, u(-n-1)$. Its ROC is $|z|<|a|$. Thus the ROC of left sided sequence is inside the circle of radius 'r'.

Property 6 : ROC of two sided sequence is the concentric ring in z-plane.

Proof : We know that ROC of $x(n) = a^n\, u(n)+b^n\, u(-n-1)$ is $|a|<|z|<|b|$, which is the concentric ring as shown in Fig. 2.2.4.

Property 7 : If $x(n)$ is finite causal sequence, then its ROC is entire z-plane except z = 0.

Proof : Consider the causal sequence, $x(n) = \{1, 2, 3\}$. Then its z-transforms will be $1+2z^{-1}+3z^{-2}$. Clearly this sequence converges in entire z-plane excepts $z = 0$.

Property 8 : The ROC of stable LTI system contains unit circle in the z-plane.

Proof : This property will be proved in section 2.7.

Property 9 : The ROC is a connected region.

Proof : The covergence of the sequence exists over certain area, rather than discrete points. Hence ROC is a connected region.

Review Questions

1. *Define one sided and two sided z-transform.*
2. *What is ROC with respect to z-transform ? What are its properties ?*

2.4 Properties of z-Transform

AU : May-05, 07, 08, 10, 11, 14, 15, 17, Dec.-06, 09, 10, 12, 13, 15

2.4.1 Linearity

If $x_1(n) \xleftrightarrow{z} X_1(z)$, ROC : $a_1 < |z| < b_1$

and $x_2(n) \xleftrightarrow{z} X_2(z)$, ROC : $a_2 < |z| < b_2$, then

$$\boxed{a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{z} a_1 X_1(z) + a_2 X_2(z)} \quad \ldots (2.4.1)$$

The ROC is intersection of ROC's of $X_1(z)$ and $X_2(z)$.

Proof :

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a_1 x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) z^{-n}$$

$$= a_1 \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \quad \text{Since } a_1 \text{ and } a_2 \text{ are constants}$$

$$= a_1 X_1(z) + a_2 X_2(z)$$

2.4.2 Time Shifting or Translation

If $x(n) \xleftrightarrow{z} X(z)$, ROC : $r_1 < |z| < r_2$ then

$$\boxed{x(n-k) \xleftrightarrow{z} z^{-k} X(z)} \quad \text{ROC : } r_1 < |z| < r_2 \quad \ldots (2.4.2)$$

Proof : $Z\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k) z^{-n}$

Let $n-k = m$. Hence $n = k+m$ and $m = -\infty$ to $+\infty$. i.e.,

$$Z\{x(n-k)\} = \sum_{m=-\infty}^{\infty} x(m)\, z^{-(k+m)}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, z^{-k} \cdot z^{-m} \quad = z^{-k} \sum_{m=-\infty}^{\infty} x(m)\, z^{-m}$$

$$= z^{-k}\, X(z)$$

2.4.3 Scaling in z-Domain or Multiplication by Exponential

Let $x(n) \xleftrightarrow{z} X(z), \quad ROC : r_1 < |z| < r_2$

then $$a^n\, x(n) \xleftrightarrow{z} X\left(\frac{z}{a}\right), \qquad ROC : \ |a|\, r_1 < |z| < |a|\, r_2 \qquad \ldots (2.4.3)$$

Proof : $$Z\{a^n\, x(n)\} = \sum_{n=-\infty}^{\infty} a^n\, x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, (a^{-1} z)^{-n} \quad = X(a^{-1} z)$$

$$= X\left(\frac{z}{a}\right), \qquad ROC : r_1 < \left|\frac{z}{a}\right| < r_2 \text{ i.e. } |a|\, r_1 < |z| < |a|\, r_2$$

2.4.4 Time Reversal

Let, $x(n) \xleftrightarrow{z} X(z), \qquad ROC : r_1 < |z| < r_2$

then $$x(-n) \xleftrightarrow{z} X\left(z^{-1}\right), \qquad ROC : \frac{1}{r_2} < |z| < \frac{1}{r_1} \qquad \ldots (2.4.4)$$

Proof : $$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)\, z^{-n}$$

with $n = -m$, $$= \sum_{m=\infty}^{-\infty} x(m)\, z^{m} = \sum_{m=-\infty}^{\infty} x(m)\, (z^{-1})^{-m}$$

$$= X(z^{-1}) \qquad \text{ROC} : r_1 < |z^{-1}| < r_2 \text{ i.e. } \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

2.4.5 Differentiation in z-Domain or Multiplication by a Ramp

Let $x(n) \xleftrightarrow{z} X(z)$, $\qquad ROC : r_1 < |z| < r_2$ then,

$$\boxed{n\,x(n) \xleftrightarrow{z} -z\frac{d}{dz}X(z)} \qquad ROC : r_1 < |z| < r_2 \qquad \dots (2.4.5)$$

Proof : $X(z) = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n}$

$$\therefore \quad \frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} \frac{d}{dz}[x(n)\,z^{-n}] = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{dz}z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\cdot(-n)\cdot\, z^{-n-1} = -\sum_{n=-\infty}^{\infty} n\,x(n)\,z^{-n}\cdot z^{-1}$$

$$= -z^{-1}\sum_{n=-\infty}^{\infty}[n\,x(n)]z^{-n} = -z^{-1}\cdot Z\{n\ x(n)\}$$

or $Z\{n\ x(n)\} = -z\frac{d}{dz}X(z)$, ROC : Same as that of $x(n)$

2.4.6 Convolution in Time Domain

Let $x_1(n) \xleftrightarrow{z} X_1(z)$, $ROC : a_1 < |z| < b_1$

and $x_2(n) \xleftrightarrow{z} X_2(z)$, $ROC : a_2 < |z| < b_2$ then

$$\boxed{x_1(n) * x_2(n) \xleftrightarrow{z} X_1(z)\cdot X_2(z)} \qquad \dots (2.4.6)$$

The ROC is intersection of ROCs of $X_1(z)$ and $X_2(z)$.

Proof : $x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\,x_2(n-k)$

$$\therefore \quad Z\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)\,x_2(n-k)\right]z^{-n}$$

Interchanging orders of summation,

$$= \sum_{k=-\infty}^{\infty} x_1(k)\left\{\sum_{n=-\infty}^{\infty} x_2(n-k)\,z^{-n}\right\}$$

Since $x_2(n-k) \xleftrightarrow{z} z^{-k}X_2(z)$,

$$= \sum_{k=-\infty}^{\infty} x_1(k)\left\{z^{-k} X_2(z)\right\}$$

$$= \left\{\sum_{k=-\infty}^{\infty} x_1(k) z^{-k}\right\} \cdot X_2(z) \quad = X_1(z) \cdot X_2(z)$$

2.4.7 Correlation of Two Sequences

Let $\quad x_1(n) \xleftrightarrow{z} X_1(z), \qquad ROC : a_1 < |z| < b_1$

and $\quad x_2(n) \xleftrightarrow{z} X_2(z), \qquad ROC : a_2 < |z| < b_2$ then

$$\boxed{\sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-l) \xleftrightarrow{z} X_1(z) X_2(z^{-1})} \quad \ldots (2.4.7)$$

Proof : Correlation of two sequences is given as,

$$r_{x_1 x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-l)$$

$$= \sum_{n=-\infty}^{\infty} x_1(n)\, x_2[-(l-n)] \quad = x_1(l) * x_2(-l)$$

$$\therefore \quad Z\left\{\sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-l)\right\} = Z\{x_1(l) * x_2(-l)\}$$

$$= Z\{x_1(l)\} \cdot Z\{x_2(-l)\}$$

$$= X_1(z) \cdot X_2(z^{-1})$$

2.4.8 Multiplication of Two Sequences or Convolution in z-Domain

Let $\quad x_1(n) \xleftrightarrow{z} X_1(z), \qquad ROC : a_1 < |z| < b_1$

and $\quad x_2(n) \xleftrightarrow{z} X_2(z), \qquad ROC : a_2 < |z| < b_2$ then

$$\boxed{x_1(n)\, x_2(n) \xleftrightarrow{z} \frac{1}{2\pi j} \oint_c X_1(v) X_2\left(\frac{z}{v}\right) v^{-1}\, dv} \quad \ldots (2.4.8)$$

Here 'c' is the closed contour. It encloses the origin and lies in the ROC which is common to both $X_1(v)$ and $X_2\left(\frac{1}{v}\right)$. Thus the ROC is intersection of ROCs of $X_1(z)$ and $X_2(z)$.

Proof : Inverse z-transform is given as, $x(n) = \frac{1}{2\pi j}\oint_c X(v)\, v^{n-1}\, dv$

Let $\quad x(n) = x_1(n)\, x_2(n)$

Putting inverse z-transform of $x_1(n)$ in above equation,

$$x(n) = \frac{1}{2\pi j}\oint_c X_1(v)\, v^{n-1}\, dv \cdot x_2(n)$$

$$\therefore \quad X(z) = \sum_{n=-\infty}^{\infty}\left\{\frac{1}{2\pi j}\oint_c X_1(v)\, v^{n-1}\, dv \cdot x_2(n)\right\} z^{-n}$$

Interchanging the order of integration and summation,

$$X(z) = \frac{1}{2\pi j}\oint_c X_1(v) \sum_{n=-\infty}^{\infty} v^n \cdot v^{-1}\, x_2(n)\, z^{-n}\, dv$$

$$= \frac{1}{2\pi j}\oint_c X_1(v)\left\{\sum_{n=-\infty}^{\infty} x_2(n)\left(\frac{z}{v}\right)^{-n}\right\} v^{-1}\, dv$$

$$= \frac{1}{2\pi j}\oint_c X_1(v) \cdot X_2\left(\frac{z}{v}\right) \cdot v^{-1}\, dv$$

2.4.9 Conjugation of a Complex Sequence

Let $\quad x(n) \overset{z}{\longleftrightarrow} X(z), \quad ROC : r_1 < |z| < r_2$

$$\boxed{x^*(n) \overset{z}{\longleftrightarrow} X^*(z^*)} \quad \ldots (2.4.9)$$

Proof : $Z\left\{x^*(n)\right\} = \sum_{n=-\infty}^{\infty} x^*(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} [x(n)(z^*)^{-n}]^*$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)(z^*)^{-n}\right]^*$$

$$= [X(z^*)]^* = X^*(z^*)$$

2.4.10 z-Transform of Real Part of a Sequence

Let $x(n) \xleftrightarrow{z} X(z)$, ROC : $r_1 < |z| < r_2$ then,

$$\boxed{Re[x(n)] \xleftrightarrow{z} \frac{1}{2}[X(z) + X^*(z^*)]} \quad \ldots (2.4.10)$$

Proof : $x(n) = Re[x(n)] + j\, Im\,[x(n)]$ and $x^*(n) = Re\,[x(n)] - j\, Im\,[x(n)]$

$$\therefore \quad Re[x(n)] = \frac{1}{2}[x(n) + x^*(n)]$$

$$\therefore \quad Z\{Re[x(n)] = Z\left\{\frac{1}{2}[x(n) + x^*(n)]\right\}$$

$$= \frac{1}{2}\{Z[x(n)] + Z[x*(n)]\}$$

$$= \frac{1}{2}[X(z) + X^*(z*)]$$

2.4.11 z-Transform of Imaginary Part of Sequence

Let $x(n) \xleftrightarrow{z} X(z)$, ROC : $r_1 < |z| < r_2$ then,

$$\boxed{Im[x(n)] \xleftrightarrow{z} \frac{1}{2j}[X(z) - X^*(z*)]} \quad \ldots (2.4.11)$$

Proof : $Im[x(n)] = \frac{1}{2j}[x(n) - x*(n)]$

$$Z\{Im[x(n)]\} = Z\left\{\frac{1}{2j}[x(n) - x*(n)]\right\} = \frac{1}{2j}\{Z[x(n) - Z[x*(n)]\}$$

$$= \frac{1}{2j}\left\{X(z) - X^*(z^*)\right\}$$

2.4.12 Parseval's Relation

Let $x_1(n) \xleftrightarrow{z} X_1(z)$, and $x_2(n) \xleftrightarrow{z} X_2(z)$ then,

$$\boxed{\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j}\oint_c X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1}\, dv} \quad \ldots (2.4.12)$$

Proof : Inverse z-transform of $X_1(z)$ is, $x_1(n) = \frac{1}{2\pi j}\oint_c X_1(v)\, v^{n-1}\, dv$

$$\therefore \sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi j} \oint_c X_1(v)\, v^{n-1}\, dv\, x_2^*(n)$$

$$= \frac{1}{2\pi j} \oint_c X_1(v) \left\{ \sum_{n=-\infty}^{\infty} x_2^*(n)\, v^{n-1} \right\} dv$$

Here $v^{n-1} = v^n \cdot v^{-1} = (v^{-1})^{-n} \cdot v^{-1} = \left(\frac{1}{v}\right)^{-n} \cdot v^{-1}$ then above equation will be,

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint_c X_1(v) \left\{ \sum_{n=-\infty}^{\infty} x_2^*(n) \cdot \left(\frac{1}{v}\right)^{-n} \cdot v^{-1} \right\} dv$$

$$= \frac{1}{2\pi j} \oint_c X_1(v) \left[\sum_{n=-\infty}^{\infty} x_2(n) \cdot \left(\frac{1}{v^*}\right)^{-n} \right]^* v^{-1} dv$$

$$= \frac{1}{2\pi j} \oint_c X_1(v) \left[X_2\left(\frac{1}{v^*}\right) \right]^* v^{-1} dv$$

$$= \frac{1}{2\pi j} \oint_c X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv$$

2.4.13 Initial Value Theorem

Let $x(n) \xleftrightarrow{z} X(z)$, then,

$$\boxed{x(0) = \lim_{z\to\infty} X(z)} \qquad \dots (2.4.13)$$

Proof : z-transform of a causal sequence is given as,

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}, \text{ since } x(n) = 0 \text{ for } n < 0$$

$$= x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + \cdots$$

$$\therefore \lim_{z\to\infty} X(z) = \lim_{z\to\infty} x(0) + \lim_{z\to\infty} x(1)\, z^{-1} + \lim_{z\to\infty} x(2)\, z^{-2} + \cdots$$

$$= x(0) + 0 + 0 + 0 + 0 \cdots$$

$$\therefore x(0) = \lim_{z\to\infty} X(z)$$

Examples for Understanding

Example 2.4.1 *Determine z-transform of following :*

i) $x_3(n) = u(-n)$

ii) $x_4(n) = n\,a^n\,u(n)$

Solution : i) $x_3(n) = u(-n)$

$$x(n) \xleftrightarrow{z} X(z), \quad ROC : r_1 < |z| < r_2$$

$$x(-n) \xleftrightarrow{z} X(z^{-1}),\ ROC : \frac{1}{r_2} < |z| < \frac{1}{r_1}, \text{ By time reversal property}$$

$$u(n) \xleftrightarrow{z} \frac{1}{1-z^{-1}},\ ROC : |z| > 1. \text{ Here } r_1 = 1$$

$$\therefore \quad u(-n) \xleftrightarrow{z} \frac{1}{1-z},\ ROC : |z| < 1, \text{ By time reversal property}$$

ii) $x_4(n) = n\,a^n\,u(n)$

$$Z\{a^n\,u(n)\} = \frac{1}{1-a\,z^{-1}}, \quad ROC : |z| > |a|$$

And $Z\{n\ x(n)\} = -z\dfrac{d}{dz}X(z)$, differentiation in z-domain property

$$\therefore \quad Z\{n \cdot a^n u(n)\} = -z\frac{d}{dz}\,\frac{1}{1-a\,z^{-1}}, \quad \text{Here } x(n) = a^n\,u(n)$$

$$= -z \cdot \frac{(1-a\,z^{-1})\dfrac{d}{dz}1 - 1\cdot\dfrac{d}{dz}(1-a\,z^{-1})}{(1-a\,z^{-1})^2}$$

$$= -z \cdot \frac{0+a\,(-1)\,z^{-2}}{(1-a\,z^{-1})^2}$$

$$= \frac{a\,z^{-1}}{(1-a\,z^{-1})^2}, \quad ROC : |z| > |a|$$

Thus ,

$$\boxed{n\,a^n\,u(n) \xleftrightarrow{z} \frac{a\,z^{-1}}{(1-a\,z^{-1})^2}, \quad ROC : |z| > |a|} \quad \dots (2.4.14)$$

Example 2.4.2 *Determine the z-transform of,*

$$x(n) = \begin{cases} a^n & \text{for } \; 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Solution : $X(z) = \sum_{n=0}^{N-1} a^n z^{-n}$ By definition of z-transform

$$= \sum_{n=0}^{N-1} (a z^{-1})^n$$

Here use $\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a}$, $N_2 > N_1$. Then above equation will be,

$$X(z) = \frac{(a z^{-1})^0 - (a z^{-1})^{N-1+1}}{1 - a z^{-1}}$$

$$= \frac{1-(a z^{-1})^N}{1 - a z^{-1}}$$

The above z-transform is convergent if $\sum_{n=0}^{N-1} |a|^n$ has finite value. i.e.,

$$\sum_{n=0}^{N-1} |a z^{-1}|^n < \infty$$

Above sum will be finite if $|a z^{-1}| < \infty$, since there are finite number of such terms in above summation. The condition $|a z^{-1}| < \infty$ implies that $|a| < \infty$ and $z \ne 0$. If 'a' is finite, then ROC will be entire z-plane except z = 0.

Example 2.4.3 *Determine z-transforms of*

i) $x(n) = \cos(\Omega_0 n) u(n)$

ii) $x(n) = \sin(\Omega_0 n) u(n)$

AU : May-05, Marks 8; May-08, Marks 6; Dec.-09, Marks 16; May-10, Marks 10, Dec.-13, Marks 6, May-17, Marks 7

Solution : i) $x(n) = \cos(\Omega_0 n) u(n)$

$$= \frac{e^{j\Omega_0 n} + e^{-j\Omega_0 n}}{2} u(n)$$

$$\therefore \quad X(z) = Z\left\{\frac{e^{j\Omega_0 n} + e^{-j\Omega_0 n}}{2}\right\} u(n)$$

$$= \frac{1}{2} Z\left\{e^{j\Omega_0 n}\, u(n)\right\} + \frac{1}{2} Z\{e^{-j\Omega_0 n} u(n)\}$$

Here use $a^n u(n) \overset{z}{\longleftrightarrow} \dfrac{1}{1 - a z^{-1}}$, $ROC\ |z| > |a|$ i.e.,

$$X(z) = \frac{1}{2} \cdot \frac{1}{1 - e^{j\Omega_0} z^{-1}} + \frac{1}{2} \frac{1}{1 - e^{-j\Omega_0} z^{-1}}, \quad ROC : |z| > |e^{j\Omega_0}| \text{ and } |z| > |e^{-j\Omega_0}|$$

Here $e^{j\Omega_0} = \cos\Omega_0 + j \sin\Omega_0$. Hence $|e^{j\Omega_0}| = \sqrt{\cos^2 \Omega_0 + \sin^2 \Omega_0} = 1$

Similarly $e^{-j\Omega_0} = \cos\Omega_0 - j \sin\Omega_0$. Hence $|e^{-j\Omega_0}| = \sqrt{\cos^2 \Omega_0 + \sin^2 \Omega_0} = 1$

$$\therefore \quad X(z) = \frac{1}{2}\left\{\frac{1}{1 - e^{j\Omega_0} z^{-1}} + \frac{1}{1 - e^{-j\Omega_0} z^{-1}}\right\} \text{ ROC} : |z| > 1$$

$$= \frac{1}{2}\left\{\frac{1 - e^{-j\Omega_0} z^{-1} + 1 - e^{j\Omega_0} z^{-1}}{(1 - e^{j\Omega_0} z^{-1})(1 - e^{-j\Omega_0} z^{-1})}\right\}$$

$$= \frac{1}{2}\left\{\frac{2 - z^{-1}(e^{j\Omega_0} + e^{-j\Omega_0})}{1 - z^{-1}(e^{j\Omega_0} + e^{-j\Omega_0}) + z^{-2}}\right\} = \frac{1}{2}\left\{\frac{2 - z^{-1} \cdot 2\cos\Omega_0}{1 - z^{-1} \cdot 2\cos\Omega_0 + z^{-2}}\right\}$$

$$= \frac{1 - z^{-1}\cos\Omega_0}{1 - 2z^{-1}\cos\Omega_0 + z^{-2}}, \ ROC : |z| > 1$$

ii) $x(n) = \sin\Omega_0\, n\, u(n)$

$$= \frac{e^{j\Omega_0 n} - e^{-j\Omega_0 n}}{2j} u(n)$$

$$\therefore \quad X(z) = Z\left\{\frac{e^{j\Omega_0 n} - e^{-j\Omega_0 n}}{2j}\right\} u(n)$$

$$= \frac{1}{2j}\Big[Z\{e^{j\Omega_0 n} u(n)\} - Z\{e^{-j\Omega_0 n} u(n)\}\Big]$$

$$= \frac{1}{2j}\left[\frac{1}{1 - e^{j\Omega_0} z^{-1}} - \frac{1}{1 - e^{-j\Omega_0} z^{-1}}\right],$$

$$ROC : |z| > |e^{j\Omega_0}| \text{ and } |z| > |e^{-j\Omega_0}| \quad \text{i.e. } |z| > 1$$

$$= \frac{1}{2j}\left[\frac{1-e^{-j\Omega_0}z^{-1}-1+e^{j\Omega_0}z^{-1}}{(1-e^{j\Omega_0}z^{-1})(1-e^{-j\Omega_0}z^{-1})}\right], \; ROC : |z|>1$$

$$= \frac{1}{2j}\left[\frac{(e^{j\Omega_0}-e^{-j\Omega_0})z^{-1}}{1-z^{-1}(e^{j\Omega_0}+e^{-j\Omega_0})+z^{-2}}\right]$$

$$= \frac{1}{2j}\left[\frac{2j\sin\Omega_0 \, z^{-1}}{1-z^{-1}\cdot 2\cos\Omega_0+z^{-2}}\right]$$

$$= \frac{z^{-1}\sin\Omega_0}{1-2z^{-1}\cos\Omega_0+z^{-2}}, \quad ROC : |z|>1$$

Thus ,

$$\boxed{\cos\Omega_0\, n\, u(n) \xleftrightarrow{z} \frac{1-z^{-1}\cos\Omega_0}{1-2z^{-1}\cos\Omega_0+z^{-2}}, \; ROC : |z|>1} \quad \ldots (2.4.15)$$

$$\boxed{\sin\Omega_0\, n\, u(n) \xleftrightarrow{z} \frac{z^{-1}\sin\Omega_0}{1-2z^{-1}\cos\Omega_0+z^{-2}}, \; ROC : |z|>1} \quad \ldots (2.4.16)$$

Example 2.4.4 *Determine the z-transform of*

i) $x(n) = a^n \cos(\Omega_0 n) u(n)$

ii) $x(n) = a^n \sin(\Omega_0 n) u(n)$

AU : May-11, Marks 5, Dec.-15, Marks 8

Solution : x(n) = $a^n \cos(\Omega_0 n) u(n)$

Let $x_1(n) = \cos(\Omega_0 n) u(n)$, hence $X_1(z) = \frac{1-z^{-1}\cos\Omega_0}{1-2z^{-1}\cos\Omega_0+z^{-2}}$

From above equations, $x(n) = a^n x_1(n)$

$\therefore$ $X(z) = Z\{a^n x_1(n)\}$

$$= X_1\left(\frac{z}{a}\right), \; ROC : |a|r_1<|z|<|a|r_2, \textit{ By scaling in z-domain}$$

Replacing z by $\frac{z}{a}$ in $X_1(z)$, $= \dfrac{1-\left(\frac{z}{a}\right)^{-1}\cos\Omega_0}{1-2\left(\frac{z}{a}\right)^{-1}\cos\Omega_0+\left(\frac{z}{a}\right)^{-2}}$, $ROC : |z| > 1\,|a|$ i.e. $|z| > |a|$

ii) $x(n) = a^n \sin(\Omega_0 n) u(n)$

Let $x_1(n) = \sin(\Omega_0 n) u(n)$, hence $X_1(z) = \frac{z^{-1}\sin\Omega_0}{1-2z^{-1}\cos\Omega_0+z^{-2}}$

From above equations, $x(n) = a^n x_1(n)$

$\therefore \quad X(z) = z\{a^n x_1(n)\}$

$= X_1\left(\frac{z}{a}\right)$, $ROC : |a| r_1 < |z| < |a| r_2$, By scaling in z-domain

Replacing z by $\frac{z}{a}$ in $X_1(z)$, $= \dfrac{\left(\frac{z}{a}\right)^{-1} \sin\Omega_0}{1-2\left(\frac{z}{a}\right)^{-1}\cos\Omega_0 + \left(\frac{z}{a}\right)^{-2}}$, $ROC : |z| > 1\,|a|$ i.e. $|z| > |a|$

$$a^n \cos(\Omega_0 n) u(n) \overset{z}{\longleftrightarrow} \frac{1-\left(\frac{z}{a}\right)^{-1}\cos\Omega_0}{1-2\left(\frac{z}{a}\right)^{-1}\cos\Omega_0 + \left(\frac{z}{a}\right)^{-2}}, \quad ROC : |z| > |a| \qquad \dots (2.4.17)$$

$$a^n \sin(\Omega_0 n) u(n) \overset{z}{\longleftrightarrow} \frac{\left(\frac{z}{a}\right)^{-1}\sin\Omega_0}{1-2\left(\frac{z}{a}\right)^{-1}\cos\Omega_0 + \left(\frac{z}{a}\right)^{-2}}, \quad ROC : |z| > |a| \qquad \dots (2.4.18)$$

Examples with Solution

Example 2.4.5 *Determine the z-transform of* $x(n) = \frac{a^n}{n!}$ *for* $n \geq 0$

Solution :

By definition, $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{(a z^{-1})^n}{n!}$$

Here use $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$, then above equation will be,

$$X(z) = e^{az^{-1}}$$

Example 2.4.6 *Determine z-transform of following sequences.*

i) $\alpha^{|n|}, 0<|\alpha|<1$

ii) $A r^{n} \cos(\Omega_0 n+\phi)\ u(n),\ 0<r<1$ **AU : May-14, Marks 8**

iii) $\alpha^{-|n|}, 0<|\alpha|<1$

iv) $e^{-n/10} \sin\left(\frac{2\pi n}{8}\right) u(n)$

v) $n^2\ u(n)$ **AU : May-15, Marks 8**

Solution : i) $x(n)=\alpha^{|n|}, 0<|\alpha|<1$

Here
$$x(n) = \alpha^n u(n)+\alpha^{-n} u(-n-1)$$

$$\therefore \quad = \alpha^n u(n)+\left(\frac{1}{\alpha}\right)^n u(-n-1)$$

$$\alpha^n u(n) \xleftrightarrow{z} \frac{1}{1-\alpha z^{-1}}, \qquad ROC: |z|>\alpha$$

and
$$\left(\frac{1}{\alpha}\right)^n u(-n-1) \xleftrightarrow{z} -\frac{1}{1-\frac{1}{\alpha} z^{-1}}, \qquad ROC: |z|<\frac{1}{\alpha}$$

$$X(z) = \frac{1}{1-\alpha z^{-1}}-\frac{1}{1-\frac{1}{\alpha} z^{-1}}, \qquad ROC: \alpha<|z|<\frac{1}{\alpha}$$

ii) $x(n)=A r^n \cos(\Omega_0 n+\phi)\ u(n),\ 0<r<1$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} A r^n \cos(\Omega_0 n+\phi)\, z^{-n}$$

$$= A\sum_{n=0}^{\infty} r^n \frac{e^{j\Omega_0 n+j\phi}+e^{-j\Omega_0 n-j\phi}}{2}\, z^{-n}$$

$$= \frac{A}{2}\sum_{n=0}^{\infty} r^n\left[e^{j\Omega_0 n}\cdot e^{j\phi}+e^{-j\Omega_0 n}e^{-j\phi}\right] z^{-n}$$

$$= \frac{A}{2}\left[e^{j\phi}\sum_{n=0}^{\infty}\left(r e^{j\Omega_0}\right)^n z^{-n}+e^{-j\phi}\sum_{n=0}^{\infty}\left(r e^{-j\Omega_0}\right)^n z^{-n}\right]$$

$$= \frac{A}{2}\left[\frac{e^{j\phi}}{1-re^{j\Omega_0}z^{-1}}+\frac{e^{-j\phi}}{1-re^{-j\Omega_0}z^{-1}}\right]$$

This equation converges for $\left|r\,e^{j\Omega_0}z^{-1}\right|<1$ or $\left|r\,e^{-j\Omega_0}z^{-1}\right|<1$. These two condition lead to ROC of $|z|>r$. The above equation can be further simplified to ,

$$X(z) = A\frac{\cos\phi - r\cos(\Omega_0-\phi)\,z^{-1}}{1-2r\cos\Omega_0 z^{-1}+r^2z^{-2}} \qquad ROC: |z|>r$$

iii) $\boldsymbol{x(n) = \alpha^{-|n|}, 0<|\alpha|<1}$

$$= \alpha^{-n}u(n)+\alpha^n u(-n-1)$$

$$= \left(\frac{1}{\alpha}\right)^n u(n)+\alpha^n u(-n-1)$$

$$\therefore \quad X(z) = \frac{1}{1-\frac{1}{\alpha}z^{-1}}-\frac{1}{1-\alpha\, z^{-1}} \qquad |z|>\frac{1}{\alpha} \text{ and } |z|<\alpha$$

$$= \frac{\left(\frac{1}{\alpha}-\alpha\right)z^{-1}}{1-\left(\frac{1}{\alpha}+\alpha\right)z^{-1}+z^{-2}}, \qquad ROC: \frac{1}{\alpha}<|z|<\alpha$$

iv) $\boldsymbol{x(n) = e^{-n/10}\sin\left(\frac{2\pi n}{8}\right)u(n)}$

$$a^n\sin(\Omega_0 n)\;u(n) \xleftrightarrow{z} \frac{\left(\frac{z}{a}\right)^{-1}\sin\Omega_0}{1-2\left(\frac{z}{a}\right)^{-1}\cos\Omega_0+\left(\frac{z}{a}\right)^{-2}}, \qquad |z|>|a|$$

Here $\quad a = e^{-1/10}$ $\qquad$ Hence $a^n = \left(e^{-1/10}\right)^n$

$\therefore \quad a = 0.9$

And $\quad \Omega_0 = \frac{2\pi}{8}=\frac{\pi}{4}$

$$\therefore \quad X(z) = \frac{\left(\frac{z}{0.9}\right)^{-1}\sin\frac{\pi}{4}}{1-2\left(\frac{z}{0.9}\right)^{-1}\cos\frac{\pi}{4}+\left(\frac{z}{0.9}\right)^{-2}}, \qquad |z|>0.9$$

$$= \frac{0.636\, z^{-1}}{1-1.272\, z^{-1}+0.81 z^{-2}}, \qquad |z| > 0.9$$

v) $\boldsymbol{x(n) = n^2\, u(n)}$

$$n\, x(n) \xleftrightarrow{z} -z\frac{d}{dz}X(z)$$

$$\therefore \qquad n^2 x(n) \xleftrightarrow{z} -z\frac{d}{dz}\left[-z\frac{d}{dz}\, X(z)\right]$$

Here
$$u(n) \xleftrightarrow{z} \frac{1}{1-z^{-1}}$$

$$\therefore \qquad n^2 u(n) \xleftrightarrow{z} -z\frac{d}{dz}\left[-z\frac{d}{dz}\, \frac{1}{1-z^{-1}}\right]$$

$$\xleftrightarrow{z} -z\frac{d}{dz}\left[\frac{z^{-1}}{\left(1-z^{-1}\right)^2}\right]$$

$$\xleftrightarrow{z} -z\frac{d}{dz}\frac{z}{(z-1)^2}$$

$$\xleftrightarrow{z} -z\left[\frac{-z^2+1}{(z-1)^4}\right]$$

$$\xleftrightarrow{z} \frac{z^3-z}{(z-1)^4}$$

$$\xleftrightarrow{z} \frac{z\left(z^2-1\right)}{(z-1)^4} \quad \text{i.e.} \quad \frac{z(z-1)(z+1)}{(z-1)^4}$$

$$\xleftrightarrow{z} \frac{z(z+1)}{(z-1)^3}$$

Example 2.4.7 *Find the z-transform of $x_1(n)$ = {3, 5, 7} and $x_2(n)$ = {3, 0, 5, 0, 7}. What is the relation between $X_2(z)$ and $X_1(z)$?* **AU : Dec.-06, Marks 6**

Solution : z-transforms of $x_1(n)$ and $x_2(n)$

$$X_1(z) = \sum_{n=0}^{2} x_1(n)\, z^{-n}$$

$$= 3 + 5z^{-1} + 7z^{-2}, \quad \text{ROC : Entire z-plane except } z \neq 0$$

And $$X_2(z) = \sum_{n=0}^{4} x_2(n) z^{-n}$$

$$= 3 + 5z^{-2} + 7z^{-4}, \quad \text{ROC : Entire z-plane except } z \neq 0$$

Relationship between $X_1(z)$ and $X_2(z)$

$$X_2(z) = \sum_{n=0}^{4} x_2(n) z^{-n}$$

Let n = 2k,

$$X_2(z) = \sum_{k=0}^{2} x_2(2k) z^{-2k}$$

$$= \sum_{k=0}^{2} x_2(2k)(z^{-2})^k$$

Here observe that $x_2(2k) = x_1(k)$. Hence above equation becomes,

$$X_2(z) = \sum_{k=0}^{2} x_1(k)(z^{-2})^k = X_1(z^{-2})$$

Example 2.4.8 *Convolute the following two sequences $x_1(n) = \{0, 1, 4, -2)$ and $x_2(n) = \{1, 2, 2, 2\}$*

AU : May-15, Marks 8

Solution : Taking z-transform of $x_1(n)$ and $x_2(n)$,

$$X_1(z) = 0 + z^{-1} + 4z^{-2} - 2z^{-3}$$

$$X_2(z) = 1 + 2z^{-1} + 2z^{-2} + 2z^{-3}$$

$$X_1(z) \,.\, X_2(z) = (z^{-1} + 4z^{-2} - 2z^{-3})\ (1 + 2z^{-1} + 2z^{-2} + 2z^{-3})$$

$$= z^{-1} + 6z^{-2} + 8z^{-3} + 6z^{-4} + 4z^{-5} - 4z^{-6}$$

By convolution theorem $x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X_1(z) \cdot X_2(z)$

Taking inverse z-transform of above equation gives convolution of $x_1(n)$ and $x_2(n)$ i.e.,

$$x_1(n) * x_2(n) = \{0, 1, 6, 8, 6, 4, -4\}$$
$$\uparrow$$

Examples for Practice

Example 2.4.9

Find the z transform and its ROC of

$x(n) = \left(\frac{-1}{5}\right)^n u(n) + 5\left(\frac{1}{2}\right)^n u(-n-1).$

AU : Dec.-12, Marks 6

[Ans. : $X(z) = \frac{1}{1+\frac{1}{5}z^{-1}} - \frac{5}{1-\frac{1}{2}z^{-1}}$, ROC : $\frac{1}{5} < |z| < \frac{1}{2}$]

Example 2.4.10 *Determine the z-transform and ROC of the signal* $x[n] = [3(4^n) - 4(2^n)]u[n]$.

AU : May-10, Marks 16

[Ans. : $X(z) = \frac{3}{1-4z^{-1}} - \frac{4}{1-2z^{-1}}$, ROC : $|z| > 4$]

Review Questions

1. *State and explain four properties of z-transform.* **AU : Dec.-10, Marks 8, May-07, Marks 16**
2. *Prove following properties of z-transform*

   i) Time shifting property ii) Convolution in time domain property. **AU : Dec.-06, Marks 8**
3. *State and prove initial value theorem of z-transform.* **AU : May-17, Marks 3**
4. *State and prove Parseval's theorem using z-transform.* **AU : May-17, Marks 3**

2.5 Inverse z-Transform

AU : May-11, 14, 15, 16, 17, Dec.-05, 06, 13, 15, 16

The inverse z-transform can be obtained by,

i) Power series expansion

ii) Partial fraction expansion

iii) Contour integration.

2.5.1 Inverse z-Transform using Power Series Expansion

- By definition z-transform of the sequence x(n) is given as,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \cdots + x(-2)\, z^2 + x(-1)\, z + x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots$$

- From above expansion of z-transform, the sequence x(n) can be obtained as,

$$x(n) = \{\cdots, x(-2), x(-1), x(0), x(1), x(2), \cdots\}$$

- The power series expansion can be obtained directly or by long division method.

2.5.2 Inverse z-Transform using Partial Fraction Expansion

Following steps are to be performed for partial fraction expansions :

Step 1 : Arrange the given X(z) as,

$$\frac{X(z)}{z} = \frac{\textit{Numerator polynomial}}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

Step 2 :

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1}+\frac{A_2}{z-p_2}+\frac{A_3}{z-p_3}+\cdots\frac{A_N}{z-p_N} \qquad \ldots(2.5.1)$$

Where, $A_k = (z-p_k)\cdot\left.\frac{X(z)}{z}\right|_{z=p_k}$, k = 1, 2, … N … (2.5.2)

If $\frac{X(z)}{z}$ has the pole of multiplicity 'n' i.e.,

$$\frac{X(z)}{z} = \frac{\textit{Numerator polynomial}}{(z-p)^n}$$

$$= \frac{A_1}{z-p}+\frac{A_2}{(z-p)^2}+\cdots\frac{A_n}{(z-p)^n}$$

Where $A_1, A_2, \cdots A_n$ are given as,

$$A_k = \frac{1}{(n-k)!}\cdot\frac{d^{n-k}}{dz^{n-k}}\left.\left\{(z-p)^n\cdot\frac{X(z)}{z}\right\}\right|_{z=p} \qquad \ldots(2.5.3)$$

k = 1, 2, 3, … n

Step 3 : Equation (2.5.1) can be written as,

$$X(z) = \frac{A_1 z}{z-p_1}+\frac{A_2 z}{z-p_2}+\cdots+\frac{A_N z}{z-p_N}$$

$$= \frac{A_1}{1-p_1 z^{-1}}+\frac{A_2}{1-p_2 z^{-1}}+\cdots+\frac{A_N}{1-p_N z^{-1}}$$

Step 4 : All the terms in above step are of the form $\frac{A_k}{1-p_k z^{-1}}$. Depending upon ROC, following standard z-transform pairs must be used.

$$p_k^n \, u(n) \xleftrightarrow{z} \frac{1}{1-p_k z^{-1}}, \; ROC : |z| > |a| \text{ i.e. causal sequence} \quad \ldots (2.5.4)$$

$$-(p_k)^n \, u(-n-1) \xleftrightarrow{z} \frac{1}{1-p_k z^{-1}}, \; ROC : |z| < |a| \text{ i.e. noncausal sequence} \quad \ldots (2.5.5)$$

Examples for Understanding

Example 2.5.1 *Determine inverse z-transform of the following :*

i) $X(z) = \dfrac{1}{1-a z^{-1}}$, *ROC :* $|z|>|a|$

ii) $X(z) = \dfrac{1}{1-a z^{-1}}$, *ROC :* $|z|<|a|$

Solution :

(i) $$X(z) = \frac{1}{1-a z^{-1}}, \; ROC : |z|>|a|$$

$$\begin{array}{r|l}
 & 1 + a z^{-1} + a^2 z^{-2} + a^3 z^{-3} \leftarrow \textit{Negative power of 'z'} \\
1 - a z^{-1} & 1 \\
 & 1 - a z^{-1} \\
 & - \quad + \\ \hline
 & a z^{-1} \\
 & a z^{-1} - a^2 z^{-2} \\
 & - \quad\quad + \\ \hline
 & a^2 z^{-2} \\
 & a^2 z^{-2} - a^3 z^{-3} \\
 & - \quad\quad + \\ \hline
 & a^3 z^{-3} \\
 & a^3 z^{-3} - a^4 z^{-4} \\
 & - \quad\quad + \\ \hline
 & a^4 z^{-4} \ldots
\end{array}$$

Thus we have, $X(z) = \dfrac{1}{1-a z^{-1}} = 1 + a z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$

Taking inverse z-transform, $x(n) = \{1, a, a^2, a^3, \cdots\}$

$$= a^n u(n)$$

ii) $$X(z) = \frac{1}{1-az^{-1}}\text{ , ROC : }|z|<|a|$$

$$= \frac{1}{-az^{-1}+1}$$ Equation rearranged to get positive powers of 'z'.

$$\begin{array}{r l}
 & -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - a^{-4}z^4 \leftarrow \textit{Positive powers of 'z'} \\
-az^{-1}+1 \big) & 1 \\
 & 1 - a^{-1}z \\
 & -\quad + \\
\hline
 & a^{-1}z \\
 & a^{-1}z - a^{-2}z^2 \\
 & -\qquad + \\
\hline
 & a^{-2}z^2 \\
 & a^{-2}z^2 - a^{-3}z^3 \\
 & -\qquad + \\
\hline
 & a^{-3}z^3 \\
 & a^{-3}z^3 - a^{-4}z^4 \\
 & \qquad + \\
\hline
 & a^{-4}z^4 \ \ldots
\end{array}$$

Thus we have, $$X(z) = \frac{1}{1-az^{-1}} = -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - a^{-4}z^4 \cdots$$

Rearranging above equation, $$= \cdots\cdots - a^{-4}z^4 - a^{-3}z^3 - a^{-2}z^2 - a^{-1}z$$

Taking inverse z-transform, $$x(n) = \{\cdots - a^{-4}, -a^{-3}, -a^{-2}, -a^{-1}\}$$
$$\uparrow$$

$$= -a^n u(-n-1)$$

Comments on Results (i) When the ROC is $|z| > |a|$, then expand $X(z)$ such that powers of 'z' are negative.

ii) When ROC is $|z| < |a|$, then expand $X(z)$ such that powers of 'z' are positive.

Example 2.5.2 *Determine inverse z-transform of* $X(z) = \dfrac{1}{1-1.5z^{-1}+0.5z^{-2}}$.

For i) ROC : |z| > 1, (ii) ROC : |z|< 0.5 and iii) ROC : 0.5 < |z|< 1

AU : May-11, Marks 8

Solution : Step 1 : First convert X(z) to positive powers of z. i.e.,

$$X(z) = \frac{z^2}{z^2-1.5z+0.5}$$

$$\therefore \quad \frac{X(z)}{z} = \frac{z}{z^2-1.5z+0.5} = \frac{z}{(z-1)(z-0.5)}$$

Step 2 :
$$\frac{X(z)}{z} = \frac{A_1}{z-1}+\frac{A_2}{z-0.5} \qquad \text{...(2.5.6)}$$

$$\therefore \quad A_1 = (z-1)\cdot\left.\frac{z}{(z-1)(z-0.5)}\right|_{z=1} = \frac{1}{1-0.5} = 2$$

and
$$A_2 = (z-0.5)\cdot\left.\frac{z}{(z-1)(z-0.5)}\right|_{z=0.5} = \frac{0.5}{0.5-1} = -1$$

Equation (2.5.3) will be, $\dfrac{X(z)}{z} = \dfrac{2}{z-1}-\dfrac{1}{z-0.5}$

Step 3 :
$$X(z) = \frac{2z}{z-1}-\frac{z}{z-0.5}$$

$$= \frac{2}{1-z^{-1}}-\frac{1}{1-0.5z^{-1}} \qquad \text{... (2.5.7)}$$

Step 4 : i) x(n) for ROC of |z| > 1

- Here the poles are at z = 1 and z = 0.5 from equation (2.5.6).
- Now ROC of |z|> 1 indicates that sequence corresponding to the term $\dfrac{2}{1-z^{-1}}$ in equation (2.5.7) must be causal.
- Fig. 2.5.1 shows the ROCs of |z|>1 and |z|> 0.5. Observe that |z|>1 includes |z|> 0.5.

- Hence the sequence corresponding to the term $\dfrac{1}{1-0.5z^{-1}}$ in equation (2.5.7) must also be causal.

Therefore from equation (2.5.7) inverse z-transform becomes,

$$x(n) = 2(1)^n u(n) - 1.(0.5)^n u(n)$$

$$= [2-(0.5)^n]u(n)$$

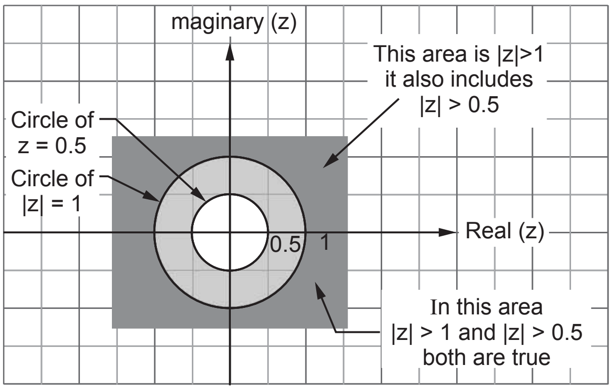


Fig. 2.5.1 ROCs of |z| > 1 and |z| > 0.5

ii) x(n) for ROC : $|z| < 0.5$

- The sequence corresponding $\dfrac{1}{1-0.5z^{-1}}$ in equation (2.5.7) will be noncausal.
- Fig. 2.5.2 shows the ROCs of $|z| < 0.5$ and $|z| < 1$. Observe that $|z| < 0.5$ includes $|z| < 1$.
- Hence sequence corresponding to $\dfrac{2}{1-z^{-1}}$ in equation (2.5.7) will be noncausal.

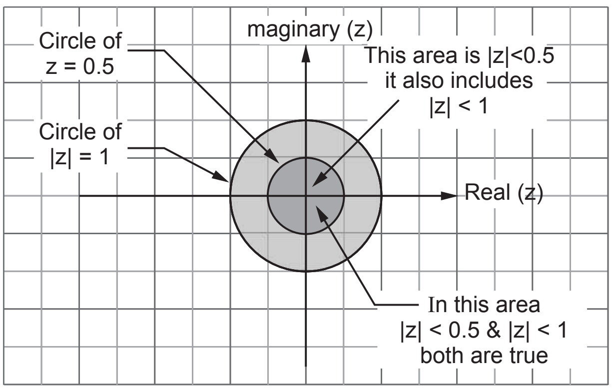


Fig. 2.5.2 ROCs of |z| < 0.5 and |z| < 1

Therefore from equation (2.5.7), inverse z-transform becomes,

$$x(n) = 2[-1^n u(-n-1)] - [-0.5^n u(-n-1)]$$

$$= [-2 + 0.5^n] u(-n-1)$$

iii) x(n) for ROC : 0.5 <|z|< 1

- This ROC can be written as $|z| > 0.5$ and $|z| < 1$.
- The sequence corresponding to $\frac{2}{1-z^{-1}}$ in equation (2.5.7) will be non causal since ROC is $|z| < 1$.
- The sequence corresponding to $\frac{1}{1-0.5z^{-1}}$ in equation (2.5.7) will be causal since ROC is $|z| > 0.5$.

Taking inverse z-transform of equation (2.5.7),

$$x(n) = 2[-1^n u(-n-1)] - (0.5)^n u(n) \quad = -2u(-n-1) - (0.5)^n u(n)$$

Example 2.5.3 *Determine inverse z-transform of*

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}, \; ROC : |z| > 1$$

Solution : Step 1 : Converting X(z) to positive powers of z,

$$X(z) = \frac{z^3}{(z+1)(z-1)^2}$$

$$\therefore \quad \frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$

Step 2 : Here there is multiple pole at z = 1. Therefore the partial fraction expansion will be,

$$\frac{X(z)}{z} = \frac{A_1}{z+1} + \frac{A_2}{(z-1)} + \frac{A_3}{(z-1)^2} \qquad \dots (2.5.8)$$

$$\therefore \quad A_1 = (z+1)\cdot\frac{X(z)}{z}\Bigg|_{z=-1} = \frac{z^2}{(z-1)^2}\Bigg|_{z=-1} = \frac{1}{4}$$

$$A_3 = (z-1)^2 \frac{X(z)}{z}\Bigg|_{z=1} = \frac{z^2}{z+1}\Bigg|_{z=1} = \frac{1}{2}$$

$$A_2 = \frac{d}{dz}\left\{(z-1)^2 \cdot \frac{X(z)}{z}\right\}\Bigg|_{z=1} \qquad \text{By equation (2.5.3)}$$

$$= \frac{d}{dz}\cdot\left(\frac{z^2}{z+1}\right)\Bigg|_{z=1} = \frac{(z+1)\,2z - z^2}{(z+1)^2}\Bigg|_{z=1} = \frac{3}{4}$$

Putting values in equation (2.5.8),

$$\frac{X(z)}{z} = \frac{1/4}{z+1} + \frac{3/4}{z-1} + \frac{1/2}{(z-1)^2}$$

Step 3 : $$X(z) = \frac{1/4\,z}{z+1} + \frac{3/4\,z}{z-1} + \frac{1/2\,z}{(z-1)^2}$$

$$\therefore \quad X(z) = \frac{1/4}{1+z^{-1}} + \frac{3/4}{1-z^{-1}} + \frac{1/2\,z^{-1}}{(1-z^{-1})^2}$$

Step 4 : ROC is $|z| > 1$. Let us use following relations :

$$p_k^n\,u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-p_k\,z^{-1}}, \; ROC : |z| > |p_k|$$

Hence inverse z-transform of first two terms of X(z) will be,

$$\text{IZT}\left\{\frac{1/4}{1+z^{-1}}\right\} = \frac{1}{4}(-1)^n\,u(n) \text{ and IZT}\left\{\frac{3/4}{1-z^{-1}}\right\} = \frac{3}{4}(1)^n\,u(n)$$

For 3rd term of X(z) let us use,

$$n\,p_k^n\,u(n) \overset{z}{\longleftrightarrow} \frac{p_k\,z^{-1}}{(1-p_k\,z^{-1})^2}, \; ROC : |z| > |p_k|$$

i.e. $$IZT\left\{\frac{1/2\,z^{-1}}{(1-z^{-1})^2}\right\} = \frac{1}{2}IZT\left\{\frac{z^{-1}}{(1-z^{-1})^2}\right\} = \frac{1}{2}n(1)^n\,u(n)$$

Putting all the sequences together,

$$x(n) = \frac{1}{4}(-1)^n\,u(n) + \frac{3}{4}(1)^n\,u(n) + \frac{1}{2}n(1)^n\,u(n) = \left[\frac{1}{4}(-1)^n + \frac{3}{4} + \frac{1}{2}n\right]u(n)$$

Example 2.5.4 *Inverse z-transform for X(z) = 1/(z – 1.5)^4; ROC : |z| > 1/4*

AU : Dec.-16, Marks 4

Solution :

$$X_o(z) = X(z)\cdot z^{n-1} = \frac{1\cdot z^{n-1}}{(z-1.5)^4} \qquad \text{ROC : } |z| > 1/4$$

The pole is at z = 1.5 has the order m = 4, and the residue is calculated as,

$$\underset{z=p_i}{\text{Res}}\ X_o(z) = \frac{1}{(m-1)!}\left\{\frac{d^{m-1}}{dz^{m-1}}(z-p_i)^m X_o(z)\right\}_{z=p_i}$$

$$\therefore \quad \underset{z=1.5}{\text{Res}}\ X_o(z) = \frac{1}{(4-1)!}\left\{\frac{d^3}{dz^3}\frac{(z-1.5)^4\cdot z^{n-1}}{(z-1.5)^4}\right\}_{z=1.5} = \frac{1}{6}\left\{\frac{d^2}{dz^2}\left\{\frac{d}{dz}z^{n-1}\right\}\right\}_{z=1.5}$$

$$= \frac{1}{6}\left\{\frac{d}{dz}\left[\frac{d}{dz}(n-1)z^{n-2}\right]\right\}_{z=1.5} = \frac{(n-1)}{6}\left\{\frac{d}{dz}[(n-2)z^{n-3}]\right\}$$

$$= \frac{(n-1)(n-2)}{6}\{(n-3)z^{n-4}\}_{z=1.5}$$

$$\underset{z=1.5}{\text{Res}}\ X_o(z) = \frac{(n-1)(n-2)(n-3)}{6}(1.5)^{n-4}$$

Since poles are inside ROC, the sequence will be causal

$$\therefore \quad x(n) = \frac{(n-1)(n-2)(n-3)}{6}(1.5)^{n-4}\ u(n)$$

Example 2.5.5 *Find the inverse z-transform of the following :*

i) $X(z) = \log\left(\frac{1}{1-az^{-1}}\right), \qquad |z|>|a|$

ii) $X(z) = \log\left(\frac{1}{1-a^{-1}z}\right), \qquad |z|<|a|$

iii) $X(z) = e^{\frac{1}{z}}$

Solution : i) $\boldsymbol{(z) = \log\left(\frac{1}{1-az^{-1}}\right), \qquad |z|>|a|}$

$$= -\log\left(1-az^{-1}\right)$$

$\log(1-p)$ is expanded with the help of power series. It is given as,

$$\log(1-p) = -\sum_{n=1}^{\infty}\frac{p^n}{n}, \qquad |p| < 1$$

$$\therefore \qquad X(z) = -\left[-\sum_{n=1}^{\infty} \frac{\left(az^{-1}\right)^n}{n}\right] \qquad \text{for } \left|a\ z^{-1}\right| < 1 \quad \text{or} \quad |z| > |a|$$

$$= \sum_{n=1}^{\infty} \frac{a^n}{n} \cdot z^{-n} \quad = \sum_{n=1}^{\infty} x(n) \cdot z^{-n}$$

Here $$x(n) = \begin{cases} \dfrac{a^n}{n} & \textit{for} \quad n \geq 1 \\ 0 & \textit{Otherwise} \end{cases}$$

i.e. $$x(n) = \frac{a^n}{n}\ u\ (n-1)$$

ii) $\boldsymbol{X(z) = \log\left(\dfrac{1}{1-a^{-1}z}\right), \qquad |z| < |a|}$

$$= -\log\left(1 - a^{-1}z\right)$$

$$= -\left[-\sum_{n=1}^{\infty} \frac{\left(a^{-1}z\right)^n}{n}\right], \qquad \left|a^{-1}z\right| < 1 \quad \text{or} \ |z| < |a|$$

$$= \sum_{n=1}^{\infty} \frac{\left(a^{-1}z\right)^n}{n}$$

Let n = – m, then above equation will be,

$$X(z) = \sum_{m=-1}^{-\infty} \frac{\left(a^{-1}z\right)^{-m}}{-m} = \sum_{m=-1}^{-\infty} -\frac{a^m}{m} \cdot z^{-m}$$

$$= \sum_{m=-1}^{-\infty} x(m)\ z^{-m}$$

Here $$x(m) = \begin{cases} -\dfrac{a^m}{m} & \textit{for} \quad m = -1 \ \ \textit{to} -\infty \\ 0 & \textit{otherwise} \end{cases}$$

$$\therefore \qquad x(n) = -\frac{a^n}{n}\ u(-n-1)$$

iii) $X(z) = e^{\frac{1}{z}}$

$$\therefore \quad \log_e X(z) = \frac{1}{z}$$

Differentiating both sides with respect to z,

$$\frac{1}{X(z)} \cdot \frac{dX(z)}{dz} = -\frac{1}{z^2}$$

$$\therefore \quad \frac{dX(z)}{dz} = -z^{-2} X(z)$$

Multiplying both sides by $-z$, we get,

$$-z \frac{d}{dz} X(z) = z^{-1} X(z)$$

Taking inverse z-transform,

$$n\, x(n) = x\,(n-1)$$

or

$$x(n) = \frac{1}{n}\, x(n-1)$$

Initial value is given as,

$$x(0) = \lim_{z \to \infty} X(z) = \lim_{z \to \infty} e^{\frac{1}{z}}$$

$$= e^0 = 1$$

$$\therefore \quad x(0) = 1 \quad \text{i.e } x(0) = \frac{1}{0!}$$

and for $n = 1$, $\quad x(1) = \frac{1}{1} x(0) = 1 = \frac{1}{1!}$

$n = 2$, $\quad x(2) = \frac{1}{2} x(1) = \frac{1}{2} = \frac{1}{2!}$

$n = 3$, $\quad x(3) = \frac{1}{3} x(2) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6} = \frac{1}{3!}$

$n = 4$, $\quad x(4) = \frac{1}{4} x(3) = \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{4!}$

Thus,

$$x(n) = \frac{1}{n!}\, u(n)$$

Example 2.5.6 *Find x(n) if* $X(z) = \dfrac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}$

AU : May-16, Marks 6

Solution :

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{2} \cdot \frac{z^{-1}}{1-\frac{1}{2}z^{-1}}$$

Here use $a^n u(n) \xleftrightarrow{z} \frac{1}{1-az^{-1}}$ and $x(n-k) \xleftrightarrow{z} z^{-k}X(z)$.

Then inverse z - transform will be,

$$x(n) = \left(\frac{1}{2}\right)^n u(n) + \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n-1} u(n-1)$$

$$= \left(\frac{1}{2}\right)^n u(n) + \frac{1}{2} \cdot \left(\frac{1}{2}\right)^n \cdot \left(\frac{1}{2}\right)^{-1} u(n-1)$$

$$= \left(\frac{1}{2}\right)^n [u(n) + u(n-1)]$$

Example 2.5.7 *Find the inverse z-transform of*

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

For i) $|z| > 1$ *ii)* $|z| < \frac{1}{3}$ *iii)* $\frac{1}{3} < |z| < 1$.

AU : May-14, 15, Marks 8, Dec.-15, Marks 8

Solution : i) ROC of $|z| > 1$

$$X(z) = \frac{z/3}{z^2 - \frac{4}{3}z + \frac{1}{3}}$$

$$\therefore \quad \frac{X(z)}{z} = \frac{1/3}{(z-1)\left(z-\frac{1}{3}\right)} = \frac{1/2}{z-1} - \frac{1/2}{z-\frac{1}{3}}$$ and poles are at $p_1 = 1$ and $p_2 = \frac{1}{3}$

$$\therefore \quad X(z) = \frac{1}{2}\left[\frac{1}{1-z^{-1}} - \frac{1}{1-\frac{1}{3}z^{-1}}\right] \quad ...(1)$$

Since ROC is $|z| > 1$, it is outside of both the poles at $z = 1$ and $z = \frac{1}{3}$. Hence time domain sequence will be causal.

$$x(n) = \frac{1}{2}\left[u(n) - \left(\frac{1}{3}\right)^n u(n)\right]$$

ii) ROC of $|z| < \frac{1}{3}$

For this ROC, both the poles lie outside the ROC of $|z| < \frac{1}{3}$. Hence sequences corresponding to poles at $p_1 = 1$ and $p_2 = \frac{1}{3}$ will be noncausal.

$$\therefore \qquad x(n) = \frac{1}{2}\left[-1+\left(\frac{1}{3}\right)^n\right]u(-n-1)$$

iii) ROC of $\frac{1}{3} < |z| < 1$:

Here for pole $p_2 = \frac{1}{3}$, ROC is outside, since $|z| > \frac{1}{3}$. Hence sequence corresponding to pole at $p_2 = \frac{1}{3}$ will be causal. And for pole $p_1 = 1$, ROC is inside, since $|z| < 1$. Hence sequence correspnding to pole at $p_1 = 1$ will be noncausal.

$$\therefore \qquad x(n) = \frac{1}{2}\left[-u(-n-1)-\left(\frac{1}{3}\right)^n u(n)\right]$$

Example 2.5.8 *Find the inverse z-transform of X(z) = (x+1)/(x + 0.2)(x − 1), |z| > 1 using residue method.* **AU : May-17, Marks 13**

Solution :

Step 1 : $X_o(z) = X(z)z^{n-1} = \dfrac{(z+1)z^{n-1}}{(z+0.2)(z-1)}$

Step 2 : Here the poles are z = 0.2 and z = 1 and for n = 0, the pole is at z = 0.

$\therefore$ for n = 0;

$$x(0) = \sum \text{residues of } \frac{z+1}{z(z+0.2)(z+1)} \text{ with poles at z = 0, z = 0.2 and z = 1.}$$

$$= z\frac{(z+1)}{z(z+0.2)(z-1)}\bigg|_{z=0} + (z+0.2)\frac{(z+1)}{(z+0.2)(z-1)}\bigg|_{z=0.2}$$

$$+ (z-1)\frac{(z+1)}{z(z+0.2)(z-1)}\bigg|_{z=1}$$

$$= 0$$

i.e. $x(0) = 0$

Now for $n \geq 1$

$$x(n) = \sum \text{residues of } \frac{z+1}{(z+0.2)(z-1)} \text{ at poles } z=-0.2 \text{ and } z=1$$

$$= \text{residue of } \frac{(z+1)z^{n-1}}{(z+0.2)(z-1)} \text{ at poles z = 1 and z = – 0.2}$$

$$= (z-1)\frac{(z+1)z^{n-1}}{(z+0.2)(z-1)}\bigg|_{z=1} + (z+0.2)\frac{(z+1)z^{n-1}}{(z+0.2)(z-1)}\bigg|_{z=-0.2} = \frac{5}{3}-\frac{2}{3}(-0.2)^{n-1}$$

$$\therefore \quad x(n) = \frac{-2}{3}(-0.2)^{n-1}u(n-1)+\frac{5}{3}u(n-1)$$

Examples for Practice

Example 2.5.9 *Find the inverse z-transform of H(z) = z (z + 2) / (z – 0.2) (z + 0.6).*

AU : Dec.-06, Marks 10

[Ans. : $h(n) = \frac{11}{4}(0.2)^n u(n) - \frac{7}{4}(-0.6)^n u(n)$]

Example 2.5.10 *Find the inverse z-transform of* $\frac{z(z+2)}{z^2+0.4z-1.2}$

AU : Dec.-05, Marks 8

[Ans. : $x(n)=1.3\ (0.9136)^n u(n) - 0.3(-1.3136)^n\ u(n)$]

2.6 Unilateral z-Transform

- The unilateral z-transform is also called as one sided z-transform.
- It is defined for $n \geq 0$, i.e. causal sequences.
- The unilateral z-transform is used to solve difference equations with initial conditions.

Definition

$$\boxed{X(z) = \sum_{n=0}^{\infty} x(n)\ z^{-n}} \quad \dots (2.6.1)$$

The inverse z-transform can be obtained using any of the methods discussed earlier. In this case all sequences will be causal. Unilateral and bilateral z-transforms are same for causal signals. For example,

$$a^n\ u(n) \quad \xleftrightarrow{Z_u} \quad \frac{1}{1-a\ z^{-1}}$$

2.6.1 Properties of Unilateral z-Transform

The unilateral z-transform has almost all the properties similar to bilateral z-transform. Except time shift property.

2.6.1.1 Time Shift

If $$x(n) \xleftrightarrow{Z_u} X(z)$$

then $$x(n-k) \xleftrightarrow{Z_u} x(-k)+x(-k+1)\, z^{-1}+.....+x(-1)\, z^{-k+1}+z^{-k}\, X(z) \qquad ...\ (2.6.2)$$

Proof :

Consider the function $y(n) = x(n-1)$. Then z-transform of this function will be,

$$Y(z) = \sum_{n=0}^{\infty} y(n)\, z^{-n} = \sum_{n=0}^{\infty} x(n-1)\, z^{-n}$$

$$= x(-1)+\sum_{n=1}^{\infty} x(n-1)\, z^{-n}$$

Put $n-1=m$ in above equation,

$$Y(z) = x(-1)+\sum_{m=0}^{\infty} x(m)\, z^{-(m+1)}$$

$$= x(-1)+\sum_{m=0}^{\infty} x(m)\, z^{-m}\, z^{-1}$$

$$= x(-1)+z^{-1}\sum_{m=0}^{\infty} x(m)\, z^{-m}$$

$$= x(-1)+z^{-1}\, X(z) \qquad ...(2.6.3)$$

Thus we obtained,

$$x(n-1) \xleftrightarrow{Z_u} x(-1)+z^{-1}\, X(z) \qquad ...(2.6.4)$$

Here $x(-1)$ is the additional term. It is initial condition. Similarly the above result can be given for $x(n-2)$ as,

$$x(n-2) \xleftrightarrow{Z_u} x(-2)+x(-1)\, z^{-1}+z^{-2}\, X(z) \qquad ...(2.6.5)$$

If above result is generalized we get equation (2.6.2). This property is useful in solving difference equations with initial conditions. This is discussed in next section.

For time advance,

$$x(n+k) \xleftrightarrow{z} -x(0)z^{k}-x(1)z^{k-1}-.....-x(k-1)z+z^{k}X(z) \qquad ...(2.6.6)$$

Above equation can be proved similarly.

2.6.1.2 Final Value Theorem

The final value is given as,

$$x(\infty) = \lim_{z\to 1}(1-z^{-1})\, X(z) \qquad \ldots(2.6.7)$$

Here $X(z)$ is the unilateral z-transform.

Proof :

With k = 1 in time advance equation given by equation (2.6.6) we get,

$$x(n+1) \xleftrightarrow{z} -x(0)z + z\,X(z)$$

i.e. $$\sum_{n=0}^{\infty} x(n+1)\, z^{-n} = -x(0)z + z\,X(z)$$

Subtracting $X(z)$ from both the sides,

$$\sum_{n=0}^{\infty} x(n+1)\, z^{-n} - X(z) = -x(0)\, z + z\,X(z) - X(z)$$

$X(z)$ of LHS can be written by definition as,

$$\sum_{n=0}^{\infty} x(n+1)\, z^{-n} - \sum_{n=0}^{\infty} x(n)\, z^{-n} = -x(0)\, z + (z-1)\, X(z)$$

Taking the limit as $z \to 1$ in above equation,

$$\sum_{n=0}^{\infty} x(n+1) - \sum_{n=0}^{\infty} x(n) = -x(0) + \lim_{z\to 1}\,(z-1)\, X(z)$$

$$\sum_{n=0}^{\infty} x(n+1) - x(0) - \sum_{n=1}^{\infty} x(n) = -x(0) + \lim_{z\to 1}\,(z-1)\, X(z)$$

$$\therefore \qquad \sum_{n=0}^{\infty} x(n+1) - \sum_{n=1}^{\infty} x(n) = \lim_{z\to 1}(z-1)\, X(z) \qquad \ldots(2.6.8)$$

Let us expand the LHS of above equation as $n \to \infty$ i.e.,

$$\lim_{n\to\infty}[x(1)+x(2)+x(3)+\ldots+x(n)+x(n+1)] - [x(1)+x(2)+x(3)+\ldots+x(n)]$$

$$= \lim_{n\to\infty}\{[x(1)-x(1)]+[x(2)-x(2)]+[x(3)-x(3)]+\ldots+[x(n)-x(n)]+x(n+1)\}$$

$$= \lim_{n\to\infty} x(n+1) = x(\infty)$$

Thus we obtained the final value as (from equation (2.6.8)),

$$x(\infty) = \lim_{z\to 1}(z-1)\, X(z)$$

Example for Understanding

Example 2.6.1 *Find the unilateral z-transform of the following x(n) :*

i) $x(n) = a^n\ u(n)$

ii) $x(n) = a^{n+1}\ u(n+1)$

Solution : i) $x(n) = a^n\ u(n)$

Unilateral and bilateral z-transforms are same for causal signals. Hence,

$$a^n\ u(n) \xleftrightarrow{Z_u} \frac{1}{1-a\,z^{-1}}$$

ii) $x(n) = a^{n+1}\ u(n+1)$

$$X(z) = \sum_{n=0}^{\infty} a^{n+1} u(n+1)\, z^{-n}$$

We know that $u(n+1) = 1$ for $n \geq -1$. Hence above equation becomes,

$$X(z) = \sum_{n=0}^{\infty} a^{n+1}\, z^{-n} = \sum_{n=0}^{\infty} a \cdot a^n z^{-n}$$

$$= \text{a} \sum_{n=0}^{\infty} a^n\, z^{-n} = a \cdot \frac{1}{1-a\ z^{-1}}$$

$$\boxed{= \frac{a}{1-a\,z^{-1}}}$$

This is the unilateral z-transform of given sequence.

Review Question

1. *State and prove final value theorem for z-transform.*

AU : May-05, 10, 11, Dec.-10, 11, 12

The convolution sum (linear convolution) relates the input, output and unit sample response of the discrete time systems.

2.7.1 Discrete Time Signal as Weighted Impulses

Here we will see how a discrete time signal can be expressed in the

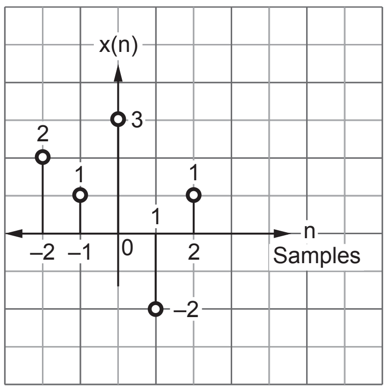


Fig. 2.7.1 Sketch of discrete time signal of equation 2.7.1

form of weighted impulses. Consider the arbitrary discrete time signal $x(n)$ of five samples :

$$x(n) = \{2,\ 1,\ \underset{\uparrow}{3},\ -2,\ 1\} \qquad \text{... (2.7.1)}$$

Here $x(-2) = 2$, $x(-1) = 1$, $x(0) = 3$, $x(1) = -2$ and $x(2) = 1$. This signal is shown graphically in Fig. 2.7.1.

Step 1 :

Now let us consider the unit sample sequence $\delta(n)$. Fig. 2.7.1 shows the sketch of $\delta(n)$. It is defined as,

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases} \qquad \text{... (2.7.2)}$$

Now if $\delta(n)$ is delayed by one sample, then above equation will be,

$$\delta(n-1) = \begin{cases} 1 & \text{for } n = 1 \\ 0 & \text{for } n \neq 1 \end{cases}$$

Similarly if $\delta(n)$ is advanced by one sample, then it can be represented as,

$$\delta(n+1) = \begin{cases} 1 & \text{for } n = -1 \\ 0 & \text{for } n \neq -1 \end{cases}$$

This can be generalized. The $\delta(n)$ function delayed by 'k' samples can be represented as,

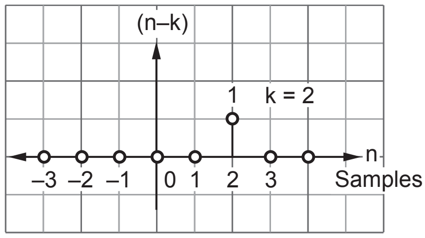


Fig. 2.7.2 Sketch of a delayed unit sample sequence. Here delay, k = 2 samples *k* = 2 samples

$$\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases} \qquad \text{... (2.7.3)}$$

Here k can be positive or negative. Fig. 2.7.2 shows the sketch of a signal $\delta(n-k)$ for $k = 2$.

Step 2 :

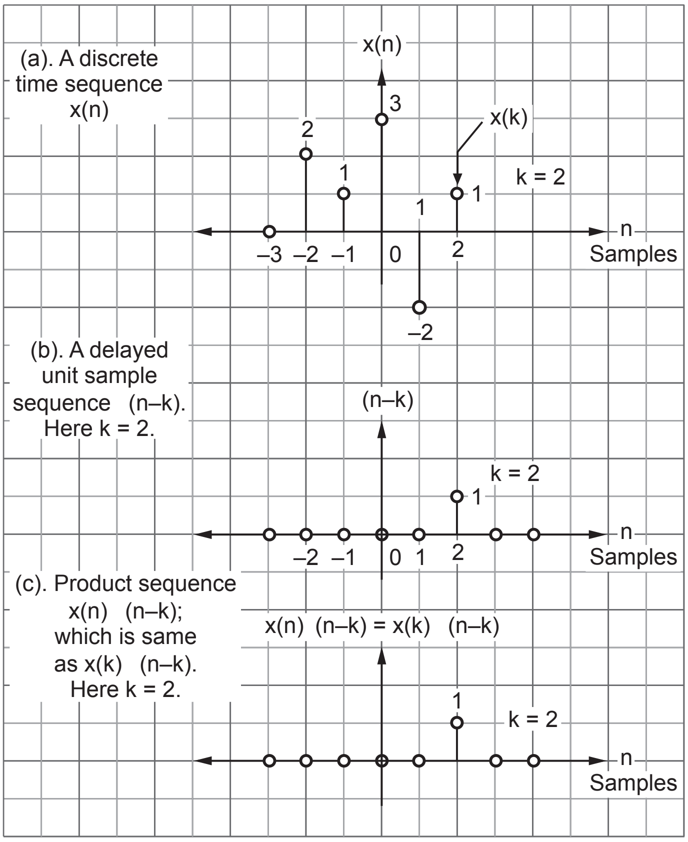


Fig. 2.7.3 The product $x(n)\,\delta(n-k) = x(k)\,\delta(n-k)$ since $\delta(n-k) = 1$ at $n = k$ and $x(n) = x(k)$

Now let us multiply $x(n)$ and $\delta(n-k)$. The value of $\delta(n-k)$ is zero everywhere except at $n = k$. Hence the result of multiplication becomes,

$$x(n)\,\delta(n-k) = x(k)\,\delta(n-k) \qquad \text{... (2.7.4)}$$

This operation is indicated in Fig. 2.7.3. Fig. 2.7.3 (a) shows the sequence $x(n)$ of Fig. 2.7.1. Fig. 2.7.3 (b) shows the delayed unit sample sequence $\delta(n-k)$ of Fig. 2.7.2. Fig. 2.7.3 (c) shows the product sequence $x(n)\,\delta(n-k)$. Thus equation 2.7.4 is clear since $x(n) = x(k)$ at $n = k$. Fig. 2.7.4 shows the various product sequences for $x(n)$ of Fig. 2.7.3.

Step 3 :

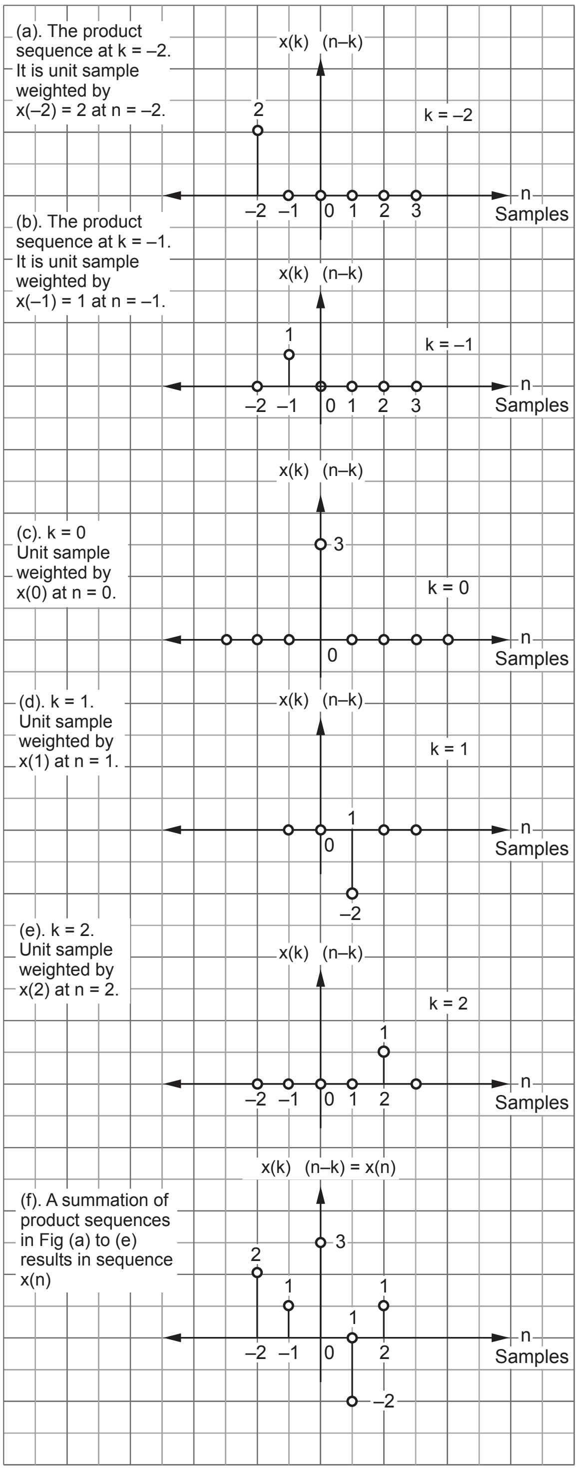


Fig. 2.7.4 A discrete time signal $x(n)$ is expressed as sum of weighted unit samples

From Fig. 2.7.4 it is clear that if we add all the product sequences $x(k)\,\delta(n-k)$, then we get $x(n)$. Here 'k' varies from –2 to +2. For the generalized case the range of 'k' will be $-\infty < k < \infty$, to accommodate all possible samples of $x(n)$. Thus,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \qquad \text{... (2.7.5)}$$

In this result the infinite number of unit sample sequences are added to get $x(n)$. The unit sample sequence $\delta(n-k)$ takes an amplitudes of $x(k)$. The result of equation (2.7.5) is an important one useful for convolution as we will see.

For the sequence $x(n)$ given by equation (2.7.1), the range of 'k' will be, $-2 \le k \le 2$. Hence equation (2.7.5) above can be written as,

$$x(n) = \sum_{k=-2}^{2} x(k)\,\delta(n-k)$$

Expanding above summation, we get,

$$\begin{aligned} x(n) &= x(-2)\ \delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) \\ &\quad + x(1)\ \delta(n-1) + x(2)\,\delta(n-2) \end{aligned}$$

Putting the values of $x(n)$ from equation (2.7.1), above equation becomes,

$$x(n) = 2\ \delta(n+2) + \delta(n+1) + 3\ \delta(n) - 2\,\delta(n-1) + \delta(n-2) \qquad \text{... (2.7.6)}$$

Thus $x(n)$ is equal to the summation of unit samples. The amplitudes of unit samples are basically sample values of $x(n)$. The unit samples represent the location of particular sample or its delay in the sequence $x(n)$.

2.7.2 Convolution Formula

Linear convolution is a very powerful technique used for the analysis of LTI systems. In the last subsection we have seen that how the sequence $x(n)$ can be expressed as sum of weighted impulses. It is given by equation (2.7.5) as,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \qquad \text{... (2.7.7)}$$

Step 1 : If $x(n)$ is applied as an input to the discrete time system, then response $y(n)$ of the system is given as,

$$y(n) = T\,[x(n)]$$

Putting for $x(n)$ in above equation from equation (2.7.7),

$$y(n) = T\left[\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right]$$

Step 2 : The above equation can be expanded as,

$$\begin{aligned} y(n) = T\,[\,&.....+x(-3)\,\delta(n+3)+x(-2)\,\delta(n+2)+x(-1)\,\delta(n+1)+x(0)\,\delta(n) \\ &+x(1)\,\delta(n-1)+x(2)\,\delta(n-2)+x(3)\,\delta(n-3)+........] \end{aligned} \quad ...\ (2.7.8)$$

Step 3 : Since the system is *linear*, the above equation can be written as,

$$\begin{aligned} y(n) =&......+T\left[x(-3)\ \delta(n+3)\right]+T\left[x(-2)\ \delta(n+2)\right]+T\left[x(-1)\ \delta(n+1)\right]+T\left[x(0)\ \delta(n)\right] \\ &+T\left[x(1)\ \delta(n-1)\right]+T\left[x(2)\ \delta(n-2)\right]+T\left[x(3)\ \delta(n-3)\right]+..... \end{aligned} \quad ...\ (2.7.9)$$

Step 4 : The linearity property states that output due to linear combination of inputs [equation (2.7.8)] is same as the sum of outputs due to individual inputs [equation (2.7.9)]. Thus we have written equation (2.7.9) from (2.7.8) with the help of linearity property. In the above equation [equation (2.7.9)] the sample values $.....x(-3),\ x(-2),\ x(-1),\ x(0),\ x(1), x(2).....$ etc. are constants. Hence with the help of *scaling* property of *linear* systems we can write equation (2.7.9) as,

$$\begin{aligned} y(n) =&......+x(-3)T\left[\ \delta(n+3)\right]+x(-2)T\left[\ \delta(n+2)\right]+x(-1)\ T\left[\ \delta(n+1)\right]+x(0)\ T\left[\ \delta(n)\right] \\ &+x(1)T\left[\ \delta(n-1)\right]+x(2)T\left[\ \delta(n-2)\right]+x(3)T\left[\ \delta(n-3)\right]+..... \end{aligned} \quad ...\ (2.7.10)$$

The above equation we have written on the basis of scaling property. It states that if $y(n) = T\,[a\,x(n)]$, then $y(n) = a\,T\,[x(n)]$ for a = constant. The above equation can be written in compact form with the help of '$\sum$' sign. i.e.,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,T\left[\delta(n-k)\right] \quad ...\ (2.7.11)$$

Step 5 : The response of the system to unit sample sequence $\delta(n)$ is given as,

$$T\left[\delta(n)\right] = h(n) \quad ...\ (2.7.12)$$

Here $h(n)$ is called unit sample response or impulse response of the system. If the discrete time system is shift invariant, then above equation can be written as,

$$T\left[\delta(n-k)\right] = h(n-k) \quad ...\ (2.7.13)$$

Here 'k' is some shift in samples. The above equation indicates that; if the excitation of the shift invariant system is delayed, then its response is also delayed by the same amount. Putting for $T\left[\delta(n-k)\right] = h(n-k)$ in equation (2.7.11) we get,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \text{... (2.7.14)}$$

This equation gives the response of linear shift invariant (LTI) system or LTI system

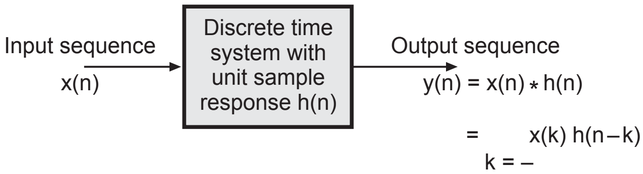


Fig. 2.7.5 Convolution of unit sample response *h*(*n*) and input sequence *x*(*n*) gives output *y*(*n*) in LTI system

to an input $x(n)$. The behaviour of the LTI system is completely characterized by the unit sample response $h(n)$. The above equation is basically linear convolution of $x(n)$ and $h(n)$. This linear convolution gives $y(n)$. Thus,

Convolution sum : $y(n) = x(n) * h(n)$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \text{... (2.7.15)}$$

$y(n)$ is the response, $x(n)$ is the input to the system and $h(n)$ depends upon characteristics of the system. Fig. 2.7.5 illustrates this relationship.

Computation of convolution sum :

Now let us see how to calculate $y(n)$ by using linear convolution of equation (2.7.15). With $n = 0$, equation (2.7.15) becomes,

$$n = 0 \Rightarrow \quad y(0) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k)$$

Here $x(k)$ and $h(-k)$ are multiplied on sample to sample basis and added together to get $y(0)$. Observe that basically $h(-k)$ is the folded sequence of $h(k)$. Similarly with $n = 1$, equation (2.7.15) becomes,

$$n = 1 \Rightarrow \quad y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k)$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h[-(k-1)] \quad \text{By rearranging the equation.}$$

Here again $x(k)$ and $h[-(k-1)]$ are multiplied on sample to sample basis and added together to give $y(1)$. Here $h[-(k-1)]$ indicates shifted (or delayed) version of $h(-k)$ by one sample since $n=1$.

Similarly $y(n)$ can be calculated for other values. Thus the operations in computation of convolution are as follows :

(i) Folding : Sequence $h(k)$ is folded at $k=0$, to get $h(-k)$

(ii) Shifting : $h(-k)$ is shifted depending upon the value of 'n' in $y(n)$.

(iii) Multiplication : $x(k)$ and $h(n-k)$ are then multiplied on sample to sample basis.

(iv) Summation : The product sequence obtained by multiplication of $x(k)$ and $h(n-k)$ is added over all values of 'k' to get value of $y(n)$.

These operations will be more clear through the following example.

Examples for Understanding

Example 2.7.1 *Convolve the following two sequences $x(n)$ and $h(n)$ to get $y(n)$*

$x(n) = \{1, 1, 1, 1\}$

$h(n) = \{2, 2\}$

Solution : Here upward arrow $(\uparrow)$ is not shown in $x(n)$ as well as $h(n)$ means, the first sample in the sequence is 0^{th} sample. Thus the sample values are :

$$x(k=0) = 1$$

$$x(k=1) = 1$$

$$x(k=2) = 1$$

$$x(k=3) = 1$$

and

$$h(k=0) = 2$$

$$h(k=1) = 2$$

The convolution of $x(n)$ and $h(n)$ is given by equation (2.7.15) as,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \text{... (2.7.15 (a))}$$

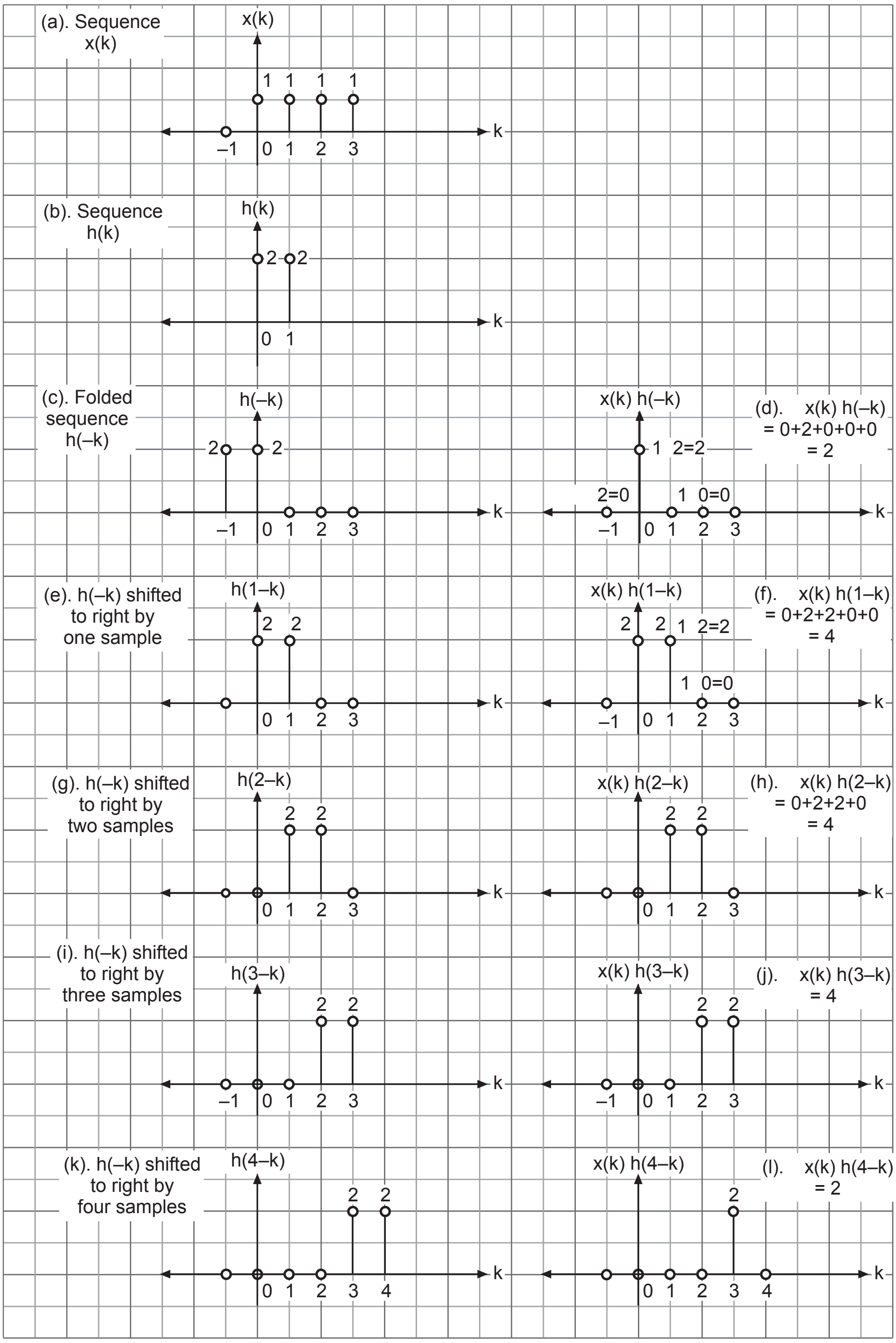


Fig. 2.7.6 Illustration of linear convolution

For $n=0$, the above equation gives 0^{th} sample of $y(n)$ i.e.,

$$n=0 \Rightarrow, \quad y(0) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k) \qquad \text{... (2.7.16)}$$

Here $h(-k)$ is obtained by folding $h(k)$ around $k=0$. Fig. 2.7.6 (a) shows $x(k)$, Fig. 2.7.6 (b) shows $h(k)$ and Fig. 2.7.6 (c) shows $h(-k)$. The product $x(k)\,h(-k)$ is obtained by multiplying sequences of Fig. 2.7.6 (a) and Fig. 2.7.6 (c) sample to sample.

The product of sequence is shown in Fig. 2.7.6 (d). (See Fig. 2.7.6 on next page.)

The summation of samples of this product sequence gives $y(0)$ i.e.,

$$y(0) = \sum_{\substack{\text{over all}\\ \text{samples}}} x(k)\, h(-k) = 0+2+0+0+0 = 2 \qquad \boxed{y(0)=2}$$

Now let $n=1$ in equation (2.7.15) to determine value of $y(1)$ i.e.,

$$n=1 \Rightarrow \quad y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k) \qquad \text{... (2.7.17)}$$

Here $h(1-k)$ can be written as $h\left[-(k-1)\right]$. This is nothing but $h(-k)$ shifted to right by one sample. This shifted sequence $h(1-k)$ is shown in Fig. 2.7.6 (e). The product sequence $x(k)\,h(1-k)$ is obtained by multiplying $x(k)$ of Fig. 2.7.6 (a) and $h(1-k)$ of Fig. 2.7.6 (e) sample to sample. This product sequence is shown in Fig. 2.7.6 (f). The summation of samples of this product sequence gives $y(1)$ i.e.,

$$y(1) = \sum_{\substack{\text{over all}\\ \text{samples}}} x(k)\, h(1-k) = 0+2+2+0+0 = 4 \qquad \boxed{y(1)=4}$$

Now let $n=2$ in equation (2.7.15) to determine value of $y(2)$ i.e.,

$$n=2 \Rightarrow \quad y(2) = \sum_{k=-\infty}^{\infty} x(k)\, h(2-k) \qquad \text{... (2.7.18)}$$

Here $h(2-k)$ can be written as $h\left[-(k-2)\right]$. This is nothing but sequence $h(-k)$ shifted to right by two samples. This delayed sequence is shown in Fig. 2.7.6 (g). The product sequence $x(k)\,h(2-k)$ is obtained by multiplying $x(k)$ of Fig. 2.7.6 (a) and $h(2-k)$ of Fig. 2.7.6 (g) sample to sample.

This product sequence is shown in Fig. 2.7.6 (h). The summation of the samples of this product sequence gives $y(2)$ [see equation (2.7.18)]. i.e.,

$$y(2) = \sum_{\substack{\text{over all}\\ \text{samples}}} x(k)\, h(2-k) = 0+0+2+2+0 = 4 \qquad \boxed{y(2)=4}$$

Similarly for $n = 3$, equation (2.7.15) becomes,

$$n = 3 \Rightarrow \quad y(3) = \sum_{k=-\infty}^{\infty} x(k)\, h(3-k)$$

$h(3-k)$ is obtained by shifting $h(-k)$ to right by 3 samples. i.e. delaying by 3 samples. Fig. 2.7.6 (i) shows $h(3-k)$. The product sequence $h(k)\, h(3-k)$ is shown in Fig. 2.7.6 (j). Hence

$$y(3) = \sum_{\substack{\text{over all}\\ \text{samples}}} x(k)\, h(3-k) = 0+0+0+2+2 = 4 \qquad \boxed{y(3) = 4}$$

This is not the last sample of $y(n)$. With $n = 4$ in equation (2.7.15) we get,

$$n = 4 \Rightarrow \quad y(4) = \sum_{k=-\infty}^{\infty} x(k)\, h(4-k)$$

$h(4-k)$ is obtained by delaying or shifting $h(-k)$ to right by four samples. This is shown in Fig. 2.7.6 (k). The product sequence $x(k)\, h(4-k)$ is shown in Fig. 2.7.6 (l). Hence

$$y(4) = \sum_{\substack{\text{over all}\\ \text{samples}}} x(k)\, h(4-k) = 0+0+0+0+2+0 = 2 \qquad \boxed{y(4) = 2}$$

Now let us see whether we get $y(5)$. That is with $n = 5$ in equation (2.7.15) we get,

$$n = 5 \Rightarrow \quad y(5) = \sum_{k=-\infty}^{\infty} x(k)\, h(5-k)$$

Fig. 2.7.7 (b) shows the delayed sequence $h(5-k)$. Here observe that $x(k) = 0$ for $k > 3$ onwards. And $h(5-k) = 0$ for $k < 4$. Hence the product sequence $x(k)\, h(5-k)$ is zero for all samples as shown in Fig. 2.7.7 (c). Thus (See Fig. 2.7.7 on next page).

$$y(5) = \sum_{\substack{\text{over all}\\ \text{samples}}} x(k)\, h(5-k) = 0 \qquad \boxed{y(n \geq 5) = 0}$$

And all next samples of $y(n)$ for $n \geq 5$ will be zero only.

Now let us see whether we get some samples of $y(n)$ for $n < 0$. Let $n = -1$ in equation (2.7.15) we get,

$$n = -1 \Rightarrow \quad y(-1) = \sum_{k=-\infty}^{\infty} x(k)\, h(-1-k)$$

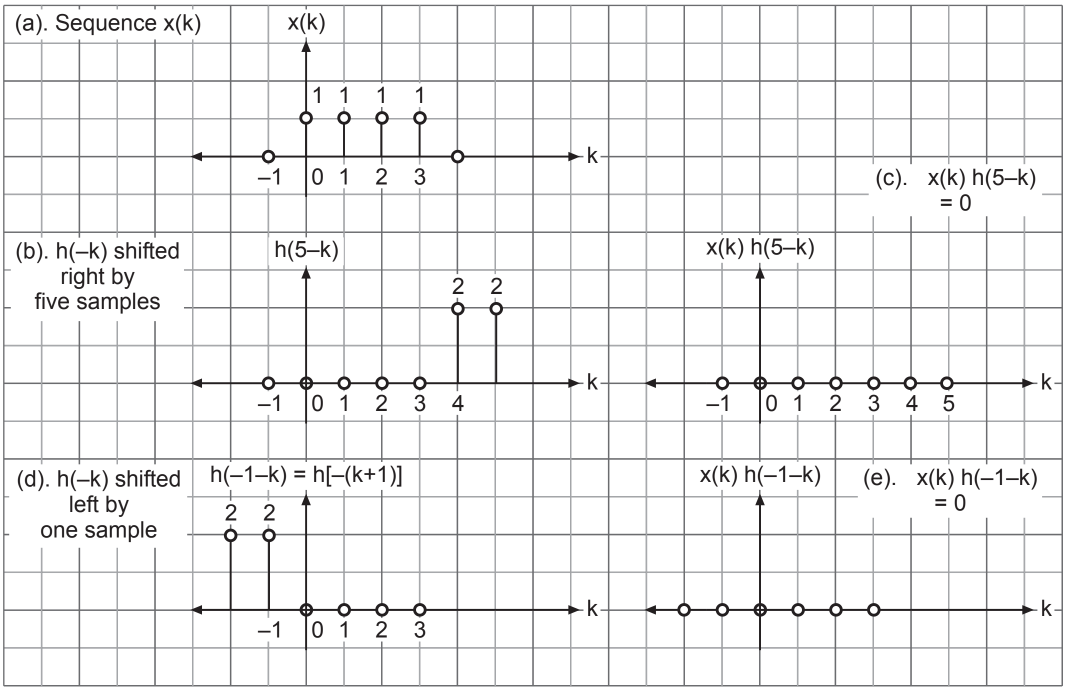


Fig. 2.7.7 Illustration of convolution

Here $h(-1-k)$ can be written as $h\left[-(k+1)\right]$. This is nothing but sequence $h(-k)$ advanced by one sample. Such sequence can be obtained by shifting $h(-k)$ of Fig. 2.7.6(c) left by one sample. This shifted sequence is shown in Fig. 2.7.7 (d). Here observe that $x(k) = 0$ for $k < 0$ and $h(-1-k) = 0$ for $k > -1$. Hence the product sequence $x(k)\,h(-1-k)$ is zero for all samples as shown in Fig. 2.7.7 (e). Thus,

$$y(-1) = \sum_{\substack{\text{over all} \\ \text{samples}}} x(k)\, h(-1-k) = 0 \qquad \boxed{y(n \leq -1) = 0}$$

Hence all next samples of $y(n)$ for $n \leq -1$ will be zero only.

Thus the sequence $y(n)$ becomes as shown below :

$$y(n) = \{\ldots 0,\ 0,\ \underset{\uparrow}{2},\ 4,\ 4,\ 4,\ 2,\ 0,\ 0, \ldots\}$$

or it can also be written as,

$$y(n) = \{2, 4, 4, 4, 2\}$$

Here note that we have used graphical sketches to evaluate convolution.

Comments :

1. Convolution involves folding, shifting, multiplication and summation operations as discussed in this example.

2. If there are 'M' number of samples in $x(n)$ and 'N' number of samples in $h(n)$; then the maximum number of samples in $y(n)$ is equal to $M+N-1$.

In this example $M=4$ and $N=2$. Hence number of samples in $y(n)$ are $4+2-1=5$. Thus there are maximum five samples in $y(n)$.

Example 2.7.2 *Recompute the sequence $y(n)$ in example 2.7.1 with the help of basic convolution equation.*

Solution : The given $x(n)$ and (n) are,

$x(0)=1 \qquad h(0) = 2$

$x(1)=1 \qquad h(1) = 2$

$x(2)=1$

$x(3)=1$

The linear convolution of $x(n)$ and $h(n)$ is given by equation (2.7.15) as,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \text{... (2.7.19)}$$

Here clearly $x(k)=0$ for $k<0$ in the given sequence. Hence lower limit on 'k' will be 0 in above equation.

$n_{xl} = 0$

x(0) = 1
x(1) = 1
x(2) = 1
x(3) = 1

$n_{xh} = 3$

$n_{hl} = 0$

h(0) = 2
h(1) = 2

$n_{hh} = 1$

Similarly $x(k)=0$ for $k>3$ in the given sequence. Hence upper limit on 'k' will be 3 in above equation. Hence equation (2.7.19) can be written as,

$$y(n) = \sum_{k=0}^{3} x(k)\, h(n-k) \qquad \text{... (2.7.20)}$$

To determine range of 'n' in $y(n)$:

Let us represent lowest index of $x(n)$ as n_{xl}, which is zero for given sequence.

Similarly let n_{xh} be the highest index in $x(n)$. For given sequence of $x(n)$, $n_{xh}=3$. Similarly let lowest index in $h(n)$ be n_{hl}; which is zero for given $h(n)$. And let n_{hh} be the highest index in $h(n)$; which is 1 for given sequence of $h(n)$.

Then lowest index of 'n' for $y(n)$ is given as,

$$n_{yl} = n_{xl} + n_{hl} = 0 + 0 = 0 \quad \text{... (2.7.21)}$$

And highest index of 'n' for $y(n)$ is given as,

$$n_{yh} = n_{xh} + n_{hh} = 3 + 1 = 4 \quad \text{... (2.7.22)}$$

Thus the range of n for $y(n)$ will be $0 \le n \le 4$.

$n = 0$ in equation (2.7.20) gives $y(0)$:

$$y(0) = \sum_{k=0}^{3} x(k)\, h(-k)$$

$$= x(0)\, h(0) + x(1)\, h(-1) + x(2)\, h(-2) + x(3)\, h(-3)$$

$$= (1 \times 2) + 0 + 0 + 0 \quad \text{since } h(-1) = h(-2) = h(-3) = 0$$

$$= 2 \qquad \boxed{y(0) = 2}$$

$n = 1$ in equation (2.7.20) gives $y(1)$:

$$y(1) = \sum_{k=0}^{3} x(k)\, h(1-k)$$

$$= x(0)\, h(1) + x(1)\, h(0) + x(2)\, h(-1) + x(3)\, h(-2)$$

$$= (1 \times 2) + (1 \times 2) + 0 + 0 \quad \text{since } h(-1) = h(-2) = 0$$

$$= 2 + 2 = 4 \qquad \boxed{y(1) = 4}$$

$n = 2$ in equation (2.7.20) gives $y(2)$:

$$y(2) = \sum_{k=0}^{3} x(k)\, h(2-k)$$

$$= x(0)\, h(2) + x(1)\, h(1) + x(2)\, h(0) + x(3)\, h(-1)$$

$$= 0 + (1 \times 2) + (1 \times 2) + 0 \quad \text{since } h(2) = h(-1) = 0$$

$$= 4 \qquad \boxed{y(2) = 4}$$

$n = 3$ in equation (2.7.20) gives $y(3)$:

$$y(3) = \sum_{k=0}^{3} x(k)\, h(3-k)$$

$$= x(0)\, h(3) + x(1)\, h(2) + x(2)\, h(1) + x(3)\, h(0)$$

$$= 0 + 0 + (1 \times 2) + (1 \times 2) \quad \text{since } h(3) = h(2) = 0$$

$$= \quad 4 \qquad\qquad \boxed{y(3) = 4}$$

$n = 4$ in equation (2.7.20) gives $y(4)$:

$$y(4) = \sum_{k=0}^{3} x(k)\,h(4-k)$$

$$= x(0)\,h(4) + x(1)\,h(3) + x(2)\,h(2) + x(3)\,h(1)$$

$$= 0 + 0 + 0 + (1 \times 2) \quad \text{since} \quad h(4) = h(3) = h(2) = 0$$

$$= \quad 2 \qquad\qquad \boxed{y(4) = 2}$$

Thus the calculated sequence due to convolution of $x(n)$ and $h(n)$ is as follows :

$$y(n) = \{2, 4, 4, 4, 2\}$$

Observe that this sequence is similar to the one calculated in example 2.7.1.

Comments :

1. The range of 'n' in $y(n)$ is given as,

   Lowest index = sum of lowest indices of sequences to be convolved.

   and Highest index = sum of highest indices of sequences to be convolved.

2. The given equation for $y(n)$ as convolution of $x(n)$ and $h(n)$ is,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad \text{... (2.7.23)}$$

If lowest index for $x(k)$ is n_{xl} and highest index is n_{xh}. Then,

$$x(k) = 0 \quad \text{for} \quad k < n_{xl} \text{ and } k > n_{xh}$$

Hence the product $x(k)\,h(n-k)$ becomes zero for $k < n_{xl}$ and $k > n_{xh}$. Hence limits of 'k' in equation (2.7.23) can be changed as $n_{xl} \le k \le n_{xh}$ i.e.,

$$y(n) = \sum_{k=n_{xl}}^{n_{xh}} x(k)\,h(n-k) \qquad \text{... (2.7.24)}$$

and $\left(n_{xl} + n_{hl}\right) \le n \le \left(n_{xh} + n_{hh}\right)$

This equation is then easily computable.

Example 2.7.3 *The convolution of $x(n)$ and $h(n)$ is given as,*

$y(n) = x(n) * h(n)$

Then show that

$$\sum_{n=-\infty}^{\infty} y(n) = \sum_{n=-\infty}^{\infty} x(n) \cdot \sum_{n=-\infty}^{\infty} h(n) \quad \text{... (2.7.25)}$$

Solution : The linear convolution of $x(n)$ and $h(n)$ is given by equation (2.7.15) as,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Let us take summation of $y(n)$ from $-\infty$ to $+\infty$ i.e.,

$$\sum_{n=-\infty}^{\infty} y(n) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} h(n-k)$$

Let $n-k=m$ in second summation term in above equation. Since limits for 'n' are $(-\infty, \infty)$, the limits for 'm' will also be $(-\infty, \infty)$. Hence above equation becomes,

$$\sum_{n=-\infty}^{\infty} y(n) = \sum_{k=-\infty}^{\infty} x(k) \sum_{m=-\infty}^{\infty} h(m)$$

In the above equation 'k' and 'm' are just indices and any alphabet like 'n' can be used. Thus equation (2.7.25) is proved. The above equation indicates that the sum of all values of $y(n)$ is equal to product of sums of $x(n)$ and $h(n)$. Hence equation (2.7.25) can also be written as follows :

$$\sum y = \sum x \sum h \quad \text{... (2.7.26)}$$

Here observe that indices are avoided.

To verify above equation for the result obtained in example 2.7.2 :

Refer to example 2.7.2. In this example $x(n)$ and $h(n)$ are given as,

$$x(n) = \{1, 1, 1, 1\}$$

$$h(n) = \{2, 2\}$$

And $y(n)$ is calculated in example 2.7.2 as,

$$y(n) = \{2, 4, 4, 4, 2\}$$

$$\sum x(n) = 1+1+1+1=4$$

$$\sum h(n) = 2+2=4$$

and $$\sum y(n) = 2+4+4+4+2=16$$

According to equation (2.7.26),

$$\sum y = \sum x \cdot \sum h = 4 \times 4$$

$$= 16$$

Which is same as calculated above. Thus this equation can be used as a 'check' to verify the results of convolution.

Example 2.7.4 *Easy method to compute convolution*

The sequences $x(n)$ and $h(n)$ are given as follows :

$$x(n) = \{1,\ 1,\ \underset{\uparrow}{0},\ 1,\ 1\}$$

$$h(n) = \{1, -2, -3, \underset{\uparrow}{4}\}$$

Compute the convolution of these two sequences

$y(n) = x(n) * h(n)$

Solution : This method is relatively easy and is based on the technique similar to multiplication. The two sequences are multiplied as we multiply multiple digit numbers. The result of this multiplication is nothing but the convolution of two sequences. The complete procedure is illustrated in Fig. 2.7.8 .

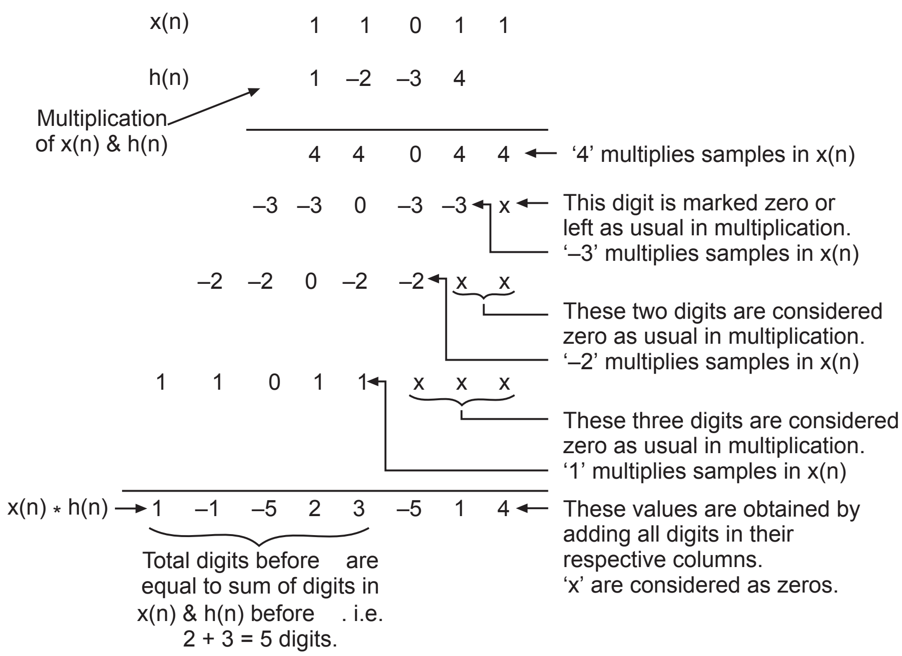


Fig. 2.7.8 Computation of convolution using multiplication

In the sequence $x(n)$ observe that there are '2' digits before the zero mark $\uparrow$ arrow. Similarly there are '3' digits before the zero mark $\uparrow$ arrow in $h(n)$. Hence there will be $2+3=5$ digits before the zero mark $\uparrow$ arrow in $y(n)$. Thus the result of convolution is obtained in Fig. 2.7.8 as,

$$y(n) = \{1, -1, -5, 2, 3, \underset{\uparrow}{-5}, 1, 4\} \qquad \text{... (2.7.27)}$$

Example 2.7.5 *Another easy method to compute convolution.*

This example illustrates another easy method to compute convolution.

$$x(n) = \{1,\ 1,\ \underset{\uparrow}{0},\ 1,\ 1\} \text{ and } h(n) = \{1, -2, -3, \underset{\uparrow}{4}\}$$

These are sequences of previous example. Sometimes this method is also called tabulation method.

Solution : The values of $x(n)$ and $h(n)$ can be written as follows :

$x(-2) = 1 \quad h(-3) = 1$

$x(-1) = 1 \quad h(-2) = -2$

$x(0) = 0 \leftarrow h(-1) = -3$

$x(1) = 1 \quad h(0) = 4 \quad \leftarrow$

$x(2) = 1$

The above values of $x(n)$ and $h(n)$ are tabulated as shown in Fig. 2.7.9.

See Fig. 2.7.9 on next page.

Thus as shown in figure $h(-3), h(-2), h(-1)$ and $h(0)$, form the columns of table. And $x(-2), x(-1), x(0), x(1), x(2)$ form the rows of the table. In the table the multiplications of $x(n)$ and $h(n)$ are written as shown. Then the multiplications are separated diagonally as shown by dotted lines in Fig. 2.7.9.

From given sequences $x(n)$ and $h(n)$ we have,

lowest index of $x(n) \Rightarrow n_{xl} = -2$

lowest index of $h(n) \Rightarrow n_{hl} = -3$

Hence lowest index of $y(n) \Rightarrow n_{yl} = n_{xl} + n_{hl} = -2 - 3$

$$\therefore \quad n_{yl} = -5$$

Thus first element in $y(n)$ will be $y(-5)$. This element is equal to top left diagonal array. It contains only one multiplication. i.e.,

$$y(-5) = x(-2)\, h(-3)$$

The other diagonal arrays are successively $y(-4), y(-3), y(-2)\ldots.$ as shown in Fig. 2.7.9. Finally the last element in the array is $y(2)$ and it is the bottom right element in table. i.e.,

$$y(2) = x(2)\, h(0)$$

Let us put values in table of Fig. 2.7.9. Such table is shown in Fig. 2.7.10.

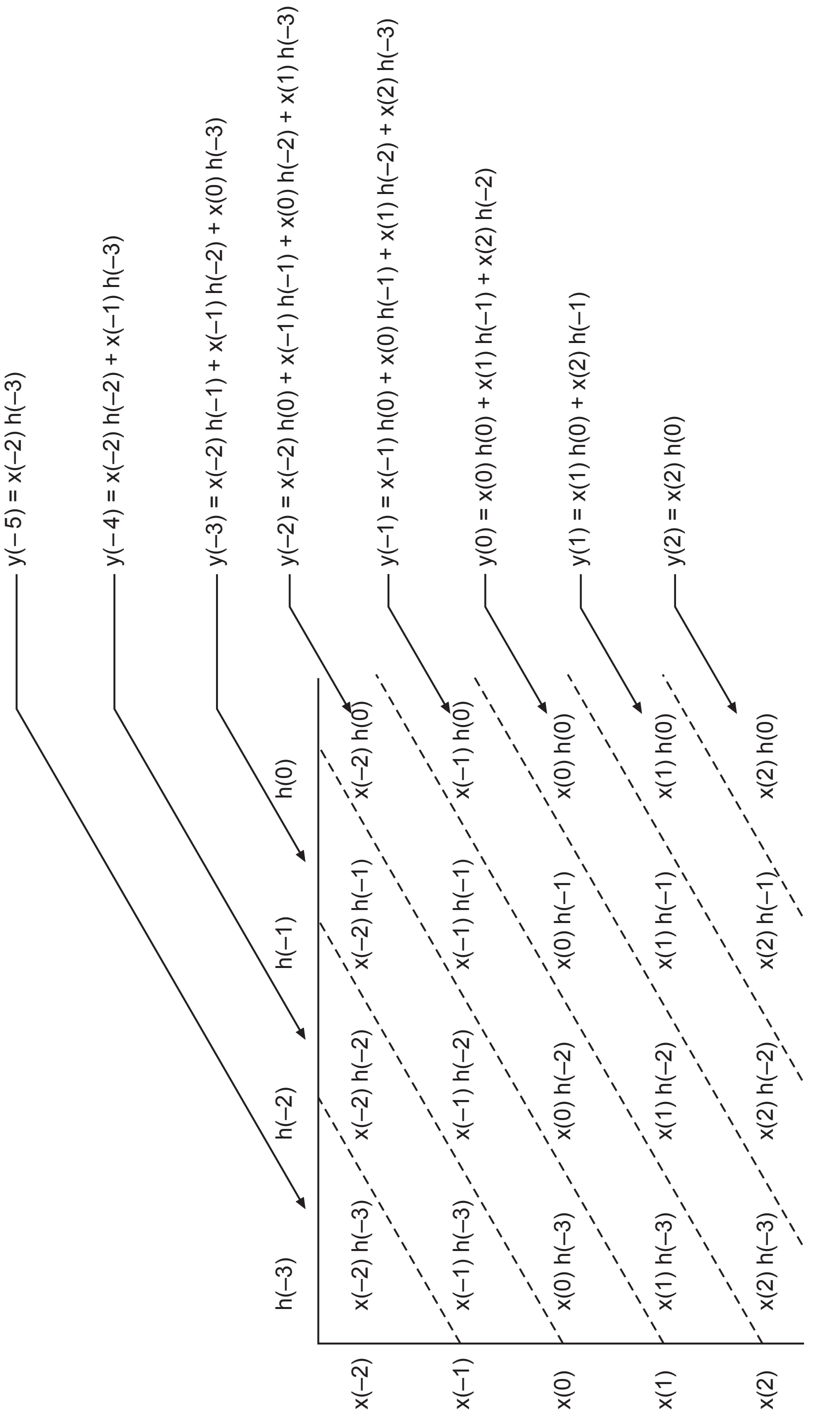


Fig. 2.7.9 Computation of convolution using tabulation

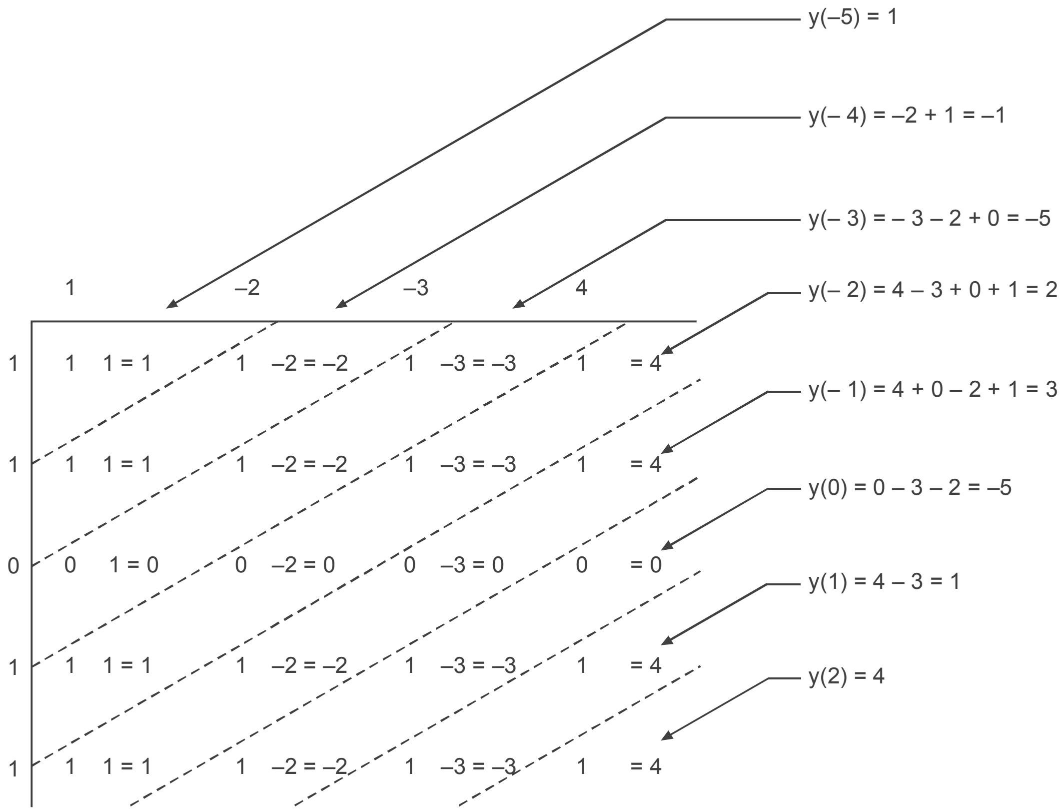


Fig. 2.7.10 Computation of convolution

Thus the computed sequence is,

$$y(n) = \{1, -1, -5, 2, 3, \underset{\uparrow}{-5}, 1, 4\}$$

This sequence is exactly similar to that calculated in last example. Thus tabulation method and multiplication methods are easier methods for computation of convolution.

Comments :

1. Convolution can be computed easily by tabulation and multiplication methods.
2. If $y(n) = x(n) * h(n)$, then result of convolution satisfies $\sum y = \sum x \cdot \sum h$. This equation can be used to check correctness of the result.
3. The multiplications and summations are arranged in specific format in the table. This makes the computations easy.
4. Observe the multiplication method in Fig. 2.7.8 carefully. Here the summations are actually summations in basic convolution computation.

Thus tabulation and multiplication techniques are derived from basic convolution method only. They provide easier and faster computations of convolution.

Examples with Solution

Example 2.7.6 *Prove that the convolution of any sequence with the unit sample sequence results in the same sequence.*

OR

$x(n) * \delta(n) = x(n)$ ... (2.7.28)

OR

$x(n) * h(n) = x(n) \quad if \quad h(n) = \{1\}$ ... (2.7.29)

Solution : We know that convolution of $x(n)$ and $h(n)$ is given as,

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Since convolution is commutative, above equation can be written as,

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \quad \text{... (2.7.30)}$$

$$h(k) = \delta(k) = \begin{cases} 1 & at \quad k=0 \\ 0 & if \quad k \neq 0 \end{cases}$$

Hence the summation of equation (2.7.30) is evaluated only at $k = 0$. i.e.,

$$\begin{aligned} x(n) * h(n) &= h(k)\, x(n-k)\big|_{k=0} \\ &= 1 \cdot x(n-0) \\ &= x(n) \end{aligned}$$

Thus $x(n) * h(n) = x(n) * \delta(n) = x(n)$ if $h(n) = \delta(n)$.

Example 2.7.7 *The impulse response of the relaxed LTI system is given as,*

$h(n) = a^n\, u(n)$ and $|a| < 1$

Determine the response of this system if it is excited by unit step sequence.

Solution :

Here $x(n) = u(n)$

and $h(n) = a^n u(n)$

$$\therefore \qquad y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \text{... By definition}$$

Putting for $x(k)$ and $h(k)$,

$$y(n) = \sum_{k=-\infty}^{\infty} u(k)\, a^{n-k}\, u(n-k) = \sum_{k=-\infty}^{\infty} u(k)\, a^{n} \cdot a^{-k}\, u(n-k)$$

$$= a^n \sum_{k=-\infty}^{\infty} a^{-k} \cdot u(k) \cdot u(n-k)$$

Here $u(k) \cdot u(n-k) = \begin{cases} 1 & for \;\; k \geq 0 \; and \;\; n \geq k \\ & i.e. \;\; 0 \leq k \leq n \\ 0 & otherwise \end{cases}$

$$\therefore \qquad y(n) = a^n \sum_{k=0}^{n} a^{-k}$$

$$= a^n \sum_{k=0}^{n} \left(\frac{1}{a}\right)^k$$

Here let us use the standard relation

$$\sum_{k=0}^{N} a^k = \frac{a^{N+1}-1}{a-1}$$

$$\therefore \qquad y(n) = a^n \cdot \frac{\left(\frac{1}{a}\right)^{n+1} - 1}{\frac{1}{a}-1}$$

$$= a^n \cdot \frac{\frac{1}{a^n} \cdot \frac{1}{a} - 1}{\frac{1}{a}-1} = \frac{\frac{1}{a} - a^n}{\frac{1}{a}-1}$$

$$= \frac{1-a^{n+1}}{1-a} \qquad \text{or} \qquad \frac{a^{n+1}-1}{a-1}$$

Example 2.7.8 *Find the convolution of $x(n) = a^n\, u(n)$, $a < 1$ with $h(n) = 1$ for $0 \le n \le N-1$.*

AU : May-05, Marks 10

Solution :

$$x(n) = a^n u(n)$$

$$h(n) = 1 \text{ for } 0 \le n \le N-1$$

$$\therefore \quad y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$\therefore \quad y(n) = \sum_{k=0}^{N-1} 1\, a^{n-k}\, u(n-k)$$

$$u(n-k) = 1 \text{ for } n \ge k \text{ or } k \le n$$

$$\therefore \quad y(n) = \sum_{k=0}^{\min(n,N-1)} a^{n-k}$$

$$= a^n \sum_{k=0}^{\min(n,N-1)} \left(\frac{1}{a}\right)^k$$

$$= a^n \cdot \frac{\left(\frac{1}{a}\right)^{p+1} - 1}{\frac{1}{a} - 1}, \qquad \text{Here } p = \text{Min}\,(n, N-1)$$

$$= a^n\, \frac{\left(\frac{1}{a}\right)^{p} - a}{1-a}$$

Example 2.7.9 *Find the convolution.* $x(n) = \left\{ \underset{\uparrow}{-1}, 1, 2, -2 \right\}, h(n) = \left\{ 0.5, \underset{\uparrow}{1}, -1, 2, 0.75 \right\}$

AU : Dec.-10, Marks 8

Solution : The convolution is as shown below :

$$\begin{array}{cccccccc}
 & & & & \underset{\uparrow}{-1} & 1 & 2 & -2 \\
 & & & 0.5 & \underset{\uparrow}{1} & -1 & 2 & 0.75 \\
\hline
 & & & & -0.75 & 0.75 & 1.5 & -1.5 \\
 & & & -2 & 2 & 4 & -4 & \times \\
 & & 1 & -1 & -2 & 2 & \times & \times \\
 & -1 & 1 & 2 & -2 & \times & \times & \times \\
-0.5 & 0.5 & 1 & -1 & \times & \times & \times & \times \\
\hline
\end{array}$$

$$x(n) * h(n) = \left\{ \begin{matrix} -0.5 & \underset{\uparrow}{-0.5} & 3 & -2 & -2.75 & 6.75 & -2.5 & -1.5 \end{matrix} \right\}$$

Examples for Practice

Example 2.7.10 *Find the linear convolution of x(n) = {2,4,6,8,10} with h(n) = {1,3,5,7,9}.*

AU : Dec.-12, Marks 6

[Ans. : $x(n) * h(n)$ = {2, 10, 28, 60,110, 148, 160, 142, 90}]

Example 2.7.11 *Obtain the linear convolution of*

$x(n) = \{3,2,1,2\}$ $h(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$ **AU : Dec.-11, Marks 3**

[Ans. : $x(n) * h(n) \Rightarrow \{3, \underset{\uparrow}{6}, 8, 12, 9, 4, 4\}$]

Example 2.7.12 *Determine the linear convolution of the following sequences*

$x_1(n) = \{1, 2, 3, 1\}$ $x_2(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$ **AU : May-11, Marks 6**

[Ans. : $x_1(n) * x_2(n) = \{1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1\}$]

Example 2.7.13 *Determine the convolution of the sequences* $x_1[n] = x_2[n] = \left\{1, \underset{\uparrow}{1}, 1\right\}$.

AU : May-10, Marks 6

[Ans. : $x_1(n) * x_2(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$]

Review Questions

1. *Explain how any discrete time signal can be expressed in terms of periodic impulses ?*
2. *Derive an expression for convolution formula.*

2.8 Properties of Impulse Response Representations for LTI Systems

AU : Dec.-12

2.8.1 Commutative Property

This property states that convolution is commutative operation. Consider the discrete time convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \text{... (2.8.1)}$$

Let us define the new index of summation as,

$m = n - k$ and hence $k = n - m$.

The limits of m will be same as k, i.e. $(-\infty, \infty)$. Hence above equation becomes,

$$y(n) = \sum_{m=-\infty}^{\infty} x(n-m)\, h(m)$$

Here observe that 'm' is the dummy index and it can be replaced by any character, the meaning remains same. Hence replacing 'm' by 'k' in the above equation we get,

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)\, h(k) \qquad \text{... (2.8.2)}$$

This equation is alternate form of convolution. In the above equation $x(k)$ is folded and shifted, whereas impulse response $h(k)$ is unaltered. Equation 2.8.1 and equation 2.8.2 are the two equations for convolution representing the same output. i.e.,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = x(n) * h(n) \qquad \text{... (2.8.3)}$$

and $$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)\, h(k) = h(n) * x(n) \qquad \text{... (2.8.4)}$$

From the above two equations, it is clear that,

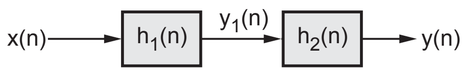


Fig. 2.8.1 Cascade connection of discrete time systems

$$\boxed{x(n) * h(n) = h(n) * x(n) = y(n)} \qquad \text{... (2.8.5)}$$

This equation shows that discrete convolution is also commutative.

2.8.2 Cascade Connection of Systems

Consider the cascade of discrete time systems,

For this system we can write,

$$y(n) = x(n) * h(n) \qquad \text{... (2.8.6)}$$

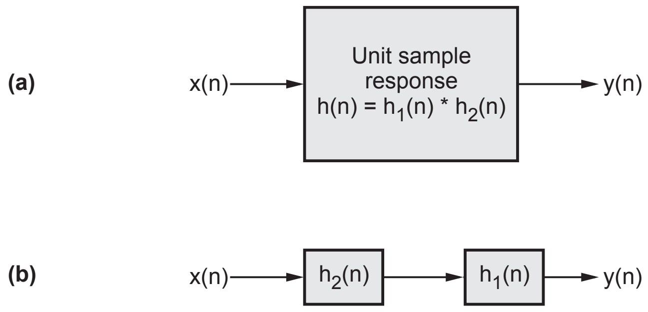


Fig. 2.8.2 (a) Equivalent of cascade connection
(b) Cascade connection is commutative

where $$h(n) = h_1(n) * h_2(n) \qquad \text{... (2.8.7)}$$

Since convolution is commutative,

$$h(n) = h_2(n) * h_1(n) \qquad \text{... (2.8.8)}$$

The equivalent representation of cascade connection of discrete time systems is shown below :

Convolution is associative, hence we can write,

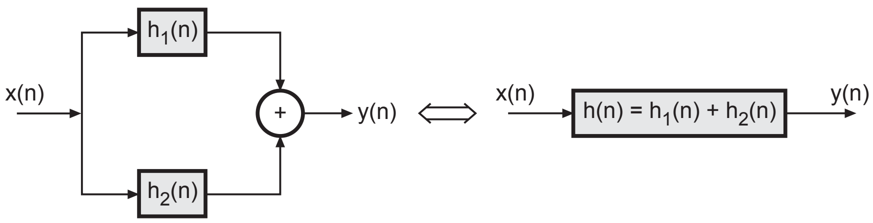


Fig. 2.8.3 Parallel connection and its equivalent for discrete time systems

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)] \qquad \text{... (2.8.9)}$$

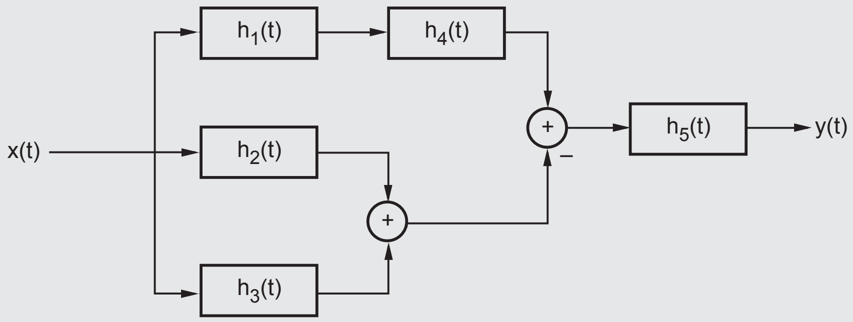


Fig. 2.8.4 Interconnection of the systems

2.8.3 Parallel Connection of Systems

Consider the two systems connected in parallel in Fig. 2.8.3. For these systems we can

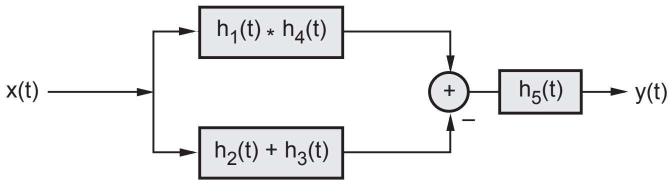


Fig. 2.8.5 Simplified version of Fig. 2.8.4

write,

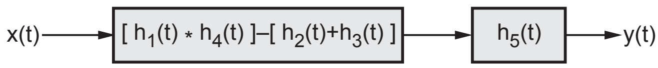


Fig. 2.8.6 Further simplified version of Fig. 2.8.5

$$x(n)*h_1(n)+x(n)*h_2(n) \quad = \quad x(n)*\{h_1(n)+h_2(n)\} \qquad \text{... (2.8.10)}$$

Examples for Understanding

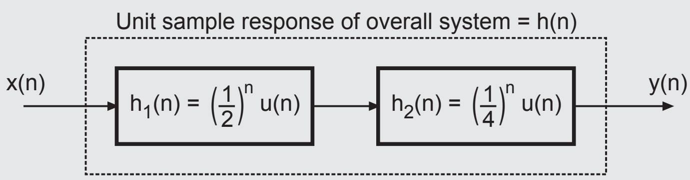


Fig. 2.8.7 Cascade connection of two discrete time systems

Example 2.8.1 *For the interconnection of the systems shown in Fig. 2.8.4, obtain the overall impulse response in terms of impulse responses of individual subsystems.*

Solution : Let us combine cascade connection of $h_1(t)$ and $h_4(t)$. Similarly combine parallel connection of $h_2(t)$ and $h_3(t)$. This is shown below.

Now let us combine parallel combination of the two systems. This is shown below.

From above figure we can write the overall impulse response as,

$$h(t) \quad = \quad \{h_1(t) * h_4(t) - [h_2(t) + h_3(t)]\} * h_5(t)$$

Example 2.8.2 *Two discrete time LTI systems are connected in cascade as shown in Fig. 2.8.7. Determine the unit sample response of this cascade connection.*

Solution :

This is cascade connection. Hence overall impulse response will be convolution of $h_1(n)$ and $h_2(n)$.

Here $h_1(n) = \left(\frac{1}{2}\right)^n u(n)$ and $h_2(n) = \left(\frac{1}{4}\right)^n u(n)$

$$h(n) = h_1(n) * h_2(n)$$

$$= \sum_{k=-\infty}^{\infty} h_1(k) \cdot h_2(n-k) \qquad \text{By definition of convolution}$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k) \cdot \left(\frac{1}{4}\right)^{n-k} u(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k) \cdot \left(\frac{1}{4}\right)^n \cdot \left(\frac{1}{4}\right)^{-k} u(n-k)$$

$$= \left(\frac{1}{4}\right)^n \cdot \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k \cdot \left(\frac{1}{4}\right)^{-k} u(k) \cdot u(n-k)$$

Here $u(k) \cdot u(n-k) = \begin{cases} 1 & \text{for } \quad k \geq 0 \quad \text{and} \quad n \geq k \\ & \quad i.e. \quad 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$

$$\therefore \qquad h(n) = \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \cdot \left(\frac{1}{4}\right)^{-k}$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \cdot (4)^k \qquad \text{since } \left(\frac{1}{4}\right)^{-k} = 4^k$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} \left(\frac{4}{2}\right)^k = \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} 2^k$$

Here let us use,

$$\sum_{k=0}^{N} a^k = \frac{a^{N+1}-1}{a-1} \quad . \quad \text{i.e.,}$$

$$h(n) = \left(\frac{1}{4}\right)^n \cdot \frac{2^{n+1}-1}{2-1}$$

$$= \left(\frac{1}{4}\right)^n \cdot \left[2^{n+1} - 1\right]$$

Example 2.8.3 *Determine the output of the LTI system whose input and unit sample response are given as follows :*

$x(n) = b^n\, u(n)$

and $h(n) = a^n\, u(n)$

Solution : Here both $x(n)$ and $h(n)$ are infinite duration sequences. We have already solved this type of convolution in example 2.8.2. By definition of convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Putting the given sequences in above equation,

$$y(n) = \sum_{k=-\infty}^{\infty} b^k\, u(k) \cdot a^{n-k}\, u(n-k) \qquad \text{... (2.8.11)}$$

Here $u(k) = \begin{cases} 1 & \text{for } k \ge 0 \\ 0 & \text{for } k < 0 \end{cases}$

Hence lower limit of summation in equation 2.8.11 becomes $k = 0$ and $u(k) = 1$. i.e.,

$$y(n) = \sum_{k=0}^{\infty} b^k\ a^{n-k}\ u(n-k) \qquad \text{... (2.8.12)}$$

In above equation $u(n-k) = \begin{cases} 1 & \text{for } n \ge k \\ 0 & \text{for } n < k \text{ or } k > n \end{cases}$

Hence upper limit of summation in equation 2.8.12 becomes 'n' and $u(n-k) = 1$, i.e.,

$$y(n) = \sum_{k=0}^{n} b^k\ a^{n-k} \qquad \text{Here } n \ge k \ge 0$$

$$= \sum_{k=0}^{n} b^k\, a^n \cdot a^{-k}$$

$$= a^n \sum_{k=0}^{n} \left(b\, a^{-1}\right)^k$$

$$y(n) = a^n \cdot \frac{\left(b\,a^{-1}\right)^{n+1} - 1}{\left(b\,a^{-1}\right) - 1}, \quad \text{since} \sum_{k=0}^{N} = \frac{a^{N+1} - 1}{a - 1}$$

$$= a^n \cdot \frac{\left(\frac{b}{a}\right)^{n+1} - 1}{\frac{b}{a} - 1}$$

$$= a^n \cdot a \frac{\left(\frac{b}{a}\right)^{n+1} - 1}{b - a}$$

$$= a^{n+1} \cdot \frac{\frac{b^{n+1}}{a^{n+1}} - 1}{b - a}$$

$$= \frac{b^{n+1} - a^{n+1}}{b - a} \quad \text{for } n \geq 0 \text{ and } a \neq b$$

2.8.4 Causality of LTI Systems

- The output of the causal system depends only upon the present and past inputs. For example, the output of the causal system at $n = n_0$ depends only upon inputs $x(n)$ for $n \leq n_0$.
- This condition for causality can be expressed in terms of unit sample response $h(n)$ for the LTI systems. The output $y(n)$ is given as convolution of $h(n)$ and $x(n)$ for LTI systems.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

At $n = n_0$, the output $y(n_0)$ will be,

$$y(n_0) = \sum_{k=-\infty}^{\infty} h(k)\, x\left(n_0 - k\right) \quad \text{... (2.8.13)}$$

Let us rearrange the $\sum_{k=-\infty}^{\infty}$ in the above equation for the values of $k \geq 0$ and $k < 0$ in two separate terms as follows :

$$y(n_0) = \sum_{k=0}^{\infty} h(k)\, x\left(n_0 - k\right) + \sum_{k=-\infty}^{-1} h(k)\, x\left(n_0 - k\right)$$

Let us expand the terms of summation in the above equation,

$$y(n_0) = \left[h(0)\,x(n_0) + h(1)\,x(n_0-1) + h(2)\,x(n_0-2) + \ldots\right]$$
$$+\left[h(-1)\,x(n_0+1) + h(-2)\,x(n_0+2) + h(-3)\,x(n_0+3) + \ldots\right]$$

Here $x(n_0)$ is present input and $x(n_0-1), x(n_0-2), \ldots$ etc. are past inputs. And $x(n_0+1), x(n_0+2), x(n_0+3), \ldots$ etc. are the future inputs. We know that the output of causal system at $n = n_0$ depends upon the inputs for $n \le n_0$. Hence for causality,

$$h(-1) = h(-2) = h(-3) \ldots = 0$$

This is because $x(n_0+1), x(n_0+2), \ldots$ etc. need not be zero compulsorily, since they are inputs. But $h(n)$ is the unit sample response of the system. In other words, it is the characteristic of the system. Hence for the system to be causal,

$$h(n) = 0 \quad \text{for} \quad n < 0 \qquad \text{for causal system}$$

We know that, $h(n) = f[\,\delta(n)]$

That is, $h(n)$ is the unit sample response of the system for unit sample $\delta(n)$ applied at $n = 0$. Hence for the system to be causal, $h(n) = 0$ for $n < 0$. Thus this condition is two fold for causality. Hence we can state,

A LTI system is causal if and only if

$$h(n) = 0 \quad \text{for} \quad n < 0 \qquad \ldots (2.8.14)$$

This is the necessary and sufficient condition for causality of the system.

Output of causal LTI system

The convolution of $x(n)$ and $h(n)$ is given as,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

For the causal LTI system, $h(n) = 0$ for $n < 0$. Hence $h(k) = 0$ for $k < 0$ in the above equation. This is just the change of index from 'n' to 'k'. Hence the convolution equation for causal LTI system becomes,

$$y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k) \qquad \ldots (2.8.15)$$

Here let us change the index as,

$$m = n-k \quad \text{hence} \quad k = n-m$$

when $\quad k = 0, \quad m = n$

and when $k = \infty, \quad m = n - \infty = -\infty$

Hence equation 2.8.15 becomes,

$$y(n) = \sum_{m=n}^{-\infty} h(n-m)\, x(m)$$

This equation can also be written simply as,

$$y(n) = \sum_{m=-\infty}^{n} x(m)\, h(n-m)$$

In the above equation replacing the index 'm' by 'k' does not change the meaning. Here we are doing it for consistent notations. Hence above equation becomes,

$$y(n) = \sum_{k=-\infty}^{n} x(k)\, h(n-k) \qquad \text{... (2.8.16)}$$

When the sequence $x(n) = 0$. For $n < 0$, then it is called causal sequence. Hence $x(k) = 0$ for $k < 0$. Hence above equation becomes,

$$y(n) = \sum_{k=0}^{n} x(k)\, h(n-k) \qquad \text{... (2.8.17)}$$

This is the equation for linear convolution. It gives output of the causal LTI system for causal input sequence. The causality of LTI system is imposed by unit sample response $h(n)$ and input sequence $x(n)$ is also causal. Hence the limits of summation in $\sum$ are changed from $(-\infty, \infty)$ to $(0, n)$. In the above equation $y(n) = 0$ for $n < 0$, hence the response of causal LTI system is also causal for causal input sequence.

2.8.5 Stability of LTI Systems

- The system is said to be stable if it produces bounded output for every bounded input. In this section we will derive the stability criteria for Linear Time Invariant (LTI) systems in terms of their unit sample response.
- The linear convolution is given as,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

Taking the absolute value of both the sides,

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \right|$$

The absolute value of the total sum is always less than or equal to sum of the absolute values of individual terms. Hence right hand side of the above equation can be written as,

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)|\, |x(n-k)| \qquad \text{... (2.8.18)}$$

If the input sequence $x(n)$ is bounded, then there exists a finite number M_x, such that

$$|x(n)| \leq M_x < \infty \qquad \text{... (2.8.19)}$$

Putting this condition for bounded input in equation 2.8.18 we get,

$$|y(n)| \leq M_x \sum_{k=-\infty}^{\infty} |h(k)|$$

Here M_x is the finite number. Then for the $|y(n)|$ to be finite in the above equation, the condition is,

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

With this condition, the sum of impulse response is finite and hence the output $|y(n)|$ is also finite. Thus bounded input $x(n)$ produces bounded output $y(n)$ in the LTI system only if,

$$\boxed{\sum_{k=-\infty}^{\infty} |h(k)| < \infty} \qquad \text{... (2.8.20)}$$

When this condition is satisfied, the system will be stable. The above condition states that the LTI system is stable if its unit sample response is absolutely summable. This is the necessary and sufficient condition for the stability of LTI system.

2.8.6 Memoryless and with Memory Systems

We know that discrete convolution is given as,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

Let us expand above equation,

$$y(n) = ...+h(-3)\,x(n+3)+h(-2)\,x(n+2)+h(-1)\,x(n+1)$$
$$+h(0)\,x(n)+h(1)x(n-1)+h(2)x(n-2)+...$$

For the memoryless system, output depends only upon present input. Hence all the terms in above equation will be zero, except $h(0)\,x(n)$. $x(n+3)$, $x(n+2)$, $x(n+1)$, $x(n-1), x(n-2)$ ... etc. cannot be necessarily zero since they are inputs. Hence the impulse response values must be zero. i.e.,

$$h(\pm 1) = h(\pm 2) = h(\pm 3) = ... = 0$$

This can also be written as,

$$h(n) = 0 \quad \text{for} \quad n \neq 0 \qquad ... (2.8.21)$$

This is the condition for unit sample response of memoryless or static system. Under the above condition, the unit sample response will be of the form of unit impulse. i.e.,

$$h(n) = c\;\delta(n) \qquad ... (2.8.22)$$

Here 'c' is arbitrary constant.

2.8.7 Step Response

Now let us determine the step response of the LTI system in terms of impulse response. Consider the discrete convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\;x(n-k) \qquad ... (2.8.23)$$

Let $\quad x(n) = u(n) = 1 \quad$ for $\quad n \geq 0$

then $\quad x(n-k) = u(n-k) = 1 \quad$ for $\quad n \geq k$

The above equation can also be written as,

$$x(n-k) = u(n-k) = 1 \quad \text{for} \quad k \leq n$$

Putting for $x(n-k)$ from above equation in equation 2.8.23 and modifying the upper limit of summation,

$$y(n) = \sum_{k=-\infty}^{n} h(k) = 1$$

The upper limit in above equation is 'n' since $u(n-k) = 0$ for $k > n$.

Thus the step response of discrete time system becomes,

$$y(n) = \sum_{k=-\infty}^{n} h(k) \qquad ... (2.8.24)$$

This equation indicates that step response is summation of the unit sample response.

Review Question

1. *Derive the necessary and sufficient condition on the impulse response of the system for causality and stability.* **AU : Dec.-12, Marks 8**

2.9 Transform Analysis of LTI Systems

AU : May-05, 06, 07, 11, 12, 15, 16, Dec.-06, 08, 10, 11, 15, 16

In this section we will use z-transforms for the analysis of LTI systems. Here we will discuss pole zero plots, system function and properties of discrete time systems in z-domain.

2.9.1 Pole-Zero Plots

Let $X(z)$ represents the rational z-transform. The rational z-transform can be expressed as,

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} \qquad \ldots (2.9.1)$$

- The poles are the values of 'z' for which $X(z) = \infty$. And zeros are the values of 'z' for which $X(z) = 0$.
- We know that ROC is the region where $X(z)$ has finite value (i.e. it converges). But poles are the locations where $X(z) = \infty$. Hence poles does not lie in the ROC of $X(z)$.
- Zeros are the locations for which $X(z) = 0$; which is finite. Hence zeros can lie in the ROC of $X(z)$. Above equation can also be written in factored form as,

$$X(z) = \frac{(z - z_1)(z - z_2) \ldots (z - z_M)}{(z - p_1)(z - p_2) \ldots (z - p_N)} \qquad \ldots (2.9.2)$$

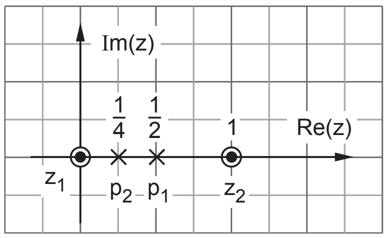


Fig. 2.9.1 Pole zero plot

It is clear from above equation that,

$$\left.\begin{aligned} X(z) &= 0 \text{ for } z = z_1, z_2, \ldots z_M \text{ i.e. zeros} \\ \text{and} \quad X(z) &= \infty \text{ for } z = p_1, p_2, \ldots p_N \text{ i.e. poles} \end{aligned}\right\} \qquad \ldots (2.9.3)$$

Example 2.9.1 *Determine the pole zero plot for the system described by difference equation.*

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) - x(n-1)$$

AU : May-07, Marks 8

Solution : $y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} Y(n-2) = x(n) - x(n-1)$

Taking z-transform,

$$Y(z)-\frac{3}{4}z^{-1}Y(z)+\frac{1}{8}z^{-2}Y(z)=X(z)-z^{-1}X(z)$$

$$\therefore \qquad H(z) = \frac{Y(z)}{X(z)}=\frac{1-z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

$$= \frac{z(z-1)}{z^2-\frac{3}{4}z+\frac{1}{8}}$$

$$= \frac{z(z-1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} \qquad \ldots (2.9.4)$$

Here zeros are, $z_1 = 0$, $z_2 = 1$

and poles are, $p_1 = \frac{1}{2}, p_2 = \frac{1}{4}$

Fig. 2.9.1 shows the pole zero plot.

2.9.2 Transfer Function of the LTI System

Consider an output of the relaxed LTI system. We have seen earlier that the output of such system is given as,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)=x(n) * h(n) \qquad \ldots (2.9.5)$$

Here $y(n)$ is the output of LTI system

$x(n)$ is the input to the system

and $h(n)$ is the impulse response of the system.

Thus output $y(n)$ is given as convolution of input $x(n)$ and impulse response $h(n)$.

Let $x(n) \overset{z}{\longleftrightarrow} X(z)$

$y(n) \overset{z}{\longleftrightarrow} Y(z)$

and $h(n) \overset{z}{\longleftrightarrow} H(z)$ be the z-transform pairs.

Taking z-transform of equation 2.9.5,

$$Y(z) = Z\left\{x(n) * h(n)\right\}$$

By convolution property of z-transform, above equation becomes,

$$Y(z) = X(z) \cdot H(z) \quad \text{... (2.9.6)}$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} \quad \text{... (2.9.7)}$$

Here $H(z)$ is called the transfer function or system function. It is the z-transform of the unit sample response $h(n)$ of the system. Thus the system is characterized by $H(z)$ in z-domain. We know that the system is described by following linear constant coefficient difference equation.

$$y(n) = -\sum_{k=1}^{N} a_k \, y(n-k) + \sum_{k=0}^{M} b_k \, x(n-k) \quad \text{... (2.9.8)}$$

Taking z-transform of this equation,

$$Y(z) = Z\left\{-\sum_{k=1}^{N} a_k \, y(n-k) + \sum_{k=0}^{M} b_k \, x(n-k)\right\}$$

By applying linearity property of z-transform,

$$Y(z) = -\sum_{k=1}^{N} a_k \, Z\{y(n-k)\} + \sum_{k=0}^{M} b_k \, Z\{x(n-k)\}$$

By applying time shifting property of z-transform, we get

$$Y(z) = -\sum_{k=1}^{N} a_k \, z^{-k} \, Y(z) + \sum_{k=0}^{M} b_k \, z^{-k} \, X(z)$$

$$\therefore \quad \left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] Y(z) = \sum_{k=0}^{M} b_k \, z^{-k} \, X(z)$$

$$\therefore \quad \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k \, z^{-k}}{1 + \sum_{k=1}^{N} a_k \, z^{-k}}$$

Since $H(z) = \frac{Y(z)}{X(z)}$ we can write above equation as,

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} \quad \text{... (2.9.9)}$$

Thus the system function of LTI system can be obtained from its difference equation. Also observe that the LTI system has rational system function.

Example 2.9.2 *A difference equation of the system is given below :*

$y(n) = 0.5\, y(n-1) + x(n)$

Determine

i) System function

ii) Pole zero plot of the system function

iii) Unit sample response of **AU : May-12**

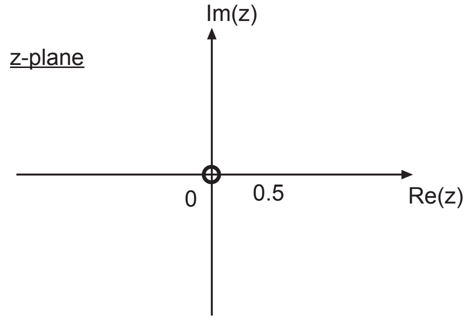


Fig. 2.9.2 Pole zero plot of the system given in example 2.9.2

Solution : (i) To obtain system function

Taking z-transform of the given difference equation,

$$Y(z) = Z\{0.5\, y(n-1) + x(n)\}$$

Applying the linearity property of z-transform,

$$Y(z) = 0.5\, Z\{y(n-1)\} + Z\{x(n)\}$$

Applying the time shift property of z-transform,

$$Y(z) = 0.5\, z^{-1} Y(z) + X(z)$$

$$\therefore \quad \left(1-0.5\, z^{-1}\right) Y(z) = X(z)$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.5\, z^{-1}} \quad \text{... (2.9.10)}$$

This is the required system function.

(ii) Pole zero plot

Let us rearrange the system functions to make powers of z positive i.e.,

$$H(z) = \frac{z}{z} \times \frac{1}{1-0.5\, z^{-1}}$$

$$= \frac{z}{z-0.5} = \frac{z-z_1}{z-p_1}$$

Thus $H(z)$ has one zero at $z_1 = 0$ i.e. at origin and $H(z)$ has one pole at $p_1 = 0.5$

Fig. 2.9.2 shows the pole zero plot.

(iii) Unit sample response

The unit sample response $h(n)$ is obtained by taking inverse z-transform of system function $H(z)$. i.e.,

$$h(n) = IZT\{H(z)\} = IZT\left\{\frac{1}{1-0.5z^{-1}}\right\} \quad \text{from equation (2.9.10)}$$

From Table 2.4.2 we can easily obtain the inverse z-transform of above equation as,

$$h(n) = (0.5)^n u(n)$$

Example 2.9.3 *Obtain the system function and impulse response of the following system* $y(n) - 5y(n-1) = x(n) + x(n-1)$ **AU : May-11, Marks 10**

Solution : Taking z-transform of given equation,

$$Y(z) - 5z^{-1}Y(z) = X(z) + z^{-1}X(z)$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-5z^{-1}} = \frac{z+1}{z-5}$$

$$\therefore \quad \frac{H(z)}{z} = \frac{z+1}{z(z-5)} = \frac{-1/5}{z} + \frac{6/5}{z-5}$$

$$\therefore \quad H(z) = -\frac{1}{5} + \frac{6/5}{1-5z^{-1}}$$

Taking inverse z-transform,

$$h(n) = -\frac{1}{5}\delta(n) + \frac{6}{5}\cdot(5)^n u(n)$$

2.9.3 Causality and Stability interms of z-Transform

Causality

The condition for LTI system to be causal is given as,

$$h(n) = 0 \qquad n < 0$$

Here $h(n)$ is the unit sample response of the LTI system. When the sequence is causal, its ROC is the exterior of the circle. Hence, *LTI system is causal if and only if the ROC of the system function is exterior of a circle of radius $r < \infty$.*

This is the condition for causality of the LTI system in terms of z-transform.

Stability

Now consider the condition for stability. A necessary and sufficient condition for the system to be BIBO stable is given as,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \quad \text{... (2.9.11)}$$

We know that the system function is given as,

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

Taking magnitudes of both the sides

$$|H(z)| = \left| \sum_{n=-\infty}^{\infty} h(n) z^{-n} \right|$$

Magnitude of overall sum is less than the sum of magnitudes of individual terms. i.e.,

$$|H(z)| < \sum_{n=-\infty}^{\infty} \left| h(n) z^{-n} \right|$$

$$\therefore \qquad |H(z)| < \sum_{n=-\infty}^{\infty} |h(n)| \left| z^{-n} \right|$$

If $H(z)$ is evaluated on the unit circle, then $\left| z^{-n} \right| = |z| = 1$. Hence above equation becomes,

$$|H(z)| < \sum_{n=-\infty}^{\infty} |h(n)|$$

If the system is BIBO stable, then $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$. This condition implies that,

$$|H(z)| < \sum_{n=-\infty}^{\infty} |h(n)| < \infty \quad \text{Evaluated on unit circle.} \quad \text{... (2.9.12)}$$

This equation gives the stability condition in terms of z-transform. This condition requires that the unit circle should be present in the ROC of $H(z)$. Thus,

> *LTI system is BIBO stable if and only if the ROC of the system function includes the unit circle.*

Causal and Stable System

For the causal LTI system we know the ROC of system function $H(z)$ is the exterior of some circle of radius 'r'.

i.e. $$|z| > r \qquad \text{... (2.9.13)}$$

We have also concluded that for the stable LTI system the ROC of $H(z)$ must include the unit circle. i.e.,

$$r < 1 \qquad \text{... (2.9.14)}$$

Thus the condition of above equation and equation 2.9.13 can be combined for causal and stable system. That is a causal and stable system must have a system function that converges for

$$|z| > r < 1 \text{ for causal and stable system} \qquad \text{... (2.9.15)}$$

We have seen that ROC of $H(z)$ does not contain any poles. Since ROC of causal and stable LTI system includes unit circle, we can state that all the poles of $H(z)$ of such system are inside the unit circle.

Example 2.9.4 *Determine the impulse response of the system described by the difference equation $y(n) = y(n-1) - \left(\frac{1}{2}\right)y(n-2) + x(n) - \frac{1}{2}x(n-1)$ using z transform and discuss its stability.* **AU : Dec.-12, Marks 10**

Solution : i) To obtain impulse response

Taking z-transform of given difference equation,

$$Y(z) = z^{-1}Y(z) - \frac{1}{2}z^{-2}Y(z) + X(z) - \frac{1}{2}z^{-1}X(z)$$

$$\therefore \quad Y(z)\left[1 - z^{-1} + \frac{1}{2}z^{-2}\right] = X(z)\left[1 - \frac{1}{2}z^{-1}\right]$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{2}z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

Compare denominator of above equation with $1-2\left(\frac{z}{a}\right)^{-1}\cos\Omega_0+\left(\frac{z}{a}\right)^{-2}$, we get,

$$\left(\frac{1}{a}\right)^{-2} = \frac{1}{2} \Rightarrow a^2 = \frac{1}{2}, \text{ hence } a = \frac{1}{\sqrt{2}}$$

$$2\cdot\left(\frac{1}{a}\right)^{-1}\cos\Omega_0 = 1$$

$$\therefore \quad 2\cdot a\cos\Omega_0 = 1$$

$$\therefore \quad 2\times\frac{1}{\sqrt{2}}\cos\Omega_0 = 1 \Rightarrow \cos\Omega_0 = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \Omega_0 = \frac{\pi}{4}$$

and

$$1-\left(\frac{z}{a}\right)^{-1}\cos\Omega_0 = 1-\left(\frac{z}{1/\sqrt{2}}\right)^{-1}\cos\frac{\pi}{4}$$

$$= 1-z^{-1}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}$$

$$= 1-\frac{1}{2}z^{-1}$$

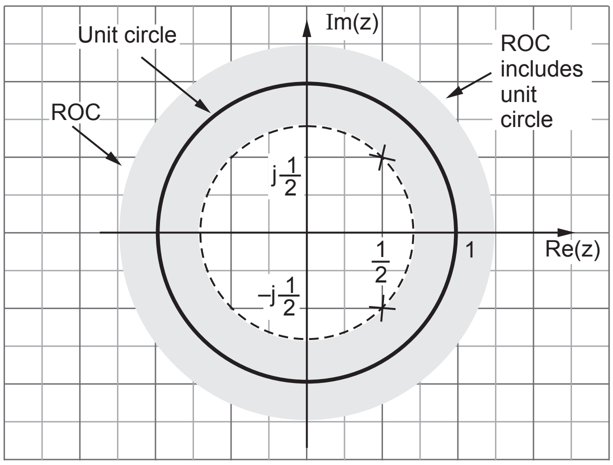


Fig. 2.9.3 Pole-zero plot

Thus, $$\frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-1}} = \frac{1-\left(\frac{z}{a}\right)^{-1}\cos\Omega_0}{1-2\left(\frac{z}{a}\right)^{-1}\cos\Omega_0+\left(\frac{z}{a}\right)^{-2}} \text{ with } a = \frac{1}{\sqrt{2}} \text{ and } \Omega_0 = \frac{\pi}{4}$$

The inverse z-transform of above equation (causing standard relations),

$$h(n) = a^n \cos\Omega_0\, n\, u(n)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{4}n\right)u(n), \qquad \text{with a} = \frac{1}{\sqrt{2}} \text{ and } \Omega_0 = \frac{\pi}{4}$$

ii) Stability of the system

The system function can be written as,

$$\text{H(z)} = \frac{z\left(z-\frac{1}{2}\right)}{z^2-z+\frac{1}{2}} = \frac{z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{2}-j\frac{1}{2}\right)\left(z-\frac{1}{2}+j\frac{1}{2}\right)}$$

The poles of the system are located at $p_1 = \frac{1}{2}+j\frac{1}{2}$ and $p_2 = \frac{1}{2}-j\frac{1}{2}$. As shown in Fig. 2.9.3, the unit circle is located in the ROC of the causal system. Hence this system is **stable**.

Example 2.9.5 *The system function of the LTI system is given as,*

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$$

Specify the ROC of $H(z)$ and determine unit sample response $h(n)$ for following conditions :

i) Stable system ii) Causal system iii) Anticausal system. **AU : Dec.-08, Marks 16**

Solution : First let us convert powers of z in $H(z)$ to positive values. i.e.,

$$H(z) = \frac{z^2}{z^2}\times\frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$$

$$= \frac{z(3z-4)}{z^2-3.5z+1.5}$$

$$\frac{H(z)}{z} = \frac{3z-4}{z^2-3.5z+1.5} \qquad \text{... (2.9.16)}$$

$$= \frac{3z-4}{(z-3)(z-0.5)} = \frac{A_1}{z-3}+\frac{A_2}{z-0.5}$$

A_1 and A_2 can be obtained as follows :

$$A_1 = (z-3)\left.\frac{H(z)}{z}\right|_{z=3} = \left.\frac{3z-4}{z-0.5}\right|_{z=3} = 2$$

and
$$A_2 = (z-0.5)\cdot\left.\frac{H(z)}{z}\right|_{z=0.5} = \left.\frac{3z-4}{z-3}\right|_{z=0.5} = 1$$

Thus the partial fraction expansion is,

$$\frac{H(z)}{z} = \frac{2}{z-3} + \frac{1}{z-0.5}$$

The system has two poles and they are at $p_1 = 3 + j\,0$ and $p_2 = 0.5 + j\,0$. Thus both the poles have real part only. We can further rearrange above equation as,

$$H(z) = \frac{2z}{z-3} + \frac{z}{z-0.5}$$

$$= \frac{2}{1-3z^{-1}} + \frac{1}{1-0.5z^{-1}} \qquad \text{... (2.9.17)}$$

(i) To determine ROC and *h* (*n*) for stable system :

Since the system is stable, the ROC of $H(z)$ must include the unit circle. The system has one pole at $p_1 = 3$ and second pole at $p_2 = 0.5$. Hence the ROC must be

ROC : $0.5 < |z| < 3$ which includes $|z| = 1$ i.e. unit circle.

Since $0.5 < |z|$ or $|z| > 0.5$, the second term of equation (2.9.17) will correspond to causal part of $h(n)$. i.e,

$$IZT\left\{\frac{1}{1-0.5z^{-1}}\right\} = (0.5)^n\, u(n) \qquad \text{for ROC : } |z| > 0.5$$

Similarly since $|z| < 3$, the first term of equation (2.9.17) will correspond to anticausal part of $h(n)$. i.e.,

$$IZT\left\{\frac{1}{1-3z^{-1}}\right\} = -(3)^n\, u(-n-1) \quad \text{for ROC : } |z| < 3$$

Hence the unit sample response can be obtained as inverse z-transform of $H(z)$ of equation (2.9.17) i.e.,

$$h(n) = IZT\{H(z)\}$$

$$= -2(3)^n\, u(-n-1) + (0.5)^n\, u(n)$$

This is required unit sample response for stable system.

(ii) Causal system :

The poles of this system are at $p_1 = 3$ and $p_2 = 0.5$. For the causal system the ROC is the exterior of the circle. Hence ROC will be $|z| > 3$. The unit sample response can be obtained by taking inverse z-transform of equation (2.9.17) i.e.,

$$h(n) = IZT\{H(z)\} = IZT\left\{\frac{2}{1-3z^{-1}}\right\} + IZT\left\{\frac{1}{1-0.5z^{-1}}\right\}$$

$$= 2(3)^n u(n) + (0.5)^n u(n)$$

(iii) Anticausal system :

The poles of the system are at $p_1 = 3$ and $p_2 = 0.5$. We know that ROC is the interior of the circle for anticausal system. Hence the ROC will be $|z| < 0.5$

$$\therefore \quad h(n) = IZT\{H(z)\} = IZT\left\{\frac{2}{1-3z^{-1}}\right\} + IZT\left\{\frac{1}{1-0.5z^{-1}}\right\}$$

$$= -2(3)^n u(-n-1) - (0.5)^n u(-n-1)$$

Example 2.9.6 *Using z–transform determine the response* $y(n)$ *for* $n \geq 0$ *if* $y(n) = \left(\frac{1}{2}\right) y(n-1) + x(n),\ x(n) = \left(\frac{1}{3}\right)^n u(n)$ *and* $y(-1) = 0$

AU : Dec.-15, Marks 16

Solution : Taking z - transform of given difference equation,

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + X(z)$$

Here $$x(n) = \left(\frac{1}{3}\right)^n u(n).$$

Hence $$X(z) = \frac{1}{1 - \frac{1}{3} z^{-1}}.$$

$$\therefore \quad Y(z) - \frac{1}{2} z^{-1} Y(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$$

$$\therefore \quad Y(z)\left[1 - \frac{1}{2} z^{-1}\right] = \frac{1}{1-\frac{1}{3}z^{-1}}$$

$$\therefore \quad Y(z) = \frac{1}{\left(1 - \frac{1}{3} z^{-1}\right)\left(1 - \frac{1}{2} z^{-1}\right)} = \frac{z^2}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{2}\right)}$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{z}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)} = \frac{-2}{z-\frac{1}{3}} + \frac{3}{z-\frac{1}{2}}$$

$$\therefore \quad Y(z) = -2 \cdot \frac{1}{1-\frac{1}{3}z^{-1}} + 3\,\frac{1}{1-\frac{1}{2}z^{-1}}$$

Taking inverse z - transform,

$$y(n) = 3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)$$

Example 2.9.7 *Compute the response of the system.*
$y(n) = 0.7\, y(n-1) - 0.12\, y(n-2) + x(n-1) + x(n-2)$ *to the input* $x(n) = n\,u(n)$. *Is the system stable ?* **AU : May-08, Marks 8**

Solution :

Step 1 : To obtain system function H(z).

The given difference equation is, (since no initial conditions are given),

$$y(n) = 0.7\, y(n-1) - 0.12\, y(n-2) + x(n-1) + x(n-2)$$

z-transform of this equation becomes,

$$Y(z) = 0.7\, z^{-1}\, Y(z) - 0.12\, z^{-2}\, Y(z) + z^{-1}\, X(z) + z^{-2}\, X(z)$$

$$\left(1 - 0.7\, z^{-1} + 0.12\, z^{-2}\right) Y(z) = \left(z^{-1} + z^{-2}\right) X(z)$$

$$\therefore \quad \frac{Y(z)}{X(z)} = \frac{z^{-1} + z^{-2}}{1 - 0.7\, z^{-1} + 0.12\, z^{-2}}$$

Since $\frac{Y(z)}{X(z)} = H(z)$ i.e. system function, we have,

$$H(z) = \frac{z^{-1} + z^{-2}}{1 - 0.7 z^{-1} + 0.12 z^{-2}}$$

$$= \frac{z+1}{z^2 - 0.7 z + 0.12}$$

Stability of the system from $H(z)$

$H(z)$ can be expressed in factored form as follows :

$$H(z) = \frac{z+1}{(z-0.4)(z-0.3)}$$

Poles $\quad p_1 = 0.4$ and $p_2 = 0.3$ and

Zeros : $\quad z = -1$

Thus both the poles p_1 and p_2 of this system are inside the unit circle $|z| = 1$. Hence this system is stable.

Step 2 : To obtain $X(z)$.

The input to the system is,

$$x(n) = n\, u(n)$$

$$\therefore \quad X(z) = \frac{z^{-1}}{(1 - z^{-1})^2} = \frac{z}{(z-1)^2}$$

Setp 3 : To obtain $Y(z)$.

The z-transform of output $Y(z)$ is given as,

$$Y(z) = H(z) \cdot X(z)$$

$$= \frac{z+1}{(z-0.4)(z-0.3)} \cdot \frac{z}{(z-1)^2}$$

Step 4 : To obain $y(n)$ by inverse z-transform.

The above equation can be written as,

$$\frac{Y(z)}{z} = \frac{z+1}{(z-0.4)(z-0.3)(z-1)^2} \quad \text{... (2.9.18)}$$

$$= \frac{A_1}{z-0.4} + \frac{A_2}{z-0.3} + \frac{A_3}{z-1} + \frac{A_4}{(z-1)^2} \quad \text{... (2.9.19)}$$

Here A_1, A_2, A_4 can be obtained

$$A_1 = (z-0.4)\left.\frac{Y(z)}{z}\right|_{z=0.4} = \left.\frac{z+1}{(z-0.3)\,(z-1)^2}\right|_{z=0.4}$$

$$= \frac{0.4+1}{(0.4-0.3)\,(0.4-1)^2} = 38.89$$

$$A_2 = (z-0.3)\left.\frac{Y(z)}{z}\right|_{z=0.3} = \left.\frac{z+1}{(z-0.4)\,(z-1)^2}\right|_{z=0.3}$$

$$= \frac{0.3+1}{(0.3-0.4)\,(0.3-1)^2} = -26.53$$

$$A_4 = (z-1)^2\cdot\left.\frac{Y(z)}{z}\right|_{z=1} = \left.\frac{z+1}{(z-0.4)\,(z-0.3)}\right|_{z=1}$$

$$= \frac{1+1}{(1-0.4)\,(1-0.3)} = 4.76$$

A_3 can be obtained by equation

$$A_3 = \frac{d}{dz}\left[(z-1)^2\cdot\frac{Y(z)}{z}\right]_{z=1} = \frac{d}{dz}\left[\frac{z+1}{(z-0.4)\,(z-0.3)}\right]_{z=1}$$

$$= \left.\frac{(z-0.4)\,(z-0.3)\,1-(z+1)\,(2\,z-0.7)}{\left(z^2-0.7\,z+0.12\right)^2}\right|_{z=1}$$

$$= -12.35$$

Thus equation (2.9.19) becomes,

$$\frac{Y(z)}{z} = \frac{38.89}{z-0.4}-\frac{26.53}{z-0.3}-\frac{12.35}{z-1}+\frac{4.76}{(z-1)^2}$$

$$\therefore \quad Y(z) = \frac{38.89z}{z-0.4}-\frac{26.53z}{z-0.3}-\frac{12.35z}{z-1}+\frac{4.76z}{(z-1)^2}$$

$$\therefore \quad Y(z) = \frac{38.89}{1-0.4z^{-1}}-\frac{26.53}{1-0.3\,z^{-1}}-\frac{12.53}{1-z^{-1}}+\frac{4.76\,z^{-1}}{\left(1-z^{-1}\right)^2}$$

$$\therefore \quad y(n) = 38.89\,(0.4)^n\,u(n)-26.53\,(0.3)^n\,u(n)-12.53\,(1)^n\,u(n)+4.76\cdot n\,(1)^n\,u(n)$$

$$= \left[38.89\,(0.4)^n - 26.53\,(0.3)^n - 12.53 + 4.76\,n\right] u\,(n)$$

Example 2.9.8 *Find the response of the causal system y(n) – y(n – 1) = x(n) + x(n – 1) to the input x(n) = u(n). Test its stability.* **AU : May-16, Marks 10**

Solution : i) Stability

Taking z - transform of given difference equation,

$$Y(z) - z^{-1}Y(z) = X(z) + z^{-1}X(z)$$

$$\therefore \quad Y(z)[1 - z^{-1}] = X(z)[1 + z^{-1}]$$

$$\therefore \quad \mathrm{H(z)} = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-z^{-1}} = \frac{z+1}{z-1}$$

Here the pole is at z = 1. Since the pole is not inside the unit circle, the system is *not stable*

ii) Response to input *x(n) = u(n)*

Since $x(n) = u(n), \qquad X(z) = \dfrac{1}{1-z^{-1}}$

$$\therefore \quad Y(z) = \frac{z+1}{z-1} \times \frac{1}{1-z^{-1}} = \frac{z(z+1)}{(z-1)^2}$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{z+1}{(z-1)^2} = \frac{A_1}{z-1} + \frac{A_2}{(z-1)^2}$$

$$\therefore \quad A_2 = (z-1)^2 \frac{(z+1)}{(z-1)^2}\bigg|_{z=1} = 1 + 1 = 2$$

$$A_1 = \frac{d}{dz}\left[(z-1)^2 \cdot \frac{(z+1)}{(z-1)^2}\right]_{z=1} = \frac{d}{dz}(z+1)\bigg|_{z=1} = 1$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{1}{z-1} + \frac{2}{(z-1)^2}$$

$$\therefore \quad Y(z) = \frac{1}{1-z^{-1}} + \frac{2z^{-1}}{(1-z^{-1})^2}$$

Taking inverse z - transform with standard z - transform pairs,

$$y(n) = u(n) + 2nu(n) = (2n+1)\,u(n)$$

Example 2.9.9 *Find the system function and the impulse response of the system described by the difference equation y(n) = x(n) + 2x (n – 1) – 4x (n – 2) + x(n – 3).*

AU : May-07, Marks 8, Dec.-16, Marks 4

Solution : i) To obtain system function

$$y(n) = x(n)+2x(n-1)-4x(n-2)+x(n-3)$$

Taking z-transform,

$$Y(z) = X(z)+2z^{-1}X(z)-4z^{-2}X(z)+z^{-3}X(z)$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = 1+2z^{-1}-4z^{-2}+z^{-3}$$

This is the system function.

ii) To obtain impulse response

The system function is,

$$H(z) = 1+2z^{-1}-4z^{-2}+z^{-3}$$

Taking inverse z-transform,

$$\therefore \quad h(n) = \delta(n)+2\,\delta(n-1)-4\,\delta(n-2)+\delta(n-3).$$

This is the impulse response. It can also represented as,

$$h(n) = \{1, 2, -4, 1\}$$

Example 2.9.10 *Determine the impulse response h(n) for a system specified by the equation,*

$$y(n)-0.6y(n-1)-0.16y(n-2) = 5x(n)$$

Assume h(0) = 5 and h(1) = 3.

Solution :

Step 1 : To obtain initial conditions

Here initial conditions are not directly given. The difference equation can be written as,

$$y(n) = 5x(n)+0.6\,y(n-1)+0.16\,y(n-2)$$

Let $x(n) = \delta(n)$, *then* $y(n) = h(n)$. Then above equation can be written as,

$$h(n) = 5\,\delta(n)+0.6\,y(n-1)+0.16\,y(n-2) \quad \text{... (2.9.22)}$$

with n = 0 in above equation,

$$h(0) = 5\,\delta(0)+0.6\,y(-1)+0.16\,y(-2)$$

We know that $\delta(0) = 1$ and $h(0) = 5$ given,

$\therefore \quad 5 = 5 + 0.6\, y(-1) + 0.16\, y(-2)$

$\therefore \quad 0.6\, y(-1) + 0.16 y(-2) = 0$ ... (2.9.23)

with n = 1 in equation (2.9.22),

$$h(1) = 5\,\delta(1) + 0.6\, y(0) + 0.16\, y(-1)$$

We know that $\delta(1) = 0$. And h(1) = 3 given, y(0) = h(0) = 5 given. Hence above equation becomes,

$$3 = 5 \times 0 + 0.6 \times 5 + 0.16\, y(-1)$$

$$\therefore \quad 3 = 3 + 0.16\, y(-1)$$

$$\therefore \quad y(-1) = 0$$

Putting this value of $y(-1)$ in equation (2.9.23),

$$0.6 \times 0 + 0.16\, y(-2) = 0$$

$$\therefore \quad y(-2) = 0$$

Thus the initial conditions are,

$$y(-1) = y(-2) = 0$$

Step 2 : To obtain $H(z)$

Taking z-transform of the difference equation,

$$Y(z) - 0.6 z^{-1} Y(z) - 0.16 z^{-2} Y(z) = 5 X(z)$$

$$\therefore \quad Y(z)\,[1 - 0.6\, z^{-1} - 0.16 z^{-2}] = 5 X(z)$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{5}{1 - 0.6 z^{-1} - 0.16 z^{-2}}$$

$$= \frac{5z^2}{z^2 - 0.6z - 0.16} = \frac{5z^2}{(z-0.8)(z+0.2)}$$

Step 3 : To obtain $h(n)$ by inverse z-transform

Here, $$\frac{H(z)}{z} = \frac{5z}{(z-0.8)\,(z+0.2)} = \frac{4}{z-0.8} + \frac{1}{z+0.2}$$

$$\therefore \quad H(z) = \frac{4}{1 - 0.8\, z^{-1}} + \frac{1}{1 + 0.2\, z^{-1}}$$

$$\therefore \quad h(n) = 4(0.8)^n u(n) + (-0.2)^n u(n)$$

$$= [4(0.8)^n + (-0.2)^n]\, u(n)$$

2.9.4

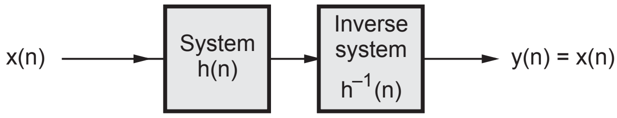


Fig. 2.9.4 System and its inverse system

Inverse Systems

As shown in Fig. 2.9.4, if we cascade the system and its inverse system, then output $y(n) = x(n)$, i.e. input. For such systems we can write,

$$h(n) * h^{-1}(n) = \delta(n)$$

Taking z-transform of the above equation,

$$H(z)\, H^{-1}(z) = 1$$

i.e. $$H^{-1}(z) = \frac{1}{H(z)} \quad \text{... (2.9.24)}$$

- Thus the transfer function of the inverse system is equal to inverse of the transfer function of the desired system.
- In this operation note that poles of $H(z)$ become zeros of $H^{-1}(z)$ and zeros of $H(z)$ become poles of $H^{-1}(z)$.
- For the inverse system to be causal and stable its poles should lie within the unit circle. These poles are zeros of $H(z)$. Hence, in order for the inverse system to be causal and stable, zeros of $H(z)$ should lie in the unit circle.

2.9.5 Deconvolution

Output $y(n)$ and impulse response $h(n)$ is known. Then the process of finding $x(n)$ is known as deconvolution.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h\,(n-k)$$

Let $y(n)$ and $h(n)$ be causal. Then,

$$y(n) = \sum_{k=0}^{n} x(k)\; h\;(n-k)$$

$$= x(0)\; h(n) + x(1)\; h(n-1) + x(2)\; h(n-2) + x(3)\; h(n-3) + \ldots$$

$$\therefore \quad y(0) = x(0)\,h(0) \qquad \dots (2.9.25)$$

$$y(1) = x(0)\,h(1) + x(1)\,h(0) \qquad \dots (2.9.26)$$

$$y(2) = x(0)\,h(2) + x(1)\,h(1) + x(2)\,h(0) \qquad \dots (2.9.27)$$

Above equations can be written in matrix from as,

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} h(0) & .. & .. & . \\ h(1) & h(0) & .. & . \\ h(2) & h(1) & h(0) & .. \\ : & : & : & .. \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \end{bmatrix}$$

or $\quad Y = HX \;$ or $\; X = H^{-1}Y.$

From equation (2.9.25), $\quad x(0) = \dfrac{y(0)}{h(0)}$

From equation (2.9.26), $\quad x(1) = \dfrac{y(1) - x(0)\,h(1)}{h(0)}$

From equation (2.9.27), $\quad x(2) = \dfrac{y(2) - x(0)\,h(2) - x(1)\,h(1)}{h(0)}$

$$= \frac{y(2) - \sum_{k=0}^{1} x(k)\,h(2-k)}{h(0)}$$

Above equation can be generalized as follows :

$$x(n) = \frac{y(n) - \sum_{k=0}^{n-1} x(k)\,h(n-k)}{h(0)} \qquad \dots (2.9.28)$$

Example 2.9.11 *What is the input signal x(n) that will generate the output sequence*

$y(n) = \{1, 5, 10, 11, 8, 4, 1\}$ *for a system with impulse response*
$h(n) = \{1, 2, 1\}$. **AU : May-16, Marks 8**

Solution : Here $x(n)$ has 'N' length, $h(n)$ has 'M' length. Then $y(n)$ has 'N + M – 1' length.

$N + M - 1 = 7$ (since there are '7' samples in $y(n)$)

$\therefore \quad N + M = 8$ and $M = 3$. Hence $N = 5$ samples.

Let us use equation (2.9.28) for n = 0 to 4 (since N = 5).

$$\therefore \qquad x(0) = \frac{y(0)}{h(0)} = \frac{1}{1} = 1$$

$$x(1) = \frac{y(1) - x(0)\,h(1)}{h(0)} = \frac{5 - 1 \times 2}{1} = 3$$

$$x(2) = \frac{y(2) - \sum_{k=0}^{1} x(k)\,h(2-k)}{h(0)} = \frac{y(2) - x(0)\,h(2) - x(1)\,h(1)}{h(0)}$$

$$= \frac{10 - 1 \times 1 - 3 \times 2}{1} = 3$$

$$x(3) = \frac{y(3) - \sum_{k=0}^{2} x(k)\,h(3-k)}{h(0)} = \frac{y(3) - x(0)\,h(3) - x(1)\,h(2) - x(2)\,h(1)}{h(0)}$$

$$= \frac{11 - 1 \times 0 - 3 \times 1 - 3 \times 2}{1} = 2$$

$$x(4) = \frac{y(4) - \sum_{k=0}^{3} x(k)\,h(4-k)}{h(0)} = \frac{y(4) - x(0)\,h(4) - x(1)\,h(3) - x(2)\,h(2) - x(3)\,h(1)}{h(0)}$$

$$= \frac{8 - 1 \times 0 - 3 \times 0 - 3 \times 1 - 2 \times 2}{1} = 1$$

Thus $x(n) = \{1, 3, 3, 2, 1\}$

Example 2.9.12 *A difference equation of the system is given as,*

$$y(n) - y(n-1) + \frac{1}{4}\,y(n-2) = x(n) + \frac{1}{4}\,x(n-1) - \frac{1}{8}\,x(n-2)$$

Determine the transfer function of the inverse system. Check whether the inverse system is causal and stable.

Solution : Taking the z-transform of the given difference equation,

$$Y(z) - z^{-1}\,Y(z) + \frac{1}{4}\,z^{-2}\,Y(z) = X(z) + \frac{1}{4}\,z^{-1}\,X(z) - \frac{1}{8}\,z^{-2}\,X(z)$$

$$\therefore \qquad Y(z)\left[1 - z^{-1} + \frac{1}{4}\,z^{-2}\right] = X(z)\left[1 + \frac{1}{4}\,z^{-1} - \frac{1}{8}\,z^{-2}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{4}\,z^{-1} - \frac{1}{8}\,z^{-2}}{1 - z^{-1} + \frac{1}{4}\,z^{-2}}$$

This is the transfer function of the given system. The transfer function of the inverse system is given by equation (2.9.24) as,

$$H^{-1}(z) = \frac{1}{H(z)}$$

$$= \frac{1}{\left[\dfrac{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}{1-z^{-1}+\frac{1}{4}z^{-2}}\right]} = \frac{1-z^{-1}+\frac{1}{4}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

$$= \frac{z^2-z+\frac{1}{4}}{z^2+\frac{1}{4}z^{-1}-\frac{1}{8}} = \frac{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{2}\right)}$$

Thus poles of inverse system are

$$z_1 = \frac{1}{4} \quad \text{and} \quad z_2 = -\frac{1}{2}$$

These poles are inside the unit circle. Hence inverse system is causal and stable.

Example 2.9.13 *Determine the step response of the system* $y(n)-\alpha\, y(n-1) = x(n),\ -1<\alpha<1$, *when the initial condition is* $y(-1) = 1$. **AU : May-12, Marks 8**

Solution : Take unilateral z-transform of given equation,

$$Y(z)-\alpha[z^{-1}Y(z)+y(-1)] = X(z)$$

$$Y(z)-\alpha[z^{-1}Y(z)+1] = X(z)$$

$$Y(z)-\alpha z^{-1}Y(z)-\alpha = X(z)$$

$$\therefore \quad Y(z)[1-\alpha z^{-1}] = X(z)+\alpha$$

For step response , $x(n) = u(n)$. Hence $X(z) = \frac{1}{1-z^{-1}}$.

$$\therefore Y(z)[1-\alpha z^{-1}] = \frac{1}{1-z^{-1}}+\alpha$$

$$\therefore \quad Y(z) = \frac{1}{(1-z^{-1})(1-\alpha z^{-1})}+\frac{\alpha}{1-\alpha z^{-1}}$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{z}{(z-1)(z-\alpha)}+\frac{\alpha}{z-\alpha} = \frac{\frac{1}{1-\alpha}}{z-1}+\frac{\frac{\alpha}{\alpha-1}}{z-\alpha}+\frac{\alpha}{z-\alpha} = \frac{\frac{1}{1-\alpha}}{z-1}+\frac{\frac{\alpha^2}{\alpha-1}}{z-\alpha}$$

$$\therefore \quad Y(z) = \frac{1}{1-\alpha}\cdot\frac{1}{1-z^{-1}}+\frac{\alpha^2}{\alpha-1}\cdot\frac{1}{1-\alpha z^{-1}},$$

Taking inverse z-transform,

$$y(n) = \frac{1}{1-\alpha}u(n)+\frac{\alpha^2}{\alpha-1}\cdot\alpha^n u(n)$$

2.9.6 Solution of Difference Equations using z-Transform

Earlier we have seen a time domain method to solve difference equation. These equations can be solved easily using z-transform. Following example illustrates the procedure.

Example 2.9.14 *Given that $y(-1) = 5$ and $y(-2) = 0$, solve the difference equation,*
$y(n) - 3\,y(n-1) - 4\,y(n-2) = 0,\ \ n \geq 0$

Solution : Taking unilateral z-transform of the given difference

$$Y(z) - 3\left[z^{-1}\,Y(z) + y(-1)\right] - 4\left[z^{-2}Y(z) + y(-1)\,z^{-1} + y(-2)\right] = 0$$

Putting the initial conditions in above equation,

$$Y(z) - 3\left[z^{-1}\,Y(z) + 5\right] - 4\left[z^{-2}\,Y(z) + 5z^{-1} + 0\right] = 0$$

i.e. $\quad Y(z)\left[1 - 3z^{-1} - 4z^{-2}\right] - 20\,z^{-1} - 15 = 0$

$$Y(z) = \frac{15 + 20\,z^{-1}}{1 - 3\,z^{-1} - 4\,z^{-2}} = \frac{z\,(15\,z + 20)}{z^2 - 3z - 4}$$

i.e. $$\frac{Y(z)}{z} = \frac{15\,z + 20}{z^2 - 3z - 4} = \frac{15z + 20}{(z+1)\,(z-4)} = -\frac{1}{z+1} + \frac{16}{z-4}$$

$\therefore$ $$Y(z) = -\frac{z}{z+1} + \frac{16z}{z-4} = -\frac{1}{1+z^{-1}} + \frac{16}{1-4\,z^{-1}}$$

$\therefore$ $$y(n) = \left[-(-1)^n + 16\,(4)^n\right] u(n)$$

This is the solution of the given difference equation.

Example 2.9.15 *Solve the difference equation using z-transform method.*
$x(n-2) - 9x(n-1) + 18x(n) = 0$. Initial conditions are $x(-1) = 1$, $x(-2) = 9$.

Solution : Consider the difference equation,

$$x(n-2) - 9x(n-1) + 18x(n) = 0$$

Taking unilateral z- transform of above equation,

$$\left[z^{-2}X(z) + x(-1)\,z^{-1} + x(-2)\right] - 9\left[z^{-1}X(z) + x(-1)\right] + 18X(z) = 0$$

$$\left[z^{-2}X(z) + z^{-1} + 9\right] - 9\left[z^{-1}X(z) + 1\right] + 18X(z) = 0$$

$$\therefore \quad X(z) = -\frac{z^{-1}}{z^{-2}-9z^{-1}+18} = -\frac{z}{1-9z+18z^2}$$

$$\therefore \quad \frac{X(z)}{z} = -\frac{1}{18\left(z^2-\frac{1}{2}z+\frac{1}{18}\right)} = -\frac{1}{18\left(z-\frac{1}{3}\right)\left(z-\frac{1}{6}\right)} = \frac{1/3}{z-\frac{1}{6}} - \frac{1/3}{z-\frac{1}{3}}$$

$$\therefore \quad X(z) = \frac{1/3}{1-\frac{1}{6}z^{-1}} - \frac{1/3}{1-\frac{1}{3}z^{-1}}$$

$$\therefore \quad x(n) = \frac{1}{3}\left(\frac{1}{6}\right)^n u(n) - \frac{1}{3}\left(\frac{1}{3}\right)^n u(n) = \left[2\left(\frac{1}{6}\right)^{n+1} - \left(\frac{1}{3}\right)^{n+1}\right]u(n)$$

Example 2.9.16 *A system is described by the difference equation* $y(n) - \left(\frac{1}{2}\right)y(n-1) = 5x(n)$. *Determine the solution, when the input* $x(n) = \left(\frac{1}{5}\right)^n u(n)$ *and the initial condition is given by* $y(-1) = 1$, *using z transform.* **AU : Dec.-12, Marks 10**

Solution : Taking unilateral z-transform of the given equation,

$$Y(z) - \frac{1}{2}[z^{-1}Y(z) + y(-1)] = 5X(z)$$

Putting for $y(-1) = 1$,

$$Y(z) - \frac{1}{2}[z^{-1}Y(z) + 1] = 5X(z)$$

$$\therefore \quad Y(z) - \frac{1}{2}z^{-1}Y(z) - \frac{1}{2} = 5X(z)$$

Here $x(n) = \left(\frac{1}{5}\right)^n u(n)$. Therefore $X(z) = \dfrac{1}{1-\frac{1}{5}z^{-1}}$

$$\therefore \quad Y(z) - \frac{1}{2}z^{-1}Y(z) - \frac{1}{2} = 5 \cdot \frac{1}{1-\frac{1}{5}z^{-1}}$$

$$\therefore \quad Y(z)\left[1-\frac{1}{2}z^{-1}\right] = \frac{5}{1-\frac{1}{5}z^{-1}} + \frac{1}{2}$$

$$\therefore \quad Y(z) = \frac{5}{\left(1-\frac{1}{5}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} + \frac{1/2}{1-\frac{1}{2}z^{-1}}$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{5z}{\left(z-\frac{1}{5}\right)\left(z-\frac{1}{2}\right)} + \frac{1/2}{z-\frac{1}{2}}$$

$$= \frac{-10/3}{z-\frac{1}{5}} + \frac{25/3}{z-\frac{1}{2}} + \frac{1/2}{z-\frac{1}{2}} = \frac{-10/3}{z-\frac{1}{5}} + \frac{53/6}{z-\frac{1}{2}}$$

$$\therefore \quad Y(z) = \frac{-10/3}{1-\frac{1}{5}z^{-1}} + \frac{53/6}{1-\frac{1}{2}z^{-1}}$$

Taking inverse z-transform,

$$y(n) = -\frac{10}{3}\left(\frac{1}{5}\right)^n u(n) + \frac{53}{6}\left(\frac{1}{2}\right)^n u(n)$$

Example 2.9.17 *Find the impulse response of a discrete time invariant system whose difference equation is given by :*
$y(n) = y(n-1) + 0.5\, y(n-2) + x(n) + x(n-1)$. **AU : May-15, Marks 12**

Solution : Taking z-transform of given difference equation,

$$Y(z) = z^{-1} Y(z) + 0.5\, z^{-2}\, Y(z) + X(z) + z^{-1} X(z)$$

$$\therefore \quad Y(z)\left[1 - z^{-1} - 0.5 z^{-2}\right] = [1 + z^{-1}]\, X(z)$$

$$\therefore \quad H(z) = \frac{1 + z^{-1}}{1 - z^{-1} - 0.5 z^{-2}} = \frac{z\,(z+1)}{z^2 - z - 0.5}$$

$$\therefore \quad \frac{H(z)}{z} = \frac{z+1}{z^2 - z - 0.5} = \frac{z+1}{(z-1.366)\ (z+0.366)}$$

$$= \frac{1.366}{z - 1.366} - \frac{0.366}{z + 0.366}$$

$$\therefore \quad H(z) = \frac{1.366}{1 - 1.366 z^{-1}} - \frac{0.366}{1 + 0.366\, z^{-1}}$$

Taking inverse z-transform,

$$h(n) = 1.366\,(1.366)^n\ u(n) - 0.366\ (-0.366)^n\ u(n)$$

Examples for Practice

Example 2.9.18 *The system function of causal LTI system is given as,*

$$H(z) = \frac{1+2\,z^{-1}+z^{-2}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}$$

(i) Determine unit sample response. ii) Determine output if input is $x(n) = e^{jn\pi/2}$

Ans. : (i) $h(n) = -2\delta(n) + \frac{1}{3}\left(-\frac{1}{2}\right)^n u(n) + \frac{8}{3}u(n)$ **(ii)** $y(n) = \frac{4}{3+j}e^{jn\pi/2}$

Example 2.9.19 *Find the impulse response of the system described by the equation.*
$y(n) = 0.7\ y(n-1) - 0.1\ y\ (n-2) + 2\ x(n) - x(n-2)$

AU : May-05, Marks 16

Ans. : $h(n) = -10\ \delta(n) - 3.33\,(0.5)^n u(n) + 15.33\,(0.2)^n u(n)$

Example 2.9.20 *Consider a causal LTI system function.*

$$H\,(z) = \frac{1-a^{-1}\,z^{-1}}{1-a\,z^{-1}} \quad \textit{where 'a' is real.}$$

Determine the value of 'a' for which the system is stable. Show graphically, for $0 < a < 1$, the pole zero plot and ROC. **Ans. :** $|z| > |a| < 1$

Example 2.9.21 *Obtain the response of the system described by the following equation.*
$y(n) = 0.85\ y(n-1) + 0.15\ x(n)$ for the inputs,
i) $x(n) = u(n)$ i.e. step response of the system
ii) $x(n) = (-1)^n u(n)$

Hint and Ans. : i) $Y(z) = \frac{-0.85}{1-0.85\ z^{-1}} + \frac{1}{1-z^{-1}}$, $y(n) = [1 - (0.85)^{n+1}]u(n)$

ii) $Y(z) = \frac{0.0689}{1-0.85\,z^{-1}} + \frac{0.08108}{1+z^{-1}}$, $y(n) = 0.0689(0.85)^n\,u(n) + 0.08108\,(-1)^n\,u(n)\,]$

Example 2.9.22 *Find the output of the system described by the input output relation.*
$y(n) = 5\ y(n-1) - 6y\ (n-2) + x\ (n-1) - x\ (n-2)$ for step input.

AU : Dec.-05, Marks 16

Hint and Ans. : $Y(z) = \frac{z^{-1} - z^{-2}}{(1-5z^{-1}+6z^{-2})\,(1-z^{-1})}$, $Y(z) = \frac{1}{1-3z^{-1}} - \frac{1}{1-2z^{-1}}$,

$y(n) = [3^n - 2^n]\ u(n)$

Example 2.9.23

If odd numbered sequence in x(n) is multiplied by – 1 what is the z-transform of new sequence ? What will happen to the location of poles and zeros ? Prove your statements. **AU : May-06, Marks 6**

Hint and Ans. : $y(n) = (-1)^n x(n),\ (-1)^n x(n) \xleftrightarrow{z} X\left(\frac{z}{-1}\right) = X(-z)$

Signs of poles and zeros will change.

Example 2.9.24 *A discrete time system is described by the following equation :*

$$y(n)+\frac{1}{4}\,y(n-1) = x(n)+\frac{1}{2}\,x(n-1)$$

Determine its impulse response. **AU : Dec.-11, Marks 10**

Ans. : $h(n) = \left(-\frac{1}{4}\right)^n u(n)+\frac{1}{2}\cdot\left(-\frac{1}{4}\right)^{n-1} u(n-1)$

Example 2.9.25 *Find the impulse response given by difference equation.*

$$y(n)-3y(n-1)-4y(n-2) = x(n)+2x(n-1)$$

AU : Dec.-10, Marks 8

Ans. : $h(n) = \frac{6\cdot(4)^n-(-1)^n}{5}\,u(n)$

Example 2.9.26 *Determine the system function and unit sample response of the system described by the difference equation,*

$$Y(n)-\frac{1}{2}\,y(n-1) = 2\,x(n),\ y(-1) = 0$$

AU : May-12, Marks 8

Ans. : $h(n) = 2\left(\frac{1}{2}\right)^n u(n)$

Review Questions

1. *Explain how causality and stability is determined in terms of z-transform?*
2. *How inverse systems are represented with the help of z-transform?*
3. *How do you characterize a discrete time LTI system in z-domain ? Derive the corresponding relation in the z-transform for an LTI system.* **AU : May-06, Marks 6**

2.10 Fourier Transform of Discrete Time Signals

AU : May-04, 06, 10, 14, 15, 16, Dec.-16

Just now we studied that discrete Fourier series is obtained for periodic discrete time signals. Fourier transform is normally obtained for discrete time nonperiodic signals.

2.10.1 Definition

Let us consider the discrete time signal $x(n)$. Its Fourier transform is denoted as $X(\omega)$. It is given as,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \quad \text{... (2.10.1)}$$

- Thus $X(\omega)$ is the Fourier transform of $x(n)$. We know that the frequency range for 'ω' is from $-\pi$ to π or equivalently as 0 to 2π . Hence $X(\omega)$ becomes periodic with period 2π . This can be verified as follows :

$$X\left(\omega + 2\pi\, k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega + 2\pi\, k)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \cdot e^{-j\, 2\pi\, kn} \quad \text{... (2.10.2)}$$

In the above equation $e^{-j2\pi\, kn}$ can be written as,

$$e^{-j2\pi\, kn} = \cos(2\pi kn) - j\sin(2\pi kn)$$

Here k and n both are integers. Hence $cos(2\pi kn) = 1$ always and $sin(2\pi kn) = 0$ always. Hence $e^{-j2\pi kn} = 1$ and equation (2.10.2) becomes,

$$X\left(\omega + 2\pi\, k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= X(\omega) \quad \text{from equation (2.10.1)}$$

This shows that Fourier transform $X(\omega)$ is periodic with period 2π.

- The inverse Fourier transform is given as,

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\; e^{j\omega n}\; d\omega \quad \text{... (2.10.3)}$$

Thus sequence $x(n)$ is obtained from $X(\omega)$ by inverse Fourier transform. The Fourier transform of equation (2.10.1) is infinite summation of terms. Fourier transform is convergent if,

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty \quad \text{... (2.10.4)}$$

This is the sufficient condition for existence of Fourier transform.

2.10.2 Properties of DTFT

Now let us discuss the properties of DTFT. These properties are similar to those of continuous time Fourier transform.

2.10.2.1 Periodicity

$X(\omega)$ is periodic with period 2π . i.e.,

$$\boxed{X(\omega+2\pi k) = X(\omega)} \qquad \text{... (2.10.5)}$$

Proof : By definition of DTFT,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad \text{... (2.10.6)}$$

Let $\omega = \omega + 2\pi k$ in above equation,

$$X(\omega+2\pi k) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega+2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\, e^{-j2\pi kn} \qquad \text{... (2.10.7)}$$

Here $\quad e^{-j2\pi kn} = \cos(2\pi kn) - j\sin(2\pi kn)$

In the above equation k and n both are integers. Hence $\cos(2\pi kn)=1$ always and $\sin(2\pi kn)=0$ always. Hence $e^{-j2\pi kn}=1$ and equation (2.10.7) becomes,

$$X(\omega+2\pi k) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= X(\omega) \qquad \text{... By equation (2.10.6)}$$

2.10.2.2 Linearity

This property states that,

If $\quad x(n) \xleftrightarrow{DTFT} X(\omega)$

and $\quad y(n) \xleftrightarrow{DTFT} Y(\omega)$

then

$$\boxed{z(n) = a\,x(n) + b\,y(n) \xleftrightarrow{DTFT} Z(\omega) = a\,X(\omega) + b\,Y(\omega)} \qquad \text{... (2.10.8)}$$

Proof : By definition of DTFT,

$$Z(\omega) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\omega n}$$

Putting for $z(n) = a\,x(n) + b\,y(n)$, i.e., linear combination of two inputs in above equation,

$$Z(\omega) = \sum_{n=-\infty}^{\infty} [a\,x(n) + b\,y(n)]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a\,x(n)\, e^{-j\omega n} + \sum_{n=-\infty}^{\infty} b\,y(n)\, e^{-j\omega n}$$

$$= a \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} + b \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n}$$

$$= a\,X(\omega) + b\,Y(\omega)$$

Thus the outputs are also linearly related. This is Superposition principle.

2.10.2.3 Time Shift

This property states that,

If $x(n) \xleftrightarrow{DTFT} X(\omega)$

then $$\boxed{y(n) = x(n-n_0) \xleftrightarrow{DTFT} Y(\omega) = e^{-j\omega n_0}\, X(\omega)} \quad \text{... (2.10.9)}$$

Proof : By definition of DTFT,

$$Y(\omega) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(n-n_0)\, e^{-j\omega n}$$

Put $m = n - n_0$. Since n varies from $-\infty$ to ∞, m will also have same range. Then above equation becomes,

$$Y(\omega) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega(m+n_0)} = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega m}\, e^{-j\omega n_0}$$

$$= e^{-j\omega n_0} \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega n} = e^{-j\omega n_0}\, X(\omega)$$

Thus delaying sequence in time domain is equivalent to multiplying its spectrum by $e^{-j\omega n_0}$.

2.10.2.4 Frequency Shift

This property states that,

If $x(n) \xleftrightarrow{DTFT} X(\omega)$

then $$\boxed{y(n) = e^{j\omega_0 n}\, x(n) \xleftrightarrow{DTFT} Y(\omega) = X(\omega - \omega_0)}$$ ... (2.10.10)

Proof : By definition of DTFT,

$$Y(\omega) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n}\, x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega-\omega_0)n} = X(\omega - \omega_0)$$

Thus shifting the frequency by ω_0 is equivalent to multiplying its sequence by $e^{j\omega_0 n}$.

2.10.2.5 Scaling

Let the discrete time sequence be scaled as,

$$y(n) = x(p\,n) \quad \text{for } p\text{-integer.}$$

In this case information in $x(n)$ is discarded. Then scaling property has no meaning with such sequences. Hence this property is applicable only to those sequences for which,

$$x(n) = 0 \quad \text{for } \frac{n}{p} \neq \text{integer.}$$

Then $x(p\,n) \neq 0$ for all n values.

Then data will not be discarded. The discarded data due to scaling will be zeros. The scaling property can be given as,

If $$x(n) \xleftrightarrow{DTFT} X(\omega)$$

$$\boxed{y(n) = x(p\,n) \xleftrightarrow{DTFT} Y(\omega) = X\left(\frac{\omega}{p}\right)}$$ ... (2.10.11)

Proof : By definition of DTFT,

$$Y(\omega) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(p\ n)\, e^{-j\omega n}$$

Here put $p\,n = m$. Since n has the range of $-\infty$ to ∞, m will also have the same range. Then above equation becomes,

$$Y(\omega) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega m/p} = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\left(\frac{\omega}{p}\right)m} = X\left(\frac{\omega}{p}\right)$$

Thus expanding in time domain is equivalent to compressing in frequency domain.

2.10.2.6 Differentiation in Frequency Domain

This property states that,

If $$x(n) \xleftrightarrow{DTFT} X(\omega)$$

then $$\boxed{-j\, n\, x(n) \xleftrightarrow{DTFT} \frac{d}{d\,\omega} X(\omega)} \qquad \text{... (2.10.12)}$$

Proof : By definition of DTFT,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Differentiating both the sides with respect to ω, we get,

$$\frac{d}{d\,\omega} X(\omega) = \frac{d}{d\,\omega} \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Changing the order of summation and differentiation,

$$\frac{d}{d\,\omega} X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \frac{d}{d\,\omega} [e^{-j\omega n}] = \sum_{n=-\infty}^{\infty} x(n)\, (-j\, n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[-j\, n\, x(n)\right] e^{-j\omega n}$$

Comparing above equation with the definition of DTFT, we find that $-j\, n\, x(n)$ has DTFT of $\frac{d}{d\,\omega} X(\omega)$. i.e.,

$$-j\, n\, x(n) \xleftrightarrow{DTFT} \frac{d}{d\,\omega} X(\omega)$$

2.10.2.7 Time Reversal

This property states that,

If $$x(n) \xleftrightarrow{DTFT} X(\omega)$$

Then $$\boxed{y(n) = x(-n) \xleftrightarrow{DTFT} Y(\omega) = X(-\omega)} \qquad \text{... (2.10.13)}$$

Proof : By definition of DTFT,

$$Y(\omega) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\omega n}$$

Put $m = -n$, then above equation becomes,

$$Y(\omega) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega(-m)} = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j(-\omega)m} = X(-\omega)$$

Thus if the sequence is folded in time, then its spectrum is also folded.

2.10.2.8 Convolution

This property states that,

If $\quad x(n) \xleftrightarrow{DTFT} X(\omega)$

and $\quad y(n) \xleftrightarrow{DTFT} Y(\omega)$

then $$\boxed{z(n) = x(n) * y(n) \xleftrightarrow{DTFT} Z(\omega) = X(\omega)\, Y(\omega)} \quad \text{... (2.10.14)}$$

Proof : By definition of DTFT,

$$Z(\omega) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\omega n}$$

Putting for $z(n) = x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)\, y(n-k)$ in the above equation,

$$Z(\omega) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, y(n-k) \right] e^{-j\omega n}$$

Changing the order of summations,

$$Z(\omega) = \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k)\, e^{-j\omega n}$$

Put $n - k = m$, then above equation becomes,

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{m=-\infty}^{\infty} y(m)\, e^{-j\omega(m+k)}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{m=-\infty}^{\infty} y(m)\, e^{-j\omega m}\, e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, e^{-j\omega k} \sum_{m=-\infty}^{\infty} y(m)\, e^{-j\omega m}$$

$$= X(\omega)\, Y(\omega)$$

Thus convolution of the two sequences is equivalent to multiplication of their spectrums.

2.10.2.9 Multiplication in Time Domain (Modulation)

This property states that

If $\quad x(n) \xleftrightarrow{DTFT} X(\omega)$

and $\quad y(n) \xleftrightarrow{DTFT} Y(\omega)$

then
$$\boxed{z(n) = x(n)\, y(n) \xleftrightarrow{DTFT} Z(\omega) = \frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]} \qquad \text{... (2.10.15)}$$

Proof : By definition of DTFT,

$$Z(\omega) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\omega n}$$

Putting for $z(n) = x(n)\, y(n)$ in above equation,

$$Z(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, y(n)\, e^{-j\omega n} \qquad \text{... (2.10.16)}$$

From the inverse DTFT of equation (2.10.3) we know that,

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda)\, e^{j\lambda n}\, d\lambda$$

Here we have used separate frequency variable λ. Putting the above expression of $x(n)$ in equation (2.10.16),

$$Z(\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda)\, e^{j\lambda n}\, d\lambda \cdot y(n)\, e^{-j\omega n}$$

Interchanging the order of summation and integration,

$$Z(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda) \sum_{n=-\infty}^{\infty} y(n)\, e^{j\lambda n} e^{-j\omega n}\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda) \left[\sum_{n=-\infty}^{\infty} y(n)\, e^{-j(\omega-\lambda)n} \right] d\lambda$$

The term in square brackets is $Y(\omega-\lambda)$, hence above equation becomes,

$$Z(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda)\, Y(\omega-\lambda)\, d\lambda$$

Above equation represents convolution of $X(\omega)$ and $Y(\omega)$. i.e.

$$Z(\omega) = \frac{1}{2\pi} \left[X(\omega) * Y(\omega) \right]$$

Thus multiplication of the sequences in time domain is equivalent to convolution of their spectrums.

2.10.2.10 Parseval's Theorem

Parseval's theorem states that,

If $\qquad x(n) \xleftrightarrow{DTFT} X(\omega),$

then energy of the signal is given as,

$$\boxed{E = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega} \qquad \text{... (2.10.17)}$$

Proof : We know that energy of the signal is given as,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$|x(n)|^2$ is also equal to $x(n)\, x^*(n)$. Hence energy becomes,

$$E = \sum_{n=-\infty}^{\infty} x(n)\, x^*(n) \qquad \text{... (2.10.18)}$$

From equation (2.10.3), we can write the inverse DTFT of $x^*(n)$ as,

$$x*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega)\, e^{-j\omega n}\, d\omega$$

Putting above expression for $x^*(n)$ in equation (2.10.18),

$$E = \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega)\, e^{-j\omega n}\, d\omega$$

Changing the order of summation and integration,

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega)\, X(\omega)\, d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$

Thus energy 'E' of the discrete time signal is,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$

The properties of DTFT are combhinely presented in Table 2.10.1.

Name of the property	**Time domain representation**	**Frequency domain representation**
	$x(n)$ $x_1(n)$ $x_2(n)$	$X(\omega)$ $X_1(\omega)$ $X_2(\omega)$
Linearity	$a_1\, x_1(n) + a_2\, x_2(n)$	$a_1\, X_1(\omega) + a_2\, X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\,\Omega k}\, X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega) \cdot X_2(\omega)$
Correlation	$x_1(l) * x_2(-l)$	$X_1(\omega)\, X_2(-\omega)$
Modulation theorem	$x(n) \cos(\omega_0 n)$	$\frac{1}{2} X(\omega + \omega_0) + \frac{1}{2} X(\omega - \omega_0)$
Frequency shifting	$e^{j\,\omega_0\, n}\, x(n)$	$X(\omega - \omega_0)$
Wiener-Khintchine theorem	$x(l)\, x(-l)$	$\|X(\omega)\|^2$

Multiplication	$x_1(n)\, x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)\, X_2(\omega-\lambda)\, d\lambda$
Differentiation in frequency domain	$-jn\, x(n)$	$\frac{d\,X(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)\, X_2^*(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)\cdot X_2^*(\omega)\, d\omega$

Sequence	Fourier transform
$x(n)$	$X(\omega)$
$X^*(n)$	$X^*(-\omega)$
$X^*(-n)$	$X^*(\omega)$
Real $x(n)$	$\begin{cases} X(\omega) = X^*(-\omega) \\ X_R(\omega) = X_R(-\omega) \\ X_I(\omega) = -X_I(-\omega) \\ \lvert X(\omega)\rvert = \lvert X(-\omega)\rvert \\ \angle X(\omega) = -\angle X(-\omega) \end{cases}$

Table 2.10.1 Properties of discrete time Fourier transform

Examples for Understanding

Example 2.10.1 *Determine Fourier transform of*

$$x(n) = a^n\, u(n) \qquad for\ \ -1 < a < 1 \qquad \text{... (2.10.19)}$$

Solution : Let us check whether the Fourier transform is convergent. i.e.,

$$\sum_{n=-\infty}^{\infty} |x(n)| = \sum_{n=0}^{\infty} |a|^n$$

$$= \frac{1}{1-|a|} \quad \text{By geometric series and } |a| < 1$$

Thus $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$. Hence Fourier transform is convergent. By definition of Fourier transform we have,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{\infty} a^n \; e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(a\, e^{-j\omega}\right)^n$$

Here $\left|a\, e^{-j\,\omega}\right| = |a| < 1$, hence we can apply geometric summation formula. i.e.,

$$X(\omega) = \frac{1}{1 - a\, e^{-j\,\omega}} \qquad \text{... (2.10.20)}$$

This is the required Fourier transform.

Example 2.10.2 *Determine the Fourier transform of the discrete time rectangular pulse of amplitude 'A' and length L. i.e.,*

$$x(n) = A \qquad \text{for } 0 \le n \le L-1$$
$$= 0 \qquad \text{otherwise} \qquad \text{... (2.10.21)}$$

Solution : Let us check whether Fourier transform is convergent. i.e.,

$$\sum_{n=-\infty}^{\infty} |x(n)| = \sum_{n=0}^{L-1} |A|$$

$$= |A|\, L < \infty$$

Thus $x(n)$ is absolutely summable and Fourier transform exists. By definition of Fourier transform,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{L-1} A \; e^{-j\omega n}$$

Here let us use the standard relation,

$$\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a} \qquad \text{... (2.10.22)}$$

$$\therefore \qquad X(\omega) = A\, \frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} \qquad \text{... (2.10.23)}$$

The above equation can be further simplified using Euler's identity as,

$$X(\omega) = A\; e^{-j\omega(L-1)/2}\, \frac{\sin(\omega L / 2)}{\sin(\omega / 2)} \qquad \text{... (2.10.24)}$$

Example 2.10.3 *Determine the Fourier transform of unit step sequence,* $x(n) = u(n)$

Solution : The unit step sequence is defined as,

$$x(n) = 1 \quad \text{for} \quad n \geq 0$$
$$= 0 \quad \text{elsewhere}$$

By definition of Fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{\infty} 1 \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(e^{-j\omega}\right)^n$$

Here let us use the relation,

$$\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a} \qquad \text{... (2.10.25)}$$

Hence $X(\omega)$ becomes,

$$X(\omega) = \frac{\left(e^{-j\omega}\right)^0 - \left(e^{-j\omega}\right)^\infty}{1-e^{-j\omega}}$$

$$= \frac{1}{1-e^{-j\omega}} \qquad \text{... (2.10.26)}$$

This relation is not convergent for $\omega = 0$.

This is because $x(n)$ is not absolutely summable sequence. However $X(\omega)$ can be evaluated for other values of ω. Let us rearrange equation (2.10.26) as,

$$X(\omega) = \frac{1}{e^{-j\omega/2} \cdot e^{j\omega/2} - e^{-j\omega/2} \cdot e^{-j\omega/2}}$$

$$= \frac{1}{e^{-j\omega/2}\left[e^{j\omega/2} - e^{-j\omega/2}\right]}$$

By Euler's identity we can write,

$$X(\omega) = \frac{1}{e^{-j\omega/2} \cdot 2j \sin\frac{\omega}{2}}$$

$$= \frac{e^{j\omega/2}}{2j \sin \frac{\omega}{2}}, \ \omega \neq 0 \qquad \text{... (2.10.27)}$$

Example 2.10.4 *Determine Fourier transform of the unit sample* $x(n) = \delta(n)$

Solution : The unit sample is defined as,

$$x(n) = 1 \quad \text{for} \quad n = 0$$

$$= 0 \quad \text{for} \quad n \neq 0$$

By definition of Fourier transform,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\ e^{-j\omega n} = 1 \cdot e^{0}$$

$$= 1 \qquad \text{for all } \omega \qquad \text{... (2.10.28)}$$

Thus Fourier transform has a value of 1 for all values of ω.

Example 2.10.5 *Determine the DTFT of* $x(n) = a^n\ u(-n)$, $|a| > 1$. **AU : May-04, Marks 8**

Solution :

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\ e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{0} a^n\ e^{-j\omega n}$$

Let n = – m

$$X(\omega) = \sum_{m=0}^{\infty} a^{-m}\ e^{j\omega m} = \sum_{m=0}^{\infty} \left(\frac{1}{a} e^{j\omega}\right)^m$$

$$= \frac{1}{1 - \frac{1}{a} e^{j\omega}}$$

2.10.3 Inverse Fourier Transform

The inverse fourier transform is given as,

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega \qquad \text{... (2.10.29)}$$

$X(\omega)$ is the continuous function of 'ω', hence there is integration in above equation. We know that the range of ω is from $-\pi$ to π , hence limits of integrations are from $-\pi$ to π . Consider the following example to illustrate inverse fourier transform.

Example 2.10.6 *Determine the discrete time sequence where fourier transform is given as,*

$$X(\omega) = 1 \quad \text{for} \quad -\omega_c \le \omega \le \omega_c$$
$$= 0 \quad \text{for} \quad \omega_c < |\omega| \le \pi \qquad \text{... (2.10.30)}$$

Solution : The inverse Fourier transform is given as,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1\cdot e^{j\omega n}\, d\omega \qquad \text{... (2.10.31)}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{jn}\right]$$

$$= \frac{1}{n\pi}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right]$$

By Euler's identity we can write above equation as,

$$x(n) = \frac{1}{n\pi}\sin(\omega_c n) \quad \text{for} \quad n \ne 0 \qquad \text{... (2.10.32)}$$

With $n = 0$ in equation (2.10.20) we get,

$$x(0) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^0\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{1}{2\pi}\,|\,\omega\,|_{-\omega_c}^{\omega_c}$$

$$= \frac{\omega_c}{\pi}$$

Thus the sequence $x(n)$ is,

$$\left.\begin{aligned} x(n) &= \frac{\omega_c}{\pi} && \text{for} \quad n = 0 \\ &= \frac{\sin \omega_c n}{n\pi} && \text{for} \quad n \ne 0 \end{aligned}\right\} \qquad \text{... (2.10.33)}$$

2.10.4 Magnitude / Phase Transfer Functions using Fourier Transform

We know that the output of LTI system is given by linear convolution i.e.,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

Let the system be excited by the sinusoid or phasor $e^{j\omega n}$ i.e.,

$$x(n) = e^{j\omega n} \quad \text{for} \quad -\infty < n < \infty$$

Here the sinusoid is complex in nature. It has unit amplitude and frequency is 'ω'. Then output $y(n)$ becomes,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)} = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega n}\, e^{-j\omega k}$$

$$= \left[\sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}\right] e^{j\omega n}$$

$$= H(\omega)\, e^{j\omega n} \qquad \text{... (2.10.34)}$$

Here, $$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k} \qquad \text{... (2.10.35)}$$

Thus $H(\omega)$ is the fourier transform of $h(k)$. And $h(k)$ is the unit sample response. $H(\omega)$ is called transfer function of the system. $H(\omega)$ is complex valued function of ω in the range $-\pi \le \omega \le \pi$. The transfer function $H(\omega)$ can be expressed in polar form as,

$$H(\omega) = |H(\omega)|\; e^{j\angle H(\omega)} \qquad \text{... (2.10.36)}$$

Here $|H(\omega)|$ is magnitude of $H(\omega)$

and $\angle H(\omega)$ is angle of $H(\omega)$.

By Euler's identity we can write $e^{\theta} = \cos\theta + j\sin\theta$. Hence equation 2.10.35 can be written as,

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)\left[\cos(\omega k) - j\sin(\omega k)\right]$$

$$= \sum_{k=-\infty}^{\infty} h(k)\cos(\omega k) - j\sum_{k=-\infty}^{\infty} h(k)\sin(\omega k) \qquad \text{... (2.10.37)}$$

Here $H_R(\omega)$ = real part of $H(\omega) = \sum_{k=-\infty}^{\infty} h(k)\cos(\omega k)$ ... (2.10.38)

and $H_I(\omega)$ = imaginary part of $H(\omega) = -\sum_{k=-\infty}^{\infty} h(k)\sin(\omega k)$ ... (2.10.39) and $$|H(\omega)| = \sqrt{H_R^2(\omega) + H_I^2(\omega)} \quad \text{... (2.10.40)}$$

and $$\angle H(\omega) = \tan^{-1}\frac{H_I(\omega)}{H_R(\omega)} \quad \text{... (2.10.41)}$$

Example 2.10.7 *The difference equation for the low pass filter is given as,*

$$y(n) = \frac{x(n) + x(n-1)}{2}$$

Obtain the magnitude/phase transfer function plots for this filter.

Solution : The given difference equation is,

$$y(n) = \frac{1}{2}x(n) + \frac{1}{2}x(n-1) \quad \text{... (2.10.42)}$$

The linear convolution of unit sample response $h(n)$ and input $x(n)$ gives output $y(n)$. i.e.,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

This equation can be expanded as,

$$y(n) = ... + h(-1)\,x(n+1) + h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + ...$$

On comparing above equation with equation (2.10.42) we find that,

$$h(0) = \frac{1}{2}$$

$$h(1) = \frac{1}{2}$$

And all the other terms are absent, hence they are considered zero. From equation (2.10.24) we know that,

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}$$

This is Fourier transform of unit sample response and it is called transfer function. Putting for $h(0)$ and $h(1)$,

$$H(\omega) = h(0)\, e^{-j\omega 0} + h(1)\, e^{-j\omega 1}$$

$$= \frac{1}{2} + \frac{1}{2}e^{-j\omega} = \frac{1}{2}\left(1 + e^{-j\omega}\right) = \frac{1}{2}\left(1 + \cos\omega - j\sin\omega\right)$$

$$= \frac{1}{2}\left(1 + \cos\omega\right) - j\frac{1}{2}\sin\omega \quad \text{... (2.10.43)}$$

$\therefore$ Real part of $H(\omega) \Rightarrow H_R(\omega) = \frac{1}{2}(1 + cos\ \omega)$

$$= cos^2 \frac{\omega}{2}$$

Imaginary part of $H(\omega) \Rightarrow H_I(\omega) = -\frac{1}{2} sin\ \omega$

$$= -sin \frac{\omega}{2} cos \frac{\omega}{2}$$

$\therefore$ Magnitude of $H(\omega) \Rightarrow |H(\omega)| = \sqrt{H_R^2(\omega) + H_I^2(\omega)}$

$$= \sqrt{\cos^4 \frac{\omega}{2} + \left(-\sin \frac{\omega}{2} \cos \frac{\omega}{2}\right)^2} = \sqrt{\cos^4 \frac{\omega}{2} + \sin^2 \frac{\omega}{2} \cos^2 \frac{\omega}{2}}$$

$$= \sqrt{\cos^2 \frac{\omega}{2}\left(\cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2}\right)} = \sqrt{\cos^2 \frac{\omega}{2}}$$

$$= \cos \frac{\omega}{2} \qquad \text{... (2.10.44)}$$

Phase of $H(\omega)$ $= \angle H(\omega) = tan^{-1} \frac{H_I(\omega)}{H_R(\omega)}$

$$= \tan^{-1}\left[\frac{-\sin \frac{\omega}{2} \cos \frac{\omega}{2}}{\cos^2 \frac{\omega}{2}}\right] = \tan^{-1}\left(-\tan \frac{\omega}{2}\right)$$

$$= -\frac{\omega}{2} \qquad \text{... (2.10.45)}$$

Table 2.10.2 shows the calculations of magnitude and phase of $H(\omega)$ for few values of ω.

ω	**Magnitude** $\|H(\omega)\| = cos \frac{\omega}{2}$	**Phase** $\angle H(\omega) = -\frac{\omega}{2}$
$-\pi$	0	$\frac{\pi}{2}$
$-\frac{2\pi}{3}$	0.5	$\frac{\pi}{3}$
$-\frac{\pi}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$

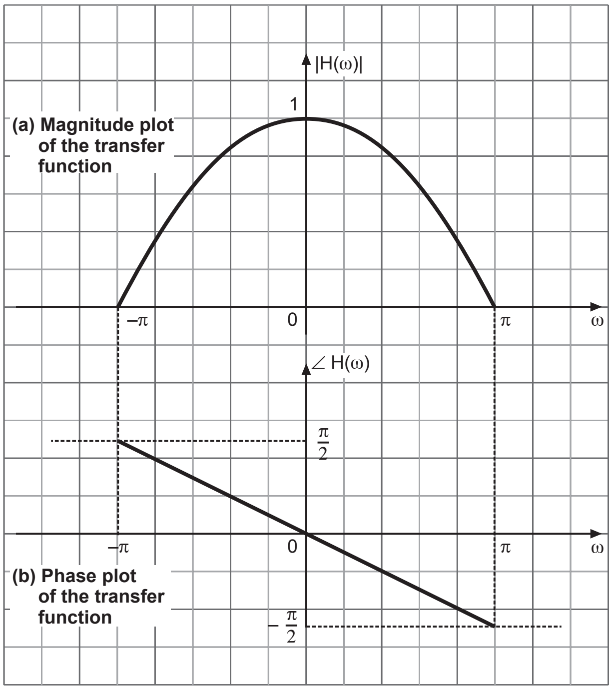


Fig. 2.10.1 Magnitude and phase plots of the transfer function $H(\omega) = \frac{1}{2}\left(1 + e^{-j\omega}\right)$

$-\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\pi}{6}$
0	1	0
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{\pi}{6}$
$\frac{\pi}{2}$	$\frac{1}{\sqrt{2}}$	$-\frac{\pi}{4}$
$\frac{2\pi}{3}$	0.5	$-\frac{\pi}{3}$
π	0	$-\frac{\pi}{2}$

Table 2.10.2 Calculation of $|H(\omega)|$ and $\angle H(\omega)$

Fig. 2.10.1 (a) shows the magnitude and Fig. 2.10.1 (b) shows the phase plot of transfer function based on calculations in above table. (See Fig. 2.10.1 on next page)

Here note that transfer function $H(\omega)$ is the continuous function of 'ω' in the range of $-\pi \le \omega \le \pi$. Hence magnitude and phase plots in above figure are continuous.

Comments on magnitude and phase plots of $H(\omega)$:

1. The magnitude and phase plots are continuous function of 'ω' for nonperiodic sequences.

2. The magnitude plot is even symmetric around $\omega = 0$, where as phase plot has odd

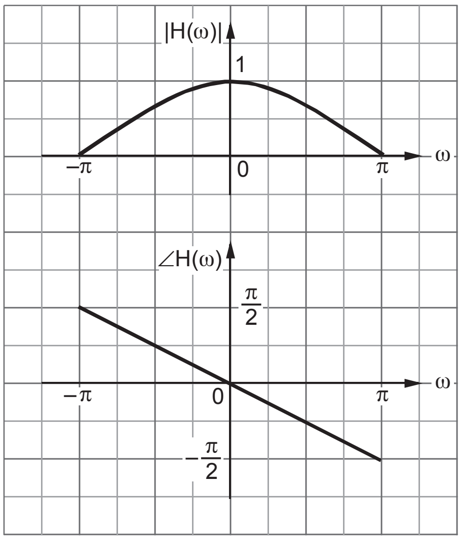


Fig. 2.10.2 Magnitude and phase plot

symmetry. i.e.,

$$\left.\begin{aligned} |H(\omega)| &= |H(-\omega)| \\ \text{and} \quad \angle H(-\omega) &= -\angle H(\omega) \end{aligned}\right\} \quad \text{... (2.10.46)}$$

3. The magnitude and phase plots are periodic with period 2π. Readers can verify this statement by actually calculating magnitude and phase transfer functions of the preceding example.

Example 2.10.8 *Find the Fourier transform of the system described by the equation.*

$$y(n) = \frac{1}{2}[\, x(n) + x(n-1)]$$

Plot the magnitude and phase spectrum. **AU : May-06, Marks 10**

Solution : Taking Fourier transform of both sides,

$$\therefore \quad Y(\omega) = \frac{1}{2}X(\omega) + \frac{1}{2}e^{-j\omega}X(\omega)$$

$$\therefore \quad H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{2} + \frac{1}{2}e^{-j\omega}$$

$$= \frac{1}{2}e^{-j\frac{\omega}{2}}e^{j\frac{\omega}{2}} + \frac{1}{2}e^{-j\frac{\omega}{2}} \cdot e^{-j\frac{\omega}{2}} = e^{-j\frac{\omega}{2}}\ \frac{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}}{2}$$

$$= \cos\frac{\omega}{2} e^{-j\frac{\omega}{2}}$$

Here $$|H(\omega)| = \cos\frac{\omega}{2}$$

and $$\angle H(\omega) = -\frac{\omega}{2}$$

Fig. 2.10.2 shows the magnitude and phase plot of system as per above equation.

2.10.5 Relationship between Fourier Transform and z-Transform

We know that fourier transform of discrete time signal is given as,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\ e^{-j\omega n} \qquad \text{... (2.10.47)}$$

And z-transform of x(n) is given as,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\ z^{-n}, \quad \text{ROC} : r_2 < |z| < r_1 \qquad \text{... (2.10.48)}$$

'z' is the complex variable and it can be expressed as

$$z = r\, e^{j\omega} \qquad \text{... (2.10.49)}$$

Here 'r' is the magnitude of z i.e. r = | z | and 'ω' is the angle of z i.e. ω = ∠z

Putting $z = r\, e^{j\omega}$ in equation (2.10.37) we can write,

$$X(z)\big|_{z=r\,e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\left[r\, e^{j\omega}\right]^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[x(n)\, r^{-n}\right] e^{-j\omega n} \qquad \text{... (2.10.50)}$$

Compare the above equation with Fourier transform of equation (2.10.50) Thus $X(z)$ is the Fourier transform of the sequence $x(n)\, r^{-n}$. Now let us evaluate equation (2.10.50) at $|z| = 1$, i.e. unit circle. Since $r = |z|$, we have $r^{-n} = 1$. Therefore equation (2.10.50) becomes,

$$X(z)\big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$\therefore \quad X(z)\big|_{z=e^{j\omega}} = X(\omega) \ \text{ at } |z| = 1 \text{ i.e. unit circle} \qquad \text{... (2.10.51)}$$

Thus Fourier transform is basically the z-transform of the sequence evaluated on unit circle. This shows that z-transform is the more general transform. And Fourier transform is the special case of z-transform on unit circle.

Example 2.10.9 *The difference equation of the system is given as,*

$$y(n) = \frac{1}{2}x(n) + \frac{1}{2}x(n-1) \quad \text{... (2.10.52)}$$

Find out (i) System function (ii) Unit sample response (iii) Pole zero plot (iv) Transfer function and its magnitude phase plot.

Solution : (i) To find system function :

Taking z-transform of given difference equation,

$$Z\{y(n)\} = Z\left\{\frac{1}{2}x(n) + \frac{1}{2}x(n-1)\right\}$$

Applying the linearity and time shift properties,

$$Y(z) = \frac{1}{2}X(z) + \frac{1}{2}z^{-1}X(z)$$

$$= \left(\frac{1}{2} + \frac{1}{2}z^{-1}\right)X(z)$$

$$\therefore \quad \frac{Y(z)}{X(z)} = \frac{1}{2} + \frac{1}{2}z^{-1}$$

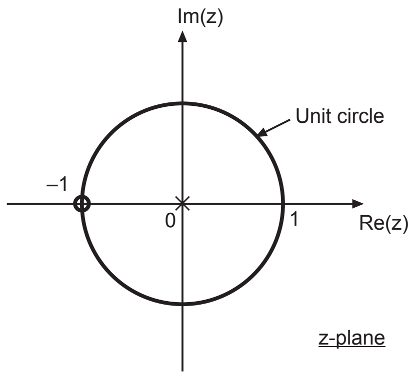


Fig. 2.10.3 Pole-zero plot of the system function of equation (2.10.46)

Hence system function $H(z) = \dfrac{Y(z)}{X(z)}$

becomes,

$$H(z) = \frac{1}{2} + \frac{1}{2}z^{-1} \quad \text{... (2.10.53)}$$

(ii) To find unit sample response :

A unit sample response $h(n)$ is obtained by taking inverse z-transform of $H(z)$. i.e.,

$$h(n) = IZT\{H(z)\} = IZT\left\{\frac{1}{2} + \frac{1}{2}z^{-1}\right\}$$

Applying the linearity and time shifting properties and from z-transform pair $\delta(n) \leftrightarrow 1$, we can write,

$$h(n) = \frac{1}{2}\delta(n) + \frac{1}{2}\delta(n-1) \quad \text{... (2.10.54)}$$

(iii) To determine pole zero plot :

Consider the system function $H(z)$,

$$H(z) = \frac{1}{2}+\frac{1}{2}z^{-1} \quad \text{... (2.10.55)}$$

Let us convert the powers of 'z' to positive by rearranging the equation as follows;

$$H(z) = \frac{1}{2}\cdot\frac{z+1}{z}$$

Hence zero is at $z=-1$ i.e. $-1+j0$

and pole is at $z=0$ i.e. origin

Fig. 2.10.2 shows the pole-zero plot.

(iv) To obtain transfer function $H(\omega)$ and its magnitude/phase plot :

The transfer function $H(\omega)$ can be obtained from system function $H(z)$ by putting $z=e^{j\omega}$. Hence equation (2.10.55) becomes,

$$H(\omega) = H(z)\big|_{z=e^{j\omega}} = \frac{1}{2}+\frac{1}{2}e^{-j\omega}$$

$$= \frac{1}{2}\left(1+e^{-j\omega}\right) \quad \text{... (2.10.56)}$$

$$= \frac{1}{2}\cdot e^{-j\omega/2}\left(e^{j\omega/2}+e^{-j\omega/2}\right) \quad \text{By rearranging the equation}$$

By Euler's identity we know that $\frac{e^{j\theta}+e^{-j\theta}}{2} = cos\,\theta$. Hence above equation can be written as,

$$H(\omega) = cos\left(\frac{\omega}{2}\right)\cdot e^{-j\omega/2} \quad \text{... (2.10.57)}$$

This is the required transfer function. This transfer function can be expressed in phasor form as,

$$H(\omega) = |H(\omega)|\, e^{j\angle H(\omega)} \quad \text{... (2.10.58)}$$

On comparing above equation with equation (2.10.57) we get,

$$\text{Magnitude of } H(\omega) \Rightarrow |H(\omega)| = cos\left(\frac{\omega}{2}\right) \quad \text{... (2.10.59)}$$

$$\text{Phase of } H(\omega) \Rightarrow \angle H(\omega) = -\frac{\omega}{2} \quad \text{... (2.10.60)}$$

'ω' is the continuous function and varies from $-\pi$ to π. Table 2.10.3 shows the values of $|H(\omega)|$ and $\angle H(\omega)$ for few values of ω.

ω	$\lvert H(\omega) \rvert = \cos\left(\frac{\omega}{2}\right)$	$\angle H(\omega) = -\frac{\omega}{2}$
$-\pi$	0	$\frac{\pi}{2}$
$-\frac{3\pi}{4}$	0.383	$\frac{3\pi}{8}$
$-\frac{\pi}{2}$	0.707	$\frac{\pi}{4}$
$-\frac{\pi}{4}$	0.924	$\frac{\pi}{8}$
0	1	0
$\frac{\pi}{4}$	0.924	$-\frac{\pi}{8}$
$\frac{\pi}{2}$	0.707	$-\frac{\pi}{4}$
$\frac{3\pi}{4}$	0.383	$-\frac{3\pi}{8}$
π	0	$-\frac{\pi}{2}$

Table 2.10.3 : Calculation of magnitude and phase response

Table 2.10.3 shows the magnitude and phase response of the transfer function.

Note We know that the range of ω is, $-\pi \leq \omega \leq \pi$. Hence useful range in above magnitude and phase plot is from $-\pi$ to π. Reader can just try out by plotting $\lvert H(\omega) \rvert$ and $\angle H(\omega)$ outside this range. If we plot $\lvert H(\omega) \rvert$ and $\angle H(\omega)$ from π to 3π, the similar response repeats.

Here observe that the magnitude/phase plot given in Fig. 2.10.3 is same as that of Fig. 2.10.1, since difference equation of this example is same as that of example 2.10.6.

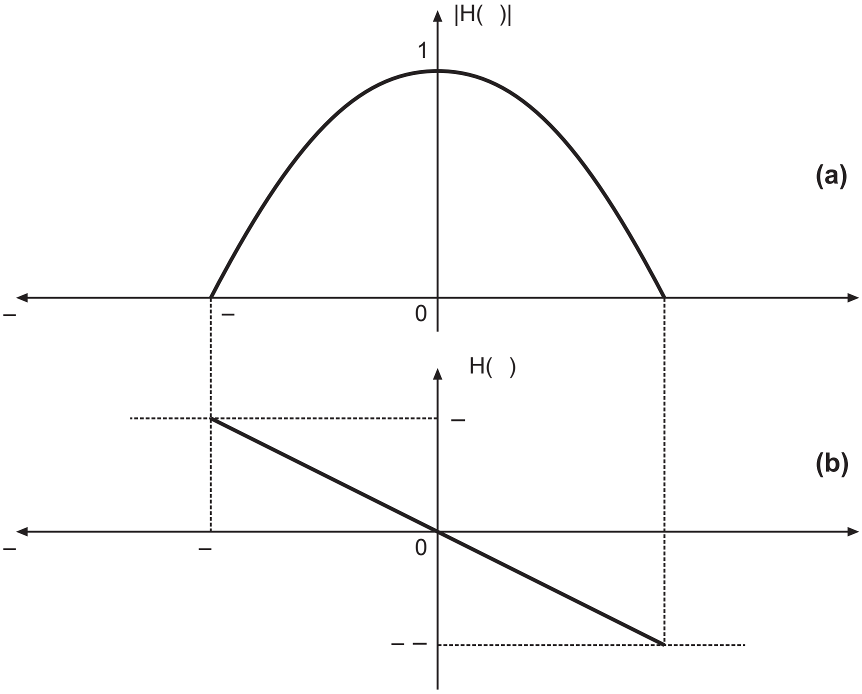


Fig. 2.10.4 Plots of magnitude and phase of transfer function

Example 2.10.10 *Determine the DTFT of the given sequence*

$x(n) = a^n[u(n)-u(n-8)]$, $|a| < 1$. **AU : May-14, Marks 8**

Solution : We know that

$$a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}}$$

By time shifting property,

$$a^n u(n-8) \xleftrightarrow{DTFT} e^{-j\omega 8} \frac{1}{1-e^{-j\omega}}$$

$$\therefore \qquad X(\omega) = \frac{1}{1-ae^{-j\omega}} - \frac{e^{-j\omega 8}}{1-e^{-j\omega}}$$

Example 2.10.11 *What is the need for frequency response analysis ? Determine the frequency response and plot he magnitude response and phase response for the system.*

$y(n) = 2x(n) + x(n-1) + y(n-2)$ **AU : Dec.-16, Marks 16**

Solution : The frequency response of a system describes nature of system i.e. characteristics of system, how signals are processed when they are given as input to the system. .

- Frequency Response is studied with help of
  (i) Magnitude response (ii) Phase response

i) Magnitude response gives the interpretation of gain/attenuation that system enforces on frequency.

ii) Where as phase response interprets by how much a particular frequency component of signal might be delayed/advanced by the system.

- $$y(n) = 2x(n)+x(n-1)+y(n-2)$$

Taking the z-transform

$$Y(z)-Y(z)z^{-2} = 2X(z)+X(z)z^{-1}$$

$$\therefore \quad Y(z)[1-z^{-2}] = 2X(z)[1+z^{-1}]$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = 2\left[\frac{1+z^{-1}}{1-z^{-2}}\right] = 2\left[\frac{(z+1)z}{(z^2-1)}\right]$$

$$H(z) = 2\frac{z}{z-1} = \frac{2}{(1-z^{-1})}$$

Taking inverse z-transform,

$$h(n) = 2u(n)$$

Frequency response $= H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = 2\dfrac{1}{1-e^{-j\omega}}$

Magnitude response :

$$H|e^{j\omega}| = \frac{2}{1-\cos\omega+j\sin\omega}$$

$$|H|e^{j\omega}||^2 = \frac{4}{(1-\cos\omega)^2+\sin^2\omega} = \frac{4}{2-2\cos\omega} = \frac{2}{1-\cos\omega}$$

$$\therefore \quad H(e^{j\omega}) = \frac{\sqrt{2}}{\sqrt{1-\cos\omega}}$$

Phase response :

$$\phi(\omega) = 2-\tan^{-1}\left(\frac{\sin\omega}{1-\cos\omega}\right)$$

Example 2.10.12 *Find the frequency response of the LTI system governed by the equation $y(n) = a_1\, y(n-1) - a_2\, y(n-2) - x(n)$.* **AU : May-15, Marks 8**

Solution : Taking z-transform of given difference equation,

$$Y(z) = a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - X(z)$$

$$\therefore \quad Y(z)[1 - a_1 z^{-1} + a_2 z^{-2}] = -X(z)$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = -\frac{1}{1 - a_1 z^{-1} + a_2 z^{-2}}$$

Frequency response is obtained by,

$$H(\omega) = H(z)\big|_{z = j\omega} = -\frac{1}{1 - a_1 e^{-j\omega} + a_2 e^{-j2\omega}}$$

Review Question

1. *State and prove linearity and frequency shifting theorems of DTFT.* **AU : May-14, Marks 8**

2.11 Short Answered Questions [2 Marks Each]

Q.1 What is an LTI system ? **Madras Univ. : April-01, Oct.-2000**

Ans. : A linear time invariant (LTI) system follows two principles : Superposition and time invariance.

For a linear system, the response due to linear combination of inputs is same as linear combination of corresponding outputs.

Time invariance means shift of time origin of input does not change the response of the system.

Q.2 Define impulse response of the system. **Madras Univ. : April-01**

Ans. : When an unit sample $\delta(n)$ is applied to the input of the system. The output of the system is called impulse response. It is denoted by $h(n)$. i.e.,

$$h(n) = T\{\delta(n)\}$$

And output of the system to any arbitrary input $x(n)$ is given as,

$$y(n) = x(n) * h(n)$$

Thus, the convolution of input and impulse response gives output. Impulse response is the characteristic of the system.

Q.3 Give the formula for discrete convolution. **Madras Univ. : April-01**

Ans. : Discrete convolution is given as,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = \sum_{k=-\infty}^{\infty} h(k)\; x(n-k)$$

It is also denoted as,

$$y(n) = x(n) * h(n)$$

Convolution is commutative and associative.

Q.4 Define system transfer function. **Madras Univ. : April-01**

Ans. : System function is the ratio of z-transform of the output to z-transform of the input. i.e.,

$$\text{System function } H(z) = \frac{Y(z)}{X(z)}$$

Here $y(n) \xleftrightarrow{z} Y(z)$ and $x(n) \xleftrightarrow{z} X(z)$. Inverse z-transform of system function gives impulse response $h(n)$. i.e.,

$$h(n) = IZT\{H(z)\}$$

Important properties such as stability, causality etc can be studied from system function.

Q.5 Write the condition for system stability. **Madras Univ. : April-2000, Oct.-97, April-96**

Ans. : A system is said to be BIBO stable if, it produces bounded output for bounded input. i.e.,

The input is bounded if, $|x(n)| \leq M_x < \infty$. Output is bounded if, $|y(n)| \leq M_y < \infty$. The stable system produces bounded output for every bounded input. For an LTI system stability is given in terms of impulse response as,

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

i.e., the impulse response must be summable.

Q.6 State the conditions for causality and stability of LTI system in z-domain.

Ans. : A system is said to be causal if ROC of its system function is exterior of some circle of radius 'r' i.e.,

$$|z| > r$$

The LTI system is stable if ROC of its system function includes the unit circle. i.e.,

$$r < 1$$

Thus combining the two conditions,

if $|z| > r < 1$, then the system will be causal and stable.

Q.7 What is z-transform of n x(n) and x(n – m) in terms of X(z) ? **AU : May-06**

Ans. : By differentiation in z-domain property,

$$n\,x(n) \xleftrightarrow{z} -z\frac{d}{dz}X(z)$$

By time shifting property

$$x(n-m) \overset{z}{\longleftrightarrow} z^{-m} X(z)$$

Q.8 Represent the condition satisfied by a stable LTI DT system in the z-domain. What is equivalent condition in time domain ? AU : May-06

Ans. : z-domain : LTI system is BIBO stable if and only if the ROC of the system function includes the unit circle.

Time domain : LTI system is BIBO stable if its impulse response is absolutely summable i.e.

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Q.9 Determine the ROC of the z-transform of the sequence y(n) = 2^n u(– n – 1). AU : May-04

Ans. : We know that, $-a^n u(-n-1) \overset{z}{\longleftrightarrow} \dfrac{1}{1-az^{-1}}$ ROC : $|z| < |a|$

Here a = – 2, $2^n u(-n-1) \overset{z}{\longleftrightarrow} \dfrac{1}{1+2z^{-1}}$ ROC : $|z| < |2|$.

Q.10 Define transfer function. AU : May-05

Ans. : The transfer function is the z-transform of unit sample response of LTI system. It is given as,

$$H(z) = Z\{h(n)\}$$

$$= \frac{Y(z)}{X(z)}$$

Q.11 Differentiate between Fourier series and Fourier transform. AU : May-05

Ans. :

Sr. No.	Fourier Series	Fourier Transform
1.	It is expansion of signal into infinite series in time domain.	It is conversion of the signal from time domain to frequency domain.
2.	$x(n) = \sum_{k=0}^{N-1} c(k)\, e^{j2\pi kn/N}$ where, $c(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$	$x(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$

Q.12 What is ROC ? **AU : Dec.-05, 11, 12, 15, May-11, 14**

Ans. : ROC : It is the region in which z-transform is convergent. In other words, z-transform can be calculated in ROC.

Q.13 Define z-transform and its ROC. **AU : Dec.-06**

Ans. : Definition of z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

ROC : It is the region where z-transform is convergent.

Q.14 What is the difference between Fourier transform of the continuous signal x(t) and the FT of the sampled signal x(n) ? **AU : Dec.-06**

Ans. :

- FT of CT signals : $X(\Omega) = \int_{-\infty}^{\infty} x(t)\; e^{-j\Omega t}\, dt$
- FT of sampled signals : $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$

Here $X(\omega)$ is periodic in 2π, but $X(\Omega)$ is not periodic.

Q.15 Let x(n) and X(jω) is Fourier transform pair. What is the Fourier transform of even sequence and odd sequence of x(n) interms of X(jω) ? **AU : Dec.-06**

Ans. : Here $x(n) = X(\omega)$

For even sequence,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= 2 \sum_{n=0}^{\infty} x(n) \cos \omega n + x(0)$$

For odd sequence,

$$X(\omega) = \mathrm{j}2 \sum_{n=0}^{\infty} x(n) \sin \omega n + x(0)$$

Q.16 What are the properties of region of convergence (ROC) ? **AU : May-07, 17, Dec.-12, 15**

Ans. :

i) ROC does not contain any poles.

ii) ROC of the causal sequence is of the form $|z| > r$

iii) ROC of the left handed sequence is of the form $|z| < r$.

Q.17 Determine the Z-transform and ROC of the following finite duration signals x(n) = {3, 2, 2, 3, 5, 0, 1} AU : May-10, 16

Ans. :

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Here $x(0) = 3,\ x(1) = 2,\ x(2) = 2,\ x(3) = 3,\ x(4) = 5,\ x(5) = 0,\ x(6) = 1$

$$\therefore \quad X(z) = x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2} + \cdots + x(6)z^{-6}$$

$$= 3 + 2z^{-1} + 2z^{-2} + 3z^{-3} + 5z^{-4} + z^{-6}$$

ROC : Entire z–plane except z = 0

Q.18 Find the Fourier transform of the signal x[n] = u[n]. (Refer example 2.10.3) AU : May-10

Q.19 Determine the discrete time Fourier transform of the sequence x[n] = {1, – 1, 1, – 1}. AU : May-10

Ans. :

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Putting for $x(n) = \{1, -1, 1, -1\}$

$$X(\omega) = 1.e^{0} - 1.e^{-j\omega} + 1.e^{-j2\omega} - 1.e^{-j3\omega}$$

$$= 1 - e^{-j\omega} + e^{-j2\omega} - e^{-j3\omega}$$

$$= (1 - e^{-j\omega}) + e^{-j2\omega}(1 - e^{-j\omega})$$

$$= (1 - e^{-j\omega})(1 + e^{-j2\omega})$$

Rearranging above equation,

$$X(\omega) = \left(e^{-j\frac{\omega}{2}} \cdot e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} \cdot e^{-j\frac{\omega}{2}}\right)\left(e^{-j\omega} \cdot e^{j\omega} + e^{-j\omega} \cdot e^{-j\omega}\right)$$

$$= e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right) \cdot e^{-j\omega}\left(e^{j\omega} + e^{-j\omega}\right)$$

$$= e^{-j\frac{3\omega}{2}} \cdot 2j\sin\frac{\omega}{2} \cdot 2\cos\omega$$

$$= e^{-j\frac{\omega}{2}} \cdot j \cdot 4\sin\frac{\omega}{2}\cos\omega$$

Q.20 Mention the relation between, Z transform and Fourier transform. AU : Dec.-10

Ans. : Fourier transform is basically the z-transform of the sequence evaluated on unit circle. i.e.,

$$X(\omega) = X(z)|_{z=e^{j\omega}}$$

Q.21 Give any two properties of linear convolution. AU : Dec.-10, 11

Ans. : 1. Linear convolution is commutative, i.e.,

$$x(n) * h(n) = h(n) * x(n)$$

2. Linear convolution is distributive, i.e.,

$$x(n)\{h_1(n) + h_2(n)\} = x(n) * h_1(n) + x(n) * h_2(n)$$

Q.22 What are the basic operations involved in convolution process ? AU : Dec.-06

Ans. : The convolution process involves four basic operations,

i) Folding i.e. $h(-k)$

ii) Shifting i.e. $h(-k)$

iii) Multiplication i.e. $x(k)\ h(n-k)$

iv) Summation i.e. $\sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$

Q.23 What is the resultant impulse response of the two systems whose impulse responses are $h_1(n)$ and $h_2(n)$ when they are in
a) Series and in b) Parallel AU : May-07

Ans. : **i) Series connection :** $h(n) = h_1(n) * h_2(n)$.

ii) Parallel connection : $h(n) = h_1(n) + h_2(n)$.

Q.24 Find the convolution of the input signal {1, 2, 1} and the impulse response {1, 1, 1} using z-transform. AU : May-17

Ans. : $x(n) = \{1, 2, 1\}$ and $h(n) = \{1, 1, 1\}$

$\therefore \quad X(z) = 1 + 2z^{-1} + z^{-2}$ and $H(z) = 1 + z^{-1} + z^{-2}$

By convolution theorem,

$$y(n) = x(n) * h(n) = IZT\{X(z) \cdot H(z)\} = IZT\{(1 + 2z^{-1} + z^{-2}) \cdot (1 + z^{-1} + z^{-2})\}$$

$$= IZT\{1 + 3z^{-1} + 4z^{-2} + 3z^{-3} + z^{-4}\} = \{1, 3, 4, 3, 1\}$$

Q.25 Write the DTFT for a) x(n) = $a^n u(n)$ b) x(n) = $4\,\delta(n) + 3\,\delta(n-1)$ AU : Dec.-11

Ans. : a) $x(n) = a^n u(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1-ae^{-j\omega}}$$

b) $x(n) = 4\delta(n)+3\delta(n-1)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = 4+3e^{-j\omega}$$

Q.26 Compute the convolution of the two sequences x(n) = {2,1,0,0.5} and h(n) = {2,2,1,1}. **AU : May-12, 16**

Ans. :

		x(n)	⇒	2	1	0	0.5
			⇒	2	2	1	1
				2	1	0	0.5
			2	1	0	0.5	×
		4	2	0	1	×	×
	4	2	0	1	×	×	×
y(n) =	{ 4	6	4	4	2	0.5	0.5 }

Q.27 Consider the signal x(n) = $|1|$ for $-1 \le n \le 1$ and 0 for all other values of n, sketch the magnitude and phase spectrum. **AU : May-12**

Ans. :

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \quad \text{Fourier transform}$$

Since $x(n) = |1|$ for $-1 \le n \le 1$,

$$X(\omega) = 1\times e^{-j\omega} + 1\times e^{0} + 1\times e^{j\omega}$$

$$= 1 + e^{j\omega} + e^{-j\omega}$$

$$= 1+2\cos\omega, \text{ since}$$

$$e^{j\theta}+e^{-j\theta} = 2\cos\theta$$

$$|X(\omega)| = 1+2\cos\omega$$

$$\angle X(\omega) = 0$$

Fig. 2.1 shows above magnitude spectrum. Phase is zero.

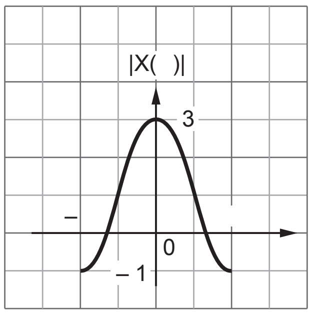


Fig. 2.1 Magnitude spectrum

Q.28 State and prove the time reversal property of Fourier transform. **AU : May-12**

Ans. : Fourier transform is given as,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Fourier transform of $x(-n)$ will be,

$$\text{F.T. } \{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\omega n}$$

Let $n = -m$,

$$\text{F.T. } \{x(-n)\} = \sum_{m=\infty}^{-\infty} x(m)\, e^{-j\omega(-m)} = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j(-\omega)m} = X(-\omega)$$

Thus $x(-n) = \xleftrightarrow{DTFT} X(-\omega)$

Q.29 Define discrete-time Fourier transform pair for a discrete sequence. **AU : Dec.-12**

Ans. : DTFT pair is defined as,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

and

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

Q.30 Determine the z-transform and ROC for the signal $x(n) = \delta(n-k) + \delta(n+k)$. **AU : Dec.-13, May-16**

OR Determine the z-transform and ROC of x(n) = $\delta(n-k)$ **AU : May-16**

Ans. : We know that $\delta(n) \xleftrightarrow{z} 1$, ROC : entire z-plane

By time shifting, $\delta(n-k) \xleftrightarrow{z} z^{-k}$, ROC : entire z-plane except z = 0.

$\delta(n+k) \xleftrightarrow{z} z^{k}$, ROC : entire z-plane except z = ∞.

∴ $X(z) = z^{-k} + z^{k}$ ROC : entire z-plane except z = 0 and z = ∞

Q.31 Prove the convolution property of z-transform. [Refer section 2.4.6] **AU : Dec.-13**

Q.32 State the initial and final value theorem of z-transform. **AU : May-14, Dec.-15**

Ans. : **Initial value theorem :** $x(0) = \lim_{z\to\infty} X(z)$

Final value theorem : $x(\infty) = \lim_{z\to 1} (z-1) X(z)$.

Q.33 Check if the system described by the difference equation y(n) = ay(n – 1) + x(n) with y(0) = 1 is stable. **AU : May-15**

Ans. : Taking unilateral z-transform of given difference equation,

$$Y(z) = a\,[y(-1) + z^{-1}Y(z)] + X(z)$$

$$= az^{-1}Y(z) + X(z)\text{ , since } y(-1) = 0 \text{ (not given)}$$

$$\therefore \quad Y(z)\,[1 - az^{-1}] = X(z)$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

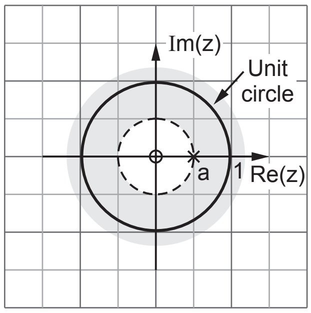


Fig. 2.2

Fig. 2.2 shows the pole zero plot for this function for $|a| < 1$. The ROC of the stable system includes unit circle. Hence the system is stable only if $|a| < 1$.

Q.34 Determine the Z-transform of x(n) = a^n. (Refer example 2.2.4) **AU : May-15**

Q.35 Determine the Fourier Transform of the signal x(t) = sin ω_0t. **AU : May-15**

Ans. : $$x(t) = \sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

$$1 \xleftrightarrow{FT} 2\pi\,\delta(\omega)$$

$$\therefore \quad \frac{1}{2} \xleftrightarrow{FT} \pi\,\delta(\omega)$$

$$\left.\begin{aligned}\therefore \quad & \frac{1}{2}e^{j\omega_0 t} \xleftrightarrow{FT} \pi\,\delta(\omega-\omega_0) \\ \text{Hence} \quad & \frac{1}{2}e^{-j\omega_0 t} \xleftrightarrow{FT} \pi\,\delta(\omega+\omega_0)\end{aligned}\right\}$$ since $e^{j\beta t}\,x(t) \xleftrightarrow{FT} X(\omega-\beta)$, by frequency shifting property

$$\therefore \quad X(\omega) = \frac{\pi}{j}\delta(\omega-\omega_0) - \frac{\pi}{j}\delta(\omega+\omega_0)$$

$$\therefore \quad \sin\omega_0\,t \xleftrightarrow{FT} \frac{\pi}{j}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$$

Q.36 Find the system transfer function H(z) if y(n) = x(n) + y(n –1). **AU : Dec.-16**

Ans. : Taking z-transform of given equation,

$$Y(z) = X(z) + z^{-1}Y(z)$$

$$\therefore \quad (1-z^{-1})Y(z) = X(z)$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-z^{-1}}$$

Q.37 Explain the relationship between s-plane and z-plane. AU : Dec.-16

Ans. : The impulse response of CT system and equivalent DT system is related as,

$$h_\delta(t) = \sum_{n=-\infty}^{\infty} h(n)\ \delta(t - nT_s)$$

z-transform of $h(n)$ is given as,

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\ z^{-n} \quad \ldots(1)$$

Laplace transform of $h_\delta(t)$ is given as,

$$H(s) = \int_{-\infty}^{\infty} h_\delta(t) e^{-st}\, dt = \sum_{n=-\infty}^{\infty} h(n)\, e^{-snT_s} \quad \ldots(2)$$

Here from equation (1) and (2) we can write,

$$H(s) = H(z)\,\big|_{z=e^{-sT_s}}$$

This is the relationship between s-plane and z-plane.

❑❑❑

3 Discrete Fourier Transform and Computation

Syllabus

Discrete Fourier Transform - Properties, Magnitude and phase representation - Computation of DFT using FFT algorithm - DIT and DIF - Using radix 2 FFT - Butterfly structure.

Contents

3.1 Introduction to DFT ***May-07, 11, 12, 14, 15, 17,***

. . . . ***Dec.-05, 06, 07, 09, 12, 15*** · Marks 12

3.2 Relationship of DFT with Other Transforms

3.3 Properties of DFT ***May-04, 06, 11, 12, 14,***

. . . . ***Dec.-07, 11, 12, 13, 15, 16,*** Marks 10

3.4 Applications of DFT

3.5 Computation of DFT using FFT Algorithms

3.6 Radix-2 FFT Algorithms ***May-04, 05, 06, 10, 11, 12, 15, 16, 17***

. . . . ***Dec.-04, 05, 06, 08,***

. . . . ***Dec.-10, 11, 12, 15, 16,*** · · · Marks 16

3.7 Inverse DFT using FFT Algorithms. . . . . ***Dec.-10, 15, May-15, 16*** · · · Marks 16

3.8 Short Answered Questions [2 Marks Each]

3.1 Introduction to DFT **AU : May-07, 11, 12, 14, 15, 17, Dec.-05, 06, 07, 09, 12**

- **Fourier Transform**

  Fourier transform is used for frequency analysis of the signal. It is given as,

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

  Here 'Ω' is the frequency and it has the continuous range of 0 to 2π.

- Fourier transform cannot be calculated on any digital processor since 'Ω' is continuous in nature. This problem can be solved by evaluating fourier transform at only discrete points of Ω.

- The total range of Ω from 0 to 2π is divided into 'N' equally spaced points. And fourier transform is calculated at those points. It is called discrete fourier transform. In discrete fourier transform Ω is represented as,

$$\Omega_k = \frac{2\pi}{N} k \ , \ k = 0, 1, \ldots N-1$$

Thus discrete fourier transform is evaluated only at Ω_k.

3.1.1 Definition of DFT and IDFT

Putting Ω_k for Ω in definition of fourier transform,

$$X\left(\frac{2\pi}{N} k\right) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \ , \ k = 0, 1, \ldots N-1 \qquad \ldots (3.1.1)$$

Here 'N' is also the length of the sequence *x(n)*. The values of $X\left(\frac{2\pi}{N} k\right)$ are addressed by 'k' only. Hence we can write $X\left(\frac{2\pi}{N} k\right)$ as *X(k)* only as a short hand notation.

$$\textbf{DFT :} \quad X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \ , \qquad k = 0, 1, \ldots N-1 \qquad \ldots (3.1.2)$$

Here 'k' indicates the index of the frequency. Above equation is called *Discrete Fourier Transform* (DFT).

The *inverse discrete fourier transform* (IDFT) is given as,

$$\textbf{IDFT :} \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N} \ , \qquad n = 0, 1, \ldots N-1 \qquad \ldots (3.1.3)$$

Here 'N' is length of the sequence *x(n)*. Note that *x(n)* and *X(k)* both contain N samples.

3.1.2 DFT as a Linear Transformation

- Let us define, $W_N = e^{-j\,2\pi/N}$. This is called twiddle factor. Hence DFT and IDFT equation can be written as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \qquad k = 0, 1, \ldots\ldots N-1 \qquad \ldots (3.1.4)$$

and
$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}, \qquad n = 0, 1, \ldots\ldots N-1 \qquad \ldots (3.1.5)$$

The above DFT and IDFT equation are obtained by putting $e^{-j2\pi/N} = W_N$ in equation 3.1.2 and equation 3.1.3.

- Let us represent sequence $x(n)$ as vector x_N of N samples. i.e.,

$$x_N = \begin{array}{r} n=0 \\ n=1 \\ : \\ : \\ n=N-1 \end{array}\begin{bmatrix} x(0) \\ x(1) \\ : \\ : \\ x(N-1) \end{bmatrix}_{N\times 1} \qquad \ldots (3.1.6)$$

and $X(k)$ can be represented as a vector X_N of 'N' samples. i.e.,

$$X_N = \begin{array}{r} k=0 \\ k=1 \\ : \\ : \\ k=N-1 \end{array}\begin{bmatrix} X(0) \\ X(1) \\ : \\ : \\ X(N-1) \end{bmatrix}_{N\times 1} \qquad \ldots (3.1.7)$$

The values of W_N^{kn} can be represented as a matrix $[W_N]$ of size $N \times N$ as follows :

$$[\mathbf{W_N}] = \begin{array}{r} \\ k=0 \\ k=1 \\ k=2 \\ \\ \\ k=N-1 \end{array}\begin{array}{ll} \qquad \mathbf{n=0} & \qquad \mathbf{n=1} \\ \left[\begin{array}{l} W_N^{kn}\Big|_{k=0,\,n=0,} = W_N^0 \\ W_N^{kn}\Big|_{k=1,\,n=0,} = W_N^0 \\ W_N^{kn}\Big|_{k=2,\,n=0,} = W_N^0 \\ \quad : \\ \quad : \\ W_N^{kn}\Big|_{k=N-1,\,n=0,} = W_N^0 \end{array}\right. & \begin{array}{l} W_N^{kn}\Big|_{k=0,\,n=1,} = W_N^0 \\ W_N^{kn}\Big|_{k=1,\,n=1,} = W_N^1 \\ W_N^{kn}\Big|_{k=2,\,n=1,} = W_N^2 \\ \quad : \\ \quad : \\ W_N^{kn}\Big|_{k=N-1,\,n=1,} = W_N^{N-1} \end{array} \end{array}$$

$$\begin{array}{ccc} \mathbf{n = 2} & & \mathbf{n = N-1} \\ \left. W_N^{kn} \right|_{k=0,n=2,} = W_N^0 & \cdots & \left. W_N^{kn} \right|_{k=0,n=N-1,} = W_N^0 \\ \left. W_N^{kn} \right|_{k=1,n=2,} = W_N^2 & \cdots & \left. W_N^{kn} \right|_{k=1,n=N-1,} = W_N^{N-1} \\ \left. W_N^{kn} \right|_{k=2,n=2,} = W_N^4 & \cdots & \left. W_N^{kn} \right|_{k=2,n=N-1,} = W_N^{2(N-1)} \\ \vdots & & \vdots \\ \left. W_N^{kn} \right|_{k=N-1,n=2,} = W_N^{2(N-1)} & \cdots & \left. W_N^{kn} \right|_{k=N-1,n=N-1,} = W_N^{(N-1)(N-1)} \end{array} \Bigg]_{N \times N}$$

$$\therefore \quad [W_n] = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}_{N \times N} \quad \text{... (3.1.8)}$$

Here the individual elements are written as W_N^{kn} with 'k' rows and 'n' columns. Then N-point DFT of equation 3.1.4 can be represented in the matrix form as,

while $$\boxed{X_N = [W_N]\, x_N} \quad \text{... (3.1.9)}$$

Similarly IDFT of equation 3.1.5 can be expressed in matrix form as,

$$\boxed{x_N = \frac{1}{N}\left[W_N^*\right] X_N} \quad \text{... (3.1.10)}$$

Here $W_N^{kn} = [W_N]$, hence $\left[W_N^*\right] = W_N^{-kn}$ as written above.

3.1.3 Periodicity Property of W_N

Calculation of W_N

Now let us calculate the values of W_N for $N = 8$. We know that W_N is given as,

$$W_N = e^{-j\frac{2\pi}{N}}$$

With $N = 8$ above equation becomes,

$$W_8 = e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}} \quad \text{... (3.1.11)}$$

Table 3.1.1 shows values of $W_8^0, W_8^1, W_8^2, \ldots W_8^{15}$

kn	$W_8^{kn} = e^{-j\frac{\pi}{4}\times kn}$	Comments
0	$W_8^0 = e^0$	Phasor of magnitude 1 and angle 0
1	$W_8^1 = e^{-j\frac{\pi}{4}\times 1} = e^{-j\frac{\pi}{4}}$	Phasor of magnitude 1 and angle $-\frac{\pi}{4}$
2	$W_8^2 = e^{-j\frac{\pi}{4}\times 2} = e^{-j\frac{\pi}{2}}$	Magnitude 1, Angle $-\frac{\pi}{2}$
3	$W_8^3 = e^{-j\frac{\pi}{4}\times 3} = e^{-j\frac{3\pi}{4}}$	Magnitude 1, Angle $-\frac{3\pi}{4}$
4	$W_8^4 = e^{-j\frac{\pi}{4}\times 4} = e^{-j\pi}$	Magnitude 1, Angle $-\pi$
5	$W_8^5 = e^{-j\frac{\pi}{4}\times 5} = e^{-j\frac{5\pi}{4}}$	Magnitude 1, Angle $-\frac{5\pi}{4}$
6	$W_8^6 = e^{-j\frac{\pi}{4}\times 6} = e^{-j\frac{3\pi}{2}}$	Magnitude 1, Angle $-\frac{3\pi}{2}$
7	$W_8^7 = e^{-j\frac{\pi}{4}\times 7} = e^{-j\frac{7\pi}{4}}$	Magnitude 1, Angle $-\frac{7\pi}{4}$
8	$W_8^8 = e^{-j\frac{\pi}{4}\times 8} = e^{-j\,2\pi}$	Magnitude 1, angle -2π. Angle of -2π is same as '0' $\therefore W_8^8 = W_8^0$
9	$W_8^9 = e^{-j\frac{\pi}{4}\times 9} = e^{-j\left(2\pi+\frac{\pi}{4}\right)}$	Magnitude 1, angle $-\left(2\pi+\frac{\pi}{4}\right)$. This angle is same as $-\frac{\pi}{4}$ $\therefore W_8^9 = W_8^1$
10	$W_8^{10} = e^{-j\frac{\pi}{4}\times 10} = e^{-j\left(2\pi+\frac{\pi}{2}\right)}$	Magnitude 1, angle $-\left(2\pi+\frac{\pi}{2}\right)$. This angle is same as $-\frac{\pi}{2}$ $\therefore W_8^{10} = W_8^2$
Similarly following values can be verified		
11	$W_8^{11} = e^{-j\frac{\pi}{4}\times 11} = e^{-j\left(2\pi+\frac{3\pi}{4}\right)}$	$W_8^{11} = W_8^3$
12	$W_8^{12} = e^{-j\frac{\pi}{4}\times 12} = e^{-j\,(2\pi+\pi)}$	$W_8^{12} = W_8^4$
13	$W_8^{13} = e^{-j\frac{\pi}{4}\times 13} = e^{-j\left(2\pi+\frac{5\pi}{4}\right)}$	$W_8^{13} = W_8^5$

:	:	:
:	:	:
:	:	For all further values the above logic is applicable

Table 3.1.1 Values of W_N for $N = 8$

In the above table observe the value of W_8^1. i.e.,

$$W_8^1 = e^{-j\frac{\pi}{4}\times 1} = e^{-j\frac{\pi}{4}}$$

$$= \text{Magnitude } e^{-j \text{ Angle}}$$

Thus magnitude = 1 and angle = $-\frac{\pi}{4}$. In the above table all phasors W_8^{kn} have magnitude '1'.

Plot of W_N

Fig. 3.1.1 shows these phasors in complex plane against unit circle.

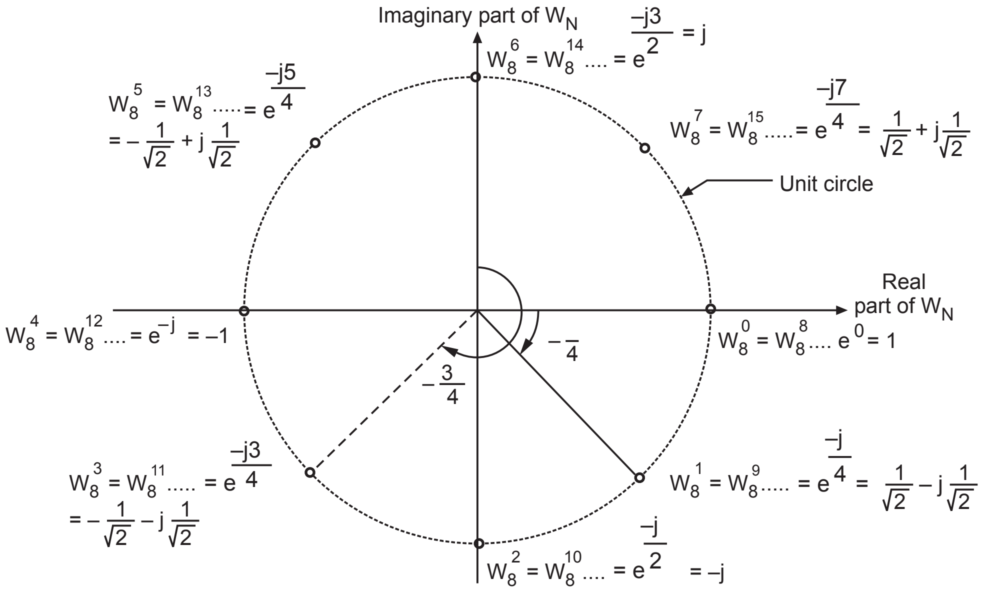


Fig. 3.1.1 Periodicity property of W_N^{kn} and values of various W_N

The values of these phasors are also shown in Fig. 3.1.1 observe that,

$$W_8^1 = W_8^9 = ... = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

Similarly other values are obtained.

Examples for Understanding

Example 3.1.1 *Compute DFT of the following sequence* $x(n) = \{0, 1, 2, 3\}$.

AU : Dec.-07, May-11, Marks 6

Solution : Here $N = 4$. Hence first evaluate W_4^0, W_4^1, W_4^2 and W_4^3. From the cyclic property of W_N and from Fig. 3.1.1 we can determine these values. They will be evenly spaced along the unit circle as shown in Fig. 3.1.2.

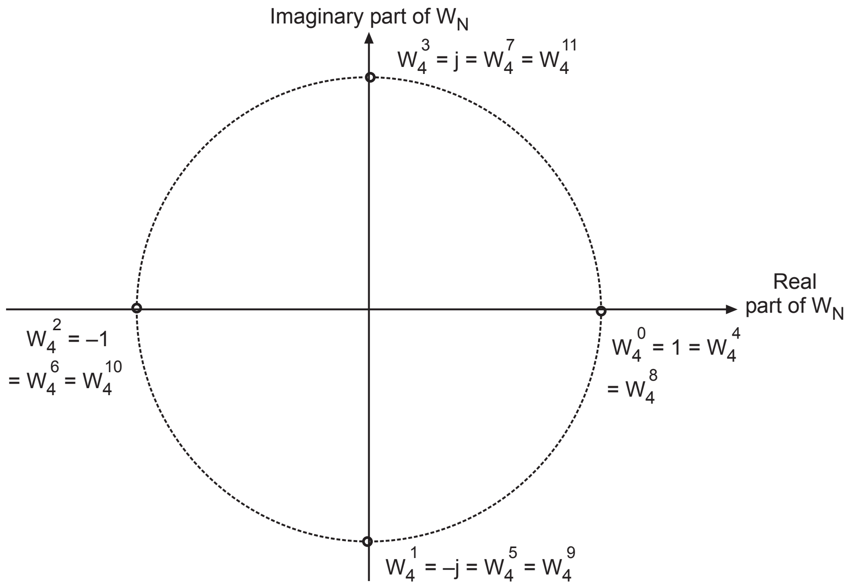


Fig. 3.1.2 Calculation of W_4^{kn} and its periodic values

The values given above can be verified easily. For example consider W_4^3.

$$W_4^3 = e^{-j\frac{2\pi}{4}\times 3} = e^{-j\frac{3\pi}{2}}$$

This has magnitude '1' and angle $-\frac{3\pi}{2}$ and,

$$W_4^3 = e^{-j\frac{3\pi}{2}} = cos\frac{3\pi}{2} - j\,sin\frac{3\pi}{2} = j$$

We have the matrix $[W_4]$ of 4×4 size. Its individual elements will be as shown below :

$$[W_4] = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} \begin{array}{c} \begin{array}{cccc} n=0 & n=1 & n=2 & n=3 \end{array} \\ \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{array}$$

In the above matrix individual elements are W_4^{kn} with $k = 0$ to 3 rows and $n = 0$ to 3 columns. From Fig. 3.1.2, the values of various elements in above matrix are,

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Key Point: *Remember this matrix byheart.*

Also we have, $x_4 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$

From equation 3.1.9 we can obtain 4 point DFT as

$$X_N = [W_N]\, x_N$$

With $N = 4, X_4 = [W_4]\, x_4$

Putting values of W_4 and x_4,

$$X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0+1+2+3 \\ 0-j-2+3j \\ 0-1+2-3 \\ 0+j-2-3j \end{bmatrix} = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

Thus we obtained 4 point DFT as,

$$X_4 = \begin{bmatrix} X(0) = 6 \\ X(1) = -2+2j \\ X(2) = -2 \\ X(3) = -2-2j \end{bmatrix}$$

This is the required DFT.

Example 3.1.2 *Calculate 8-point DFT of the following signal $x(n) = \{1, 1, 1, 1\}$ Assume imaginary part is zero. Also calculate magnitude and phase of $X(k)$.*

Solution :

Key Point *When the length of sequence is less than 'N', then append it with zeros (zero padding).*

- The given sequence is of length '4'. Since 8-point DFT is asked, we should make length of the sequence to be '8'. This is done by appending four zeros at the end of $x(n)$. i.e.,

$$x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$$

Thus appending zeros at the end of sequence does not change its meaning. This is also called as *zero padding*.

- DFT is given by equation 3.1.4 as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \qquad k = 0, 1, \ldots. N-1$$

For 8 point DFT, above equation becomes,

$$X(k) = \sum_{n=0}^{7} x(n)\, W_8^{kn}, \qquad k = 0, 1, \ldots.7$$

This equation is formulated in the matrix form by equation 3.1.9 as,

$$X_N = [W_N]\, x_N$$

with $N = 8$, $X_8 = [W_8]\, x_8$

These matrices can be expanded as,

$$X_8 = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} \quad \text{and} \quad x_8 = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

And the 8×8 matrix $[W_8]$ can be written from equation 3.1.8 as follows,

$$[W_8] = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix}$$

In Table 3.1.1 and Fig. 3.1.1 we have obtained the values of W_N^{kn} for $N = 8$. From Table 3.1.1 and Fig. 3.1.1 we obtain following.

$$W_8^0 = W_8^8 = W_8^{16} = W_8^{24} = W_8^{32} = W_8^{40} = \ldots = 1$$

$$W_8^1 = W_8^9 = W_8^{17} = W_8^{25} = W_8^{33} = W_8^{41} = W_8^{49} = \ldots = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = W_8^{10} = W_8^{18} = W_8^{26} = W_8^{34} = W_8^{42} = \ldots. = -j$$

$$W_8^3 = W_8^{11} = W_8^{19} = W_8^{27} = W_8^{35} = W_8^{43} = \ldots. = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^4 = W_8^{12} = W_8^{20} = W_8^{28} = W_8^{36} = W_8^{44} = \ldots. = -1$$

$$W_8^5 = W_8^{13} = W_8^{21} = W_8^{29} = W_8^{37} = W_8^{45} = \ldots. = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$W_8^6 = W_8^{14} = W_8^{22} = W_8^{30} = W_8^{38} = W_8^{46} = \ldots. = j$$

$$W_8^7 = W_8^{15} = W_8^{23} = W_8^{31} = W_8^{39} = W_8^{47} = \ldots. = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

- Putting different values in the equation for X_8, we obtain,

$$X_8 = [W_8] \cdot x_8 \quad \text{i.e.,}$$

Key Point *Remember* $[W_8]$ *byheart.*

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 1 + 1 + 1 + 0 + 0 + 0 + 0 \\ 1 + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} - j - \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} + 0 + 0 + 0 + 0 \\ 1 - j - 1 + j + 0 + 0 + 0 + 0 \\ 1 - \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} + j + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} + 0 + 0 + 0 + 0 \\ 1 - 1 + 1 - 1 + 0 + 0 + 0 + 0 \\ 1 - \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} - j + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} + 0 + 0 + 0 + 0 \\ 1 + j - 1 - j + 0 + 0 + 0 + 0 \\ 1 + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} + j - \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} + 0 + 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 - j(1 + \sqrt{2}) \\ 0 \\ 1 + j(1 - \sqrt{2}) \\ 0 \\ 1 - j(1 - \sqrt{2}) \\ 0 \\ 1 + j(1 + \sqrt{2}) \end{bmatrix}$$

This is the required DFT of the given sequence.

- Its real and imaginary parts can be separated as follows :

$$\begin{bmatrix} X_R(0) \\ X_R(1) \\ X_R(2) \\ X_R(3) \\ X_R(4) \\ X_R(5) \\ X_R(6) \\ X_R(7) \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X_I(0) \\ X_I(1) \\ X_I(2) \\ X_I(3) \\ X_I(4) \\ X_I(5) \\ X_I(6) \\ X_I(7) \end{bmatrix} = \begin{bmatrix} 0 \\ -(1+\sqrt{2}) \\ 0 \\ (1-\sqrt{2}) \\ 0 \\ -(1-\sqrt{2}) \\ 0 \\ (1+\sqrt{2}) \end{bmatrix}$$

Now 8-point magnitude and phase of $X(k)$ can be obtained as,

$$|X(k)| = \sqrt{[X_R(k)]^2 + [X_I(k)]^2}$$

and $$\angle X(k) = tan^{-1} \frac{X_I(k)}{X_R(k)}$$

Thus we have,

$$\begin{bmatrix} |X(0)| \\ |X(1)| \\ |X(2)| \\ |X(3)| \\ |X(4)| \\ |X(5)| \\ |X(6)| \\ |X(7)| \end{bmatrix} = \begin{bmatrix} 4 \\ 2.613 \\ 0 \\ 1.082 \\ 0 \\ 1.082 \\ 0 \\ 2.613 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \angle X(0) \\ \angle X(1) \\ \angle X(2) \\ \angle X(3) \\ \angle X(4) \\ \angle X(5) \\ \angle X(6) \\ \angle X(7) \end{bmatrix} = \begin{bmatrix} 0 \\ -1.178 \\ \textit{Not calculated} \\ -0.392 \\ \textit{Not calculated} \\ 0.392 \\ \textit{Not calculated} \\ 1.178 \end{bmatrix}$$

The magnitude and phase of DFT $X(k)$ is as given above. By taking more number of points plots can be obtained. Observe that $\angle X(2)$, $\angle X(4)$ and $\angle X(6)$ is not calculated since both real and imaginary values are zero.

Example 3.1.3 *Determine the 8-point DFT of the sequence x(n) = {1, 1, 1, 1, 1, 1, 0, 0}.*

AU : May-07, Dec.-09, Marks 16, May-14, Marks 12, May-17, Marks 13

Solution : Here $x(n) = x_8 = \{1, 1, 1, 1, 1, 1, 0, 0\}$

DFT is given as,

$$X_N = [W_N]\, x_N$$

For N = 8, $$X_8 = [W_8]\, x_8$$

Putting the values of $[W_8]$ and x_8,

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & \frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+1+1+1+1+1+0+0 \\ 1+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}-j-\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}-1-\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}+0+0 \\ 1-j-1+j+1-j+0+0 \\ 1-\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}+j+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}-1+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}+0+0 \\ 1-1+1-1+1-1+0+0 \\ 1-\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}-j+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}-1+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}+0+0 \\ 1+j-1-j+1+j+0+0 \\ 1+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}+j-\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}-1-\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}+0+0 \end{bmatrix} = \begin{bmatrix} 6 \\ -\frac{1}{\sqrt{2}}-j\left(1-\frac{1}{\sqrt{2}}\right) \\ 1-j \\ \frac{1}{\sqrt{2}}+j\left(1-\frac{1}{\sqrt{2}}\right) \\ 0 \\ \frac{1}{\sqrt{2}}-j\left(1-\frac{1}{\sqrt{2}}\right) \\ 1+j \\ \frac{1}{\sqrt{2}}+j\left(1+\frac{1}{\sqrt{2}}\right) \end{bmatrix}$$

Example 3.1.4 *Compute DFT of unit sample $\delta(n)$.*

Solution : The unit sample $\delta(n)$ is given as,

$$\begin{aligned} x(n) &= 1 \quad \text{for} \quad n=0 \\ &= 0 \quad \text{for} \quad n \neq 0 \end{aligned}$$

DFT is given by equation 3.1.2 as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

Putting for $x(n)$ in above equation,

$$X(k) = x(0)\,e^{0} = 1 \times 1$$

$$= 1 \qquad \text{... (3.1.12)}$$

Thus DFT of unit sample sequence is $X(k) = 1$ for all values of 'k'.

Example 3.1.5 *Find the inverse DFT of* $X(K) = \left\{7, -\sqrt{2} - j\sqrt{2}, -j, \sqrt{2} - j\sqrt{2}, 1, \sqrt{2} + j\sqrt{2}, j, -\sqrt{2} + j\sqrt{2}\right\}$.

AU : Dec.-12, Marks 10, Dec.-15, Marks 12

Solution : IDFT is given as,

$$x_8 = W_8^* X_8$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\ 1 & -j & -1 & j & 1 & -j & -1 & -j \\ 1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 7 \\ -\sqrt{2} - j\sqrt{2} \\ -j \\ \sqrt{2} - j\sqrt{2} \\ 1 \\ \sqrt{2} + j\sqrt{2} \\ j \\ -\sqrt{2} + j\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1.5 \\ 1 \\ 1 \\ 1 \\ -0.5 \end{bmatrix}$$

Example 3.1.6 *Find the X(K) for the given sequence x(n) = {1,2,3,4,1,2,3,4}.*

AU : May-12, Marks 16

Solution : Here $X_8 = W_8 \; x_8$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 20 \\ 0 \\ -4+j4 \\ 0 \\ -4 \\ -4-j4 \\ 0 \end{bmatrix}$$

Example 3.1.7 *Determine the DFT of the sequence* $x\,(n) = \begin{cases} \frac{1}{4}, & \text{for } 0 \le n \le 2 \\ 0, & \text{otherwise} \end{cases}$.

AU : May-15, Marks 8

Solution : Here $x(n) = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$ Let us calculate 4 point DFT of $x(n)$, Hence,

$$x(n) = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right\}$$

$$\therefore \qquad X_4 = [W_4]\, x_4$$

$$\begin{bmatrix} X\,(0) \\ X\,(1) \\ X\,(2) \\ X\,(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/4 \\ -j/4 \\ 1/4 \\ j/4 \end{bmatrix}$$

Example 3.1.8 *The analog signal has a bandwidth of 4 kHz. If we use N point DFT with $N = 2^m$ (m is an integer) to compute the spectrum of the signal with resolution less than or equal to 25 Hz. Determine the minimum sampling rate, minimum number of required examples and minimum lengh of the analog signal. What is the step size required for quantize this signal.* **AU : May-17, Marks 15**

Solution : a) The minimum sampling rate,

$$f_s \geq 2W$$

$$f_s = 2 \times 4kHz = 8\,kHz$$

b) Here

$$\Delta f = \frac{2W}{N}$$

$$25 = \frac{2 \times 4000}{N}$$

$$N = \frac{2 \times 4000}{25}$$

$$\text{No. of samples} = 320 \text{ samples}$$

c) The minimum length of analog signal is

$$t = 1/\Delta f = \frac{1}{25} = 0.04 \text{ sec.}$$

3.1.4 Comparison between DTFT and DFT

Sr. No.	DTFT	DFT
1.	Equation : $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\,\omega n}$	Equation : $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi\, kn/N}$ k = 0, 1, ... N − 1
2.	Here $X(\omega)$ is continuous function of ω.	Here $X(k)$ is defined for k = 0, 1, ... N − 1.
3.	DTFT cannot be evaluated on digital computer.	DFT can be evaluated on digital computer.
4.	DTFT is double sided.	DFT is single sided.
5.	The sequence $x(n)$ need not be periodic.	The sequence $x(n)$ is assumed to be periodic.

Table 3.1.2 Comparison between DTFT and DFT

Disadvantages of DFT over DTFT

1. DFT is evaluated only at discrete points. Hence chance of time domain aliasing is there.
2. The spectrum obtained by DFT is discrete.

Examples for Practice

Example 3.1.9 *Find the DFT of the sequence*
$x(n) = 0.5^n u(n); \quad 0 \le n \le 3$

[Ans. : X(k) = $\frac{0.937}{1-0.5e^{-j\pi/2k}}$]

Example 3.1.10 *Find the 4-point DFT of the sequence* $x(n) = \cos\left(\frac{n\pi}{4}\right)$.

[Ans. : X(k) = { 1, 1– j 1.414, 1, 1 + j 1.414 }]

Example 3.1.11 *Find the DFT of the sequence x(n) = {1, 0, 1, 0, 1, 0, 1, 0}.*

AU : Dec.-05, Marks 16

[Ans. : X(k) = {4, 0, 0, 0, 4 0, 0, 0}]

Example 3.1.12 *Find the DFT coefficients of the input sequence*
x(n) = {1, 1, 1, 1, 1, 1, 1, 1}

AU : Dec.-05, Marks 16

[Ans. : X(k) = {8, 0, 0, 0, 0, 0, 0, 0}]

Example 3.1.13 *Compute the N-point DFT of*
$x(n) = a^n \quad$ *for* $\quad 0 \le n \le N-1$

[Ans. : X (k) = $\frac{1-a^N}{1-a\ e^{-j2\pi k/N}}$]

Review Question

1. *Define DFT and IDFT.* **AU : Dec.-06, Marks 8**

3.2 Relationship of DFT with Other Transforms

3.2.1 Relationship between DFT and z-Transform

- The z-transform of sequence $x(n)$ is given as,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \qquad \text{... (3.2.1)}$$

- Let us sample $X(z)$ at 'N' equally spaced points on the unit circle. These points will be,

$$z_k = e^{j\,2\pi\, k/N} \quad \text{and} \quad k = 0, 1, \ldots\ldots N \qquad \text{... (3.2.2)}$$

- If we evaluate z-transform at these points,

$$X(z)\big|_{z_k = e^{j2\pi k}} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi kn/N}$$

If $x(n)$ is causal sequence and has 'N' number of samples, then above equation becomes,

$$X(z)\big|_{z_k = e^{j2\pi k/N}} = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad \text{... (3.2.3)}$$

The RHS of above equation is nothing but DFT $X(k)$. Thus the relationship between DFT and z-transform is,

$$\boxed{\textbf{Relationship :}\ X(k) = X(z)\big|_{z_k = e^{j2\pi k/N}}} \qquad \text{... (3.2.4)}$$

Explanation :

This means if z-transform is evaluated on unit circle at evenly spaced points only, then it becomes DFT. Earlier we have seen that if z-transform is evaluated on unit circle, then it becomes Fourier Transform.

3.2.2 Relationship between DFT and DFS

- The Discrete Fourier Series (DFS) coefficients for a periodic sequence $x_p(n)$ over period N are given as,

$$c(k) = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots.. N-1 \qquad \text{... (3.2.5)}$$

- It is shown in next section that,

$$x(n) \xleftrightarrow[N]{DFT} X(k) \qquad \text{By equation 3.3.6}$$

and

$$x_p(n) \xleftrightarrow[N]{DFT} X(k) \qquad \text{By equation 3.3.7}$$

Here $x_p(n)$ is the periodic version of $x(n)$. $x(n)$ as well as $x_p(n)$ have the same DFT i.e. $X(k)$. By definition of DFT we can write,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots N-1$$

- Since $x(n)$ and $x_p(n)$ have same DFT we can write above equation as,

$$X(k) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots. N-1$$

Putting this value in equation 3.2.5 we have,

$$c(k) = \frac{1}{N} \cdot X(k), \qquad k = 0, 1, \dots N-1 \qquad \dots (3.2.6)$$

Relationship : $X(k) = N \cdot c(k), \qquad k = 0, 1, \dots N-1$ ... (3.2.7)

This equation gives the relationship between DFT and DFS coefficients. If we know the DFS coefficients, then DFT can be obtained by above equation.

Consider the IDFT formula,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j 2\pi kn/N}, \quad n = 0, 1, \dots. N-1$$

From equation 3.2.6 we know that $c(k) = \frac{1}{N} X(k)$, hence above equation becomes,

$$x(n) = \sum_{k=0}^{N-1} c(k)\; e^{j 2\pi kn/N}, \quad n = 0, 1, \dots N-1$$

We know that $x(n)$ is basically periodic with period N. Observe that this equation is basically DFS equation. Thus IDFT is same as DFS equation.

3.2.3 Relationship between DFT and DTFT

- The DTFT is given as,

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

- Let us evaluate this DTFT at $\Omega = \frac{2\pi}{N} k$, where k = 0, 1, 2, ... $N-1$. Then we get,

$$X\left(\frac{2\pi}{N} k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\pi kn/N}$$

- Here note that $X\left(\frac{2\pi}{N} k\right)$ gives the DFT coefficients of periodic sequence.

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

Thus $x_p(n)$ is the sequence which is periodic with period N. Hence if sequence $x(n)$ has the length less than 'N', then $x_p(n) = x(n)$.

Relationship : $X\left(\frac{2\pi}{N}k\right) = X(k)$, i.e. DFT of $x(n)$

If length of $x(n)$ is greater than N, then DFT and DTFT will have no direct relationship.

Review Questions

1. *What is the relationship between DFT and DFS ?*
2. *Explain the relationship between z-transform and DFT.*

3.3 Properties of DFT

AU : May-04, 06, 11, 12, 14, Dec.-07, 11, 12, 13, 16

We have defined DFT with the help of some examples. A signal $x(n)$ and its DFT $X(k)$ is represented as a pair by shorthand notation as,

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

Here 'N' represents 'N' point DFT. Various properties of DFT are described in this subsection.

3.3.1 Periodicity

Let $x(n)$ and $X(k)$ be the DFT pair.

Then, if $x(n+N) = x(n)$ for all n, then

$$\therefore \quad \boxed{X(k+N) = X(k)} \quad \text{for all k} \qquad \ldots (3.3.1)$$

Proof :

Step 1 : By definition DFT is given as,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_n^{kn} \qquad \ldots(3.3.2)$$

Step : 2 Let us replace k by k + N, then above equation becomes,

$$X(k+N) = \sum_{n=0}^{N-1} x(n) W_N^{(k+N)n} = \sum_{n=0}^{N-1} x(n) W_N^{kn} \cdot W_N^{Nn} \qquad \ldots (3.3.3)$$

Step 3 : We know that, $W_N = e^{-j\frac{2\pi}{N}}$

$$\therefore \quad W_N^{Nn} = e^{-j\frac{2\pi}{N}\cdot Nn} = e^{-j2\pi n} = \cos(2\pi\, n) - j \sin 2\pi\, n$$

$$= 1 - j\,0 = 1 \text{ always}$$

This is because for all values of 'n', $\cos(2\pi\, n) = 1$ and $\sin(2\pi\, n) = 0$. Hence equation 3.3.3 becomes,

$$X(k+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= X(k) \qquad \text{by equation 3.3.2}$$

Similarly by using IDFT formula it can be shown that $x(N+n) = x(n)$.

3.3.2 Linearity

The linearity property of DFT states that

if $\quad x_1(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_1(k)$

and $\quad x_2(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_2(k)$ then,

$$\boxed{a_1\, x_1(n) + a_2\, x_2(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} a_1\, X_1(k) + a_2\, X_2(k)} \qquad \text{... (3.3.4)}$$

Here a_1 and a_2 are constants.

Proof :

Step : 1 By definition of DFT, we can write,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

Step : 2 Let $x(n) = a_1\, x_1(n) + a_2\, x_2(n)$, then above equation becomes,

$$X(k) = \sum_{n=0}^{N-1} \left[a_1\, x_1(n) + a_2\, x_2(n)\right] W_N^{kn} = \sum_{n=0}^{N-1} a_1\, x_1(n)\, W_N^{kn} + \sum_{n=0}^{N-1} a_2\, x_2(n)\, W_N^{kn}$$

$$= a_1 \sum_{n=0}^{N-1} x_1(n)\, W_N^{kn} + a_2 \sum_{n=0}^{N-1} x_2(n)\, W_N^{kn}$$

$$= a_1\, X_1(k) + a_2\, X_2(k)$$

3.3.3 Circular Symmetries of a Sequence

- Till now we have seen that $x(n)$ is the sequence containing 'N' samples. And $X(k)$ is the 'N' point DFT. When we take IDFT we get a periodic sequence $x_p(n)$ which is related to $x(n)$ as,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \quad \text{... (3.3.5)}$$

Thus $x_p(n)$ is nothing but periodic repetition of $x(n)$. And $x_p(n)$ contains such infinite periodic repetitions. On the basis of this conclusion we can say that if we take DFT of $x_p(n)$ we get the same DFT $X(k)$ which is obtained due to $x(n)$. i.e.,

Key Point:

$$\left.\begin{aligned} x(n) &\overset{DFT}{\underset{N}{\longleftrightarrow}} X(k) \\ x_p(n) &\overset{DFT}{\underset{N}{\longleftrightarrow}} X(k) \end{aligned}\right\} \textit{Thus both sequences have same DFT} \quad \begin{aligned}&\text{...(3.3.6)}\\ &\text{...(3.3.7)}\end{aligned}$$

And $x(n)$ and $x_p(n)$ are related by equation 3.3.5.

- Or we can write,

$$x(n) = \begin{cases} x_p(n) & \text{for } 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{... (3.3.8)}$$

- Now let $x_p(n)$ be shifted by 'k' units to the right. Let this new sequence be $x'_p(n)$. i.e.

$$x'_p(n) = x_p(n-k) \quad \text{... (3.3.9)} \quad = \sum_{l=-\infty}^{\infty} x(n-k-lN) \quad \text{... (3.3.10)}$$

- Then the corresponding sequence $x'(n)$ can be obtained from equation 3.3.8 as,

$$x'(n) = \begin{cases} x'_p(n) & \text{for } 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{... (3.3.11)}$$

This sequence $x'(n)$ is related to $x(n)$ by the circular shift. This concept is illustrated in Fig. 3.3.1. Fig. 3.3.1 (a) shows sequence $x(n)$ containing N = 4 samples. The periodic repetition of $x(n)$ is $x_p(n)$ and this sequence is shown in Fig. 3.3.1 (b). Fig. 3.3.1 (c) shows the sequence $x'_p(n)$ which is obtained by shifting $x_p(n)$ to the right by two samples. Fig. 3.3.1 (d) shows sequence $x'(n)$ as per equation 3.3.11. The sequence $x'(n)$ is basically one period of $x'_p(n)$ from $0 \le n \le 3$.

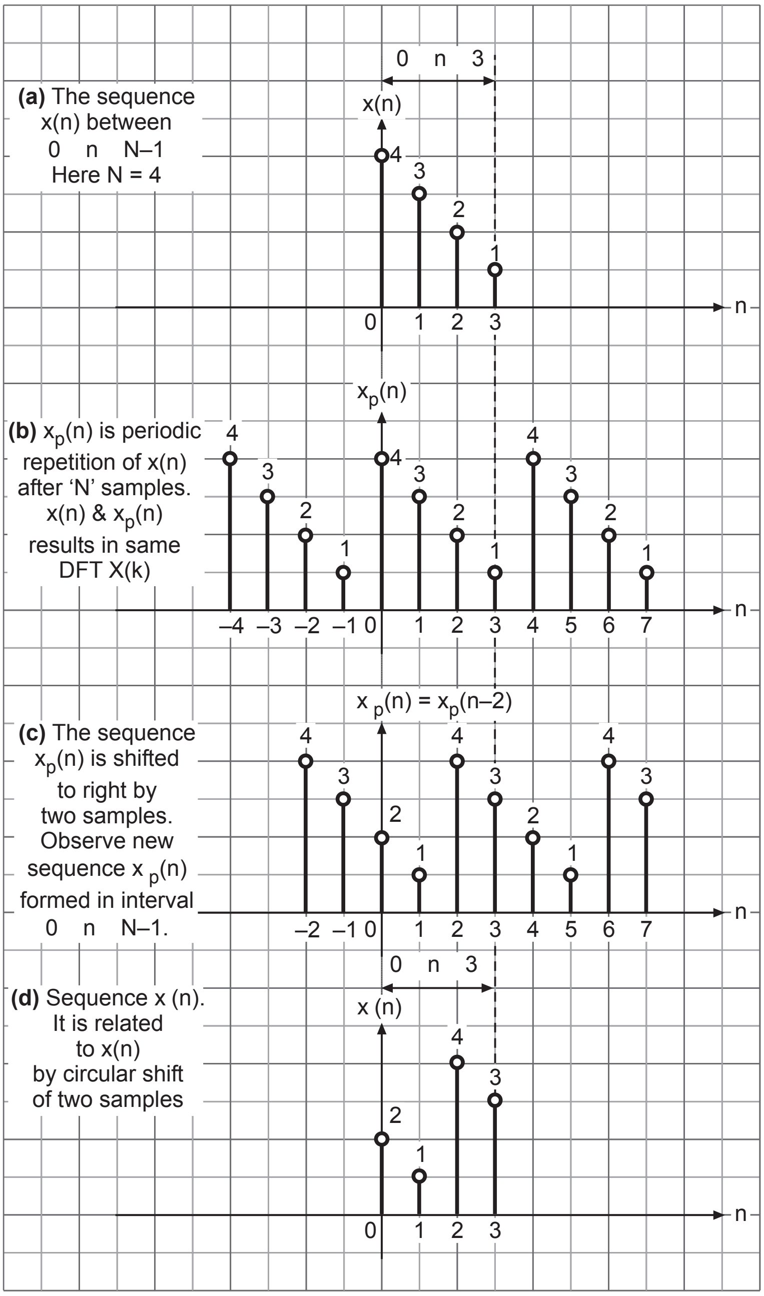


Fig. 3.3.1 Circular shift of sequence $x\,(n)$. Here $x'(n)$ is obtained by shifting $x\,(n)$ circularly by two samples

- The sequence $x'(n)$ is related to $x(n)$ by a circular shift. It is represented as follows :

$$x'(n) \;=\; x(n-k,\ \text{modulo}\ \ \text{N}) \qquad \ldots (3.3.12)$$

The shorthand notation for $(n-k$, modulo N) is $((n-k))_N$ i.e.,

$$x'(n) = x((n-k))_N \qquad \dots (3.3.13)$$

Now let us evaluate samples of $x'(n)$ as per above equation with $k=2$ and $N=4$. Then above equation becomes,

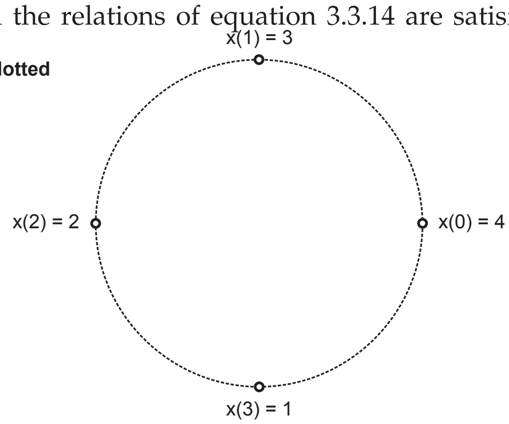
$$x'(n) = x((n-2))_4$$

Hence,

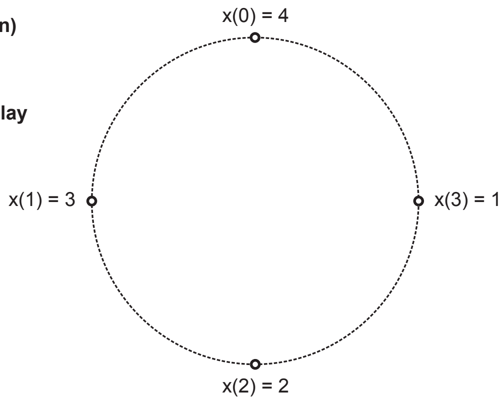
$$\left.\begin{aligned} x'(0) &= x((-2))_4 = x(2) \\ x'(1) &= x((-1))_4 = x(3) \\ x'(2) &= x((0))_4 = x(0) \\ x'(3) &= x((1))_4 = x(1) \end{aligned}\right\} \qquad \dots (3.3.14)$$

The above relation indicates that $x'(n)$ is obtained by shifting $x(n)$ circularly by two samples. Such relationship can be better understood by plotting $x(n)$ anticlockwise along the circle as shown in Fig. 3.3.2 (a). Fig. 3.3.2 (b) indicates the sequence $x(n)$ delayed circularly by one sample, i.e. $x((n-1))_4$. This shift is anticlockwise. Fig. 3.3.2 (c) indicates $x(n)$ delayed circularly by two samples, i.e. $x((n-2))_4$. This is sequence $x'(n)$ as per equation 3.3.14. Observe that all the relations of equation 3.3.14 are satisfied.

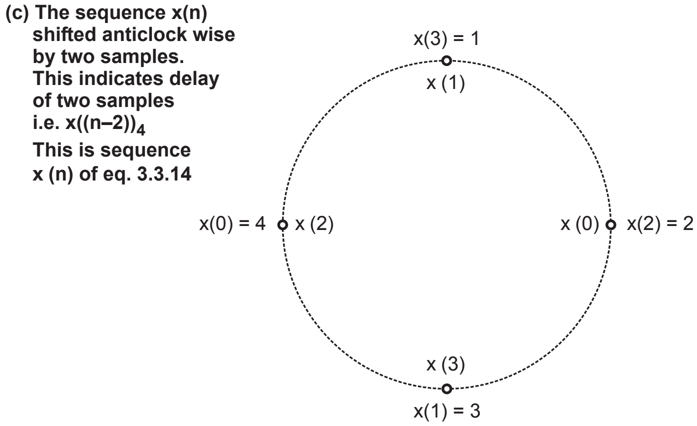
(a) The samples of sequence x(n) plotted circularly - anticlock wise i.e. $x((n))_4$



(b) The sequence x(n) shifted circularly by one sample anticlock wise. This indicates delay of one sample i.e. $x((n-1))_4$



... Continued on next page



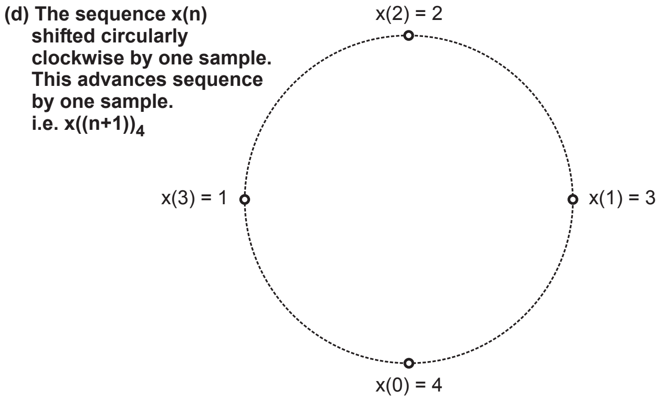


Fig. 3.3.2 Circular shifts of the sequence $x\,((n-k))_N$ indicates shifting sequence $x\,(n)$ anticlockwise by 'k' samples. And $((n-k))_N$ means shifting sequence $x\,(n)$ clockwise by 'k' samples

Fig. 3.3.2 (d) indicates the sequence $x(n)$ advanced by one sample, i.e. $x\big((n+1)\big)_4$. This shift is clockwise. .

Circularly even sequence :

A sequence is said to be circularly even if it is symmetric about the point zero on the circle i.e.,

$$\textbf{Definition :}\ x\,(N-n) = x\,(n) \qquad 1 \le n \le N-1 \qquad \text{... (3.3.15)}$$

Consider the sequence $x\,(n) = \{4,\ 6,\ 8,\ 6\}$

Here $x\,(4-1) = x\,(1)$ i.e. $x\,(3) = x\,(1)$

$x\,(4-2) = x\,(2)$ i.e. $x\,(2) = x\,(2)$

$x\,(4-3) = x\,(3)$ i.e. $x\,(1) = x\,(3)$

Thus the sequence is circularly even. Fig. 3.3.3 shows that this sequence is symmetric about the point zero on the circle.

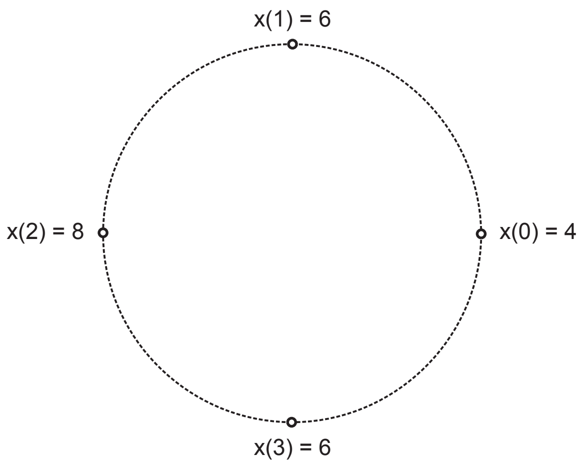


Fig. 3.3.3 Circularly even sequence. This sequence is symmetric around *x* (0)

Circularly odd sequence :

A sequence is said to be circularly odd if it is antisymmetric about point x (0) on the circle. i.e.

$$\textbf{Definition : } x\,(N-n) = -x\,(n) \qquad 1 \le n \le N-1 \qquad \dots (3.3.16)$$

Consider the sequence $x\,(n) = \{4,\ -6,\ 8,\ 6\}$

Here
$$x\,(4-1) = -x\,(1) \quad \text{i.e.} \quad x\,(3) = -x\,(1)$$
$$x\,(4-2) = -x\,(2) \quad \text{i.e.} \quad x\,(2) = -x\,(2)$$
$$x\,(4-3) = -x\,(3) \quad \text{i.e.} \quad x\,(1) = -x\,(3)$$

Thus the given sequence is circularly odd.

Circularly folded sequence :

A circularly folded sequence is represented as $x\left((-n)\right)_N$. It is obtained by plotting x (n) in clockwise direction along the circle. It is represented as,

$$\textbf{Definition : } x\left((-n)\right)_N = x\,(N-n) \qquad 0 \le n \le N-1 \qquad \dots (3.3.17)$$

Fig. 3.3.4 (a) shows the 4 point sequence x (n).

The samples of x (n) are plotted anticlockwise in Fig. 3.3.4 (a). Fig. 3.3.4 (b) shows circularly folded sequence $x\,((-n))_4$. In this figure, x (n) is circularly folded.

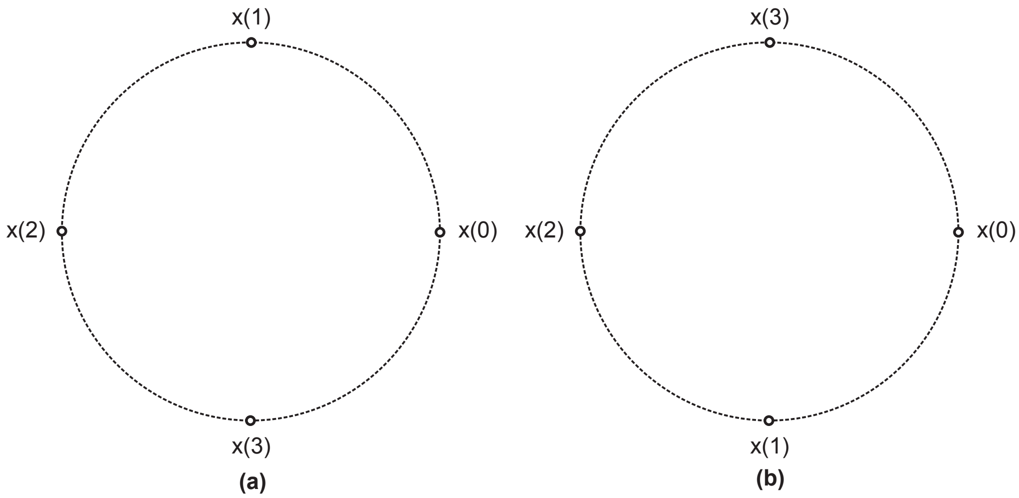


Fig. 3.3.4 (a) The sequence $x(n)$ plotted across the circle
(b) The sequence $x(n)$ folded circularly to get $x((-n))_4$

Hence the samples of $x(n)$ are plotted clockwise in Fig. 3.3.4 (b). The circular folding and shifting operations are useful in circular convolution which will be discussed next. These operations are summarized below in Table 3.3.1.

$x((n))_N$	N-point sequence plotted across the circle anticlockwise. Anticlockwise means positive direction.
$x((n-k))_N$	Sequence $x(n)$ shifted anticlockwise (positive direction) by 'k' samples. Anticlockwise shift indicates delay.
$x((n+k))_N$	Sequence $x(n)$ shifted clockwise (negative direction) by 'k' samples. Clockwise shift indicates advancing operation.
$x((-n))_N$	Circular folding. Sequence $x(n)$ plotted across circle in clockwise direction. i.e. negative direction.

Table 3.3.1 : Circular properties of sequences

3.3.4 Symmetry Properties

Let the sequence $x(n)$ be complex valued and expressed as,

$$x(n) = X_R(n) + j\,x_I(n), \qquad 0 \le n \le N-1 \qquad \text{... (3.3.18)}$$

Let the DFT of $x(n)$ be complex valued and expressed as,

$$X(k) = X_R(n) + j\,X_I(n), \qquad 0 \le k \le N-1 \qquad \text{... (3.3.19)}$$

By definition of DFT,
$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad \text{since } W_N = e^{-j\frac{2\pi}{N}}$$

$$= \sum_{n=0}^{N-1} \left[x_R(n) + j\, x_I(n)\right] e^{-j2\pi kn/N} \quad \text{by putting for } x(n)$$

$$= \sum_{n=0}^{N-1} \left[x_R(n) + j\, x_I(n)\right]\left[\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$= \sum_{n=0}^{N-1}\left[x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$-j\sum_{n=0}^{N-1}\left[x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cos\left(\frac{2\pi kn}{N}\right)\right] \quad \text{... (3.3.20)}$$

We know that $X(k) = X_R(n) + j X_I(n)$. Hence comparing with above equation we obtain real and imaginary parts of $X(k)$ as follows.

$$X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right] \quad \text{... (3.3.21)}$$

and
$$X_I(k) = -\sum_{n=0}^{N-1}\left[x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cos\left(\frac{2\pi kn}{N}\right)\right] \quad \text{... (3.3.22)}$$

Similarly the real and imaginary parts of sequence $x(n)$ can be obtained in terms of its DFTs as follows (using IDFT formula) :

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\cos\left(\frac{2\pi kn}{N}\right) - X_I(k)\sin\left(\frac{2\pi kn}{N}\right)\right] \quad \text{... (3.3.23)}$$

and
$$x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\sin\left(\frac{2\pi kn}{N}\right) + X_I(k)\cos\left(\frac{2\pi kn}{N}\right)\right] \quad \text{... (3.3.24)}$$

(i) Symmetry property for real valued $x(n)$:

This property states that if $x(n)$ is real, then

$$\boxed{X(N-k) = X^*(k) = X(-k)} \quad \text{... (3.3.25)}$$

Proof :

By definition of DFT we have,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad \text{... (3.3.26)}$$

Now $X(N-k)$ can be obtained as,

$$X(N-k) = \sum_{n=0}^{N-1} x(n) W_N^{(N-k)n} = \sum_{n=0}^{N-1} x(n) W_N^{Nn} \cdot W_N^{-kn}$$

Here $W_N^{Nn} = e^{-j\frac{2\pi}{N}\cdot Nn} = e^{-j2\pi n} = cos(2\pi n) - j\, sin(2\pi n)$

$= 1$ always

$$\therefore \quad X(N-k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn}$$

$$= X(-k) \qquad \text{... (3.3.27)}$$

Also complex conjugate of $X(k)$, i.e. $X^*(k)$ is obtained from equation 3.3.26 as,

$$X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn} \qquad \text{... (3.3.28)}$$

Thus from above equation and equation 3.3.27,

$$X(N-k) = X(-k) = X^*(k)$$

(ii) Real and even sequence :

This property states that if the sequence is real and even i.e.,

$$x(n) = x(N-n) \qquad 0 \le n \le N-1$$

then DFT becomes,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, cos\left(\frac{2\pi\, kn}{N}\right), \quad 0 \le k \le N-1 \qquad \text{... (3.3.29)}$$

The above equation is obtained from equation 3.3.21 since $x_I(n) = 0$ and $X(k) = X_R(k)$. $X_I(k)$ of equation 3.3.22 is zero since $x_I(n) = 0$ and 'sin' function is odd function. This can be verified by taking an example.

Similarly $x(n)$ can be obtained from equation 3.3.23,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cos\left(\frac{2\pi kn}{N}\right), \quad 0 \le n \le N-1 \qquad \text{... (3.3.30)}$$

And $x_I(n)$ of equation 3.3.24 will be zero since $X_I(k)$ is zero and 'sin' function is odd function.

(iii) Real and odd sequence :

A real and odd sequence is expressed as,

$$x(n) = -x(N-n), \qquad 0 \le n \le N-1$$

Then DFT of such sequence becomes,

$$X(k) = -j\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right), \quad 0 \le k \le N-1 \qquad \text{... (3.3.31)}$$

For real and odd sequence $X_R(k)$ of equation 3.3.21 will be zero. This is because $x_I(n)$ is zero since sequence is real, and terms of $x_R(n)\cos\left(\frac{2\pi kn}{N}\right)$ in equation 3.3.21 cancel each other since the sequence $x_R(n)$ is odd and 'cosine' is even function. Equation 3.3.31 is obtained from equation 3.3.22 with $X(k) = X_I(k)$ and $x_I(n)$ is zero. Similarly IDFT is given from equation 3.3.23 as,

$$x(n) = j\frac{1}{N}\sum_{k=0}^{N-1} X(k)\sin\left(\frac{2\pi kn}{N}\right), \ 0 \le n \le N-1 \qquad \text{... (3.3.32)}$$

(iv) Purely imaginary sequence :

The purely imaginary sequence is given as,

$$x(n) = j\, x_I(n)$$

Then DFT can be obtained from equation 3.3.21 and 3.3.22 as,

$$\left.\begin{aligned} X_R(k) &= \sum_{n=0}^{N-1} x_I(n)\sin\left(\frac{2\pi kn}{N}\right) \\ \text{and} \qquad X_I(k) &= \sum_{n=0}^{N-1} x_I(n)\cos\left(\frac{2\pi kn}{N}\right) \end{aligned}\right\} \qquad \text{... (3.3.33)}$$

Here if $x_I(n)$ is odd, then $X_I(k)$ becomes zero since 'cos' is even function. Similarly if $x_I(n)$ is even, then $X_R(k)$ becomes zero. Since 'sin' is odd function.

3.3.5 Circular Convolution

This property states that if

$$x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$$

and $$x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$$

Then $$\boxed{x_1(n) \textcircled{N} x_2(n) \xleftrightarrow[N]{DFT} X_1(k)\, X_2(k)} \quad \text{... (3.3.34)}$$

Here $x_1(n)$ $x_2(n)$ means circular convolution of $x_1(n)$ and $x_2(n)$. This property states that multiplication of two DFTs is equivalent to circular convolution of their sequences in time domain.

Proof :

Step 1 : By definition two DFTs, $X_1(k)$ and $X_2(k)$ are given as,

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N}, \qquad k = 0, 1, \ldots N-1 \quad \text{... (3.3.35)}$$

and $$X_2(k) = \sum_{l=0}^{N-1} x_2(l)\, e^{-j2\pi kl/N}, \qquad k = 0, 1, \ldots N-1 \quad \text{... (3.3.36)}$$

Here observe that we have used different indices for summations. This is because when $X_1(k)$ and $X_2(k)$ are multiplied those indices of summations are independent and different.

Step 2 : Let $X_3(k)$ be equal to multiplication of $X_1(k)$ and $X_2(k)$, i.e.,

$$X_3(k) = X_1(k) \cdot X_2(k) \quad \text{... (3.3.37)}$$

Step 3 : Let $x_3(m)$ be the sequence whose DFT is $X_3(k)$. Then $x_3(m)$ can be obtained from $X_3(k)$ by taking IDFT. i.e.,

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k)\, e^{j2\pi km/N}$$

Here observe that we have used index 'm' for $x_3(\ )$ to make it different from $x_1(\ )$ and $x_2(\ )$.

Step 4 : Putting for $X_3(k)$ in above equation from equation 3.3.37 we get,

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \cdot X_2(k)\, e^{j2\pi km/N} \quad \text{... (3.3.38)}$$

Step 5 : Let us substitute for $X_1(k)$ and $X_2(k)$ in above equation from equation 3.3.35 and 3.3.36,

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N}\right]\left[\sum_{l=0}^{N-1} x_2(l)\, e^{-j2\pi kl/N}\right] e^{j2\pi km/N}$$

Here observe that all the three summations have different indices, since they are independent. Rearranging the summations and terms in above equation as follows,

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left\{\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right\} \quad ...(3.3.39)$$

Step 6 : Here let us consider the following standard relation,

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & \text{for } a=1 \\ \dfrac{1-a^N}{1-a} & \text{for } a \neq 1 \end{cases} \quad ... (3.3.40)$$

Now let $\quad a = e^{j2\pi(m-n-l)/N}$

Here when $(m-n-l) = N, 2N, 3N, ...$ i.e. multiple of N; a = 1. This is because,

$$a^{j2\pi N/N} = e^{j2\pi\, 2N/N} = e^{j2\pi\, 3N/N} = ... = 1$$

Thus in equation 3.3.40 we can write (for a = 1 i.e. first condition),

$$\sum_{k=0}^{N-1} a^k = \sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N} = N, \quad \text{When } (m-n-l) \text{ is multiple of N} \quad ...(3.3.41)$$

Now consider the second condition in equation 3.3.40 i.e. $a \neq 1$. Then we have,

$$\sum_{k=0}^{N-1} a^k = \sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}$$

$$= \frac{1-a^N}{1-a} \quad \text{when } a \neq 1 \text{ i.e. when } (m-n-l) \text{ is not multiple of N.}$$

$$= \frac{1-e^{j2\pi k(m-n-l)}}{1-e^{j2\pi k(m-n-l)/N}}$$

Here $e^{j2\pi k(m-n-l)} = 1$ always. Hence above equation becomes,

$$\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N} = \frac{1-1}{1-e^{j2\pi k(m-n-l)/N}}$$

$$= 0 \qquad \text{when } (m-n-l) \text{ is not multiple of N.} \qquad \text{... (3.3.42)}$$

Thus we obtained from above equation and equation 3.3.41 as,

$$\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N} = \begin{cases} N & \text{when } (m-n-l) \text{ is multiple of N} \\ 0 & \text{otherwise} \end{cases} \qquad \text{... (3.3.43)}$$

Putting this value in equation 3.3.39 we get,

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \cdot N \qquad \text{When (m − n − l) is multiple of N}$$

$$= \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \qquad \text{When (m − n − l) is multiple of N} \qquad \text{... (3.3.44)}$$

Step 7 : In the above equation $(m-n-l)$ multiple of N can be written as,

$$(m-n-l) = pN \qquad \text{Here p is some integer.}$$

Since integer multiple can be positive or negative both, we can write above condition for convenience as,

$$m-n-l = -pN$$

$$\therefore \qquad l = m-n+pN \qquad \text{... (3.3.45)}$$

When 'l' is given by above equation, the condition of equation 3.3.44 is satisfied. Hence putting for 'l' from above equation in equation 3.3.44 we get,

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n+pN) \qquad \text{... (3.3.46)}$$

Here observe that the summation for $x_2(\ )$ is dropped since it is redundant because of change of index. There is no 'l' term in above equation.

In the above equation $x_2(m-n+pN)$ represents it is periodic sequence with period N. This periodic sequence is delayed by 'n' samples. Such type of sequences we have treated in the begining of section 3.3.3.

$x_2(m-n+pN)$ represents sequence $x_2(m)$ shifted circularly by 'n' samples. This concept is explained in Fig. 3.3.1. Such sequence is represented by equation 3.3.12 as,

$$x_2(m-n+pN) = x_2(m-n, \text{ modulo N}) \qquad \text{... (3.3.47)}$$

or by equation 3.3.13 above sequence can be represented as,

$$x_2(m-n+pN) = x_2((m-n))_N \quad \text{... (3.3.48)}$$

Putting this sequence in equation 3.3.46 we get,

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2\left((m-n)\right)_N, \quad m = 0, 1, \ldots N-1 \qquad \ldots (3.3.49)$$

Let us compare the above equation with linear convolution which is given as,

$$y(n) = \sum_{k=\infty}^{\infty} x(k)\, h(n-k)$$

Observe that equation 3.3.49 appears like convolution operation. But sequence $x_2(\ )$ is shifted circularly in equation 3.3.49. Hence it is called circular convolution. The circular convolution takes place because of circular shift of sequence $x_2(\ )$.

Thus the property of circular convolution is proved. Circular convolution of $x_1(n)$ and $x_2(n)$ is denoted by $x_1(n)$ (N) $x_2(n)$ and it is given by equation 3.3.49 as,

$$x_3(m) = x_1(n) \text{ (N) } x_2(n) = \sum_{n=0}^{N-1} x_1(n)\, x_2\left((m-n)\right)_N, \ \text{m = 0, 1, ... N – 1} \qquad \ldots (3.3.50)$$

Thus circular convolution property is proved.

The comparison of linear, periodic and circular convolutions is shown in Table 3.3.2.

Sr. No.	Parameter	Linear convolution	Periodic convolution	Circular convolution
1	Time domain equation	$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$ OR $x_3(n) = \sum_{m=-\infty}^{\infty} x_1(m)\, x_1(n-m)$	$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2(n-m)$	$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N$ $m = 0, 1, \ldots N-1$
2	Frequency domain equation	$x_1(n) * x_2(n) \overset{FT}{\longleftrightarrow} X_1(\omega) \cdot X_2(\omega)$	$c_3(k) = c_1(k) \cdot c_2(k)$	$x_1(n)$ (N) $x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_1(k)\, X_2(k)$
3	Range of time index	$-\infty \le n \le \infty$	$0 \le m \le N-1$	$0 \le m \le N-1$
4	Range of frequency index	$0 \le \omega \le 2\pi$	$0 \le k \le N-1$	$0 \le k \le N-1$

5	Type of sequences	Sequences are non periodic and must be of finite length.	The sequences are periodic.	The sequences are of length 'N'. They may be periodic.
6	Shifting of sequences	Sequences are shifted linearly.	Sequences are shifted linearly.	Sequences are shifted circularly.
7	Convolution sum	Convolution sum is of infinite length.	Convolution sum is of length 'N'.	Convolution sum is of length 'N'.
8	Name of the transform domain	Fourier transform.	Discrete fourier series.	Discrete fourier transform.

Table 3.3.2 Comparison of linear convolution, periodic convolution and circular convolution

Example 3.3.1 *The two sequences* $x_1(n)$ and $x_2(n)$ *are given as follows :*

$$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

and $$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

Find out the sequence $x_3(m)$ *which is equal to circular convolution of above two sequences. i.e.*

$$x_3(m) = x_1(n) \; Ⓝ \; x_2(n)$$

AU : May-14, Marks 4

Solution : Here we have to perform circular convolution of $x_1(n)$ and $x_2(n)$ which is given by equation 3.3.50 as (with N = 4),

$$x_3(m) = \sum_{n=0}^{3} x_1(n) \cdot x_2((m-n))_4, \qquad m = 0, 1, 2, 3.... \qquad ... (3.3.51)$$

To find x_3 (0), Put m = 0 in above equation :

Putting m = 0 in above equation we get

$$x_3(0) = \sum_{n=0}^{3} x_1(n)\, x_2((-n))_4, \qquad ... (3.3.52)$$

Fig. 3.3.5 (a) shows sequence $x_1(n)$ plotted across the circle anticlockwise. Fig. 3.3.5 (b) shows $x_2(n)$ plotted across the circle anticlockwise. The sequence $x_2((-n))_4$ is obtained by circular folding of sequence $x_2(n)$. This is obtained by plotting $x_2(n)$ across the circle in clockwise direction as shown in Fig. 3.3.5 (c).

Please refer Fig. 3.3.5 on next page.

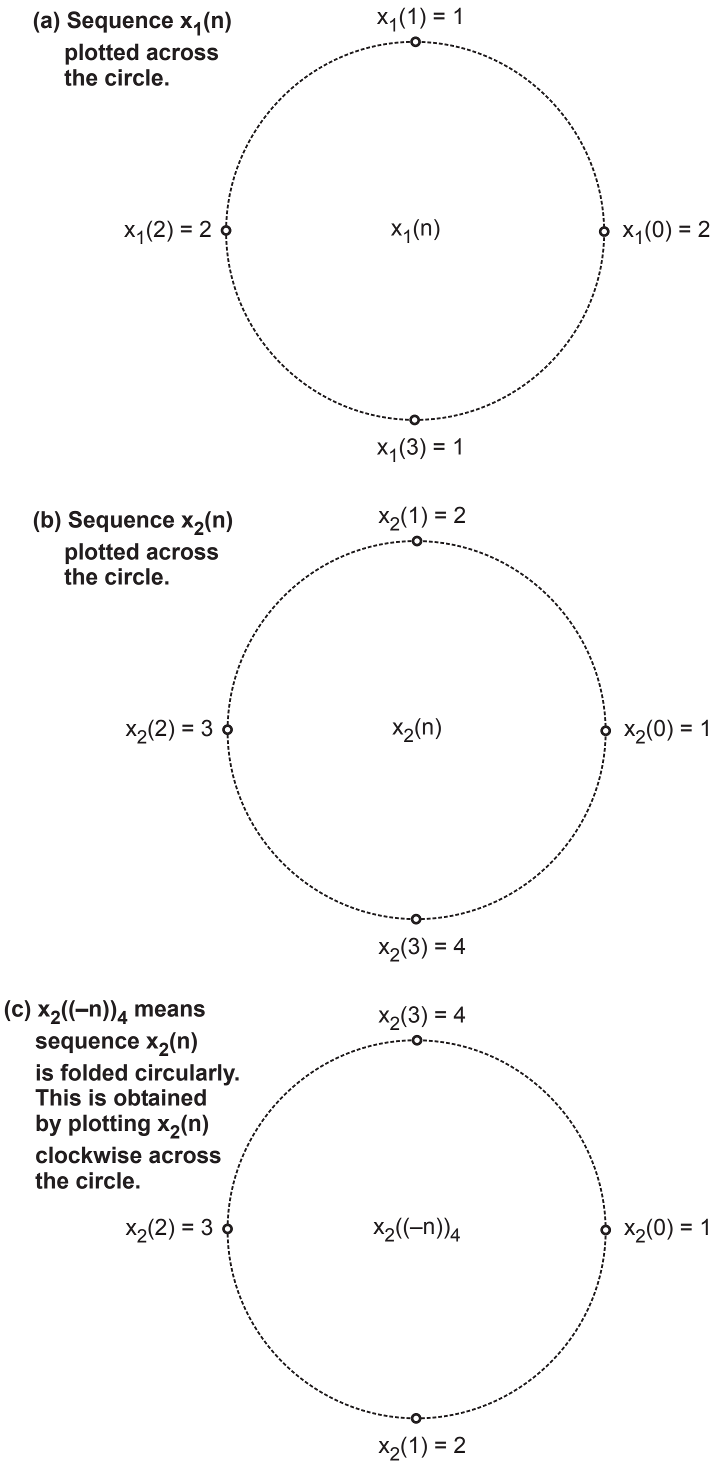

(a) Sequence $x_1(n)$ plotted across the circle.
$x_1(1) = 1$
$x_1(2) = 2$
$x_1(n)$
$x_1(0) = 2$
$x_1(3) = 1$
(b) Sequence $x_2(n)$ plotted across the circle.
$x_2(1) = 2$
$x_2(2) = 3$
$x_2(n)$
$x_2(0) = 1$
$x_2(3) = 4$
(c) $x_2((-n))_4$ means sequence $x_2(n)$ is folded circularly. This is obtained by plotting $x_2(n)$ clockwise across the circle.
$x_2(3) = 4$
$x_2(2) = 3$
$x_2((-n))_4$
$x_2(0) = 1$
$x_2(1) = 2$

Fig. 3.3.5

Fig. 3.3.6 shows $x_1(n)$ of Fig. 3.3.5 (a) and $x_2((-n))_4$ of Fig. 3.3.5 (c) plotted on two concentric circles. $x_1(n)$ is plotted on inner circle and $x_2((-n))_4$ is plotted on outer circle. The individual values of product $x_1(n)\,x_2((-n))_4$ are obtained by multiplying those two sequences point by point as shown in Fig. 3.3.6. And finally $x_3(0)$ of equation 3.3.52 is obtained by adding all these products i.e.,

$$x_3(0) = 2+4+6+2$$

$$= 14 \qquad \text{Thus} \quad \boxed{x_3(0) = 14}$$

Because of concentric representation of Fig. 3.3.6, the products can be calculated easily.

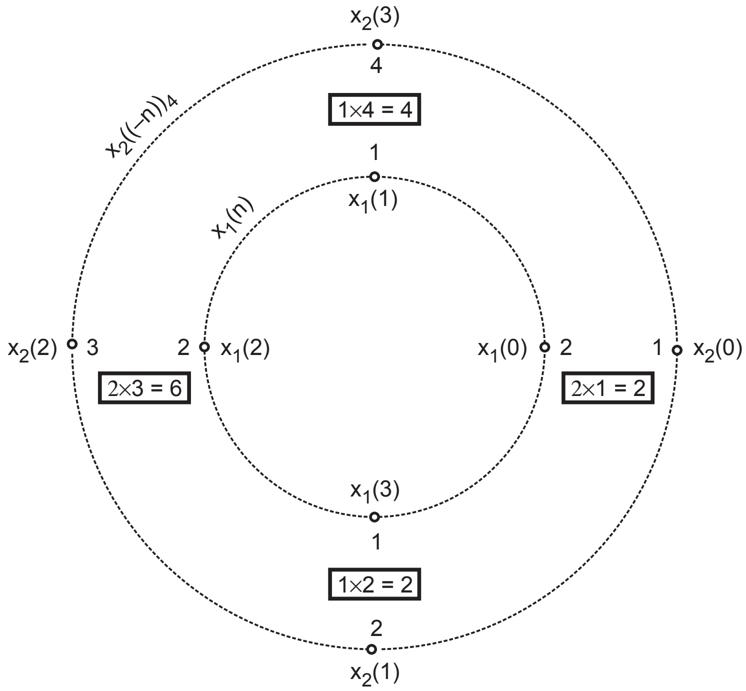


Fig. 3.3.6 $x_1(n)$ and $x_2((-n))_4$ plotted on two concentric circles

To obtain $x_3(1)$, put m = 1 in equation 3.3.51 :

Putting m = 1 in equation 3.3.51 we get,

$$x_3(1) = \sum_{n=0}^{3} x_1(n)\,x_2((1-n))_4 \qquad \text{... (3.3.53)}$$

Here the sequence $x_2((1-n))_4$ can be written as $x_2((-[n-1]))_4$. This means $x_2((-n))_4$ delayed by one sample. The delay of one sample is equivalent to shifting $x_2((-n))_4$ anticlockwise by one sample. [See equation 3.3.13 and its relevant description). Thus $x_2((1-n))$ is obtained by shifting $x_2((-n))_4$ anticlockwise by one position or sample. This sequence is shown in Fig. 3.3.7 (b). Fig. 3.3.7 (a) shows $x_2\left((-n)\right)_4$ for reference.

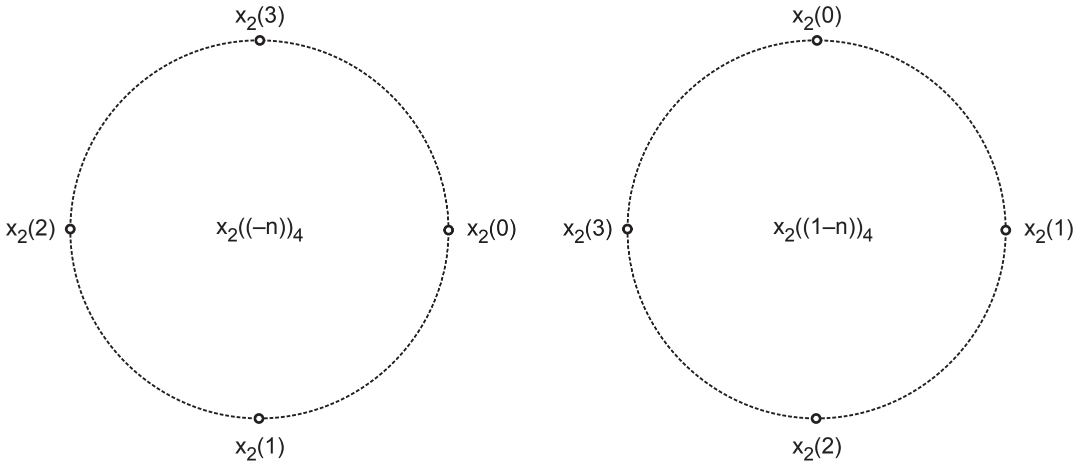


Fig. 3.3.7 (a) The sequence of x_2 $((-n))_4$ (b) The sequence x_2 $((1-n))_4$ is obtained by shifting x_2 $((-n))_4$ anticlockwise by one sample position

Now we have to obtain products $x_1(n)\ x_2\left((1-n)\right)_4$ and their sum. This is obtained easily by plotting $x_1(n)$ and $x_2\left((1-n)\right)_4$ on concentric circles as shown in Fig. 3.3.8. The

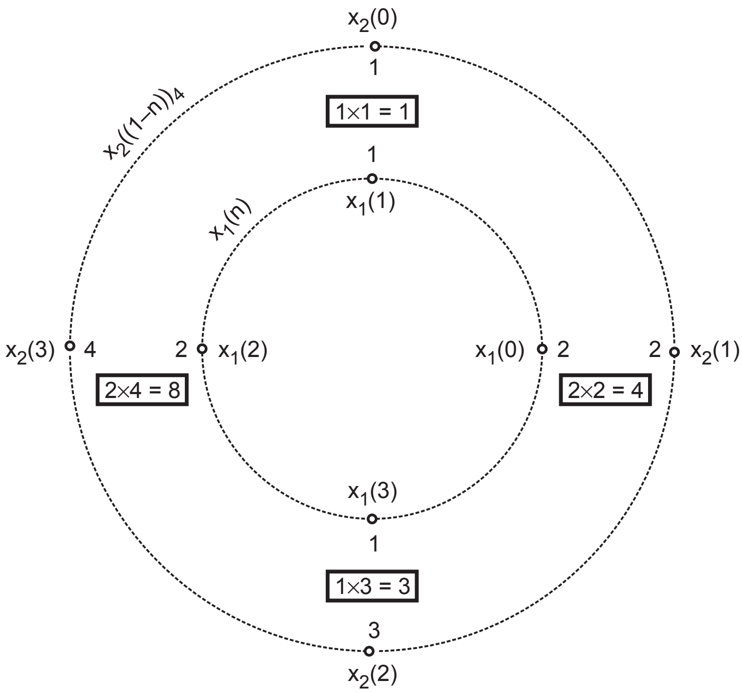


Fig. 3.3.8 Sequences $x_1(n)$ plotted on inner circle and x_2 $((1-n))_4$

point by point products of the two sequences are also shown in Fig. 3.3.8. Hence $x_3(1)$ of equation 3.3.53 becomes,

$$x_3(1) = 4 + 1 + 8 + 3$$

$$= 16$$

Thus $\boxed{x_3(1) = 16}$

To find $x_3(2)$, put m = 2 in equation 3.3.51 :

Putting m = 2 in equation 3.3.51 we get

$$x_3(2) = \sum_{n=0}^{3} x_1(n)\, x_2\left((2-n)\right)_4 \quad \text{... (3.3.54)}$$

We have seen earlier that $x_2\left((1-n)\right)_4$ is obtained by shifting $x_2\left((-n)\right)_4$ anticlockwise by one sample position. Hence $x_2\left((2-n)\right)_4$ is obtained by shifting $x_2\left((1-n)\right)_4$ anticlockwise by one sample position. This is equivalent to shifting $x_2\left((-n)\right)_4$ anticlockwise by two sample positions. Fig. 3.3.9 (b) shows $x_2\left((2-n)\right)_4$. Fig. 3.3.9 (a) shows $x_2\left((1-n)\right)_4$ for reference.

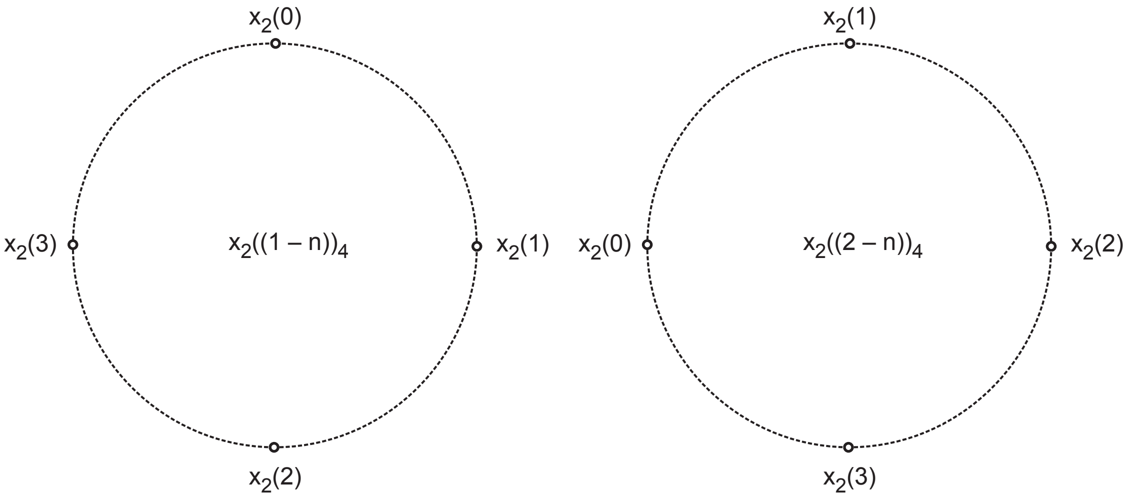


Fig. 3.3.9 (a) The sequence of $x_2((1-n))_4$ (b) The sequence $x_2((2-n))_4$ is obtained by shifting $x_2((1-n))_4$

Now we have to obtain the products $x_1(n)\, x_2\left((2-n)\right)_4$ and their sum. This is obtained easily by plotting $x_1(n)$ and $x_2\left((2-n)\right)_4$ on concentric circles as shown in Fig. 3.3.10. The point by point products are also shown in the figure.

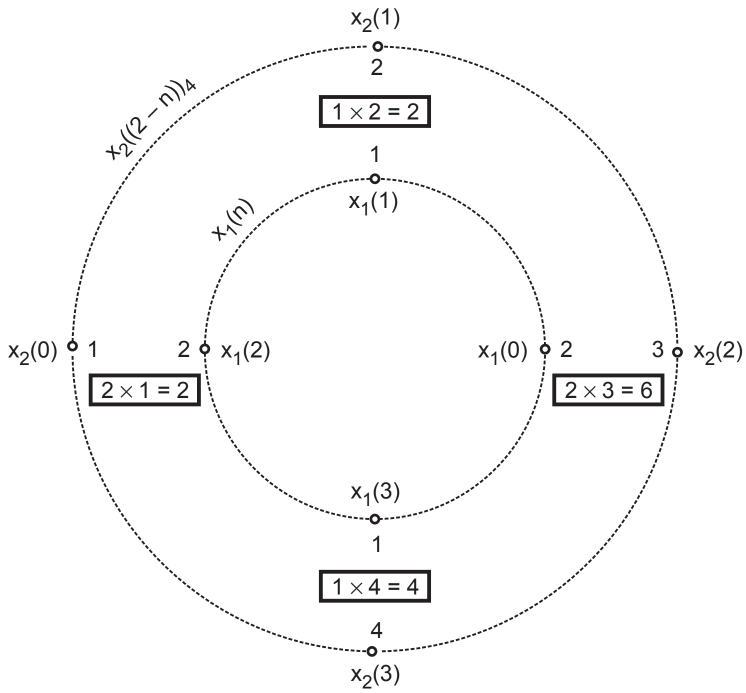


Fig. 3.3.10 Sequence $x(n)$ plotted on inner circle and $x_2((2-n))_4$ plotted on outer circle

Thus $x_3(2)$ of equation 3.3.54 can be obtained from above figure as,

$$x_3(2) = 6+2+2+4$$

$$= 14$$

Thus $\boxed{x_3(2) = 14}$

To find $x_3(3)$, put m = 3 in equation 3.3.51 :

Putting m = 3 in equation 3.3.51 we get,

$$x_3(3) = \sum_{n=0}^{3} x_1(n)\, x_2\left((3-n)\right)_4 \qquad \text{... (3.3.55)}$$

Here $x_2\left((3-n)\right)_4$ is obtained by shifting $x_2\left((2-n)\right)_4$ anticlockwise by one sample position. This is equivalent to shifting $x_2\left((-n)\right)_4$ by three sample positions. Now here let us use the sequences plotted on concentric circles in Fig. 3.3.10. Sequence $x_1(n)$ is plotted on inner circle, and it will remain as it is. This sequence is shown in Fig. 3.3.11 (inner circle). The $x_2\left((2-n)\right)_4$ plotted on outer circle in Fig. 3.3.10 should be shifted anticlockwise by one sample position to get $7x_2\left((3-n)\right)_4$. This new shifted sequence is shown in Fig. 3.3.11 on outer circle.

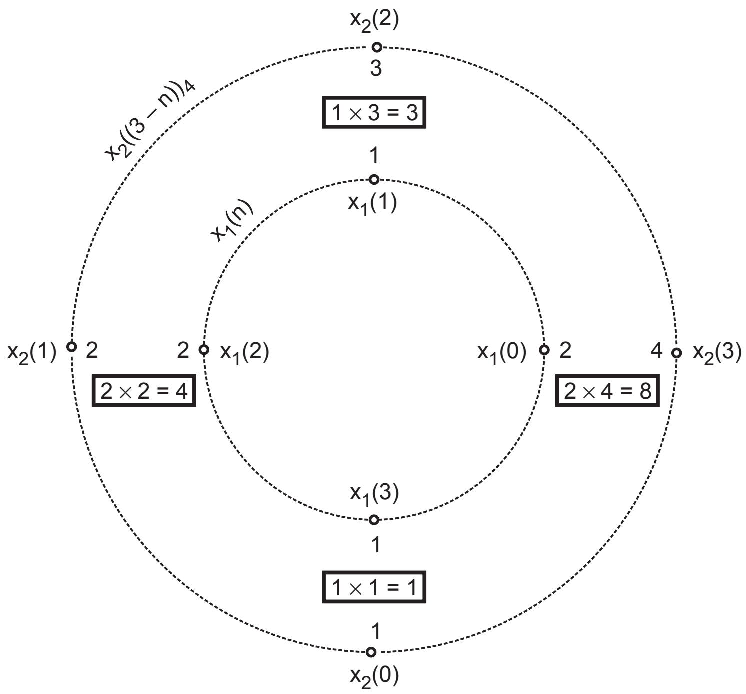


Fig. 3.3.11 To obtain x_3 (3)

The point by point products $x_1(n)\, x_2\left((3-n)\right)_4$ are shown in above figure. Then $x_3(3)$ of equation 3.3.55 can be obtained by adding all these product terms, i.e.,

$$x_3(3) = 8 + 3 + 4 + 1$$

$$= 16$$

Thus $\boxed{x_3(3) = 16}$

As per equation 3.3.51, 'm' varies from 0 to 3. This means there will be four samples in sequence $x_3(\ )$ i.e. $x_3(0)$, $x_3(1)$, $x_3(2)$ and $x_3(3)$ as we calculated. Thus sequence $x_3(\ )$ is,

$$x_3(n) = \{\underset{\uparrow}{14}, 16, 14, 16\} \qquad \text{... (3.3.56)}$$

This is the required sequence obtained due to circular convolution of $x_1(n)$ and $x_2(n)$.

Important steps to be followed for circular convolution

Step 1 : Plot $x_1(n)$ anticlockwise on inner circle.

Step 2 : Plot $x_2(n)$ clockwise on outer circle.

Key Point : *Match* $x_1(0)$ *and* $x_2(0)$ *while plotting.*

Step 3 : Multiply point-to-point samples on two circles.

Step 4 : Add all the multiplications. This gives first value of circular convolution.

Step 5 : For next value of circular convolution, shift outer circle anticlockwise by one sample position.

Step 6 : Repeat steps 3 to 5 till all values are calculated.

Example 3.3.2 *Recompute the circular convolution of example 3.3.2 using DFT and IDFT.*

AU : Dec.-07, Marks 10

Solution : The sequences given in example 3.3.1 are,

$$x_1(n) = \{2, 1, 2, 1\}$$

$$x_2(n) = \{1, 2, 3, 4\}$$

In this example we have to calculate the following,

$$x_1(n) \xleftrightarrow[N=4]{DFT} X_1(k)$$

$$x_2(n) \xleftrightarrow[N=4]{DFT} X_2(k)$$

Multiplication of two DFTs is equivalent to circular convolution of corresponding time domain sequences. Hence,

$$X_3(k) = X_1(k) X_2(k)$$

$$x_3(n) = \text{IDFT}\{X_3(k)\}$$

Here $x_3(n)$ is the circular convolution of $x_1(n)$ and $x_2(n)$. We know that the DFT is given as,

$$X_N = [W_N]\, x_N \qquad \text{from equation 3.1.9}$$

Here X_N is $N \times 1$ matrix of DFTs

W_N is $N \times N$ matrix of twiddle factors

x_N is $N \times 1$ matrix of sequence $x(n)$

For $N = 4$, above equation becomes,

$$X_4 = [W_4]\ x_4 \qquad \text{... (3.3.57)}$$

Using equation 3.1.6, equation 3.1.7 and equation 3.1.8 we can expand the above equation as,

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} \qquad \text{... (3.3.58)}$$

(i) To compute DFT of $x_1(n)$:

We know that the given sequence $x_1(n)$ is,

$$x_1(n) = \{2, 1, 2, 1\}$$

Using equation 3.3.58 we can write the DFT of $x_1(n)$ as,

$$\begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix} \qquad \text{... (3.3.59)}$$

Here we have,

$$\begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

We have evaluated $[W_4]$ matrix in example 3.1.1. Referring to this example $[W_4]$ matrix is given as,

$$[W_4] = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad \text{... (3.3.60)}$$

Putting the values in equation 3.3.59 we get,

$$\begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1+2+1 \\ 2-j2-2+j2 \\ 2-1+2-1 \\ 2+j-2-j \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix} \qquad \text{... (3.3.61)}$$

(ii) To compute DFT of $x_2(n)$:

We know that the given sequence $x_2(n)$ is,

$$x_2(n) = \{1, 2, 3, 4\}$$

This sequence can be written as,

$$\begin{bmatrix} x_2(0) \\ x_2(1) \\ x_2(2) \\ x_2(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Using equation 3.3.58 we can write DFT of $x_2(n)$ as,

$$\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x_2(0) \\ x_2(1) \\ x_2(2) \\ x_2(3) \end{bmatrix}$$

Putting the values of $[W_4]$ from equation 3.3.60 and sequence $x_2(n)$ in above equation,

$$\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+3+4 \\ 1-j2-3+j4 \\ 1-2+3-4 \\ 1+j2-3-j4 \end{bmatrix} = \begin{bmatrix} 10 \\ -2+j2 \\ -2 \\ -2-j2 \end{bmatrix} \qquad \text{... (3.3.62)}$$

(iii) To multiply the two DFTs $X_1(k)$ and $X_2(k)$:

Now let us multiply the sequences $X_1(k)$ and $X_2(k)$ to get $X_3(k)$ i.e.,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \\ X_3(2) \\ X_3(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \cdot X_2(0) \\ X_1(1) \cdot X_2(1) \\ X_1(2) \cdot X_2(2) \\ X_1(3) \cdot X_2(3) \end{bmatrix}$$

Putting the values of $X_1(k)$ and $X_2(k)$ obtained earlier,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \\ X_3(2) \\ X_3(3) \end{bmatrix} = \begin{bmatrix} 6 \times 10 \\ 0 \times (-2+j2) \\ 2 \times -2 \\ 0 \times (-2-j2) \end{bmatrix} = \begin{bmatrix} 60 \\ 0 \\ -4 \\ 0 \end{bmatrix} \quad \text{... (3.3.63)}$$

(iv) To obtain $x_3(n)$ by IDFT of $X_3(k)$:

The sequence $x_3(n)$ can be obtained by taking Inverse DFT of $X_3(k)$. $x_3(n)$ is the circular convolution of $x_1(n)$ and $x_2(n)$.

From equation 3.1.10 we know that IDFT is given as,

$$x_N = \frac{1}{N}\left[W_N^*\right] X_N$$

Here x_N is the $N \times 1$ matrix of time domain sequence

$\left[W_N^*\right]$ is the $N \times N$ complex conjugate matrix of $\left[W_N\right]$.

X_N is the $N \times 1$ matrix of DFTs

For $N = 4$ above equation can be written as,

$$x_4 = \frac{1}{4}\left[W_4^*\right] X_4$$

This equation can be expanded for sequence $x_3(n)$ as follows,

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \\ W_4^* \\ \\ \end{bmatrix} \begin{bmatrix} X_3(0) \\ X_3(1) \\ X_3(2) \\ X_3(3) \end{bmatrix} \quad \text{... (3.3.64)}$$

Here matrix $\left[W_4^*\right]$ is complex conjugate of $\left[W_4\right]$ and they are related as,

$$\left[W_4^*\right] = \left[W_4^{-1}\right]$$

From equation 3.3.60 $\left[W_4\right]$ is given as,

$$\left[W_4\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$\left[W_4^*\right]$ or $\left[W_4^{-1}\right]$ can be obtained by inverting the signs of imaginary values in above equation. i.e.

$$\therefore \quad \left[W_4^*\right] = \left[W_4^{-1}\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \quad \textbf{Key Point} : \textit{Only signs of 'j' are changed} \quad \ldots (3.3.65)$$

Putting the values in equation 3.3.64 we get,

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 60 \\ 0 \\ -4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 60+0-4+0 \\ 60+0+4+0 \\ 60+0-4+0 \\ 60+0+4+0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 56 \\ 64 \\ 56 \\ 64 \end{bmatrix}$$

$$= \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix}$$

Thus the sequence $x_3(n)$ is,

$$x_3(n) = \{14, 16, 14, 16\}$$

Observe that this sequence is same as that obtained in example 3.3.1 and given by equation 3.3.56. This shows that circular convolution can be computed using DFT and IDFT. Computationally this method is more efficient since DFT and IDFT can be computed fast using FFT algorithms.

Matrix approach to the computation of circular convolution

This method is actually obtained from the previous one only. The circular shifted sequences are combined to make matrix. Then circular convolution is obtained by matrix multiplication. Let there be two sequences $h(n)$ and $x(n)$ of length 'N'. Then the N-point

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ : \\ : \\ y(N-2) \\ y(N-1) \end{bmatrix}_{N \times 1} = \underbrace{\begin{bmatrix} h(0) & h(N-1) & h(N-2) & .... & h(2) & h(1) \\ h(1) & h(0) & h(N-1) & .... & h(3) & h(2) \\ h(2) & h(1) & h(0) & .... & h(4) & h(3) \\ : & : & : & & : & : \\ : & : & : & & : & : \\ h(N-2) & h(N-3) & h(N-4) & .... & h(0) & h(N-1) \\ h(N-1) & h(N-2) & h(N-3) & .... & h(1) & h(0) \end{bmatrix}}_{\text{Circular shifts are represented by this matrix}}{}_{N \times N} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ : \\ : \\ x(N-2) \\ x(N-1) \end{bmatrix}_{N \times 1}$$

circular convolution of these two sequences will be,

$$y(n) \;=\; h(n) \qquad x(n), \quad n = 0, 1, ... N-1 \qquad ...\ (3.3.66)$$

This equation can be formulated in matrix form as follows :

…(3.3.67)

Example 3.3.3 *Using the matrix approach obtain the circular convolution of*
$h(n) = \{0, 1, 2, 3, 0, 0, 0, 0\}$
and $x(n) = \{1, 0.5, 1, 0.5, 1, 0.5, 1, 0.5\}$

Solution : Putting above values in equation 3.3.67 we get,

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 1 \\ 0.5 \\ 1 \\ 0.5 \\ 1 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} (0\times 1)+(0\times 0.5)+(0\times 1)+(0\times 0.5)+(0\times 1)+(3\times 0.5)+(2\times 1)+(1\times 0.5) \\ (1\times 1)+(0\times 0.5)+(0\times 1)+(0\times 0.5)+(0\times 1)+(0\times 0.5)+(3\times 1)+(2\times 0.5) \\ (2\times 1)+(1\times 0.5)+(0\times 1)+(0\times 0.5)+(0\times 1)+(0\times 0.5)+(0\times 1)+(3\times 0.5) \\ (3\times 1)+(2\times 0.5)+(1\times 1)+(0\times 0.5)+(0\times 1)+(0\times 0.5)+(0\times 1)+(0\times 0.5) \\ (0\times 1)+(3\times 0.5)+(2\times 1)+(1\times 0.5)+(0\times 1)+(0\times 0.5)+(0\times 1)+(0\times 0.5) \\ (0\times 1)+(0\times 0.5)+(3\times 1)+(2\times 0.5)+(1\times 1)+(0\times 0.5)+(0\times 1)+(0\times 0.5) \\ (0\times 1)+(0\times 0.5)+(0\times 1)+(3\times 0.5)+(2\times 1)+(1\times 0.5)+(0\times 1)+(0\times 0.5) \\ (0\times 1)+(0\times 0.5)+(0\times 1)+(0\times 0.5)+(3\times 1)+(2\times 0.5)+(1\times 1)+(0\times 0.5) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 1.5 & + & 2 & + & 0.5 \\ 1 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 3 & + & 1 \\ 2 & + & 0.5 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 1.5 \\ 3 & + & 1 & + & 1 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 \\ 0 & + & 1.5 & + & 2 & + & 0.5 & + & 0 & + & 0 & + & 0 & + & 0 \\ 0 & + & 0 & + & 3 & + & 1 & + & 1 & + & 0 & + & 0 & + & 0 \\ 0 & + & 0 & + & 0 & + & 1.5 & + & 2 & + & 0.5 & + & 0 & + & 0 \\ 0 & + & 0 & + & 0 & + & 0 & + & 3 & + & 1 & + & 1 & + & 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 4 \\ 5 \\ 4 \\ 5 \\ 4 \\ 5 \end{bmatrix}$$

Thus we obtained sequence $y(n)$ as,

$$y(n) = \{4, 5, 4, 5, 4, 5, 4, 5\}$$

Example 3.3.4 *Consider the sequences : $x_1(n) = \{0, 1, 2, 3, 4\}$, $x_2(n) = \{0, 1, 0, 0, 0\}$ and $s(n) = \{1, 0, 0, 0, 0\}$.*

i) Determine a sequence $y(n)$ so that $Y(k) = X_1(k) \cdot X_2(k)$

ii) Is there a sequence $X_3(n)$ such that $S(k) = X_1(k) \cdot X_3(k)$. **AU : Dec.-13, Marks 16**

Solution : i) To obtain y(n) so that $Y(k) = X_1(k) \cdot X_2(k)$

We know that

$$x_1(n) \; N \; x_2(n) \xleftrightarrow[N]{DFT} X_1(k) \cdot X_2(k)$$

$$\therefore \quad y(n) = x_1(n) \; 4 \; x_2(n) = \begin{bmatrix} 0 & 4 & 3 & 2 & 1 \\ 1 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

ii) To obtain $x_3(n)$ so that $S(k) = X_1(k) \cdot X_3(k)$

By standard DFT techniques we obtain,

$$S(k) = \text{DFT}\{s(n)\} = [1, 1, 1, 1]$$

$$X_1(k) = \text{DFT}\{x_1(n)\} = [10, -2.5 + j3.441, -2.5 + j0.8123, -2.5 - j0.8123, -2.5 - j3.441]$$

Since $S(k) = X_1(k) \cdot X_3(k)$

$$X_3(k) = \frac{S(k)}{X_1(k)}$$

Dividing on sample to sample basis,

$$X_3(k) = [0.1, -0.1382 - j0.1902, -0.3618 - j0.1176, -0.3618 + j0.1176, -0.1382 + j0.1902]$$

Taking IDFT of above DFT we get,

$$x_3(n) = X_3(k) = \{-0.18, 0.22, 0.02, 0.02, 0.02\}$$

3.3.6 Time Reversal of a Sequence

This property states that if

$$x(n) \xleftrightarrow[N]{DFT} X(k) \quad \text{then,}$$

$$\boxed{x\left((-n)\right)_N = x\left(N-n\right) \xleftrightarrow[N]{DFT} X\left((-k)\right)_N = X\left(N-k\right)} \quad \text{... (3.3.68)}$$

This means when the sequence is circularly folded (reversed), its DFT is also circularly folded.

Proof :

The circularly folded sequence is basically represented as,

$$x\left((-n)\right)_N = x\left(N-n\right) \qquad \text{by equation 3.3.17}$$

By definition DFT of such sequence is given as,

$$DFT\left\{x\left(N-n\right)\right\} = \sum_{n=0}^{N-1} x(N-n)\, e^{-j2\pi kn/N} \quad \text{... (3.3.69)}$$

Let us change the index to $m = N - n$ then limits of summation will be,

$n = 0 \quad \therefore \quad m = N$ and

$n = N-1 \quad \therefore \quad m = N - N + 1 = 1$

Then equation 3.3.69 can be written as,

$$DFT\left\{x(N-n)\right\} = \sum_{m=N}^{1} x(m)\, e^{-j2\pi k(N-m)/N} \quad \text{... (3.3.70)}$$

We know that sequence $x(N-n)$ is circular, and DFT is periodic. Basically, the summation of equation 3.3.69 is performed from 0 to $N-1$ i.e. 'N' samples. If we perform summation from $(0+N)$ to $(N-1+N)$ i.e. next period ; the DFT will remain same since, the sequence is circular. The limits of $(0+N)$ to $(N-1+N)$ is equivalent to N to 1. These are the limits of equation 3.3.70. This shows that even if index is changed, the limits of summation are indirectly same because of circular nature of sequence and DFT is periodic. Hence equation 3.3.70 can be written as,

$$DFT\ \left\{x(N-n)\right\} = \sum_{m=0}^{N-1} x(m)\ e^{-j2\pi k(N-m)/N}$$

$$= \sum_{m=0}^{N-1} x(m)\ e^{-j2\pi k} \cdot e^{j2\pi km/N} \qquad \text{... (3.3.71)}$$

Here

$$\left.\begin{aligned} e^{-j2\pi k} &= cos\,(2\pi k) - j\ sin\,(2\pi k) \\ &= 1 \qquad \text{for all values of k} \end{aligned}\right\} \qquad \text{...(3.3.72)}$$

Hence equation 3.3.71 can be written as,

$$DFT\ \{x(N-n)\} = \sum_{m=0}^{N-1} x(m)\ e^{j2\pi km/N} \qquad \text{... (3.3.73)}$$

On the basis of equation 3.3.72 we can write,

$$e^{-j2\pi m} = cos\,(2\pi m) - j\ sin\,(2\pi m)$$

$$= 1 \quad \text{for all values of m}$$

Hence if we multiply RHS of equation 3.3.73 by $e^{-j2\pi m}$, its meaning will not change. i.e.,

$$DFT\ \left\{x(N-n)\right\} = \sum_{m=0}^{N-1} x(m)\ e^{j2\pi km/N} \cdot e^{-j2\pi m}$$

Here let us rearrange $e^{-j2\pi m}$ as $e^{-j2\pi mN/N}$, then above equation becomes,

$$DFT\ \left\{x(N-n)\right\} = \sum_{m=0}^{N-1} x(m)\ e^{j2\pi km/N} \cdot e^{-j2\pi mN/N}$$

Now combining exponential terms in above equation we get,

$$DFT\ \left\{x(N-n)\right\} = \sum_{m=0}^{N-1} x(m)\ e^{-j2\pi m(N-k)/N}$$

By definition of DFT, RHS of above equation is $X(N-k)$. i.e.,

$$DFT\ \left\{x(N-n)\right\} = X(N-k)$$

Since DFT is periodic over period N, $X(N-k)$ is equivalent to folding $X(k)$. Thus,

$$DFT\ \left\{x(N-n)\right\} = X(N-k)$$

$$= X\left((-k)\right)_N$$

3.3.7 Circular Time Shift of a Sequence

This property states that if

$$x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k) \quad \text{then}$$

$$\boxed{x\left((n-l)\right)_N \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)\ e^{-j2\pi kl/N}} \qquad \text{... (3.3.74)}$$

Thus shifting the sequence circularly by 'l' samples is equivalent to multiplying its DFT by $e^{-j2\pi kl/N}$.

Proof : By definition DFT of $x\left((n-l)\right)$ is given as,

$$DFT\left\{x\left((n-l)\right)_N\right\} = \sum_{n=0}^{N-1} x\left((n-l)\right)_N\ e^{-j2\pi kn/N} \qquad \text{... (3.3.75)}$$

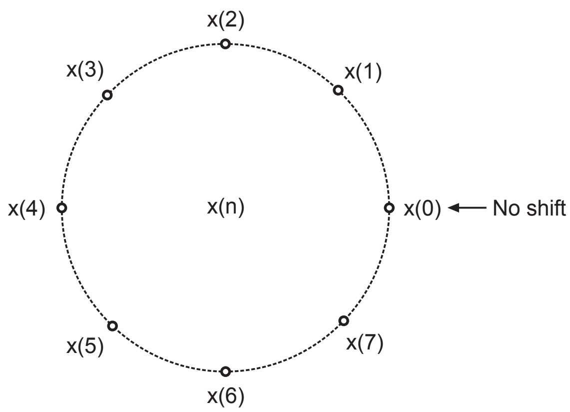


Fig. 3.3.12 A sequence having eight (N = 8) samples plotted across the circle. This sequence has no shift

Here on the basis of circular shift we can split summation in above equation. Fig. 3.3.12 below shows the sequence with eight samples i.e. $N = 8$.

Now the sequence $x(n)$ as given above is delayed (shifted to positive side) by $l = 2$ sample positions. This is equivalent to shifting samples anticlockwise by '2' sample positions. This new shifted sequence $x\left((n-l)\right)_N$ with $l = 2$ and $N = 8$ is shown in Fig. 3.3.13. The DFT of this circularly shifted sequence can be splitted into two parts. It is given as [see two DFTs in Fig. 3.3.13],

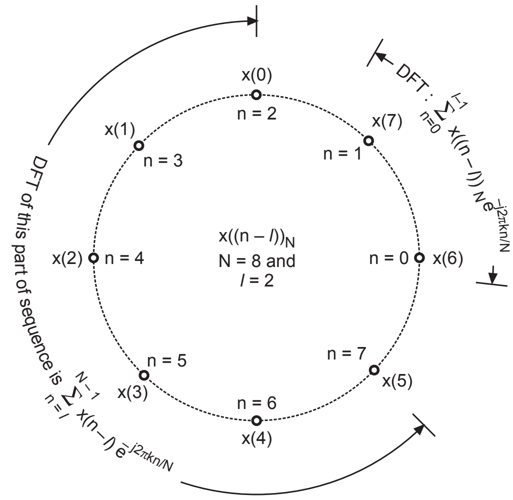


Fig. 3.3.13 DFT of a circularly shifted sequence is splitted in two parts as shown

$$DFT\left\{x\left((n-l)\right)_N\right\} = \sum_{n=l}^{N-1} x(n-l)\, e^{-j2\pi kn/N} + \sum_{n=0}^{l-1} x\left((n-l)\right)_N \; e^{-j2\pi kn/N} \qquad \text{... (3.3.76)}$$

Here $x\left((n-l)\right)_N$ can be written as $x(N-l+n)$ since this is circular shift. Hence second summation in above equation becomes,

$$\sum_{n=0}^{l-1} x\left((n-l)\right)_N \; e^{-j2\pi kn/N} = \sum_{n=0}^{l-1} x\left(N-l+n\right) e^{-j2\pi kn/N}$$

Let $m = N-l+n$ in RHS of above equation. Then limits of summation becomes,

When $n=0,\quad m=N-l$ and

when $n=l-1,\quad m=N-1$ i.e.,

$$\sum_{n=0}^{l-1} x\left((n-l)\right)_N \; e^{-j2\pi kn/N} = \sum_{m=N-l}^{N-1} x(m)\; e^{-j2\pi k\,(m+l-N)/N}$$

$$= \sum_{m=N-l}^{N-1} x(m)\; e^{-j2\pi k\,(m+l)/N} \cdot e^{j2\pi k}$$

Here $\quad e^{j2\pi k} = \cos(2\pi k) + j\sin(2\pi k)$

$= 1$ always

Hence $$\sum_{n=0}^{l-1} x\left((n-l)\right)_N \; e^{-j2\pi kn/N} = \sum_{m=N-l}^{N-1} x(m)\; e^{-j2\pi k\,(m+l)/N} \qquad \text{... (3.3.77)}$$

Now consider first summation in equation 3.3.76, i.e., $\sum_{n=l}^{N-1} x(n-l)\ e^{-j2\pi kn/N}$.

Let us put $n-l=m$. Then limits of summation become,

When $n=l,\quad m=0$ and

When $n=N-1,\quad m=N-1-l$. Thus,

$$\sum_{n=l}^{N-1} x(n-l)\ e^{-j2\pi kn/N} = \sum_{m=0}^{N-1-l} x(m)\ e^{-j2\pi k(m+l)/N} \qquad \text{... (3.3.78)}$$

Now let us put values of above summation and summation of equation 3.3.77 in equation 3.3.76 i.e.,

$$DFT\left\{x\left((n-l)\right)_N\right\} = \sum_{m=0}^{N-1-l} x(m)\ e^{-j2\pi k\,(m+l)/N} + \sum_{m=N-l}^{N-1} x(m)\ e^{-j2\pi k\,(m+l)/N} \qquad \text{... (3.3.79)}$$

The above two summations can be combined into single one. Note that even though we have assumed two different values for 'm', it is just an index. First summation in above equation is performed from 0 to $(N-1-l)$ and second summation is from $(N-l)$ to $(N-1)$. Thus the overall summation is done from 0 to $N-1$. At the same time the quantities inside both the summations are same. Hence the summations can be combined. Then equation 3.3.79 becomes,

$$\begin{aligned} DFT\left\{x\left((n-l)\right)_N\right\} &= \sum_{m=0}^{N-1} x(m)\ e^{-j2\pi k\,(m+l)/N} \\ &= \sum_{m=0}^{N-1} x(m)\ e^{-j2\pi km/N} \cdot e^{-j2\pi kl/N} \\ &= X(k)\, e^{-j2\pi kl/N} \qquad \text{by definition of DFT} \end{aligned}$$

Since $W_N = e^{-j2\pi/N}$, above equation can be written as,

$$DFT\ \left\{x\left((n-l)\right)_N\right\} = X(k)\, W_N^{k\,l}$$

Similarly if l is positive,

$$DFT\ \left\{x\left((n+l)\right)_N\right\} = X(k)\, e^{j2\pi kl/N}$$

$$= X(k)\, W_N^{-kl}$$

3.3.8 Circular Frequency Shift

This property is similar to circular time shift. This property states that if,

$$x(n) \xleftrightarrow[N]{DFT} X(k) \quad \text{then,}$$

$$\boxed{x(n)\, e^{j2\pi ln/N} \xleftrightarrow[N]{DFT} X\big((k-l)\big)_N} \qquad \text{... (3.3.80)}$$

Thus, shifting the frequency components of DFT circularly is equivalent to multiplying the time domain sequence by $e^{j2\pi ln/N}$

Proof :

This property can be proved by the same method that is used to prove circular time shift property. Here we have to start from IDFT formula.

3.3.9 Complex Conjugate Properties

This property states that if,

$$x(n) \xleftrightarrow[N]{DFT} X(k) \quad \text{then}$$

$$\boxed{x^*(n) \xleftrightarrow[N]{DFT} X^*\big((-k)\big)_N = X^*(N-k)} \qquad \text{... (3.3.81)}$$

and $$x^*\big((-n)\big)_N = x^*(N-k) \xleftrightarrow[N]{DFT} X^*(k) \qquad \text{... (3.3.82)}$$

Proof : By definition DFT of $x^*(n)$ will be given as,

$$DFT\left\{x^*(n)\right\} = \sum_{n=0}^{N-1} x^*(n)\; e^{-j2\pi kn/N} \qquad \text{... (3.3.83)}$$

We know that,

$$e^{j2\pi nN/N} = e^{j2\pi n} = \cos(2\pi n) + j\sin(2\pi n) = 1 \quad \text{always.}$$

Hence multiplying equation 3.3.83 by $e^{j2\pi nN/N}$ will not change the meaning.

Hence,

$$DFT\left\{x^*(n)\right\} = \sum_{n=0}^{N-1} x^*(n)\; e^{-j2\pi kn/N} \cdot e^{j2\pi nN/N}$$

$$= \sum_{n=0}^{N-1} x^*(n)\, e^{j2\pi n(N-k)/N} = \left[\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi n(N-k)/N}\right]^*$$

$$= \left[X(N-k)\right]^* = X^*(N-k)$$

Thus equation (3.3.81) is proved. Starting from IDFT formula and using the same method as above equation (3.3.82) can be proved.

3.3.10 Circular Correlation

The circular correlation property states that if,

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

and $\quad y(n) \xleftrightarrow[N]{DFT} Y(k)$ then,

$$\boxed{\tilde{r}_{xy}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xy}(k) = X(k)\, Y^*(k)} \qquad \text{... (3.3.84)}$$

Here $\tilde{r}_{xy}(l)$ is the circular cross correlation, which is given as,

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*\big((n-l)\big)_N \qquad \text{... (3.3.85)}$$

This means multiplication of DFT of one sequence and conjugate DFT of another sequence is equivalent to circular cross correlation of these two sequences in time domain.

Proof : Consider the circular cross correlation given by equation 3.3.85 i.e.,

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*\big((n-l)\big)_N$$

The term $y^*\big((n-l)\big)_N$ can be written as $y^*\Big(\big(-[l-n]\big)\Big)_N$. Hence above equation becomes,

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*\Big(\big(-[l-n]\big)\Big)_N$$

Compare the above equation with circular convolution given by equation (3.3.50). The above equation is circular convolution of $x(l)$ and $y^*(-l)$. i.e.,

$$\tilde{r}_{xy}(l) = x(l) \; Ⓝ \; y^*(-l) \quad \text{... (3.3.86)}$$

From the circular convolution property of equation (3.3.34) we can write,

$$DFT\left\{\tilde{r}_{xy}(l)\right\} = DFT\left\{x(l)\right\} \cdot DFT\left\{y^*(-l)\right\}$$

i.e. $$\tilde{R}_{xy}(k) = X(k) \cdot DFT\left\{y^*(-l)\right\} \quad \text{... (3.3.87)}$$

By definition of DFT we can write,

$$DFT\left\{y^*(-l)\right\} = \sum_{l=0}^{N-1} y^*(-l)\, e^{-j2\pi kl/N} \quad \text{... (3.3.88)}$$

Let $n = -l$, then limits of summation will be,

When $l = 0$, $n = 0$ and

When $l = N-1$, $n = -(N-1)$

Then equation 3.3.88 becomes,

$$DFT\left\{y^*(-l)\right\} = \sum_{n=0}^{-(N-1)} y^*(n)\, e^{j2\pi kn/N}$$

The sequence $y^*(n)$ is circular in nature. Hence summation from 0 to $-(N-1)$ will be same as from 0 to $(N-1)$. Hence above equation can be written as,

$$DFT\left\{y^*(-l)\right\} = \sum_{n=0}^{N-1} y^*(n)\, e^{j2\pi kn/N}$$

$$= \left[\sum_{n=0}^{N-1} y(n)\, e^{-j2\pi kn/N}\right]^*$$

$$= \left[Y(k)\right]^* \quad \text{...by definition of DFT}$$

$$= Y^*(k)$$

Putting this value in equation 3.3.87 we get,

$$\tilde{R}_{xy}(k) = X(k)\, Y^*(k) \quad \text{... (3.3.89)}$$

Thus equation 3.3.84 is proved. When $x(n) = y(n)$ we get circular autocorrelation. Then equation 3.3.84 can be written as,

$$\tilde{r}_{xx}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xx}(k) = X(k)\, X^*(k) \quad \text{... (3.3.90)}$$

We know that $X(k)\, X^*(k) = |X(k)|^2$, Hence above equation becomes,

$$\tilde{r}_{xx}(l) \overset{DFT}{\underset{N}{\longleftrightarrow}} \tilde{R}_{xx}(k) = |X(k)|^2 \qquad \text{... (3.3.91)}$$

3.3.11 Multiplication of Two Sequences

This property states that if

$$x_1(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_1(k) \quad \text{and}$$

$$x_2(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_2(k) \quad \text{then,}$$

$$\boxed{x_1(n)\, x_2(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} \frac{1}{N} X_1(k) \; Ⓝ \; X_2(k)} \qquad \text{... (3.3.92)}$$

This means multiplication of two sequences in time domain results in circular convolution of their DFTs in frequency domain.

Proof : This property can be proved by the same method which is discussed for circular convolution. The proof can be started with DFT of $x_3(m)$ which is product of $x_1(n)$ and $x_2(n)$. Then putting for IDFTs of $x_1(n)$ and $x_2(n)$, the derived equation 3.3.92 can be obtained.

3.3.12 Parseval's Theorem

Consider the complex valued sequences $x(n)$ and $y(n)$. Then if,

$$x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k) \quad \text{and}$$

$$y(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} Y(k) \quad \text{then}$$

$$\boxed{\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)} \qquad \text{... (3.3.93)}$$

When $y(n) = x(n)$ above equation becomes,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2 \qquad \text{... (3.3.94)}$$

The above two equations are relations of Parseval's Theorem. Above equation give energy of finite duration sequence in terms of its frequency components.

Proof : Circular correlation is given by equation 3.3.85 as,

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*\big((n-l)\big)_N \quad \text{... (3.3.95)}$$

For $l = 0$ above equation becomes,

$$\tilde{r}_{xy}(0) = \sum_{n=0}^{N-1} x(n)\, y^*(n) \quad \text{... (3.3.96)}$$

From equation 3.3.84 we know that

$$DFT\left\{\tilde{r}_{xy}(l)\right\} = X(k)\, Y^*(k)$$

i.e.

$$\tilde{r}_{xy}(l) = IDFT\left\{X(k)\, Y^*(k)\right\}$$

By definition of IDFT,

$$\tilde{r}_{xy}(l) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)\, e^{j2\pi kl/N}$$

With $l = 0$, above equation becomes,

$$\tilde{r}_{xy}(0) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k) \quad \text{... (3.3.97)}$$

Equating above equation with equation 3.3.96 we have,

$$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)$$

This is the required relation.

The properties of DFT are summarized below in Table 3.3.3.

Sr. No.	Name of the property	Time domain representation	Frequency domain representation
1	Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
2	Linearity	$a_1\, x_1(n) + a_2\, x_2(n)$	$a_1\, X_1(k) + a_2\, X_2(k)$

3	Symmetry properties	$x^*(n)$	$X^*(N-k)$
		$x^*(N-n)$	$X^*(k)$
		For real valued $x(n)$	$X(k) = X^*(N-k)$ $X_R(k) = X_R(N-k)$ $X_I(k) = -X_I(N-k)$ $\lvert X(k)\rvert = \lvert X(N-k)\rvert$ $\angle X(k) = -\angle X(N-k)$
4	Circular convolution	$x_1(n)$ $x_2(n)$	$X_1(k)\, X_2(k)$
5	Circular time reversal	$x((-n))_N = x(N-n)$	$X((-k))_N = X(N-k)$
6	Circular time shift	$x((n-l))_N$	$X(k)\ e^{-j2\pi kl/N}$
7	Circular frequency shift	$x(n)\ e^{j2\pi ln/N}$	$X((k-l))_N$
8	Complex conjugate properties	$x^*(n)$	$X^*((-k))_N = X^*(N-k)$
		$x^*((-n))_N = x^*(N-k)$	$X^*(k)$
9	Circular correlation	$\tilde{r}_{xy}(l) = x(l)$ $y^*(-l)$	$X(k)\, Y^*(k)$
10	Multiplication of two sequences	$x_1(n)\, x_2(n)$	$\frac{1}{N} X_1(k)$ $X_2(k)$
11	Parsevals Theorem	$\sum_{n=0}^{N-1} \lvert x(n)\rvert^2$	$\frac{1}{N}\sum_{k=0}^{N-1} \lvert X(k)\rvert^2$
		$\sum_{n=0}^{N-1} x(n)\, y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, Y^*(k)$

Table 3.3.3 Properties of DFT

Example for Understanding

Example 3.3.5 *The first five points of the eight point DFT of a real valued sequence are (0.25, 0,125 – j 0.3018, 0, 0.125 – j 0.0518) Determine the remaining three points.*

AU : Dec.-15, Marks 4

Solution : Here

$$X(0) = 0.25, \quad X(1) = 0, X(2) = 0.125 - j0.3018,$$
$$X(3) = 0, \quad X(4) = 0.125 - j0.0518$$

The symmetry property for a real valued sequence is given as,

$$X(N-k) = X^*(k)$$

$$X(8-k) = X^*(k)$$

Let $k = 3, 2, 1$ in above equation i.e.,

For $k = 3, X(8-3) = X^*(3)$

$\therefore \quad X(5) = X^*(3) = 0$

For $k = 2, X(8-2) = X^*(2)$

$\therefore \quad X(6) = X^*(2) = 0.125 + j\,0.3018$

For $k = 1, X(8-1) = X^*(1)$

$\therefore \quad X(7) = X^*(1) = 0$

Examples with Solution

Example 3.3.6 *An analog signal is sampled at 10 kHz and the DFT of 512 samples is computed. Determine the frequency spacing between spectral samples of the DFT.*

Solution : We know that the range of 'ω' in $X(\omega)$ is from 0 to 2π. In DFT, $X(\omega)$ is sampled in this range (0 to 2π) by 'N' number of samples. Hence the frequency resolution in the DFT samples will be,

$$\text{Frequency resolution} = \frac{\text{Frequency range}}{\text{Number of DFT samples}}$$

$$= \frac{2\pi}{N} \text{ radians per sample} \qquad \text{... (3.3.98)}$$

This frequency resolution may be denoted by $\Delta\omega$ and it is in radians per sample. Similarly frequency resolution in cycles per sample may be denoted by Δf and it will be given as,

$$\Delta f = \frac{\Delta\omega}{2\pi} \text{ cycles per sample} \qquad \text{... (3.3.99)}$$

The above relationship is written on the basis of $\omega = 2\pi f$. Here the resolution Δf is in discrete time. It is related to continuous time frequencies as,

$$\Delta f = \frac{\Delta F}{F_s} \qquad \text{...(3.3.100)}$$

Here ΔF is the frequency resolution in hertz and F_s is the sampling frequency. We have written the above equation on the basis of $f = \frac{F}{F_s}$. The above equation can be written as,

Frequency resolution in Hertz, $\Delta f = \Delta f \times f_s$ putting for Δf from equation 3.3.100 in above equation,

$$\Delta F = \frac{\Delta \omega}{2\pi} F_s \qquad \ldots(3.3.101)$$

The frequency resolution $\Delta \omega$ is given by equation 3.3.101. Putting this value in above equation,

$$= \frac{\frac{2\pi}{N}}{2\pi} F_s$$

$$= \frac{1}{N} F_s \qquad \ldots(3.3.102)$$

In this example it is given that $N = 512$ and F_s = 10 kHz, Hence above equation becomes,

Frequency resolution $\Delta F = \frac{1}{512} \times 10{,}000$ = 19.53125 Hz

This means the frequency spacing between the samples of DFT will be 19.53125 Hz.

Example 3.3.7 *G (k) and H (k) are 6-point DFTs of sequences g(n) and h(n) respectively. The DFT G (k) is given as,*

$G(k) = \{1 + j, -2.1 + j\,3.2, -1.2 - j\,2.4, 0, 0.9 + j\,3.1, -0.3 + j\,1.1\}$

The sequences g (n) and h (n) are related by the circular time shift as,

$h(n) = g\,((n-4))_6$

Determine H (k), without computing the DFT.

Solution : Here consider the circular time shift property of equation 3.3.74 which is given as,

$$x\,((n-l))_N \xleftrightarrow[N]{DFT} X(k)\, e^{-j 2\pi k l / N}$$

We know that,

$$H(k) = DFT\,\{h(n)\}$$

$$= DFT\,\{g\,((n-4))_6\} \text{ since } h(n) = g\,((n-4))_6 \qquad \ldots(3.3.103)$$

Using the circular time shift property we can write the DFT of $\{g\,((n-4))_6\}$ as,

$$DFT\,\{g\,((n-4))_6\} = G(k)\, e^{-j 2\pi k 4/6}$$

$$= G(k)\, e^{-j 4\pi k / 3}$$

Hence $H(k)$ of equation 3.3.103 can be written as,

$$H(k) = G(k)\, e^{-j 4\pi k / 3} \qquad \ldots(3.3.104)$$

This is the relationship between $G(k)$ and $H(k)$. Hence $H(k)$ can be obtained from $G(k)$ without computation of DFT.

With $k = 0$ in equation 3.3.104 we get,

$$H(0) = G(0)\, e^{-j4\pi 0/3}$$

$$= G(0) = 1 + j \qquad \therefore \boxed{\mathbf{H(0) = 1 + j}}$$

With $k = 1$ in equation 3.3.104 we get,

$$H(1) = G(1)\, e^{-j4\pi 1/3}$$

Putting for $G(1) = -2.1 + j\,3.2$ in above equation we get,

$$H(1) = (-2.1 + j\,3.2)\, e^{-j4\pi/3} \qquad \text{... (3.3.105)}$$

We know that

$$e^{-j4\pi/3} = 1 \angle -4\pi/3 \text{ or}$$

$$= 1 \angle -240° \qquad \text{... (3.3.106)}$$

This can be verified using Euler's identity very easily. We know that $e^{-j\theta} = cos\,\theta - j\,sin\,\theta$.

Hence,

$$e^{-j4\pi/3} = \cos\left(\frac{4\pi}{3}\right) - j\sin\left(\frac{4\pi}{3}\right) = -0.5 + j\,0.866$$

The polar value of above can be obtained using rectangular to polar utility as,

$$e^{-j4\pi/3} = 1 \angle 120°$$

The above value can also be written as,

$$e^{-j4\pi/3} = 1 \angle -240°$$

This is because 120° is equivalent to −240°. Observe that the above value is same as that of equation 3.3.106. Hence equation 3.3.106 becomes,

$$H(1) = (-2.1 + j\,3.2)(1 \angle -240°)$$

$$= (3.827 \angle 123°)(1 \angle -240°)$$

$$= (3.827 \times 1) \angle (123° - 240°)$$

$$= 3.827 \angle -117°$$

$$= -1.738 - j3.4 \qquad \therefore \boxed{\mathbf{H(1) = -1.738 - j\,3.4}}$$

With $k = 2$ in equation 3.3.104 we get,

$$H(2) = G(2)\, e^{-j\, 4\pi\, 2/3} = G(2)\, e^{-j\, 8\pi/3}$$

$$= G(2)\, \left(1 \angle -8\pi/3\right) = G(2)\, \left(1 \angle -480°\right)$$

Putting the value of $G(2) = -1.2 - j\,2.4$ in above equation,

$$H(2) = \left(-1.2 - j\,2.4\right) \left(1 \angle -480°\right)$$

$$= \left(2.683 \angle -116.56°\right) \left(1 \angle -480°\right)$$

$$= \left(2.683 \times 1\right) \left(\angle -116.56° - 480°\right)$$

$$= 2.683 \angle -596.56$$

$$= -1.478 + j\,2.238 \qquad \therefore \boxed{\boldsymbol{H(2) = -1.478 + j\,2.238}}$$

With $k = 3$ in equation 3.3.104 we get,

$$H(3) = G(3)\, e^{-j\, 4\pi\, 3/3}$$

Putting value of $G(3) = 0$ in above equation,

$$H(3) = 0 \cdot e^{-j\, 4\pi}$$

$$= 0 \qquad \therefore \boxed{\boldsymbol{H(3) = 0}}$$

With $k = 4$ in equation 3.3.104 we get,

$$H(4) = G(4)\, e^{-j\, 4\pi\, 4/3} = G(4)\, e^{-j\, 16\pi/3}$$

$$= G(4)\, \left(1 \angle -16\pi/3\right) = G(4)\, \left(1 \angle -960°\right)$$

Putting value of $G(4)$,

$$H(4) = \left(0.9 + j\,3.1\right) \left(1 \angle -960°\right)$$

$$= \left(3.228 \angle 73.81°\right) \left(1 \angle -960°\right)$$

$$= 3.228 \angle -886.189°$$

$$= -3.13 - j\,0.77 \qquad \therefore \boxed{\mathbf{H(4) = -3.13 - j\,0.77}}$$

With $k = 5$ in equation 3.3.104 we get,

$$H(5) = G(5)\, e^{-j\, 4\pi\, 5/3} = G(5)\, e^{-j\, 20\pi/3}$$

$$= G(5)\, \left(1 \angle -20\pi/3\right) = G(5)\, \left(1 \angle -1200°\right)$$

Putting value of $G(5)$,

$$H(5) = \left(-0.3 + j\,1.1\right) \left(1 \angle -1200°\right)$$

$$= \left(1.14 \angle 105.25°\right) \left(1 \angle -1200°\right)$$

$$= 1.14 \angle -1094.74°$$

$$= 1.1 - j\,0.29 \qquad \therefore \quad \boxed{H(5) = 1.1 - j\,0.29}$$

Thus the DFT $H(k)$ is,

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \end{bmatrix} = \begin{bmatrix} 1+j \\ -1.738 - j\,3.4 \\ -1.478 + j\,2.238 \\ 0 \\ -3.13 - j\,0.77 \\ 1.1 - j\,0.29 \end{bmatrix}$$

Example 3.3.8 *Compute the following if x_1 = [–1, –1, –1, 2]; x_2 = [–2 –1, –1, –2]*

i) Linear and circular convolution of a sequence.

ii) $x_1; x_2$ subject to addition and multiplication. **AU : Dec.-16, Marks 16**

Solution : i) Linear convolution :

			– 1	– 1	–1	2
			– 2	– 1	–1	–2
			2	2	2	– 4
		1	1	1	– 2	*
	1	1	1	– 2	*	*
2	2	2	– 4	*	*	*
2	3	4	0	1	0	– 4

$$\therefore \qquad x_1 * x_2 = \{2, 3, 4, 0, 1, 0, -4\}$$

ii) Circular convolution :

Step 1 : DFTs of $x_1(n)$ and $x_2(n)$

$$X_4 = [W_4]x_4$$

$$\begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0+j3 \\ -3 \\ 0-3j \end{bmatrix}$$

Similarly DFT of $x_2(n)$,

$$\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -6 \\ -1-j \\ 0 \\ -1+j \end{bmatrix}$$

Step 2 : Multiplication of two DFTs

$$X(k) = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} -1 \\ 3j \\ -3 \\ -3j \end{bmatrix} \begin{bmatrix} -6 \\ -1-j \\ 0 \\ -1+j \end{bmatrix} = \begin{bmatrix} 6 \\ 3-3j \\ 0 \\ 3+3j \end{bmatrix}$$

Step 3 : Use convolution theorem

$$x(n) = x_1(n) \text{ N } x_2(n) \xleftrightarrow[N]{DFT} X_1(k) X_2(k) = X(k)$$

Thus IDFT of $X(k)$ will give circular convolution,

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4}[W^*{}_4] \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 6 \\ 3-3j \\ 0 \\ 3+3j \end{bmatrix}$$

$$x(n) = \{3, 3, 0, 0\}$$

Examples for Practice

Example 3.3.9 *Determine 8-point DFT of the signal.*
$x(n) = \{1, 1, 1, 1, 1, 1, 0, 0\}$
Also sketch its magnitude and phase.

[Ans. : $X(k) = \left\{6, -\frac{1}{\sqrt{2}} - j\left(1 + \frac{1}{\sqrt{2}}\right), 1 - j, \frac{1}{\sqrt{2}} + j\left(1 - \frac{1}{\sqrt{2}}\right),\right.$
$\left. 0, \frac{1}{\sqrt{2}} + j\left(-1 + \frac{1}{\sqrt{2}}\right), 1 + j, -\frac{1}{\sqrt{2}} + j\left(1 + \frac{1}{\sqrt{2}}\right)\right\}$

$|X(k)| = \{6, 1.847, 1.414, 0.765, 0, 0.765, 1.414, 1.847\}$

$\angle X(k) = \{0, -112.5°, -45°, 22.5°, 0°, -22.5°, 45°, 112.5°\}$**]**

Example 3.3.10 *Determine N-point DFT of the sequence which is given as,* $x(n) = \delta(n - n_0)$ **[Ans. :** $X(k) = e^{-j2\pi k n_0 / N}$**]**

Example 3.3.11 *The five samples of the 9-point DFT are given as follows :*
$X(0) = 23, X(1) = 2.242 - j, \quad X(4) = -6.379 + j\,4.121$
$X(6) = 6.5 + j\,2.59 \quad X(7) = -4.153 + j\,0.264$
Determine the remaining samples of DFT if the corresponding time domain sequence is real. **[Ans. :** $X(2) = -4.153 - j\,0.264$

$X(3) = 6.5 - j\,2.59$

$X(5) = -6.379 - j\,4.121$

$X(8) = 2.42 + j$**]**

Example 3.3.12 *Compute circular convolution of the following two sequences :*
$x_1(n) = \{\underset{\uparrow}{1}, 2, 0, 1\}$

$x_2(n) = \{\underset{\uparrow}{2}, 2, 1, 1\}$

[Ans. : $x_1(n) \text{ N } x_2(n) = \{6, 7, 6, 5\}$**]**

Example 3.3.13 *Compute the circular convolution of the following sequences using DFT and IDFT.*

$x_1(n) = \{1, 2, 3, 1\}$

$x_2(n) = \{4, 3, 2, 2\}$ **[Ans. : $x_1(n) N\ x_2(n) = \{17, 19, 22, 19\}$]**

Example 3.3.14 *Find circular convolution of two sequences :*

$x(n) = \{2, 4, 0, 2\}$ and $h(n) = \{4, 4, 2, 2\}$ **AU : May-06, Marks 8**

[Ans. : x(n) 4 h(n) = {24, 28, 24, 20}]

Example 3.3.15 *The first five points of the 8-point DFT of a real valued sequence are $\{0.25 - j\,0.3018,\ 0,\ 0,\ 0.125,\ -j\,0.0518\}$. Determine remaining three points of the DFT.* **AU : Dec.-13, Marks 4**

$$X(k) = \{0.25,\ -j\,0.3018,\ 0,\ 0,\ 0.125 - j\,0.0518,\ 0,\ 0,\ j\,0.3018\}$$

Review Questions

1. *Explain circular convolution. What is the difference between circular convolution, linear convolution and periodic convolution ?* **AU : May-11, Marks 10, Dec.-11, Marks 5, Dec.-12, Marks 6**
2. *State and prove following properties of DFT.*
   i) Circular time reversal *ii) Circular time shift*
   iii) Circular frequency shift *iv) Circular correlation.* **AU : Dec.-11, Marks 5, May-12, Marks 8**
3. *State amd prove the following properties of DFT.*
   i) Time shifting *ii) Conjugate symmetry.* **AU : May-11, Marks 10**
4. *Prove* $\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |x(k)|^2$, *where X(k) is the DFT of x(n).* **AU : May-04, Marks 8**

3.4 Applications of DFT

3.4.1 Linear Filtering using DFT (Linear Convolution using DFT)

Linear filtering operation is implemented with the help of linear convolution. The output y(n) is obtained by convolving impulse response h(n) with input x(n). Now let us see how DFT can be used to implement linear convolution. Hence linear convolution can be computed efficiently with the help of DFT.

Let the unit sample response of the LTI system be $h(n)$ of length M, i.e. $h(0), h(1), \ldots h(M-1)$. Let the input to this LTI system be $x(n)$ of length L, i.e. $x(0), x(1), \ldots. x(L-1)$. Such system is shown in Fig. 3.4.1.

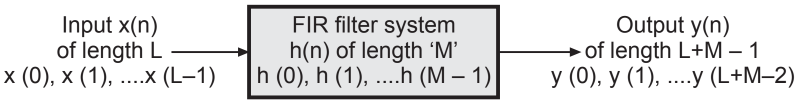


Fig. 3.4.1 FIR filter which uses linear convolution. The output $y(n)$ is linear convolution of input $x(n)$ and unit sample response $h(n)$

The output of the LTI system as shown in above figure is $y(n)$. This output $y(n)$ is the linear convolution of input $x(n)$ and unit sample response $h(n)$. i.e.,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \qquad \text{... (3.4.1)}$$

Since the length of $h(n)$ is 'M' and that of $x(n)$ is 'L', the length of $y(n)$ will be $L+M-1$. Let us consider the Fourier transform of above equation,

$$Y(\omega) = F\left\{ \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \right\} \qquad \text{... (3.4.2)}$$

Convolution property of Fourier transform states that,

$$F\{x_1(n) * x_2(n)\} = X_1(\omega)\, X_2(\omega)$$

Hence RHS of equation 3.4.2 can be written as,

$$Y(\omega) = H(\omega)\, X(\omega) \qquad \text{... (3.4.3)}$$

Here $H(\omega)$ is Fourier transform of $h(n)$ and

$X(\omega)$ is Fourier transform of $x(n)$.

We know that $y(n)$ has $L+M-1$ samples. If the DFT of $y(n)$ has $L+M-1$ samples, then only it will represent $y(n)$ uniquely. Thus $Y(k)$; the DFT of size $N \geq L+M-1$ represents $y(n)$ uniquely in time domain. We know that $Y(k)$ can be obtained from $Y(\omega)$ as,

$$Y(k) = Y(\omega)\Big|_{\omega = \frac{2\pi k}{N}}, \quad k = 0, 1, \ldots N-1$$

Putting for $Y(\omega)$ from equation 3.4.3 we have,

$$Y(k) = X(\omega) \cdot H(\omega)\Big|_{\omega = \frac{2\pi}{N} k}, \qquad k = 0, 1, \ldots\ldots N-1$$

Since $X(\omega)\Big|_{\omega = \frac{2\pi}{N} k} = X(k)$ and $H(\omega)\Big|_{\omega = \frac{2\pi}{N} K} = H(k)$, above equation can be written as,

$$Y(k) = X(k) \cdot H(k), \qquad k = 0, 1, \ldots\ldots N-1 \qquad \text{... (3.4.4)}$$

This shows that multiplying the N-point DFTs of $x(n)$ and $h(n)$, we get DFT $Y(k)$. This DFT represent $y(n)$ uniquely if $N \geq L+M-1$. Hence $y(n)$ can be obtained by taking IDFT of $Y(k)$. i.e.,

$$\begin{aligned} y(n) &= IDFT\,\{Y(k)\} \text{ or} \\ &= IDFT\,\{X(k) \cdot H(k)\}\,, \qquad k = 0, 1, \ldots.. N-1 \end{aligned} \qquad \ldots (3.4.5)$$

Here obsereve that $y(n)$ can be obtained with the help of DFT. Thus linear convolution is implemented by DFT. This operation is illustrated in Fig. 3.4.2.

Fig. 3.4.2 (a) shows straight forward linear convolution of $x(n)$ and $h(n)$. Fig. 3.4.2 (b) shows how this operation can be implemented with the help of DFT and IDFT. We have seen that the DFT should be of length $N = L+M-1$. Hence the lengths of $x(n)$ and $h(n)$ are first appended to 'N' by zero padding as shown in Fig. 3.4.2 (b) above. This zero padding does not change the meaning of the sequence. The N-point DFTs $X(k)$ and $H(k)$ are then computed. The two DFTs $X(k)$ and $H(k)$ are multiplied to give $Y(k)$. The N-point IDFT of $Y(k)$ then gives $y(n)$ as shown in Fig. 3.4.2 (b).

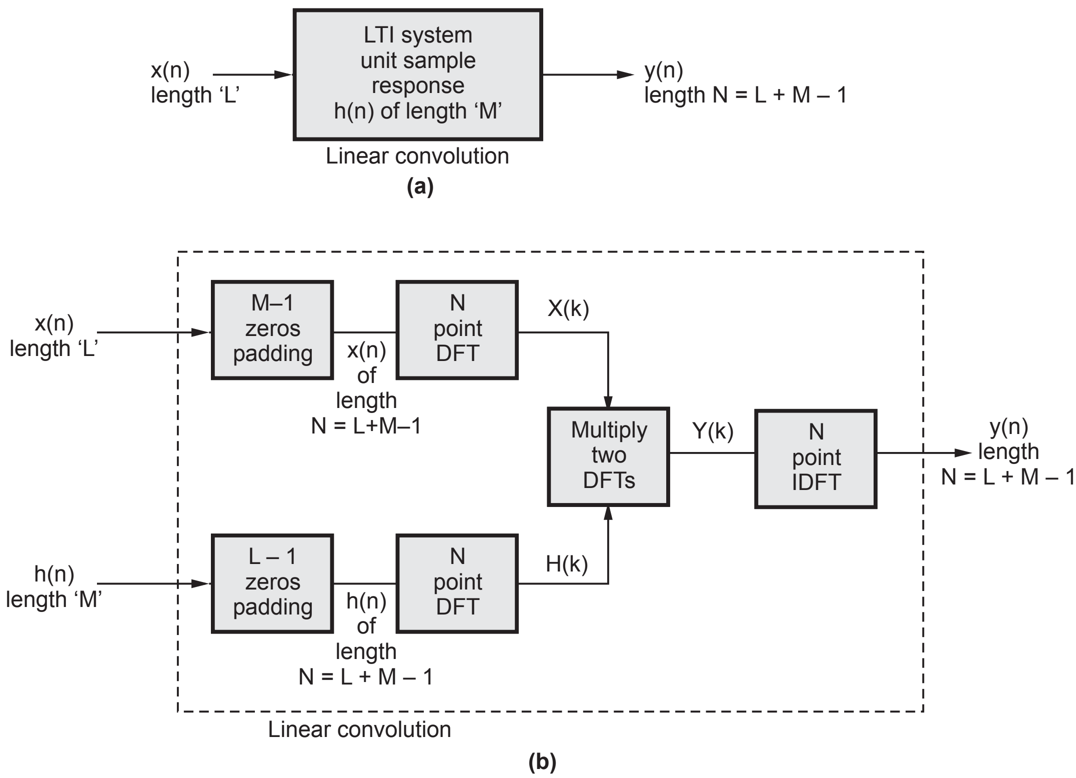


Fig. 3.4.2 Linear convolution of $x(n)$ and $h(n)$ to get $y(n)$

To implement linear convolution using circular convolution :

Consider equation 3.4.4 i.e.,

$$Y(k) = X(k) \cdot H(k), \qquad k = 0, 1, \ldots. N-1 \qquad \ldots (3.4.6)$$

From the circular convolution property of DFT we know that

$$x_1(n) \, (\mathrm{N}) \, x_2(n) \xleftrightarrow[N]{DFT} X_1(k) X_2(k)$$

This means multiplication of two DFTs is equivalent to circular convolution of corresponding time domain sequences. Applying this property to equation 3.4.6 we get,

$$y(n) = x(n) \, (\mathrm{N}) \, h(n), \qquad n = 0, 1, \ldots.. N-1 \qquad \ldots (3.4.7)$$

Thus N-point circular convolution of $x(n)$ and $h(n)$ gives sequence $y(n)$. Thus linear convolution can be obtained by circular convolution if $N = L + M - 1$. This operation is illustrated in Fig. 3.4.3.

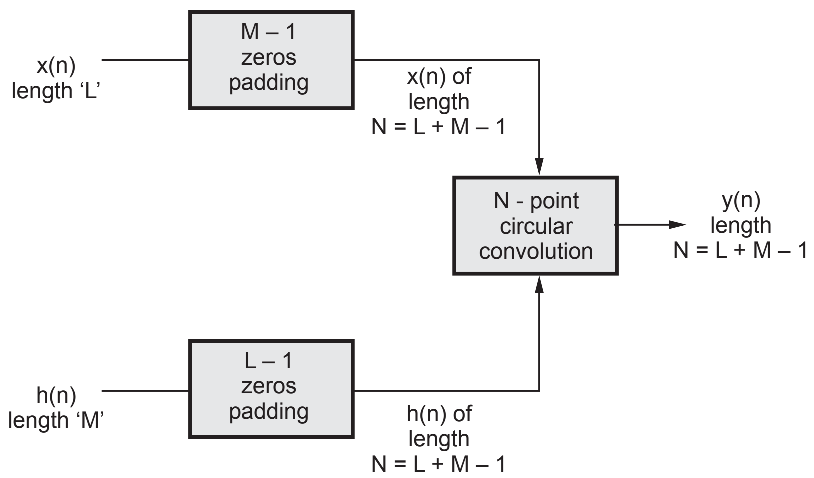


Fig. 3.4.3 Linear convolution implemented by circular convolution

Comments :

1. Linear convolution operation can be implemented by DFT and IDFT. These calculations are carried out in frequency domain.
2. Linear convolution operation can be implemented by circular convolution. The sequences are appended by zeros to the length N. This avoids aliasing effect in output sequence $y(n)$. The calculations of circular convolution are carried out in time domain.

Examples with Solution

Example 3.4.1 *An FIR filter has the impulse response of* $h(n) = \{\underset{\uparrow}{1}, 2, 3\}$. *Determine the response of the filter to the input sequence* $x(n) = \{\underset{\uparrow}{1}, 2\}$. *Use DFT and IDFT and verify the result using direct computation of linear convolution.*

Solution : Direct computation of $y(n)$ **using linear convolution :**

Let us first calculate $y(n)$ using linear convolution. We know that $y(n)$ i.e. output of the FIR filter is obtained by linear convolution of $h(n)$ and $x(n)$. We have $h(n)$ and $x(n)$ as,

$$h(n) = \{1, 2, 3\}$$
$$x(n) = \{1, 2\}$$

Fig. 3.4.4 shows linear convolution of $h(n)$ and $x(n)$ using multiplication method.

$$\begin{array}{lcccc}
 & & \downarrow & & \\
h(n) \Rightarrow & & 1 & 2 & 3 \\
x(n) \Rightarrow & & 1 & 2 & \\
 & & \uparrow & & \\
\hline
 & & 2 & 4 & 6 \\
 & 1 & 2 & 3 & \times \\
\hline
y(n) \Rightarrow & \underset{\uparrow}{1} & 4 & 7 & 6
\end{array}$$

Fig. 3.4.4 Linear convolution of $h(n)$ **and** $x(n)$

Observe that there are $L = 2$ samples in $x(n)$ and $M = 3$ samples in $h(n)$. Hence there will be $L + M - 1 = 2 + 3 - 1 = 4$ samples in $y(n)$. Above figure shows calculation of $y(n)$ using linear convolution. The sequence $y(n)$ is,

$$y(n) = \{\underset{\uparrow}{1}, 4, 7, 6\} \qquad \text{... (3.4.8)}$$

Observe that there are 4 samples in $y(n)$.

Computation of $y(n)$ **using DFT and IDFT :**

(i) Length 'N' of DFT and IDFT :

Key Point *Make lengths of two sequences same for DFT and IDFT.*

We know that there are 4 samples in the sequences $y(n)$. Hence we should consider $N = 4$. That is 4-point DFT and IDFT will give correct $y(n)$.

(ii) Zero padding to make length $N = 4$:

We should pad zeros in $h(n)$ and $x(n)$ such that their lengths will be $N = 4$. i.e.,

$$h(n) = \{1, 2, 3, 0\} \qquad \text{... (3.4.9)}$$

$$x(n) = \{1, 2, 0, 0\} \qquad \text{... (3.4.10)}$$

Here observe that $h(n)$ contains three samples hence one zero is padded at the end. And $x(n)$ has two samples hence two zeros are padded at the end. Note that these zeros does not change the meaning.

(iii) Calculation of DFTs $H(k)$ and $X(k)$:

Now let us calculate the 4-point DFTs of $h(n)$ and $x(n)$.

The N-point DFT is given in matrix form by equation 3.1.9 as,

$$X_N = [W_N]\, x_N \qquad \text{... (3.4.11)}$$

Here X_N is the $N \times 1$ matrix of DFTs

W_N is the $N \times N$ matrix of twiddle factors

and x_N is the $N \times 1$ matrix of sequence $x(n)$.

For $N = 4$ above equation becomes,

$$X_4 = [W_4]\, x_4$$

The above equation can be expanded as [see equation 3.1.6, equation 3.1.7 and equation 3.1.8].

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} \qquad \text{... (3.4.12)}$$

The values of W_4^0, W_4^1, W_4^2, ...W_4^9 etc are obtained in Fig. 3.1.2. Let us put these values in above equation. And putting for $x(n)$ from equation 3.4.10 in above equation we get,

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+0+0 \\ 1-j2+0+0 \\ 1-2+0+0 \\ 1+j2+0+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1-j2 \\ -1 \\ 1+j2 \end{bmatrix} \quad \text{... (3.4.13)}$$

Now let us consider the DFT of $h(n)$. In the matrix form it can be written as,

$$H_4 = [W_4]\, h_4$$

The above equation is written on the basis of equation 3.4.11 with $N = 4$. Similarly the above equation can be expanded in matrix form as,

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \end{bmatrix}$$

Putting the values of W_4^0, W_4^1, W_4^2, ... etc and $h(n)$ we get,

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+3+0 \\ 1-j2-3+0 \\ 1-2+3+0 \\ 1+j2-3+0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2-j2 \\ 2 \\ -2+j2 \end{bmatrix} \quad \text{... (3.4.14)}$$

(iv) To multiply two DFTs i.e. $Y(k) = H(k) \cdot X(k)$ *:*

The DFT of $y(n)$ be $Y(k)$ this DFT can be obtained as,

$$Y(k) = H(k) \cdot X(k)$$

Putting the values of $H(k)$ and $X(k)$ from equation 3.4.14 and equation 3.4.13 we get,

$$\begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{bmatrix} = \begin{bmatrix} (6) \times (3) \\ (-2-j2) \times (1-j2) \\ (2) \times (-1) \\ (-2+j2) \times (1+j2) \end{bmatrix}$$

Converting the values of their polar equivalents

$$\begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{bmatrix} = \begin{bmatrix} 6 \times 3 \\ 2.82 \angle -135^\circ \times 2.23 \angle -63.45^\circ \\ 2 \times -1 \\ 2.82 \angle 135^\circ \times 2.23 \angle 63.45^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 18 \\ 6.288 \angle -198.45^\circ \\ -2 \\ 6.288 \angle 198.45^\circ \end{bmatrix}$$

Now let us convert the above polar DFT values to their equivalent rectangular values i.e.,

$$\begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{bmatrix} = \begin{bmatrix} 18 \\ -6+j2 \\ -2 \\ -6-j2 \end{bmatrix} \qquad \text{... (3.4.15)}$$

This is the 4-point DFT of $y(n)$

(v) To obtain $y(n)$ *from* $Y(k)$ *by IDFT :*

Now let us calculate Inverse DFT (IDFT) to get $y(n)$. The IDFT is given by equation 3.1.10 as,

$$x_N = \frac{1}{N}\left[W_N^*\right] X_N$$

Similarly for y_N above equation can be written as,

$$y_N = \frac{1}{N}\left[W_N^*\right] Y_N \qquad \text{... (3.4.16)}$$

for $N = 4$, $$y_4 = \frac{1}{4}\left[W_4^*\right] Y_4 \qquad \text{... (3.4.17)}$$

Here Y_4 is 4×1 matrix of DFT. From equation 3.4.15 we have,

$$Y_4 = \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{bmatrix} = \begin{bmatrix} 18 \\ -6+j2 \\ -2 \\ -6-j2 \end{bmatrix}$$

$\left[W_4^*\right]$ is complex conjugate of $\left[W_4\right]$. We have $\left[W_4\right]$ as,

$$[W_4] = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \quad \text{... (3.4.18)}$$

Now complex conjugate of $[W_4]$ can be obtained by inverting signs of imaginary values. i.e.,

$$[W_4^*] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \quad \text{... (3.4.19)}$$

Putting the values in equation 3.4.17 we get,

$$y_4 = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 18 \\ -6+j2 \\ -2 \\ -6-j2 \end{bmatrix}$$

$$= \begin{bmatrix} 18-6+j2-2-6-j2 \\ 18-j6-2+2+j6-2 \\ 18+6-j2-2+6+j2 \\ 18+j6+2+2-j6+2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 \\ 16 \\ 28 \\ 24 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 6 \end{bmatrix}$$

Thus the sequence $y(n)$ is

$$y(n) = \{1, 4, 7, 6\}$$

Observe that this result is same as that obtained by direct calculation of linear convolution in equation 3.4.8. Thus linear convolution can be obtained using DFT and IDFT.

Example 3.4.2 *Determine the response of the FIR filter whose unit sample response is given as,*

$$h(n) = \{\underset{\uparrow}{1}, 2\}$$

When input applied is, $x(n) = \{2, 1\}$. *Use circular convolution and verify your result using linear convolution.*

Solution : The given sequences are,

$$h(n) = \{\underset{\uparrow}{1}, 2\} \text{ with } M = 2 \text{ and}$$

$$x(n) = \{\underset{\uparrow}{2}, 1\} \quad \text{with } L = 2$$

The output of the FIR filter can be obtained by linear convolution of $x(n)$ and $h(n)$ i.e.

$$y(n) = x(n) * h(n)$$

Fig. 3.4.5 below illustrates linear convolution of $x(n)$ and $h(n)$ using multiplication.

$$\begin{array}{rccc}
x(n) \Rightarrow & & \underset{\uparrow}{2} & 1 \\
h(n) \Rightarrow & & \underset{\uparrow}{1} & 2 \\
\hline
 & & 4 & 2 \\
 & 2 & 1 & \times \\
\hline
y(n) \Rightarrow & \underset{\uparrow}{2} & 5 & 2
\end{array}$$

Fig. 3.4.5 Linear convolution of $x(n)$ and $h(n)$

Thus output of FIR filter is,

$$h(n) = \{\underset{\uparrow}{2}, 5, 2\} \qquad \text{... (3.4.20)}$$

Key Point *The output contains $L + M - 1 = 2 + 2 - 1 = 3$ samples. If we compute the circular convolution of these two sequences for three or more points we will get the same output sequence.*

Circular convolution of $x(n)$ and $h(n)$ with $N = 4$:

Since output has '3' samples, we should perform DFT or circular convolution for at least '3' points or more. Hence let us take $N = 4$ points. Hence $x(n)$ and $h(n)$ should have 4 samples. Therefore these sequences are appended with zeros to get 4 samples. This is called zero padding. These sequences are,

$$\left.\begin{array}{l} h(n) = \{\underset{\uparrow}{1}, 2, 0, 0\} \\ x(n) = \{\underset{\uparrow}{2}, 1, 0, 0\} \end{array}\right\} \qquad \text{... (3.4.21)}$$

Observe that zero padding is done at the end, hence meaning of sequences remain unchanged. Fig. 3.4.6 shows $x(n)$ and $h(n)$ plotted across the circle.

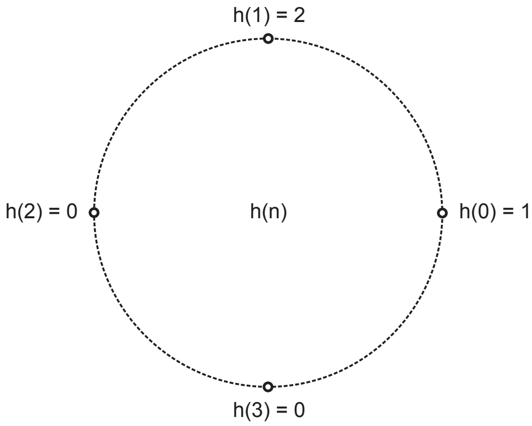
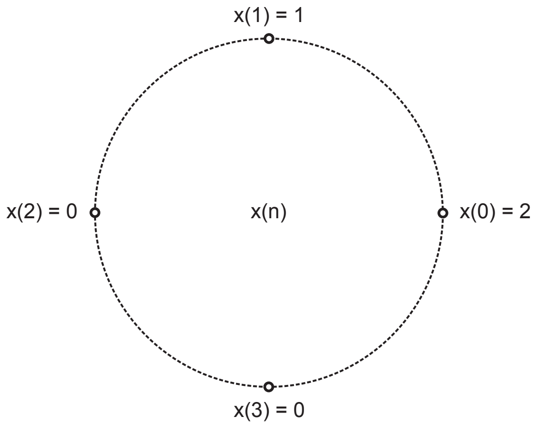


Fig. 3.4.6 (a) Sequence $h(n)$



(b) Sequence $x(n)$

The circular convolution of $x(n)$ and $h(n)$ is given as,

$$y(m) = \sum_{n=0}^{3} h(n)\, x\left((m-n)\right)_4, \qquad m = 0, 1, 2, 3$$

$$y(0) = \sum_{n=0}^{3} h(n)\, x\left((-n)\right)_4$$

Fig. 3.4.7 shows $h(n)$ and $x\left((-n)\right)_4$ plotted on concentric circles to obtain $y(0)$. $x\left((-n)\right)_4$ is obtained by circular folding of $x(n)$ of Fig. 3.4.6 (b).

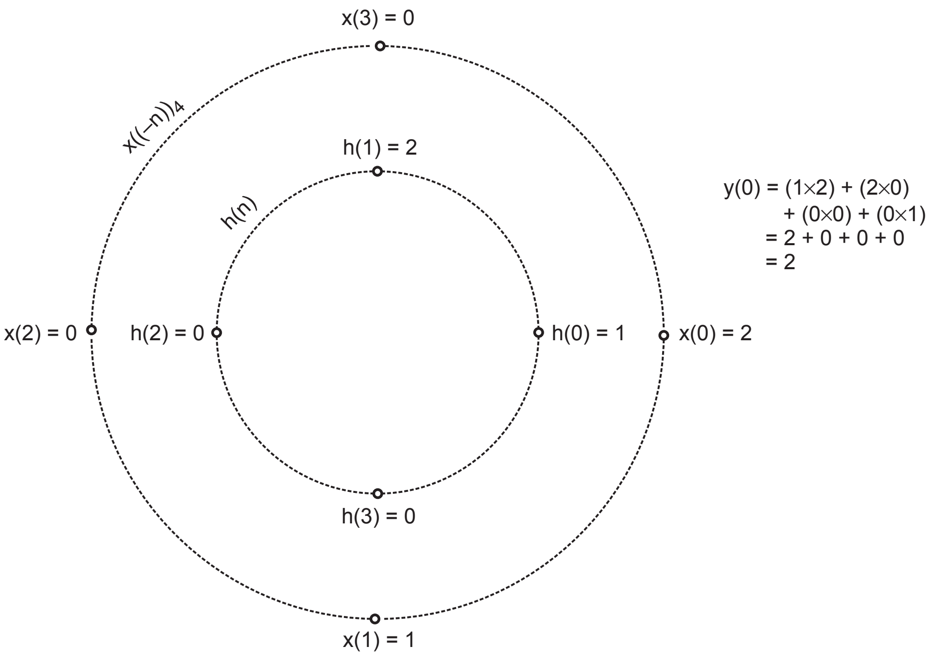


Fig. 3.4.7 Circular convolution to obtain $y(0)$

As shown in Fig. 3.4.7 the two sequences are multiplied point by point and the products are added. The value of $y(0) = 2$ as obtained above.

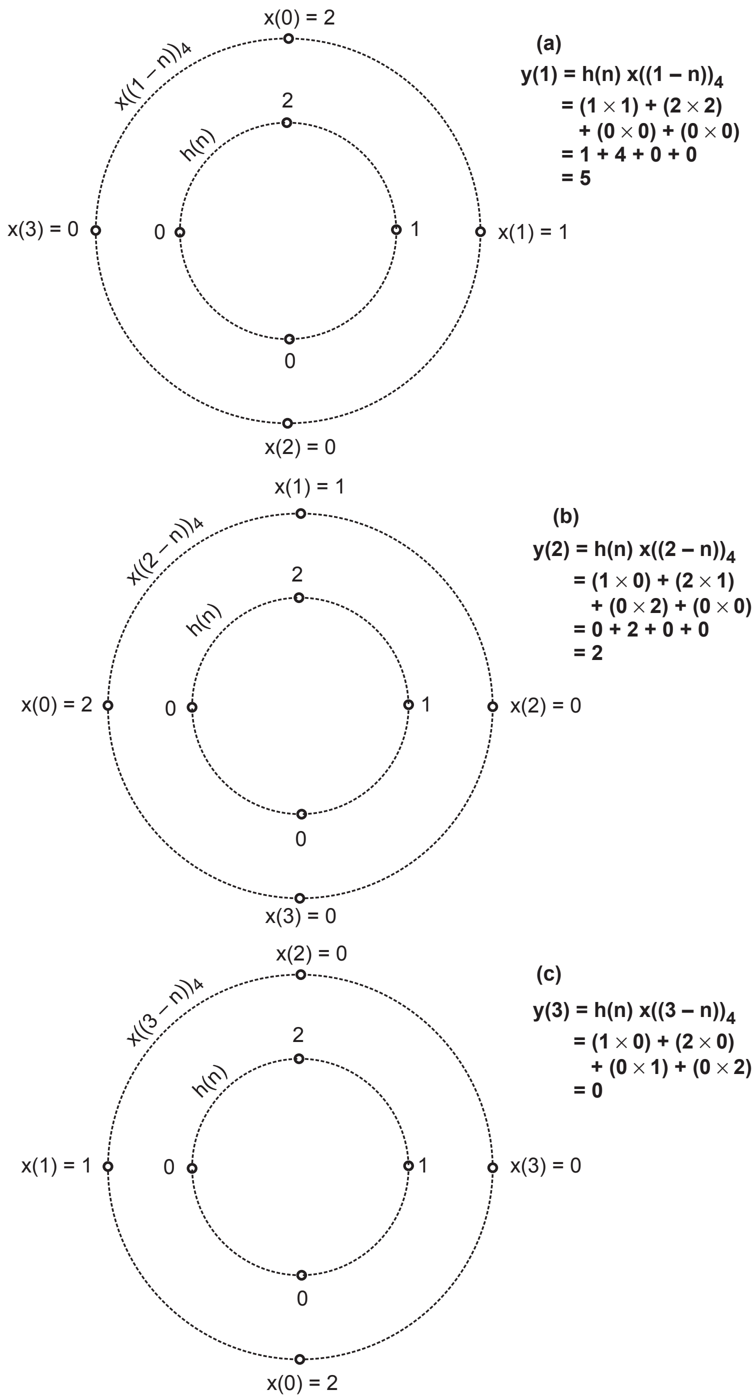


Fig. 3.4.8

Fig. 3.4.8 (a), (b) and (c) show how $y(1), y(2)$ and $y(3)$ are obtained using the same method. The sequence $x\left((-n)\right)_4$ plotted on outer circle is shifted anticlockwise by one sample successively. (See Fig. 3.4.8 on previous page.)

Thus from Fig. 3.4.8 and Fig. 3.4.7 the sequence $y(n)$ due to circular convolution is,

$$\begin{aligned} y(n) &= \{2, 5, 2, 0\} \\ &= \{\underset{\uparrow}{2}, 5, 2\} \end{aligned}$$

This sequence is same as that obtained by linear convolution in equation 3.4.20. Thus circular convolution provides linear convolution.

This similar operation can be performed by taking 4-point DFTs of $x(n)$ and $h(n)$ of equation 3.4.21. Then 4-point DFT of output is given by point by point multiplication of $X(k)$ and $H(k)$ i.e.

$$Y(k) = H(k) \cdot X(k), \qquad k = 0, 1, 2, 3$$

The output sequence $y(n)$ can be obtained by taking IDFT of $Y(k)$ i.e.

$$y(n) = IDFT\ \{Y(k)\}$$

3.4.2 DFT for Linear Filtering of Long Duration Sequences

Consider the FIR filter system of Fig. 3.4.9 for linear filtering. The input sequence is $x(n)$ and unit sample response of the filter is $h(n)$. We know that the output $y(n)$ of the filter is obtained by linear convolution of $x(n)$ and $h(n)$. We have also seen that linear convolution can be implemented using DFT. Sometimes the input data sequence $x(n)$ is a real time signal such as speech, ECG etc. Such signals are very long in duration. Linear filtering of such long duration sequences can be implemented using DFT. The long duration sequence is segmented in the short duration blocks (subsegments). Since the filtering is linear, the successive input sequence blocks are filtered one at a time using DFT. The corresponding output blocks are fitted together to generate combined output sequence.

Advantages in using DFT :

1. Complicated calculations are involved in linear convolution. Hence DFT provides more simpler approach.
2. Linear filtering using DFT is computationally efficient because of FFT algorithms.

Two methods are available based on this approach. These methods are overlap save method and overlap add method. These methods are discussed next.

3.4.3 Overlap Save Method for Linear Filtering

For linear filtering we discussed the use of DFT. When the input data sequence is long, then it requires large time to get the output sequence. Hence other techniques are used to filter long data sequences. These techniques are overlap save method and overlap add method.

Instead of finding the output of complete input sequence, it is broken into small length sequences. The outputs due to these small length input sequences are computed fast. Since the filtering is linear, the outputs due to these small length sequences are fitted one after another (concatenated) to get the final output sequence.

Let the unit sample response of FIR filter has length 'M'. Let the input data sequence be segmented into blocks of 'L' samples.

Overlap save method :

In the overlap save method 'L' samples of the current segment and $(M-1)$ samples of the previous segment forms the input data block.

Thus the input data blocks will be,

$$x_1(n) = \left\{ \underbrace{0, 0, 0, \ldots\ldots 0}_{\text{first block padded with } M-1 \text{ zeros}}, \quad \underbrace{x(0), x(1), \ldots\ldots x(L-1)}_{\text{'L' samples of data sequence } x(n)} \right\} \qquad \ldots (3.4.22)$$

$$x_2(n) = \left\{ \underbrace{x(L-M+1), \ldots\ldots, x(L-1)}_{(M-1) \text{ data samples of sequence } x_1(n)}, \quad \underbrace{x(L), x(L+1), \ldots.. x(2L-1)}_{\text{Next 'L' data samples of sequence } x(n)} \right\} \qquad \ldots (3.4.23)$$

$$x_3(n) = \left\{ \underbrace{x(2L-M+1), \ldots\ldots, x(2L-1)}_{(M-1) \text{ data samples of sequence } x_2(n)}, \quad \underbrace{x(2L), x(2L+1), \ldots x(3L-1)}_{\text{Next 'L' data samples of sequence } x(n)} \right\}$$

$$\ldots (3.4.24)$$

Thus $x_1(n), x_2(n), x_3(n), \ldots$ etc blocks of $N = L+M-1$ samples are formed for block by block filtering. We know that unit sample response $h(n)$ contains 'M' samples. Hence its length is made 'N' by padding $L-1$ zeros as shown below.

$$h(n) = \left\{ \underbrace{h(0), h(1), \ldots. h(M-1)}_{\text{'M' samples of unit sample response}}, \quad \underbrace{0, 0, \ldots..(L-1\ zeros)}_{(L-1) \text{ zeros are padded to make } N = L+M-1 \text{ total samples}} \right\}$$

$$\ldots (3.4.25)$$

Thus as shown above $h(n)$ also contains $N = L+M-1$ samples. The N-point DFT of this $h(n)$ will be $H(k)$. Let the DFT of m^{th} input data block be $X_m(k)$ and corresponding DFT of output be $\hat{Y}_m(k)$. i.e.,

$$\hat{Y}_m(k) = H(k)\, X_m(k), \quad k = 0, 1, \ldots. N-1 \qquad \ldots (3.4.26)$$

The sequence $\hat{y}_m(n)$ can be obtained by taking N-point IDFT of $\hat{Y}_m(k)$. Then the individual samples of this sequence can be represented as,

$$\hat{y}_m(n) = \left\{\hat{y}_m(0), \hat{y}_m(1), \ldots. \hat{y}_m(M-1), \hat{y}_m(M), \hat{y}_m(M+1), \ldots \hat{y}_m(N-1)\right\} \qquad \ldots (3.4.27)$$

This type of sequence will be obtained due to input data blocks $x_1(n), x_2(n), x_3(n), \ldots$ etc. The sequence $\hat{y}_m(n)$ contains $N = L+M-1$ samples. Observe that in $x_1(n), x_2(n), x_3(n), \ldots$ etc. initial $M-1$ samples are taken from previous segment i.e. overlap, and last 'L' samples are actual input samples. Because of this overlap of initial $(M-1)$ samples in input data block, there is aliasing in initial $(M-1)$ samples in the corresponding output data block i.e. $\hat{y}_m(n)$. The aliasing effect occurs because of circular shift and overlap of samples in computation of DFT. Hence the initial $(M-1)$ samples of $\hat{y}_m(n)$ must be discarded. The last 'L' samples of $\hat{y}_m(n)$ are the correct output samples. Hence the actual output samples to be considered in every output block are,

$$\hat{y}_m(n) = y_m(n), \quad \text{for} \quad n = M, M+1, \ldots. N-1 \qquad \ldots (3.4.27(a))$$

Such blocks are fitted one after another to get the final output. The overlap save method discussed above is illustrated in Fig. 3.4.9. (See Fig. 3.4.9 on next page.)

3.4.4 Overlap Add Method for Linear Filtering

In this method the data blocks of length $N = L+M-1$ are formed by taking 'L' samples from input sequence and padding $M-1$ zeros as shown below.

$$x_1(n) = \left\{ \underbrace{x(0), x(1), \ldots. x(L-1)}_{\text{'L' samples of input data sequence } x(n)}, \underbrace{0, 0, \ldots\ldots 0}_{(M-1) \text{ zeros are padded at the end}} \right\} \qquad \ldots (3.4.28)$$

$$x_2(n) = \left\{ \underbrace{x(L), x(L+1), \ldots.. x(2L-1)}_{\text{Next 'L' samples of input sequence } x(n)}, \underbrace{0, 0, \ldots\ldots.. 0}_{(M-1) \text{ zeros are padded at the end}} \right\} \qquad \ldots (3.4.29)$$

$$x_3(n) = \left\{ \underbrace{x(2L), x(2L+1), \ldots x(3L-1)}_{\text{Next 'L' samples of input sequence } x(n)}, \underbrace{0, 0, \ldots\ldots. 0}_{(M-1) \text{ zeros}} \right\} \qquad \ldots (3.4.30)$$

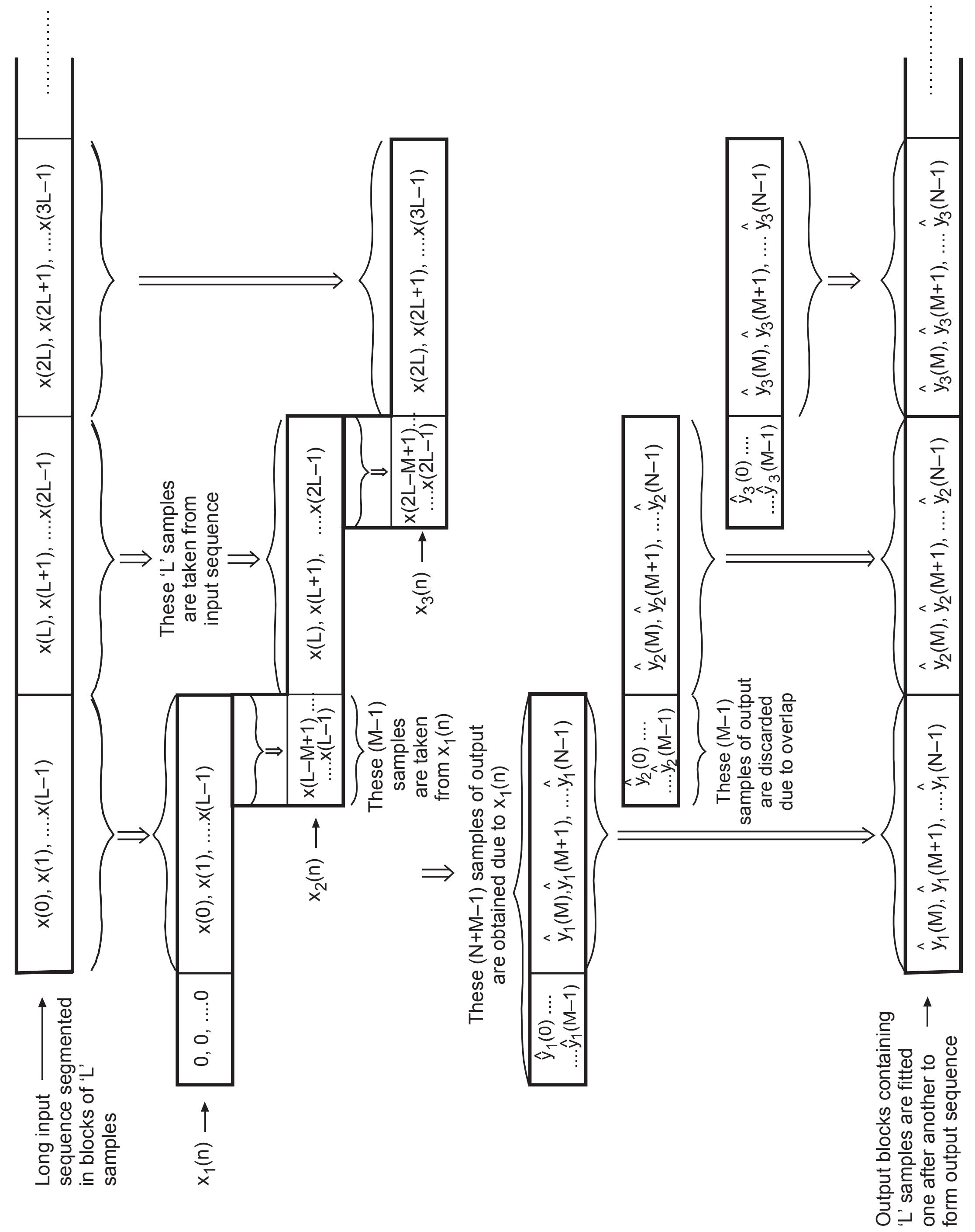


Fig. 3.4.9 Overlap save method for filtering of long data sequences using DFT

Thus each data block is of length 'N'. The N-point DFT $Y_m(k)$ of the output is obtained by multiplying $H(k)$ and $X_m(k)$. i.e.,

$$Y_m(k) = H(k) \cdot X_m(k), \quad k = 0, 1, ... N-1 \qquad ... (3.4.31)$$

Here $H(k)$ is N-point DFT of unit sample response $h(n)$ and $X_m(k)$ is DFT of m^{th} data block.

The sequence $y_m(n)$ is obtained by taking N-point IDFT of $Y_m(k)$. Thus samples of sequence $y_m(n)$ will be,

$$y_1(n) = \{y_1(0), y_1(1), \ldots y_1(L-1), y_1(L), y_1(L+1), \ldots y_1(N-1)\} \qquad ... (3.4.32)$$

$$y_2(n) = \{y_2(0), y_2(1), \ldots y_2(L-1), y_2(L), y_2(L+1), \ldots y_2(N-1)\} \qquad ... (3.4.33)$$

Similarly other sequences are obtained. We know that each data block is terminated with $M-1$ zeros. Hence the last $M-1$ samples of each output sequence must be overlapped and added to first $M-1$ samples of succeeding output sequence. This is done since the sequence is appended with zeros at the end. And hence the name overlap and add is given. For example the output $y(n)$ due to overlap and adding of $y_1(n)$ and $y_2(n)$ is given as follows :

$$y(n) = \{y_1(0), y_1(1), \ldots y_1(L-1), [\, y_1(L) + y_2(0)\,], [\, y_1(L+1) + y_2(1)\,],$$
$$..... [\, y_1(N-1) + y_2(M-1)\,],\ y_2(M), ... y_2(N-1)\} \qquad ... (3.4.34)$$

This process continues till the end of input sequence. This algorithm is illustrated in Fig. 3.4.10. (See Fig. 3.4.10 on next page.)

Here note that the $(M-1)$ overlapping samples of the output blocks are not discarded. This is because the input data blocks are padded with $(M-1)$ zeros to make their length equal to 'N'. Hence there will be no aliasing (effect due to circular shifting and overlap in DFT) in the output data blocks. Therefore for the last $(M-1)$ samples of current output block must be added to the first $(M-1)$ samples of next output block. This is the major difference between this method and overlap save method. In overlap save method the initial $(M-1)$ samples of every output block are discarded since they are aliased due to overlap.

Comments :

Eventhough overlap save and overlap add methods seem to be complicated, the computations involved actually are less compared to linear convolution. This is because DFT can be computed fast using FFT algorithms. These algorithms compute DFT with very small number of computations. These algorithms are discussed in next section.

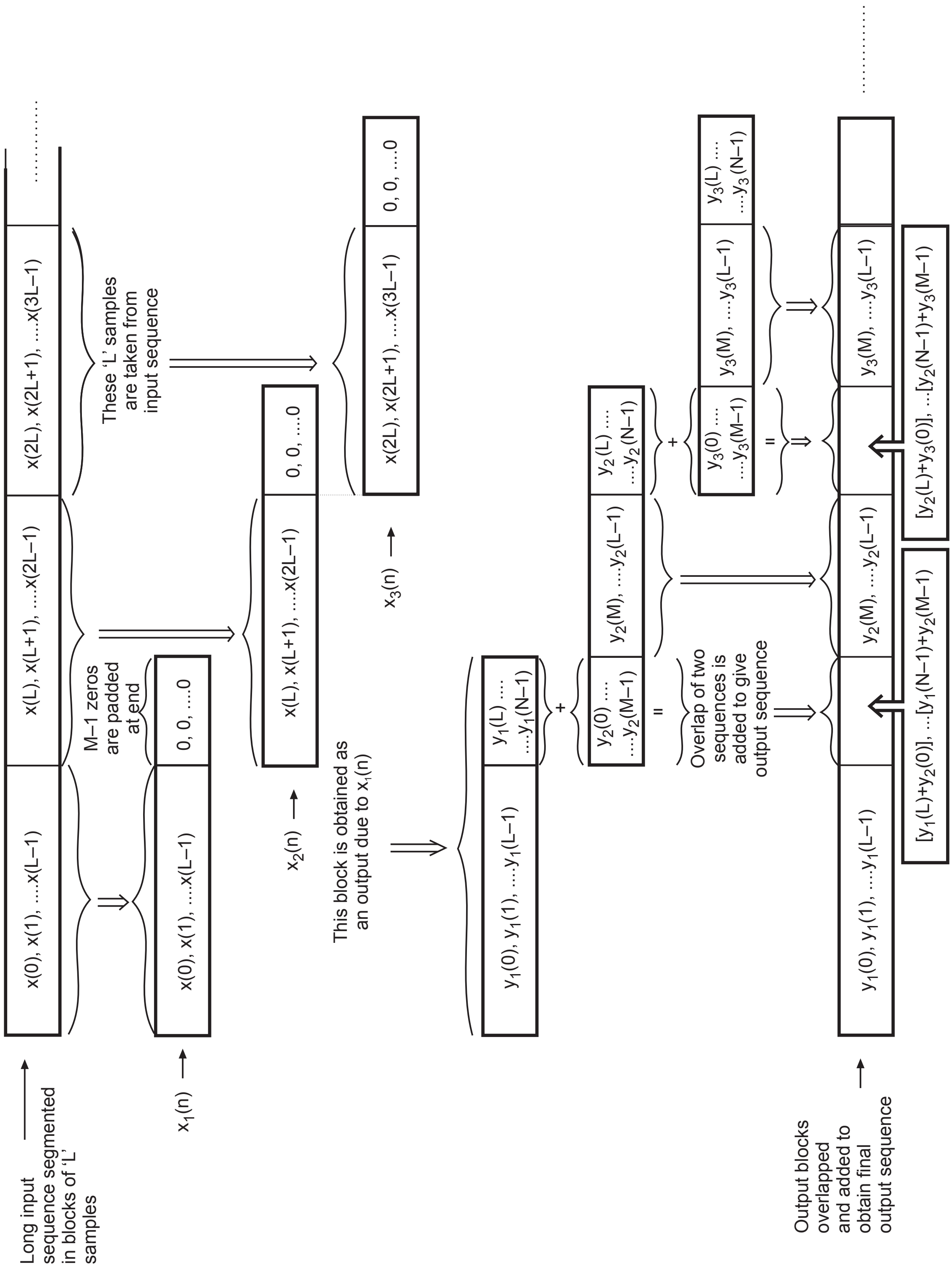


Fig. 3.4.10 Overlap add method for filtering of long data sequences using DFT

3.4.5 Spectrum Analysis using DFT

Principle

When we want to analyze the spectrum of the signal its fourier transform is taken. For example consider the sequence $x(n)$ and its fourier transform $X(\omega)$. Since $X(\omega)$ is continuous function of 'ω', it is not possible to analyze $X(\omega)$ on digital computer or digital signal processor. Hence DFT of the signal can be used for spectrum analysis.

The signal to be analyzed is passed through the antialiasing filter and sampled at the rate of $F_s \geq 2F_{max}$. Hence highest frequency in the sampled signal is $\frac{F_s}{2}$. For spectrum analysis some finite length of the sequence is taken.

Technique of spectrum analysis

Let 'T' be the sampling interval and 'L' number of samples of input sequence are taken. Then the time interval of the sequence will be $T_0 = LT$. It can be shown that the time interval of the sequence should be as large as possible. Because the smallest frequency resolution is given as $\frac{1}{T_0}$.

Let the 'L' number of samples of the sequence $x(n)$ be obtained by multiplying $x(n)$ by rectangular window $w(n)$ of length 'L'. i.e.,

$$\hat{x}(n) = x(n)\, w(n) \qquad \ldots (3.4.35)$$

Here $w(n)$ is the rectangular window which is given as,

$$w(n) = \begin{cases} 1, & \text{for} \quad 0 \leq n \leq L-1 \\ 0 & \text{otherwise} \end{cases} \qquad \ldots (3.4.36)$$

Let $x(n)$ be the cosine wave consisting single frequency which is given as,

$$x(n) = \cos(\omega_0 n) \qquad \ldots (3.4.37)$$

Then the finite length sequence $\hat{x}(n)$ becomes,

$$\hat{x}(n) = \cos(\omega_0 n) \quad \text{for} \quad 0 \leq n \leq L-1 \qquad \ldots (3.4.38)$$

Here we have taken only 'L' samples of cosine wave.

Then by definition of DFT,

$$\hat{X}(k) = \sum_{n=0}^{L-1} \hat{x}(n)\, e^{-j2\pi kn/N} \qquad \ldots (3.4.39)$$

Here $k = 0, 1, \ldots N-1$. Actually $N \geq L$. But spectrum is better if we take large value of 'N' compared to 'L'. The frequency spectrum can be analyzed as $\hat{X}(k)$ with respect to k.

The total frequency range of '2π' is divided into 'N' points. Hence the frequency resolution is $\frac{2\pi}{N}$. The individual frequency components are numbered by 'k' in the range from 0 to 2π .

Advantages of using DFT for spectral analysis :

1. DFT can be computed on digital computer or digital signal processor.
2. DFT can be computed quickly using FFT algorithms hence fast processing is done.
3. Other digital signal processing operations such as power spectrum estimation or calculation of harmonics can be done easily.
4. More accurate resolution can be obtained by increasing number of samples i.e. N.

Limitations of using DFT for spectral analysis :

1. Since finite length of the input sequence is taken, spectrum is not perfect.
2. Because of the windowing, the power leaks out (i.e. spreads) in the entire frequency range. Hence selection of windows depends upon type of signal to be resolved.
3. For more accurate spectrum, proper window function and large values of 'N' and 'L' are required. This increase processing time.

The advantages outweigh the disadvantages for using DFT in spectrum analysis. Hence DFT is used extensively for such applications.

Example for Practice

Example 3.4.3 *Using circular convolution obtain linear convolution between the sequences*

$$x(n) = \left(\frac{1}{2}\right)^n, \quad 0 \le n \le 3$$

$$h(n) = \left(\frac{1}{4}\right)^n, \quad 0 \le n \le 3$$

[Ans. : $y(n) = \left\{1, \frac{3}{4}, \frac{7}{16}, \frac{15}{64}, \frac{7}{128}, \frac{3}{256}, \frac{1}{512}\right\}$]

Review Questions

1. *Explain how linear convolution can be obtained using DFT and IDFT. Also explain how linear convolution can be obtained using circular convolution.*
2. *With appropriate diagrams describe*

   i) Overlap-save method. *ii) Overlap-add method.*

3.5 Computation of DFT using FFT Algorithms

We studied Discrete Fourier Transform (DFT) earlier. The DFT is used in large number of applications of DSP such as filtering, correlation analysis, spectrum analysis etc. But the direct computation of DFT involves large number of computations. Hence the processor remain busy. Special algorithms have been developed to compute DFT quickly. These algorithms exploit the periodicity and symmetry properties of twiddle factors (phase factors). Hence DFT is computed fast using such algorithms compared to direct computation. These algorithms are collectively called as Fast Fourier Transform (FFT) algorithms. These algorithms are very efficient in terms of computations. As the value of 'N' increases, the computational efficiency of FFT algorithms increases.

3.5.1 Direct Computation of DFT

Before directly starting study of FFT algorithms, let us see the complexity of computations in the direct computation of DFT. By definition, the DFT is given as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\ W_N^{kn}, \qquad k = 0, 1, \ldots.. N-1$$

Here $x(n)$ is the input sequence which can be real or complex. And W_N is the twiddle factor or phase factor which is complex number. Thus computation of $X(k)$ involves the multiplications and summations of complex numbers. Fig. 3.5.1 shows the expansion of summation of the above equation.

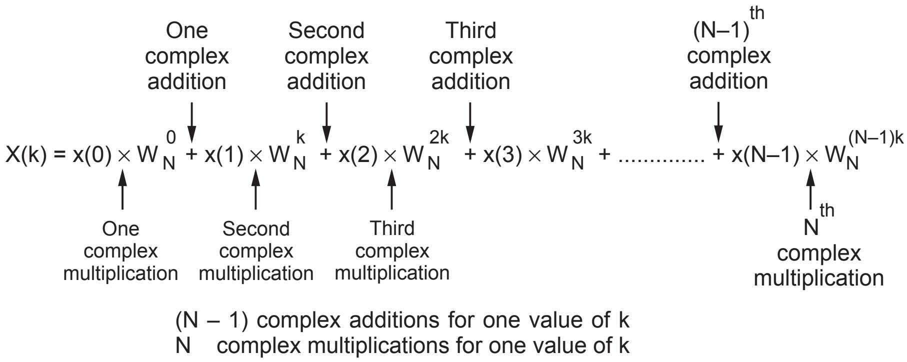


Fig. 3.5.1 Number of complex multiplications and additions in direct computation of DFT

For any value of 'k' in this equation, observe the multiplications and additions involved. 'N' number of complex multiplications and '$(N-1)$' number of complex additions are required to calculate $X(k)$ for one value of k. We know that there are such $k = 0, 1, \ldots N-1$; i.e. 'N' number of values of $X(k)$.

Hence,

$$\left.\begin{array}{l}\text{Number of complex}\\ \text{multiplications required for}\\ \text{calculating X (k) for}\\ k = 0, 1, \ldots, N-1\end{array}\right\} = N \times N = N^2 \qquad \ldots (3.5.1)$$

And,

$$\left.\begin{array}{l}\text{Number of complex}\\ \text{additions required for}\\ \text{calculating X (k) for}\\ k = 0, 1, \ldots N-1\end{array}\right\} = (N-1) \times N = N^2 - N \qquad \ldots (3.5.2)$$

Thus if we want to evaluate the 1024 point DFT of the sequence, then N = 1024,

$$\text{Complex multiplications} = N^2 = (1024)^2 \approx 1 \times 10^6$$

$$\text{Complex additions} = N^2 - N = (1024)^2 - 1024 \approx 1 \times 10^6$$

Here let us assume that the processor executes one complex multiplication in 1 microsecond. Let the one complex addition is also executed in 1 microsecond. Then the time required for computations will be,

Time = (complex multiplications) × (time for one multiplication)

+ (complex additions) × (time for one addition)

$$= \left(1 \times 10^6 \times 1 \times 10^{-6}\right) + \left(1 \times 10^6 \times 1 \times 10^{-6}\right)$$

$$= 1 + 1 = 2 \text{ seconds.}$$

Thus two seconds of the time is required for computations of 1024 point DFT. In terms of processors, this is large time. This is because processors has to do lot of other work such as fetching and storing data in the memory, handling data inputs and outputs, displays etc. Hence real time computation of DFT for large values of 'N' becomes practically impossible by direct computation. We will see further that FFT algorithms are extremely fast and their computation speed increases as 'N' increases. Hence FFT algorithms are used always to compute DFT.

Example 3.5.1 *In the direct computation of N-point DFT of a sequence, how many multiplications, additions and trigonometric function evaluation are required ?*

Solution : Number of complex multiplications and complex additions :

By equation 3.5.1 we know that N^2 complex multiplications are required in the direct computation of N-point DFT. Similarly by equation 3.5.2 we know that $\left(N^2 - N\right)$ complex additions are required in the direct computation of N-point DFT.

To determine number of real multiplications, real additions and trigonometric functions :

We know that DFT is given as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j 2\pi kn/N}, \qquad k = 0, 1, \ldots, N-1$$

Here $e^{-j\frac{2\pi}{N}} = W_N$ as we have seen. Hence above equation can be written as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \qquad k = 0, 1, \ldots, N-1 \qquad \text{... (3.5.3)}$$

We know that both $x(n)$ and W_N^{kn} are complex valued. Let their real and imaginary parts be expressed as,

$$x(n) = x_R(n) + j\, x_I(n)$$

and

$$W_N^{kn} = W_{RN}^{kn} + j\, W_{IN}^{kn}$$

Here $x_R(n)$ is real part of $x(n)$

$x_I(n)$ is imaginary part of $x(n)$

W_{RN}^{kn} is real part of W_N^{kn}

W_{IN}^{kn} is imaginary part of W_N^{kn}

Then equation 3.5.3 can be written as,

$$X(k) = \sum_{n=0}^{N-1} \left[x_R(n) + j\, x_I(n)\right]\left[W_{RN}^{kn} + j\, W_{IN}^{kn}\right]$$

$$= \sum_{n=0}^{N-1} \left\{\left[x_R(n)\, W_{RN}^{kn} - x_I(n)\, W_{IN}^{kn}\right] + j\left[x_R(n)\, W_{IN}^{kn} + x_I(n)\, W_{RN}^{kn}\right]\right\} \qquad \text{... (3.5.4)}$$

Here compare equation 3.5.3 with above equation. Observe that one complex multiplication $x(n)\,W_N^{kn}$ of equation 3.5.3 is converted as follows in Fig. 3.5.2.

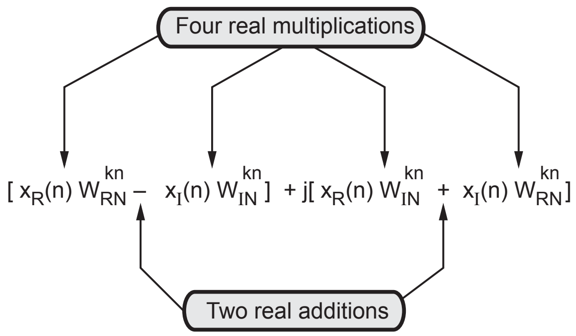


Fig. 3.5.2 Conversion of one complex multiplication

Important Notes :

(1) Subtraction is also counted as addition in signal processing, since it requires almost same time as addition.

(2) The complex numbers are expressed as,

(Real part) + j (Imaginary part)

This addition is never executed since it is just way of representing complex numbers.

(3) 'Real part' and 'Imaginary part' of the complex number are basically real numbers.

From Fig. 3.5.2 we have,

One complex multiplication is converted to $\Rightarrow$ 4 Real multiplications, 2 Real additions ... (3.5.5)

We know that for each value of k, there are 'N' complex multiplications. Hence we can write,

For each value of k, N complex multiplications are converted to $\Rightarrow$ $4N$ Real multiplications, $2N$ Real additions ... (3.5.6)

We know that k is varied from 0 to $N-1$ for computing DFT of $x(n)$. In other words, k takes 'N' different values for computation of DFT. Hence there are N^2 complex multiplications. These are converted as follows :

For complete DFT, $(N \times N)$ complex multiplications are converted to $\Rightarrow$

$4N \times N = 4N^2$ Real multiplications

$2N \times N = 2N^2$ Real additions ... (3.5.7)

Now let us see how complex additions are converted. Let us consider the two complex numbers $a + jb$ and $c + jd$. The addition of two complex numbers is shown in Fig. 3.5.3.

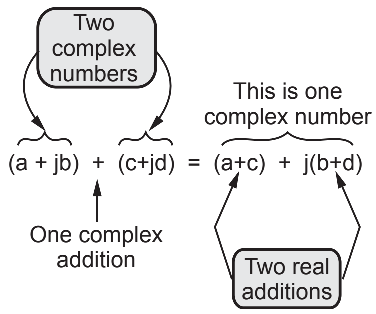


Fig. 3.5.3 Conversion of one complex addition. Here a, b, c and d are real numbers

As shown in Fig. 3.5.3 above, one complex addition is converted to two real additions. i.e.,

One complex addition is converted to $\Rightarrow$ Two real additions ... (3.5.8)

We know that for each value of 'k' there are $(N-1)$ complex additions. Hence we can write,

For each value of *k*, $(N-1)$ complex additions are converted to $\Rightarrow 2 \times (N-1) = 2(N-1)$ real additions ... (3.5.9)

We know that k varies from 0 to $N-1$ for computing DFT of $x(n)$. i.e. k takes $'N'$ different values. Hence $(N-1)N$ complex additions are converted as,

For complete *DFT* $(N-1)$ complex additions are converted to $\Rightarrow$ $2 \times (N-1)N = 2N^2 - 2N$ real additions …(3.5.10)

Hence from equation 3.5.7 and above equation, total number of real additions are,

Total real additions in computation of DFT = $2N^2 + 2N^2 - 2N$

$$= 4N^2 - 2N$$

$$= N(4N-2) \quad \text{... (3.5.11)}$$

And total number of real multiplications are (see equation 3.5.7),

Total real multiplications in computation of DFT = $4N^2$ ... (3.5.12)

We know that, $$W_N^{kn} = e^{-j2\pi kn/N}$$

By Euler's identity $e^{-j\theta} = cos\,\theta - j\,sin\,\theta$, we can write above equation as,

$$W_N^{kn} = cos\frac{2\pi\,kn}{N} - j\,sin\frac{2\pi\,kn}{N}$$

Here $$W_{RN}^{kn} = cos\left(\frac{2\pi\,kn}{N}\right) \text{ i.e. real part of } W_N^{kn}$$

and $$W_{IN}^{kn} = -\,sin\left(\frac{2\pi\,kn}{N}\right) \text{ i.e. imaginary part of } W_N^{kn}$$

Here note that two trigonometric values are executed for every value of W_N^{kn}. In equation 3.5.3 or equation 3.5.4 observe that n varies from 0 to $N - 1$ and hence W_N^{kn} takes 'N' different values. Similarly 'k' varies from 0 to $N-1$ and hence W_N^{kn} takes $'N'$ different values. Hence for simultaneous variation of 'k' and 'n', W_N^{kn} takes $N \times N = N^2$ different values. Hence,

Number of trigonometric values evaluated in computation of DFT $= 2 \times N \times N = 2N^2$ ... (3.5.13)

Table 3.5.1 summarizes the results of this example :

Sr. No.	Operation	Number of computations
1	Complex multiplications	N^2
2	Complex additions	$N^2 - N$
3	Real multiplications	$4N^2$
4	Real additions	$4N^2 - 2N$
5	Trigonometric functions	$2N^2$

Table 3.5.1 Computational complexity of direct computation of N-point DFT

3.5.2 Properties of W_N

Now let us consider the properties of W_N. These properties are used by FFT algorithms to reduce the number of calculations.

We know the N-point DFT of sequence $x(n)$ is given as,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0, 1, \ldots.. N-1 \qquad \ldots (3.5.14)$$

Here W_N is called twiddle factor and it is defined as,

$$W_N = e^{-j\frac{2\pi}{N}} \qquad \ldots (3.5.15)$$

This twiddle factor exhibit symmetry and periodicity properties.

(i) Periodicity property of W_N :

$$\boxed{W_N^{k+N} = W_N^k} \qquad \ldots (3.5.16)$$

Proof :

We know that $\quad W_N = e^{-j\frac{2\pi}{N}}$

$$\therefore \quad W_N^{k+N} = e^{-j\frac{2\pi}{N}(k+N)}$$

$$= e^{-j\frac{2\pi}{N}\cdot k - j2\pi} \qquad \ldots (3.5.17)$$

$$= e^{-j\frac{2\pi}{N}k} \cdot e^{-j2\pi}$$

Here $\quad e^{-j2\pi} = \cos 2\pi - j \sin 2\pi$

$= 1 - j\,0 = 1$ always

Hence equation 3.5.17 becomes,

$$W_N^{k+N} = e^{-j\frac{2\pi}{N}k} = \left(e^{-j\frac{2\pi}{N}}\right)^k$$

$$= W_N^k \quad \text{since} \quad e^{-j\frac{2\pi}{N}} = W_N$$

This shows that W_N^k is periodic with period 'N'.

This property was illustrated earlier in Fig. 3.5.1. Observe that for $N = 8$,

$$W_8^{k+8} = W_8^k$$

(ii) Symmetry property of W_N :

$$\boxed{W_N^{k+\frac{N}{2}} = -W_N^k} \quad \text{... (3.5.18)}$$

Proof :

We know that $W_N = e^{-j\frac{2\pi}{N}}$

$$\therefore \quad W_N^{k+\frac{N}{2}} = e^{-j\frac{2\pi}{N}\left(k+\frac{N}{2}\right)} = e^{-j\frac{2\pi}{N}k - j\pi}$$

$$= e^{-j\frac{2\pi}{N}k} \cdot e^{-j\pi} \quad \text{... (3.5.19)}$$

Here $e^{-j\pi} = \cos\pi - j\sin\pi$

$= -1 - j\,0$

$= -1$ always

Hence equation 3.5.19 becomes,

$$W_N^{k+\frac{N}{2}} = e^{-j\frac{2\pi}{N}k}$$

$$= -W_N^k, \quad \text{since} \quad e^{-j\frac{2\pi}{N}} = W_N \quad \text{... (3.5.20)}$$

(iii) Prove that $\boxed{W_N^2 = W_{N/2}}$ ... (3.5.21)

We know that $W_N = e^{-j\frac{2\pi}{N}}$

In this equation replace 'N' by N/2, i.e.

$$W_{N/2} = e^{-j\frac{2\pi}{N/2}} = e^{-j\frac{2\pi}{N}\cdot 2}$$

$$= W_N^2 \quad \text{since} \quad e^{-j\frac{2\pi}{N}} = W_N$$

which is same as equation 3.5.21.

The direct computation of DFT does not use these properties of W_N. The FFT algorithms exploit these properties of W_N to reduce calculations of DFT as discussed next.

3.5.3 Classification of FFT Algorithms

The FFT algorithms are based on two basic methods. The first one is divide and conquer approach. In this method the 'N' point DFT is divided successively to 2-point DFTs to reduce calculations. In this method, Radix-2, Radix-4, decimation in time, decimation in frequency etc type of FFT algorithms are developed.

The second one is based on linear filtering. Based on this method, there are two algorithms. Goertzel algorithm and the chirp-z transform algorithm. This complete classification is listed below in Fig. 3.5.4.

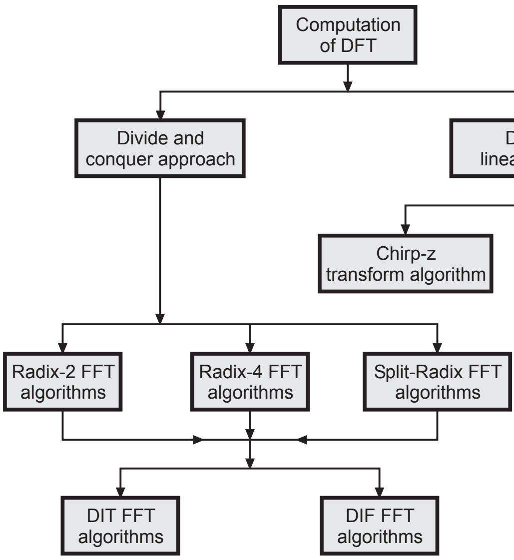


Fig. 3.5.4 Various FFT algorithms and their classification

Review Question

1. *Classify FFT algorithms.*

3.6 Radix-2 FFT Algorithms

AU : May-04, 05, 06, 10, 11, 12, 15, 17, Dec.-04, 05, 06, 08, 10, 11, 12, 15, 16

The radix-2 FFT algorithms are based on divide and conquer approach. In this approach the N-point DFT is successively decomposed into smaller DFTs. Because of this decomposition, the number of computations are reduced.

Let value of 'N' be selected such that $N = 2^v$. This N-point DFT is decomposed successively such that smallest DFT will be of size $N = 2$. Hence this type of algorithms are called as Radix-2, or radix of these algorithms is '2'.

3.6.1 Radix-2 DIT-FFT Algorithm

Principle

To express N-point DFT in terms of two $\frac{N}{2}$ - point DFTs.

Step 1 : Here DIT means Decimation in Time. Let the N-point data sequence $x(n)$ be splitted into two $\frac{N}{2}$ point data sequences $f_1(n)$ and $f_2(n)$. Let $f_1(n)$ contain even numbered samples of $x(n)$ and $f_2(n)$ contain odd numbered samples of $x(n)$. Thus we can write,

$$\left.\begin{aligned} f_1(n) &= x(2n) \quad , \quad n = 0, 1, \ldots, \frac{N}{2}-1 \\ f_2(n) &= x(2n+1) \ , \quad n = 0, 1, \ldots, \frac{N}{2}-1 \end{aligned}\right\} \qquad \ldots (3.6.1)$$

Key Point *Here time domain sequence $x(n)$ is splitted into two sequences. This splitting operation is called decimation. Since it is done on time domain sequence it is called Decimation in Time (DIT).*

Step 2 : We know that N-point DFT is given as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad , \quad k = 0, 1, \ldots N-1 \qquad \ldots (3.6.2)$$

Since the sequence $x(n)$ is splitted into even numbered and odd numbered samples, above equation can be written as,

$$X(k) = \sum_{n\,even} x(n)\, W_N^{kn} + \sum_{n\,odd} x(n)\, W_N^{kn}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} x(2m)\, W_N^{2mk} + \sum_{m=0}^{\frac{N}{2}-1} x(2m+1)\, W_N^{k(2m+1)}$$

Step 3 : From equation 3.6.1 we know that $f_1(m) = x(2m)$ and $f_2(m) = x(2m+1)$, hence above equation becomes,

$$X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m) \left(W_N^2\right)^{km} + \sum_{m=0}^{\frac{N}{2}-1} f_2(m) \left(W_N^2\right)^{km} \cdot W_N^k$$

In the above equation we have rearranged W_N factors.

Step 4 : From equation 3.5.21 we know that $W_N^2 = W_{N/2}$. Hence above equation can be written as,

$$X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m)\, W_{N/2}^{km} + W_N^k \sum_{m=0}^{\frac{N}{2}-1} f_2(m)\, W_{N/2}^{km} \qquad \text{... (3.6.3)}$$

Step 5 : Comparing above equation with the definition of DFT of equation 3.5.21, we find that first summation represent $\frac{N}{2}$ - point DFT of $f_1(m)$ and second summation represents $\frac{N}{2}$ - point DFT of $f_2(m)$ i.e.,

$$X(k) = F_1(k) + W_N^k\, F_2(k)\,, \qquad k = 0, 1, \ldots, N-1 \qquad \text{... (3.6.4)}$$

Thus $F_1(k)$ is $\frac{N}{2}$ - point DFT of $f_1(m)$ and $F_2(k)$ is $\frac{N}{2}$ point DFT of $f_2(m)$.

Step 6 : Since $F_1(k)$ and $F_2(k)$ are $\frac{N}{2}$ - point DFTs, they are periodic with period $\frac{N}{2}$. i.e.,

$$\left.\begin{aligned} F_1\left(k+\frac{N}{2}\right) &= F_1(k) \quad \text{and} \\ F_2\left(k+\frac{N}{2}\right) &= F_2(k) \end{aligned}\right\} \qquad \text{... (3.6.5)}$$

Hence replacing 'k' by $k+\frac{N}{2}$ in equation 3.6.4 we get,

$$X\left(k+\frac{N}{2}\right) = F_1\left(k+\frac{N}{2}\right) + W_N^{k+\frac{N}{2}}\, F_2\left(k+\frac{N}{2}\right)$$

Step 7 : From equation 3.5.20 we know that $W_N^{k+\frac{N}{2}} = -W_N^k$, and from equation 3.6.5 we can write above equation is,

$$X\left(k+\frac{N}{2}\right) = F_1(k) - W_N^k\, F_2(k) \qquad \text{... (3.6.6)}$$

Here observe that $X(k)$ is N-point DFT. By taking $k = 0$ to $\frac{N}{2}-1$, we can calculate $F_1(k)$ and $F_2(k)$ since they are $\frac{N}{2}$ - point DFTs. The N-point DFT $X(k)$ can be combinely obtained from equation 3.6.4 and equation 3.6.6 above by taking $k = 0$ to $\frac{N}{2}-1$. i.e., These equations express N - point DFT in terms of two $\frac{N}{2}$ - point DFTs

$$\left\{ \begin{aligned} X(k) &= F_1(k) + W_N^k\, F_2(k), k = 0, 1, \ldots, \frac{N}{2}-1 \qquad \ldots(3.6.7) \\ X\left(k+\frac{N}{2}\right) &= F_1(k) - W_N^k\, F_2(k), k = 0, 1, \ldots, \frac{N}{2}-1 \qquad \ldots(3.6.8) \end{aligned} \right.$$

The above two equations show that N-point DFT can be obtained by two $\frac{N}{2}$ - point DFTs.

An example of 8 point DIT FFT :

Let us consider 8-point DIT FFT for better understanding of this discussion.

Fig. 3.6.1 shows that 8-point DFT can be computed directly and hence no reduction in computation. It is shown symbolically as a single block which computes 8-point DFT directly.

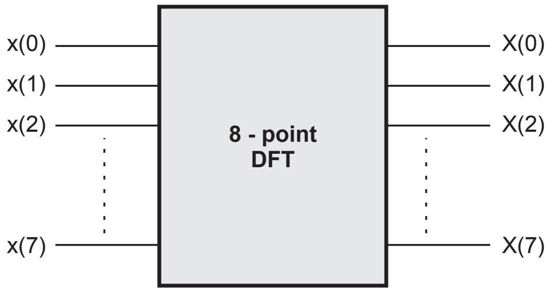


Fig. 3.6.1 Direct computation of 8-point DFT

According to equation 3.6.7 and equation 3.6.8, $X(k)$ can be obtained from $F_1(k)$ and $F_2(k)$. Here $F_1(k)$ and $F_2(k)$ are two 4-point DFTs. Fig. 3.6.2 shows the symbolic diagram for this operation. In Fig. 3.6.2 observe that for 8-point $x(n)$, the sequences $f_1(m)$ and $f_2(m)$ are given as (see equation 3.6.1),

$$\left. \begin{aligned} f_1(m) &= x(2n) = \{x(0), x(2), x(4), x(6)\} \\ f_2(m) &= x(2n+1) = \{x(1), x(3), x(5), x(7)\} \end{aligned} \right\} - \frac{N}{2} \textit{ point sequences} \qquad \ldots (3.6.9)$$

As shown in Fig. 3.6.2 two $\frac{N}{2}$ - point DFTs are to be computed separately and then they are combined as per equation 3.6.7 and equation 3.6.8 to get N-point DFT.

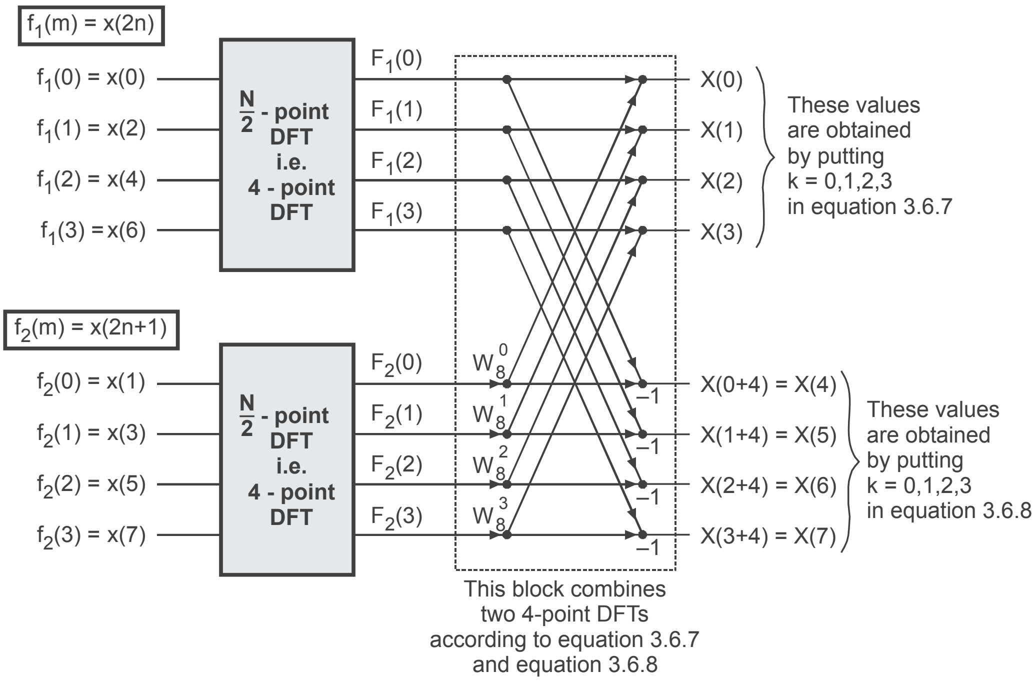


Fig. 3.6.2 8-point DFT of $X(k)$ is obtained by combining two 4-point DFTs $F_1(k)$ and $F_2(k)$

Second stage of decimation on f_1 (n) and f_2 (n)

We know that $f_1(n)$ and $f_2(n)$ are $\frac{N}{2}$ - point sequences. Let $f_1(n)$ be splitted into even numbered samples and odd numbered samples as follows :

$$\left.\begin{aligned} v_{11}(n) &= f_1(2n), \qquad n = 0, 1, \ldots \frac{N}{4}-1 \\ \text{and} \quad v_{12}(n) &= f_1(2n+1), \qquad n = 0, 1, \ldots \frac{N}{4}-1 \end{aligned}\right\} \quad \ldots (3.6.10)$$

Here observe that $v_{11}(n)$ is even numbered sequence of $f_1(n)$ and $v_{12}(n)$ is odd numbered sequence of $f_1(n)$. Since $f_1(n)$ contain $N/2$ samples. $v_{11}(n)$ and $v_{12}(n)$ contain $N/4$ samples.

Similarly $f_2(n)$ can be splitted into even and odd numbered sequences as follows :

$$\left.\begin{aligned} v_{21}(n) &= f_2(2n), \qquad n = 0, 1, \ldots \frac{N}{4}-1 \\ \text{and} \quad v_{22}(n) &= f_2(2n+1), \qquad n = 0, 1, \ldots \frac{N}{4}-1 \end{aligned}\right\} \quad \ldots (3.6.11)$$

$v_{21}(n)$ is even numbered sequence of $f_2(n)$ and

$v_{22}(n)$ is odd numbered sequence of $f_2(n)$. Both $v_{21}(n)$ and $v_{22}(n)$ contain $\frac{N}{4}$ samples each.

We obtained $X(k)$ and $X\left(k+\frac{N}{2}\right)$ of equation 3.6.7 and equation 3.6.8 from $F_1(k)$ and $F_2(k)$ with $f_1(n)$ and $f_2(n)$ as decimated sequences. The length of DFT was $\frac{N}{2}$. By using the similar method we can obtain $F_1(k)$ and $F_1\left(k+\frac{N}{4}\right)$ from $V_{11}(k)$ and $V_{12}(k)$. Here $V_{11}(k)$ is $\frac{N}{4}$- point DFT of $v_{11}(n)$ and $V_{12}(k)$ is $\frac{N}{4}$ - point DFT of $v_{12}(n)$ of equation 3.6.10. i.e.,

$$F_1(k) = V_{11}(k) + W_{N/2}^{k} V_{12}(k), \qquad k = 0, 1, \ldots, \frac{N}{4}-1 \qquad \ldots (3.6.12)$$

$$F_1\left(k+\frac{N}{4}\right) = V_{11}(k) - W_{N/2}^{k} V_{12}(k), \qquad k = 0, 1, \ldots, \frac{N}{4}-1 \qquad \ldots (3.6.13)$$

Compare the above two equations with equation 3.6.7 and equation 3.6.8. In equation 3.6.7 and equation 3.6.8 N-point DFT is obtained from two $\frac{N}{2}$ - point DFTs. In the above equations $\frac{N}{2}$ - point DFT (i.e. $F_1(k)$) is obtained from two $\frac{N}{4}$ - point DFTs.

Similarly we can write equations for $F_2(k)$ as follows,

$$F_2(k) = V_{21}(k) + W_{N/2}^{k} V_{22}(k), \qquad k = 0, 1, \ldots, \frac{N}{4}-1 \qquad \ldots (3.6.14)$$

$$F_2\left(k+\frac{N}{4}\right) = V_{21}(k) - W_{N/2}^{k} V_{22}(k), \qquad k = 0, 1, \ldots, \frac{N}{4}-1 \qquad \ldots (3.6.15)$$

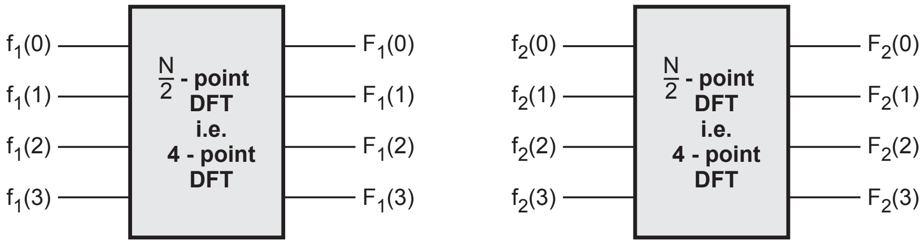
Here $V_{21}(k)$ is $\frac{N}{4}$ - point DFT of $v_{21}(n)$ and $V_{22}(k)$ is $\frac{N}{4}$ - point DFT of $v_{22}(n)$ of equation 3.6.11.

Recall the example of 8-point DFT

Now let us recall of example of 8-point DFT. As shown in Fig. 3.6.3, $F_1(k)$ and $F_2(k)$ are two 4-point DFTs computed directly. These two DFTs are represented by single blocks symbolically as shown below in Fig. 3.6.3.

$F_1(k)$ can be obtained from $V_{11}(k)$ and $V_{12}(k)$ as per equation 3.6.12 and equation 3.6.13.

For $N = 8$, $V_{11}(k)$ and $V_{12}(k)$ will be $\frac{N}{4}$ i.e. 2-point DFTs.



(a) A symbolic representation of 4 - point DFT $F_1(k)$ using direct computation

(b) A symbolic representation of 4 - point DFT $F_2(k)$ using direct computation

Fig. 3.6.3

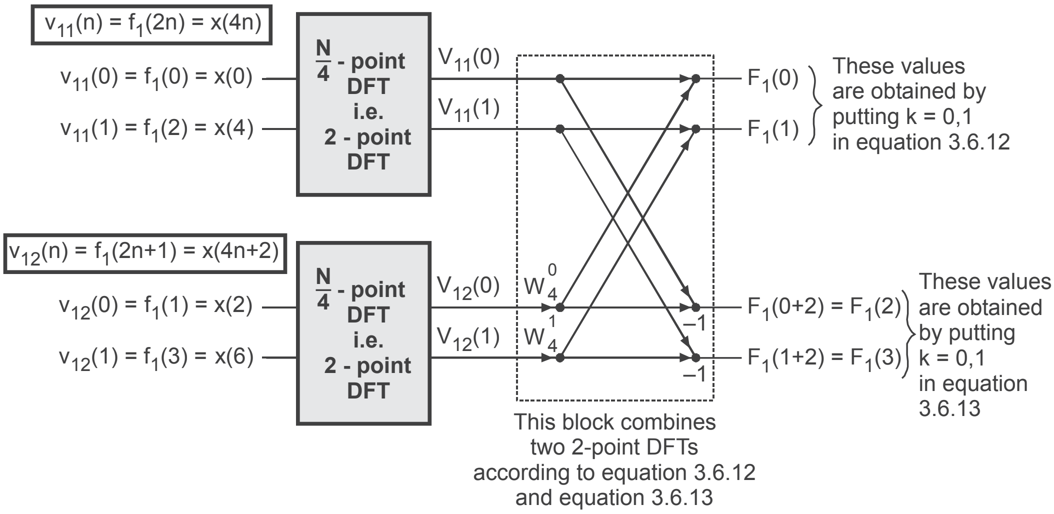


Fig. 3.6.4 (a) 4-point DFT of $F_1(k)$ is obtained by combining two 2-point DFTs of $V_{11}(k)$ and $V_{12}(k)$

Fig. 3.6.4 (a) shows the symbolic diagram for this operation. In this figure observe that for 4-point $f_1(n)$ the sequences $v_{11}(n)$ and $v_{12}(n)$ are given as (see equation 3.6.10),

$$\left.\begin{aligned} v_{11}(n) &= f_1(2n) = x(4n) = \{x(0), x(4)\}, & n &= 0,1 \\ v_{12}(n) &= f_1(2n+1) = x(4n+2) = \{x(2), x(6)\}, & n &= 0,1 \end{aligned}\right\} - \frac{N}{4} \text{ point sequences} \qquad \text{... (3.6.16)}$$

Thus as shown in Fig. 3.6.4 (a) the two $\frac{N}{4}$ - point DFTs are to be computed separately and then they are combined as per equation 3.6.12 and equation 3.6.13 to get $\frac{N}{2}$ - point DFT of $F_1(k)$.

Fig. 3.6.4 (b) shows the computation of $F_2(k)$ as per equation 3.6.14 and equation 3.6.15. In this figure observe that the two $\frac{N}{4}$ - point DFTs are combined. In this figure observe that the $\frac{N}{4}$ - point sequences $v_{21}(n)$ and $v_{22}(n)$ sequences are given as (see equation 3.6.11),

$$\left.\begin{aligned} v_{21}(n) &= f_2(2n) = x(4n+1) = \{x(1), x(5)\},\ n = 0, 1 \\ v_{22}(n) &= f_2(2n+1) = x(4n+3) = \{x(3), x(7)\},\ n = 0, 1 \end{aligned}\right\} - \frac{N}{4} \text{ point sequences} \quad \ldots(3.6.17)$$

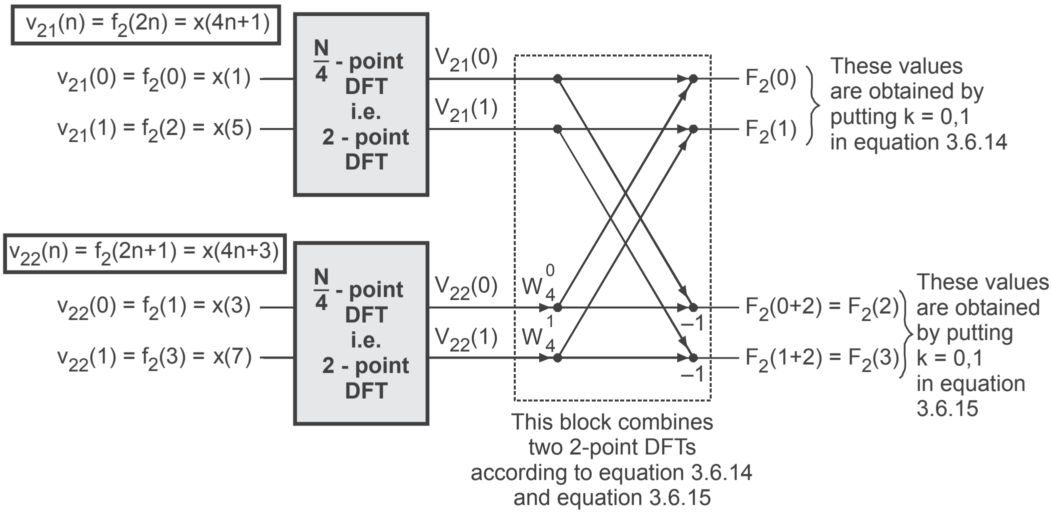


Fig. 3.6.4 (b) 4-point DFT of $F_2(k)$ is obtained by combining two 2-point DFTs of $V_{21}(k)$ and $V_{22}(k)$

Computation of 2-point DFTs

The $\frac{N}{4}$ - point sequences of equation 3.6.10 and equation 3.6.11 are further splitted in their even and odd parts. Hence we get next stage of decimation and the sequences will be of length $\frac{N}{8}$. The procedure discussed till now can be repeated further to compute the DFTs, till we reach to 2-point DFT. Since $N = 2^v$, we reach to 2-point DFT after $(v-1)$ decimations. For example for $N = 8$, we performed $3-1 = 2$ (since $v = 3$) decimations and we reached to 2-point DFTs as shown in Fig. 3.6.4. Now let us see how 2-point DFTs can be evaluated.

By definition of N-point DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \qquad k = 0, 1, \ldots, N-1 \qquad \ldots (3.6.18)$$

Let us see the computation of 2-point DFT $V_{11}(k)$ of Fig. 3.6.4 (a).

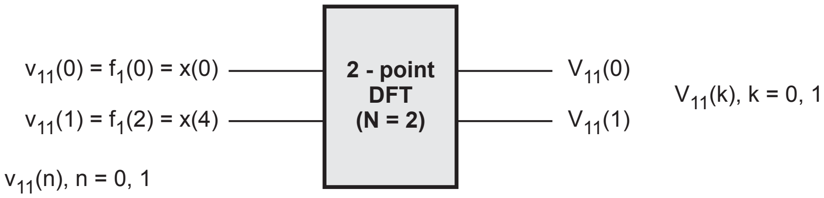


Fig. 3.6.5 Computation of 2-point DFT $V_{11}(k)$ of Fig. 3.6.4 (a)

From Fig. 3.6.5 we can write equation 3.6.18 as,

$$V_{11}(k) = \sum_{n=0}^{1} v_{11}(n) W_2^{kn}, \quad k = 0, 1 \qquad \text{... (3.6.19)}$$

For $k = 0$, $V_{11}(0) = \sum_{n=0}^{1} v_{11}(n) W_2^0 = \sum_{n=0}^{1} v_{11}(n)$ since $W_2^0 = 1$

$$= v_{11}(0) + v_{11}(1) \qquad \text{... (3.6.20)}$$

For $k = 1$ equation 3.6.19 becomes,

$$V_{11}(1) = \sum_{n=0}^{1} v_{11}(n) W_2^n$$

$$= v_{11}(0) W_2^0 + v_{11}(1) W_2^1 \qquad \text{... (3.6.21)}$$

Here $W_2^0 = 1$ and

$$W_2^1 = e^{-j\frac{2\pi}{2}\cdot 1}, \quad \text{since} \quad W_N = e^{-j\frac{2\pi}{N}}$$

$$= e^{-j\pi}$$

$$= \cos\pi - j\sin\pi = -1 - j\,0$$

$$= -1 \text{ always}$$

Thus equation 3.6.21 becomes,

$$V_{11}(1) = v_{11}(0) - v_{11}(1) \qquad \text{... (3.6.22)}$$

We know that $W_2^0 = 1$. Let us multiply $v_{11}(1)$ by W_2^0 in equation 3.6.20 and equation 3.6.22. Note that this does not make any difference. Now equation 3.6.20 and equation 3.6.22 becomes,

$$\left.\begin{aligned} V_{11}(0) &= v_{11}(0) + W_2^0\, v_{11}(1) \\ V_{11}(1) &= v_{11}(0) - W_2^0\, v_{11}(1) \end{aligned}\right\} \qquad \text{... (3.6.23)}$$

This computation is illustrated below in Fig. 3.6.6 with the help of signal flow graph.

Similar equations and signal flow graphs can be obtained for other 2-point DFTs of Fig. 3.6.4 (a) and Fig. 3.6.4 (b).

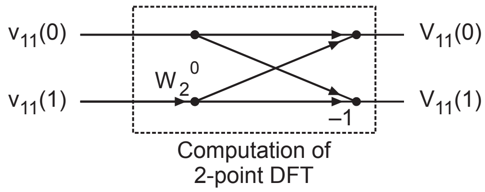


Fig. 3.6.6 Basic computation of 2-point DFT

Signal flow graph for 8-point DIT FFT

The signal flow graph indicates the flow of computations. Fig. 3.6.6 shows the signal flow graph for computation of 2-point DFT. Observe carefully the representation of equation 3.6.23 by this signal flow graph of Fig. 3.6.6. Fig. 3.6.2 shows signal flow graph for first stage of decimation. Fig. 3.6.4 (a) and (b) shows-signal flow graph for second stage of decimation. And Fig. 3.6.6 above shows signal flow graph for 3^{rd} stage which is not decimated since it is 2-point DFT. Fig. 3.6.7 shows the complete signal flow graph which is obtained by interconnection of these individual stagewise signal flow graphs.

(See Fig. 3.6.7 on next page)

We know that for radix-2 algorithms value of N is given as,

$$N = 2^v \quad \text{... (3.6.24)}$$

Hence value of v can be obtained as,

$$v = \frac{log_{10} N}{log_{10} 2}$$

$$= log_2 N \quad \text{... (3.6.25)}$$

If $N = 8$, the value of $v = log_2 8 = 3$. For N-point DFT, there are 'v' number of stages of decimation. For example in Fig. 3.6.7 observe that for $N = 8$, there are $v = 3$ stages of decimation, i.e. $v = 1,2,3$. Similarly there are v-stages of computation. These stages of computation are shown by dotted blocks ($\boxed{}$) in Fig. 3.6.7. In this Fig. 3.6.7 observe that the last block combines two 4-point DFTs to give 8-point DFT. The middle block combine 2-point DFTs to give 4-point DFTs separately. The first blocks combine (or calculate) 2-point DFTs.

From equation 3.5.21 we know that,

$$W_{N/2} = W_N^2$$

$$\therefore \quad W_4^1 = W_8^2$$

In Fig. 3.6.7 observe that we have written $W_4^1 = W_8^2$. Similarly $W_4^0 = W_8^0 = 1$ always. Thus the notation for twiddle factors can be made constant. This type of notation is useful for implementing FFT algorithms, and computations are also reduced.

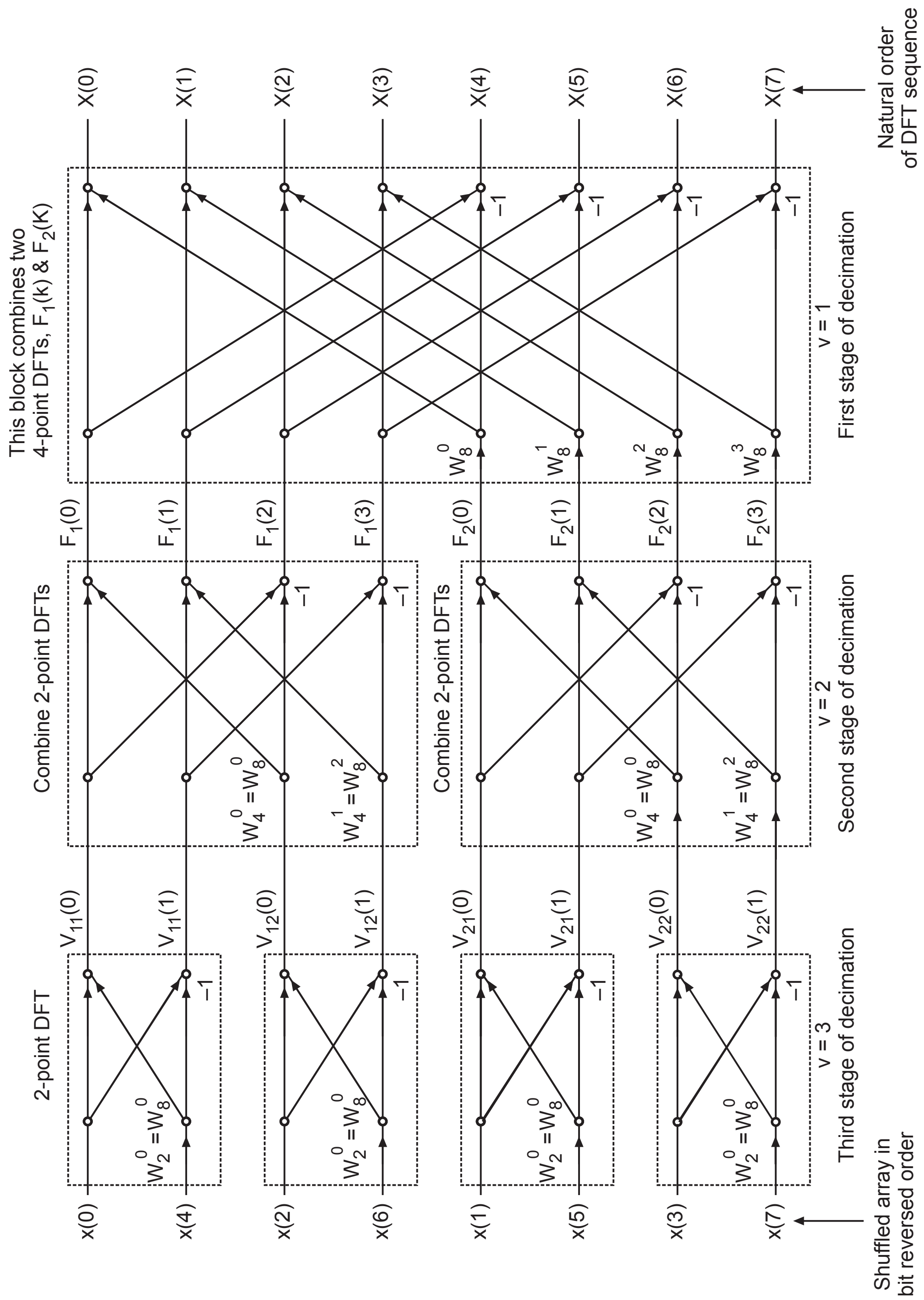


Fig. 3.6.7 Signal flow graph and stages of computation of Radix-2 DIT-FFT algorithm for N = 8

3.6.1.1 Butterfly Computation

This is the fundamental or basic computation in FFT algorithms. Fig. 3.6.8 shows this operation.

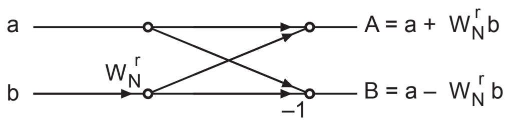


Fig. 3.6.8 Butterfly computation in FFT

In the Fig. 3.6.8 observe that two values '*a*' and '*b*' are available as input. From these two values '*A*' and '*B*' are computed. This operation is called butterfly operation.

In Fig. 3.6.7 observe that the complete signal flow graph is made up of such butterflies of Fig. 3.6.8. For example $V_{11}(0)$ and $V_{11}(1)$ in Fig. 3.6.7 is computed from $x(0)$ and $x(4)$ which is butterfly computation. Similarly $X(1)$ and $X(5)$ are computed from $F_1(1)$ and $F_2(1)$ which is also butterfly computation. Thus butterfly computation becomes the base of implementing FFT algorithms.

3.6.1.2 Computational Complexity Compared to Direct Computation

Computational complexity of FFT algorithm :

Now let us see how this FFT algorithm reduces number of computations. The computations involved in the butterfly operation are shown in Fig. 3.6.9.

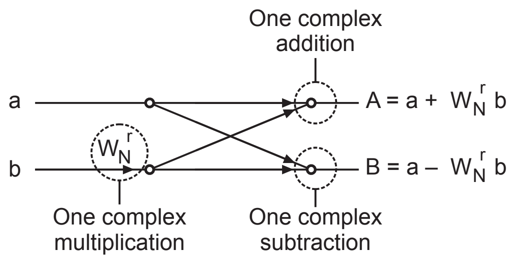


Fig. 3.6.9 Computations in one butterfly operation. Observe that 'subtraction' operation avoids multiplications by '−1'

In the Fig. 3.6.9 'b' is multiplied by 'W_N^r' which results in one complex multiplication. $A = a + W_N^r\, b$; this results in one complex addition. $B = a - W_N^r\, b$; this is obtained by subtraction. Hence there is no need to multiply $W_N^r\, b$ by −1. Computational complexity of addition is same as subtraction, hence we can say that butterfly operation requires two complex additions and one complex multiplication.

In Fig. 3.6.7 observe that there are $\frac{N}{2} = 4$ (since N = 8) butterflies in every stage of decimation. There are $v = 3$ (since $v = log_2\, N = log_2\, 8$) stages of decimation. Hence total number of butterflies in signal flow graph of Fig. 3.6.7 are, Total number of butterfly operations $= \frac{N}{2} \times v$

$$= \frac{N}{2} \times \log_2 N \qquad \text{... (3.6.26)}$$

$$= 4 \times 3 \qquad (\text{for } N = 8)$$

$$= 12$$

Hence number of additions and multiplications will be,

$$\text{Complex multiplications} = \frac{N}{2} \cdot \log_2 N \qquad \text{... (3.6.27)}$$

This is because one butterfly needs one multiplication.

$\therefore$ Complex multiplications = 12 for N = 8

And two additions are required for one butterfly.

Hence,

$$\text{Complex additions} = 2 \times \frac{N}{2} \cdot \log_2 N$$

$$= N \log_2 N \qquad \text{... (3.6.28)}$$

Complex additions = 24 for N = 8

Computational complexity of direct computation :

The direct computation of DFT is given by definition as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \qquad k = 0, 1, \ldots.., N-1$$

For one value of 'k' observe that the multiplication of $x(n)$ and W_N^{kn} is done for 'N' times, since $n = 0$ to $N-1$. That is there are 'N' complex multiplications for one value of k. Since 'k' also has 'N' values (since $k = 0, 1, \ldots, N-1$), the total complex multiplications for computation of $X(k)$ will be,

$$\text{Complex multiplications} = N \times N = N^2 \qquad \text{... (3.6.29)}$$

Similarly for one value of 'k', $\sum_{n=0}^{N-1}$ is performed on $x(n)\, W_N^{kn}$. Thus to add total 'N' (0 to $N-1$) terms, '$N-1$' complex additions are required. Since 'k' also has 'N' values, the total complex additions for computations of $X(k)$ will be,

$$\text{Complex additions} = N(N-1) = N^2 - N \qquad \text{... (3.6.30)}$$

Table 3.6.1 shows the comparison of computational complexities of direct computation and DIT FFT algorithm.

Number of points N	Direct computation		DIT FFT algorithm		Improvement in processing speed for multiplications
	Complex multiplications N^2	Complex additions $N^2 - N$	Complex multiplications $\frac{N}{2} \cdot \log_2 N$	Complex additions $N \log_2 N$	$\frac{N^2}{N \log_2 N}$
8	64	52	12	24	5.3 times
16	256	240	32	64	8 times
256	65536	65280	1024	2048	6.4 times
1024	1048576	1047552	5120	10240	204.8 times

Table 3.6.1 Computational complexity of DIT FFT algorithm compared to direct method

The Table 3.6.1 shows comparison of computational complexity of Direct computation and FFT algorithm for few commonly used values of 'N'. The last column shows improvement in processing speed for multiplications. Observe that for N = 8, the improvement in processing speed is 5.3 times. And for N = 1024, the processing speed increases by 200 times ! This is the strength of FFT algorithms.

3.6.1.3 Memory Requirement and Inplace Computations

Observe the butterfly computation of Fig. 3.6.8 used in FFT. From values 'a' and 'b' new values 'A' and 'B' are computed. Once 'A' and 'B' are computed, there is no need to store 'a' and 'b'. Thus same memory locations can be used to store $A - B$ where $a - b$ were stored. Since A, B or a, b are complex numbers, they need two memory locations each. Thus for computation of one butterfly, four memory locations are required, i.e. two for 'a' or 'A' and two for 'b' or 'B'.

$$\text{Memory locations for one butterfly} = 2 \times 2 = 4 \qquad \text{... (3.6.31)}$$

In such computation, 'A' is stored in place of 'a' and 'B' is stored in place of 'b'. This is called *in place computation*. The advantage of in place computation is that it reduces memory requirement.

Now observe that signal flow graph of Fig. 3.6.7 for N = 8. Normally the computations are performed stagewise. In every stage observe that there are $\frac{N}{2}$ (i.e. 4 for N = 8) butterflies. Hence number of memory locations required for one stage will be,

$$\text{Memory locations for one stage} = 4 \times \frac{N}{2} = 2N \qquad \text{... (3.6.32)}$$

Thus '2N' locations are required for one stage. Since stages are computed successively, these locations can be shared. Hence total 2N locations are required for computation of N-point DFT. In the signal flow graph of Fig. 3.6.7 observe that in every stage there are $\frac{N}{2}$ (i.e. 4 for N = 8) twiddle factors required. Hence maximum storage requirement of N-point DFT including twiddle factors will be $\left(2N+\frac{N}{2}\right)$, i.e.,

Memory requirement of N-point DFT = 2N ... (3.6.33)

Maximum memory requirement of N-point DFT including twiddle factors $= \left(2N+\frac{N}{2}\right)$ ... (3.6.34)

3.6.1.4 Bit Reversal

Consider again the signal flow graph of 8-point DIT FFT algorithm. Observe that the sequence of input data is shuffled as $x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)$. And the DFT sequence $X(k)$ at the output is in proper order, i.e. $x(0), x(1), \ldots x(7)$. The shuffling of the input sequence has well defined format. It is illustrated in Table 3.6.2.

Memory address of $x(n)$ in decimal i.e.	Memory address of $x(n)$ in binary			Memory address in bit reversed order			New memory address of $x(n)$ according to reversed order of bits
n	n_2	n_1	n_0	n_0	n_1	n_2	m
0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	4
2	0	1	0	0	1	0	2
3	0	1	1	1	1	0	6
4	1	0	0	0	0	1	1
5	1	0	1	1	0	1	5
6	1	1	0	0	1	1	3
7	1	1	1	1	1	1	7

Table 3.6.2 Shuffling of input sequence in bit reversed order

In the table observe that the data point $x(1) \equiv x(001)$ is to be placed at $m=4$ i.e. $(100)^{th}$ position in the decimated array. This is true in the signal flow graph of Fig. 3.6.7. $x(1)$ is on the 4^{th} position as an input in the decimated array. The last column of above table shows the order in which the data is required.

Thus the input data should be stored in the bit reversed order then the DFT will be obtained in natural sequence. In the FFT algorithms bit reversal is always one of the important parts.

Examples for Understanding

Example 3.6.1 *Use the 8-point radix-2 DIT-FFT algorithm to find the DFT of the sequence,*

$x(n) = \{0.707, 1, 0.707, 0, -0.707, -1, -0.707, 0\}$

Solution : We have discussed 8-point DIT-FFT earlier. The signal flow diagram for 8-point DIT-FFT algorithm is shown in Fig. 3.6.7. We have to proceed as per this signal flow diagram to obtain the DFT.

Computation of twiddle factors or phase factors

As per the signal flow diagram of Fig. 3.6.7, we need W_8^0, W_8^1, W_8^2 and W_8^3 for calculations. Hence let us first evaluate these phase factors.

We know that,

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\therefore \quad W_8 = e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}}$$

Hence

$$W_8^0 = e^0 = 1$$

$$W_8^1 = e^{-j\frac{\pi}{4}} = cos\left(\frac{\pi}{4}\right) - j\,sin\left(\frac{\pi}{4}\right)$$

$$= 0.7071 - j\,0.7071$$

$$W_8^2 = e^{-j\frac{\pi}{4}\times 2} = e^{-j\frac{\pi}{2}}$$

$$= cos\left(\frac{\pi}{2}\right) - j\,sin\,\frac{\pi}{2} = -j$$

$$W_8^3 = e^{-j\frac{\pi}{4}\times 3} = e^{-j\frac{3\pi}{4}}$$

$$= cos\left(\frac{3\pi}{4}\right) - j\,sin\left(\frac{3\pi}{4}\right)$$

$$= -0.7071 - j\,0.7071$$

These phase factors are tabulated below for easy reference.

Sr.No.	Phase factor or twiddle factor	Value
1	W_8^0	1
2	W_8^1	$0.7071 - j\,0.7071 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$
3	W_8^2	$-j$
4	W_8^3	$-0.7071 - j\,0.7071 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

Table 3.6.3 Phase factors required for 8-point DIT-FFT algorithms

Computation of 2-point DFTs :

Observe the signal flow graph of Fig. 3.6.7. The two point DFTs are first computed. This part of Fig. 3.6.7 is shown below in Fig. 3.6.10.

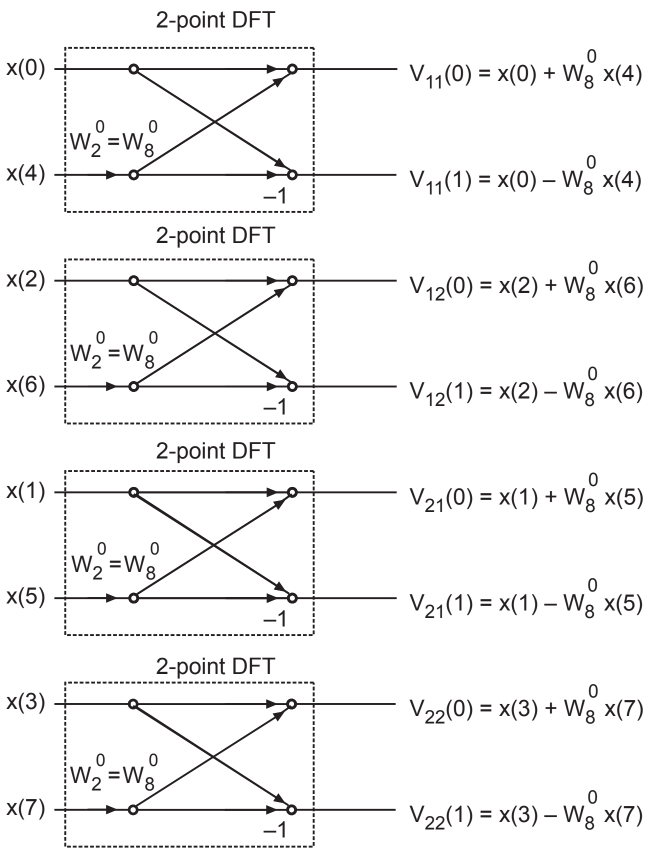


Fig. 3.6.10 Computation of 2-point DFTs

Let us calculate the various values of 2-point DFTs in Fig. 3.6.10. The given sequence is,

$x(0) = 0.707$ $\quad$ $x(4) = -0.707$

$x(1) = 1$ $\quad$ $x(5) = -1$

$x(2) = 0.707$ $\quad$ $x(6) = -0.707$

$x(3) = 0$ $\quad$ $x(7) = 0$

Calculations of $V_{11}(0)$ *and* $V_{11}(1)$

$V_{11}(0)$ is given as (see Fig. 3.6.10),

$$V_{11}(0) = x(0) + W_8^0\, x(4)$$

Putting the values,

$$V_{11}(0) = 0.707 + (1) \times (-0.707)$$

$$= 0 \qquad \therefore \boxed{V_{11}(0) = 0}$$

$V_{11}(1)$ is given as (see Fig. 3.6.10),

$$V_{11}(1) = x(0) - W_8^0\, x(4)$$

Putting the values,

$$V_{11}(1) = 0.707 - (1) \times (-0.707)$$

$$= 1.414 \qquad \therefore \boxed{V_{11}(1) = 1.414}$$

Calculations of $V_{12}(0)$ *and* $V_{12}(1)$

Using the equations in Fig. 3.6.10,

$$V_{12}(0) = x(2) + W_8^0\, x(6) = 0.707 + (1) \times (-0.707)$$

$$= 0 \qquad \therefore \boxed{V_{12}(0) = 0}$$

$$V_{12}(1) = x(2) - W_8^0\, x(6) = 0.707 - (1) \times (-0.707)$$

$$= 1.414 \qquad \therefore \boxed{V_{12}(1) = 1.414}$$

Calculations of $V_{21}(0)$ *and* $V_{21}(1)$

From Fig. 3.6.10 $V_{21}(0)$ and $V_{21}(1)$ are given as,

$$V_{21}(0) = x(1) + W_8^0\, x(5) = 1 + (1) \times (-1)$$

$$= 0 \qquad \therefore \boxed{V_{21}(0) = 0}$$

$$V_{21}(1) = x(1) - W_8^0\, x(5) = 1 - (1) \times (-1)$$

$$= 2 \qquad \therefore \boxed{V_{21}(1) = 2}$$

Calculations of $V_{22}(0)$ *and* $V_{22}(1)$

From Fig. 3.6.10 $V_{22}(0)$ and $V_{22}(1)$ are given as,

$$V_{22}(0) = x(3) + W_8^0 \, x(7)$$

$$= 0 + 1 \times 0 = 0 \qquad \therefore \quad \boxed{V_{22}(0) = 0}$$

$$V_{22}(1) = x(3) - W_8^0 \, x(7)$$

$$= 0 - 1 \times 0 = 0 \qquad \therefore \quad \boxed{V_{22}(1) = 0}$$

Thus we obtained the 2-point DFTs of the given input sequence as per signal flow graph of Radix-2 DIT FFT algorithm.

Combining 2-point DFTs

The next stage is to combine two point DFTs to get 4-point DFTs. This part of the signal flow graph of Fig. 3.6.7 is shown in Fig. 3.6.11.

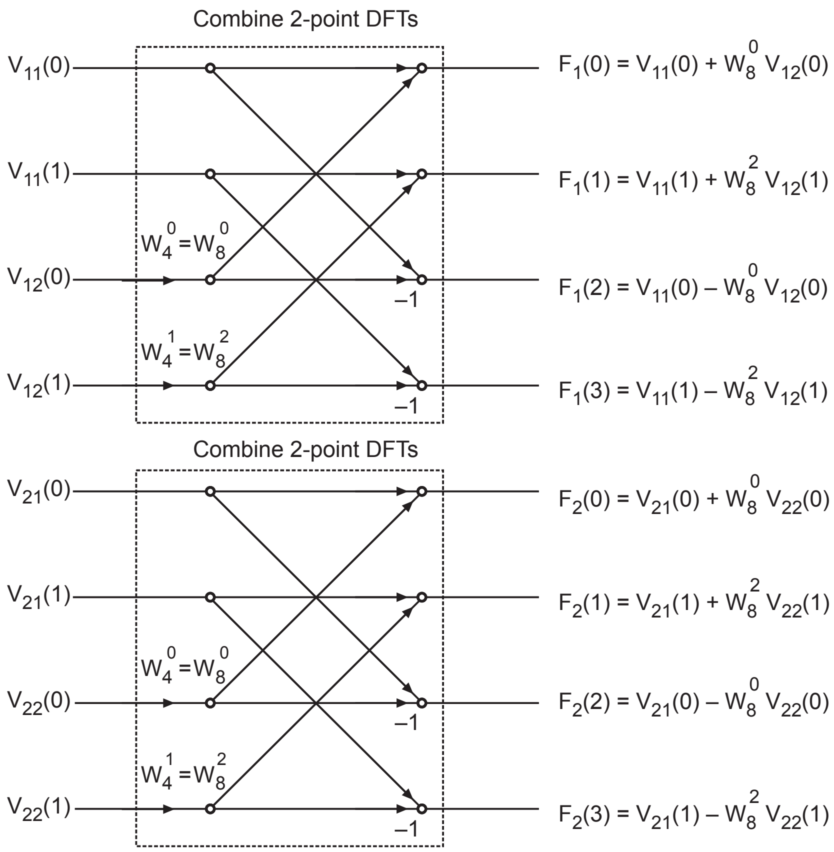


Fig. 3.6.11 Combining 2-point DFTs

Calculation of $F_1(0), F_1(1), F_1(2)$ *and* $F_1(3)$

Observe the signal flow graph shown in Fig. 3.6.11. It combines two-point DFTs to give 4-point DFTs. The equations for $F_1(0), F_1(1), F_1(2)$ and $F_1(3)$ are given in Fig. 3.6.11. i.e.,

$$F_1(0) = V_{11}(0) + W_8^0 V_{12}(0)$$

Putting the values in above equation,

$$F_1(0) = 0 + 1 \times (0) = 0 \qquad \therefore \boxed{F_1(0) = 0}$$

$$F_1(1) = V_{11}(1) + W_8^2 V_{12}(1)$$

Putting the values,

$$F_1(1) = 1.414 + (-j) \times 1.414$$

$$= 1.414 - j\,1.414 \qquad \therefore \boxed{F_1(1) = 1.414 - j\,1.414}$$

$$F_1(2) = V_{11}(0) - W_8^0 V_{12}(0)$$

Putting the values,

$$F_1(2) = 0 - 1 \times (0) = 0 \qquad \therefore \boxed{F_1(2) = 0}$$

$$F_1(3) = V_{11}(1) - W_8^2 V_{12}(1)$$

Putting the values,

$$F_1(3) = 1.414 - (-j) \times 1.414$$

$$= 1.414 + j\,1.414 \qquad \therefore \boxed{F_1(3) = 1.414 + j\,1.414}$$

Calculation of $F_2(0), F_2(1), F_2(2)$ *and* $F_2(3)$

The equations for $F_2(0), F_2(1), F_2(2)$ and $F_2(3)$ are given in Fig. 3.6.11. Let us use these equations. i.e.,

$$F_2(0) = V_{21}(0) + W_8^0 V_{22}(0)$$

$$= 0 + 1 \times (0) = 0 \qquad \therefore \boxed{F_2(0) = 0}$$

$$F_2(1) = V_{21}(1) + W_8^2 V_{22}(1) = 2 + (-j) \times 0$$

$$= 2 \qquad \therefore \boxed{F_2(1) = 2}$$

$$F_2(2) = V_{21}(0) - W_8^0 \, V_{22}(0)$$

$$= 0 - 1 \times (0) = 0 \qquad \therefore \boxed{F_2(2) = 0}$$

$$F_2(3) = V_{21}(1) - W_8^2 \, V_{22}(1)$$

$$= 2 - (-j) \times 0 \qquad \therefore \boxed{F_2(3) = 2}$$

$$= 2$$

Combining 4-point DFTs $F_1(k)$ and $F_2(k)$

The next stage is to combine the two 4-point DFTs to get 8-point DFT. This part of the signal flow graph of Fig. 3.6.7 is shown in Fig. 3.6.12.

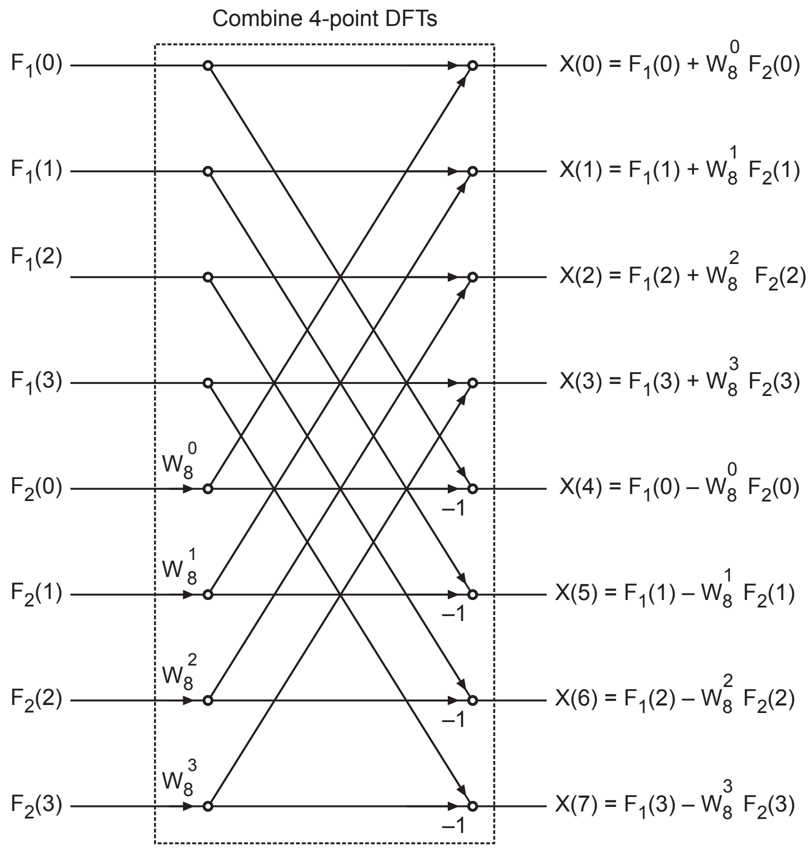


Fig. 3.6.12 Combining 4-point DFTs

The equations for combining 4-point DFTs are given in Fig. 3.6.12. Let us use these equations to obtain the values of DFT $X(0), X(1), X(2), \ldots X(7)$ etc. i.e.,

$$X(0) = F_1(0) + W_8^0 F_2(0)$$

$$= 0 + 1 \times 0 = 0 \qquad \therefore \boxed{X(0) = 0}$$

$$X(1) = F_1(1) + W_8^1 F_2(1)$$

$$= 1.414 - j\,1.414 + (0.7071 - j\,0.7071) \times 2$$

$$= 1.414 - j\,1.414 + 1.4142 - j\,1.4142$$

$$= 2.8284 - j\,2.8284 \qquad \therefore \boxed{X(1) = 2.8284 - j\,2.8284}$$

$$X(2) = F_1(2) + W_8^2 F_2(2)$$

$$= 0 + (-j) \times 0 = 0 \qquad \therefore \boxed{X(2) = 0}$$

$$X(3) = F_1(3) + W_8^3 F_2(3)$$

$$= 1.414 + j\,1.414 + (-0.7071 - j\,0.7071) \times 2$$

$$= 1.414 + j\,1.414 - 1.414 - j\,1.414$$

$$= 0 \qquad \therefore \boxed{X(3) = 0}$$

$$X(4) = F_1(0) - W_8^0 F_2(0)$$

$$= 0 - 1 \times 0 = 0 \qquad \therefore \boxed{X(4) = 0}$$

$$X(5) = F_1(1) - W_8^1 F_2(1)$$

$$= 1.414 - j\,1.414 - (0.7071 - j\,0.7071) \times 2$$

$$= 1.414 - j\,1.414 - 1.414 + j\,1.414$$

$$= 0 \qquad \therefore \boxed{X(5) = 0}$$

$$X(6) = F_1(2) - W_8^2 F_2(2)$$

$$= 0 - (-j) \times 0 = 0 \qquad \therefore \boxed{X(6) = 0}$$

$$X(7) = F_1(3) - W_8^3 F_2(3)$$

$$= 1.414 + j\,1.414 - (-0.7071 - j\,0.7071) \times 2$$

$$= 1.414 + j\,1.414 + 1.414 + j\,1.414$$

$$= 2.8284 + j\,2.8284 \qquad \therefore \boxed{X(7) = 2.8284 + j\,2.8284}$$

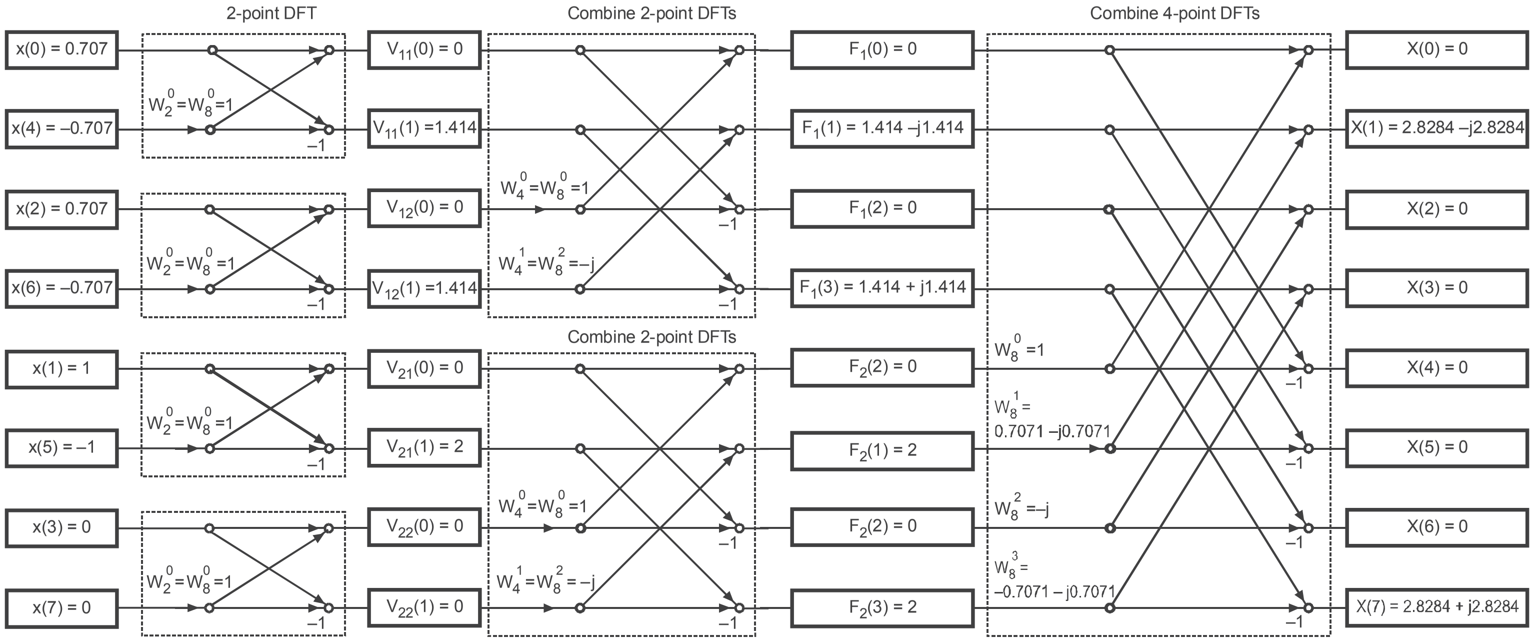


Fig. 3.6.13 Complete signal flow graph of example 3.6.1. It shows input sequence, output DFT and intermediate values in Radix-2 DIT-FFT algorithm

Thus we obtained the 8-point DFT of the given sequence as,

$$X(k) = \{0, 2.8284 - j\,2.8284, 0, 0, 0, 0, 0, 2.8284 + j\,2.8284\} \quad \ldots \quad (3.6.35)$$

The complete signal flow graph with all the intermediate values is shown in Fig. 3.6.13. If this algorithm is written for processor, then inplace computations will need storage of only eight complex numbers and four phase factors. This is how FFT algorithms operate fast and reduce memory requirement.

Refer Fig. 3.6.13 on previous page.

Examples with Solution

Example 3.6.2 *Given x(n) = n + 1, and N = 8, find X(K) using DIT, FFT algorithm.*

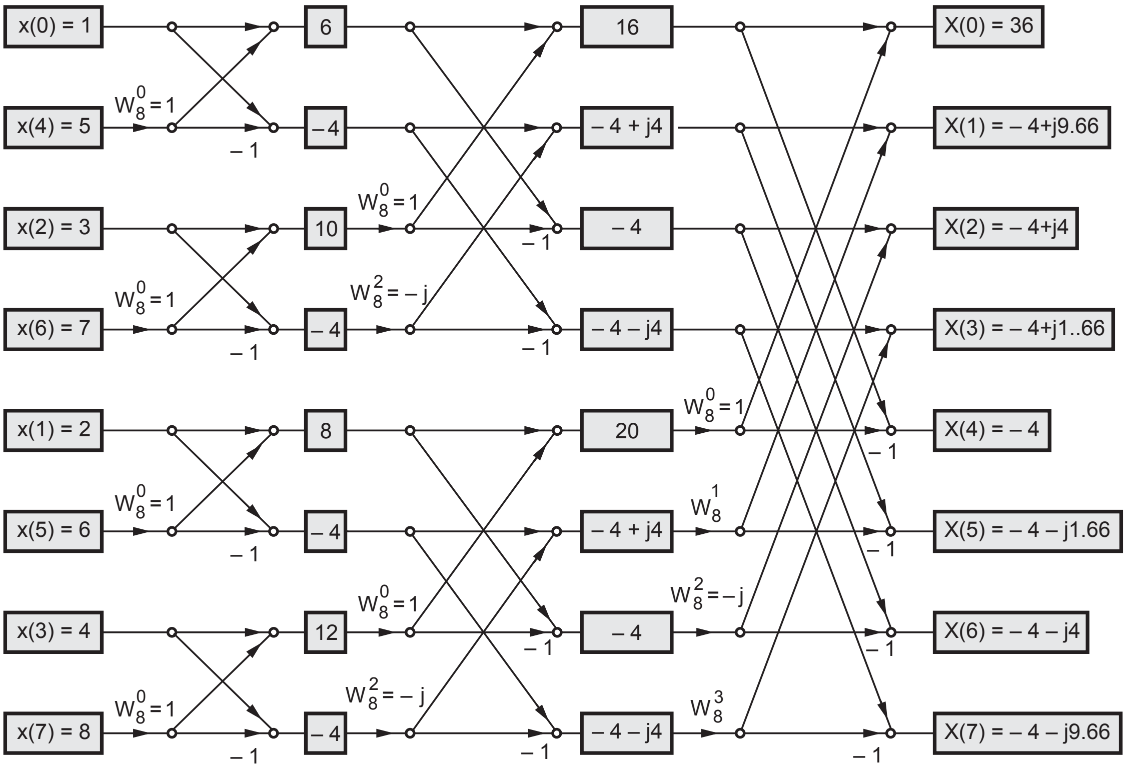
AU : May-15, Marks 8

Solution : Following table shows the calculations :

Bit reversed inputs	Decimation stage - 1	Decimation stage - 2	Decimation stage - 3 Final output
x(0) = 1	$V_{11}(0) = x(0) + W_8^0 x(4)$ $= 1 + 5 = 6$	$F_1(0) = V_{11}(0) + W_8^0 V_{12}(0)$ $= 6 + 1 \times 10 = 16$	$X(0) = F_1(0) + W_8^0 F_2(0)$ $= 16 + 1 \times 20 = 36$
x(4) = 5	$V_{11}(1) = x(0) - W_8^0 x(4)$ $= 1 - 5 = -4$	$F_1(1) = V_{11}(1) + W_8^2 V_{12}(1)$ $= -4 + (-j)(-4)$ $= -4 + j4$	$X(1) = F_1(1) + W_8^1 F_2(1)$ $= -4 + j4 + \left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(-4 + j4)$ $= -4 + j\,9.66$
x(2) = 3	$V_{12}(0) = x(2) + W_8^0 x(6)$ $= 3 + 7 = 10$	$F_1(2) = V_{11}(0) - W_8^0 V_{12}(0)$ $= 6 - 1 \times 10 = -4$	$X(2) = F_1(2) + W_8^2 F_2(2)$ $= -4 + (-j)(-4)$ $= -4 + j4$
x(6) = 7	$V_{12}(1) = x(2) - W_8^0 x(6)$ $= 3 - 7 = -4$	$F_1(3) = V_{11}(1) - W_8^2 V_{12}(1)$ $= -4 - (-j)(-4)$ $= -4 - j4$	$X(3) = F_1(3) + W_8^3 F_2(3)$ $= -4 - j4 + \left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(-4 - j4)$ $= -4 + j\,1.66$
x(1) = 2	$V_{21}(0) = x(1) + W_8^0 x(5)$ $= 2 + 6 = 8$	$F_2(0) = V_{21}(0) + W_8^0 V_{22}(0)$ $= 8 + 1 \times 12 = 20$	$X(4) = F_2(0) - W_8^0 F_2(0)$ $= 16 - 1 \times 20 = -4$
x(5) = 6	$V_{21}(1) = x(1) - W_8^0 x(5)$ $= 2 - 6 = -4$	$F_2(1) = V_{21}(1) + W_8^2 V_{22}(1)$ $= -4 + (-j)(-4)$ $= -4 + j4$	$X(5) = F_1(1) - W_8^1 F_2(1)$ $= -4 + j4 - \left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$ $(-4 + j4)$ $= -4 - j1.66$

$x(3) = 4$	$V_{22}\ (0) = x(3) + W_8^0\, x(7)$ $= 4 + 8 = 12$	$F_2(2) = V_{21}\ (0) - W_8^0\ V_{22}\ (0)$ $= 8 - 1 \times 12 = -4$	$X(6) = F_1(2) - W_8^2\ F_2(2)$ $= -4 - (-j)\ (-4)$ $= -4 - j4$
$x(7) = 8$	$V_{22}\ (1) = x\ (3) - W_8^0\, x(7)$ $= 4 - 8 = -4$	$F_2(3) = V_{21}\ (1) - W_8^2\ V_{22}\ (1)$ $= -4 - (-j)\ (-4)$ $= -4 - j4$	$X\ (7) = F_1(3) - W_8^3\ F_2(3)$ $= -4 - j4 -$ $\left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(-4 - j4)$ $= -4 - j\ 9.66$

Fig. 3.6.15 shows the signal flow graph of radix-2 DIT-FFT for above calculations.



$$W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \text{ and } W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

Fig. 3.6.15 Radix-2 DIT-FFT

Example 3.6.3 *For the given sequence x[n] = {0, 1, 2, 3, 4, 5, 6, 7} find x[k] using decimation-in-time FFT algorithm.* **AU : May-10, Marks 16**

OR

Compute 8 point DFT of the given sequence using DIT algorithm $x(n) = \begin{Bmatrix} n & n \leq 7 \\ 0 & otherwise \end{Bmatrix}$

AU : May-17, Marks 13

Solution : The signal flow graph of DIT-FFT along with stage wise calculations is shown in Fig. 3.6.14.

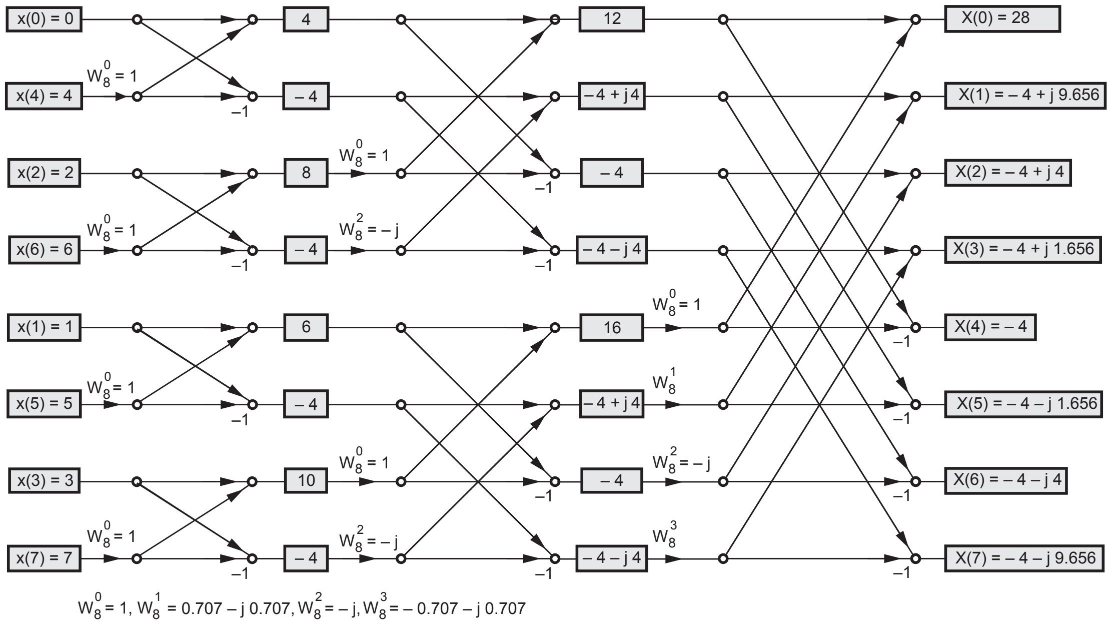


Fig. 3.6.14 DIT-FFT

Example 3.6.4 *An 8-point sequence is given by* $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$. *Compute 8-point DFT of* $x(n)$ *by radix-2 DIT-FFT method also sketch the magnitude and phase.*

AU : Dec.-10, Marks 16, Dec.-12, Marks 8

Solution : The signal flow graph of DIT-FFT along with stage-wise calculations is shown in Fig. 3.6.15. (See Fig. 3.6.15 on next page).

Magnitude and phase

The DFT, its magnitude and phase are calculated below. The sketch is shown in Fig. 3.6.16. (See Fig. 3.6.16 on next page).

k	X(k)	|X(k)|	∠ X(k)
0	12	12	0
1	1 – j 2.414	2.613	– 1.178

2	0	0	0
3	1 – j 0.414	1.082	– 0.392
4	0	0	0
5	1 + j 0.414	1.082	0.392
6	0	0	0
7	1 + j 2.414	2.613	1.178

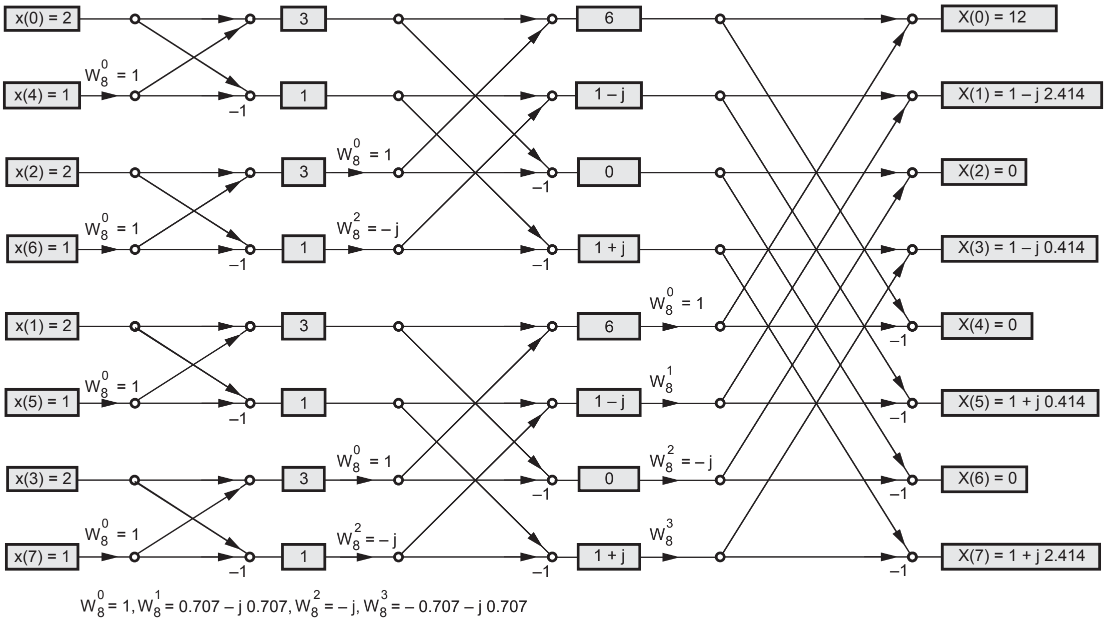


Fig. 3.6.15 DIT-FFT

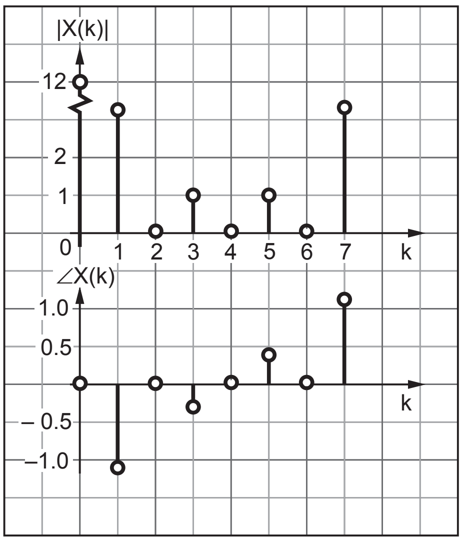


Fig. 3.6.16 Magnitude and phase response

Example 3.6.5 *Using FFT algorithm compute the DFT of x(n) = {2, 2, 2, 2}.*

AU : Dec.-15, Marks 12

Solution : Fig. 3.6.17 shows the signal flow graph of radix - 2 DIT - FFT with stage - rise results.

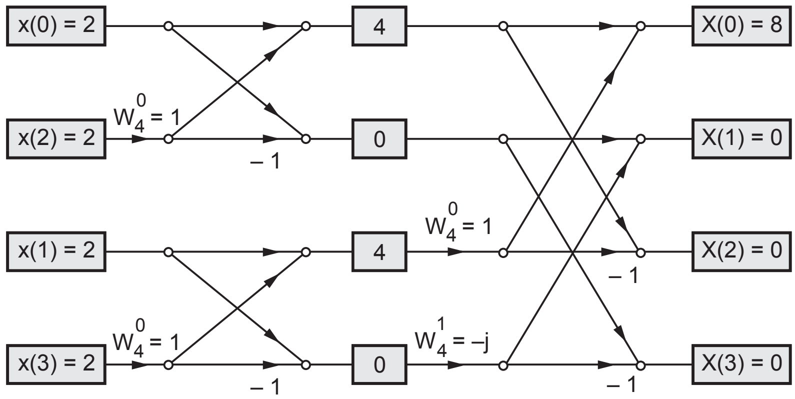


Fig. 3.6.17 Radix-2 DIT-FFT

Thus, $X(k) = \{ 8, 0, 0, 0 \}$

Example 3.6.6 *For a sequence x(n) = {4, 3, 2, 1, –1, 2, 3, 4} obtain the 8 point FFT computation using DIT method.*

AU : Dec.-16, Marks 12

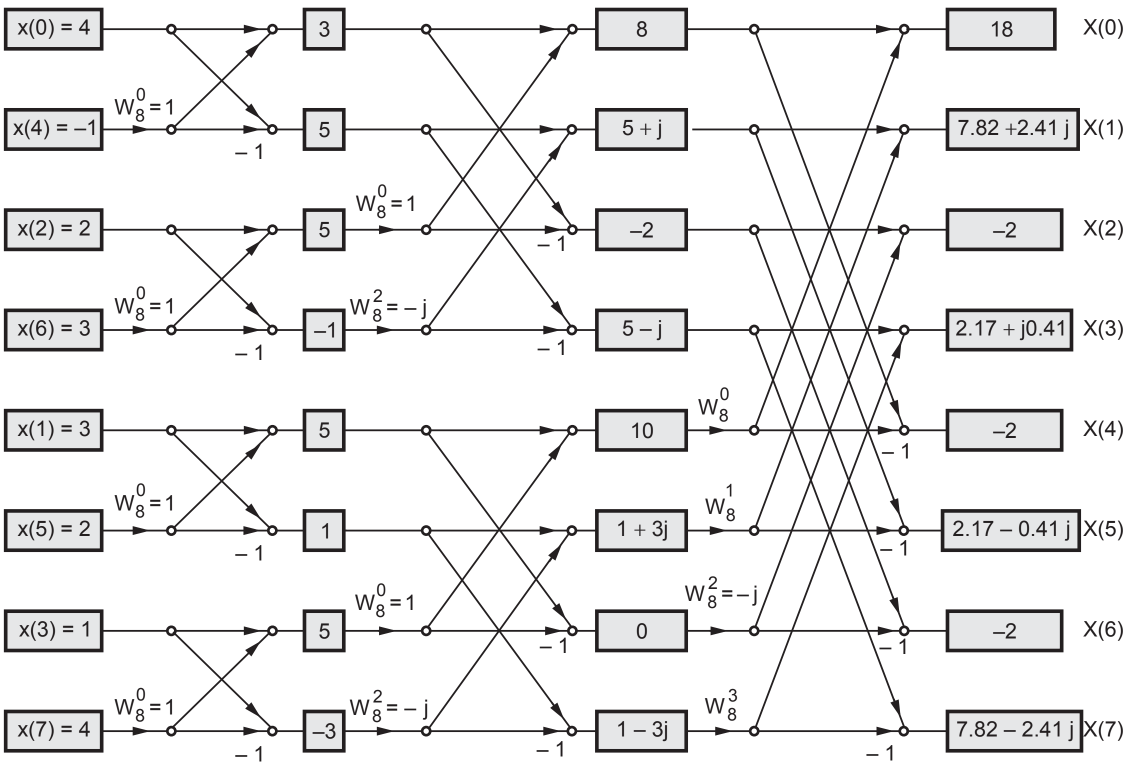


Fig. 3.6.17 (a) Radix-2 DIT-FFT

3.6.2 Radix-2 DIF FFT Algorithm

Principle : To express N-point sequence in terms of two $\frac{N}{2}$ - point sequences.

Step 1 : We know that by definition N-point DFT is given as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad \dots (3.6.36)$$

Let us split this formula into two summations. The first summation will be for first $\frac{N}{2}$ data points and second summation will be for remaining $\frac{N}{2}$ data points as shown below :

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n)\, W_N^{kn} \quad \dots (3.6.36(a))$$

Step 2 : Let us rearrange the second summation in the above equation as follows :

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{k\left(n+\frac{N}{2}\right)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + W_N^{kN/2} \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{kn} \quad \dots (3.6.37)$$

Step 3 : In the above equation $W_N^{kN/2}$ can be expressed as,

$$W_N^{kN/2} = e^{-j\frac{2\pi}{\mathrm{N}}\cdot\frac{kN}{n}} \quad \text{since} \quad W_N = e^{-j\frac{2\pi}{\mathrm{N}}}$$

$$= e^{-j\pi k} = \left(e^{-j\pi}\right)^k = (\cos\pi - j\sin\pi)^k = (-1-j\,0)^k$$

$$= (-1)^k \quad \dots (3.6.38)$$

Hence equation 3.6.37 can be written as,

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + (-1)^k \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^k \, x\left(n + \frac{N}{2}\right) \right] W_N^{kn} \qquad \text{... (3.6.39)}$$

Step 4 : Now let us split $X(k)$ into even and odd numbered samples. i.e.,

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2k} \, x\left(n + \frac{N}{2}\right) \right] W_N^{2kn} \qquad \text{... (3.6.40)}$$

$$k = 0, 1, \ldots \frac{N}{2} - 1$$

and
$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2k+1} \, x\left(n + \frac{N}{2}\right) \right] W_N^{(2k+1)n} \qquad \text{... (3.6.41)}$$

$$k = 0, 1, \ldots \frac{N}{2} - 1$$

Step 5 : In equation 3.6.40 above, $(-1)^{2k} = 1$ always. And W_N^{2kn} will be,

$$W_N^{2kn} = \left(W_N^2\right)^{kn}$$

From equation 3.5.21 we know that $W_N^2 = W_{N/2}$. Hence above equation becomes,

$$W_N^{2kn} = \left(W_{N/2}\right)^{kn} = W_{N/2}^{kn} \qquad \text{... (3.6.42)}$$

Hence equation 3.6.40 becomes,

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n + \frac{N}{2}\right) \right] W_{N/2}^{kn} \qquad \text{... (3.6.43)}$$

$$k = 0, 1, \ldots \frac{N}{2} - 1$$

Step 6 : Now consider equation 3.6.41. In this equation, $(-1)^{2k+1} = (-1)^{2k}\,(-1) = -1$ always. And $W_N^{(2k+1)n} = W_N^{2kn+n} = W_N^{2kn} \cdot W_N^n$. i.e.,

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n + \frac{N}{2}\right) \right] W_N^{2kn} \cdot W_N^n$$

In the above equation $W_N^{2kn} = W_{N/2}^{kn}$ from equation 3.6.42. Hence,

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left\{\left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^n\right\} W_{N/2}^{kn} \quad \text{... (3.6.44)}$$

Equation 3.6.43 and above equation combinely gives DFT $X(k)$. Let us define the two $\frac{N}{2}$ point sequences $g_1(n)$ and $g_2(n)$ as,

These equations express N – point sequence interms of two $\frac{N}{2}$ point sequences

$$\begin{cases} g_1(n) = x(n) + x\left(n+\frac{N}{2}\right), & n = 0, 1, \ldots, \frac{N}{2}-1 \\ g_2(n) = \left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^n, & n = 0, 1, \ldots, \frac{N}{2}-1 \end{cases}$$

....(3.6.45)

...(3.6.46)

Then equation 3.6.43 and equation 3.6.44 can be written as,

Calculation of N – point DFT from $\frac{N}{2}$ – point sequences

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} g_1(n)\, W_{N/2}^{kn}, \quad k = 0, 1, \ldots, \frac{N}{2}-1 \quad \text{....(3.6.47)}$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} g_2(n)\, W_{N/2}^{kn}, \quad k = 0, 1, \ldots, \frac{N}{2}-1 \quad \text{...(3.6.48)}$$

Key Point *Here observe that the N-point DFT is splitted into two $\frac{N}{2}$ points DFTs. Since $X(k)$ is decimated with 'k' even and 'k' odd, this is called Decimation in Frequency (DIF) FFT. 'k' represents frequency components in the range of 0 to 2π .*

An example of N = 8 point DFT

Now let us consider an example of 8-point DFT. The values of $g_1(n)$ and $g_2(n)$ of equation 3.6.45 and equation 3.6.46 becomes,

$$g_1(n) = x(n) + x(n+4) \quad \text{from equation 3.6.45 with } N = 8$$

$$\therefore \quad \left.\begin{aligned} g_1(0) &= x(0) + x(4) \\ g_1(1) &= x(1) + x(5) \\ g_1(2) &= x(2) + x(6) \\ g_1(3) &= x(3) + x(7) \end{aligned}\right\} \quad \text{... (3.6.49)}$$

And $g_2(n) = [x(n) - x(n+4)]\, W_8^n$ from equation 3.6.46 with $N = 8$

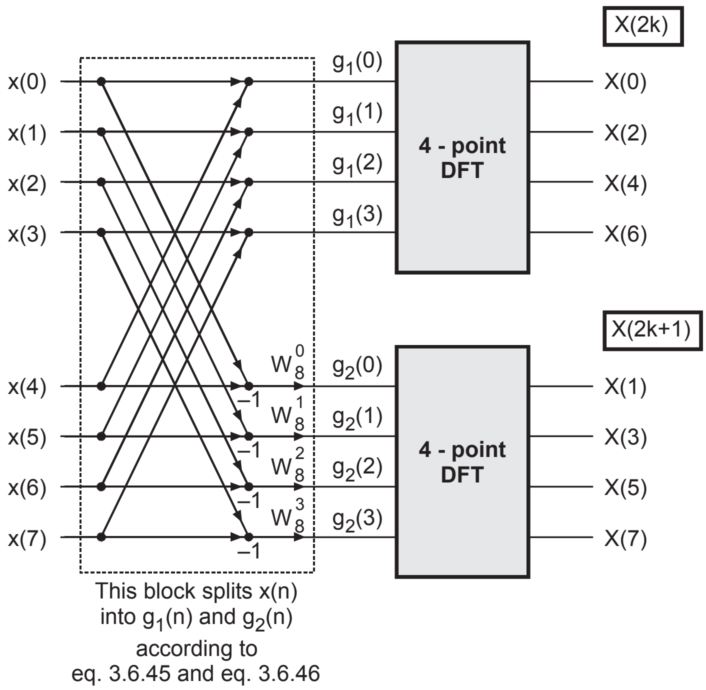


Fig. 3.6.18 First stage of decimation of DIF FFT algorithm for N = 8

$$\therefore \quad \left.\begin{aligned} g_1(0) &= [x(0)-x(4)]\, W_8^0 \\ g_1(1) &= [x(1)-x(5)]\, W_8^1 \\ g_1(2) &= [x(2)-x(6)]\, W_8^2 \\ g_1(3) &= [x(3)-x(7)]\, W_8^3 \end{aligned}\right\} \qquad \ldots (3.6.50)$$

Fig. 3.6.18 shows the partial signal flow graph for computation of $g_1(n)$ and $g_2(n)$ according to above set of equations. This Fig. 3.6.18 also shows that from $g_1(n)$, the 4-point DFT $X(2k)$ is obtained. Similarly from $g_2(n)$, the 4-point DFT $X(2k+1)$ is obtained.

In the Fig. 3.6.14 observe that $X(k)$ is splitted into even numbered values and odd numbered values. These two sets of DFTs are computed separately by 4-point DFTs. For these 4-point DFTs sequences $g_1(n)$ and $g_2(n)$ are required. The signal flow graph for computation of $g_1(n)$ and $g_2(n)$ from $x(n)$ is shown in above figure.

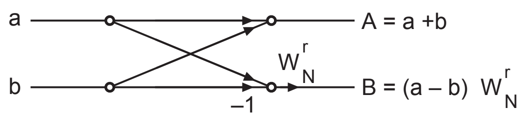


Fig. 3.6.19 Butterfly operation in DIF-FFT

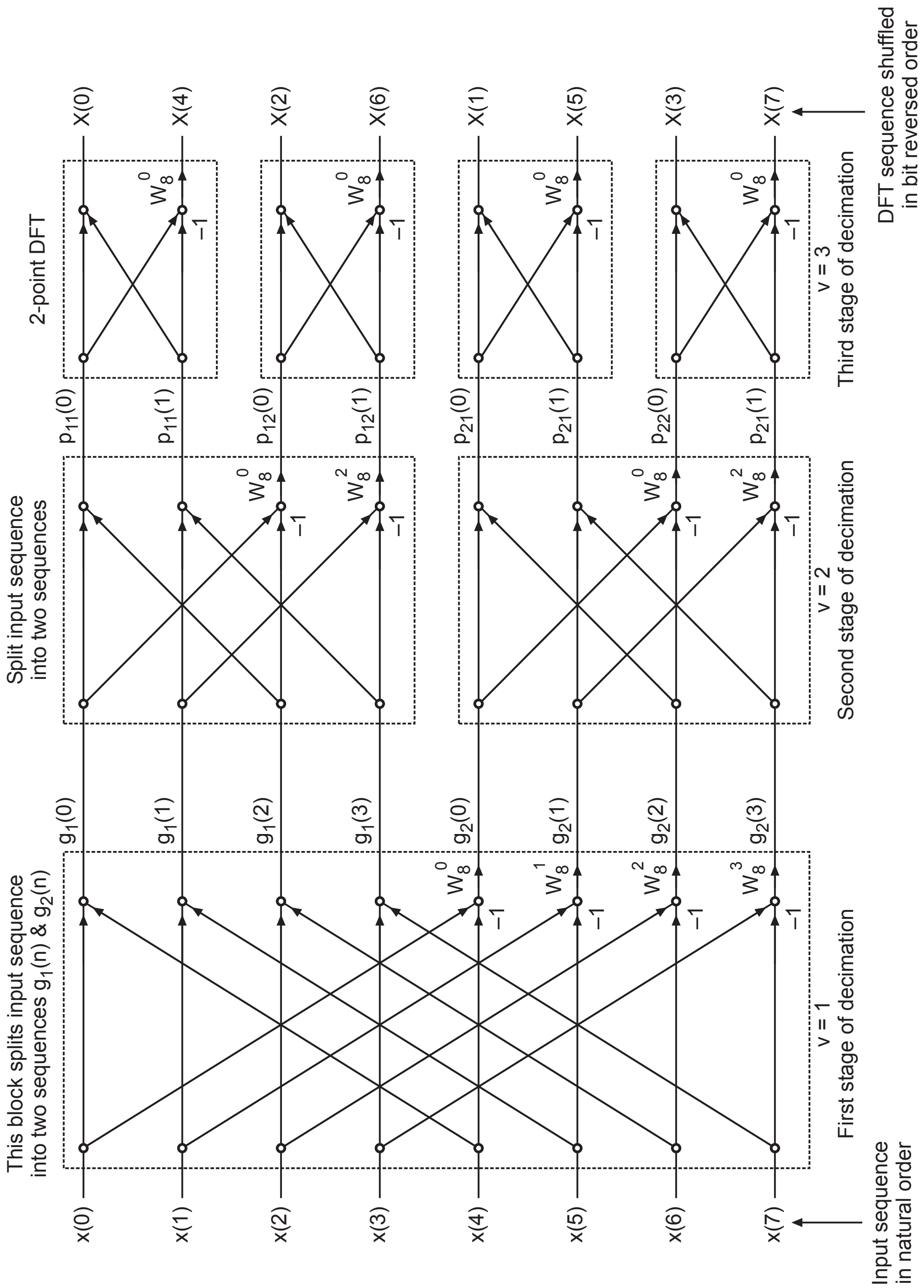


Fig. 3.6.20 Signal flow graph and stages of computation of Radix-2 DIF-FFT algorithm for N = 8

Observe that the computation of Fig. 3.6.18 is opposite to that shown in Fig. 3.6.2 for DIT-FFT.

3.6.2.1 Butterfly Operation in DIF-FFT

The basic butterfly operation in DIF-FFT algorithm is shown in Fig. 3.6.19. Observe that the butterflies in Fig. 3.6.18 are similar to the one shown in Fig. 3.6.19.

The sequences $X(2k)$ and $X(2k+1)$ are further decimated into their even and odd numbered values. Thus for N-point DFT, there will be $v = log_2 N$ decimation stages. The procedure is exactly similar to that of DIT-FFT, but opposite in direction.

Signal flow graph for N = 8 point DIF-FFT

Now let us consider how the signal flow graph for $N = 8$ point DIF-FFT will be obtained. The signal flow graph of Fig. 3.6.18 is developed further and is shown in Fig. 3.6.20. (See Fig. 3.6.20 on previous page)

Observe that this signal flow graph is similar to that of DIT-FFT (Fig. 3.6.20) but opposite in direction. The input sequence is in natural order. Observe that the DFT sequence $X(0), X(4), X(2), X(6), X(1), X(5), X(3), X(7)$ is shuffled in bit reversed order. Since $N = 8$, there are $v = log_2 N = log_2 8 = 3$ stages of decimation. Each stage contains $\frac{N}{2}$ i.e. 4 for $N = 8$ butterfly operations.

Observe the signal flow graph of 8-point DIF-FFT given in Fig. 3.6.19. The input sequence $x(n)$ is applied in natural order. This 8-point sequence is splitted into two 4-point sequences $g_1(n)$ and $g_2(n)$. The sequences $g_1(n)$ and $g_2(n)$ can be computed with the help of equation 3.6.49 and equation 3.6.50. Similarly, the 4-point sequence $g_1(n)$ can be splitted into two 2-point sequences $p_{11}(n)$ and $p_{12}(n)$. The equations for $p_{11}(n)$ and $p_{12}(n)$ can be written as follows :

$$\left.\begin{aligned} p_{11}(0) &= g_1(0) + g_1(2) \\ p_{11}(1) &= g_1(1) + g_1(3) \\ p_{12}(0) &= \left[g_1(0) - g_1(2)\right] W_8^0 \\ p_{12}(1) &= \left[g_1(1) - g_1(3)\right] W_8^0 \end{aligned}\right\} \quad \text{... (3.6.51)}$$

Similarly, the 4-point sequence $g_2(n)$ can be splitted into two 2-point sequences $p_{21}(n)$ and $p_{22}(n)$. The equations for $p_{21}(n)$ and $p_{22}(n)$ can be written as follows :

$$\left.\begin{aligned} p_{21}(0) &= g_2(0) + g_2(2) \\ p_{21}(1) &= g_2(1) + g_2(3) \\ p_{22}(0) &= \left[g_2(0) - g_2(2)\right] W_8^0 \\ p_{22}(1) &= \left[g_2(1) - g_2(3)\right] W_8^2 \end{aligned}\right\} \quad \text{... (3.6.52)}$$

Here observe that equation 3.6.51 and above equation can be derived on the same logic which is used for equation 3.6.49 and equation 3.6.50. The equation 3.6.51 and equation 3.6.52 can also be written from the signal flow graph of Fig. 3.6.20 directly.

The next stage is to compute 2-point DFTs directly. These equations can be written simply as follows :

$$\left.\begin{aligned} X(0) &= p_{11}(0) + p_{11}(1) \\ X(4) &= \left[p_{11}(0) - p_{11}(1)\right] W_8^0 \\ X(2) &= p_{12}(0) + p_{12}(1) \\ X(6) &= \left[p_{12}(0) - p_{12}(1)\right] W_8^0 \\ X(1) &= p_{21}(0) + p_{21}(1) \\ X(4) &= \left[p_{21}(0) - p_{21}(1)\right] W_8^0 \\ X(3) &= p_{22}(0) + p_{22}(1) \\ X(7) &= \left[p_{22}(0) - p_{22}(1)\right] W_8^0 \end{aligned}\right\} \quad \text{... (3.6.53)}$$

Important note : Compare the signal flow graph of DIT-FFT of Fig. 3.6.7 and that of DIF-FFT of Fig. 3.6.20 carefully. The direction of DIF-FFT is opposite to that of DIT-FFT. These signal flow diagrams can be prepared for higher value of 'N' using the same logic. There is no need to go through all the mathematics for every stage. The logic used for first stage of decimation remains same for next stages. It is possible to actually write the 'C' program or MATLAB program using these signal flow diagrams.

3.6.2.2 Computational Complexity

Observe that the butterfly operation of DIF-FFT given in Fig. 3.6.19 is similar to that of DIT-FFT given in Fig. 3.6.8 except multiplication by factor W_N^r. Hence the butterfly of Fig. 3.6.19 also needs one complex multiplication and two complex additions. Since there are $\frac{N}{2}$ butterflies per stage and there are $v = log_2 N$ stages, there are total $\frac{N}{2} \times v$ butterflies in the computation. Hence we can write,

$$\text{Number of complex multiplications} = \frac{N}{2} \times v$$

$$= \frac{N}{2} \log_2 N \quad \text{... (3.6.54)}$$

$$\text{Number of complex additions} = 2 \times \left(\frac{N}{2} \times v \right) = N \log_2 N \quad \text{... (3.6.55)}$$

Key Point *Thus the computational complexity of DIT-FFT and DIF-FFT algorithms is same (see equation 3.6.27 and equation 3.6.28).*

Thus the computational complexity of DIT-FFT and DIF-FFT algorithms is same (see equation 3.6.27 and equation 3.6.28).

3.6.2.3 Memory Requirement and Inplace Computation

The butterfly operation of this algorithm can be implemented similar to that of DIT FFT. The value of 'A' can be stored where 'a' was stored and 'B' can be stored where 'b' was stored. This is called inplace computation. $A-a$ and $B-b$ are complexed valued. Hence two memory locations are required to store each of them. Therefore the memory locations required for in place computation of one butterfly are [see equation 3.6.31].

$$\text{Memory locations for one butterfly} = 2\times 2 = 4 \quad \text{... (3.6.56)}$$

The computations are performed stagewise. There are $\frac{N}{2}$ butterflies in one stage. Hence memory requirement for computation of N-point DFT becomes [see equation 3.6.33],

$$\text{Memory requirement of N-point DFT} = 4\times \frac{N}{2} = 2N \quad \text{... (3.6.57)}$$

Key Point *Thus the memory requirement of DIF-FFT is same as that of DIT-FFT. The maximum memory requirement including storage of twiddle factors will be* $2N+\frac{N}{2}$.

3.6.2.4 Bit Reversal

In the signal flow graph of Fig. 3.6.20 observe that the input sequence is in natural order but output DFT sequence is in bit reversed order. In DIT FFT, the input sequence is in bit reversed order. Hence to get DFT in natural order it should be read in bit reversed order. The common bit reversal algorithm can be developed. This algorithm can be used to reshuffle the input sequence in bit reversed order. Such algorithm can be used for DIF as well as DIT algorithms.

Example for Understanding

Example 3.6.7 *Obtain the 8-point DFT of the following sequence using Radix-2 DIF-FFT algorithm. Show all the results along signal flow graph.*

$x(n) = \{2,1,2,1\}$

Verify your result using direct computation of DFT.

Solution : The given sequence has 4-samples. Since we want 8-point DFT, we should pad 4-zeros at the end of $x(n)$. i.e.,

$$x(n) = \{2, 1, 2, 1, 0, 0, 0, 0\}$$

This sequence should be used as input sequence to DIF-FFT algorithm. The individual samples are

$$x(0) = 2 \qquad x(4) = 0$$
$$x(1) = 1 \qquad x(5) = 0$$
$$x(2) = 2 \qquad x(6) = 0$$
$$x(3) = 1 \qquad x(7) = 0$$

We need phase factors for the computation of DFT. These phase factors are calculated in example 3.6.1. From Table 3.6.3, these phase factors are reproduced below for convenience.

$$W_8^0 = 1$$
$$W_8^1 = 0.707 - j\ 0.707$$
$$W_8^2 = -j$$
$$W_8^3 = -0.707 - j\ 0.707$$

Now let us calculate DFT using signal flow graph given in Fig. 3.6.20.

To split 8-point input sequence $x(n)$ into two 4-point sequences $g_1(n)$ and $g_2(n)$:

In the first stage, the input sequence is splitted into two 4-point sequences $g_1(n)$ and $g_2(n)$. Table 3.6.4 shows the calculation of $g_1(n)$ and $g_2(n)$. The part of the signal flow graph of Fig. 3.6.20 for this calculation is also shown in the table. Equation 3.6.49 and equation 3.6.50 are used for calculation. Actually the signal flow graph also shows how $g_1(n)$ and $g_2(n)$ can be obtained.

To split the 4-point sequences $g_1(n)$ and $g_2(n)$ into 2-point sequences $p_{11}(n)$, $p_{12}(n)$, $p_{21}(n)$ and $p_{22}(n)$:

We have obtained the two 4-point sequences $g_1(n)$ and $g_2(n)$ in the first stage of DIF-FFT algorithm. The next stage is to decompose these 4-point sequences into 2-point sequences. Table 3.6.5 below shows this computation. The table shows part of the signal flow graph of Fig. 3.6.20 used for this computation.

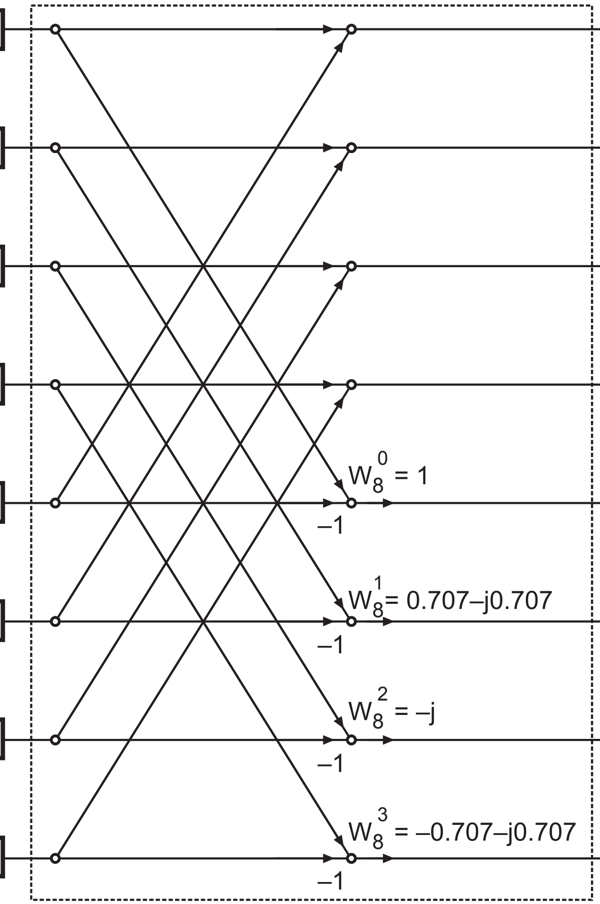
Inputs	Signal flow graph for splitting x(n) into $g_1(n)$ and $g_2(n)$	Calculations of $g_1(n)$ and $g_2(n)$ according to 8-point DIF-FFT algorithm and signal flow graph	
$x(0) = 2$		$g_1(0) = x(0)+x(4) = 2+0 = 2$	$g_1(0) = 2$
$x(1) = 1$		$g_1(1) = x(1)+x(5) = 1+0 = 1$	$g_1(1) = 1$
$x(2) = 2$		$g_1(2) = x(2)+x(6) = 2+0 = 2$	$g_1(2) = 2$
$x(3) = 1$		$g_1(3) = x(3)+x(7) = 1+0 = 1$	$g_1(3) = 1$
$x(4) = 0$		$g_2(0) = [x(0)-x(4)]W_8^0 = [2-0]\ 1 = 2$	$g_2(0) = 2$
$x(5) = 0$		$g_2(1) = [x(1)-x(5)]W_8^1 = [1-0](0.707-j0.707) = 0.707-j0.707$	$g_2(1) = 0.707-j0.707$
$x(6) = 0$		$g_2(2) = [x(2)-x(6)]W_8^2 = [2-0]\ (-j) = -j2$	$g_2(2) = -j2$
$x(7) = 0$		$g_2(3) = [x(3)-x(7)]W_8^3 = [1-0](-0.707-j0.707) = -0.707-j0.707$	$g_2(3) = -0.707-j0.707$

Table 3.6.4 Splitting 8-point input sequence into two 4-point sequences $g_1(n)$ and $g_2(n)$

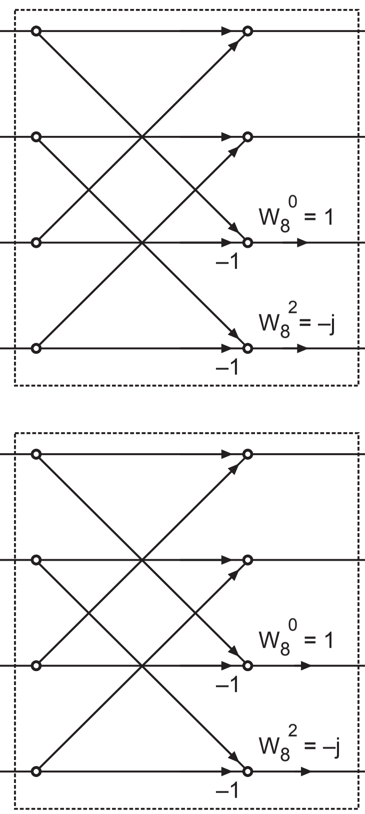
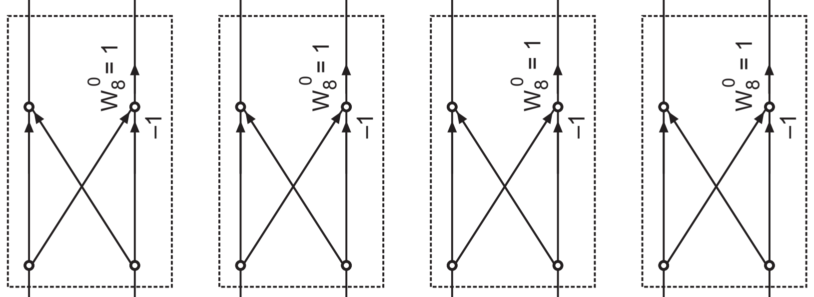
Inputs	Signal flow graph for splitting 4 point sequences into 2-point sequences	Calculations for 2-point sequences	
$g_1(0) = 2$		$p_{11}(0) = g_1(0) + g_1(2)$ $= 2+2$ $= 4$	$p_{11}(0) = 4$
$g_1(1) = 1$		$p_{11}(1) = g_1(1) + g_1(3)$ $= 1+1$ $= 2$	$p_{11}(1) = 2$
$g_1(2) = 2$		$p_{12}(0) = [g_1(0) - g_1(2)]\, W_8^0$ $= (2-2)\ 1$ $= 0$	$p_{12}(0) = 0$
$g_1(3) = 1$		$p_{12}(1) = [g_1(1) - g_1(3)] W_8^2$ $= (1-1)\ (-j)$ $= 0$	$p_{12}(1) = 0$
$g_2(0) = 2$		$p_{21}(0) = g_2(0) + g_2(2)$ $= 2 - j2$	$p_{21}(0) = 2-j2$
$g_2(1) = 0.707-j0.707$		$p_{21}(1) = g_2(1) + g_2(3)$ $= 0.707-j0.707-0.707-j0.707$ $= -j1.414$	$p_{21}(1) = -j1.414$
$g_2(2) = -j2$		$p_{22}(0) = [g_2(0) - g_2(2)] W_8^0$ $= [2-(-j2)]\ 1$ $= 2 + j2$	$p_{22}(0) = 2+j2$
$g_2(3) = -0.707-j0.707$		$p_{22}(1) = [g_2(1) - g_2(3)] W_8^1$ $= [0.707-j0.707+0.707+j0.707](-j)$ $= -j1.414$	$p_{22}(1) = -j1.414$

Table 3.6.5 Splitting 4-point sequences into 2-point sequences

To calculate the 2-point DFTs :

In the previous stage we obtained the 2-point sequences. According to the signal flow graph of Fig. 3.6.20, the next stage is to calculate 2-point DFTs. This part of the signal flow graph of Fig. 3.6.20 is shown below in Table 3.6.6. The table also shows relevent calculations of these 2-point DFTs.

Signal flow graph for calculation of 2-point DFT



Inputs	Calculations	
$p_{11}(0) = 4$	$X(0) = p_{11}(0) + p_{11}(1) = 4+2 = 6$	$X(0) = 6$
$p_{11}(1) = 2$	$X(4) = [p_{11}(0) - p_{11}(1)]\, W_8^0 = (4-2)\ 1 = 2$	$X(4) = 2$
$p_{12}(0) = 0$	$X(2) = p_{12}(0) - p_{12}(1) = 0+0 = 0$	$X(2) = 0$
$p_{12}(1) = 0$	$X(6) = [p_{12}(0) - p_{12}(1)]W_8^0 = (0-0)\ 1 = 0$	$X(6) = 0$
$p_{21}(0) = 2-j2$	$X(1) = p_{21}(0) + p_{21}(1) = 2-j2-j1.414 = 2-j3.414$	$X(1) = 2-j3.414$
$p_{21}(1) = -j1.414$	$X(5) = [p_{21}(0) - p_{21}(1)]W_8^0 = [2-j2-(-j1.414)]\ 1 = 2-j0.586$	$X(5) = 2-j0.586$
$p_{22}(0) = 2+j2$	$X(3) = p_{22}(0) + p_{22}(1) = 2+j2-j1.414 = 2+j\,0.586$	$X(3) = 2+j0.586$
$p_{22}(1) = -j1.414$	$X(7) = [p_{22}(0) - p_{22}(1)]W_8^0 = [2+j2-(-j1.414)]\ 1 = 2+j3.414$	$X(7) = 2+j3.414$

Table 3.6.6 Calculation of 2-point DFTs

The complete signal flow graph of DIF FFT algorithm :

Fig. 3.6.21 shows the complete signal flow graph of this DIF-FFT algorithm along with intermediate results.

Please refer Fig. 3.6.21 on next page.

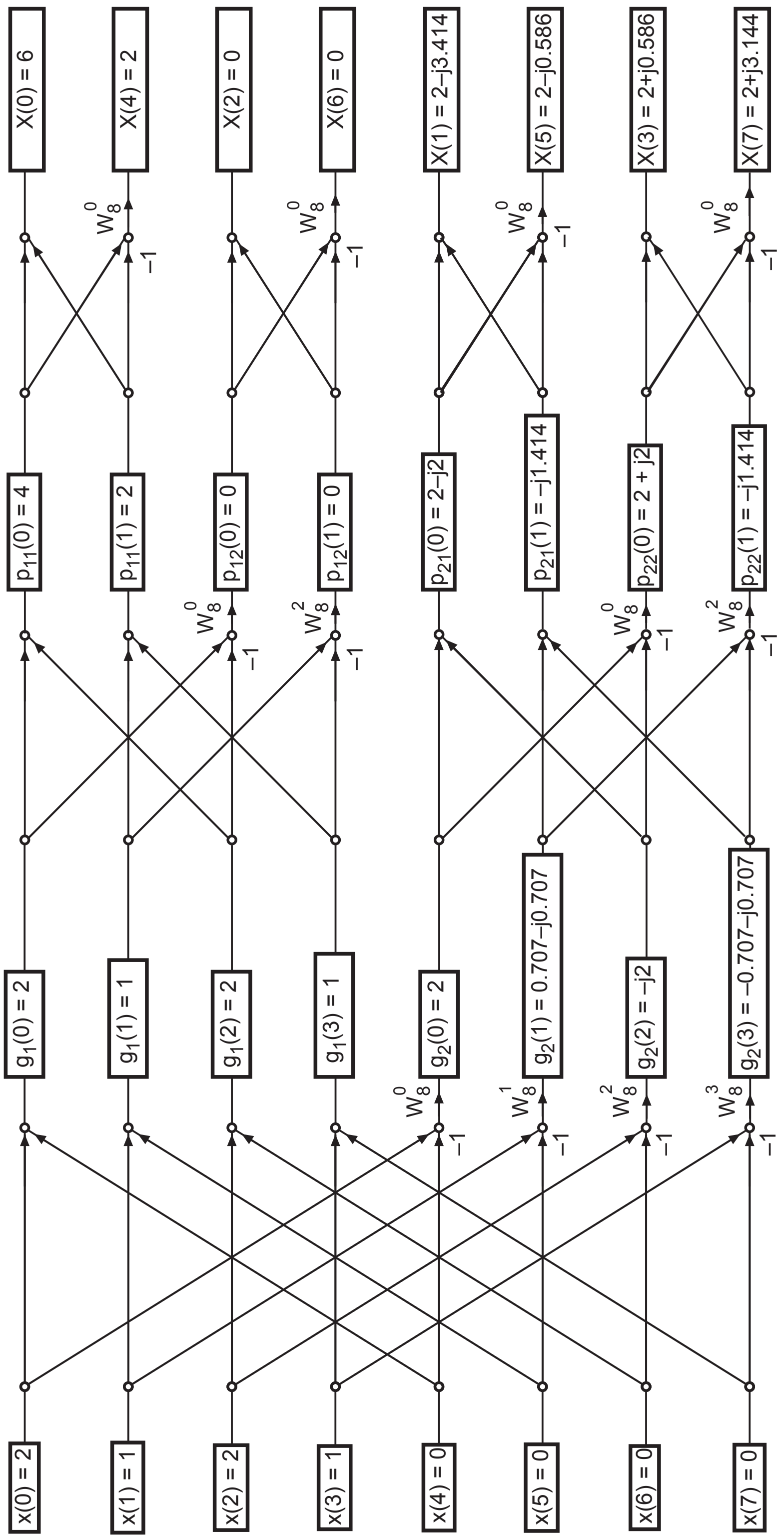


Fig. 3.6.21 Signal flow graph of example 3.6.2. It shows computation of DFT using DIF-FFT algorithm. Input sequence, output DFT and intermediate results are shown in the above diagram

Thus we obtained the DFT of the sequence $x(n)$ using DIF-FFT algorithm as follows :

$$\left.\begin{array}{l} X(0) = 6 \\ X(1) = 2 - j\,3.414 \\ X(2) = 0 \\ X(3) = 2 + j\,0.586 \\ X(4) = 2 \\ X(5) = 2 - j\,0.586 \\ X(6) = 0 \\ X(7) = 2 + j\,3.414 \end{array}\right\} \quad \ldots (3.6.58)$$

Examples with Solution

Example 3.6.8 *Compute DFT of the sequence* $x(n) = \cos\frac{n\pi}{2}$, *where N = 4 using DIF-FFT algorithm.*

Solution : The sequence $x(n)$ can be obtained by putting n = 0, 1, 2, 3 in

$$x(n) = \cos\frac{n\pi}{2}. \text{ i.e.,}$$

$$x(0) = \cos(0) = 1$$

$$x(1) = \cos\frac{\pi}{2} = 0$$

$$x(2) = \cos\frac{2\pi}{2} = -1$$

$$x(3) = \cos\frac{3\pi}{2} = 0$$

Table 3.6.7 shows the calculations and signal flow graph of the DIF-FFT algorithm.

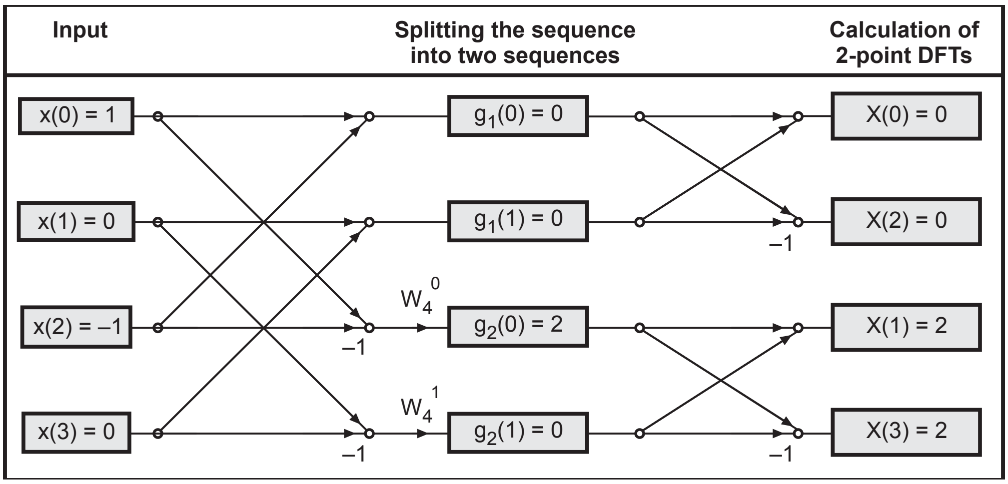


Table 3.6.7 Calculation of DFT

Thus the DFT is, $X(k) = \{ 0, 2, 0, 2 \}$

Example 3.6.9 *Obtain the 8 point DFT using DIFFFT algorithm for*

$x(n) = \{\underset{\uparrow}{1},1,1,1,1,1,1,1\}$

Solution : The signal flow graph of DIF-FFT along with stage wise values is shown in Fig. 3.6.22.

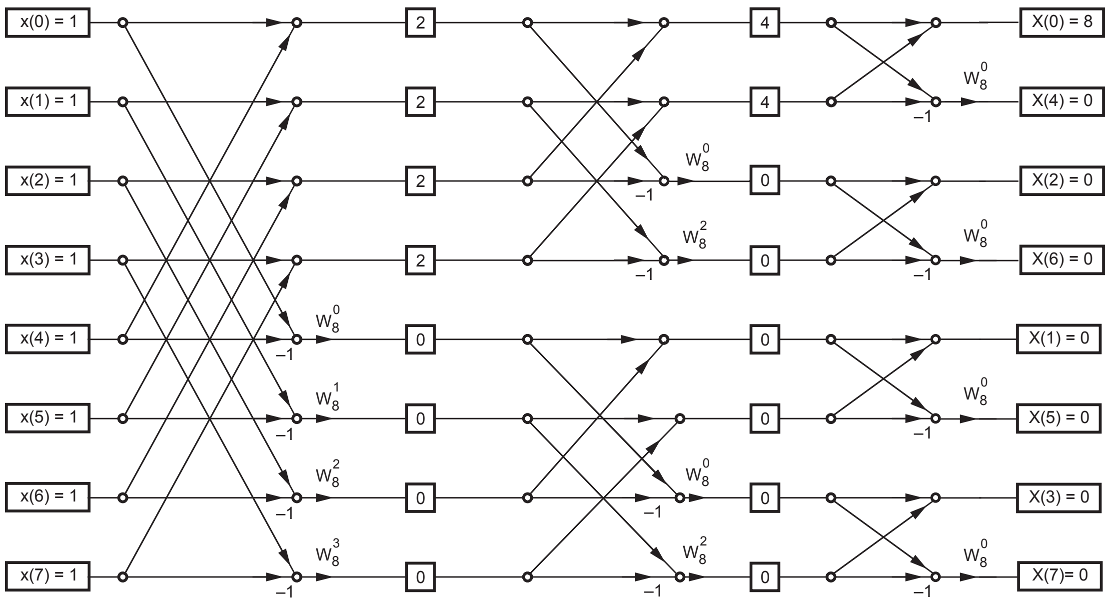


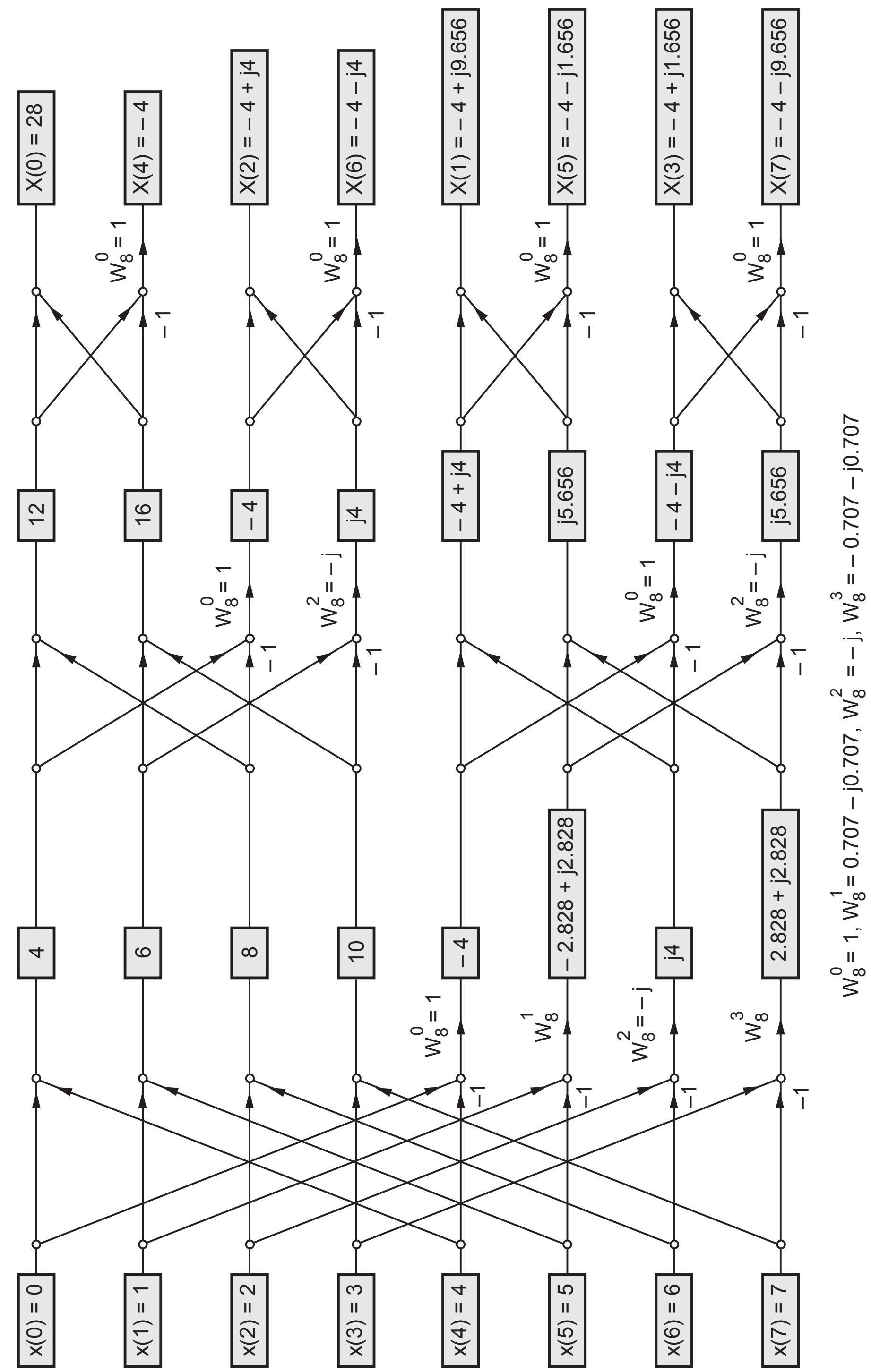
Fig. 3.6.22 DIF-FFT

Example 3.6.10 *Compute the eight point DFT of the sequence x = {0, 1, 2, 3, 4, 5, 6, 7} using DIF FFT algorithm.* **AU : Dec.-15, Marks 12**

Solution : Fig. 3.6.23 shows the signal flow graph of radix - 2 DIF - FFT algorithm along with stage wise results. (See Fig. 3.6.23 on next page)

Thus we obtained,

$$X(k) = \{28, -4 + j\,9.656, -4 + j\,4, -4 + j\,1.656, -4 - j\,1.656, -4 - j4, -4 - j\,9.656\}$$



x(0) = 0
x(1) = 1
x(2) = 2
x(3) = 3
x(4) = 4
x(5) = 5
x(6) = 6
x(7) = 7
4
6
8
10
– 4
– 2.828 + j2.828
j4
2.828 + j2.828
12
16
– 4
j4
– 4 + j4
j5.656
– 4 – j4
j5.656
X(0) = 28
X(4) = – 4
X(2) = – 4 + j4
X(6) = – 4 – j4
X(1) = – 4 + j9.656
X(5) = – 4 – j1.656
X(3) = – 4 + j1.656
X(7) = – 4 – j9.656
$W_8^0 = 1$
W_8^1
$W_8^2 = -j$
W_8^3
–1
$W_8^0 = 1, W_8^1 = 0.707 - j0.707, W_8^2 = -j, W_8^3 = -0.707 - j0.707$

Fig. 3.6.23

Example 3.6.11 *Compute the 4 point DFT of the sequence x(n) = {0, 1, 2, 3} using DIT and DIF algorithm.* **AU : May-16, Marks 8**

Solution : i) DFT using DIT - FFT

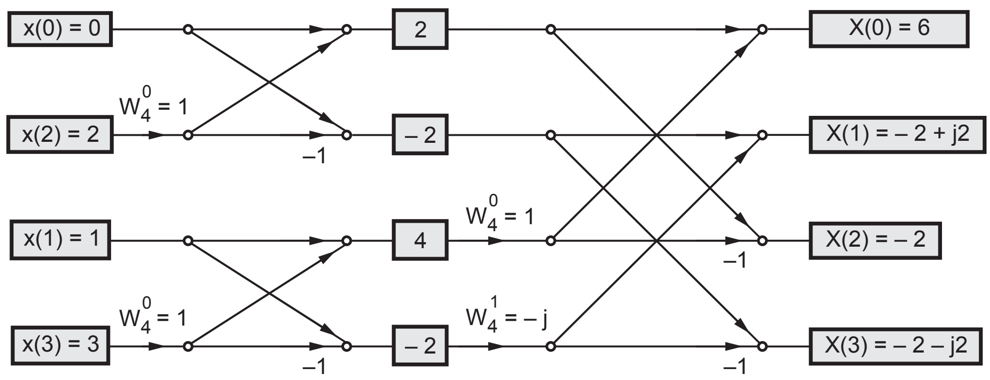


Fig. 3.6.24 DIT - FFT algorithm

ii) DFT using DIF – FFT

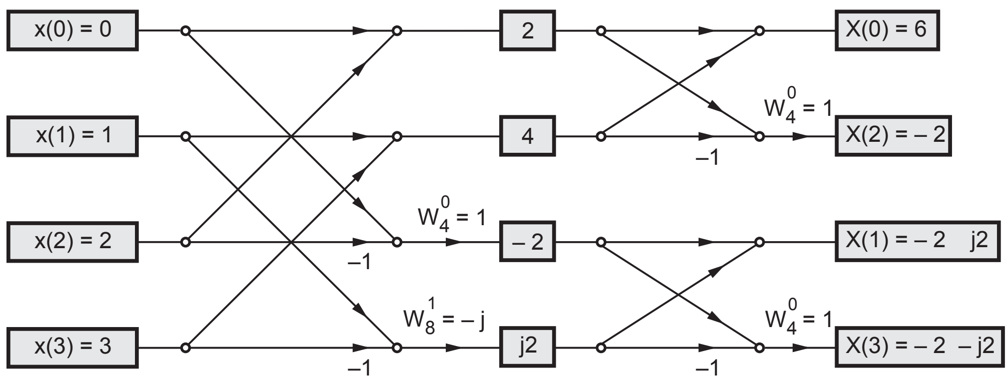


Fig. 3.6.25 DIF - FFT algorithm

3.6.3 Comparison between Radix-2 DIT and DIF FFT Algorithms

Following table lists the comparison.

Sr. No.	DIT FFT	DIF FFT
1.	The time domain sequence is decimated.	The DFT X(k) is decimated.
2.	Input sequence is to be given in bit reversed order.	The DFT at the output is in bit reversed order.
3.	First calculates 2-point DFTs and combines them.	Decimates the sequence step by step to 2-point sequence and calculates DFT.
4.	Suitable for calculating inverse DFT.	Suitable for calculating DFT.

Table 3.6.8 Comparison between DIT and DIF FET

Examples for Practice

Example 3.6.12 *Draw the flow graph for the implementation of 8-point DIT-FFT of the following sequence.*

$x(n) = \{0.5, 0.5, 0.5, 0.5, 0, 0, 0, 0\}$

[Ans. : $X(k) = \{2, 0.5 - j\,1.206, 0, 0.5 - j\,0.206, 0, 0.5 + j\,0.206, 0, 0.5 + j\,1.206\}$]

Example 3.6.13 *Obtain the 8-point FFT of the following pulse signal using flow diagram :*

$x(0) = x(1) = x(2) = x(3) = 1$

$x(4) = 0$

$x(5) = x(6) = x(7) = 1$

Use DIF-FFT algorithm. **[Ans. : $X(k) = \{7, 1, -1, 1, -1, 1, -1, 1\}$]**

Example 3.6.14 *Determine the DFT of the following sequence using DIF-FFT algorithm.*

$x_1(n) = \{1, 1, 1, 0, 0, 1, 1, 1\}$

Using the DFT of $x_1(n)$, findout the DFT of the sequence,

$x_2(n) = \{1, 1, 1, 1, 1, 0, 0, 1\}$

[Ans. : $X_1(k) = \{6, 1.707 + j\,0.707, -1 - j, 0.293 + j\,0.707, 0, 0.293 - j\,0.707, -1 + j, 1.707 - j\,0.707\}$

$X_2(k) = \left\{\begin{matrix} 6, 0.707 - j\,1.707, 1 + j, -0.707 + j\,0.293, 0, -0.707 \\ -j\,0.293, 1 - j, 0.707 + j\,1.707 \end{matrix}\right\}$]

Review Questions

1. *Explain Radix-2 DIT-FFT algorithm. Explain how calculations are reduced.* **AU : Dec.-04, 08, Marks 16, Dec.-12, May-06, 12, Marks 8, May-04, 05, Marks 12**
2. *Explain Radix-2 DIF-FFT algorithm. Compare it with DIT-FFT algorithms.* **AU : Dec.-06, May-11, Marks 8, Dec.-11, Marks 10**
3. *Write short notes on the following :*

   i) Butterfly computation. ii) Inplace computations. iii) Bit reversal. **AU : May-04, Marks 4**
4. *Draw the flowchart for 8-point DIT-FFT.* **AU : Dec.-16, Marks 4**
5. *How do you obtain linear convolution using FFT ?* **AU : May-06, Marks 4**
6. *How do you calculate circular convolution using FFT ?* **AU : Dec.-06, Marks 4**
7. *Draw the flow graph of an 8-point DIF-FFT algorithm and explain.* **AU : May-15, Marks 8**
8. *Summerize the steps of radix-2 DIT-FFT algorithm.* **AU : May-16, Marks 8**

3.7 Inverse DFT using FFT Algorithms **AU : Dec.-10, May-15**

- The DFT is given as,

DFT : $$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

and IDFT is given as,

IDFT : $$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

- Above equations shows that the FFT algorithm developed for DFT can also be used to calculate IDFT with following minor changes :

i) The twiddle factor becomes W_N^{-kn} for IDFT.

ii) Divide the result by N in last stage.

Fig. 3.7.1 shows the signal flow graph of FFT algorithm to calculate IDFT. Observe that the input is DFT, *X(k)* in bit reversed order. Hence it is DIF - FFT algorithm. In the last stage the output is divided by N = 8. And the signs of exponents of 'W_8' are negative.

See Fig. 3.7.1 on next page.

Example for Understanding

Example 3.7.1 *Determine the response of LTI system when the input sequence is $x(n) = \{-1, 1, 2, 1, -1\}$ using radix-2 DIF FFT. The impulse response is $h(n) = \{-1, 1, -1, 1\}$.*

AU : Dec.-10, Marks 16

Solution : Step 1 : Defining lengths of DFTs

- Here length of *x(n)* is $M = 5$ and that of $h(n)$ is $L = 4$. Hence convolution of *x(n)* and *h(n)* will have a length of $N = L + M - 1 = 5 + 4 - 1 = 8$.
- Let us make length of *x(n)* and *h(n)* be '8' by appending necessary zeroes at the end. i.e.,

$$x(n) = \{-1,\ 1,\ 2,\ 1,\ -1,\ 0,\ 0,\ 0\}$$

$$h(n) = \{-1, 1, -1, 1, 0, 0, 0, 0\}$$

In such case circular convolution of above two sequences will be same as linear convolution.

Step 2 : DFTs of *x(n)* and *h(n)*

Fig. 3.7.2 shows the signal flow graph to obtain the DFT of *x(n)* using radix-2 DIF-FFT.

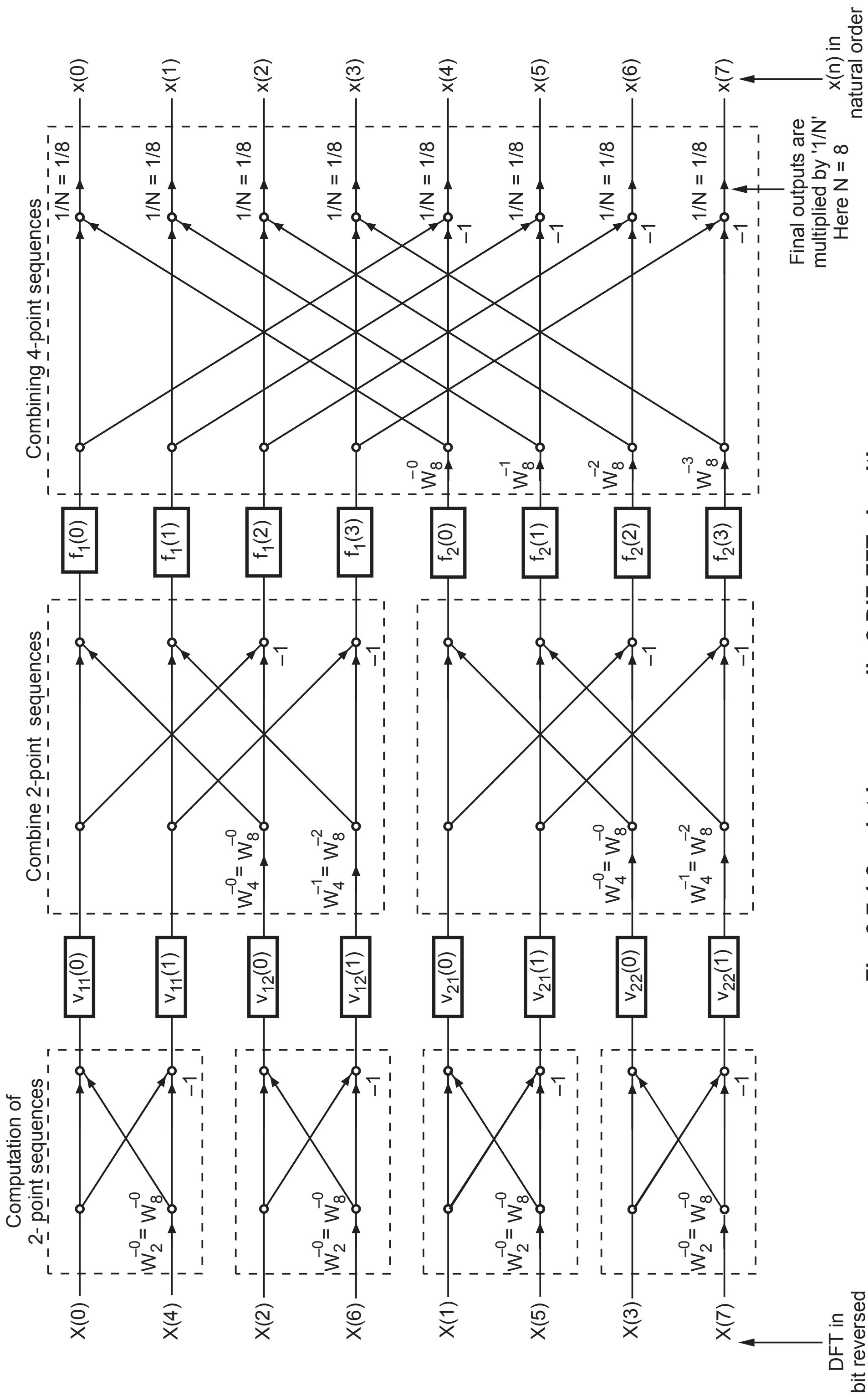


Fig. 3.7.1 8-point Inverse radix-2 DIF-FFT algorithm.

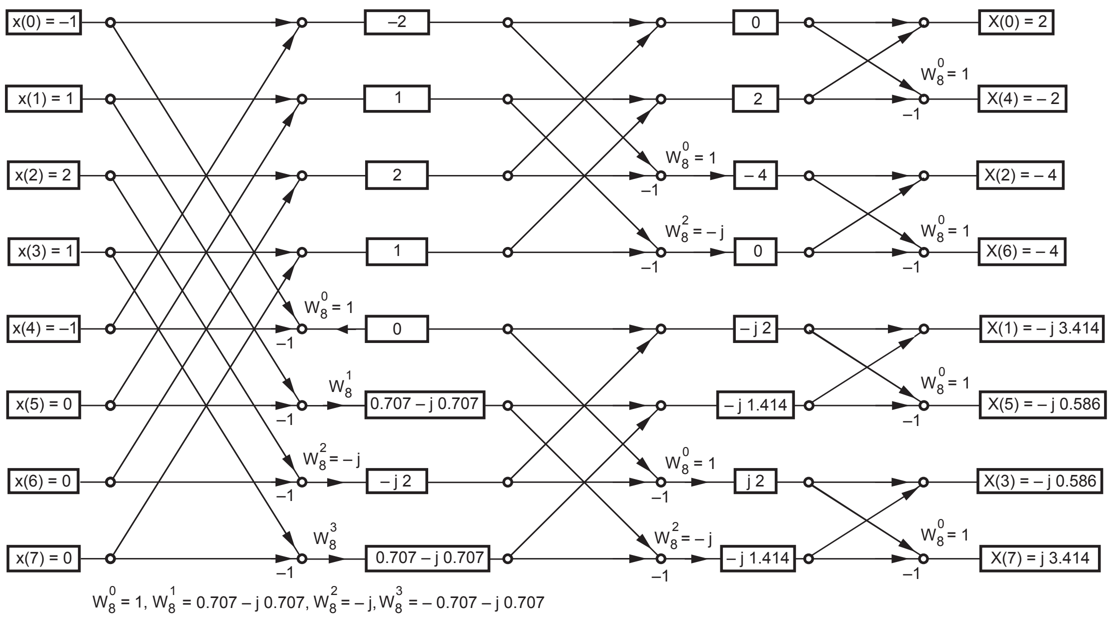


Fig. 3.7.2 DFT of x(n) using radix-2 DIF-FFT

Fig. 3.7.3 shows the signal flow graph to obtain the DFT of *h(n)* using radix-2 DIF-FFT.

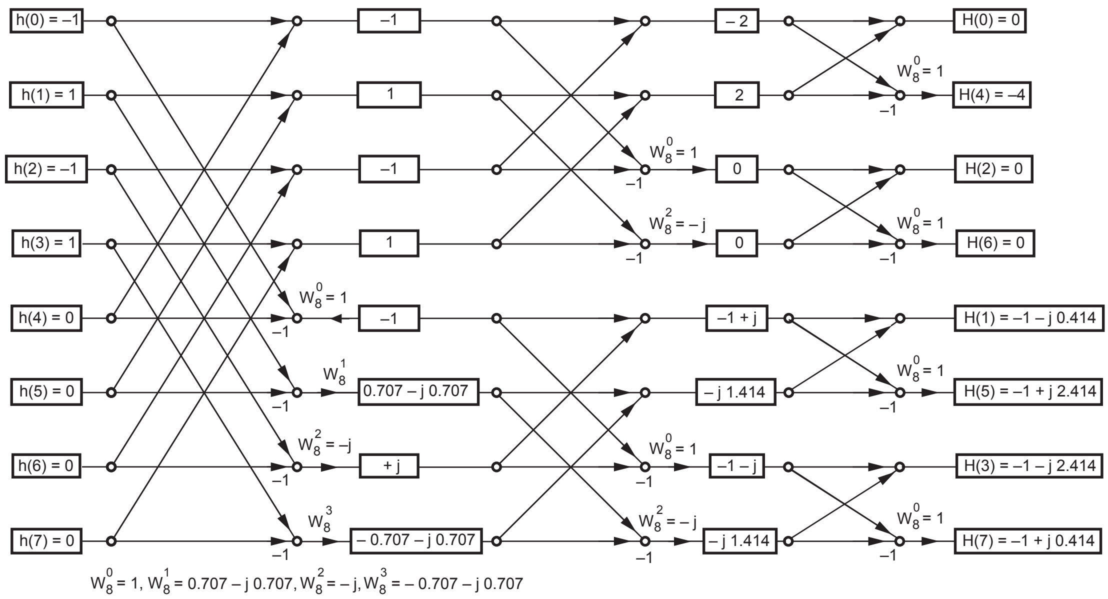


Fig. 3.7.3 DFT of h(n) using radix-2 DIF-FFT

Step 3 : Multiplication of DFTs

The DFTs *X(k)* and *H(k)* are multiplied to obtain *Y(k)*.

k	X(k)	H(k)	Y(k)
0	2	0	0
4	– 2	– 4	8
2	– 4	0	0
6	– 4	0	0
1	– j 3.414	– 1 – j 0.414	– 1.413 + j 3.414
5	– j 0.586	– 1 + j 2.414	1.414 + j 0.586
3	j 0.586	– 1 – j 2.414	1.414 – j 0.586
7	j 3.414	– 1 + j 0.414	– 1.413 – j 3.414

Step 4 : IDFT of $X(k) \cdot H(k)$

- As per properties of DFT, $x(n) \; N \; h(n) \xleftrightarrow[N]{DFT} X(k) \cdot H(k)$. Thus IDFT of $Y(k) = X(k) \cdot H(k)$ will be circular convolution of *x(n)* and *y(n)*. Thus,

$$y(n) = x(n) \; 8 \; h(n)$$

- Fig. 3.7.4 shows the radix-2 DIF-FFT to obtain IDFT of *Y(k)* i.e.,

$$y(n) = x(n) \; 8 \; h(n) = \{1, -2, 0, -1, 1, 0, 2, -1\}$$

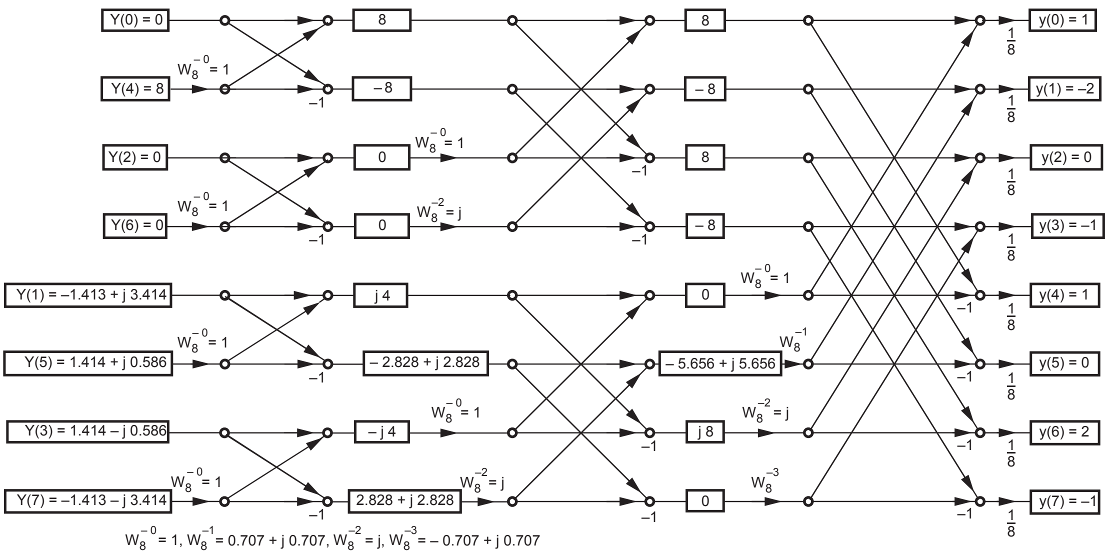


Fig. 3.7.4 Radix-2 DIF-FFT to obtain IDFT of Y(k)

- This sequence has length $N = 8$. It is basically the linear convolution of $x(n) = \{-1, 1, 2, 1, -1\}$ and $h(n) = \{-1, 1, -1, 1\}$, since 8-point circular convolution of *x(n)* and *h(n)* is obtained by appending 'zeroes' to both the sequences.
- The response of the LTI system is given by linear convolution of input sequence with impulse response. i.e.,

$$y(n) = x(n) * h(n) = \{1, -2, 0, -1, 1, 0, 2, -1\}$$

Example 3.7.2 *Use 4 - point inverse FFT for the DFT result {6, –2 + j2, –2, –2 – j2}*

AU : May-15, Marks 8

Solution : Fig. 3.7.5 shows the signal flow graph to calculate inverse DFT.

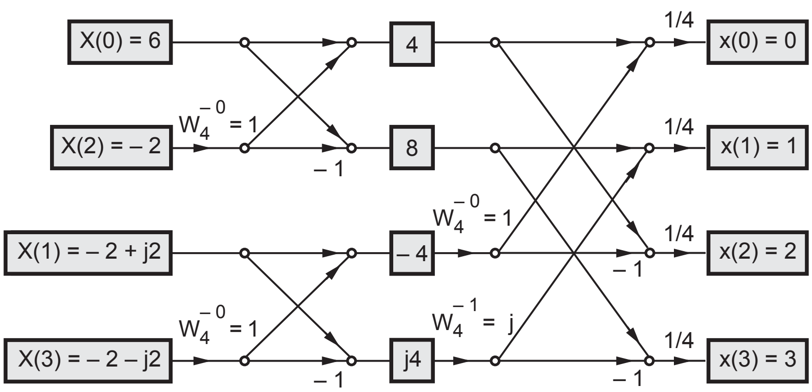


Fig. 3.7.5 : Radix-2 DIF-FFT to obtain IDFT

Thus the time domain signal is $x(n) = \{0, 1, 2, 3\}$.

Example 3.7.3 *Find the IDFT of the sequence*

X(K)={4, 1 – j 2.414, 0, 1 – j 0.414, 0, 1 + j 0.414, 0, 1 + j 2.414} Using DIF algorithm

AU : May-16, Marks 16

Solution : Following signal flow graph shows radix - 2 DIF - FFT signal flow graph to calculate IDFT along with stage – wise results. (See Fig. 3.7.6 on next page)

From above signal flow graph,

$$x(n) = \{ 1, 1, 1, 1, 0, 0, 0, 0 \}$$

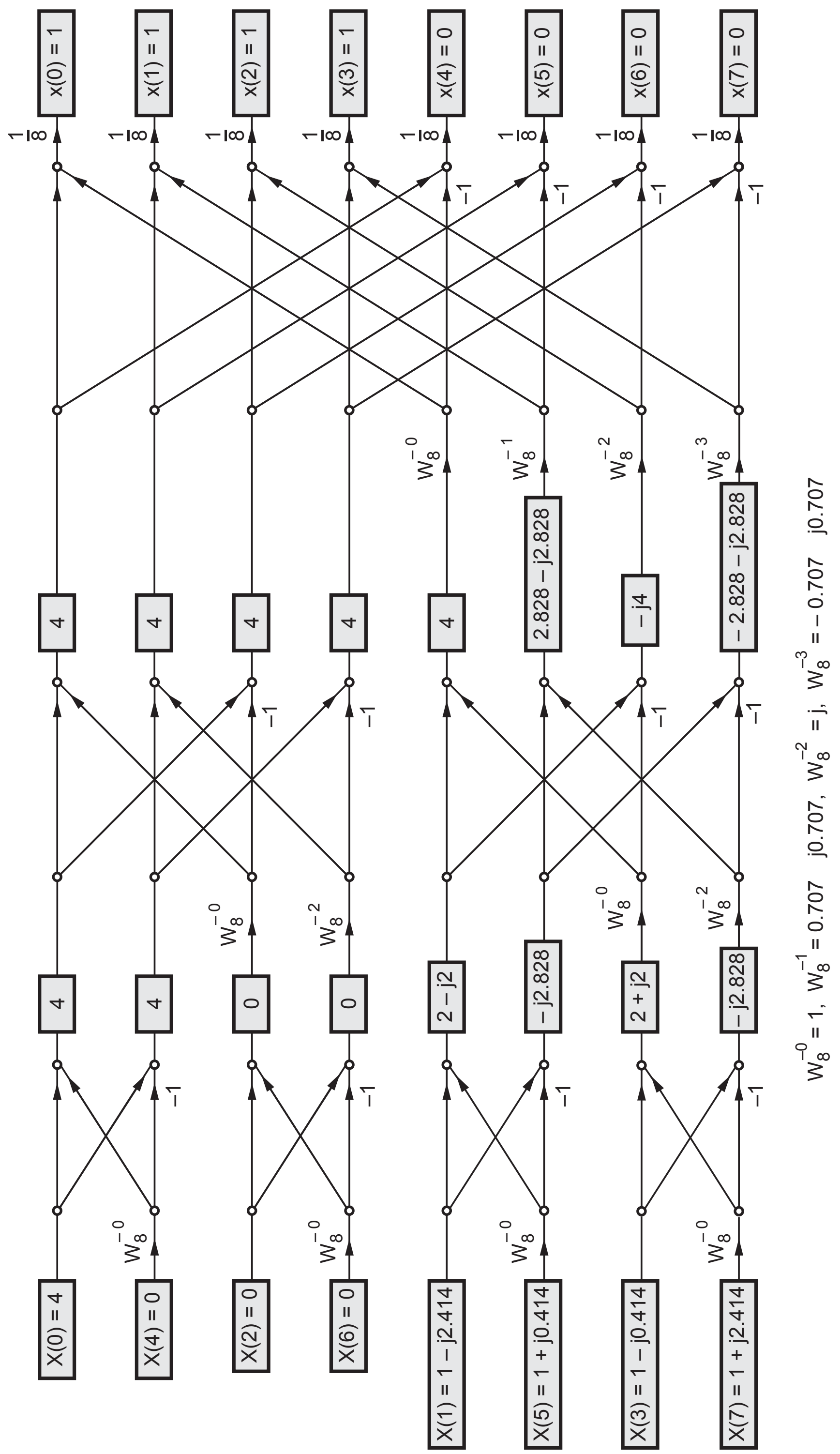


Fig. 3.7.6 IDFT using DIF-FFT algorithm

Examples for Practice

Example 3.7.4 *The DFT X(k) of sequence is given as*

$$X(k) = \left\{0, 2\sqrt{2}(1-j), 0, 0, 0, 0, 0, 2\sqrt{2}(1+j)\right\}$$
$$\uparrow$$

Determine the corresponding time sequence x(n) and write its signal flow graph. **[Ans. : x(n) = { 0.707, 1, 0.707, 0, – 0.707, – 1, –0.707, 0 }]**

Example 3.7.5 *If* $x_1(n) = \{1,2,0,1\}$, $x_2(n) = \{1,3,3,1\}$ *obtain* $x_1(n)$ N $x_2(n)$ *by using DIT-FFT algorithm.* **[Ans. : $x_3(n)$ = { 6, 8, 10, 8 }]**

Review Question

1. *Explain inverse radix algorithms.*

3.8 Short Answered Questions [2 Marks Each]

Q.1 What are the advantages of FFT over DFTs ? **AU : Dec.-12**

Ans. :

1. FFTs are the algorithms used to compute DFT fast.
2. FFT algorithms are computationally efficient than direct computation of DFT.
3. FFT algorithms exploit periodicity and symmetry properties of DFT.

Q.2 Give the relationship between z-domain and frequency domain.

Madras Univ. : April-01, Oct.-2000

What is the relationship between DTFT and z-transform ? **AU : May-12**

Ans. : Frequency domain representation is given by Fourier transform. If z-transform is evaluated on unit circle, it is same as Fourier transform. i.e.,

$$X(z)\big|_{z=e^{j\omega}} = X(\omega) \text{ at } |z| = 1 \text{ i.e. unit circle.}$$

Thus Fourier transform is a special case of z-transform. The angles from 0 to 2π on unit circle represent the frequencies from 0 to 2π.

Q.3 What is the difference between DFT and DTFT ? **Madras Univ. : April-01; Dec.-08**

Ans. : Discrete time Fourier transform is given as,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

This is the DTFT of discrete time signals. Here 'ω' takes on continuous values from 0 to 2π.

Discrete Fourier transform (DFT) is given as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}, \quad k = 0, 1, 2, \ldots N-1$$

Here $X(k)$ is DFT and it is basically sampled version of $X(\omega)$. It takes only discrete values of 'k'.

Q.4 What is the use of Fourier transform ? **Madras Univ. : April-01**

Ans. : Fourier transform converts time domain signal to frequency domain. Fourier transform is useful to study the frequency domain nature of periodic as well as nonperiodic signals. Most of the spectrum and frequency related aspects of the signals are studied with the help of Fourier transform.

Q.5 Draw the butterfly diagram for DIF-FFT algorithm. **Madras Univ. : Oct.-2000**

Ans. : Fig. 3.8.1 shows the butterfly operation of DIF-FFT algorithm. 'a' and 'b' are input values and A and B are output DFT values.

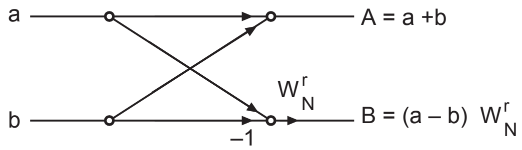


Fig. 3.8.1 Butterfly operation in DIF-FFT algorithm

Q.6 What do you understand by periodic convolution ? **Madras Univ. : Oct.-2000**

Ans. : Let x_1(n) and x_2(n) be two periodic sequences having Fourier coefficients of c_1(k) and c_2(k) respectively. Let these coefficients be mutliplied to give c_3(k)

i.e., $$c_3(k) = c_1(k) \cdot c_2(k)$$

If c_3(k) are the Fourier coefficients of the sequence x_3(n), then

$$x_3(\text{n}) = \sum_{m=0}^{N-1} x_1(m)\, x_2\,(n-m)$$

Thus x_3(n) is the convolution of x_1(n) and x_2(n). In other words, multiplication of the Fourier coefficients is equivalent to convolution of the corresponding sequences. Since the sequence x_3(n) is also periodic, the convolution is performed only over 'N' samples. Therefore it is called periodic convolution.

Q.7 : Distinguish between Fourier series and Fourier transform. **Madras Univ. : Oct.-2000**

Ans. : Fourier series expands the signal in terms of sinusoidal orthogonal basis functions. Any periodic signal can be expressed in terms of infinite number of sine and cosine terms. Fourier transform converts the signal from time domain to frequency domain. Fourier transform is mainly used for nonperiodic signals.

Q.8 : Distinguish between discrete Fourier series and discrete Fourier transform. **Madras Univ. : April-2000**

Ans. : The discrete Fourier series of a discrete time signal is given as,

$$x(n) = \sum_{k=0}^{N-1} c(k) e^{j2\pi kn/N}$$

Here c(k) are the coefficients of series expansion. They are given as,

$$c(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

And DFT is given as,

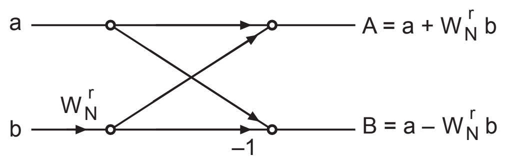
$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

Thus DFS coefficients are basically DFT, and they are scaled down by N.

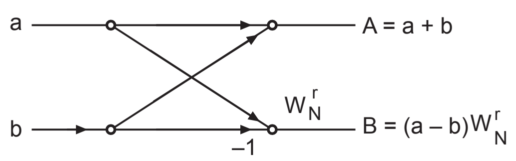
Q.9 Draw the basic structure of DIT and DIF-FFT flow chart of radix-2.

AU : Dec.-11, 13, May-06, 07, 08, 11, 15

Ans. : Following figures show the basic structures.



(a) Butterfly operation in DIT - FFT



(b) Butterfly operation in DIF - FFT

Fig. 3.8.2

Q.10 Compute DFT of the sequence, $x(n)$ = {1, 2} **AU : May-04**

Ans. : Here $x(0) = 1$ and $x(1) = 2$. The 2-point DFT is given as,

$$X(0) = x(0) + x(1)$$

and $$X(1) = x(0) - x(1)$$

$\therefore$ $$X(0) = 1 + 2 = 3$$

$$X(1) = 1 - 2 = -1$$

Thus, $$X(k) = \{3, -1\}$$

Q.11 Perform circular convolution of x(n) = {1, 2, 2, 1} and w(n) = {1, 2, 3, 4}.

AU : May-04

Ans. : The circular convolution is given as follows :

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} w(0) & w(3) & w(2) & w(1) \\ w(1) & w(0) & w(3) & w(2) \\ w(2) & w(1) & w(0) & w(3) \\ w(3) & w(2) & w(1) & w(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 15 \\ 13 \\ 15 \end{bmatrix}$$

Q.12 How is FFT faster ? **AU : May-05**

OR

How many multiplications and additions are required to compute N-point DFT using radix-2 FFT? **AU : Dec.-09**

Ans. : FFT is faster because it requires less number of complex multiplications and complex additions compared to direct computation of DFT.

Operation	FFT	DFT
Complex multiplications	$\frac{N}{2} \log_2 N$	N^2
Complex additions	$N \log_2 N$	$N^2 - N$

Q.13 Define DFT and Inverse DFT. **AU : Dec.-05**

OR

Define DFT pair. **AU : May-05**

Ans. : DFT : $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0,1,\ldots.N-1$

IDFT : $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\ e^{j2\pi kn/N}, \quad n = 0,1,\ldots.N-1$

Q.14 What is the difference between circular convolution and linear convolution ? **AU : Dec.-05**

Ans. : Following table shows the difference between circular and linear convolution :

Sr. No.	Parameter	Linear convolution	Circular convolution
1.	Shifting of sequences.	Sequences are shifted linearly.	Sequences are shifted circularly.
2.	Convolution sum.	Convolution sum is of infinite length.	Convolution sum is of length 'N'.
3.	Types of sequences.	Sequences are non periodic and must be of finite length.	Sequences are of length 'N' and they are periodic.

Table 3.8.1 Difference between circular and linear convolution

Q.15 Calculate DFT of $x(n) = \{1, 1, -2, -2\}$. **AU : Dec.-10**

Ans. :

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3-j3 \\ 0 \\ 3+j3 \end{bmatrix}$$

Q.16 Differentiate between DIF and DIT. **AU : Dec.-10, May-14**

Ans. :

Sr. No.	DIT	DIF
1.	Time domain sequence is decimated.	Frequency domain sequence or DFT is decimated.
2.	The N-point DFT is split in two $\frac{N}{2}$ - point DFTs.	The N-point sequence is split in two $\frac{N}{2}$ - point sequences.

Q.17 What is the relation between DFT and z-transform ? **AU : May-11**

Ans. : If z-transform is evaluated on unit circle at evenly spaced points, then it becomes DFT.

$$X(k) = X(z)\big|_{z_k} = e^{j2\pi k/N}$$

Q.18 Find the 4-point DFT of the sequence x(n) = {1,1}. **AU : Dec.-12, May-15**

Ans. : $x(n) = \{1, 1, 0, 0\}$

$$X_4 = W_4 \; x_4$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

Q.19 In 8-point DIT-FFT, what is the gain of the signal path that goes from x(7) to x(2) ? **AU : Dec.-13**

Ans. : From the signal flow diagram (Fig. 3.6.7) of 8-point DIT-FFT, the path from x(7) to x(2) have gains as follows :

$$\text{Gain from x(7) to x(2)} = \left(W_8^0\right)\left(W_8^0\right)(-1)\left(W_8^2\right)$$

$$= 1 \times 1 \times (-1) \times (-j) = j$$

Q.20 State the circular frequency shift property of DFT. **AU : May-14, Dec.-15**

Ans. : If $x(n) \xleftrightarrow[N]{DFT} X(k)$ then,

$$x(n)e^{j2\pi ln/N} \xleftrightarrow[N]{DFT} X((k-l))_N .$$

Q.21 Calculate the percentage saving in calculation in a 256 point radix–2 FFT when compared to direct FFT. **AU : Dec.-15**

Ans. : From Table 3.6.1, we have,

Direct computation : Multiplications = 65536 and additions = 65280

256 - point FFT : Multiplications = 1024 and additions = 2048

∴ Percentage saving in multiplications

$$= \frac{65536-1024}{65536} \times 100 = 98.43 \ \%$$

∴ Percentage saving in additions

$$= \frac{65280-2048}{65280} \times 100 = 96.86 \ \%$$

Q.22 Draw the flow graph of a point radix–2 DIT–FFT butterfly structure for DFT. **AU : May-16**

Ans. : Following Fig. 3.8.3 shows the signal flow graph

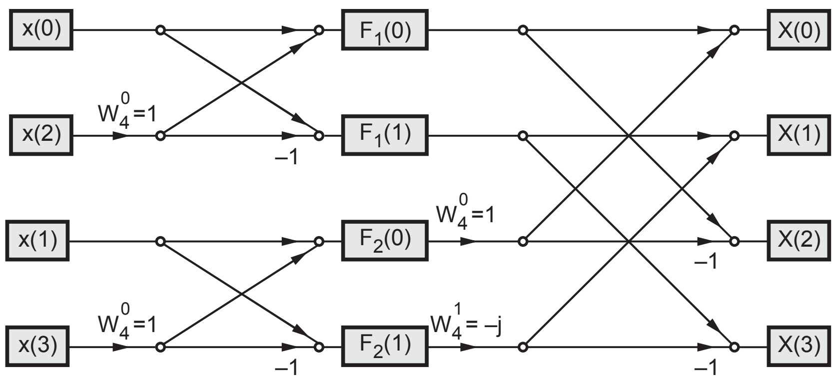


Fig. 3.8.3 4-point radix-2 DIT-FFT butterfly structure

Q.23 What are the applications of FFT algorithm ? **AU : May-16**

Ans. :

i) Linear filtering of long sequences

ii) Frequency spectrum analysis

iii) Power spectral analysis

iv) Correlation analysis

Q.24 Define twiddle factor. Write its magnitude and phase angle. AU : May-17

Ans. : $W_N = e^{-j\frac{2\pi}{N}}$ is called twiddle factor.

It's magnitude is '1' and phase $-\frac{2\pi}{N}$

Q.25 Compute the number of multiplications and additions for 32 point DFT and FFT AU : May-17

Ans. : Following table illustrates additions and multiplications.

N	DFT		FFT	
	Multiplications N^2	Additions $N^2 - N$	Multiplications $\frac{N}{2}\log_2 N$	Additions $N \log_2 N$
32	$32^2 = 1024$	$32^2 - 32 = 992$	$\frac{32}{2}\log_2 32 = 80$	$32\log_2 32 = 160$

Q.26 Why is it required to do zero padding in DFT analysis. AU : Dec.-16

Ans. : Zero padding is required for two purposes :

i) To increase the frequency resolution of DFT, zeros are appended at the end of $x(n)$ to increase the value of N.

ii) Some times zeros are appended in $x(n)$ so as to make DFT length some power of 2. This means $\log_2 N$ must be integer.

❑❑❑

Notes

4 Design of Digital Filters

Syllabus

FIR and IIR filter realization - Parallel and cascade forms. FIR design : Windowing techniques - Need and choice of windows - Linear phase characteristics. Analog filter design - Butterworth and Chebyshev approximations; IIR filters, Digital design using impulse invariant and bilinear transformation - Warping, Prewarping.

Contents

4.1 FIR and IIR Filter Realization

*4.2 Basic FIR Filter Structures . . . . **May-05, 10, 15** . . . . Marks 16*

*4.3 Basic IIR Filter Structures **May-04, 05, 06, 11, 16***

*. . . . . **Dec.-06, 12, 13, 16,** . . . . . . Marks 12*

4.4 Properties of FIR Digital Filters

*4.5 Different Types of Windows . . . . **May-10, 11, Dec.-16,** . . . . . . Marks 8*

4.6 Design of Linear Phase FIR Filters using Windows

*. **May-07, 10, 11, 12, 14, 15, 16, 17***

*. **Dec.-05, 08, 10, 11, 12, 13, 15***

*. . . . . **16,** . . . . Marks 16*

*4.7 Analog Filter Design . . . . **May-04, 16, 17, Dec.-16,** . . . Marks 8*

*4.8 Design of IIR Filters from Analog Filters **May-04, 05, 06, 07, 10, 11, 12, 16***

*. . . . . **Dec.-05, 06, 12, 13, 16,** . . . Marks 16*

4.9 Design of IIR Filters using Butterworth and Chebyshev Approximations

*. **Dec.-11,15,***

*. . . . . **May-14, 15, 17** . . . . Marks 16*

4.10 Comparison of FIR and IIR Filters

4.11 Short Answered Questions [2 Marks Each]

4.1 FIR and IIR Filter Realization

- The filter realization structures can be of two types. If the number of delays in the structure is equal to order of the difference equation or order of the transfer function, then it is called **Canonic** form realization.
- If the number of delays in the structure are not same as order, then it is called **non-canonic** realization.

4.1.1 Describing Equation

We know that the LTI systems are described by the general difference equation as,

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k) \qquad \text{... (4.1.1)}$$

Taking z-transform of above equation,

$$Y(z) = -\sum_{k=1}^{N} a_k\, z^{-k}\, Y(z) + \sum_{k=0}^{M} b_k\, z^{-k}\, X(z)$$

$$\therefore \qquad Y(z)\left[1+\sum_{k=1}^{N} a_k\, z^{-k}\right] = \sum_{k=0}^{M} b_k\, z^{-k}\, X(z)$$

We know that system function $H(z) = \dfrac{Y(z)}{X(z)}$, hence from above equation we get,

$$H(z) = \frac{\sum_{k=0}^{M} b_k\, z^{-k}}{1+\sum_{k=1}^{N} a_k\, z^{-k}} \qquad \text{... (4.1.2)}$$

This is the rational form of system function which is expressed as the ratio of two polynomials in z^{-1}.

4.1.2 Elementary Blocks for Filter Realization

A discrete time system is realized by implementing equation 4.1.1 or equation 4.1.2. These equations can be implemented in number of ways. We know that discrete time system is implemented with the help of delay elements, multipliers and adders as elementary blocks. A brief summary of elements is given in Table 4.1.1.

Sr. No.	Name of the block	Symbol
1.	Adder	
2.	Constant multiplier	
3.	Signal multiplier	
4.	Delay elements	
5.	Time advance elements	

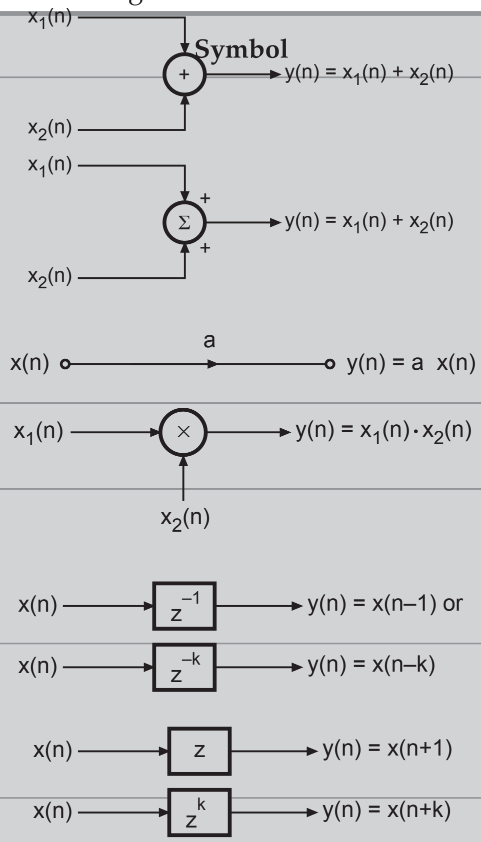


Table 4.1.1 Summary of elementary blocks used to represent discrete time systems

Equation 4.1.1 and equation 4.1.2 are realized through various blocks as shown in Table 4.1.1. Realization of the system means its actual implementation.

4.2 Basic FIR Filter Structures

AU : May-05, 10, 15

An FIR system does not have feedback. Hence the past outputs term $y(n-k)$ will be absent in equation 4.1.1. Hence output of FIR system is given as,

$$y(n) = \sum_{k=0}^{M} b_k \, x(n-k)$$

If there are 'M' coefficients, then above equation becomes,

$$y(n) = \sum_{k=0}^{M-1} b_k \, x(n-k) \qquad \text{... (4.2.1)}$$

Note that we are considering equation for 'M' coefficients for simplicity. It does not change the meaning of the equation. Taking z-transform of above equation we get,

$$Y(z) = \sum_{k=0}^{M-1} b_k \, z^{-k} \, X(z)$$

Hence system function $H(z) = \frac{Y(z)}{X(z)}$ becomes,

$$H(z) = \sum_{k=0}^{M-1} b_k \, z^{-k} \qquad \text{... (4.2.2)}$$

This is the system function of FIR system. Taking inverse z-transform of above equation we get unit sample response of FIR system. i.e.,

$$h(n) = \begin{cases} b_n & \text{for} \quad 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{... (4.2.3)}$$

Now let us see the realization structures for FIR systems using above equations.

4.2.1 Direct Form Structure of FIR System

This realization structure is obtained by implementing equation 4.2.1 directly. Since $h(n) = b_n$ from equation 4.2.3 we can write equation 4.2.1 as,

$$y(n) = \sum_{k=0}^{M-1} h(k) \, x(n-k) \qquad \text{... (4.2.4)}$$

The above equation is nothing but convolution of $h(n)$ and $x(n)$. Let us expand the summation in the above equation as,

$$y(n) = h(0)\, x(n) + h(1)\, x(n-1) + h(2)\, x(n-2) + h(3)\, x(n-3) + \ldots\ldots + h(M-1)\, x(n-M+1)$$

... (4.2.5)

Now let us consider the implementation of individual terms of above equation using symbols given in Table 4.1.1.

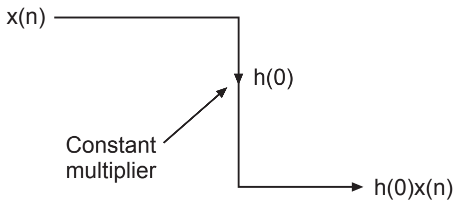


Fig. 4.2.1 Implementation of *h* (0) *x* (*n*)

Implementation of *h* (0) *x* (*n*) :

We have $x(n)$ as input and $h(0)$ is the constant multiplier. Hence this term can be implemented as follows :

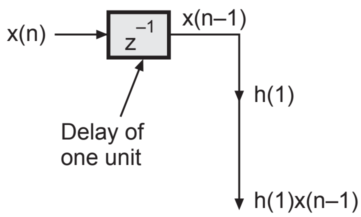


Fig. 4.2.2 Implementation of *h* (1) *x* (*n* – 1). It uses one unit delay block

The figure is drawn purposely to suit further requirements.

Implementation of *h* (1) *x* (*n* – 1) :

Here $x(n)$ is delayed by one sample. The delayed value $x(n-1)$ is multiplied by the constant multiplier. The implementation of this term is as follows :

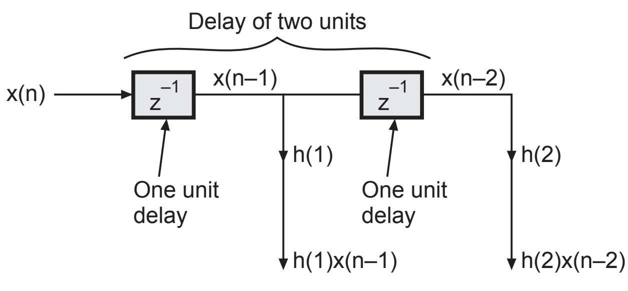


Fig. 4.2.3 Implementation of *h* (1) *x* (*n* – 1) and *h* (2) *x* (*n* – 2) simultaneously

Implementation of *h* (2) *x* (*n* – 2) :

Here $x(n)$ is delayed by two samples. And the delayed value $x(n-2)$ is multiplied by constant multiplier. We know that $x(n)$ is delayed by one unit and we get $x(n-1)$ in Fig. 4.2.2. If we delay $x(n-1)$ by one unit we will get $x(n-2)$. This is shown below in Fig. 4.2.3.

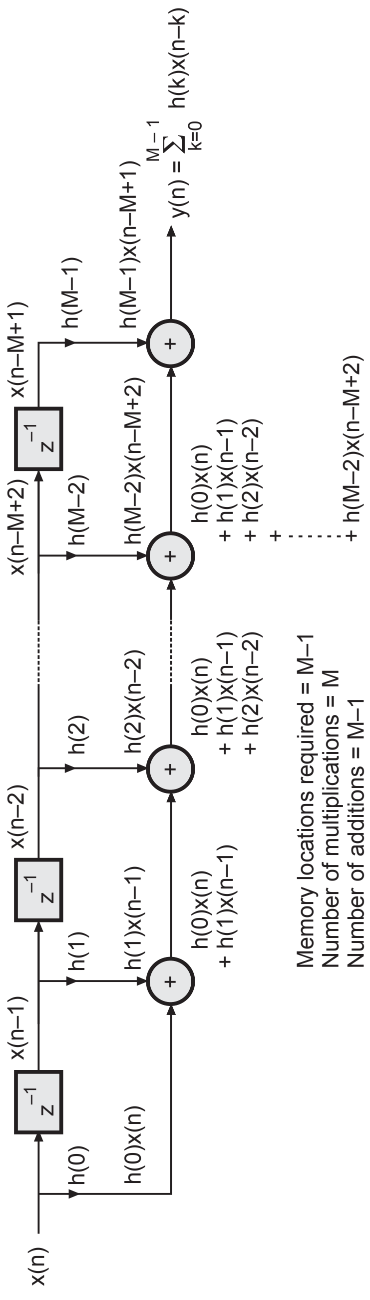


Fig. 4.2.4 Direct form realization of FIR system

On the same lines we can implement other terms of equation 4.2.5.

Implementation of equation 4.2.5 :

To get the overall implementation of equation 4.2.5, we have to add the individual outputs in Fig. 4.2.1, 4.2.2, 4.2.3 and so on. Such system is shown in Fig. 4.2.4.

In the Fig. 4.2.4 observe that there are $M-1$ unit delay blocks. Normally one unit delay block requires one memory location. Hence the above structure requires '$M-1$' memory locations. The multiplications of $h(k)$ and $x(n-k)$ is performed for 0 to $M-1$ terms. Hence there are 'M' multiplications. To add 'M' terms, '$M-1$' additions are required.

The direct form structure presented above looks like tapped delay line or a transversal system. Hence direct form realization is also called transversal or tapped delay line filter.

4.2.2 Linear Phase FIR Structures

The FIR filter has linear phase if its unit sample response satisfies the following condition :

$$h(n) = h(M-1-n) \qquad \text{... (4.2.6)}$$

The above condition states that the unit sample response is symmetric about its origin. Here the unit sample response is $h(0), h(1), \ldots, h(M-1)$ having length of 'M' samples. The above condition can be used in the realization of FIR filters. The z-transform of the unit sample response is given as,

$$H(z) = \sum_{n=0}^{M-1} h(n)\; z^{-n} \qquad \text{... (4.2.7)}$$

i) For even '*M*' :

For the linear phase FIR filters $h(n) = h(M-1-n)$. Hence above equation can be written as,

$$H(z) = \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[z^{-n} + z^{-(M-1-n)}\right] \qquad \text{... (4.2.8)}$$

We know that $H(z) = \frac{Y(z)}{X(z)}$. Hence above equation can be written as,

$$\frac{Y(z)}{X(z)} = \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[z^{-n} + z^{-(M-1-n)}\right]$$

$$\therefore \qquad Y(z) = \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[z^{-n} + z^{-(M-1-n)}\right] X(z)$$

Expanding the summation of above equation,

$$Y(z) = h(0)\left[1+z^{-(M-1)}\right] X(z)+ h\ (1)\ \left[z^{-1}+z^{-(M-2)}\right] X(z)\ + \ldots\ldots$$

$$\ldots\ldots+h\ \left(\frac{M}{2}-1\right)\left[z^{-\left(\frac{M}{2}-1\right)}+z^{-\frac{M}{2}}\right]\ X\ (z) \qquad \text{... (4.2.9)}$$

The inverse z-transform of above equation becomes,

$$y\ (n) = h(0)\ \{x\,(n) + x\,[n-(M-1)]\} + h\,(1)\ \{x\,(n-1) + x\,[n-(M-2)]\} + \ldots\ldots.$$

$$\ldots\ldots.. + h\ \left(\frac{M}{2}-1\right)\left\{x\left[n-\left(\frac{M}{2}-1\right)\right]+ x\ \left(n-\frac{M}{2}\right)\right\} \qquad \text{... (4.2.10)}$$

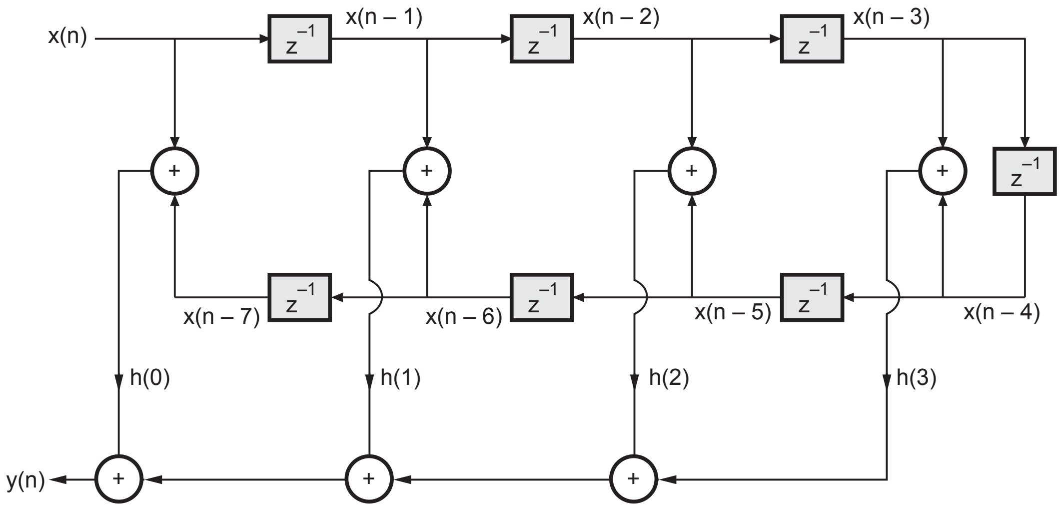


Fig. 4.2.5 Linear phase FIR structure for *M* = 8 (Even)

Let us write the above equation for $M = 8$. i.e.,

$$y\,(n) = h\,(0)\ \{x\,(n) + x\,(n-7)\} + h(1)\ \{x(n-1)+x\ (n-6)\}$$

$$+\ h\ (2)\ \{x(n-2)+x(n-5)\} + h(3)\ \{x(n-3)+x(n-4)\} \qquad \text{... (4.2.11)}$$

Fig. 4.2.5 shows the direct form FIR realization of above equation.

In the above figure observe that there are only four multiplications for $M = 8$. Thus the multiplications are reduced to half in linear phase FIR structure. This is the main advantage of these structures over other structures. Fig. 4.2.6 shows the implementation of generalized equation (i.e. equation 4.2.10) of linear phase FIR structures. (See Fig. 4.2.6 on next page)

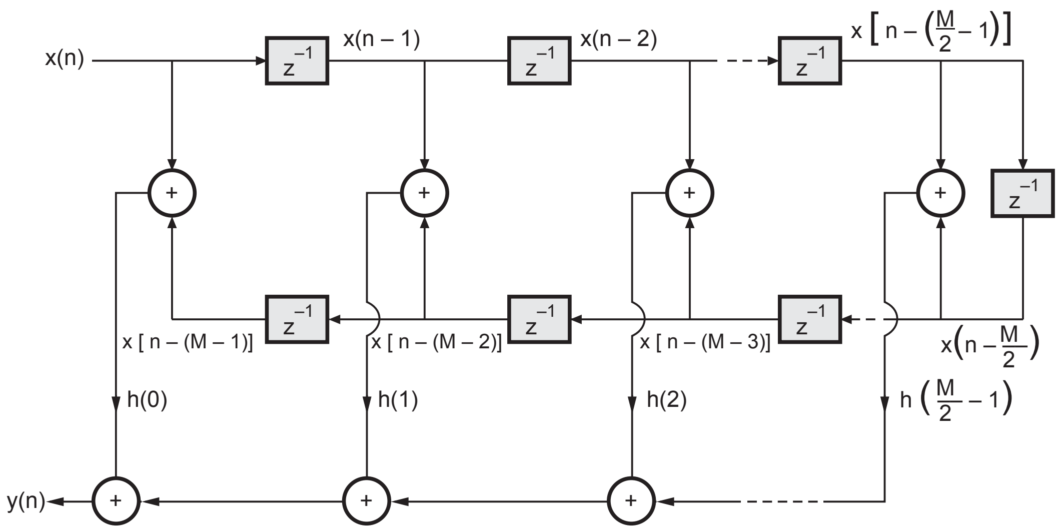


Fig. 4.2.6 Linear phase FIR structure for even '*M*'

ii) For odd '*M*' :

Now let us see the case when 'M' is odd. For linear phase FIR filter we know that $h(n) = h(M-1-n)$. For odd value of 'M' we can write equation 4.2.7 as follows :

$$H(z) = h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left[z^{-n} + z^{-(M-1-n)}\right] \qquad \text{... (4.2.12)}$$

In the above figure observe that ; the $\left(\frac{M-1}{2}\right)^{th}$ term separately written, since 'M' is odd. We know that $H(z) = \frac{Y(z)}{X(z)}$. Hence above equation becomes,

$$\frac{Y(z)}{X(z)} = h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left[z^{-n} + z^{-(M-1-n)}\right]$$

$$\therefore \qquad Y(z) = h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} X(z) + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left[z^{-n} + z^{-(M-1-n)}\right] X(z)$$

Let us expand the summation of the above equation,

$$Y(z) = h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} X(z) + h(0)\left[1+z^{-(M-1)}\right] X(z) + h(1)\left[z^{-1}+z^{-(M-2)}\right] X(z)$$

$$+ \ldots\ldots + h\left(\frac{M-3}{2}\right)\left[z^{-\left(\frac{M-3}{2}\right)} + z^{-\left(\frac{M+1}{2}\right)}\right] X(z) \quad \ldots (4.2.13)$$

The inverse z-transform of above equation becomes,

$$y(n) = h\left(\frac{M-1}{2}\right) x\left[n-\left(\frac{M-1}{2}\right)\right] + h(0)\left\{x(n)+x[n-(M-1)]\right\}$$

$$+h(1)\left\{x(n-1)+x[n-(M-2)]\right\} + \ldots + h\left(\frac{M-3}{2}\right)\left\{x\left[n-\left(\frac{M-3}{2}\right)\right]+x\left[n-\left(\frac{M+1}{2}\right)\right]\right\}$$

$$\ldots (4.2.14)$$

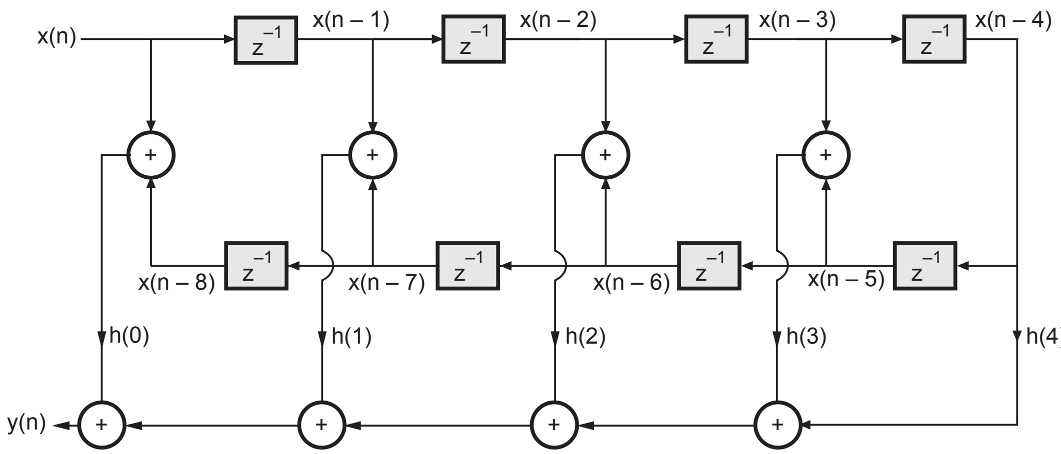


Fig. 4.2.7 Linear phase FIR structure for $M = 9$ (odd)

Let us write the above equation for $M = 9$, i.e.,

$$y(n) = h(4)\,x(n-4) + h(0)\left[x(n)+x(n-8)\right] + h(1)\left[x(n-1)+x(n-7)\right]$$

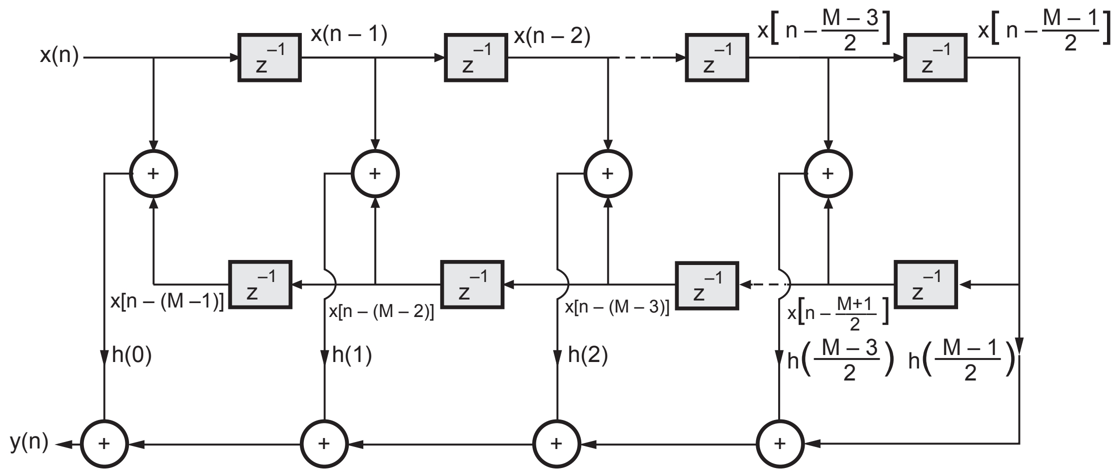


Fig. 4.2.8 Linear phase FIR structure for odd '*M*'

$$h(2)\left[x(n-2)+x(n-6)\right] + h(3)\left[x(n-3)+x(n-5)\right]$$

Let us rearrange above equation as,

$$y(n) = h(0)\left[x(n)+x(n-8)\right] + h(1)\left[x(n-1)+x(n-7)\right]$$
$$+ h(2)\left[x(n-2)+x(n-6)\right] + h(3)\left[x(n-3)+x(n-5)\right] + h(4)\, x(n-4)$$

Fig. 4.2.7 shows the implementation of above equation.

The structures of higher orders can be derived similarly. Fig. 4.2.8 shows the generalized implementation of equation 4.2.14. (See Fig. 4.2.8 on next page)

Example 4.2.1 *Realize a linear phase FIR filter with the following impulse response. Give necessary equations :*

$$h(n) = \delta(n)+\frac{1}{2}\delta(n-1)-\frac{1}{4}\delta(n-2)+\delta(n-4)+\frac{1}{2}\delta(n-3)$$

Solution : Method - I :

Let us rewrite the given impulse response as,

$$h(n) = \delta(n)+\frac{1}{2}\delta(n-1)-\frac{1}{4}\delta(n-2)+\frac{1}{2}\delta(n-3)+\delta(n-4) \quad \text{... (4.2.15)}$$

Taking z-transform of above equation,

$$H(z) = 1+\frac{1}{2}z^{-1}-\frac{1}{4}z^{-2}+\frac{1}{2}z^{-3}+z^{-4}$$

We know that $Y(z) = H(z)\,X(z)$. Hence above equation can be written as,

$$Y(z) = \left(1+\frac{1}{2}z^{-1}-\frac{1}{4}z^{-2}+\frac{1}{2}z^{-3}+z^{-4}\right)X(z)$$

$$\therefore \quad Y(z) = X(z) + \frac{1}{2} z^{-1} X(z) - \frac{1}{4} z^{-2} X(z) + \frac{1}{2} z^{-3} X(z) + z^{-4} X(z)$$

Taking inverse z-transform of above equation,

$$y(n) = x(n) + \frac{1}{2} x(n-1) - \frac{1}{4} x(n-2) + \frac{1}{2} x(n-3) + x(n-4)$$

Let us rearrange the above equation as follows :

$$y(n) = [x(n) + x(n-4)] + \frac{1}{2}[x(n-1) + x(n-3)] - \frac{1}{4} x(n-2) \quad \text{... (4.2.16)}$$

The above equation is similar to equation 4.2.14 and it can be realized by the structure shown in Fig. 4.2.8.

Method - II :

The given sequence $h(n)$ is the impulse weighted sequence as can be seen from equation 4.2.15 and hence it can also be written as follows :

$$h(n) = \left\{ \underset{\uparrow}{1}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{2}, 1 \right\}$$

Here, $h(0) = 1$

$h(1) = \frac{1}{2}$

$h(2) = -\frac{1}{4}$

$h(3) = \frac{1}{2}$

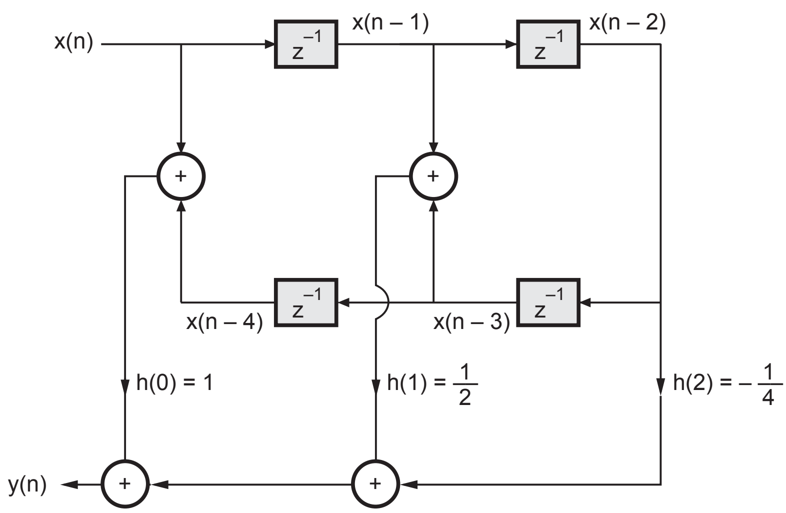


Fig. 4.2.9 Linear phase FIR structure of example 4.2.1

$$h(4) = 1$$

Here $M = 5$, and above impulse response is symmetric. i.e. above $h(n)$ satisfies the following condition :

$$h(n) = h(M-1-n)$$

i.e. $h(0) = h(4)$, $h(1) = h(3)$ etc. Since 'M' is odd, we have to use the linear phase structure of Fig. 4.2.8. It is shown in Fig. 4.2.9.

Observe that the above structure truely represents equation 4.2.16.

Example 4.2.2 *Realize the following system function by linear phase FIR structure.* $H(z) = \frac{2}{3}z + 1 + \frac{2}{3}z^{-1}$.

AU : May-15, Marks 2

Solution : The given system function can be written as,

$$H(z) = 1 + \frac{2}{3}\left(z + z^{-1}\right)$$

Since $Y(z) = H(z) \cdot X(z)$, we can write,

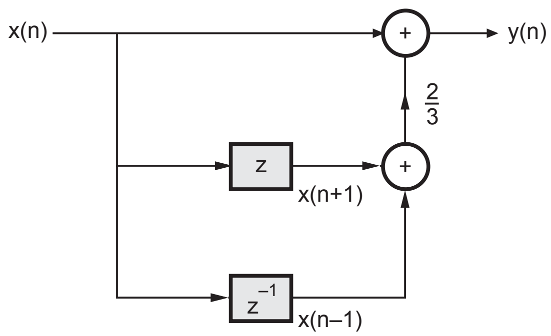


Fig. 4.2.10 Linear phase structure of example 4.2.2

$$Y(z) = \left[1 + \frac{2}{3}\left(z + z^{-1}\right)\right] X(z)$$

$$= X(z) + \frac{2}{3}\left(z + z^{-1}\right) X(z)$$

Taking inverse z-transform of above equation,

$$y(n) = x(n) + \frac{2}{3}\left[x(n+1) + x(n-1)\right]$$

Fig. 4.2.10 shows the linear phase FIR structure based on above equation.

4.2.3 Cascade Form Structure for FIR System

The system function of the FIR system is given by equation 4.2.2 as,

$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k} \quad \text{... (4.2.17)}$$

This is a polynomial in z^{-1}. This polynomial can be arranged as multiplication of multiple second order polynomials. i.e.,

i.e. $$H(z) = H_1(z) \times H_2(z) \times H_3(z) \times \ldots \times H_K(z) \quad \text{... (4.2.18)}$$

where,

$$H_K(z) = b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}, k = 0, 1, \ldots . K \quad \text{... (4.2.19)}$$

Thus $H_K(z)$ are second order systems. Normally complex conjugate roots of $H(z)$ are combined together in each second order section. Hence the coefficients $\{b_{ki}\}$ are real valued.

We know that $H_K(z) = \dfrac{Y_K(z)}{X_K(z)}$, then equation 4.2.19 can be written as,

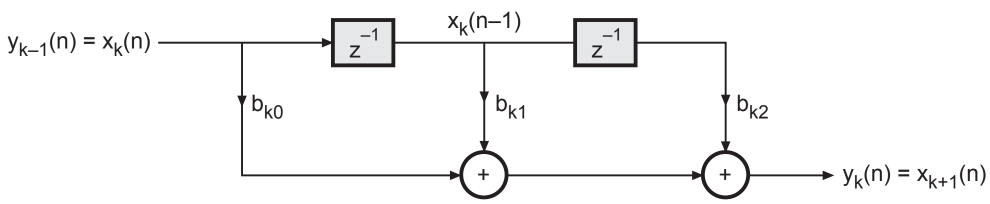


Fig. 4.2.11 Direct form realization of second order section of equation 4.2.20

$$\frac{Y_K(z)}{X_K(z)} = b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}$$

$$Y_k(z) = \left[b_{k0} + b_{k1} z^{-1} + b_k z^{-2}\right] X(z)$$

$$= b_{k0} X(z) + b_{k1} z^{-1} X(z) + b_{k2} z^{-2} X(z)$$

Taking inverse z-transform of above equation,

$$y_k(n) = b_{k0} x_k(n) + b_{k1} x_k(n-1) + b_{k2} x_k(n-2) \quad \text{... (4.2.20)}$$

Fig. 4.2.11 below shows direct form implementation of this equation.

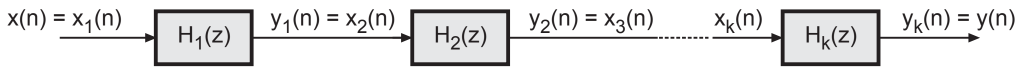


Fig. 4.2.12 Cascade form realization of FIR system

The above realization can be directly obtained from equation 4.2.19 also. There is no need to derive equation 4.2.20. Observe the second order system function of equation 4.2.19 carefully. The coefficient b_{k0} will be multiplied to the input without any delay. The coefficient b_{k1} has multiplier of z^{-1}. Hence it will be multiplied to one unit delayed input. Similarly the coefficient b_{k2} has multiplier of z^{-2}. Hence it will be multiplied to the input which is delayed by two units. Clearly we get the same realization in Fig. 4.2.11 above.

We know that when system functions are multiplied, it is cascading of actual systems. The system function $H(z)$ is multiplication of second order system function as given by equation 4.2.18. Hence the realization of the discrete time system can be obtained by cascading these second order sections as shown in Fig. 4.2.12.

Each $H_1(z), H_2(z), ...$ etc in above figure is a second order section, and it is realized by the direct form as shown in Fig. 4.2.11. The output of one section becomes input of next section.

Example 4.2.3 *Realize the following system function in cascade form.*

$$H(z) = 1+\frac{3}{4}z^{-1}+\frac{17}{8}z^{-2}+\frac{3}{4}z^{-3}+z^{-4} \qquad ... (4.2.21)$$

Solution : The given system function is an all zero function of order 4. Hence it represents FIR filter. We know that cascade realization can be obtained by connecting the second order sections in cascade. i.e.,

$$H(z) = H_1(z) \cdot H_2(z) \cdot H_3(z) \; ..... \; H_k(z)$$

Each section is of order 2, hence there will be two subsections in cascade to get order 4. i.e.,

$$H(z) = H_1(z) \cdot H_2(z)$$

From equation 4.2.19, putting for $H_k(z)$ for $k=1$ and 2 in above equation,

$$H(z) = \left(b_{10}+b_{11}z^{-1}+b_{12}z^{-2}\right)\left(b_{20}+b_{21}z^{-1}+b_{22}z^{-2}\right) \qquad ... (4.2.22)$$

$$= b_{10}b_{20}+\left(b_{10}b_{21}+b_{11}b_{20}\right)z^{-1}+\left(b_{10}b_{22}+b_{11}b_{21}+b_{12}b_{20}\right)z^{-2}$$

$$+\left(b_{11}b_{22}+b_{12}b_{21}\right)z^{-3}+b_{12}b_{22}z^{-4}$$

Comparing above equation with equation 4.2.21 we get,

$$b_{10}\, b_{20} = 1$$

$$b_{10}\, b_{21} + b_{11}\, b_{20} = \frac{3}{4}$$

$$b_{10}\, b_{22} + b_{11}\, b_{21} + b_{12}\, b_{20} = \frac{17}{8}$$

$$b_{11}\, b_{22} + b_{12}\, b_{21} = \frac{3}{4}$$

$$b_{12}\, b_{22} = 1$$

On solving the above equations we get,

$$b_{10} = 1, \quad b_{11} = \frac{1}{2}, \quad b_{12} = 1$$

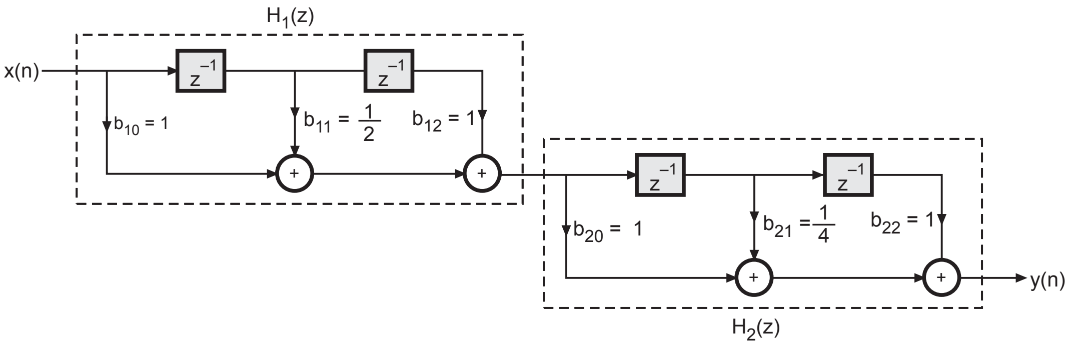


Fig. 4.2.13 Cascade form realization of FIR filter given in example 4.2.3

and $$b_{20} = 1, \quad b_{21} = \frac{1}{4}, \quad b_{22} = 1$$

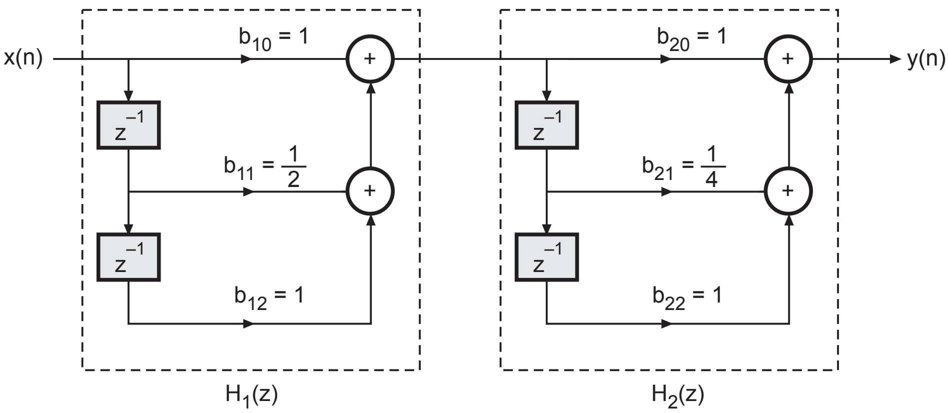


Fig. 4.2.14 Cascade form structure of Fig. 4.2.13 redrawn alternately

Hence equation 4.2.22 can be written as,

$$H(z) = \left(1+\frac{1}{2}z^{-1}+z^{-2}\right)\left(1+\frac{1}{4}z^{-1}+z^{-2}\right)$$

Here, $H_1(z) = 1+\frac{1}{2}z^{-1}+z^{-2}$

and $H_2(z) = 1+\frac{1}{4}z^{-1}+z^{-2}$

The cascade realization of $H(z) = H_1(z) \cdot H_2(z)$ is shown in Fig. 4.2.13. It is prepared with the help of subsection shown in Fig. 4.2.13.

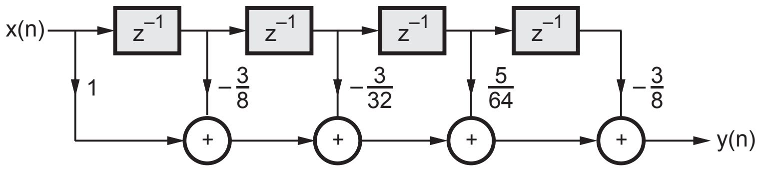


Fig. 4.2.15 Direct form realization

The cascade realization shown in Fig. 4.2.13 above can be redrawn as shown in Fig. 4.2.14.

Example 4.2.4 *Obtain direct form and cascade form realization for the transfer function of an FIR system given by* $H(z) = \left(1-1/4\, z^{-1}+3/8\, z^{-2}\right)\left(1-1/8\, z^{-1}-1/2\, z^{-2}\right)$

AU : May-10, Marks 16

Solution : Direct form realization

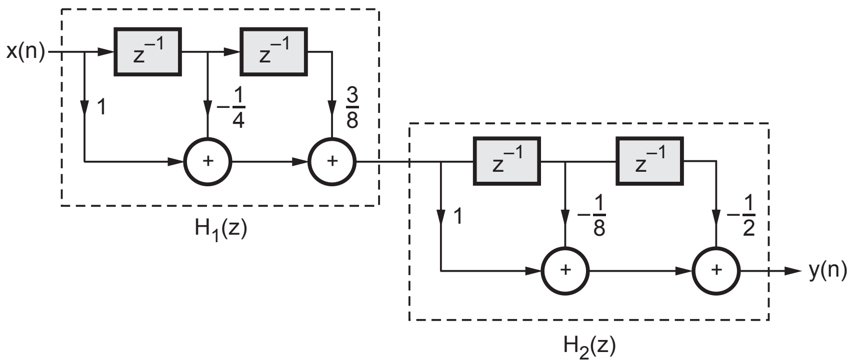


Fig. 4.2.16 Cascade form realization

$$H(z) = \left(1-\frac{1}{4}z^{-1}+\frac{3}{8}z^{-2}\right)\left(1-\frac{1}{8}z^{-1}-\frac{1}{2}z^{-2}\right)$$

$$= 1-\frac{3}{8}z^{-1}-\frac{3}{32}z^{-2}+\frac{5}{64}z^{-3}-\frac{3}{8}z^{-4}$$

Fig. 4.2.15 shows the direct form realization,

Cascade form realization

$$H(z) = \left(1-\frac{1}{4}z^{-1}+\frac{3}{8}z^{-2}\right)\left(1-\frac{1}{8}z^{-1}-\frac{1}{2}z^{-2}\right)$$

$$= H_1(z)\cdot H_2(z)$$

Here $H_1(z) = 1-\frac{1}{4}z^{-1}+\frac{3}{8}z^{-2}$ and $H_2(z) = 1-\frac{1}{8}z^{-1}-\frac{1}{2}z^{-2}$

Fig. 4.2.16 shows cascade of $H_1(z)$ and $H_2(z)$.

Review Question

1. *Draw the structure of FIR system.* **AU : May-05, Marks 6**

4.3 Basic IIR Filter Structures **AU : May-04, 05, 06, 11, 16, Dec.-06, 12, 13, 16**

As we studied the realization structures for FIR systems, similar structures are also possible for IIR systems. In addition to cascade and direct forms of realization, there is one more structure possible for IIR systems, i.e. parallel form realization. These structures are explained next.

4.3.1 Direct Form Structures for IIR Systems

We know that IIR system can be described by a generalized difference equation given by equation (4.1.1) or a system function of equation (4.1.2). Let us consider the system function of equation (4.1.2) i.e.,

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} \quad \text{... (4.3.1)}$$

Here let $H_1(z) = \sum_{k=0}^{M} b_k z^{-k}$ ... (4.3.2) and $$H_2(z) = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}} \quad \text{... (4.3.3)}$$

Hence we can write equation (4.3.1) as,

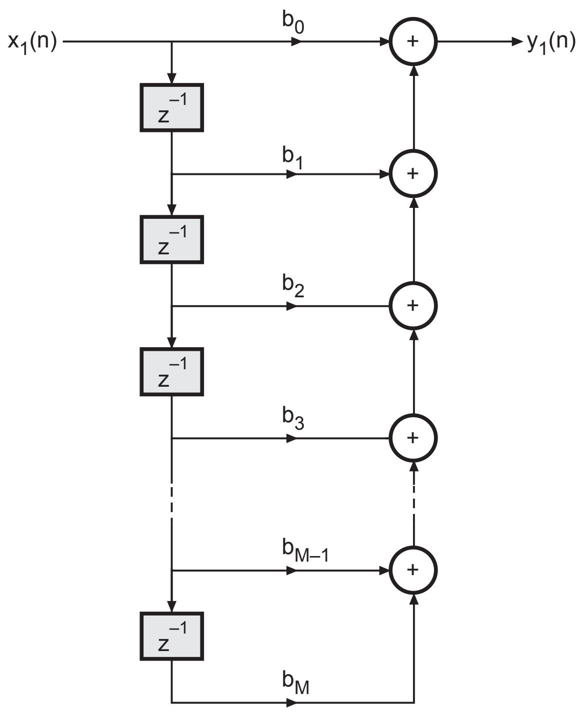


Fig. 4.3.1 Direct form realization of system function $H_1(z)$ [equation 4.3.2 or equation 4.3.5]. This is all zero system of $H(z)$ of equation 4.3.1

$$H(z) = H_1(z) \cdot H_2(z) \quad \text{... (4.3.4)}$$

Thus overall IIR system can be realized as cascade of two functions $H_1(z)$ and $H_2(z)$. Here observe that $H_1(z)$ represents zeros of $H(z)$. It is all zero system. Similarly $H_2(z)$ represents poles of $H(z)$. It is all pole system.

4.3.1.1 Direct Form-I Structure of IIR System

Let us first prepare the direct form structures of $H_1(z)$ and $H_2(z)$. We can write $H_1(z)$ of equation (4.3.2) as,

$$H_1(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}$$

Since $H_1(z) = \frac{Y_1(z)}{X_1(z)}$ above equation can be written as,

$$Y_1(z) = b_0 X_1(z) + b_1 z^{-1} X_1(z) + b_2 z^{-2} X_1(z) + \ldots + b_M z^{-M} X_1(z)$$

Taking inverse z-transform of above equation,

$$y_1(n) = b_0 x_1(n) + b_1 x_1(n-1) + b_2 x_1(n-1) + b_2 x_1(n-2) + \ldots + b_M x_1(n-M) \quad \ldots(4.3.5)$$

Fig. 4.3.1 shows the direct form realization of above equation. Observe that this realization is obtained in the similar manner as shown in Fig. 4.2.4. Only the realization of Fig. 4.3.1 is drawn vertical.

Next consider the realization of $H_2(z)$ of equation 4.3.3. This is all pole system. We will prepare the direct form realization of $H_2(z)$. Consider $H_2(z)$ given by equation 4.3.3 i.e.,

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}} \quad \ldots (4.3.6)$$

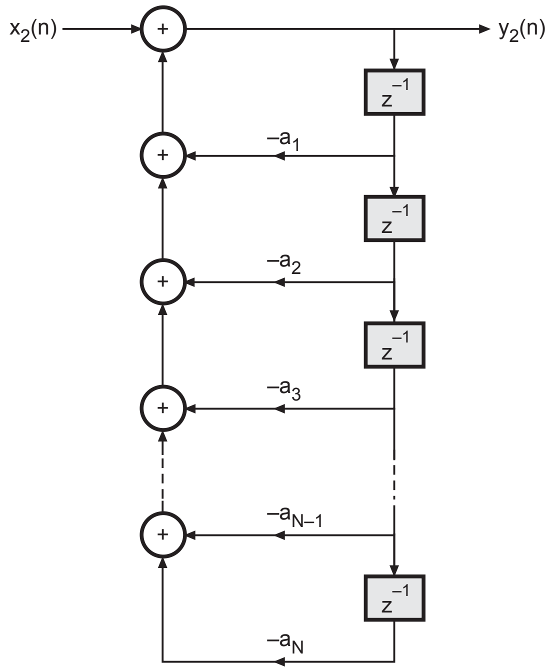


Fig. 4.3.2 Direct form realization of system function $H_2(z)$ [equation 4.3.6 or equation 4.3.7]. This is all pole system

We know that $H_2(z) = \frac{Y_2(z)}{X_2(z)}$, hence above equation becomes,

$$\frac{Y_2(z)}{X_2(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

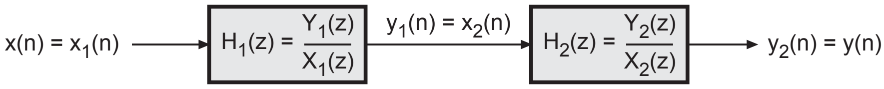


Fig. 4.3.3 $H(z) = H_1(z) \cdot H_2(z)$ **represents cascading of two systems**

$$\therefore \qquad Y_2(z)\left[1+\sum_{k=1}^{N} a_k z^{-k}\right] = X_2(z)$$

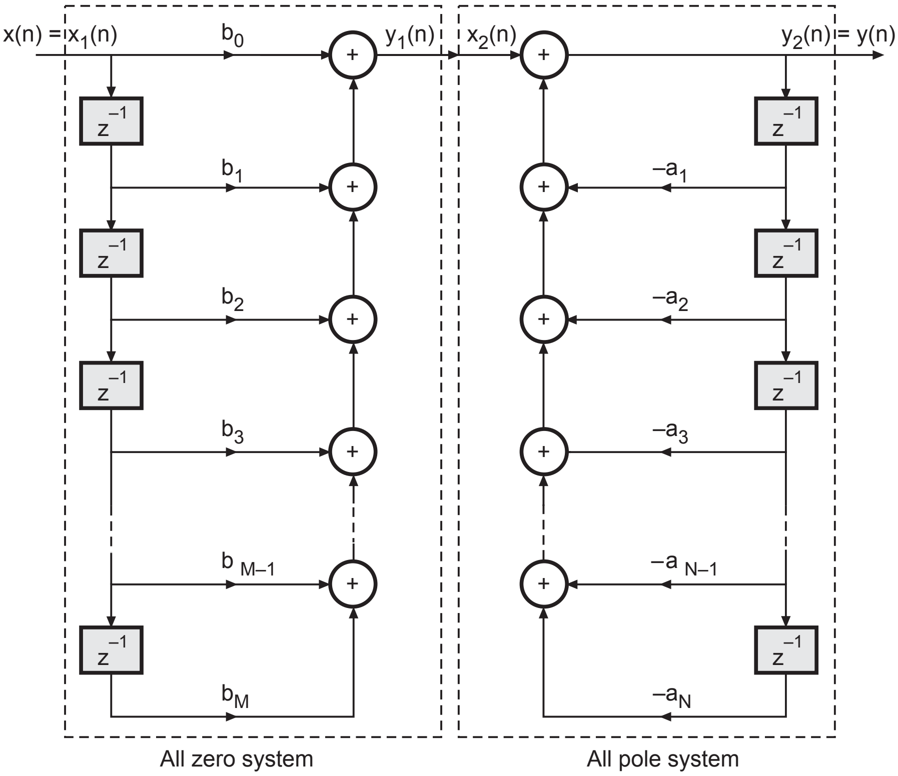


Fig. 4.3.4 Direct form-I realization of IIR system

$$\therefore \qquad Y_2(z) = -\sum_{k=1}^{N} a_k z^{-k} Y_2(z) + X_2(z)$$

Expanding summation of this equation we get,

$$Y_2(z) = a_1 z^{-1} Y_2(z) - a_2 z^{-2} Y_2(z) - a_3 z^{-3} Y_2(z) - \ldots - a_N z^{-N} Y_2(z) + X_2(z)$$

Taking inverse z-transform of above equation,

$$y_2(n) = a_1 y_2(n-1) - a_2 y_2(n-2) - a_3 y_2(n-3) - \ldots - a_N y_2(n-N) + x_2(n) \quad \ldots (4.3.7)$$

Fig. 4.3.2 shows the direct form realization of above difference equation. Here observe that the feedback terms are also present.

From equation 4.3.4 we know that,

$$H(z) = H_1(z) \cdot H_2(z)$$

This represents cascading of two systems $H_1(z)$ and $H_2(z)$. This cascading is shown in Fig. 4.3.3.

We have prepared the direct form realization for $H_1(z)$ [Fig. 4.3.1] and $H_2(z)$ [Fig. 4.3.2]. By connecting them in cascade we will get direct form realization for $H(z)$ as per above figure. This connection is shown in Fig. 4.3.4.

Observe that this realization requires $(M+N+1)$ number of multiplications, $(M+N)$ number of additions and $(M+N+1)$ number of memory locations.

4.3.1.2 Direct Form-II Structure for IIR System

We have seen that the overall system function can be connected as a cascade of $H_1(z)$ and $H_2(z)$. Recall $H(z)$ of equation 4.3.1 i.e.,

$$H(z) = \frac{\sum_{k=0}^{m} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \quad \ldots (4.3.8)$$

Let $H(z)$ be written as,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} \quad \text{By rearranging terms}$$

$$= \frac{W(z)}{X(z)} \cdot \frac{Y(z)}{W(z)} = H_1(z) \cdot H_2(z)$$

Now let $H_1(z)$ be given as,

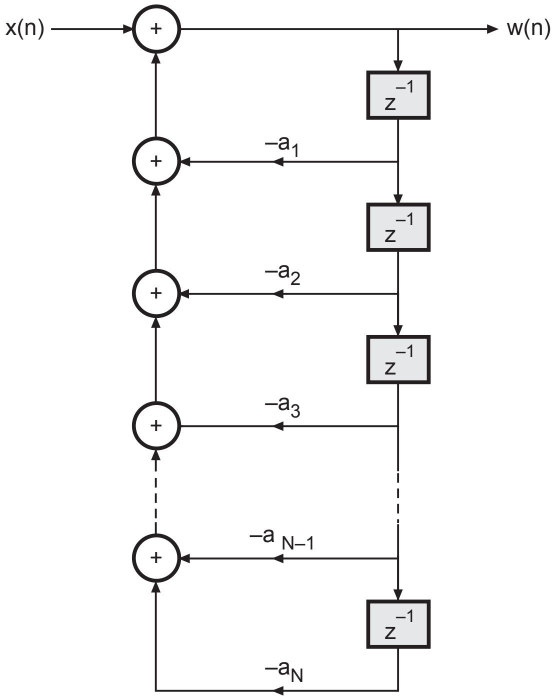


Fig. 4.3.5 Direct form implementation of equation 4.3.11. All pole system

$$H_1(z) = \frac{W(z)}{X(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}} \qquad \text{... (4.3.9)}$$

and $$H_2(z) = \frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k} \qquad \text{... (4.3.10)}$$

Cross multiply the terms of equation 4.3.9. Then we get,

$$W(z)\left[1+\sum_{k=1}^{N} a_k z^{-k}\right] = X(z)$$

$$\therefore \qquad W(z) = X(z) - \sum_{k=1}^{N} a_k z^{-k} W(z)$$

$$= X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) - a_3 z^{-3} W(z) - \ldots\ldots\ldots - a_N z^{-N} W(z)$$

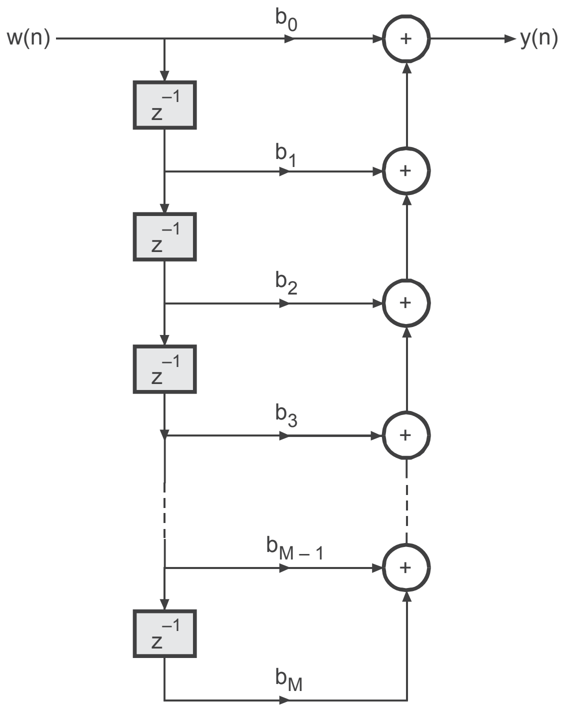


Fig. 4.3.6 Direct form implementation of equation 4.3.12. All zero system

Taking inverse z-transform of this equation,

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) - .... - a_N w(n-N) \quad ... (4.3.11)$$

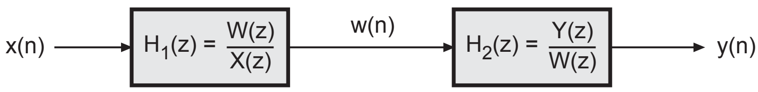


Fig. 4.3.7 Cascade connection of $H_1(z)$ and $H_2(z)$

Fig. 4.3.5 shows the direct form implementation of above equation.

In the realization of Fig. 4.3.5 observe that $x(n)$ is the input and $w(n)$ is the output.

Now let us obtain the realization of $H_2(z)$ of equation 4.3.10. From this equation we obtain,

$$Y(z) = \sum_{k=0}^{M} b_k z^{-k} W(z)$$

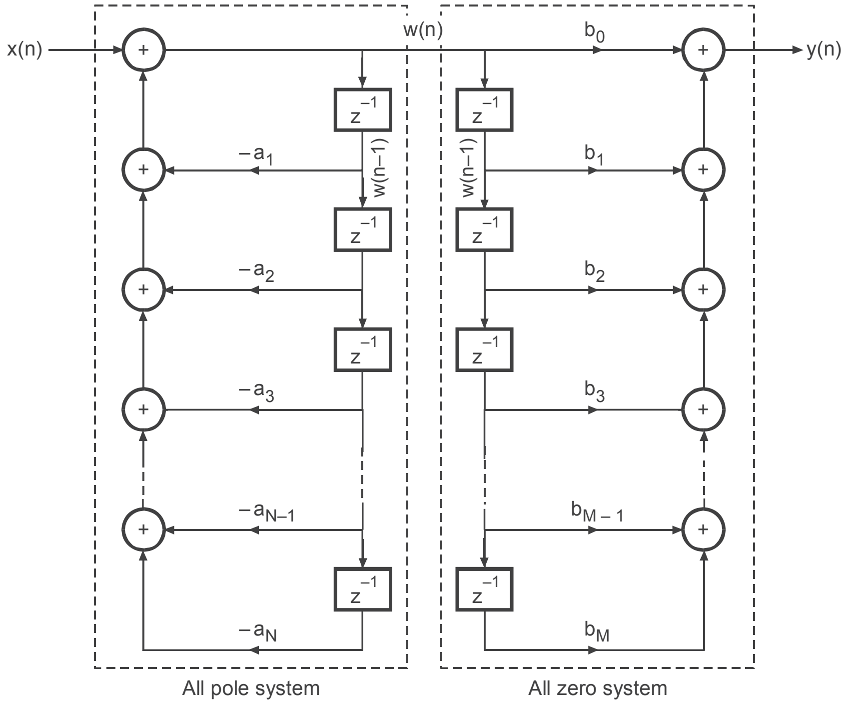


Fig. 4.3.8 Direct form realization of IIR system

$$= b_0 \; W(z) + b_1 \, z^{-1} \; W(z) + b_2 \; z^{-2} \; W(z) \; +$$

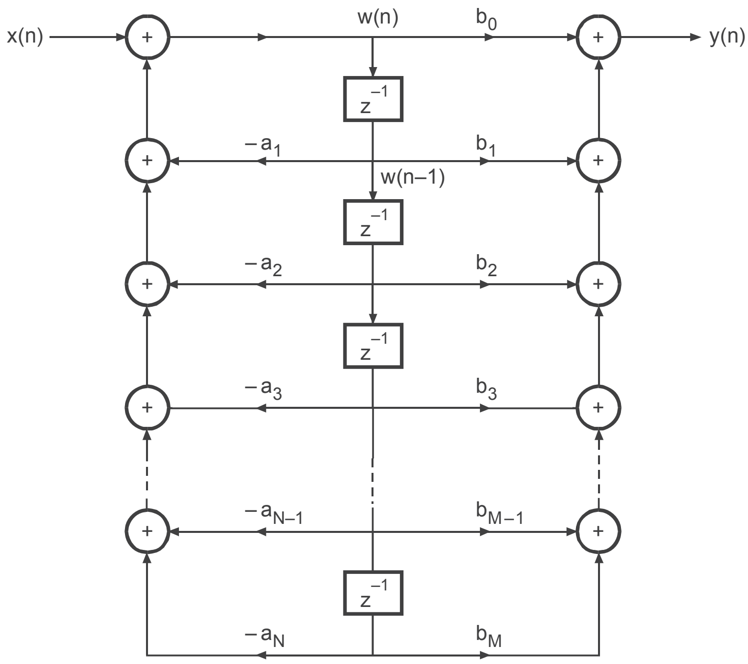


Fig. 4.3.9 Direct form-II realization of IIR system. Here $N = M$ is considered

$$........ + b_M \, z^{-M} \, W(z)$$

Taking inverse z-transform of this equation we get,

$$y(n) = b_0 \, w(n) + b_1 \, w(n-1) + b_2 \, w(n-2) + ... + b_M \, w(n-M). \quad .. (4.3.12)$$

Fig. 4.3.6 shows the direct form implementation of above equation.

See Fig. 4.3.6 on next page.

Observe that $H_1(z)$ of equation 4.3.9 is all pole system function. Hence its realization of Fig. 4.3.5 is also all pole system. Similarly $H_2(z)$ of equation 4.3.10 is all zero system function. Hence its realization given in Fig. 4.3.6 is also all zero system. In this figure. $w(n)$ is the input and $y(n)$ is the output.

Since $H(z) = H_1(z) \cdot H_2(z)$, the realization for $H(z)$ can be obtained by cascading $H_1(z)$ and $H_2(z)$ as shown in Fig. 4.3.7.

Then the direct form realization for $H(z)$ can be obtained by cascading the realization of $H_1(z)$ [Fig. 4.3.5] and realization of $H_2(z)$ [Fig. 4.3.6]. This connection is shown in Fig. 4.3.8. (See Fig. 4.3.8 on next page)

Observe Fig. 4.3.8 carefully. $w(n)$ is delayed and used in all pole system. Similarly it is delayed and used in all zero system. But we have used two separate delay elements. Observe that single delay element can be used to get $w(n-1)$. This can be used by all pole system as well as all zero system.

This means the two delay elements of all pole and zero system can be merged into single delay element. The resulting realization is shown below in Fig. 4.3.9.

In the realization shown in figure, we have assumed $N = M$. This is called Direct form-II realization of IIR system. This structure has reduced memory requirement compared to direct form-I structure. This structure requires maximum of $\{M, N\}$ memory locations. Since direct form-II structure reduces memory locations it is called canonic form. The direct form-II structure requires same number of multiplications (i.e. $M + N + 1$) and additions (i.e. $M + N$) as that of direct form-I structure.

Example 4.3.1 *A system is represented by a transfer function $H(z)$ is given by,*

$$H(z) = 3 + \frac{4z}{z - \frac{1}{2}} - \frac{2}{z - \frac{1}{4}}$$

i) Does this $H(z)$ represent a FIR or IIR filter ? Why ?

ii) Give a difference equation realization of this system using direct form-I.

iii) Draw the block diagram for the direct form-II canonic realization, and give the governing equations for implementation.

Solution : i) To check whether FIR or IIR filter :

Consider the given system function. It can be written as,

$$H(z) = 3+\frac{4z}{z-0.5}-\frac{2}{z-0.25}$$

$$= \frac{7z^2-5.25z+1.375}{z^2-0.75z+0.125} \qquad \text{... (4.3.13)}$$

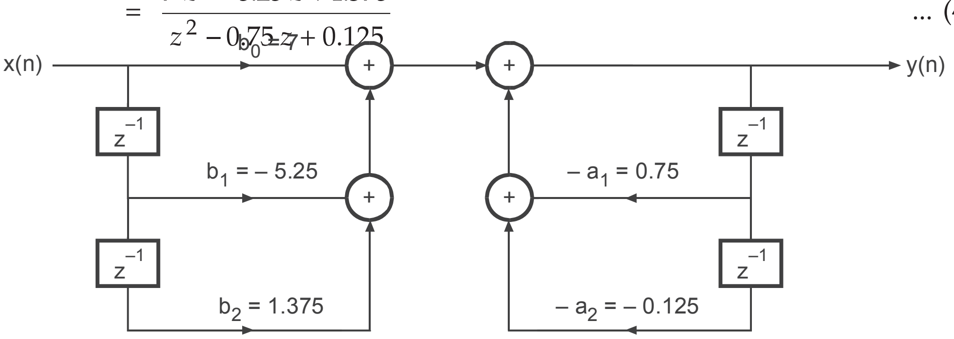


Fig. 4.3.10 Direct form-I realization of $H(z)$ of example 4.3.1

Observe that the system function has numerator polynomial of order 2 as well as denominator polynomial of order 2.

The system function has poles as well as zeros. Hence it represents IIR filter. The FIR filter has all zero system function.

ii) Direct form-I realization :

We can write equation 4.3.13 as follows :

$$H(z) = \frac{7-5.25z^{-1}+1.375z^{-2}}{1-0.75z^{-1}+0.125z^{-2}} \qquad \text{... (4.3.14)}$$

Let us expand summations of equation 4.3.1 for $M = N = 2$. i.e.,

$$H(z) = \frac{b_0+b_1 z^{-1}+b_2 z^{-2}}{1+a_1 z^{-1}+a_2 z^{-2}}$$

The direct form-I realization of above equation for generalized M and N is shown in Fig. 4.3.4. On comparing above equation with equation 4.3.14 we get,

$b_0 = 7,$ $\quad b_1 = -5.25,$ $\quad b_2 = 1.375$ and

$a_1 = -0.75,$ $\quad a_2 = 0.125$

Based on the above values and Fig. 4.3.4, the direct form-I realization is given in Fig. 4.3.10.

iii) Direct form-II canonic realization :

We can write equation 4.3.13 as follows :

$$H(z) = \frac{7-5.25\,z^{-1}+1.375\,z^{-2}}{1-0.75\,z^{-1}+0.125\,z^{-2}} \quad \text{... (4.3.15)}$$

We know that $H(z)=\dfrac{Y(z)}{X(z)}$. Hence above equation will be,

$$\frac{Y(z)}{X(z)} = \frac{7-5.25\,z^{-1}+1.375\,z^{-2}}{1-0.75\,z^{-1}+0.125\,z^{-2}}$$

Let us rearrange the above equation as,

$$\frac{Y(z)}{W(z)}\cdot\frac{W(z)}{X(z)} = \frac{7-5.25\,z^{-1}+1.375\,z^{-2}}{1-0.75\,z^{-1}+0.125\,z^{-2}}$$

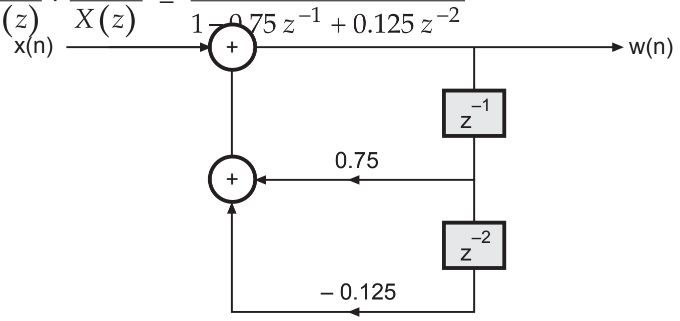


Fig. 4.3.11 Direct form realization of equation 4.3.19 i.e. $H_1(z)$

i.e. $$\frac{W(z)}{X(z)}\cdot\frac{Y(z)}{W(z)} = \frac{7-5.25\,z^{-1}+1.375\,z^{-2}}{1-0.75\,z^{-1}+0.125\,z^{-2}}$$

Let, $$H_1(z)\cdot H_2(z) = \frac{1}{1-0.75\,z^{-1}+0.125\,z^{-2}}\left(7-5.25\,z^{-1}+1.375\,z^{-2}\right) \quad \text{... (4.3.16)}$$

$\therefore$ $$\text{Let } H_1(z)=\frac{W(z)}{X(z)} = \frac{1}{1-0.75\,z^{-1}+0.125\,z^{-2}} \quad \text{... (4.3.17)}$$

and $$H_2(z) = \frac{Y(z)}{W(z)} = 7-5.25\,z^{-1}+1.375\,z^{-2} \quad \text{... (4.3.18)}$$

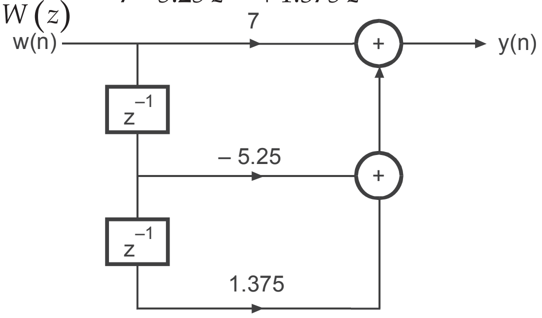


Fig. 4.3.12 Direct form realization of equation 4.3.20, i.e. $H_2(z)$

Cross multiplying in equation 4.3.17 (all pole function) we get,

$W(z)\left[1-0.75\, z^{-1}+0.125\, z^{-2}\right]=X(z)$

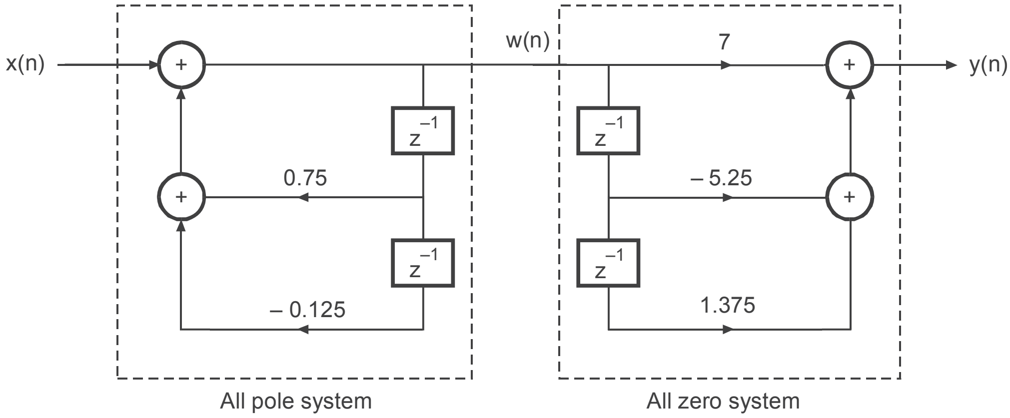


Fig. 4.3.13 Realization of $H(z)=H_1(z)\cdot H_2(z)$

$$\therefore \quad W(z) = X(z)+0.75\, z^{-1}\, W(z)-0.125\, z^{-2}\, W(z)$$

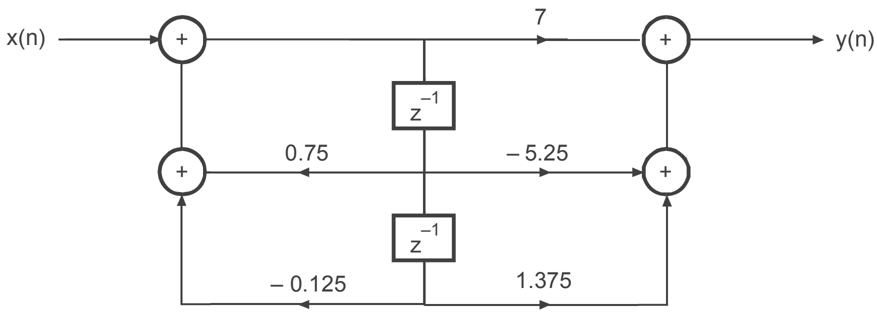


Fig. 4.3.14 Direct form-II canonic realization

Taking inverse z-transform of above equation,

$$w(n) = x(n)+0.75\, w(n-1)-0.125\, w(n-2) \qquad \text{... (4.3.19)}$$

Fig. 4.3.11 shows the direct form realization of above equation.

Cross multiply the terms in equation 4.3.18, we get,

$$Y(z) = \left(7-5.25\, z^{-1}+1.375\, z^{-2}\right) W(z) \qquad \text{... (4.3.20)}$$

$$= 7\,W(z) - 5.25\,z^{-1}\,W(z) + 1.375\,z^{-2}\,W(z)$$

Taking inverse z-transform of above equation,

$$y(n) = 7\,w(n) - 5.25\,w(n-1) + 1.375\,w(n-2)$$

Fig. 4.3.12 shows the direct form implementation of above equation.

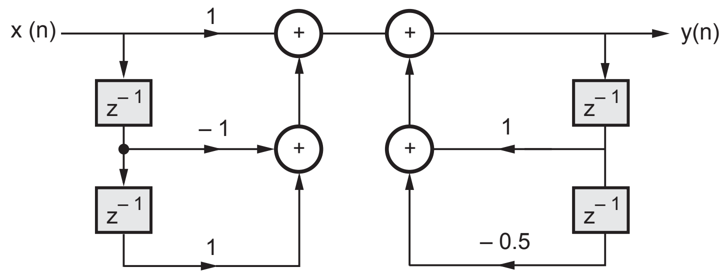


Fig. 4.3.15 Direct form - I

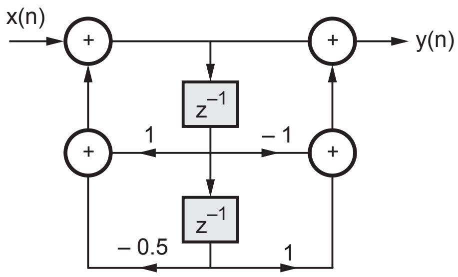


Fig. 4.3.16 Direct form - II

Here note that Fig. 4.3.11 is realization of $H_1(z)$ and Fig. 4.3.12 shows realization of $H_2(z)$. The realization of $H(z) = H_1(z) \cdot H_2(z)$ [i.e. equation 4.3.16] can be obtained by cascading realizations of Fig. 4.3.11 and Fig. 4.3.12. Such realization is shown in Fig. 4.3.13.

In the Fig. 4.3.13 observe that the delays can be combined into two. Then the realization becomes as shown in Fig. 4.3.14.

In the above figure observe that there are two delay elements. The order of $H(z)$ is also two. Hence it is canonic realization. It is also called as direct form II realization of IIR filters.

The direct form-II canonic realization for generalized N and M is given in Fig. 4.3.9. Earlier we obtained the values of b_0, b_1, b_2, a_1 and a_2. By putting these values in Fig. 4.3.9 we can directly obtain the realization of Fig. 4.3.14.

Example 4.3.2 *Draw the direct form - I and direct form - II structures for the given difference equation $y(n) = y(n-1) - 0.5\, y(n-2) + x(n) - x(n-1) + x(n+2)$.*

AU : Dec.-13, Marks 8

Solution : Fig. 4.3.15 and 4.3.16 shows the direct form - I and direct form - II structures

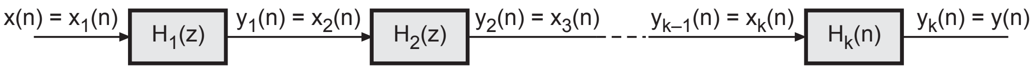


Fig. 4.3.17 Cascade form realization of IIR systems

respectively.

4.3.2 Cascade Form Structure for IIR Systems

Consider the rational system function of equation 4.1.2 i.e.,

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_z z^{-2} + \dots + a_N z^{-N}} \quad \text{... (4.3.21)}$$

The numerator and denominator polynomials of above equation can be expressed as multiplication of second order polynomials. i.e.,

$$H(z) = H_1(z) \times H_2(z) \times H_3(z) \times \dots \times H_k(z) \quad \text{... (4.3.22)}$$

where, $$H_k(z) = \frac{b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}, \quad k = 1, 2, \dots, K \quad \text{... (4.3.23)}$$

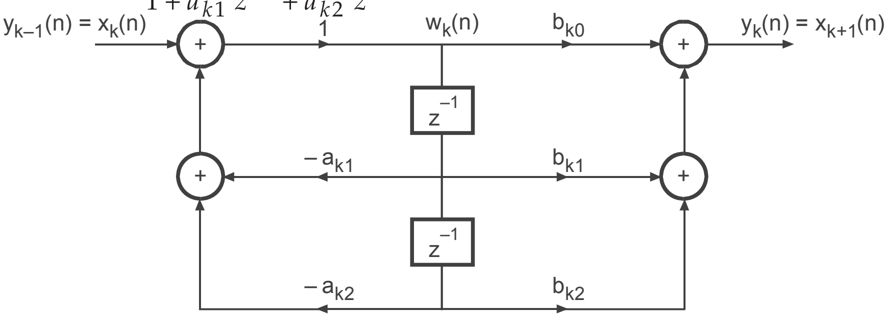


Fig. 4.3.18 Direct form-II realization of second order subsystems used in cascade connection of IIR systems

We know that the system functions $H_1(z), H_2(z)$ etc of equation 4.3.22 can be connected in cascade to obtain realization of $H(z)$. This is shown in Fig. 4.3.17.

Now each $H_1(z), H_2(z), \ldots..$ etc can be realized by direct form I or II structures. We know that $H_k(z) = \frac{Y_k(z)}{X_k(z)}$. This can also be written as,

$$H_k(z) = \frac{W_k(z)}{X_k(z)} \cdot \frac{Y_k(z)}{W_k(z)}$$

$$= H_{k1}(z) \cdot H_{k2}(z)$$

Let, $$H_{k1}(z) = \frac{W_k(z)}{X_k(z)} = \frac{1}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}} \quad \ldots (4.3.24)$$

This is all pole second order subsystem and,

$$H_{k2}(z) = \frac{Y_k(z)}{W_k(z)} = b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2} \quad \ldots (4.3.25)$$

We have discussed the procedure for obtaining direct form II in last subsection. Proceeding on the same lines we can obtain the direct form-II structure for $H_k(z)$, which is splitted into two functions given by equation 4.3.24 and equation 4.3.25. This direct form-II structure is shown below in Fig. 4.3.18.

This cascade structure is described by following equations

$$w_k(n) = -a_{k1}\, w_k(n-1) - a_{k2}\, w_k(n-2) + y_{k-1}(n) \qquad \text{... (4.3.26)}$$

$$y_k(n) = b_{k0}\, w_k(n) + b_{k1}\, w_k(n-1) + b_{k2}\, w_k(n-2) \qquad \text{... (4.3.27)}$$

This equations represent the second order subsystem of Fig. 4.3.18. And cascading is represented by following equations :

$$\left.\begin{aligned} y_{k-1}(n) &= x_k(n) \\ \therefore \quad y_k(n) &= x_{k+1}(n) \\ y_0(n) &= x(n) \\ y(n) &= x_k(n) \end{aligned}\right\} \text{...(4.3.28)}$$

The above equations represent the cascading of second order subsystems as shown in Fig. 4.3.17.

Example 4.3.3 *Realize the following system function in cascade form.*

$$H(z) = \frac{1+\frac{1}{4}z^{-1}}{\left(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right)\left(1+\frac{1}{2}z^{-2}\right)}$$

AU : May-16, Marks 2

Solution : The given transfer function can be written as the product of two functions. i.e.,

$$H(z) = H_1(z)\cdot H_2(z) = \underbrace{\frac{1}{1+\frac{1}{2}z^{-1}}}_{H_1(z)} \cdot \underbrace{\frac{1+\frac{1}{4}z^{-1}}{1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}}_{H_2(z)} \qquad \text{... (4.3.29)}$$

The above two equations are in the form of equation 4.3.23. They are written as follows :

$$H_1(z) = \frac{b_{10}}{1+a_{11}z^{-1}} = \frac{1}{1+\frac{1}{2}z^{-1}} \qquad \text{... (4.3.30)}$$

$$H_2(z) = \frac{b_{20}+b_{21}z^{-1}}{1+a_{21}z^{-1}+a_{22}z^{-2}} = \frac{1+\frac{1}{4}z^{-1}}{1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}} \qquad \text{... (4.3.31)}$$

Realization of equation 4.3.23 is given in Fig. 4.3.18. The above two equations can be realized by using Fig. 4.3.18. Fig. 4.3.19 shows the cascade realization of $H(z) = H_1(z)\cdot H_2(z)$.

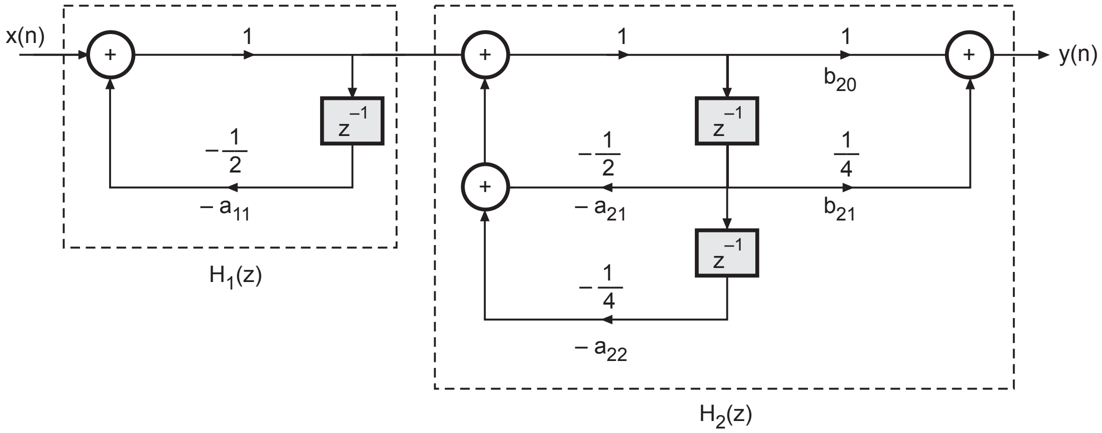


Fig. 4.3.19 Cascade realization of $H(z)$ of example 4.3.3

4.3.3 Parallel Form Structure for IIR Systems

We know that the rational system function of IIR system is given as,

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots. + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots. + a_N z^{-N}} \quad \ldots (4.3.32)$$

The above system function can be expanded in partial fractions as follows :

$$H(z) = C + H_1(z) + H_2(z) + \ldots. + H_k(z) \quad \ldots (4.3.33)$$

Here 'C' is constant and each $H_1(z), H_2(z)$, ... etc is the second order subsystem which is given as,

$$H_k(z) = \frac{b_{k0} + b_{k1} z^{-1}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}} \quad \ldots (4.3.34)$$

These second order subsystems are formed by combining complex conjugate poles. Because of this, the coefficients $b_{k0}, b_{k1}, a_{k1}, a_{k2}$ are real.

We know that addition of system functions results in parallel connection. Then the realization of $H(z)$ of equation 4.3.33 becomes as shown in Fig. 4.3.20.

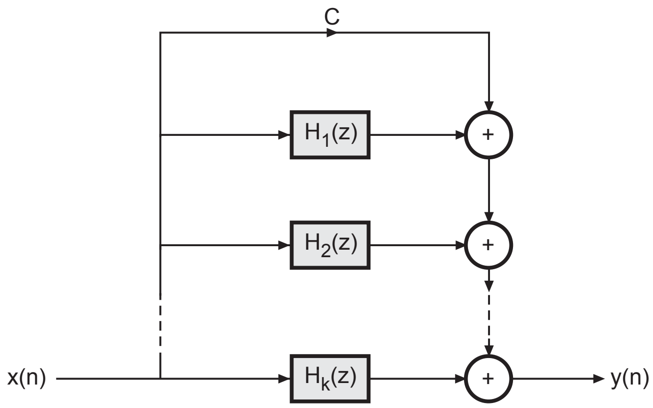


Fig. 4.3.20 Parallel form realization structure for IIR systems

Here each $H_1(z)$, $H_2(z)$ ...... etc can be realized by direct form-I or direct form-II Fig. 4.3.21 shows the direct form-II realization of $H_k(z)$ of equation 4.3.34.

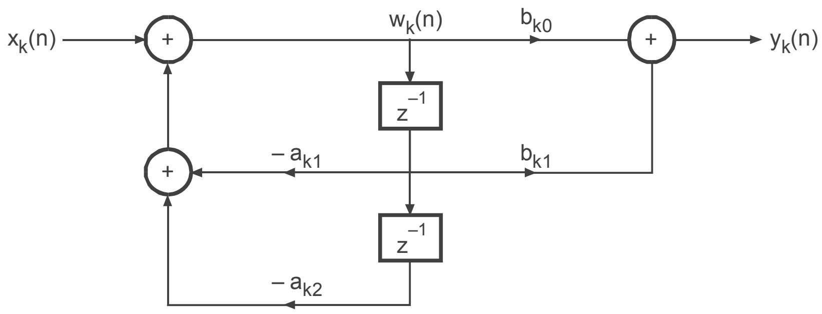


Fig. 4.3.21 Direct form-II realization of second order subsystem of equation 4.3.34

The parallel form structure discussed here can be described by following equations.

$$\left.\begin{aligned} w_k(n) &= a_{k1}\, w_k(n-1) - a_{k2}\, w_k(n-2) + x(n) \\ y_k(n) &= b_{k0}\, w_k(n) + b_{k1}\, w_k(n-1) \\ y(n) &= C\,x(n) + \sum_{k=1}^{k} y_k(n) \end{aligned}\right\} \quad \ldots (4.3.35)$$

Example 4.3.4 *Realize the following system function in parallel form.*

$$H(z) = \frac{1-\frac{2}{3}z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}} \cdot \frac{1+\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

Solution : Let us write the given system function as,

$$H(z) = \frac{z\left(z-\frac{2}{3}\right)}{z^2-\frac{7}{8}z+\frac{3}{32}} \cdot \frac{z^2-\frac{7}{4}z-\frac{1}{2}}{z^2-z+\frac{1}{2}}$$

i.e.
$$\frac{H(z)}{z} = \frac{z-\frac{2}{3}}{z^2-\frac{7}{8}z+\frac{3}{32}} \cdot \frac{z^2-\frac{7}{4}z-\frac{1}{2}}{z^2-z+\frac{1}{2}}$$

$$= \frac{z-\frac{2}{3}}{\left(z-\frac{3}{4}\right)\left(z-\frac{1}{8}\right)} \cdot \frac{z^2-\frac{7}{4}z-\frac{1}{2}}{\left(z-\frac{1}{2}-j\frac{1}{2}\right)\left(z-\frac{1}{2}+j\frac{1}{2}\right)}$$

$$= \frac{A_1}{z-\frac{3}{4}} + \frac{A_2}{z-\frac{1}{8}} + \frac{A_3}{z-\frac{1}{2}-j\frac{1}{2}} + \frac{A_4}{z-\frac{1}{2}+j\frac{1}{2}}$$

Calculating the values of A_1, A_2, A_3 and A_4 we get,

$$\frac{H(z)}{2} = \frac{2.933}{z-\frac{3}{4}} - \frac{2.947}{z-\frac{1}{8}} + \frac{2.507-j\,10.45}{z-\frac{1}{2}-j\frac{1}{2}} + \frac{2.507+j\,10.45}{z-\frac{1}{2}+j\frac{1}{2}}$$

Let us combine the first two terms and last two terms. Because of this, the complex values will be combined into real coefficients. i.e.,

$$\frac{H(z)}{z} = \frac{-0.014\,z+1.843}{z^2-\frac{7}{8}z+\frac{3}{32}} + \frac{5.02\,z+7.743}{z^2-z+\frac{1}{2}}$$

$$\therefore \quad H(z) = \frac{-0.014\,z^2+1.843\,z}{z^2-\frac{7}{8}z+\frac{3}{32}} + \frac{5.02\,z^2+7.743\,z}{z^2-z+\frac{1}{2}}$$

The above equation can also be written as,

$$H(z) = \frac{-0.014+1.843\,z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}} + \frac{5.02+7.743\,z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}} \qquad \text{... (4.3.36)}$$

The above equation has two terms, they can be called as,

$$H_1(z) = \frac{b_{10}+b_{11}\,z^{-1}}{1+a_{11}\,z^{-1}+a_{12}\,z^{-2}} = \frac{-0.014+1.843\,z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}} \qquad \text{... (4.3.37)}$$

and $$H_2(z) = \frac{b_{20} + b_{21} z^{-1}}{1 + a_{21} z^{-1} + a_{22} z^{-2}} = \frac{5.02 + 7.743 z^{-1}}{1 - z^{-1} + \frac{1}{2} z^{-2}} \quad \text{... (4.3.38)}$$

Observe that the above two equations are in the form of equation 4.3.34. The realization of equation 4.3.34 is shown in Fig. 4.3.21. The realization of $H_1(z)$ and $H_2(z)$ in parallel is shown in Fig. 4.3.22.

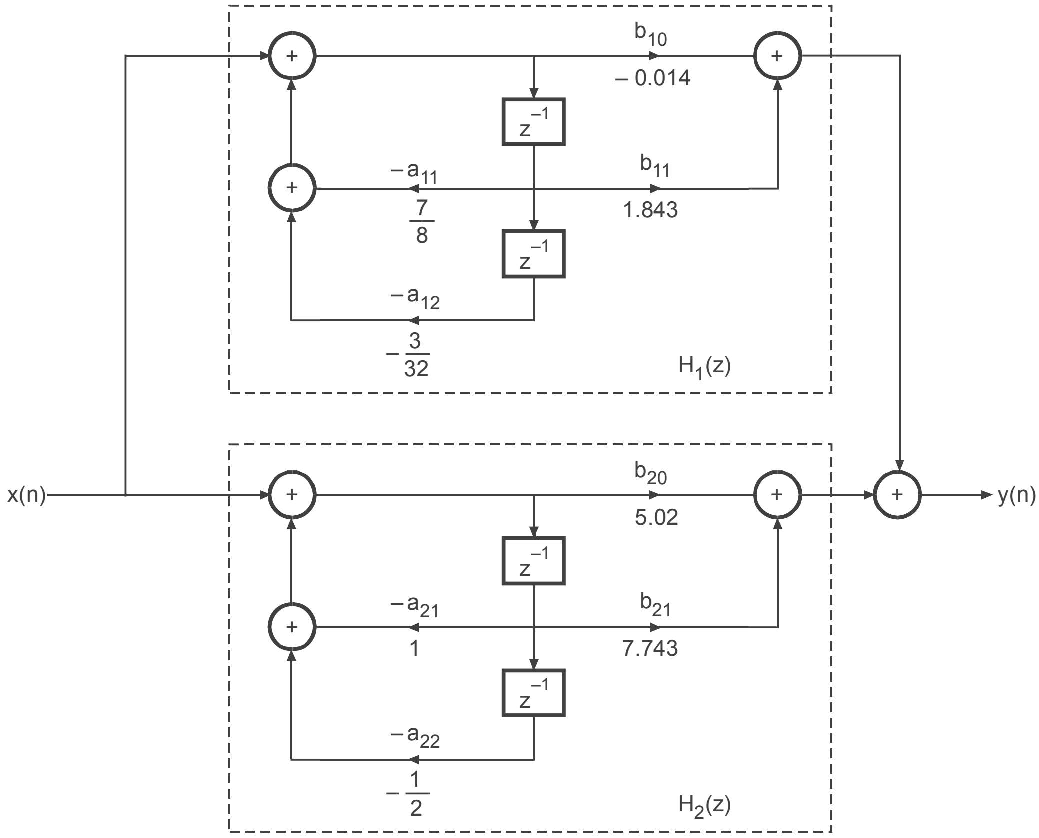


Fig. 4.3.22 Parallel realization of $H(z)$ of example 4.3.4

Example 4.3.5 *Obtain parallel realization using first and second order sections for the transfer function*

$$H(z) = \frac{2\ (z-1)(z^2 + 1.414z + 1)}{(z + 0.5)\ (z^2 - 0.9z + 0.81)}$$

AU : May-04, Marks 8

Solution : Above equation can be written as,

$$H(z) = \frac{2z^3 + (0.828)z^2 - 0.828z - 2}{z^3 - 0.4z^2 + 0.36z + 0.405}$$

$$\therefore \qquad \frac{H(z)}{z} = \frac{2z^3 + 0.828z^2 - 0.828z - 2}{(z^3 - 0.4z^2 + 0.36z + 0.405)z}$$

$$= \frac{2.39 - j0.35}{(z - 2.45 - j\,0.78)} + \frac{2.39 + j0.35}{(z - 0.45 + j\,0.78)} + \frac{2.15}{z + 0.5} - \frac{4.93}{z}$$

$$= \frac{4.78z - 1.6}{z^2 - 0.9z + 0.81} + \frac{2.15}{z + 0.5} + \frac{4.93}{z}$$

$$\therefore \qquad H(z) = \frac{4.78 - 1.6z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}} + \frac{2.15}{1 + 0.5z^{-1}} - 4.93$$

$$= H_1(z) + H_2(z) + H_3(z)$$

Here $$H_1(z) = \frac{4.78 - 1.6z^{-1}}{1 - 0.9z^{-1} + (0.81)z^{-2}}$$

$$H_2(z) = \frac{2.15}{1 + 0.5z^{-1}} \quad \text{and} \quad H_3(z) = -4.93$$

Putting above values in standard first and second order sections, we get the following realization.

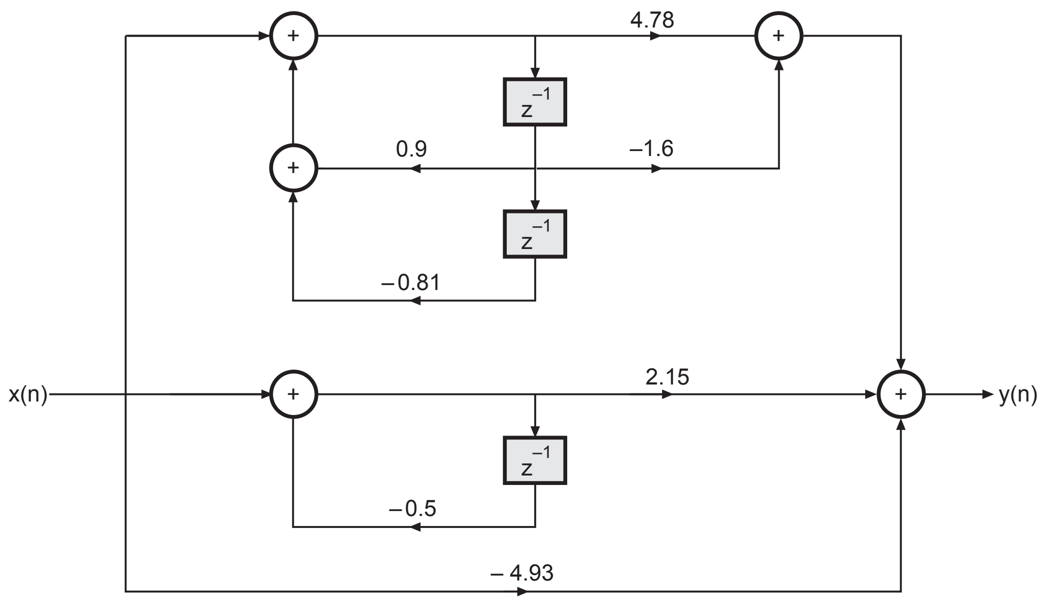


Fig. 4.3.23 Parallel form realization

Example 4.3.6 *Realize the following using cascade and parallel form.*

$$H(z) = \frac{3+3.6\,z^{-1}+0.6\,z^{-2}}{1+0.1\,z^{-1}-0.2\,z^{-2}}.$$

AU : Dec.-12, Marks 8, May-11, Marks 12

Solution : i) Cascade form realization

$$H(z) = \frac{3(1+1.2z^{-1}+0.2z^{-2})}{1+0.1z^{-1}-0.2z^{-2}} = \frac{3(z^2+1.2z+0.2)}{(z^2+0.1z-0.2)}$$

$$= \frac{3(z+1)(z+0.2)}{(z+0.5)(z-0.4)} \quad \ldots (4.3.39)$$

$$= \underbrace{\frac{3(1+z^{-1})}{(1+0.5z^{-1})}}_{H_1(z)} \cdot \underbrace{\frac{1+0.2z^{-1}}{(1-0.4z^{-1})}}_{H_2(z)}$$

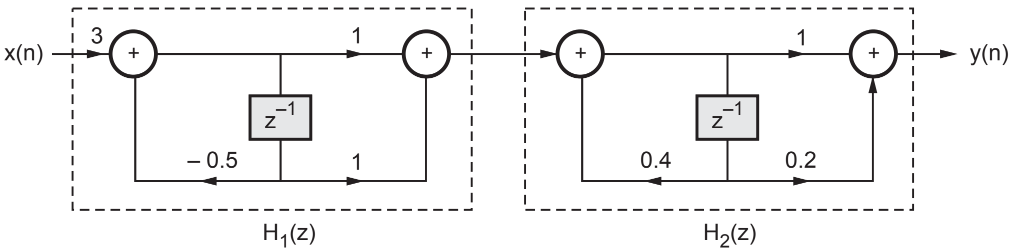


Fig. 4.3.24 Cascade form realization

Here $H_1(z)$ and $H_2(z)$ are realized in cascade as shown in Fig. 4.3.24.

ii) Parallel form realization

Consider equation (4.3.39)

$$H(z) = \frac{3(z+1)(z+0.2)}{(z+0.5)(z-0.4)}$$

$$\therefore \quad \frac{H(z)}{z} = \frac{3(z+1)(z+0.2)}{z(z+0.5)(z-0.4)} = -\frac{3}{z}-\frac{1}{z+0.5}+\frac{7}{z-0.4}$$

$$\therefore \quad H(z) = \underbrace{-3}_{H_1(z)} - \underbrace{\frac{1}{1+0.5z^{-1}}}_{H_2(z)} + \underbrace{\frac{7}{1-0.4z^{-1}}}_{H_3(z)}$$

Here $H_1(z), H_2(z)$ and $H_3(z)$ are realized in the parallel form as shown in Fig. 4.3.25.

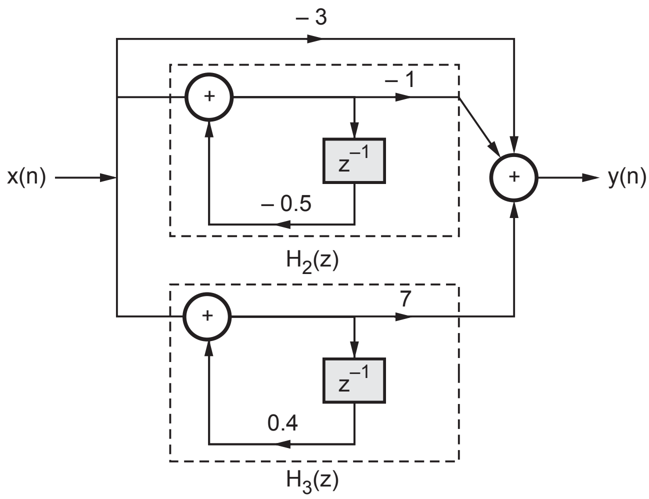


Fig. 4.3.25 Parallel form realization

Example 4.3.7 *A difference equation describing a filter is given by*
$y(n) - 2y(n-1) + y(n+2) = x(n) + 1/2\ x(n-1)$ obtain direct form II structure.

AU : Dec.-16, Marks 8

Solution : Here $b_0 = 1, b_1 = y_2, \quad a_1 = +2, \ a_2 = -1$

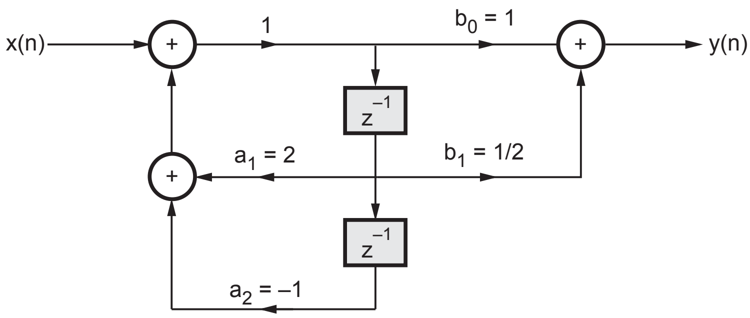


Fig. 4.3.26 Direct form - II

Example for Practice

Example 4.3.8 *Realize the transfer function*
$H(z) = (0.7 - 0.252\ z^{-2})/(1 + 0.1\ z^{-1} - 0.72\ z^{-2})$ using parallel realization.

AU : May-06, Marks 10

Ans. :

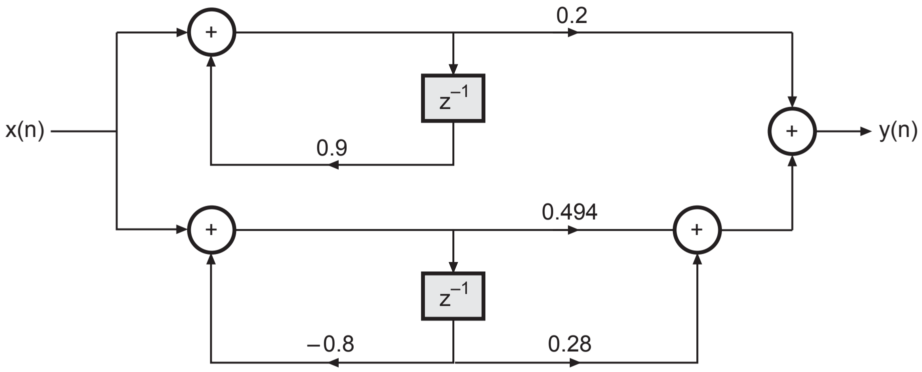


Fig. 4.3.27 Parallel form realization

Review Questions

1. *In IIR filter, where the cascade structure is unique ? Why ? Explain with an example.* **AU : Dec.-06, Marks 8**
2. *Draw direct form - I and II of an IIR system.* **AU : May-05, Marks 10, Dec.-06, Marks 6**

4.4 Properties of FIR Digital Filters

The FIR filters have following important properties, that make them different from IIR filters.

i) FIR filters are inherently stable.

ii) FIR filters have linear phase.

iii) FIR filters need higher orders for similar magnitude response compared to IIR filters.

These properties are discussed in detail in next sections.

4.4.1 Inherent Stability of FIR Filters

FIR filters have very important characteristic that they are inherently stable. We know that the difference equation of FIR filter of length M is given as

$$y(n) = \sum_{k=0}^{M-1} b_k\, x(n-k) \qquad \text{... (4.4.1)}$$

And the coefficients b_k are related to unit sample response as

$$h(n) = \begin{cases} b_n & \text{for} \quad 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{... (4.4.2)}$$

We can expand equation (4.4.1) as,

$$y(n) = b_0\, x(n) + b_1\, x(n-1) + \ldots + b_{M-1}\, x(n-M+1) \qquad \text{... (4.4.3)}$$

The BIBO stability states that if the system produces bounded output for every bounded input, then it is stable system. Here observe that the coefficients $h(n)=\{b_0, b_1, \ldots b_{M-1}\}$ of the FIR filter are stable (i.e. bounded). Then in equation (4.4.3), output $y(n)$ is bounded if input $x(n)$ is bounded. This means FIR filter produces bounded output for every bounded input according to equation (4.4.3). Hence FIR filters are always or inherently stable filters.

4.4.2 Magnitude and Phase Response of FIR Filters and Linear Phase Property

- The symmetry/antisymmetry of the unit sample response of FIR filters is related to their linearity of phase.
- The unit sample response of FIR filters is symmetric if it satisfies following condition.

$$h(n) = h(M-1-n), \qquad n = 0,1,\ldots M-1 \qquad \text{... (4.4.4)}$$

- The unit sample response of FIR filters is antisymmetric if it satisfies following condition :

$$h(n) = -h\,(M-1-n), \qquad n = 0,1,\ldots M-1 \qquad \text{... (4.4.5)}$$

- The phase of FIR filters is piecewise linear if its unit sample response symmetric or antisymmetric. This can be proved separately for even and odd lengths of FIR filters. Consider the Fourier transform of unit sample response,

$$H(\omega) = \sum_{n=0}^{M-1} h\,(n)\; e^{-j\omega n} \qquad \text{... (4.4.6)}$$

Let us assume that **'M' is odd**. Then let us split the above equation as follows :

$$H(\omega) = \sum_{n=0}^{\frac{M-3}{2}} h(n)\; e^{-j\omega n} + h\left(\frac{M-1}{2}\right) e^{-j\omega\left(\frac{M-1}{2}\right)} + \sum_{n=\frac{M+1}{2}}^{M-1} h(n)\; e^{-j\omega n} \qquad \text{... (4.4.7)}$$

Consider the last summation term in above equation. We know that $h(n) = h(M-1-n)$ for symmetric FIR filter. With this condition the last summation term in above equation becomes,

$$\sum_{n=\frac{M+1}{2}}^{M-1} h(n)\, e^{-j\omega n} = \sum_{n=\frac{M+1}{2}}^{M-1} h\,(M-1-n)\, e^{-j\omega n} \qquad \dots (4.4.8)$$

Let us put $M-1-n=k$ in the right handside. Then we have $n=M-1-k$ and limits of summation will be,

When $\quad n = \frac{M+1}{2}, \qquad k = M-1-\frac{M+1}{2} = \frac{M-3}{2}$

and when $\quad n = M-1, \qquad k = M-1-M+1 = 0$

Therefore equation (4.4.8) becomes,

$$\sum_{n=\frac{M+1}{2}}^{M-1} h(n)\, e^{-j\omega n} = \sum_{k=0}^{\frac{M-3}{2}} h(k)\, e^{-j\omega(M-1-k)}$$

Since 'k' is just an index we can write 'n' in place of 'k'. Hence above equation becomes,

$$\sum_{n=\frac{M+1}{2}}^{M-1} h(n)\, e^{-j\omega n} = \sum_{n=0}^{\frac{M-3}{2}} h(n)\, e^{-j\omega(M-1-n)} \qquad \dots (4.4.9)$$

Putting value of the 3rd summation term as obtained above in equation 4.4.7 we get,

$$H(\omega) = \sum_{n=0}^{\frac{M-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{M-1}{2}\right) e^{-j\omega\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\, e^{-j\omega(M-1-n)}$$

$$= h\left(\frac{M-1}{2}\right) e^{-j\omega\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left[e^{-j\omega n} + e^{-j\omega(M-1-n)}\right] \qquad \dots (4.4.10)$$

We can rearrange the exponential terms in square brackets in above equation as follows :

$$e^{-j\omega n} = e^{-j\omega n} \cdot e^{j\omega\left(\frac{M-1}{2}\right)} \cdot e^{-j\omega\left(\frac{M-1}{2}\right)}$$

$$= e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot e^{-j\omega\left(n-\frac{M-1}{2}\right)} \qquad \dots (4.4.11)$$

Similarly,

$$e^{-j\omega(M-1-n)} = e^{-j\omega(M-1)} \cdot e^{j\omega n}$$

$$= e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot e^{j\omega n}$$

$$= e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot e^{j\omega\left(n-\frac{M-1}{2}\right)} \quad \text{... (4.4.12)}$$

From equation (4.4.11) and above equation we can write,

$$\left[e^{-j\omega n} - e^{-j\omega(M-1-n)}\right] = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[e^{-j\omega\left(n-\frac{M-1}{2}\right)} + e^{j\omega\left(n-\frac{M-1}{2}\right)}\right]$$

With the help of Euler's identity $e^{j\theta} + e^{-j\theta} = 2\cos\theta$ we can write above equation as,

$$\left[e^{-j\omega n} - e^{-j\omega n(M-1-n)}\right] = e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot 2\cos\omega\left(n-\frac{M-1}{2}\right) \quad \text{... (4.4.13)}$$

Putting the result of above equation into equation (4.4.10) we get,

$$H(\omega) = h\left(\frac{M-1}{2}\right) e^{-j\omega\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n) \cdot e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot 2\cos\omega\left(n-\frac{M-1}{2}\right)$$

$$= e^{-j\omega\left(\frac{M-1}{2}\right)}\left\{h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega\left(n-\frac{M-1}{2}\right)\right\} \quad \text{... (4.4.14)}$$

The polar form of $H(\omega)$ can be expressed as,

$$H(\omega) = |H(\omega)|\, e^{j\angle H(\omega)} \quad \text{... (4.4.15)}$$

Here $|H(\omega)|$ is magnitude of $H(\omega)$ and

$\angle H(\omega)$ is angle or phase of $H(\omega)$.

Comparing equation (4.4.15) with equation (4.4.14) we get,

$$\text{Magnitude } |H(\omega)| = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega\left(n-\frac{M-1}{2}\right) \quad \text{... (4.4.16)}$$

This magnitude can have negative values hence it is also called pseudomagnitude. And phase or angle of $H(\omega)$ is given as,

$$\angle H(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & \text{for } |H(\omega)| > 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi & \text{for } |H(\omega)| < 0 \end{cases} \quad \text{... (4.4.17)}$$

In the above equation observe that $\frac{M-1}{2}$ is constant. Hence phase $\angle H(\omega)$ is linear function of 'ω'. Thus phase is linearly proportional to frequency. When $|H(\omega)|$ changes sign, phase changes by π. Hence the phase is said to be piecewise linear. Thus FIR filters are linear phase filters.

For **even value of *M*** we can write equation (4.4.14) directly as,

$$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)} \left\{ 2 \sum_{n=0}^{\frac{M}{2}-1} h(n) \cos \omega \left(n - \frac{M-1}{2}\right) \right\} \quad \text{... (4.4.18)}$$

Comparing above equation with the polar form of $H(\omega)$ of equation (4.4.15) we get,

Magnitude $$|H(\omega)| = 2 \sum_{n=0}^{\frac{M}{2}-1} h(n) \cos \omega \left(n - \frac{M-1}{2}\right) \quad \text{... (4.4.19)}$$

And phase $\angle H(\omega)$ is given as,

$$\angle H(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & \text{for } |H(\omega)| > 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi & \text{for } |H(\omega)| < 0 \end{cases} \quad \text{... (4.4.20)}$$

The above equation show that phase is piecewise linear. Similar results can be obtained for antisymmetric unit sample response.

For linear phase FIR filter, $$h(n) = \pm h(M-1-n) \quad \text{... (4.4.21)}$$

Linear phase is the most important feature of FIR filters. IIR filters cannot be designed with linear phase. The linear phase in FIR filters can be obtained if unit sample response satisfies equation (4.4.21). Many applications need linear phase filtering. For example the speech related applications require linear phase. Similarly in data transmission applications, linear phase prevents pulse dispersion and the detection becomes more accurate. Hence whenever linear phase is desired, FIR filtering is used.

4.4.3 Magnitude Characteristics and Order of FIR Filter

Fig. 4.4.1 shows the magnitude specifications from which FIR filter is to be designed.

The magnitude response given in Fig. 4.4.1 can be expressed mathematically as,

$$\left.\begin{array}{lll} 1-\delta_1 \le |H(\omega)| \le 1+\delta_1 & \text{for} & 0 \le \omega \le \omega_p \\ 0 \le |H(\omega)| \le \delta_2 & \text{for} & \omega_s \le \omega \le \pi \end{array}\right\} \quad \text{... (4.4.22)}$$

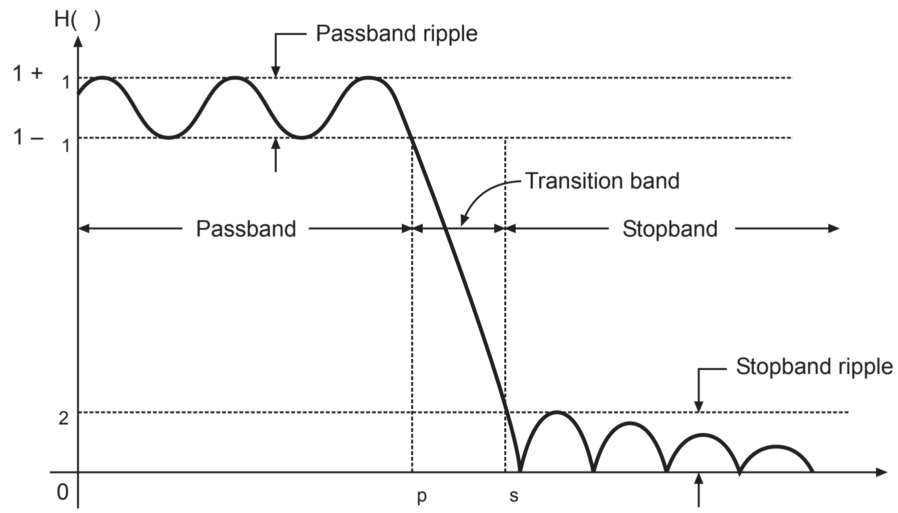


Fig. 4.4.1 Magnitude specifications used for FIR filter design

The approximate empirical formula for order N is given as,

$$N = \frac{-10\, log_{10}\,(\delta_1 \cdot \delta_2) - 15}{14\,\Delta f} \qquad \text{... (4.4.23)}$$

Here $\Delta f = \dfrac{\omega_s - \omega_p}{2\pi}$ is the transition band.

or $\Delta f = f_s - f_p$, where $\omega_s = 2\pi\, f_s$ and $\omega_p = 2\pi\, f_p$.

And length of the filter i.e., $M = N$.

There is another approximate formula used to calculate order of the FIR filter. It is given as follows :

$$M \text{ or } N = k\left(\frac{2\pi}{\omega_2 - \omega_1}\right) \qquad \text{... (4.4.24)}$$

Here value of k can be obtained from width of the main lobe. It is given as,

$$\text{Width of main lobe} = k\left(\frac{2\pi}{M}\right) \qquad \text{... (4.4.25)}$$

For the similar magnitude specifications the FIR filters have higher order than IIR filters. This is because FIR filters do not use feedback, hence they need long sequences for $h(n)$ (i.e. high order) to get sharp cutoff filters. Because of increased number of coefficients, FIR filters require large time for processing. This processing time can be reduced by using FFT algorithms.

Review Question

1. *What are the properties of FIR filters ? State their importance.*

4.5 Different Types of Windows

AU : May-10, 11, Dec.-16

Window functions are used for filter design. We will consider the time and frequency domain representation of various windows :

4.5.1 Rectangular Window

The rectangular window of length 'M' is given as,

$$w_R(n) = \begin{cases} 1 & \text{for} \quad n = 0, 1, \ldots M-1 \\ 0 & \text{otherwise} \end{cases} \quad \ldots(4.5.1)$$

Fig. 4.5.1 shows the sketch of this window.

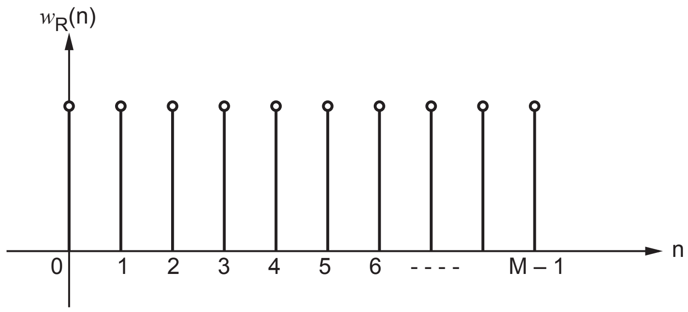


Fig. 4.5.1 Rectangular window

Now let us consider the fourier transform of rectangular window. It can be obtained as,

$$W_R(\omega) = \sum_{n=0}^{M-1} w_R(n)\, e^{-j\omega n} \quad \ldots (4.5.2)$$

Putting for $w_R(n)$ from equation (4.5.1) Then above equation becomes,

$$W_R(\omega) = \sum_{n=0}^{M-1} e^{-j\omega n} \quad \ldots (4.5.3)$$

Here let us use the standard series relation,

$$\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a} \quad \ldots (4.5.4)$$

With $a = e^{-j\omega}$ equation (4.5.3) becomes,

$$W_R(\omega) = \frac{1-e^{-j\omega M}}{1-e^{-j\omega}} \qquad \text{... (4.5.5)}$$

Let us rearrange above equation as,

$$W_R(\omega) = \frac{e^{-j\omega\frac{M}{2}} \cdot e^{j\omega\frac{M}{2}} - e^{-j\omega\frac{M}{2}} \cdot e^{-j\omega\frac{M}{2}}}{e^{-j\frac{\omega}{2}} \cdot e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} \cdot e^{-j\frac{\omega}{2}}}$$

$$= \frac{e^{-j\omega\frac{M}{2}}\left(e^{j\omega\frac{M}{2}} - e^{-j\omega\frac{M}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}$$

We know that $e^{j\theta} - e^{-j\theta} = 2\, sin\, \theta$. Hence above equation becomes,

$$W_R(\omega) = \frac{e^{-j\omega\frac{M}{2}} \cdot 2\, sin\left(\omega\frac{M}{2}\right)}{e^{-j\frac{\omega}{2}} \cdot 2\, sin\left(\frac{\omega}{2}\right)} = e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot \frac{sin\left(\frac{\omega M}{2}\right)}{sin\left(\frac{\omega}{2}\right)} \qquad \text{... (4.5.6)}$$

The polar form of $W_R(\omega)$ is given as $|W_R(\omega)|\, e^{j\angle W_R(\omega)}$. Comparing with above equation we get magnitude response of rectangular window as,

$$|W_R(\omega)| = \frac{\left|sin\left(\frac{\omega M}{2}\right)\right|}{\left|sin\left(\frac{\omega}{2}\right)\right|} \qquad \text{... (4.5.7)}$$

Fig. 4.5.2 shows the response of rectangular window given by above equation for $M = 50$.

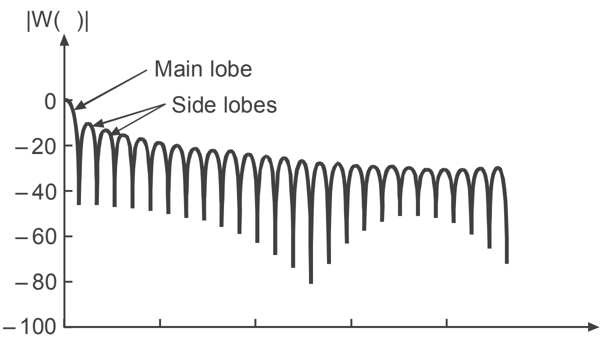


Fig. 4.5.2 Magnitude spectrum of rectangular window for *M* = 50

- In the magnitude spectrum of rectangular window given above, observe that there is one main lobe and many side lobes. As 'M' increases the main lobe becomes narrower.
- The area under the side lobes remain same irrespective of changes in 'M'.

4.5.2 Bartlett (Triangular) Window

Bartlett window is also called triangular window. It is expressed mathematically as,

$$w_T(n) = \begin{cases} 1-\dfrac{2\left|n-\dfrac{M-1}{2}\right|}{M-1} & \text{for} \quad n=0,1,\ldots M-1 \\ 0 & \text{otherwise} \end{cases} \qquad \ldots (4.5.8)$$

Fourier transform of this window can be obtained on same lines as discussed for rectangular window. Fig. 4.5.3 (a) shows the sketch of this window. Fig. 4.5.3 (b) shows the magnitude response of this window.

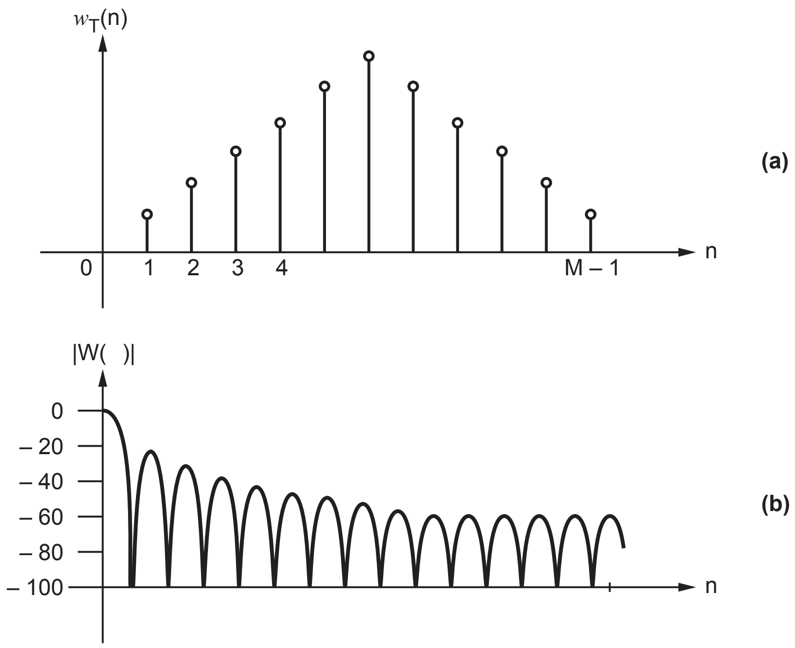


Fig. 4.5.3 Bartlett window (a) Time domain sketch (b) Magnitude response

4.5.3 Blackmann Window

The Blackmann window has a bell like shape of its time domain samples. It is expressed mathematically as,

$$w_B(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.8\cos\dfrac{4\pi n}{M-1} & for \qquad n = 0, 1, \ldots M-1 \\ 0 & otherwise \end{cases} \qquad \ldots (4.5.9)$$

Fig. 4.5.4 (a) shows the sketch of Blackmann window.

Fig. 4.5.4 (b) shows its magnitude response. Observe that the width of the main lobe is increased. But it has very small sidelobes.

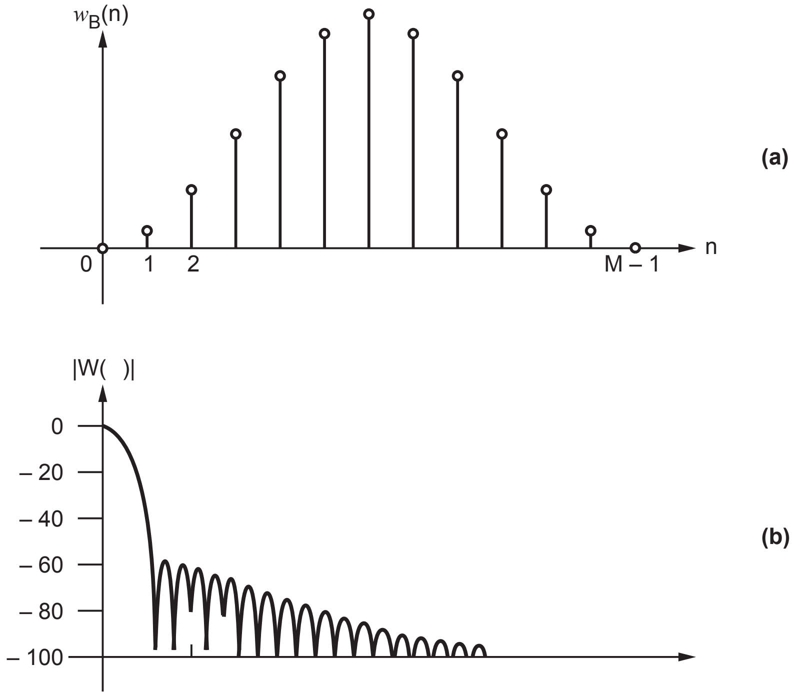


Fig. 4.5.4 Blackmann window (a) Time domain sketch
(b) Magnitude response

4.5.4 Hamming Window

Hamming window is most commonly used window in speech processing. It is given as,

$$w_H(n) = \begin{cases} 0.54 - 0.46 \ \cos\dfrac{2\pi n}{M-1} & for \qquad n = 0, 1, \ldots M-1 \\ 0 & otherwise \end{cases} \qquad \ldots (4.5.10)$$

This window also has bell like shape. Its first and last samples are not zero. Fig. 4.5.5 (a) shows the sketch of hamming window. Fig. 4.5.5 (b) shows the magnitude response of this window. It has reduced sidelobes but slightly increased main lobe. The sidelobes are higher than Blackmann window.

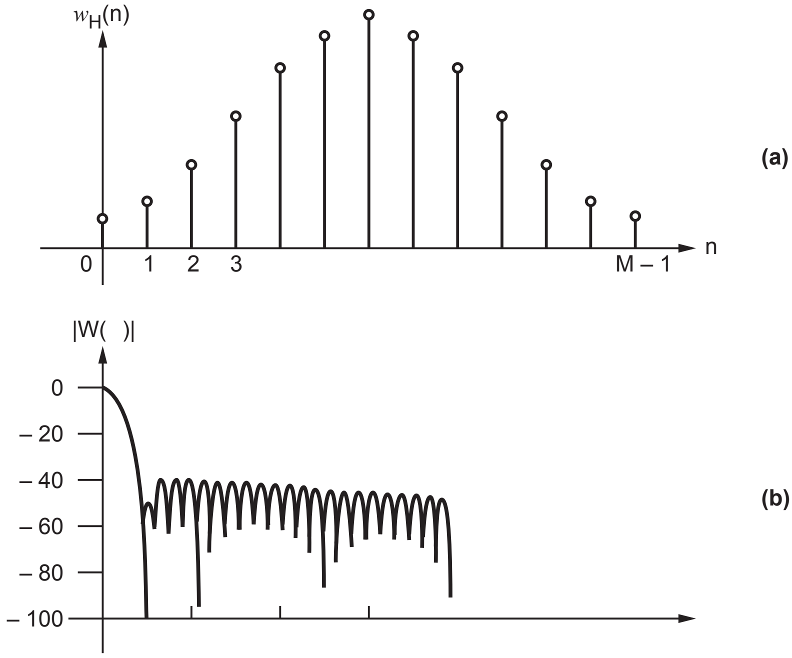


Fig. 4.5.5 Hamming window (a) Time domain sketch (b) Magnitude response

4.5.5 Hanning Window

Hanning window has shape similar to those of Blackmann and Hamming. Its first and last samples are zero. It is given as,

$$w_{HN}(n) = \begin{cases} \frac{1}{2}\left(1-\cos\frac{2\pi n}{M-1}\right) & \text{for} \qquad n = 0,1,\ldots M-1 \\ 0 & \text{otherwise} \end{cases} \qquad \ldots (4.5.11)$$

Fig. 4.5.6 (a) shows the sketch of this window. This window is commonly used for spectrum analysis, speech and music processing. Fig. 4.5.6(b) shows the magnitude response of Hanning window. Observe that it has narrow main lobe, but first few sidelobes are significant. Then sidelobe reduce rapidly. (See Fig. 4.5.6 on next page.)

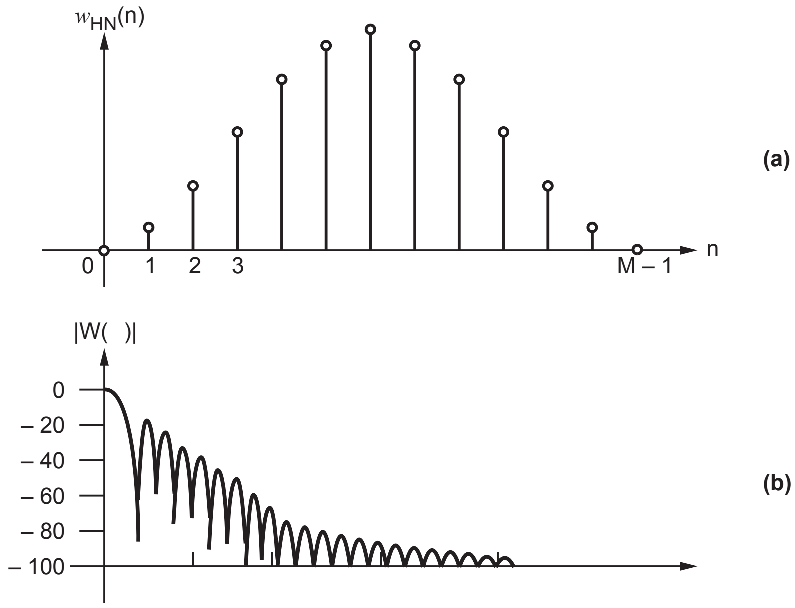


Fig. 4.5.6 : Hanning window (a) Time domain sketch (b) Magnitude response

4.5.6 Attenuation and Transition Width of Windows

Table 4.5.1 shows the attenuation and transition widths of FIR filters designed with various windows.

Window	Transition width of the main lobe	Minimum stopband attenuation	Relative amplitude of sidelobe	Passband ripple
Rectangular	$\frac{4\pi}{M+1}$	−21 dB	−13 dB	0.7416 dB
Bartlett	$\frac{8\pi}{M}$	−25 dB	−25 dB	0.1428 dB
Hanning	$\frac{8\pi}{M}$	−44 dB	− 31 dB	0.0546 dB
Hamming	$\frac{8\pi}{M}$	−53 dB	−41 dB	0.0194 dB
Blackmann	$\frac{12\pi}{M}$	−74 dB	−57 dB	0.0017 dB

Table 4.5.1 Attenuation and transition widths of commonly used windows

Characteristics of Kaiser window are not mentioned in above table. They vary according to values of β and M.

Review Questions

1. *Compare various windows for design of FIR filters.*
2. *Write a note on need and choice on windows.* **AU : May-11, Marks 4**
3. *Explain the characteristics of any four window functions used in FIR filter design.* **AU : May-10, Marks 8**
4. *Compare and explain on the choice and type of windows selection for signal analysis.* **AU : Dec.-16, Marks 6**

4.6 Design of Linear Phase FIR Filters using Windows

AU : May-07, 10, 11, 12, 14, 16, 17, Dec.-05, 08, 10, 11, 12, 13, 15, 16

Let us consider that the digital filter which is to be designed have the frequency response $H_d(\omega)$. This is also called desired frequency response. Let the corresponding unit sample response (desired) be $h_d(n)$. We know that $H_d(\omega)$ is Fourier transform of $h_d(n)$. i.e.,

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n)\, e^{-j\omega n} \qquad \text{... (4.6.1)}$$

And $h_d(n)$ can be obtained by taking inverse fourier transform of $H_d(\omega)$. i.e.,

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega \qquad \text{... (4.6.2)}$$

That is the desired unit sample response is obtained from desired frequency response by above equation.

- Generally the unit sample response obtained by equation 4.6.2 is infinite in duration.
- Since we are designing a finite impulse response filter, the length of $h_d(n)$ should be made finite.
- If we want the unit sample response of length 'M', then $h_d(n)$ is truncated to length 'M'. This is equivalent to multiplying $h_d(n)$ by a window sequence $\omega(n)$. This concept can be best explained by considering a particular type of window sequence. Here we consider rectangular window.

4.6.1 Rectangular Window for FIR Filter Design

The rectangular window is defined as,

$$w_R(n) = \begin{cases} 1 & \text{for} \quad n = 0, 1, \ldots, M-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{... (4.6.3)}$$

The unit sample response of the FIR filter is obtained by multiplying $h_d(n)$ by $w_R(n)$. i.e.,

$$h(n) = h_d(n)\, w_R(n) \quad \text{... (4.6.4)}$$

From equation (4.6.3), putting for $w_R(n)$ above equation becomes,

$$h(n) = \begin{cases} h_d(n) & \text{for} \quad n = 0, 1, \ldots, M-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{... (4.6.5)}$$

Thus $h_d(n)$ is truncated to length 'M' to give $h(n)$. This truncation is obtained by multiplication of $h_d(n)$ and $w_R(n)$. This is also called windowing since function $w_R(n)$ acts like a window for $f_d(n)$.

The fourier transform of rectangular window is given by equation (4.5.6). And its magnitude response is given by equation (4.5.7) as,

$$|W_R(\omega)| = \frac{\left| \sin\left(\frac{\omega M}{2}\right) \right|}{\left| \sin\left(\frac{\omega}{2}\right) \right|} \quad \text{... (4.6.6)}$$

Fig. 4.5.2 shows the magnitude frequency response given by above equation.

- From equation (4.6.4) we know that unit sample response of FIR filter is given as,

  $$h(n) = h_d(n)\, w(n)$$

  Here $w(n)$ represents generalized window function.
- The frequency response of FIR filter can be obtained by taking fourier transform of above equation. i.e.,

  $$H(\omega) = F.T. \left\{ h_d(n) \cdot w(n) \right\}$$
- We know that Fourier transform of multiplication of two signals is equal to convolution of their individual Fourier transforms. Then above equation becomes,

  $$H(\omega) = H_d(\omega) * W(\omega) \quad \text{... (4.6.7)}$$

This equation shows that the response of FIR filter is equal to convolution of desired frequency response with that of window function. Because of convolution, $H(\omega)$ has the smoothing effect. The sidelobes of $W(\omega)$ create undesirable ringing effects in $H(\omega)$.

4.6.2 Gibbs Phenomenon

Consider the example of lowpass filter having desired frequency response $H_d(\omega)$ as shown in Fig. 4.6.1 (a). This response has the cut-off frequency of ω_c.

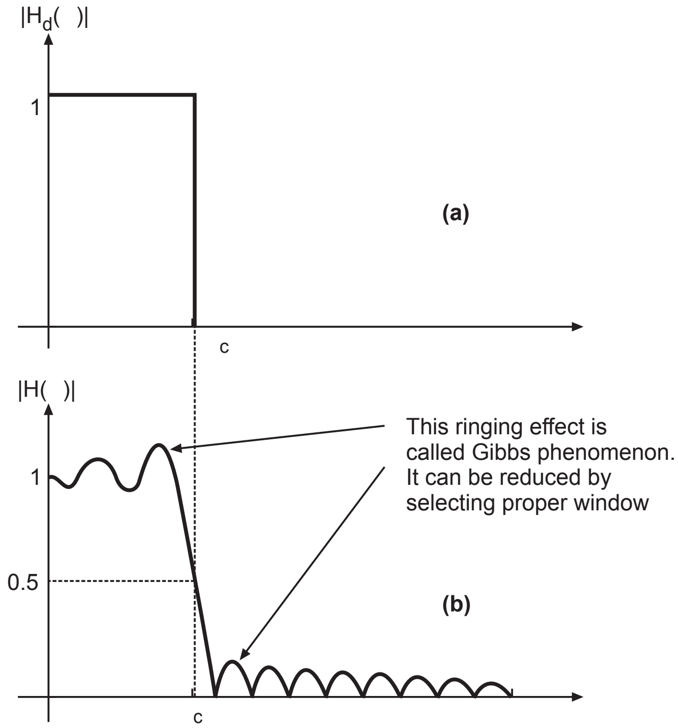


Fig. 4.6.1 (a) The desired frequency response $H_d(\omega)$ (b) The frequency response of FIR filter obtained by windowing. It has smoothing and ringing effect because of windowing

In the above figure observe that oscillations or ringing takes place near band edge (i.e. ω_c) of the filter. These oscillations or ringing is generated because of sidelobes in the frequency response $W(\omega)$ of the window function. This oscillatory behaviour (i.e. ringing effect) near the band edge of the filter is called *Gibbs phenomenon.*

- Thus the ringing effect takes place because of sidelobes in $W(\omega)$. These sidelobes are generated because of abrupt discontinuity (in case of rectangular window) of the window function.
- In case of rectangular window the sidelobes are larger in size since the discontinuity is abrupt. Hence ringing effect is maximum in rectangular window.
- Hence different window functions are developed which contain taper and decays gradually toward zero. This reduces sidelobes and hence ringing effect in $H(\omega)$.

4.6.3 Design using Other Window Functions

Different types of window functions are available which reduce ringing effect. The particular window is selected depending upon the application. The characteristics of these windows are discussed in last section.

Fig. 4.6.2 shows the frequency response of lowpass filter with cut-off frequency ω_c designed using various types of windows. The magnitude response is shown in dB scale.

- In the figure observe that FIR filter designed using hamming window (Fig. 4.6.2 (b)) has reduced sidelobes compared to rectangular window.
- Blackmann window (Fig. 4.6.2 (c)) has smallest sidelobes but width of main lobe is increased.
- Thus there should be compromise between attenuation of sidelobes and width of main lobe. Kaiser window provides this flexibility to the designer. Hence it is more commonly used window for FIR filter design.

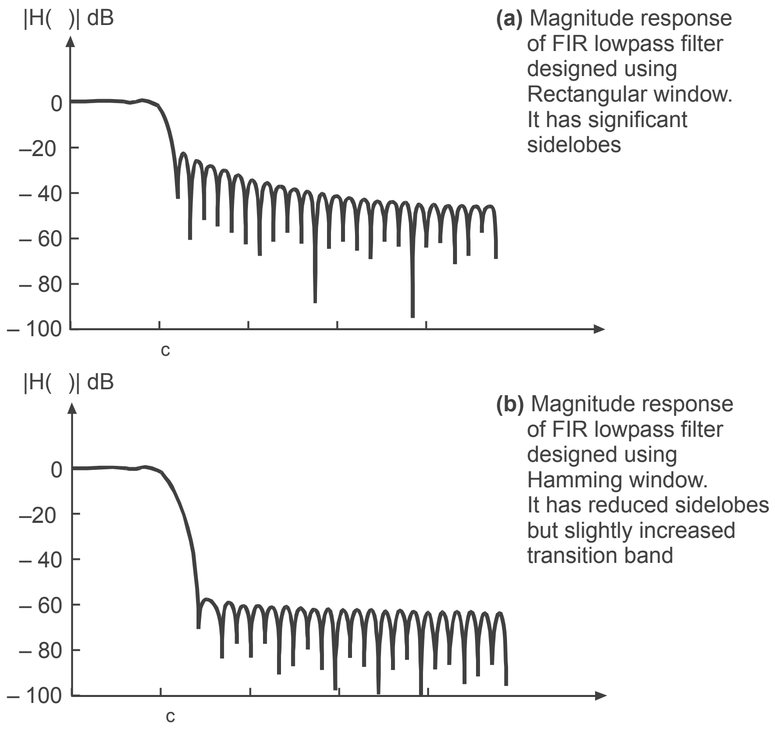


Fig. 4.6.2 Magnitude response of lowpass FIR filter designed by various windows

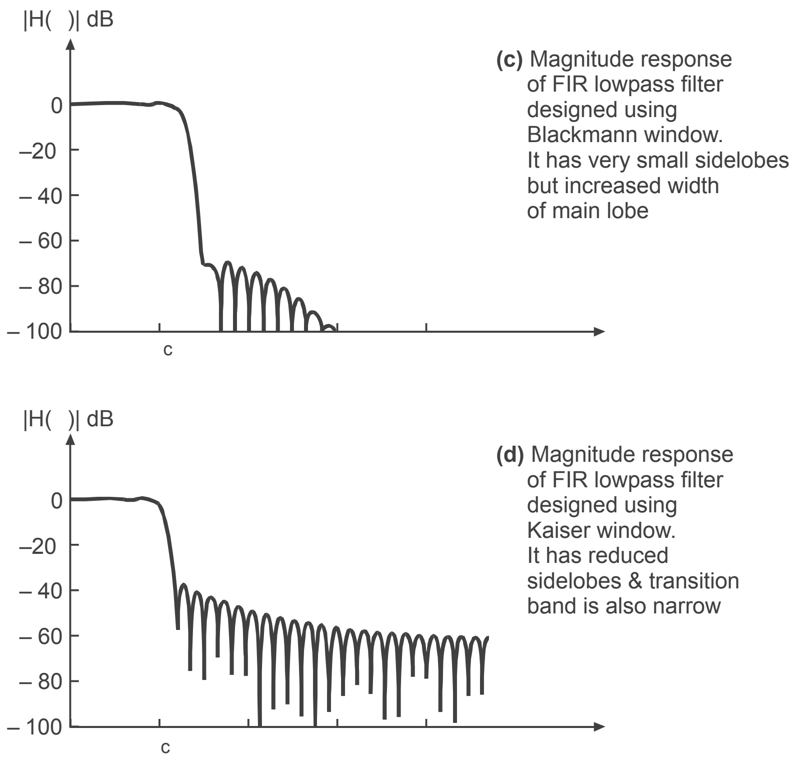


Fig. 4.6.2 Magnitude response of lowpass FIR filter designed by various windows

Example 4.6.1 *Using hanning window technique design a LPF with a passband gain of unity cutoff frequency 1000 Hz and working sampling frequency of 5 kHz. The length of the filter should be 7.* **AU : May-14, Marks 16**

Solution : Given data :

F_c = 1000 Hz, F_{SF} = 5000 Hz, M = 7

Window : Hanning.

Step 1 : Specifications of equivalent digital filter

$$f_c = \frac{F_c}{F_{SF}} = \frac{1000}{5000} = 0.2 \text{ cycles/sample}$$

$$\therefore \qquad \omega_c = 2\pi f_c = 2\pi \times 0.2 = 0.4\pi \text{ rad/sample}$$

Step 2 : Desired frequency response of an ideal filter

An ideal LPF magnitude and frequency response is given as

Here $|H_d(e^{j\omega})| = |H_d(\omega)| = \begin{cases} 1 & \text{for } |\omega| < \omega_c \\ 0 & \text{for } \omega_c < |\omega| < \pi \end{cases}$

And $\angle H_d(e^{j\omega}) = \angle H_d(\omega) = -\omega\tau$

Fig. 4.6.3 shows above response. The system function of such ideal LPF can be expressed as,

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & \text{for } |\omega| < \omega_c \\ 0 & \text{for } \omega_c < |\omega| < \pi \end{cases} \quad \dots (4.6.8)$$

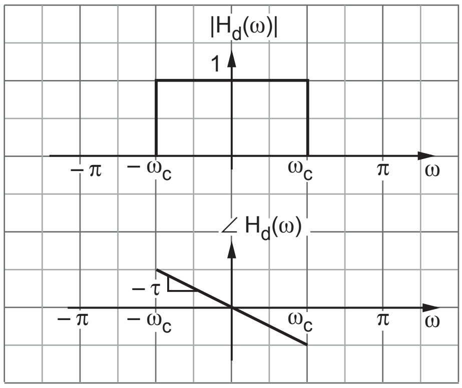


Fig. 4.6.3 An ideal LPF

Step 3 : To obtain $h_d(n)$ by inverse DTFT

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega \qquad \text{by inverse DTFT}$$

Putting from equation (4.6.8),

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\tau)}\, d\omega \quad \dots (4.6.9)$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_c}^{\omega_c} = \frac{\sin[\omega_c\,(n-\tau)]}{\pi\,(n-\tau)} \quad \text{for } n \neq \tau \quad \dots (4.6.10)$$

For $n = \tau$ equation (4.6.9) will be,

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi}$$

From equation (4.6.10) and above equation,

$$h_d(n) = \begin{cases} \dfrac{\sin\omega_c\,(n-\tau)}{\pi\,(n-\tau)} & \text{for } n \neq \tau \\ \dfrac{\omega_c}{\pi} & \text{for } n = \tau \end{cases} \quad \dots (4.6.11)$$

Step 4 : To obtain value of τ

The FIR filter being designed is symmetric,

$$\therefore \quad h(n) = h(M-1-n)$$

$$\text{Since} \quad h(n) = h_d(n) \cdot w(n),$$

$$h_d(n) \cdot w(n) = h_d(M-1-n) \cdot w(n)$$

$$\therefore \quad h_d(n) = h_d(M-1-n)$$

$$\therefore \quad \frac{\sin \omega_c (n-\tau)}{\pi (n-\tau)} = \frac{\sin \omega_c (M-1-n-\tau)}{\pi (M-1-n-\tau)} \qquad \text{By equation (4.6.11)}$$

Above equation is satisfied if,

$$-(n-\tau) = (M-1-n-\tau)$$

$$\text{or} \quad \tau = \frac{M-1}{2} \qquad \ldots (4.6.12)$$

Step 5 : To obtain value of $h_d(n)$

For M = 7, equation (4.6.12) will be,

$$\tau = \frac{7-1}{2} = 3$$

Hence for $\tau = 3$ and $\omega_c = 0.4\pi$ equation (4.6.11) will be,

$$h_d(n) = \begin{cases} \dfrac{\sin 0.4\,\pi (n-3)}{\pi (n-3)} & \text{for } n \neq 3 \\ \dfrac{0.4\pi}{\pi} = 0.4 & \text{for } n = 3 \end{cases}$$

$h_d(n)$ is obtained for n = 0, 1, 2, 3, 4, 5, 6 from above equation.

n	0	1	2	3	4	5	6
$h_d(n)$	–0.0623	0.0935	0.3027	0.4	0.3027	0.0935	–0.0623

Step 6 : To obtain values of window function

Hanning window is given as,

$$w(n) = \frac{1}{2}\left(1-\cos\frac{2\pi n}{M-1}\right)$$

$$= \frac{1}{2}\left(1-\cos\frac{2\pi n}{6}\right) \text{ for M = 7}$$

$$= \frac{1}{2}\left(1-\cos\frac{\pi n}{3}\right)$$

$w(n)$ is calculated for n = 0, 1, 2, 3, 4, 5, 6 from above equation.

n	0	1	2	3	4	5	6
$\omega(n)$	0	1/4	3/4	1	3/4	1/4	0

Step 7 : To obtain *h*(*n*) by windowing

$$h(n) = h_d(n)\cdot w(n)$$

Values are calculated as follows :

n	0	1	2	3	4	5	6
$h_d(n)$	–0.0623	0.0935	0.3027	0.4	0.3027	0.0935	–0.0623
$w(n)$	0	1/4	3/4	1	3/4	1/4	0
$h(n)$	0	0.0234	0.227	0.4	0.227	0.0234	0

In the above table, observe that,

$h(0) = h(6)$, $h(1) = h(5)$, $h(2) = h(4)$. This means,

$$h(n) = h(6-n) \quad \text{or} \quad h(n) = h(M-1-n)$$

Above condition is satisfied. It indicates a linear phase filter.

Example 4.6.2 *Design a length-5 FIR band reject filter with a lower cut-off frequency of 2 kHz, an upper cut-off frequency of 2.4 kHz, and a sampling rate 8000 Hz using hamming window.* **AU : Dec.-12, Marks 8**

Solution : Here $M = 5$, $\omega_{c_1} = \frac{2000}{8000} = 0.25$ cycles/sample

$$\omega_{c_2} = \frac{2400}{8000} = 0.3 \text{ cycles/sample}$$

i) To obtain described impulse response

Fig. 4.6.4 shows the magnitude response of band reject filter.

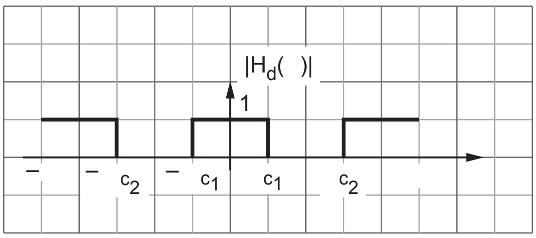


Fig. 4.6.4 Band reject filter

$$\therefore \quad H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau} & \text{for } 0 \le |\omega| < \omega_{c1} \text{ and } \omega_{c2} \le |\omega| \le \pi \\ 0 & \text{otherwise.} \end{cases}$$

Hence impulse response of above type of filter will be,

$$h_d(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{-\omega_{c2}} e^{-j\omega\tau}\cdot e^{j\omega n}\,d\omega + \int_{-\omega_{c1}}^{\omega_{c1}} e^{-j\omega\tau}\cdot e^{j\omega n}\,d\omega + \int_{\omega_{c1}}^{\pi} e^{-j\omega\tau}\cdot e^{j\omega n}\,d\omega\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\omega_{c2}} e^{j(n-\tau)\omega}\,d\omega + \int_{-\omega_{c1}}^{\omega_{c1}} e^{j(n-\tau)\omega}\,d\omega + \int_{\omega_{c1}}^{\pi} e^{j(n-\tau)\omega}\,d\omega\right] \quad \text{... (4.6.13)}$$

$$= \frac{\sin\omega_{c1}(n-\tau) - \sin\omega_{c2}(n-\tau) + \sin\pi(n-\tau)}{\pi(n-\tau)} \quad \text{for } n \ne \tau$$

with $n = \tau$ in equation (4.6.13),

$$h_d(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{-\omega_{c2}} d\omega + \int_{-\omega_{c1}}^{\omega_{c1}} d\omega + \int_{\omega_{c1}}^{\pi} d\omega\right] = \frac{\pi - \omega_{c2} + \omega_{c1}}{\pi} \quad \text{for } n = \tau$$

$$\text{Thus,} \quad h_d(n) = \begin{cases} \dfrac{\sin\omega_{c1}(n-\tau) - \sin\omega_{c2}(n-\tau) + \sin\pi(n-\tau)}{\pi(n-\tau)} & \text{for } n \ne \tau \\ \dfrac{\pi - \omega_{c2} + \omega_{c1}}{\pi} & \text{for } n = \tau \end{cases} \quad \text{... (4.6.14)}$$

Here $\tau = \frac{M-1}{2} = \frac{5-1}{2} = 2$. Putting for ω_{c1} and ω_{c2} in above equation,

$$h_d(n) = \begin{cases} \dfrac{\sin 0.25(n-2) - \sin 0.3(n-2) + \sin\pi(n-2)}{\pi(n-2)} & \text{for } n \ne 2 \\ \dfrac{\pi - 0.3 + 0.25}{\pi} = 0.984 & \text{for } n = 2 \end{cases} \quad \text{... (4.6.15)}$$

ii) To obtain $h_d(n)$, $w_H(n)$ and $h(n)$

Hamming window is given as,

$$w_H(n) = 0.54 - 0.46\cos\frac{2\pi n}{M-1} \quad \text{for } n = 0,1,\ldots.M-1$$

$$w_H(n) = 0.54 - 0.46\cos\frac{2\pi n}{4} \quad \text{with } M = 5$$

$$\therefore \quad w_H(n) = 0.54 - 0.46\cos\frac{2\pi n}{2} \quad \text{... (4.6.16)}$$

Following table shows the calculations for $h_d(n)$ of equation (4.6.15), $w_H(n)$ of equation (4.6.16) and $h(n) = h_d(n)\cdot w_H(n)$.

n	$h_d(n)$	$w_H(n)$	$h(n)$
0 4	– 0.0135	0.08	– 0.00108
1 3	– 0.0153	0.54	– 0.00826
2	0.984	1	0.984

Example 4.6.3 *Design a low pass filter using rectangular window by taking 9 samples of w (n) with cut-off sequence of 1.2 radians/sec also draw the filter.* **AU : Dec.-10, Marks 16**

Solution : Design of the filter

For linear phase FIR filter, the desired frequency response is given as,

$$H_d(\omega) = \begin{cases} 1 \cdot e^{-j\omega\left(\frac{M-1}{2}\right)} & \text{for } |\omega| < \omega_c \\ 0 & \text{elsewhere} \end{cases}$$

The impulse response of the desired filter can be obtained by,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega \qquad \text{by inverse fourier transform}$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\left(\frac{M-1}{2}\right)} e^{j\omega n}\, d\omega$$

$$= \begin{cases} \dfrac{\sin \omega_c \left(n - \frac{M-1}{2}\right)}{\pi \left(n - \frac{M-1}{2}\right)} & \text{for } n \neq \dfrac{M-1}{2} \\ \dfrac{\omega_c}{\pi} & \text{for } n = \dfrac{M-1}{2} \end{cases}$$

Putting for ω_c = 1.2 and M = 9 (since window length is 9),

$$= \begin{cases} \dfrac{\sin 1.2(n-4)}{\pi\,(n-4)} & \text{for } n \neq 4 \\ \dfrac{1.2}{\pi} \text{ or } 0.382 & \text{for } n = 4 \end{cases} \qquad \ldots (4.6.17)$$

Since the rectangular window is selected,

$$h(n) = h_d(n) \text{ for n = 0, 1, … 8}$$

The values of $h(n) = h_d(n)$ are calculated as follows by putting n = 0, 1, … 8 in equation (4.6.17).

$n = 0, \quad h(0) = h_d(0) = -0.079$

$n = 1, \quad h(1) = h_d(1) = -0.047$

$n = 2, \quad h(2) = h_d(2) = 0.107$

$n = 3, \quad h(3) = h_d(3) = 0.296$

$n = 4, \quad h(4) = h_d(4) = 0.382$

$n = 5, \quad h(5) = h_d(5) = 0.296$

$n = 6, \quad h(6) = h_d(6) = 0.107$

$n = 7, \quad h(7) = h_d(7) = -0.047$

$n = 8, \quad h(8) = h_d(8) = -0.079$

This is the required impulse response of the filter.

Realization using direct form

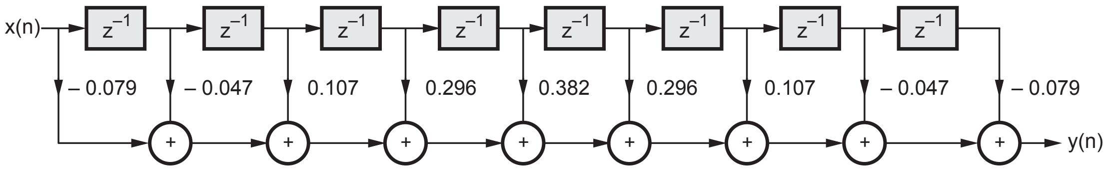


Fig. 4.6.5 Realization in direct form

Example 4.6.4 *Design the bandpass linear phase FIR filter having cutoff frequencies of $\omega_{c1} = 1$ rad/sample and $\omega_{c2} = 2$ rad / sample. Obtain the unit sample response through following window :*

$$w(n) = \begin{cases} 1 & \text{for} \quad 0 \le n \le 6 \\ 0 & \text{otherwise} \end{cases}$$

Also obtain the magnitude / frequency response. **AU : Dec.-08, Marks 16**

Solution : i)Given data : It is required to design a bandpass digital FIR filter having cutoff frequencies ω_{c1} and ω_{c2}. The desired magnitude function $H_d(\omega)$ can be written as,

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & \text{for} \quad \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0 & \text{otherwise} \end{cases}$$

Here $\tau = \dfrac{M-1}{2}$ for a linear phase FIR filter.

ii) To obtain $h_d(n)$:

The desired unit sample response $h_d(n)$ can be obtained from $H_d(\omega)$ by taking inverse fourier transform. i.e.,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\tau}\, e^{j\omega n}\, d\omega + \int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\tau}\, e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega(n-\tau)}\, d\omega + \int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega(n-\tau)}\, d\omega\right] \quad \text{... (4.6.18)}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{e^{j\omega(n-\tau)}}{n-\tau}\right]_{-\omega_{c2}}^{-\omega_{c1}} + \left[\frac{e^{j\omega(n-\tau)}}{n-\tau}\right]_{\omega_{c1}}^{\omega_{c2}}\right\}$$

$$= \frac{\sin\,\omega_{c2}\,(n-\tau) - \sin\,\omega_{c1}\,(n-\tau)}{\pi\,(n-\tau)} \quad \text{... (4.6.19)}$$

The above result stands for $n \neq \tau$. When $n = \tau$ in equation (4.6.18) we get,

$$h_d(n) = \frac{1}{2\pi}\left[\int_{-\omega_{c2}}^{-\omega_{c1}} d\omega + \int_{\omega_{c1}}^{\omega_{c2}} d\omega\right]$$

$$= \frac{\omega_{c2} - \omega_{c1}}{\pi}$$

Thus we obtained the desired unit sample response of the bandpass filter as,

$$h_d(n) = \begin{cases} \dfrac{\sin\,\omega_{c2}\,(n-\tau) - \sin\,\omega_{c1}\,(n-\tau)}{\pi\,(n-\tau)} & \text{for } n \neq \tau \\[2ex] \dfrac{\omega_{c2} - \omega_{c1}}{\pi} & \text{for } n = \tau \end{cases} \quad \text{... (4.6.20)}$$

iii) Window function and length 'M' of the filter :

The window function is given in this example as,

$$w(n) = \begin{cases} 1 & \text{for } 0 \le n \le 6 \\ 0 & \textit{elsewhere} \end{cases} \quad \text{... (4.6.21)}$$

From above equation it is clear that the given window is rectangular. Values of 'n' varies from 0 to 6, this means,

$$0 \le n \le M-1 \quad \text{i.e.} \quad 0 \le n \le 6$$

$$\therefore \quad M - 1 = 6 \quad \text{or} \quad M = 7$$

Thus length of the filter is 7.

iv) To obtain $h(n)$:

The unit sample response is given by windowing technique as,

$$h(n) = h_d(n) \cdot w(n)$$

From equation (4.6.20) and equation (4.6.21), above equation becomes,

$$h(n) = \begin{cases} \dfrac{\sin \omega_{c2}(n-\tau) - \sin \omega_{c1}(n-\tau)}{\pi(n-\tau)} & \text{for } n \neq \tau \\ \dfrac{\omega_{c2}-\omega_{c1}}{\pi} & \text{for } n = \tau \end{cases} \quad \ldots \text{ for } 0 \leq n \leq 6$$

For linear phase FIR filter, $\tau = \frac{M-1}{2} = 3$, and putting for $\omega_{c1} = 1$ rad/sample and $\omega_{c2} = 2$ rad/sample, above equation becomes,

$$h(n) = \begin{cases} \dfrac{\sin 2(n-3) - \sin(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ \dfrac{1}{\pi} & \text{for } n = 3 \end{cases} \quad \ldots (4.6.22)$$

$$\ldots \text{ for } 0 \leq n \leq 6$$

$$h(0) = h(6) = -0.04462$$
$$h(1) = h(5) = -0.26517$$
$$h(2) = h(4) = 0.02159$$
$$h(3) = 0.31831$$

v) To obtain magnitude response :

Here length of the unit sample response is $M = 7$, i.e. odd. Magnitude response is given by equation (4.6.16) for odd M i.e.,

$$|H(\omega)| = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n) \cos \omega \left(n - \frac{M-1}{2}\right)$$

For $M = 7$, $$|H(\omega)| = h(3) + 2\sum_{n=0}^{2} h(n) \cos \omega(n-3)$$

$$= h(3) + 2\left[h(0) \cos \omega(-3) + h(1) \cos \omega(1-3) + h(2) \cos \omega(2-3)\right]$$
$$= h(3) + 2\left[h(0) \cos(3\omega) + h(1) \cos(2\omega) + h(2) \cos(\omega)\right]$$

Putting values in above equation,

$$|H(\omega)| = 0.31831 + 2\left[-0.04462 \cos(3\omega) - 0.26517 \cos(2\omega) + 0.02159 \cos(\omega)\right]$$

$$= 0.31831 - 0.08924\ cos\ (3\omega) - 0.53034\ cos\ (2\omega) + 0.04318\ cos\ (\omega)$$

The frequency response can be obtained by putting $-\omega \le \omega \le \pi$ in the above equation.

Example 4.6.5 *Design an ideal high pass filter with* $H_d(e^{j\omega}) = \begin{cases} 1 & \pi/4 \le |\omega| < \pi \\ 0 & |\omega| \le \pi/4 \end{cases}$ *using hamming window with N = 11.* **AU : May-12, Marks 16**

Solution : i) To obtain $h_d(n)$: High pass filter has following frequency response as shown in Fig. 4.6.6.

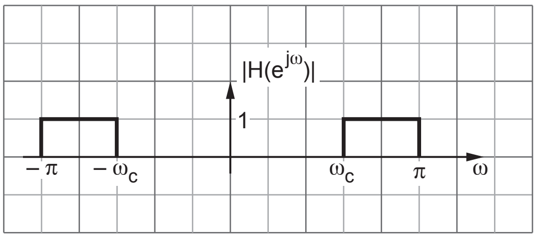


Fig. 4.6.6 High pass filter

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau} & for\ \omega_c \le |\omega| \le \pi \\ 0 & otherwise. \end{cases}$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega \quad \text{By inverse Fourier transform.}$$

$$= \frac{1}{2\pi} \left\{ \int_{-\pi}^{-\omega_c} e^{-j\omega\tau}\, e^{j\omega n}\, d\omega + \int_{\omega_c}^{\pi} e^{-j\omega\tau}\, e^{j\omega n}\, d\omega \right.$$

$$= \frac{1}{2\pi} \left\{ \int_{-\pi}^{-\omega_c} e^{j\omega(n-\tau)} d\omega + \int_{\omega_c}^{\pi} e^{j\omega(n-\tau)} d\omega \right.$$

$$= \begin{cases} \dfrac{\sin \pi\ (n-\tau) - \sin \omega_c\ (n-\tau)}{\pi(n-\tau)} & for\ n \ne \tau \\ 1 - \dfrac{\omega_c}{\pi} & for\ n = \tau \end{cases}$$

Here $M = N = 11$. Hence $\tau = \dfrac{M-1}{2} = \dfrac{11-1}{2} = 5$.

$$\therefore \quad h_d(n) = \begin{cases} \dfrac{\sin \pi\ (n-5) - \sin \dfrac{\pi}{4}(n-5)}{\pi(n-5)} & , \quad n \ne 5 \\ 1 - \dfrac{\pi/4}{\pi} = 0.75 & \quad n = 5 \end{cases}$$

ii) To obtain $h_d(n)$ $w_H(n)$ and $h(n)$

Hamming window is given as,

$$w_H(n) = 0.54 - 0.46 \cos\frac{2\pi n}{M-1} \quad \text{for } n = 0, 1, .. .M-1$$

$$= 0.54 - 0.46 \cos\frac{2\pi n}{10} \quad \text{for } n = 0, 1,...9$$

$$= 0.54 - 0.46 \cos\frac{\pi n}{5}$$

Following table shown the calculations of $h_d(n), w_H(n)$ and $h(n) = h_d(n) \cdot w_H(n)$

n		$h_d(n)$	$w_H(n)$	$h(n)$
0	10	0.045	0.08	0.0036
1	9	0	0.167	0
2	8	– 0.075	0.397	– 0.0297
3	7	– 0.159	0.682	– 0.108
4	6	– 0.225	0.912	– 0.205
	5	0.75	1	0.75

Examples with Solution

Example 4.6.6 *Design a FIR filter of order 7 using triangular window.*

$$H(j\omega) = \begin{cases} 1 & \text{for} \quad |\omega| < \frac{\pi}{2} \\ 0 & \text{for} \quad \frac{\pi}{2} < |\omega| < \pi \end{cases}$$

AU : Dec.-05, Marks 16

Solution : For the filter to be linear phase,

$$H_d(\omega) = \begin{cases} 1 . e^{-j\omega\left(\frac{M-1}{2}\right)} & \text{for } |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega\left(\frac{M-1}{2}\right)} e^{j\omega n}\, d\omega$$

$$= \begin{cases} \dfrac{\sin \omega_c \left(n - \dfrac{M-1}{2} \right)}{\pi \left(n - \dfrac{M-1}{2} \right)} & for \quad n \neq \dfrac{M-1}{2} \\ \dfrac{\omega_c}{\pi} & for \quad n = \dfrac{M-1}{2} \end{cases}$$

Putting the value of M = 7 and $\omega_c = \dfrac{\pi}{2}$

$$h_d(n) = \begin{cases} \dfrac{\sin \dfrac{\pi}{2}(n-3)}{\pi\,(n-3)} & for \quad n \neq 3 \\ \dfrac{1}{2} & for \quad n = 3 \end{cases}$$

The equation for triangular window is given as,

$$w(n) = \begin{cases} 1 - \dfrac{2\left| n - \dfrac{M-1}{2} \right|}{M-1} & for \quad n = 0, 1, \ldots M-1 \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} 1 - \dfrac{|n-3|}{2} & for \quad n = 0, 1, \ldots 6 \\ 0 & otherwise \end{cases}$$

Following table lists the values of $h_d(n)$, $w(n)$ and $h(n)$

n	$h_d(n) = \dfrac{\sin \dfrac{\pi}{3}(n-3)}{\pi(n-3)}$	$w(n) = \quad - \dfrac{n-3}{\quad}$	$h(n) = h_d(n)\ w(n)$
0	0	– 0.5	0
1	0.1378	0	0
2	0.275	0.5	0.1378
3	0.5	1	0.5
4	0.275	0.5	0.1378
5	0.1378	0	0
6	0	– 0.5	0

Table 4.6.1 Calculation of FIR filter coefficients

Example 4.6.7 *Design a filter with*

$$H_d(e^{-j\omega}) = e^{-j3\omega} \qquad -\frac{\pi}{4} \le \omega \le \frac{\pi}{4}$$
$$= 0 \qquad \frac{\pi}{4} < |\omega| \le \pi$$

using a Hanning window with N = 7. **AU : May-07, Marks 16, May-11, Marks 12**

Solution :

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & \text{for } -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0 & \text{for } \frac{\pi}{4} \le \omega \le \pi \end{cases}$$

Step 1 : To obtain $h_d(n)$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j3\omega}\cdot e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\omega(n-3)} d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-3)}}{j(n-3)}\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{\pi(n-3)} \frac{e^{j\frac{\pi}{4}(n-3)} - e^{-j\frac{\pi}{4}(n-3)}}{j2}$$

$$= \frac{\sin\frac{\pi}{4}(n-3)}{\pi(n-3)} \quad \text{for n} \ne 3$$

For n = 3, $h_d(n) = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\omega(3-3)d\omega}$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} d\omega = \frac{1}{4}$$

$$\text{Thus, } h_d(n) = \begin{cases} \dfrac{\sin\frac{\pi}{4}(n-3)}{\pi\,(n-3)} & \text{for } n \ne 3 \\ \dfrac{1}{4} & \text{for } n = 3 \end{cases}$$

Step 2 : To apply windowing to $h_d(n)$ to get $h(n)$

The Hanning window is given as,

$$W_{Hann}(n) = \frac{1}{2}\left(1-\cos\frac{2\pi n}{M-1}\right) \text{ for n = 0, 1, 2, ... M – 1}$$

with M = 7, $W_{Hann}(n) = \frac{1}{2}\left(1-\cos\frac{\pi n}{3}\right)$

Following table shows $h_d(n)$, $W_{Hann}(n)$ and $h(n) = h_d(n) \cdot W_{Hann}(n)$.

n	$h_d(n) = \begin{cases} \dfrac{\sin\frac{\pi}{4}(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ \dfrac{-}{4} & \text{for } n = 3 \end{cases}$	$W_{Hann}(n) = -\left(-\cos\frac{\pi n}{3}\right)$	$h(n) = h_d(n) \cdot W_{Hann}(n)$
0	0.075	0	0
1	0.159	1/4	0.03975
2	0.225	3/4	0.16875
3	0.25	1	0.25
4	0.225	3/4	0.16875
5	0.159	1/4	0.159
6	0.075	0	0

Example 4.6.8 *The desired response of a low pass filter is*

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & -3\pi/4 < \omega \leq 3\pi/4 \\ 0 & 3\pi/4 < |\omega| \leq \pi \end{cases}$$

Determine $H(e^{j\omega})$ *for M = 7 using a Hamming window.*

OR *Design above filter using Hanning window.*

AU : May-10, Dec.-11, 13, 15 Marks 16

Solution : i) Given data

$$M = 7$$

$$\omega_c = \frac{3\pi}{4}$$

$$\alpha = 3 \quad \text{i.e.} \quad \frac{M-1}{2} = \frac{7-1}{2} = 3$$

ii) To obtain desired unit sample response

The desired unit sample for ideal lowpass filter is given as,

$$h_d(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left(n-\dfrac{M-1}{2}\right)\right]}{\pi\left(n-\dfrac{M-1}{2}\right)} & \text{for } n \neq \dfrac{M-1}{2} \\ \dfrac{\omega_c}{\pi} & \text{for } n = \dfrac{M-1}{2} \end{cases}$$

$$= \begin{cases} \dfrac{\sin\left[\dfrac{3\pi}{4}(n-3)\right]}{\pi(n-3)} & \text{for } n \neq 3 \\ \dfrac{3\pi/4}{\pi} & \text{for } n = 3 \end{cases}$$

$$= \begin{cases} \dfrac{\sin[2.356(n-3)]}{\pi(n-3)} & \text{for } n \neq 3 \\ 0.75 & \text{for } n = 3 \end{cases}$$

iii) To obtain window coefficients and samples of unit sample response

Here Hamming window is to be used. It is given as,

$$w_H(n) = \begin{cases} 0.54 - 0.46 \cos \dfrac{2\pi n}{M-1} & \text{for } n = 0, 1, \ldots.. M-1 \\ 0 & \text{otherwise} \end{cases}$$

For $M = 7$, $$w_H(n) = 0.54 - 0.46 \cos \frac{2\pi n}{6} \quad \text{for} \quad n = 0, 1, 2, \ldots 6$$

$$= 0.54 - 0.46 \cos \frac{\pi n}{3} \quad \text{for} \quad n = 0, 1, 2, \ldots 6$$

The samples of unit sample response are given as,

$$h(n) = h_d(n)\, w_H(n) \qquad \text{for} \quad n = 0, 1, 2, \ldots 6.$$

Following table shows $h_d(n)$, $w_H(n)$ and $h(n)$ as per above equations.

n	$h_d = \dfrac{\sin\ 2.356\ (n-3)}{\pi(n-3)}$ **= 0.75 for n = 3**	$w_H(n) = 0.54 - 0.46 \cos\dfrac{n\pi}{3}$	**h(n)** $= h_d(n) \cdot w_H(n)$
0	0.075	0.08	0.006
1	– 0.156	0.31	– 0.0483
2	0.225	0.77	0.1732
3	0.75	1	0.75

4	0.225	0.77	0.1732
5	– 0.156	0.31	– 0.0483
6	0.075	0.08	0.006

Thus,

$$h(0) = h(6) = 0.006$$

$$h(1) = h(5) = -0.0483$$

$$h(2) = h(4) = 0.1732$$

$$h(3) = 0.75$$

iv) To obtain magnitude and phase response

For odd value of 'M' magnitude response is given as,

$$|H(\omega)| = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega\left(n-\frac{M-1}{2}\right)$$

$$= h(3) + 2\sum_{n=0}^{2} h(n)\cos\omega(n-3)$$

$$= h(3) + 2\{h(0)\cos 3\omega + h(1)\cos 2\omega + h(2)\cos\omega\}$$

$$= 0.75 + 2\{0.006\cos 3\omega - 0.0483\cos 2\omega + 0.1732\cos\omega\}$$

This is the magnitude response.

Phase response is given as,

$$\angle H(\omega) = \begin{cases} -\omega\left(\dfrac{M-1}{2}\right) & \text{for } |H(\omega)| > 0 \\ -\omega\left(\dfrac{M-1}{2}\right) + \pi & \text{for } |H(\omega)| < 0 \end{cases}$$

$$= \begin{cases} -3\omega & \text{for } |H(\omega)| > 0 \\ -3\omega + \pi & \text{for } |H(\omega)| < 0 \end{cases}$$

V) To obtain window coefficients and samples of unit sample response

Hanning window is given as,

$$W_{Hann}(n) = \frac{1}{2}\left(1 - \cos\frac{2\pi n}{M-1}\right) \text{ for n = 0, 1, 2, … M – 1}$$

with $\quad M = 7, W_{Hann}(n) = \frac{1}{2}\left(1-\cos\frac{\pi n}{3}\right)$ for n = 0, 1, 2, … 6

$= \{ 0, 0.25, 0.75, 1, 0.75, 0.25, 0 \}$

In part (iii) $h_d(n)$ is obtained as,

$$h_d(n) = \{0.075, -0.156, 0.225, 0.75, 0.225, -0.156, 0.075\}$$

Hence $\quad h(n) = h_d(n) \cdot W_{Hann}(n)$

$= \{ 0, -0.039, 0.16875, 0.75, 0.16875, -0.039, 0 \}$

Example 4.6.9 *A low pass filter is to be designed with the following desired frequency response.* $H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega}, & \frac{-\pi}{4} \le |\omega| \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| \le \pi \end{cases}$

Determine the filter coefficients $h_d(n)$ *if the window function is defined as*

$\omega(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$

AU : May-15, Marks 16

Solution : This example is similar to example 4.6.7.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j2\omega} \cdot e^{j\omega n}\, d\omega$$

$$= \frac{\sin\frac{\pi}{4}(n-2)}{\pi(n-2)} \quad \text{for } n \neq 2$$

and
$$h_d(n) = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j2\omega} \cdot e^{j\omega\times 2}\, d\omega = \frac{1}{4} = 0.75$$

$$\therefore \quad h_d(n) = \begin{cases} \dfrac{\sin\frac{\pi}{4}(n-2)}{\pi(n-2)} & \text{for} \quad n \neq 2 \\ 0.75 & \text{for} \quad n = 2 \end{cases}$$

Since $\omega(n)$ is rectangular window function,

$h(n) = h_d(n) \cdot \omega(n) = h_d(n)$ for $n = 0, 1, 2, 3, 4.$

$\therefore \quad h(0) = 0.159$

$h(1) = 0.225$

$h(2) = 0.75$

$h(3) = 0.225$

$h(4) = 0.159$

Example 4.6.10 *Design an ideal low pass filter with a frequency response.*

$$H_d(e^{jw}) = 1 \quad \text{for} \quad \frac{-\pi}{2} \le \omega \le \frac{\pi}{2}$$

$$= 0 \quad \text{for} \quad \frac{\pi}{2} \le |\omega| \le \pi$$

Find the values of h(n) for N = 11. Find H(z) and the filter coefficients. **AU : May-16, Marks 16**

Solution : Refer example 4.6.1. The impulse response of the designed filter is obtained as,

$$h_d(n) = \begin{cases} \dfrac{\sin \omega_c (n-\tau)}{\pi (n-\tau)} & \text{for} \quad n \ne \tau \\ \dfrac{\omega_c}{\pi} & \text{for} \quad n = \tau \end{cases}$$

And $\tau = \frac{M-1}{2} = \frac{11-1}{2} = 5$ Since N = M = 11

Putting $\tau = 5$ and $\omega_c = \frac{\pi}{2}$ in above equation,

$$h_d(n) = \begin{cases} \dfrac{\sin \frac{\pi}{2}(n-5)}{\pi (n-5)} & \text{for} \quad n \ne 5 \\ \dfrac{\pi/2}{\pi} = 0.5 & \text{for} \quad n = 5 \end{cases}$$

For rectangular window,

$$h(n) = h_d(n)$$

$$\therefore \quad h(n) = \begin{cases} \dfrac{\sin \frac{\pi}{2}(n-5)}{\pi (n-5)} & \text{for} \quad n \ne 5 \\ 0.5 & \text{for} \quad n = 5 \end{cases}$$

$$= \{ 0.064, 0, -0.106\ 0, 0.318, 0.5, 0.318, 0, -0.106, 0, 0.064 \}$$

These are the filter coefficients. Taking z–transform of above sequence,

$$H(z) = 0.064 - 0.106 z^{-2} + 0.318 z^{-4} + 0.5 z^{-5} + 0.318 z^{-6} - 0.106 z^{-8} + 0.064 z^{-10}$$

$$= 0.064 (1 + z^{-10}) - 0.106 (z^{2} + 2^{-8}) + 0.318 (z^{-4} + z^{-6}) + 0.5 z^{-5}$$

Example 4.6.11 ***Frequency sampling method :*** *Determine the coefficients of a linear phase FIR filter of length M=15, which has a symmetric unit sample response. The frequency response satisfies the function,*

$$H_k\left(\frac{2\pi k}{15}\right) = \begin{cases} 1 & k = 0,1,2,3 \\ 0.4 & k = 4 \\ 0 & k = 5,6,7 \end{cases}$$

Solution : Here $H(\omega)$ can be written as,

$$H(\omega) = \begin{cases} 1 \cdot e^{-j\omega\left(\frac{M-1}{2}\right)} \\ 0.4 \cdot e^{-j\omega\left(\frac{M-1}{2}\right)} \\ 0 \end{cases}$$

With $\omega = \frac{2\pi k}{M} = \frac{2\pi k}{15}$ and $M = 15$ above equation becomes,

$$\therefore \quad H(k) = \begin{cases} 1\, e^{-j\frac{2\pi k}{15}.7} & \text{for } k = 0,1,2,3 \\ 0.4\, e^{-j\frac{2\pi k}{15}.7} & \text{for } k = 4 \\ 0 & \text{for } k = 5,6,7 \end{cases}$$

$$= \begin{cases} e^{-j\frac{14\pi k}{15}} & \text{for } k = 0,1,2,3 \\ 0.4\, e^{-j\frac{14\pi k}{15}} & \text{for } k = 4 \\ 0 & \text{for } k = 5, 6, 7 \end{cases}$$

$$h(n) = \frac{1}{M}\left\{H(0) + 2\sum_{k=1}^{p} Re\left[H(k)\, e^{j2\pi k\frac{n}{M}}\right]\right\}$$

Here $p = \frac{M-1}{2} = \frac{15-1}{2} = 7$, and $H(k) = 0$ for $k = 5,6,7$ hence above equation will be,

$$\therefore \quad h(n) = \frac{1}{15}\left\{1 + 2\sum_{k=1}^{4} Re\left[H(k) e^{j2\pi k\frac{n}{15}}\right]\right\}$$

$$= \frac{1}{15}\left\{1 + 2Re\left[e^{-j\frac{14\pi}{15}} \cdot e^{j\frac{2\pi n}{15}}\right] + 2Re\left[e^{-j\frac{28\pi}{15}} \cdot e^{j\frac{4\pi n}{15}}\right] + 2Re\left[e^{-j\frac{42\pi}{15}} \cdot e^{j\frac{6\pi n}{15}}\right] + 2Re\left[0.4 e^{-j\frac{56\pi}{15}} \cdot e^{j\frac{8\pi n}{15}}\right]\right\}$$

$$= \frac{1}{15}\left\{1 + 2\cos\frac{2\pi(n-7)}{15} + 2\cos\frac{4\pi(n-7)}{15} + 2\cos\frac{6\pi(n-7)}{15} + 0.8\cos\frac{8\pi(n-7)}{15}\right\}$$

Putting n = 0, 1, 2....... 7 in above equation we get $h(0), h(1), \ldots\ldots h(7)$. Since $h(n) = h(M-1-n)$ i.e. $h(n) = h(14-n)$ we have,

$$h(0) = h(14) = -0.014128$$
$$h(1) = h(13) = -0.0019$$
$$h(2) = h(12) = 0.04$$
$$h(3) = h(11) = 0.0122$$
$$h(4) = h(10) = -0.0913$$
$$h(5) = h(9) = -0.018$$
$$h(6) = h(8) = 0.3133$$
$$h(7) = 0.52$$

Examples for Practice

Example 4.6.12 : *Design a normalized linear phase FIR filter having the phase delay of $\tau = 4$ and at least 40 dB attenuation in the stopband. Also obtain the magnitude/frequency response of the filter.*

[Hint and Ans. : $\omega_c = 1$ rad/sample
M = 9, Hanning window

$$h_d(n) = \begin{cases} \dfrac{\sin(n-4)}{\pi(n-4)} & \text{for } n \neq 4 \\ \dfrac{1}{\pi} & \text{for } n = 4 \end{cases}, \quad |H(\omega)| = h(4) + 2\sum_{n=0}^{3} h(n)\cos\omega(n-4)]$$

Example 4.6.13 : *An analog signal contains frequencies upto 10 kHz. This signal is sampled at 50 kHz. Design the FIR filters having linear phase characteristic and the transition band of 5 kHz. The filter should provide minimum 50 dB attenuation at the end of the transition band.*

$$\textbf{Ans. :}\ h(n) = \begin{cases} \dfrac{\sin[0.4\pi(n-20)]}{\pi(n-20)} \cdot \left[0.54 - 0.46\cos\left(\dfrac{\pi n}{20}\right)\right] & \text{for } n \neq 20 \\ 0.4 & \text{for } n = 20 \end{cases}$$

... for $0 \leq n \leq 40$

Example 4.6.13 : *Design a highpass linear phase FIR filter having cutoff frequency ω_c and window function of,*

$$w(n) = \begin{cases} 1 & \text{for } 0 \leq n \leq 6 \\ 0 & \text{elsewhere} \end{cases}$$

$$\textbf{Ans. :}\ h(n) = \begin{cases} \dfrac{\sin\pi(n-3) - \sin\omega_c(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ 1 - \dfrac{\omega_c}{\pi} & \text{for } n = 3 \end{cases}$$

Review Questions

1. *Explain the designing of FIR filters using windows.* **AU : Dec.-16, Marks 6**

2. *Explain the designing of FIR filters using Kaiser window. What are the advantages of this method ?*

4.7 Analog Filter Design

AU : May-04, 16, Dec.-16

4.7.1 Analog Filter Design using Butterworth Approximation

- Why filter approximations ?

The ideal low pass filter response shown in Fig. 4.7.1 is not physically realizable. Hence the response is approximated with the help of standard functions such as Butterworth, Chebyshev, elliptic etc.

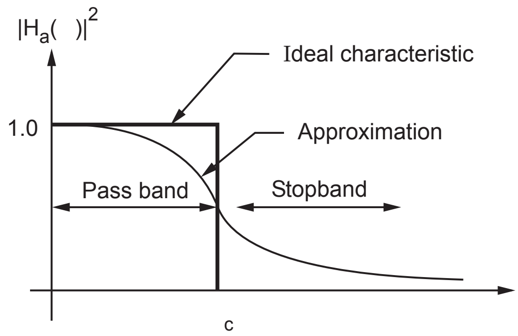


Fig. 4.7.1 Ideal lowpass filter

- The magnitude squared frequency response of the Butterworth filter is given as,

$$|H_a(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \qquad \text{... (4.7.1)}$$

Here N is the order and Ω_c is – 3 dB cut-off frequency of the filter.

- **Monotonically reducing magnitude response :**

Fig. 4.7.2 shows the plot of equation (4.7.1). It is plotted for different values of N (order).

Remember : 1) Characteristic is close to ideal response when order is increased.

2) Response is monotonically reducing.

3) $|H_a(\Omega)|^2 = 0.5$ for $\Omega = \Omega_c$ for all N.

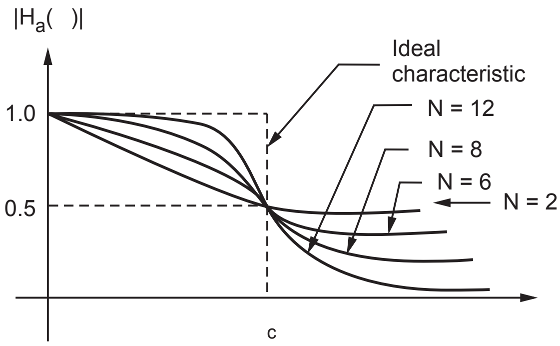


Fig. 4.7.2 Butterworth characteristic

Poles of $H_a(s)$

$H_a(s)$ is the system function of analog filter. The poles of Butterworth approximation can be calculated by following equation,

$$\boxed{p_k = \pm\,\Omega_c\, e^{j(2k+N+1)\pi/2N}\ ,\ \ k = 0, 1, \ldots N-1} \qquad \text{... (4.7.2)}$$

Note *Above equation actually gives poles of $H_a(s)$ and $H_a(-s)$. The poles of $H_a(s)$ are those which lie in the left half of the s-plane.*

Poles of Butterworth filter lie on the circle in s-plane.

Order of the Butterworth Filters

Fig. 4.7.3 shows the different attenuations and frequencies represented in Butterworth filter approximation. The filter specifications are mentioned as follows :

$$A_p \leq |H_a(\Omega)| \leq 1 \quad \text{for} \quad 0 \leq \Omega \leq \Omega_p$$

$$|H_a(\Omega)| \leq A_s \quad \text{for} \quad \Omega_s \leq \Omega$$

Here A_p is passband attenuation,

A_s is stopband attenuation,

Ω_p is passband edge frequency.

Ω_s is stopband edge frequency.

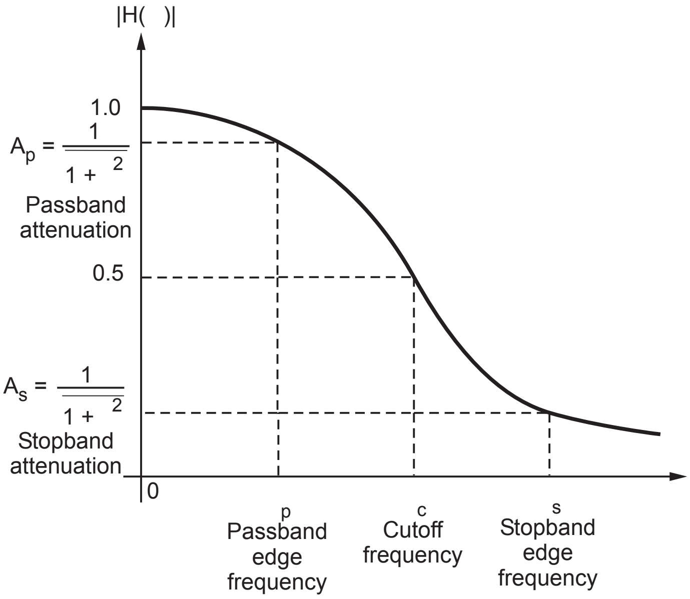


Fig. 4.7.3 Frequencies and attenuations in Butterworth approximation

Note that $A_p = \dfrac{1}{\sqrt{1+\varepsilon^2}}$ and $A_s = \dfrac{1}{\sqrt{1+\delta^2}}$ ... (4.7.3)

Note *In above equation, A_p and A_s are not in dB, they are linear.*

The order 'N' of the butterworth filter is given as,

$$N = \begin{cases} \dfrac{\log \sqrt{\left(\dfrac{1}{A_s^2}-1\right) \Big/ \left(\dfrac{1}{A_p^2}-1\right)}}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} & \text{for } A_s, A_p \text{ linear} \quad \text{... (4.7.4)} \\[2em] \dfrac{\log \sqrt{\dfrac{10^{0.1A_s}-1}{10^{0.1A_p}-1}}}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} & \text{for } A_p, A_s \text{ in dB} \quad \text{... (4.7.5)} \end{cases}$$

Above equation can be expressed in terms of 'ε' and 'δ' are as follows :

$$N = \frac{\log\left(\frac{\delta}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \quad \text{... (4.7.6)}$$

Cut-off Frequency of the Butterworth Filter

The cut-off frequency is required to calculate poles of Butterworth filter. It is given as,

$$\Omega_c = \frac{1}{2}\left\{\underbrace{\frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)^{\frac{1}{2N}}} + \frac{\Omega_s}{\left(\frac{1}{A_s^2}-1\right)^{\frac{1}{2N}}}}_{A_p \text{ and } A_s \text{ linear}}\right\} = \frac{1}{2}\left\{\underbrace{\frac{\Omega_p}{(10^{0.1A_p}-1)^{\frac{1}{2N}}} + \frac{\Omega_s}{(10^{0.1A_s}-1)^{\frac{1}{2N}}}}_{A_p \text{ and } A_s \text{ in } dB}\right\} \quad \text{... (4.7.7)}$$

or $$\Omega_c = \frac{1}{2}\left\{\frac{\Omega_p}{\varepsilon^{1/N}} + \frac{\Omega_s}{\delta^{1/N}}\right\} \quad \text{... (4.7.8)}$$

Normalized Butterworth Filter (Prototype Filter)

For normalized butterworth filter $\Omega_c = 1$ rad/sec. For normalized filter standard tables of poles or system functions are available. Normalized filter can be applied the frequency transformation to get the desired filter.

System Function $H_a(s)$ of Butterworth Filter

The poles obtained by equation (4.7.2) are arranged in complex conjugate pairs. Let these poles be $s_1, s_1^*, s_2, s_2^*, \ldots$

Then the system function of Butterworth filter can be expressed as,

$$H_a(s) = \underbrace{\frac{\Omega_c^N}{(s-s_1)(s-s_1^*)(s-s_2)(s-s_2^*)\ldots}}_{\text{Factored form}} = \underbrace{\frac{\Omega_c^N}{s^N + b_{N-1}s^{N-1} + \ldots + b_1 s + b_0}}_{\text{Polynomial form}} \quad \text{... (4.7.9)}$$

- For normalized Butterworth filter, $\Omega_c = 1$ in the above equation. The polynomial values for normalized Butterworth filters are obtained further in example 4.7.2.

Examples for Understanding

Example 4.7.1 *Design Butterworth filter for following specifications :*

$$0.8 \le |H_a(s)| \le 1 \quad \text{for} \quad 0 \le F \le 1000\,\text{Hz}$$

$$|H_a(s)| \le 0.2 \quad \text{for} \quad F \ge 5000\,\text{Hz}$$

Solution : Here $A_p = 0.8$ and $F_p = 1000$ Hz

and $A_s = 0.2$ and $F_s = 5000$ Hz

Since $\Omega_p = 2\pi F_p = 2\pi \times 1000\,\text{Hz} = 2000\pi$ rad/sec

and $\Omega_s = 2\pi F_s = 2\pi \times 5000\,\text{Hz} = 10{,}000\pi$ rad/sec

Step 1 : To obtain order of the filter

Order of the filter is given by equation (4.7.4) as,

$$N = \frac{\log \sqrt{\left(\frac{1}{A_s^2}-1\right) \Big/ \left(\frac{1}{A_p^2}-1\right)}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\log \sqrt{\left(\frac{1}{0.2^2}-1\right) \Big/ \left(\frac{1}{0.8^2}-1\right)}}{\log\left(\frac{10,000\pi}{2000\pi}\right)} = 1.167$$

Since the order of the filter must be an integer, we have to take nearest higher integer. Here, $N \approx 2$.

Step 2 : To obtain cutoff frequency

Cutoff frequency is given by equation (4.7.7) as,

$$\Omega_c = \frac{1}{2}\left\{\frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)^{\frac{1}{2N}}} + \frac{\Omega_s}{\left(\frac{1}{A_s^2}-1\right)^{\frac{1}{2N}}}\right\} = \frac{1}{2}\left\{\frac{2000\pi}{\left(\frac{1}{0.8^2}-1\right)^{\frac{1}{2\times 2}}} + \frac{10,000\pi}{\left(\frac{1}{0.2^2}-1\right)^{\frac{1}{2\times 2}}}\right\}$$

$$= 10724.5 \text{ rad/sec} = 3413.7\,\pi \text{ rad/sec.}$$

Step 3 : To obtain poles of Butterworth filter

Poles are given by equation (4.7.2) as,

$$p_k = \pm\Omega_c\, e^{j(2k+N+1)\pi/2N}, \quad k = 0,1,\ldots N-1$$

For $\Omega_c = 10724.5$ and $N = 2$ above equation becomes,

$$p_k = \pm 10724.5\, e^{j(2k+2+1)\pi/(2\times 2)}, \quad k = 0,1$$

$$= \pm 10724.5\, e^{j(2k+3)\pi/4}, \quad k = 0,\ 1 \qquad \text{... (4.7.10)}$$

$k = 0 \Rightarrow \quad p_0 = \pm 10724.5\, e^{j3\pi/4} = -7583 + j7583$ and $7583 - j\,7583$

$k = 1, \quad p_1 = \pm\ 10724.5\, e^{j(2\times 1+3)\pi/4}$ with $k=1$ in equation (4.7.10)

$$= \pm 10724.5\, e^{j5\pi/4} = -7583 - j7583 \text{ and } 7583 + j7583$$

Here observe that we obtained four poles. The poles lying in left half of s-plane are as follows.

$s_1 = -7583 + j7583$ and $s_1^* = -7583 - j7583$

Step 4 : To obtain system function $H_a\ s$

The system function of the Butterworth filter is given by equation (4.7.9) as,

$$H_a(s) = \frac{\Omega_c^N}{(s-s_1)(s-s_1^*)(s-s_2)(s-s_2^*)\ldots} = \frac{10724.5^2}{(s+7583-j7583)(s+7583+j7583)}$$

$$= \frac{10724.5^2}{(s+7583)^2+(7583)^2} = \frac{10724.5^2}{s^2+15166\,s+2\times(7583)^2}$$

This is the system function of required Butterworth filter.

Example 4.7.2 *Obtain the poles and system function for normalized Butterworth filter of 1^{st}, 2^{nd}, 3^{rd} orders.*

Solution : The poles of Butterworth filter are given by equation (4.7.2) as,

$$p_k = \pm\, \Omega_c\, e^{j(2k+N+1)\pi/2N}, \quad k = 0,\ 1,\ \ldots\ N-1$$

For normalized Butterworth filter $\Omega_c = 1$ rad/sec. Hence above equation will be,

$$p_k = e^{j(2k+N+1)\pi/2N}, \quad k = 0,1,\ldots N-1 \qquad \text{... (4.7.11)}$$

Poles and system function for $N = 1$ (1^{st} order filter)

With $N = 1$, equation (4.7.11) becomes,

$$p_k = e^{j(2k+1+1)\pi/(2\times 1)}, \quad k = 0 \ = e^{j(2k+2)\pi/2}, \quad k = 0$$

with $k = 0$ above equation will be,

$$p_0 = e^{j\pi} = -1$$

Thus $\quad s_1 = p_0 = -1$

The system function of normalized Butterworth filter is represented by $H_{an}(s)$. It is given as,

$$H_{an}(s) = \frac{\Omega_c^N}{(s-s_1)(s-s_1^*)(s-s_2)(s-s_2^*)\ldots}$$

Since $\Omega_c = 1$ in above equation,

$$H_{an}(s) = \frac{1}{(s-s_1)(s-s_1^*)(s-s_2)(s-s_2^*)\ldots} \quad \ldots (4.7.12)$$

$$= \frac{1}{s-s_1} \quad \text{for} \quad N=1$$

$$\boxed{H_{an}(s) = \frac{1}{s+1}} \quad \text{By putting } s_1 = p_0 = -1 \text{ for } N=1 \quad \ldots (4.7.13)$$

Poles and system function for $N = 2$ (2nd order filter)

With $N = 2$, equation (4.7.11) becomes,

$$p_k = e^{j(2k+2+1)\pi/(2\times 2)}, \quad k = 0, 1$$

$$= e^{j(2k+3)\pi/4}, \quad k = 0,1 \quad \ldots (4.7.14)$$

With $k = 0$ in above equation,

$$p_0 = e^{j3\pi/4} = -0.707 + j\,0.707$$

With $k = 1$ in equation (4.7.14),

$$p_1 = e^{j(2\times 1+3)\pi/4} = e^{j5\pi/4} = -0.707 - j\,0.707$$

Thus we obtained two poles which are complex conjugates of each other. i.e.,

$$s_1 = p_0 = -0.707 + j\,0.707 \quad \text{and} \quad s_1^* = p_1 = -0.707 - j\,0.707$$

Putting values in equation (4.7.12) we get,

$$H_{an}(s) = \frac{1}{(s+0.707-j\,0.707)(s+0.707+j\,0.707)} = \frac{1}{(s+0.707)^2+(0.707)^2}$$

i.e. $$\boxed{H_{an}(s) = \frac{1}{s^2+\sqrt{2}\,s+1}} \quad \ldots (4.7.15)$$

This is the system function for 2nd order normalized Butterworth filter. For higher orders it is given as,

For $N = 3$, $$H_{an}(s) = \frac{1}{(s+1)(s^2+s+1)}$$

or $$\boxed{H_{an}(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}}$$ ... (4.7.16)

For $N = 4$, $$H_{an}(s) = \frac{1}{(s^2 + 0.765\,s + 1)(s^2 + 1.847\,s + 1)} = \frac{1}{s^4 + 2.613\,s^3 + 3.414\,s^2 + 2.613\,s + 1}$$ ... (4.7.17)

For $N = 5$, $$H_{an}(s) = \frac{1}{(s+1)(s^2 + 0.618\,s + 1)(s^2 + 1.618\,s + 1)}$$

$$= \frac{1}{s^5 + 3.236\,s^4 + 5.236\,s^3 + 5.236\,s^2 + 3.236\,s + 1}$$ ... (4.7.18)

Similarly other higher order system functions can be calculated. Following tables list the Butterworth polynomials.

N	Factors
1	$s + 1$
2	$s^2 + \sqrt{2s} + 1$
3	$(s^2 + s + 1)(s + 1)$
4	$(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)$
5	$(s + 1)(s^2 + 0.6180\,s + 1)(s^2 + 1.6180\,s + 1)$
6	$(s^2 + 0.5176\,s + 1)(s^2 + \sqrt{2s} + 1)(s^2 + 1.9318\,s + 1)$
7	$(s + 1)(s^2 + 0.4450\,s + 1)(s^2 + 1.2456\,s + 1)(s^2 + 1.8022\,s + 1)$
8	$(s^2 + 0.3986\,s + 1)(s^2 + 1.1110\,s + 1)(s^2 + 1.6630\,s + 1)(s^2 + 1.9622\,s + 1)$

Table 4.7.1 Butterworth polynomials (Factored form)

N	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
1	1							
2	$\sqrt{2}$							
3	2	2						
4	2.613	3.414	2.613					
5	3.236	5.236	5.236	3.236				
6	3.864	7.464	9.141	7.464	3.864			
7	4.494	10.103	14.606	14.606	10.103	4.494		
8	5.126	13.138	21.848	25.691	21.848	12.138	5.126	1

Table 4.7.2 Butterworth polynomials

In the above tables value of $b_N = b_0 = 1$ always.

4.7.2 Analog Filter Design using Chebyshev Approximation

- There are two types of Chebychev approximation : Type-I and Type-II.

Type - I Chebyshev filters : These are all pole filters. They have equiripple characteristic in passband and monotonic characteristic in stopband. Fig. 4.7.4 (a) shows the magnitude characteristic of this type of filter.

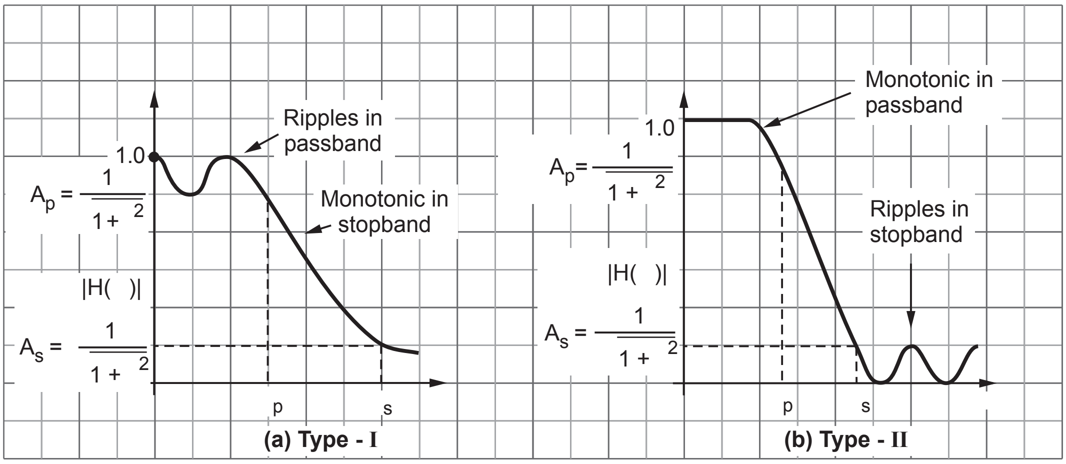


Fig. 4.7.4 Magnitude response of Chebyshev filter

Type - II Chebyshev filter : These filters have poles as well as zeros. They have monotonic characteristic in passband and equiripple characteristic in stopband. Fig. 4.7.4 (b) shows the magnitude characteristic of this type of filter.

Chebyshev Polynomials

The chebyshev approximation is implemented with the help of chebyshev polynomials. They are defined as follows :

$$\left.\begin{aligned} C_N(\Omega) &= \cos(N\cos^{-1}(\Omega)) \quad |\Omega| \le 1 \\ &= \cosh(N\cosh^{-1}(\Omega)) \quad |\Omega| > 1 \end{aligned}\right\} \quad \text{... (4.7.19)}$$

For $n = 0 \Rightarrow \quad C_0(\Omega) = \cos(0) = 1$ ... (4.7.20)

For $n = 1 \Rightarrow \quad C_1(\Omega) = \cos(1\cos^{-1}(\Omega)) = \Omega$ ... (4.7.21)

The higher order Chebyshev polynomials are obtained by following recursive formula :

$$\boxed{C_N(\Omega) = 2\Omega C_{N-1}(\Omega) - C_{N-2}(\Omega)} \quad \text{... (4.7.22)}$$

Example 4.7.3 *Find the Chebyshev polynomials for $n = 2, 3$ and 4.*

Solution : Using equation (4.7.22) we get,

$$C_N(\Omega) = 2\Omega C_{N-1}(\Omega) - C_{N-2}(\Omega)$$

i) For $n = 2$ $\quad C_2(\Omega) = 2\Omega\, C_1(\Omega) - C_0(\Omega)$

Putting the values from equation (4.7.20) and equation (4.7.21),

$$C_2(\Omega) = 2\Omega\,(\Omega) - 1 = 2\Omega^2 - 1$$

ii) For $n = 3$ $\quad C_3(\Omega) = 2\Omega C_2(\Omega) - C_1(\Omega) = 2\Omega\left(2\Omega^2 - 1\right) - \Omega = 4\Omega^3 - 3\Omega$

iii) For $n = 4$ $\quad C_4(\Omega) = 2\Omega C_3(\Omega) - C_2(\Omega) = 2\Omega\left(4\Omega^3 - 3\Omega\right) - \left(2\Omega^2 - 1\right)$

$$= 8\Omega^4 - 6\Omega^2 - 2\Omega^2 + 1 = 8\Omega^4 - 8\Omega^2 + 1$$

Table 4.7.3 lists the Chebyshev polynomials for orders upto $n = 10$.

N	**Chebyshev polynomials $C_n(\Omega) = \cos\ (n \cos^{-1}\ \Omega)$**
0	1
1	Ω
2	$2\Omega^2 - 1$
3	$4\Omega^3 - 3\Omega$
4	$8\Omega^4 - 8\Omega^2 + 1$
5	$16\Omega^5 - 20\Omega^3 + 5\Omega$
6	$32\Omega^6 - 48\Omega^4 + 18\Omega^2 - 1$
7	$64\Omega^7 - 112\Omega^5 + 56\Omega^3 - 7\Omega$
8	$128\Omega^8 - 256\Omega^6 + 160\Omega^4 - 32\Omega^2 + 1$
9	$256\Omega^9 - 576\Omega^7 + 432\Omega^5 - 120\Omega^3 + 9\Omega$
10	$512\Omega^{10} - 1280\Omega^8 + 1120\Omega^6 - 400\Omega^4 + 50\Omega^2 - 1$

Table 4.7.3

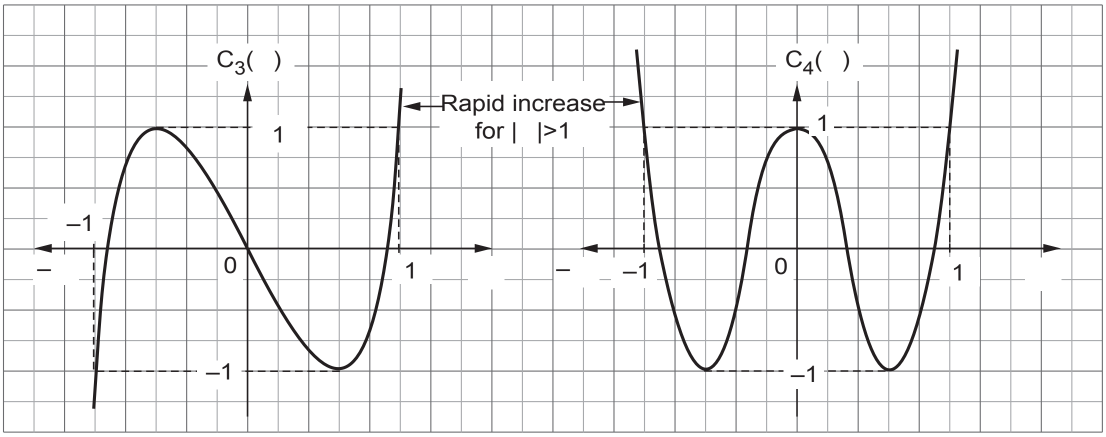
Properties of Chebyshev Polynomials

There are two ranges of $|\Omega|$ in Chebyshev polynomials :

i) $|\Omega| \geq 1$ and ii) $|\Omega| > 1$. Fig. 4.7.5 shows the plots of $C_3(\Omega)$ and $C_4(\Omega)$ for $-1 \leq \Omega \leq 1$.

1) In the above figures observe that $C_N(\Omega)$ never exceeds unity for $-1 \leq \Omega \leq 1$.

2) For $|\Omega| > 1$, $C_N(\Omega)$ increases rapidly.



(a) Plot of C_3 (Ω) **(b) Plot of C_4 (Ω)**

Fig. 4.7.5

3) $|C_n\ (\Omega)|$ increases monotonically for $|\Omega| \geq 1$. The high frequency roll-off is 20 n dB/decade.

4) In above figure observe that the zeros of $C_N\ (\Omega)$ are located in $-1 \leq \Omega \leq 1$.

5) In Table 4.7.3 observe that $C_n(\Omega)$ is an

(i) Odd polynomial if n = odd and

(ii) Even polynomial if n = even.

Magnitude Function of Chebyshev Filter

The squared magnitude function of the Chebyshev filter is given as,

$$|H_a\ (j\Omega)|^2 = \frac{1}{1+\varepsilon^2\ C_N^2\ (\Omega)} \quad \text{... (4.7.23)}$$

Few observations can be made from the above function :

i) For $|\Omega| \leq 1$, $|H_a\ (j\ \Omega)|^2$ varies between 1 and $\frac{1}{1+\varepsilon^2}$. This is the passband description of $|H\ (j\Omega)|$. i.e.,

$$1 \leq |H_a\ (j\Omega)| \leq \frac{1}{\sqrt{1+\varepsilon^2}} \quad \text{for} \quad |\Omega| \leq 1 \quad \text{... (4.7.24)}$$

And, ripple in passband $= 1 - \dfrac{1}{\sqrt{1+\varepsilon^2}}$ ... (4.7.25)

ii) At $\Omega = 1$, $C_n^2\ (1) = 1$ always. This can be easily verified for the polynomials of Table 4.7.1. Equation (4.7.23) can be written for $\Omega = 1$ as follows :

$$|H_a\ (j\Omega)|^2 = \frac{1}{1+\varepsilon^2}$$

or $$\left|H_a\,(j\Omega)\right| = \frac{1}{\sqrt{1+\varepsilon^2}} \qquad \text{... (4.7.26)}$$

Thus at cut-off frequency $\Omega = 1$, magnitude is $\frac{1}{\sqrt{1+\varepsilon^2}}$.

Order of the Chebyshev Filter

Fig. 4.7.4 shows various attenuations and frequencies in the Chebyshev filter response.

The specifications of Chebyshev filter are given as,

$$A_p \le |H_a(\Omega)| \le 1 \quad \text{for} \quad 0 \le \Omega \le \Omega_p$$

$$|H_a(\Omega)| \le A_s \quad \text{for} \quad \Omega_s \le \Omega$$

Here A_p is passband attenuation

A_s is stopband attenuation

Ω_p is passband edge frequency

Ω_s is stopband edge frequency

Here note that $$A_p = \frac{1}{\sqrt{1+\varepsilon^2}} \quad \text{and} \quad A_s = \frac{1}{\sqrt{1+\delta^2}} \qquad \text{... (4.7.27)}$$

Hence ε and δ can be expressed with the help of above relations as follows :

$$\varepsilon = \sqrt{\frac{1}{A_p^2}-1} \quad \text{and} \quad \delta = \sqrt{\frac{1}{A_s^2-1}} \qquad \text{... (4.7.28)}$$

If A_p and A_s are given in dB, then above values are written as,

$$\varepsilon = \sqrt{10^{0.1A_p}-1} \quad \text{and} \quad \delta = \sqrt{10^{0.1A_s}-1} \qquad \text{... (4.7.29)}$$

The order 'N' of the Chebyshev filter is given as,

$$N = \begin{cases} \boxed{\dfrac{\cosh^{-1}\sqrt{\left(\dfrac{1}{A_s^2}-1\right)\Big/\left(\dfrac{1}{A_p^2}-1\right)}}{\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)}} & \text{For } A_s, A_p \text{ linear} \quad \text{... (4.7.30)} \\[4ex] \boxed{\dfrac{\cosh^{-1}\sqrt{\dfrac{10^{0.1A_s}-1}{10^{0.1A_p}-1}}}{\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)}} & \text{For } A_s, A_p \text{ in dB} \quad \text{... (4.7.31)} \end{cases}$$

With the help of equation (4.7.29), above equation can be expressed in terms of 'ε' and 'δ' as follows,

$$\boxed{N = \frac{\cosh^{-1}\left(\frac{\delta}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}} \quad \text{... (4.7.32)}$$

Note *Here* $\cosh^{-1}(x)$ *can be evaluated using following equation,*

$$\cosh^{-1}(x) = ln\,[x+\sqrt{x^2-1}] \quad \text{... (4.7.33)}$$

Poles of $H_a(s)$

$H_a(s)$ is the system function of analog filter. The poles of Chebyshev approximation can be calculated by following equation :

$$p_k = \underbrace{\sigma_k}_{\text{Real part of the pole}} + j\underbrace{\Omega_k}_{\text{Imaginary part of the pole}}, \quad k = 0, 1, \ldots N-1 \quad \text{... (4.7.34 (a))}$$

Here $\sigma_k = a\cos\phi_k$ and $\Omega_k = b\sin\phi_k$

where $a = \Omega_p\left(\frac{\mu^{\frac{1}{N}} - \mu^{-\frac{1}{N}}}{2}\right)$

$b = \Omega_p\left(\frac{\mu^{\frac{1}{N}} + \mu^{-\frac{1}{N}}}{2}\right)$ ... (4.7.34 (b))

Here $\mu = \frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}$

$\phi_k = (2k+N+1)\pi / 2N, \quad k = 0,1,\ldots N-1$

Note *Above equation is used to calculate real part* (σ_k) *and imaginary part* (Ω_k) *separately. They are combine as* $s_k = \sigma_k + j\Omega_k$ *to form a pole.*

Poles of chebyshev filter lie on an ellipse in s-plane.

System Function $H_a(s)$ of Chebyshev Filter

The poles obtained by equation (4.7.2) are arranged in complex conjugate pairs. Let these poles be $s_1, s_1^*, s_2, s_2^* \ldots$ Then the system function of Chebyshev filter can be expressed as,

$$H_a(s) = \frac{k}{(s-s_1)(s-s_1^*)(s-s_2)(s-s_2^*)\ldots} \quad \ldots (4.7.35)$$

or

$$\boxed{H_a(s) = \frac{k}{s^N + b_{N-1}s^{N-1} + \ldots + b_1 s + b_0}} \quad \ldots (4.7.36)$$

Here the constant 'k' is given as,

$$k = \begin{cases} b_0 & \text{for 'N' odd} \\ \dfrac{b_0}{\sqrt{1+\varepsilon^2}} & \text{for 'N' even} \end{cases} \quad \ldots (4.7.37)$$

Normalized Chebyshev Filter

For normalized chebyshev filter Ω_p = 1 rad/sec. Standard tables of poles or polynomials for various values of 'ε' are available. Tables 4.7.3 lists the polynomial values and Table 4.7.4 lists poles-zeros of system function of normalized Chebyshev filters. (Refer all Table on page 4 - 89 to 4 - 94)

Examples for Understanding

Example 4.7.4 *Design the Chebyshev filter with following specifications.*

A_p = 2.5 dB, Ω_p = 20 rad/sec

A_s = 30 dB, Ω_s = 50 rad/sec

Solution : Step 1 : To obtain order of the filter

When the specifications are given in dB, the order of chebyshev filter is given by equation (4.7.31) as,

$$N = \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1A_s}-1}{10^{0.1A_p}-1}}}{\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\times 30}-1}{10^{0.1\times 2.5}-1}}}{\cosh^{-1}\left(\dfrac{50}{20}\right)} = 2.7265$$

The order of the filter must be an integer. Hence above value of 'N' must be rounded to next higher integer.

$\therefore \quad N = 3$

N	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
1	2.8627752									
2	1.5162026	1.4256245								
3	0.7156938	1.5348954	1.2529130							
4	0.3790506	1.0254553	1.7168662	1.1973856						
5	0.1789234	0.7525181	1.3095747	1.9373675	1.1724909					
6	0.0947626	0.4323669	1.1718613	1.5897635	2.1718446	1.1591761				
7	0.0447309	0.2820722	0.7556511	1.6479029	1.8694079	2.4126510	1.1512176			
8	0.0236907	0.1525444	0.5735604	1.1485894	2.1840154	2.1492173	2.6567498	1.1460801		
9	0.0111827	0.0941198	0.3408193	0.9836199	1.6113880	2.7814990	2.4293297	2.9027337	1.425705	
10	0.0059227	0.0492855	0.2372688	0.6269689	1.5274307	2.1442372	3.4409268	2.7097415	3.1498757	1.1400664

Table 4.7.4 : Normalized chebyshev polynomials

Table 4.7.4 (a) : Ripple = 0.5 dB i.e. $\varepsilon = 0.349$

N	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
1	1.9652267									
2	1.1025103	1.0977343								
3	0.4913067	1.2384092	0.9883412							
4	0.2756276	0.7426194	1.4539248	0.9528114						
5	0.1228267	0.5805342	0.9743961	1.6888160	0.9368201					
6	0.0689069	0.3070808	0.9393461	1.2021409	1.9308256	0.9282510				
7	0.0307066	0.2136712	0.5486192	1.3575440	1.4287930	2.1760778	0.9231228			
8	0.0172267	0.1073447	0.4478257	0.8468243	1.8369024	1.6551557	2.4230264	0.9198113		
9	0.0067767	0.0706048	0.2441864	0.7863109	1.2016071	2.3781188	1.8814798	2.6709468	0.9175476	
10	0.0043067	0.0344971	0.1824512	0.4553892	1.2444914	1.6129856	2.9815094	2.1078524	2.9194657	0.9159320

Table 4.7.4 (b) : Ripple = 1 dB = i.e. $\varepsilon = 0.508$

N	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
1	1.3075603									
2	0.6367681	0.8038164								
3	0.3268901	1.0221903	0.7378216							
4	0.2057651	0.5167981	1.2564819	0.7162150						
5	0.0817225	0.4593491	0.6934770	1.4995433	0.7064606					
6	0.0514413	0.2102706	0.7714618	0.8670149	1.7458587	0.7012257				
7	0.0204228	0.1660920	0.3825056	1.1444390	1.0392203	1.9935272	0.6978929			
8	0.0128603	0.0729373	0.3587043	0.5982214	1.5795807	1.2117121	2.2422529	0.6960646		
9	0.0051076	0.0543756	0.1684473	0.6444677	0.8568648	2.0767479	1.3837464	2.4912897	0.6946793	
10	0.0032151	0.0233347	0.1440057	0.3177560	1.0389104	1.1585287	2.6362507	1.5557424	2.7406032	0.6936904

Table 4.7.4 (c) Ripple = 2 dB i.e. ε = 0.764

N	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
1	1.0023773									
2	0.7079478	0.6448996								
3	0.2505943	0.9283480	0.5972404							
4	0.1769869	0.4047679	1.1691176	0.5815799						
5	0.0626391	0.4079421	0.5488626	1.4149847	0.5744296					
6	0.0442467	0.1634299	0.6990977	0.6906098	1.6628481	0.5706979				
7	0.0156621	0.1461530	0.3000167	1.0518448	0.8314411	1.9115507	0.5684201			
8	0.0110617	0.0564813	0.3207646	0.4718990	1.4666990	0.9719473	2.1607148	0.5669476		
9	0.0039154	0.0475900	0.1313851	0.5834984	0.6789075	1.9438443	1.1122863	2.4101346	0.5659234	
10	0.0027654	0.0180313	0.1277560	0.2492043	0.9499208	0.9210659	2.4834205	1.2526467	2.6597378	0.5652218

Table 4.7.4 (d) Ripple = 3 dB i.e. ε = 0.997

N = 1	*N* = 2	*N* = 3	*N* = 4	*N* = 5	*N* = 6	*N* = 7	*N* = 8	*N* = 9	*N* = 10
− 2.8627752	− 0.7128122 ± j1.0040425	− 0.6264565	− 0.1753531 ± j1.0162529	− 0.3623196	− 0.0776501 ± j1.0084608	− 0.2561700	− 0.0436210 ± j1.0050021	− 0.1984053	− 0.0278994 ± j1.0032732
		− 0.3132282 ± j0.219275	− 0.4233398 ± j0.4209457	− 0.1119629 ± j1.0115574	− 0.2121440 ± j0.7382446	− 0.0570032 ± j1.0064085	− 0.1242195 ± j0.8519996	− .0344527 ± j1.0040040	− 0.0809672 ± j0.9050658
				− 0.2931227 ± j0.0947626	− 0.2897940 ± j0.2702162	− 0.1597194 ± j0.8070770	− 0.1859076 ± j0.5692879	− 0.0992026 ± j0.8829063	− 0.1261094 ± j0.7182643
						− 0.2308012 ± j0.4478939	− 0.2192929 ± j0.1999073	− 0.1519873 ± j0.6553170	− 0.1589072 ± j0.4611541
								− 0.1864400 ± j0.3486869	− 0.1761499 ± j0.1589029

Table 4.7.5 : Poles of the normalized chebyshev filters

Table 4.7.5 (a) : Ripple = 0.5 dB i.e. ε = 0.349

N = 1	*N* = 2	*N* = 3	*N* = 4	*N* = 5	*N* = 6	*N* = 7	*N* = 8	*N* = 9	*N* = 10
– 1.9652267	– 0.5488672 ± j0.8951286	– 0.4941706	– 0.1395360 ±j0.98333792	– 0.2894933	– 0.0621810 ± j0.9934115	0.2054141	– 0.0350082 ± j0.9964513	– 0.1593305	– 0.0224144 ± j0.9977755
		– 0.2470853 ± j0.9659987	– 0.3368697 ± j0.4073290	– 0.0894584 ± j0.9901071	– 0.1698817 ± j0.7272275	– 0.0457089 ± j0.9952839	– 0.0996950 ± j0.8447506	– 0.0276674 ± j0.9972297	– 0.1013166 ± j0.7143234
				– 0.2342050 ± j0.6119198	– 0.2320627 ± j0.2661837	– 0.1280736 ± j0.7981557	– 0.1492041 ± j0.5644443	– 0.0796652 ± j0.8769490	– 0.0650493 ± j0.9001063
						– 0.1850717 ± j0.4429430	– 0.1759983 ± j0.1982065	– 0.1220542 ± j0.6508954	– 0.1276664 ± j0.4586271
								– 0.1497217 ± j0.3463342	– 0.1415193 ± j0.1580321

Table 4.7.5 (b) : Ripple = 1 dB i.e. ε = 0.508

N **= 1**	***N*** **= 2**	***N*** **= 3**	***N*** **= 4**	***N*** **= 5**	***N*** **= 6**	***N*** **= 7**	***N*** **= 8**	***N*** **= 9**	***N*** **= 10**
− 1.3075603	− 0.4019082 ± j0.6893750	− 0.3689108	− 0.1048872 ± j0.9579530	− 0.2183083	− 0.0469732 ± j0.9817052	− 0.1552958	− 0.0264924 ± j0.9897870	− 0.1206298	− 0.0169758 ± j0.9934868
		− 0.1844554 ± j0.9230771	− 0.2532202 ± j0.3967971	− 0.0674610 ± j0.9734557	− 0.1283332 ± j0.7186581	− 0.0345566 ± j0.9866139	− 0.0754439 ± j0.8391009	− 0.0209471 ± j0.9919471	− 0.0767332 ± j0.7112580
				− 0.1766151 ± j0.6016287	− 0.1753064 ± j0.2630471	− 0.0968253 ± j0.7912029	− 0.1129098 ± j0.5606693	− 0.0603149 ± j0.8723036	− 0.0492657 ± j0.8962374
						− 0.1399167 ± j0.4390845	− 0.1331862 ± j0.1968809	− 0.0924078 ± j0.6474475	− 0.0966894 ± j0.4566558
								− 0.1133549 ± j0.3449996	− 0.1071810 ± j0.1573528

Table 4.7.5 (c) : Ripple = 2 dB i.e. ε = 0.764

N = 1	N = 2	N = 3	N = 4	N = 5	N = 6	N = 7	N = 8	N = 9	N = 10
1.0023773	0.2224498 ± j0.7771576	− 0.2986202	− 0.0851704 ± j0.9464844	− 0.1775085	− 0.0382295 ± j0.9764060	− 0.1264854	− 0.0215782 ± j0.9867664	− 0.0982716	− 0.0138320 ± j0.9915418
		− 0.1493101 ± j0.9038144	− 0.2056195 ± j0.3920467	− 0.0548531 ± j0.9659238	− 0.1044450 ± j0.7147788	− 0.0281456 ± j0.9826957	− 0.0614494 ± j0.8365401	− 0.0170647 ± j0.9895516	− 0.0401419 ± j0.8944827
				− 0.1436074 ± j0.5969738	− 0.1426745 ± j0.2616272	− 0.0788623 ± j0.7880608	− 0.0919655 ± j0.5589582	− 0.0491358 ± j0.8701971	− 0.0625225 ± j0.7098655
						− 0.1139594 ± j0.4373407	− 0.1084807 ± j0.1962800	− 0.0752804 ± j0.6458839	− 0.0787829 ± j0.4557617
								− 0.0923451 ± j0.3436677	− 0.0873316 ± j0.1570448

Table 4.7.5 (d) : Ripple = 3 dB i.e. ε = 0.997

Step 2 : To obtain poles of the filter

From equation (4.7.29) we can calculate 'ε' from A_p. i.e.,

$$\varepsilon = \sqrt{10^{0.1A_p}-1} = \sqrt{10^{0.1\times 2.5}-1} = 0.882$$

$$\therefore \quad \mu = \frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon} \qquad \text{By equation (4.7.34 (b))}$$

$$= \frac{1+\sqrt{1+0.882^2}}{0.882} = 2.645$$

$$\therefore \quad a = \Omega_p\left(\frac{\mu^{\frac{1}{N}}-\mu^{-\frac{1}{N}}}{2}\right) \qquad \text{By equation (4.7.34 (b))}$$

$$= 20\left(\frac{2.645^{\frac{1}{3}}-2.645^{-\frac{1}{3}}}{2}\right) = 6.6$$

And $$b = \Omega_p\left(\frac{\mu^{\frac{1}{N}}+\mu^{-\frac{1}{N}}}{2}\right) \qquad \text{By equation (4.7.34 (b))}$$

$$= 20\left(\frac{2.645^{\frac{1}{3}}+2.645^{-\frac{1}{3}}}{2}\right) = 21$$

Also, $$\phi_k = \frac{(2k+N+1)\pi}{2N}, \quad \text{for} \quad k=0,1,\ldots N-1 \qquad \text{By equation (4.7.14 (b))}$$

For $N=3$, $$\phi_k = \frac{(2k+3+1)\pi}{2\times 3}, \quad \text{for} \quad k=0,1,2$$

$$= \frac{(2k+4)\pi}{6}, \quad k=0,1,2$$

$$\therefore \quad \phi_0 = \frac{4\pi}{6}, \quad \phi_1=\pi \quad \text{and} \quad \phi_2 = \frac{8\pi}{6}$$

From equation (4.7.34 (b)), the real and imaginary parts of the poles are given as,

$$\sigma_k = a\cos\phi_k \quad \text{and} \quad \Omega_k = b\sin\phi_k$$

For $k=0$, $\sigma_0 = a\cos\phi_0$ and $\Omega_0 = b\sin\phi_0$

$$= 6.6\cos\left(\frac{4\pi}{6}\right) \qquad = 21\sin\left(\frac{4\pi}{6}\right)$$

$$= -3.3 \qquad = 18.19$$

Thus, $p_0 = \sigma_0 + j\Omega_0 = -3.3 + j\,18.19$

For $k=1$, $\sigma_1 = a\cos\phi_1$ and $\Omega_1 = b\sin\phi_1$

$$= 6.6\cos\pi \qquad = 21\sin\pi$$

$$= -6.6 \qquad = 0$$

Thus, $p_1 = \sigma_1 + j\Omega_1 = -6.6$

For $k=2$, $\sigma_2 = a\cos\phi_2$ and $\Omega_2 = b\sin\phi_2$

$$= 6.6\cos\left(\frac{8\pi}{6}\right) \qquad = 21\sin\left(\frac{8\pi}{6}\right)$$

$$= -3.3 \qquad = -18.19$$

Thus, $p_2 = \sigma_2 + j\Omega_2 = -3.3 - j\,18.19$

The complex conjugate poles are combined and written as follows :

$$s_1 = p_1 = -6.6$$

$$s_2 = p_0 = -3.3 + j\,18.19 \quad \text{and} \quad s_2^* = -3.3 - j\,18.19$$

Step 3 : System function $H_a(s)$

The system function of Chebyshev filter is given by equation (4.7.35) as,

$$H_a(s) = \frac{k}{(s-s_1)(s-s_1^*)(s-s_2)(s-s_2^*)\ldots} = \frac{k}{(s+6.6)(s+3.3-j\,18.19)(s+3.3+j\,18.19)}$$

$$= \frac{k}{(s+6.6)[(s+3.3)^2+(18.19)^2]} = \frac{k}{(s+6.6)(s^2+6.6\,s+341.7)} \qquad \ldots (4.7.38)$$

$$\boxed{b_0 = 6.6\times 341.7 = 2255.2}$$

$$= \frac{k}{s^3+13.2s^2+385.26\,s+2255.2} = \frac{k}{s^3+b_2\,s^2+b_1\,s+b_0}$$

Comparing above two equations we find that $b_0 = 2255.2$. Actually same b_0 can be directly obtained from equation (4.7.38) i.e.,

$$b_0 = 6.6\times 341.7 = 2255.2$$

Since the order of this filter is $N = 3$, i.e. odd, the value of k is given by equation (4.7.37) as,

$$k = b_0 = 2255.2 \quad \text{for odd 'N'}$$

Hence system function given by equation (4.7.38) will be,

$$H_a(s) = \frac{2255.2}{(s+6.6)(s^2+6.6s+341.7)}$$

Example 4.7.5 *Design an analog filter that exhibits the magnitude response as shown in Fig. 4.7.6.*

Solution : From Fig. 4.7.6, observe that the filter has equiripple behaviour in passband and monotonic behaviour in the stopband. Hence this is chebyshev filter. Its specifications can be expressed as,

$$A_p = 0.9, \quad F_p = 200 \text{ Hz}, \quad \text{hence } \Omega_p = 2\pi F_p = 2\pi \times 200 = 400\pi \text{ rad/sec}$$

$$A_s = 0.1, \quad F_s = 500 \text{ Hz}, \quad \text{hence } \Omega_s = 2\pi F_s = 2\pi \times 500 = 1000\pi \text{ rad/sec}$$

Step 1 : To obtain order of chebyshev filter

From equation (4.7.28) values of 'ε' and 'δ' are given as,

$$\varepsilon = \sqrt{\frac{1}{A_p^2} - 1} \quad \text{and} \quad \delta = \sqrt{\frac{1}{A_s^2} - 1}$$

$$= \sqrt{\frac{1}{0.9^2} - 1} \qquad = \sqrt{\frac{1}{0.1^2} - 1}$$

$$= 0.484 \qquad = 9.95$$

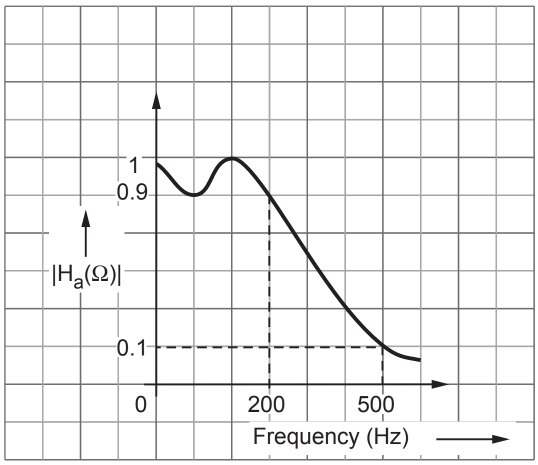


Fig. 4.7.6 Response of analog filter

Order, $$N = \frac{\cosh^{-1}(\delta/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)}$$

By equation (4.7.32)

$$= \frac{\cosh^{-1}\left(\frac{9.95}{0.484}\right)}{\cosh^{-1}\left(\frac{1000\pi}{400\pi}\right)} = 2.37 \approx 3 \quad \text{(Integer should be taken)}$$

Step 2 : To obtain poles of the filter

$$\mu = \frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon} \qquad \text{By equation (4.7.34 (b))}$$

$$= \frac{1+\sqrt{1+0.484^2}}{0.484} = 4.36$$

Values of $'a'$ and $'b'$ can be calculated from equation (4.7.34 (b)) as follows :

$$a = \Omega_p \left(\frac{\mu^{\frac{1}{N}} - \mu^{-\frac{1}{N}}}{2} \right) \quad \text{and} \quad b = \Omega_p \left(\frac{\mu^{\frac{1}{N}} + \mu^{-\frac{1}{N}}}{2} \right)$$

$$= 400\pi \left(\frac{4.36^{\frac{1}{3}} - 4.36^{-\frac{1}{3}}}{2} \right) \qquad = 400\pi \left(\frac{4.36^{\frac{1}{3}} + 4.36^{-\frac{1}{3}}}{2} \right)$$

$$= 641.85 \qquad = 1411.07$$

By equation (4.7.34 (b)) ,
$$\phi_k = \frac{(2k+N+1)\pi}{2N}, \quad k = 0, 1, \dots N-1$$

For $N = 3$
$$\phi_k = \frac{(2k+3+1)\pi}{2\times 3}, \quad k = 0, 1, 2$$

$$= \frac{(2k+4)\pi}{6}, \quad k = 0, 1, 2$$

Following table lists the calculations of poles for different values of k.

k	$\phi_k = \frac{(2k+4)\pi}{6}$	$\sigma_k = a\cos\phi_k$	$\Omega_k = b\sin\phi_k$	$p_k = \sigma_k + j\Omega_k$
0	$\phi_0 = \frac{(0+4)\pi}{6}$ $= \frac{4\pi}{6}$	$\sigma_0 = 641.85 \cos\left(\frac{4\pi}{6}\right)$ $= -320.92$	$\Omega_0 = 1411.07 \sin\left(\frac{4\pi}{6}\right)$ $= 1222$	$p_0 = -320.92 + j1222$
1	$\phi_1 = \frac{(2+4)\pi}{6}$ $= \pi$	$\sigma_1 = 641.85 \cos\pi$ $= -641.85$	$\Omega_1 = 1411.07 \sin\pi$ $= 0$	$p_1 = -641.85$
2	$\phi_2 = \frac{(2\times 2+4)\pi}{6}$ $= \frac{8\pi}{6}$	$\sigma_2 = 641.85 \cos\left(\frac{8\pi}{6}\right)$ $= -320.92$	$\Omega_2 = 1411.07 \sin\left(\frac{8\pi}{6}\right)$ $= -1222$	$p_2 = -320.92 - j1222$

Table 4.7.6 Calculations of poles of example 4.7.5

From the table, complex conjugate poles are combined as follows :

$$s_1 = p_1 = -\,641.85$$

$$s_2 = p_0 = -\,320.92 + j\,1222 \quad \text{and} \quad s_2^* = -320.92 - j\,1222$$

Step 3 : To obtain system function $H_a(s)$

$$H_a(s) = \frac{k}{(s-s_1)(s-s_1^*)(s-s_2)(s-s_2^*)\ldots} \qquad \text{By equation (4.7.35)}$$

$$= \frac{k}{(s+641.85)(s+320.92-j\,1222)(s+320.92+j\,1222)}$$

$$= \frac{k}{(s+641.85)(s^2+641.84\,s+1596273.64)}$$

Here $b_0 = 641.85 \times 1596273.64 = 1.024 \times 10^9$

From equation (4.7.37), $k = b_0 = 1.024 \times 10^9$, since order '$N$' is odd.

Then above system function will be,

$$\therefore \quad H_a(s) = \frac{1.024 \times 10^9}{(s+641.85)(s^2+641.84\,s+1596273.64)}$$

4.7.3 Analog Frequency Transformations

Fig. 4.7.7 shows the magnitude response of the lowpass filters designed using Butterworth or Chebyshev approximations. Here 'Ω_p' is the passband edge frequency of the lowpass filter from which frequency transformation is executed to the desired filter.

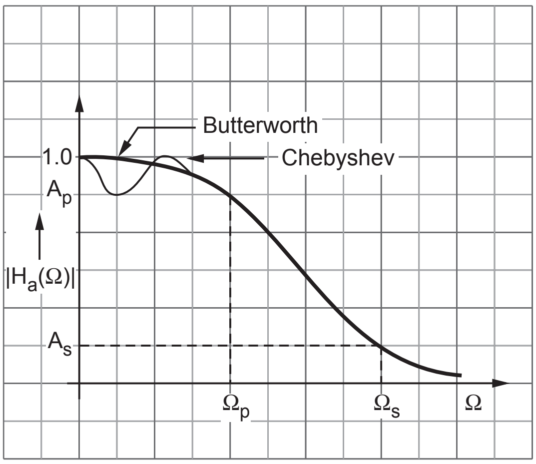
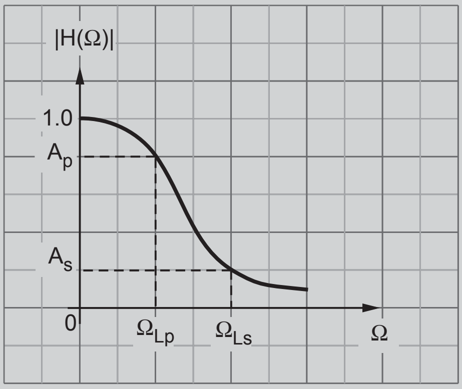
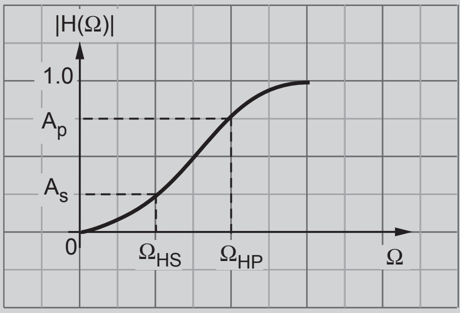
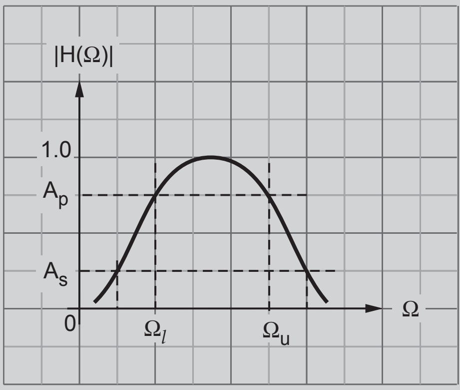


Fig. 4.7.7 Response of lowpass filter

Sr. No.	Types of transformation	Magnitude response of the transformed (new) filter	Transformation equation
1.	Lowpass to lowpass conversion		$s \rightarrow \frac{\Omega_p}{\Omega_{LP}} s$ Ω_{LP} = Passband edge frequency of the desired filter
2.	Lowpass to highpass conversion		$s \rightarrow \frac{\Omega_p \Omega_{HP}}{s}$ Ω_{HP} = Passband edge frequency of the highpass filter.
3.	Lowpass to bandpass conversion		$s \rightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$ Ω_l = Lower band edge frequency Ω_u = Upper band edge frequency

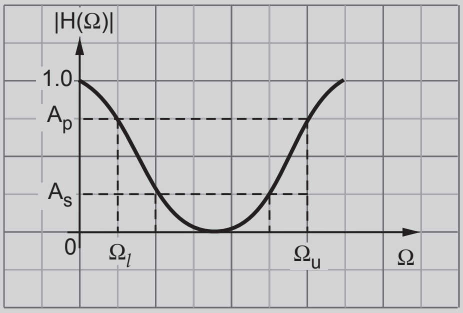
4.	Lowpass to bandstop conversion		$s \rightarrow \Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}$ Ω_l = Lower band edge frequency Ω_u = Upper band edge frequency

Table 4.7.7 Analog frequency transformations

4.7.4 Digital Frequency Transformations

Let us consider that we have a lowpass digital filter with passband edge frequency of ω_p. This filter can be transformed to any other digital filter by transformation in digital domain.

Sr. No.	Type of conversion	Transformation equations
1.	Lowpass to lowpass conversion	$z^{-1} \rightarrow \frac{z^{-1} - a}{1 - az^{-1}}$, Here $a = \frac{\sin\left(\frac{\omega_p - \omega_{LP}}{2}\right)}{\sin\left(\frac{\omega_p + \omega_{LP}}{2}\right)}$ ω_{LP} is passband edge frequency of new lowpass filter.
2.	Lowpass to highpass conversion	$z^{-1} \rightarrow -\frac{z^{-1} + a}{1 + az^{-1}}$, Here $a = -\frac{\cos\left(\frac{\omega_p + \omega_{HP}}{2}\right)}{\cos\left(\frac{\omega_p - \omega_{HP}}{2}\right)}$ ω_{HP} is passband edge frequency of new highpass filter.
3.	Lowpass to bandpass conversion	$z^{-1} \rightarrow -\frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$, Here $a_1 = \frac{2\alpha k}{k+1}$, $a_2 = \frac{k-1}{k+1}$ $\alpha = \frac{\cos\left(\frac{\omega_u + \omega_l}{2}\right)}{\cos\left(\frac{\omega_u - \omega_l}{2}\right)}$, $k = \cot\left(\frac{\omega_u + \omega_l}{2}\right) \tan\frac{\omega_p}{2}$ ω_u is upper cutoff frequency and ω_l is lower cutoff frequency
4.	Lowpass to bandstop conversion	$z^{-1} \rightarrow \frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$, Here $a_1 = \frac{z\alpha k}{k+1}$, $a_2 = \frac{1-k}{1+k}$ $\alpha = \frac{\cos\left(\frac{\omega_u + \omega_l}{2}\right)}{\cos\left(\frac{\omega_u - \omega_l}{2}\right)}$ and $k = \tan\left(\frac{\omega_u - \omega_l}{2}\right) \tan\frac{\omega_p}{2}$ ω_u is upper cutoff frequency and ω_l is lower cutoff frequency

Table 4.7.8 Digital frequency transformations

Examples for Understanding

Example 4.7.6 *The system function of the first order normalized lowpass filter is given as,* $H_{an}(s) = \frac{1}{s+1}$ *. Obtain the system function of second order bandpass filter having passband from 1 kHz to 2 kHz.*

Solution : From Table 4.7.7 lowpass to bandpass conversion is given as,

$$s \rightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

- Here note that after frequency transformation, the RHS becomes second order. Thus N^{th} order filter is converted to $2N^{\text{th}}$ order. Hence second order bandpass filter is obtained from 1^{st} order filter.
- Here the passband frequencies are given as,

$$F_l = 1 \text{ kHz} \quad \text{and} \quad F_u = 2 \text{ kHz}$$

$$\therefore \quad \Omega_l = 2\pi F_l = 2\pi \times 1000 = 2000\pi \text{ rad/sec} \qquad \Omega_u = 2\pi F_u = 2\pi \times 2000 = 4000\pi \text{ rad/sec}$$

- Hence the filter transformation equation becomes,

$$s \rightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

$$\therefore \quad s \rightarrow \Omega_p \frac{s^2 + 2000\pi \times 4000\pi}{s(4000\pi - 2000\pi)}$$

$$\therefore \quad s \rightarrow \Omega_p \frac{s^2 + 8\times 10^6 \pi^2}{2000\pi s}$$

- The desired bandpass filter is obtained by,

$$H_{BP}(s) = H_{an}(s)\Big|_{s \rightarrow \Omega_p \frac{s^2+8\times 10^6\pi^2}{2000\pi s}}$$

Here Ω_p is the passband edge frequency of the filter represented by $H_{an}(s)$.
Since it is normalized lowpass filter, $\Omega_p = 1$ rad/sec. Then above equation will be,

$$H_{BP}(s) = H_{an}(s)\Big|_{s \rightarrow \frac{s^2+8\times 10^6\pi^2}{2000\pi s}} = \frac{1}{s+1}\Big|_{s \rightarrow \frac{s^2+8\times 10^6\pi^2}{2000\pi s}} = \frac{1}{\frac{s^2 + 8\times 10^6 \pi^2}{2000\pi s} + 1}$$

$$\therefore \quad H_{BP}(s) = \frac{2000\pi s}{s^2 + 2000\pi s + 8\times 10^6 \pi^2}$$

This is the system function of the desired filter.

Example 4.7.7 *The digital lowpass filter has following system function,* $H(z) = \frac{1+2z^{-1}}{4-z^{-1}}$, *with* ω_p *= 1.12 rad. Obtain a highpass filter with cutoff frequency of 2 rad.*

Solution : Here $\omega_p = 1.12$ rad (given)

and $\omega_{HP} = 2$ rad

The lowpass to highpass digital frequency transformation is given as,

$$z^{-1} \rightarrow -\frac{z^{-1}+a}{1+az^{-1}}$$

Here
$$a = -\frac{\cos\left(\frac{\omega_p+\omega_{HP}}{2}\right)}{\cos\left(\frac{\omega_p-\omega_{HP}}{2}\right)} = -\frac{\cos\left(\frac{1.12+2}{2}\right)}{\cos\left(\frac{1.12-2}{2}\right)} = -0.012$$

$\therefore$
$$z^{-1} \rightarrow -\frac{z^{-1}-0.012}{1-0.012z^{-1}}$$

Applying this transformation to given digital filter,

$$H_{HP}(z) = H(z)\Big|_{z^{-1}\rightarrow -\frac{z^{-1}-0.012}{1-0.012z^{-1}}} = \frac{1+2z^{-1}}{4-z^{-1}}\Bigg|_{z^{-1}\rightarrow -\frac{z^{-1}-0.012}{1-0.012z^{-1}}} = \frac{1-2\left[\frac{z^{-1}-0.012}{1-0.012z^{-1}}\right]}{4+\frac{z^{-1}-0.012}{1-0.012z^{-1}}}$$

Upon simplifying above equation,

$$H_{HP}(z) = \frac{0.255(1-1.965z^{-1})}{1+0.237z^{-1}}$$

Example 4.7.8 *Given the specifications* α_p *= 3 dB,* α_s *= 10 dB,* f_p *= 1 kHz and* f_s *= 2 kHz. Determine the order of the filter using Chebyshev approximation. Find H(s).*

AU : May-16, Marks 8

Solution : Here $f_p = 1000$ Hz. Hence $\Omega_p = 2\pi f_p = 2000\pi$ red / sec

and $f_s = 2000$ Hz, hence $\Omega_s = 2\pi f_s = 4000\pi$ *rad* / sec.

Thus, $A_p = 3$ dB, $\Omega_p = 2000\pi$ rad/sec

$A_3 = 10$ dB, $\Omega_3 = 4000\pi$ rad/sec

$$N = \frac{\cos h^{-1}\sqrt{\dfrac{10^{0.1A_s}-1}{10^{0.1A_p}-1}}}{\cos h^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\times10}-1}{10^{0.1\times3}-1}}}{\cos h^{-1}\left(\dfrac{4000\pi}{2000\pi}\right)} = 1.34 \approx 2$$

$$\varepsilon = \sqrt{10^{0.1A_p}-1} = \sqrt{10^{0.1\times3}-1} = 0.998$$

$$\mu = \frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon} = \frac{1+\sqrt{1+0.998^2}}{0.998} = 2.417$$

$$a = \Omega_p\left(\frac{\mu^{\frac{1}{N}}-\mu^{-\frac{1}{N}}}{2}\right) = 2000\pi\left(\frac{2.417^{\frac{1}{2}}-2.417^{-\frac{1}{2}}}{2}\right) = 2863.4$$

$$b = \Omega_p\left(\frac{\mu^{\frac{1}{N}}+\mu^{-\frac{1}{N}}}{2}\right) = 2000\pi\left(\frac{2.417^{\frac{1}{2}}+2.417^{-\frac{1}{2}}}{2}\right) = 6904.8$$

Following table calculates the poles for k = 0 and 1

k	$\phi_k = \frac{(2k+N+1)\pi}{2N}$	$\sigma_k = a\cos\phi_k$	$\Omega_k = b\sin\phi_k$	$p_k = \sigma_k + j\Omega_k$
0	$\phi_0 = \frac{(2\times0+2+1)\pi}{2\times2} = \frac{3\pi}{4}$	$\sigma_0 = 2863.4\cos\frac{3\pi}{4} = -2024.7$	$\Omega_0 = 6904.8\sin\frac{3\pi}{4} = 4882.4$	$p_0 = -2024.7 + j4882.4$
1	$\phi_1 = \frac{(2\times1+2+1)\pi}{2\times2} = \frac{5\pi}{4}$	$\sigma_1 = 2863.4\cos\frac{5\pi}{4} = -2024.7$	$\Omega_1 = 6904.8\sin\frac{5\pi}{4} = -4882.4$	$p_1 = -2024.7 - j4882.4$

$$H_a(s) = \frac{k}{(s-p_0)(s-p_1)} = \frac{k}{(s+2024.7-j4882.4)(s+2024.7+j4882.4)}$$

$$= \frac{k}{(s+2024\cdot7)^2+(4882.4)^2}$$

Here $b_0 = 2024.7^2+4882.4^2 = 27937239.85$

And $k = \dfrac{b_0}{\sqrt{1+\varepsilon^2}}$ for 'N' even

$$= \frac{27937239.85}{\sqrt{1+0.998^2}} = 19.77\times10^6$$

$$\therefore \qquad H_a(s) = \frac{19.77\times10^6}{(s+2024.7)^2+(4882.4)^2}$$

This is the system function of chebyshev filter.

Example 4.7.9 *Convert the single pole low passs filter with systm function* $H(z) = \dfrac{0.5(1+z^{-1})}{1\text{ - }0.302\,z^{-3}}$ *into hand pass filter with upper and lower cutoff frequence* $\omega\,u$ *and* ωl *respectively. The lowpass filter has 3dB bandwidth and* $\omega_p = \pi/6$ *and* $\omega_u = 3\pi/4$, $\omega_l = \pi/4$ *and draw its realization in direct form II.* **AU : May-17, Marks 15**

Solution : The digital to digital LPF to band pass filter is given by,

$$z^{-1} \rightarrow -\left[\frac{z^{-2}-a_1z^{-1}+a_2}{a_2z^{-2}-a_1z^{-1}+1}\right] \qquad \text{Here } a_1 = \frac{2\alpha k}{k+1},\ a_2 = \frac{k-1}{k+1}$$

$$\text{and a} = \frac{\cos\left(\dfrac{\omega_u+\omega_l}{2}\right)}{\cos\left(\dfrac{\omega_u-\omega_l}{2}\right)}, \qquad \text{k} = \cot\left(\frac{\omega_u-\omega_l}{2}\right)\tan\frac{\omega_p}{2}$$

where ω_u = Upper cut off frequency and ω_l is lower cutoff frequency

$$\omega_u = \frac{3\pi}{4} \qquad\qquad \omega_l = \frac{\pi}{4}$$

$$\therefore \qquad \text{k} = \cos\left(\frac{\frac{3\pi}{4}-\frac{\pi}{4}}{2}\right)\tan\frac{\pi}{12} = 0.268$$

$$\alpha = \frac{\cos\left(\dfrac{\frac{3\pi}{4}+\frac{\pi}{4}}{2}\right)}{\cos\left(\dfrac{\frac{3\pi}{4}-\frac{\pi}{4}}{2}\right)} = 0$$

Putting the values in transformation, we get

$$z^{-1} \rightarrow \frac{-(z^{-2}-0.577)}{-0.577z^{-2}+1}$$

Now the transfer function of bandpass filter can be obtained by substituting the above transformation in *H(z)*,

$$\therefore \qquad H(z) = 0.5\frac{\left[1+\dfrac{-z^{2}+0.577}{1-0.577z^{-2}}\right]}{1-0.302\left[\dfrac{-z^{-2}+0.577}{1-0.577z^{-2}}\right]}$$

$$= \frac{0.955(1-z^{-2})}{1-0.333z^{-2}} \text{ is required transfer function}$$

$$H(z) = \frac{0.955-0.955z^{-2}}{1-0.333z^{-2}}$$

Fig. 4.7.8 shows the direct form-II realization of above eqution.

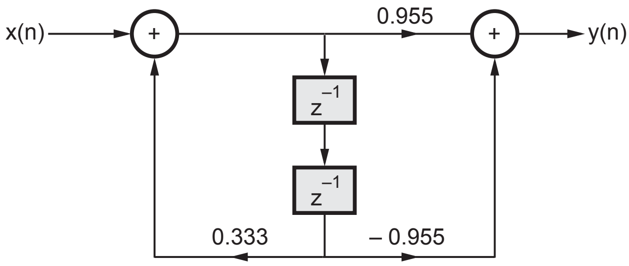


Fig. 4.7.8 Direct form-II realization

4.7.5 Comparison between Butterworth and Chebyshev Approximations

Following table lists the comparison between Butterworth and Chebyshev filter approximations :

Sr. No.	Parameter	Butterworth filter	Chebyshev filter
1.	Frequency response	Monotonically decreasing.	Ripples in passband and monotonic in stopband.
2.	Order for given set of specifications	Higher than Chebyshev.	Lower than Butterworth.
3.	Transition band	Transition band is broader than Chebyshev for given order.	Transition band is narrower than Butterworth for given order.
4.	Phase response	Fairly linear phase response. It is better than Chebyshev.	Relatively nonlinear phase response. It is inferior to Butterworth filter.
5.	Poles of $H_a(s)$ (location)	Poles of $H_a(s)$ lie on the circle of radius Ω_c in the s-plane	Poles of $H_a(s)$ lie on the ellipse in the s-plane
6.	System function $H_a(s)$	$\dfrac{(\Omega_c)^N}{(s-s_1)(s-s_2)\ldots(s-s_N)}$ (Ω_c = 1 for normalized filter)	$\dfrac{K}{(s-s_1)(s-s_2)\ldots(s-s_N)}$

Table 4.7.9 Butterworth and Chebyshev approximations

Review Questions

1. *Explain in detail Butterworth filter approximation.* **AU : May-04, Marks 8**
2. *Compare two methods used in the design of IIR digital filters based on analog filter approximations with respect to their passband, transition band and stopband characteristics. What is the selection criterion for these approximation methods?*
3. *Explain in detail Chebyshev filter approximation.*
4. *What are the types of Chebyshev filters ?*
5. *Compare Butterworth and Chebyshev approximation methods.* **AU : Dec.-16, Marks 3**

4.8 Design of IIR Filters from Analog Filters

AU : May-04, 05, 06, 07, 10, 11, 12, 16, Dec.-05, 06, 12, 13, 16

The analog filter design theory is well developed. Hence IIR digital filters are designed from analog filters. The analog filter is described by following system function.

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k} \quad \text{... (4.8.1)}$$

The system function $H_a(s)$ can also be obtained from impulse response *h(t)*, i.e.,

$$H_a(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt \quad \text{... (4.8.2)}$$

Thus laplace transform of *h(t)* gives system function. The analog filter can also be described by a linear constant coefficient differential equation i.e.,

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k} \quad \text{... (4.8.3)}$$

Here *x(t)* is input to the filter and

y(t) is the output of the filter.

Normally the design of IIR digital filter is started from specifications of analog filter. The system function of the analog filter is then obtained. The system function of the digital filter is then obtained through some transformation. We know that the analog filter is stable if its poles fall in the left half of the *s*-plane. Hence the transformation techniques should have following desirable properties :

1) The $j\Omega$ axis of s-plane should map on the unit circle in the z-plane. This provides the direct relationship between the frequencies in s-and z-plane.
2) The left half plane of s-plane should be mapped inside of the unit circle in z-plane. Because of this, the stable analog filter is converted to a stable digital filter.

Further, we will consider few important techniques for design of digital filters from analog filters.

4.8.1 IIR Filter Design by Solution of Differential Equations (Approximation of Derivatives)

4.8.1.1 $s \rightarrow z$ Transformation

In this method, the differential equation of analog filter is approximated by an equivalent difference equation of the digital filter. For the derivative $\frac{dy(t)}{dt}$ at $t = nT$, following substitution is made,

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT)-y(nT-T)}{T}$$

Here T is the sampling interval and $y(n) \equiv y(nT)$, hence above equation can be written as,

$$\frac{dy(t)}{dt} = \frac{y(n)-y(n-1)}{T} \quad \text{... (4.8.4)}$$

The system function of the differentiator having output $\frac{dy(t)}{dt}$ is,

$$\text{H(s)} = s \quad \text{... (4.8.5)}$$

The system function of the digital filter which produces output $\frac{y(n)-y(n-1)}{T}$ is

$$H(z) = \frac{1-z^{-1}}{T} \quad \text{... (4.8.6)}$$

Thus the analog domain to digital domain transformation can be obtained (from equation (4.8.4), equation (4.8.5) and equation (4.8.6)as

$$\text{s} = \frac{1-z^{-1}}{T} \quad \text{... (4.8.7)}$$

Similarly it can be shown that,

$$s^k = \left(\frac{1-z^{-1}}{T}\right)^k \quad \text{... (4.8.8)}$$

Here 'k' represents the order of the derivative. Thus the system function of the digital filter can be obtained from the system function of the analog filter by approximation of derivatives as follows.

$$\text{H(z)} = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}} \quad \text{... (4.8.9)}$$

4.8.1.2 s-plane to *z*-plane Mapping

From equation (4.8.6) we have,

$$\text{z} = \frac{1}{1-sT} \quad \text{... (4.8.10)}$$

We know that $s = \sigma + j\,\Omega$, hence above equation becomes,

$$\text{z} = \frac{1}{1-(\sigma+j\Omega)T} = \frac{1}{1-\sigma T - j\Omega T} = \frac{1-\sigma T + j\Omega T}{(1-\sigma T)^2 + (\Omega T)^2}$$

$$= \frac{1-\sigma T}{(1-\sigma T)^2 + (\Omega T)^2} + j\,\frac{\Omega T}{(1-\sigma T)^2 + (\Omega T)^2} \quad \text{... (4.8.11)}$$

Let us see how $j\Omega$ axis is mapped in z-plane. For this, substitute $\sigma = 0$ in above equation. Hence we get,

$$\text{z} = \frac{1}{1+(\Omega T)^2} + j\,\frac{\Omega T}{1+(\Omega T)^2} \quad \text{... (4.8.12)}$$

The above equation shows that complete $j\Omega$ axis $(-\infty \; to + \infty)$ is mapped on the circle of radius 1/2 and center at $z = \frac{1}{2}$. This is shown in Fig. 4.8.1. This circle is inside the unit circle. From equation (4.8.11) it can be shown that left hand plane of $j\Omega$ axis maps inside the circle of radius $\frac{1}{2}$ centred at $z = \frac{1}{2}$. And right hand plane of $j\Omega$ axis maps outside this circle. This is shown in Fig. 4.8.1. Thus a stable analog filter is converted to stable digital filter.

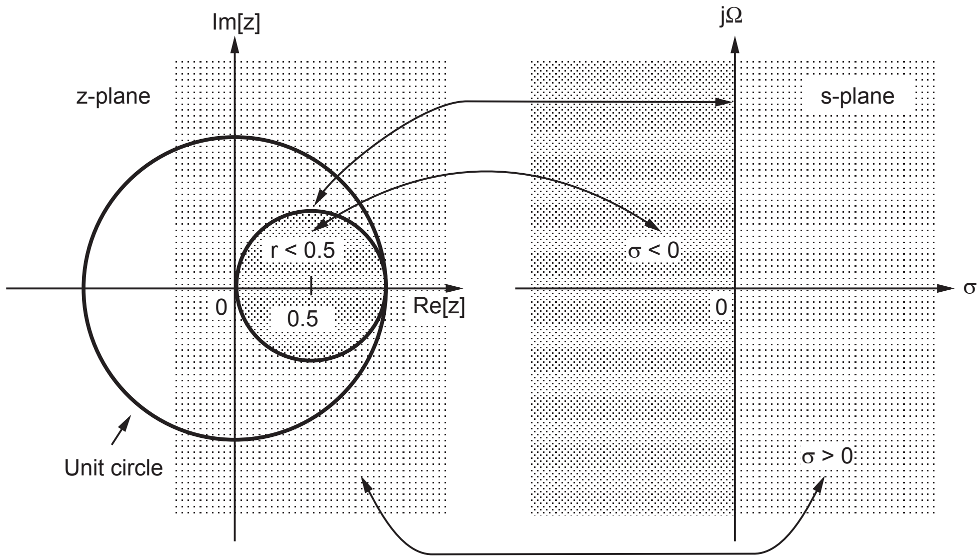


Fig. 4.8.1 Mapping from *s*-plane to *z*-plane for s $= \dfrac{1 - z^{-1}}{T}$
i.e. approximation of derivatives method

Example 4.8.1 *Obtain the system function of the digital filter by approximation of derivatives if the system function of the analog filter is as follows :*

$$H_a(s) = \frac{1}{(s+0.1)^2 + 9} \qquad \text{... (4.8.13)}$$

Solution : The derivative approximation is obtained by putting,

$$s = \frac{1 - z^{-1}}{T}$$

Hence equation (4.8.13) becomes,

$$\text{H(z)} = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}} = \frac{1}{\left(\dfrac{1-z^{-1}}{T} + 0.1\right)^2 + 9}$$

$$= \frac{\dfrac{T^2}{(1+0.2T+9.01T^2)}}{1 - \dfrac{2(1+0.1T)}{1+0.2T+9.01T^2} z^{-1} + \dfrac{1}{1+0.2T+9.01T^2} z^{-2}}$$

This is the required system function.

Review Question

1. *Explain the design of IIR filter by solution of differential equations.*

4.8.2 IIR Filter Design by Impulse Invariance

4.8.2.1 $s \to z$ Transformation

Let the impulse responses of the analog filter be $h_a(t)$. Then the unit sample response of the corresponding digital filter is obtained by uniformly sampling the impulse response of the analog filter. We know that sampled signal is obtained by putting $t = nT$ i.e.,

$$h(n) = h_a(nT), \quad n = 0, 1, 2, \ldots \qquad \text{... (4.8.14)}$$

Here $h(n)$ is the unit sample response of digital filter and T is the sampling interval.

Let the system function of the analog filter be denoted as $H_a(s)$. Let us assume that the poles of analog filter are distinct. Then its partial fraction expansion can be written as,

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k} \qquad \text{... (4.8.15)}$$

Here $\{p_k\}$ are the poles of the analog filter and $\{c_k\}$ are the coefficients of partial fraction expansion. The impulse response of the analog filter i.e. $h_a(t)$ can be obtained by taking inverse laplace transform of the system function $H_a(s)$ given by equation (4.8.15). From the standard relations of Laplace transform we can obtain $h_a(t)$ from equation (4.8.15) as,

$$h_a(t) = \sum_{k=1}^{N} c_k \, e^{p_k t} \qquad t \geq 0 \qquad \text{... (4.8.16)}$$

The unit sample response of the digital filter is obtained by uniform sampling of $h_a(t)$. i.e.,

$$h(n) = h_a(t)\Big|_{t=nT} = h_a(nT)$$

$$= \sum_{k=1}^{N} c_k \, e^{p_k nT} \qquad \text{... (4.8.17)}$$

Now the system function of the IIR filter can be obtained by taking z-transform of $h(n)$. i.e.,

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n} \quad \text{By definition of } z\text{-transform}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=1}^{N} c_k \, e^{p_k nT} \right] z^{-n} = \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} \left[e^{p_k T} z^{-1} \right]^n$$

We know that,

$\sum_{n=0}^{\infty} a^k = \frac{1}{1-a}$ using this standard relation we can write above equation of $H(z)$ as,

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_k T} z^{-1}} \qquad \text{... (4.8.18)}$$

Thus we have obtained the transformation of analog system function [equation 4.8.15] to digital system function [equation 4.8.18]. i.e.,

$$\frac{1}{s-p_k} \rightarrow \frac{1}{1-e^{p_k T} z^{-1}} \qquad \text{... (4.8.19)}$$

Above result can be extended to multiple and complex conjugate poles. The $s \rightarrow z$ transformation relationships are as follows :

$$\frac{1}{(s+p_k)^m} \rightarrow \frac{(-1)^{m-1}}{(m-1)!} \cdot \frac{d^{m-1}}{d\, p_k^{m-1}} \cdot \frac{1}{1-e^{-p_k T} z^{-1}} \qquad \text{... (4.8.20)}$$

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT)\, z^{-1}}{1-2\, e^{-aT}(\cos bT)\, z^{-1}+e^{-2aT} z^{-2}} \qquad \text{... (4.8.21)}$$

$$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}(\sin bT)\, z^{-1}}{1-2\, e^{-aT}(\cos bT)\, z^{-1}+e^{-2aT} z^{-2}} \qquad \text{... (4.8.22)}$$

Note that zeros of the system are not mapped according to the transformation discussed here.

4.8.2.2 *s*-plane to *z*-plane Mapping

From equation (4.8.18) observe that the system function $H(z)$ has poles which are located as,

$$z_k = e^{p_k T}, \qquad k = 1, 2, \ldots.. N \qquad \text{... (4.8.23)}$$

This means the analog pole at $s = p_k$ is transformed into a digital pole at $z = e^{p_k T}$. Hence the poles of analog filter are related to the corresponding poles of digital filter by following relation,

$$z = e^{sT} \qquad (\text{Here } s = p_k) \qquad \text{... (4.8.24)}$$

This relation also gives the transformation from s-plane to z-plane. We know that $s = \sigma + j\Omega$, and z can be expressed in polar form as $z = r\, e^{j\omega}$. Hence above equation can be written as,

$$r\, e^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$\therefore \qquad r\, e^{j\omega} = e^{\sigma T} \cdot e^{j\Omega T}$$

Separating the real and imaginary parts of above equation we get,

$$r = e^{\sigma T} \qquad \text{... (4.8.25)}$$

and

$$\omega = \Omega T \qquad \text{... (4.8.26)}$$

From above equations we have,

i) If $\sigma < 0$ then $0 < r < 1$ ii) If $\sigma > 0$ then $r > 1$ iii) If $\sigma = 0$ then $r = 1$

The first statement indicates that left side of s-plane (i.e. $\sigma < 0$) is mapped inside the unit circle (i.e. $0 < r < 1$).

The second statement indicates that right side of s-plane (i.e. $\sigma > 0$) is mapped outside the unit circle (i.e. $r > 1$).

The third statement indicates that the $j\Omega$ axis in s-plane (i.e. $\sigma = 0$) is mapped on the unit circle (i.e. $r = 1$).

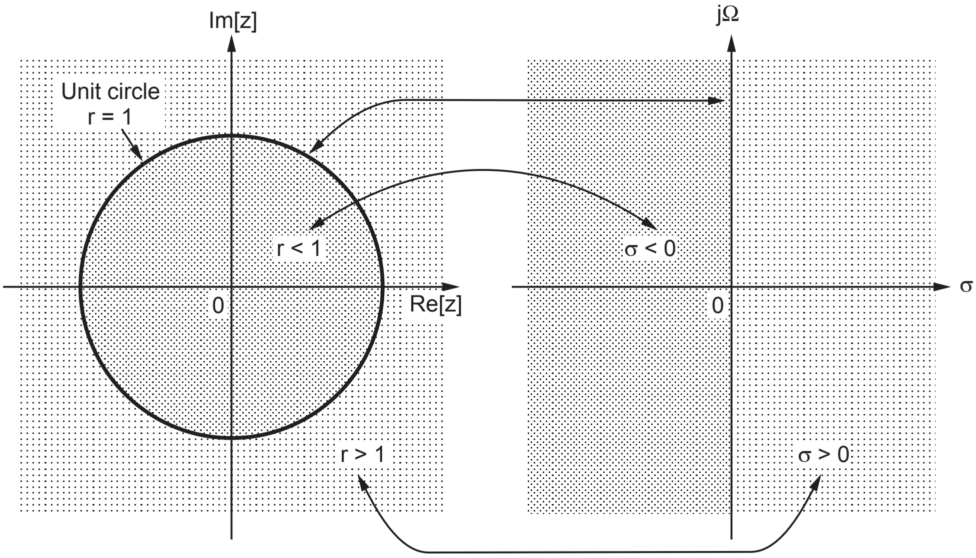


Fig. 4.8.2 Mapping of poles from *s*-plane to *z*-plane by $z = e^{sT}$

The mapping is shown below in Fig. 4.8.2.

We have seen that the discrete time system is stable if its poles lie inside the unit circle. The analog system is stable if its poles lie in left side plane of $j\Omega$ axis. Since left side plane of analog system is mapped inside the unit circle, a stable analog system is converted into stable digital system.

4.8.2.3 Aliasing in Frequency Domain

The entire $j\Omega$ axis is mapped on the unit circle. Hence this mapping is not one to one. We know that range of 'ω' is $-\pi \le \omega \le \pi$. Hence corresponding range of 'Ω' is,

$$\omega = \Omega T$$

$$\therefore \quad -\pi \le \omega \le \pi \text{ corresponds to } -\pi \le \Omega T \le \pi \quad \text{i.e.} \quad -\frac{\pi}{T} \le \Omega \le \frac{\pi}{T}.$$

Thus $-\frac{\pi}{T} \le \Omega \le \frac{\pi}{T}$ maps on $-\pi \le \omega \le \pi$.

We know that $\pi \le \omega \le 3\pi$ is same as $-\pi \le \omega \le \pi$ on the unit circle.

$\therefore$ $\pi \le \omega \le 3\pi$ corresponds to $\pi \le \Omega T \le 3\pi$ i.e. $\frac{\pi}{T} \le \Omega \le \frac{3\pi}{T}$.

Thus $\frac{\pi}{T} \le \Omega \le \frac{3\pi}{T}$ maps on $-\pi \le \omega \le \pi$. This shows that the mapping of $j\Omega$ axis is many to one on unit circle. The segments $\frac{(2k-1)}{T}\pi \le \Omega \le \frac{(2k+1)\pi}{T}$ of $j\Omega$ axis are all mapped on the unit circle $-\pi \le \omega \le \pi$. This effect takes place because of sampling. It is basically aliasing in frequency domain. This aliasing is the major drawback of impulse invariance transformation.

4.8.2.4 Frequency Response of Digital Filter Designed using Impulse Invariance

Always we are required to find frequency response of digital filter. When the digital filter is designed using impulse invariance its frequency response is related to that of analog filter by following relation,

$$H(\omega) = \frac{1}{T} H_a\left(j\frac{\omega}{T}\right) \quad -\pi \le \omega \le \pi \qquad \text{... (4.8.27)}$$

The proof of this relation is not presented here just to avoid complex mathematics.

Example 4.8.2 *The system function of the analog filter is given as,*

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}$$

Obtain the system function of the IIR digital filter by using impulse invariance method.

AU : Dec.-13, Marks 8, May-10, Marks 16

Solution : The denominator of $H_a(s)$ has roots at

$$p_1 = -0.1 + j3, \quad p_2 = -0.1 - j3$$

$$\therefore \quad H_a(s) = \frac{s+0.1}{(s+0.1-j3)(s+0.1+j3)}$$

Let us expand $H_a(s)$ is partial fractions,

$$H_a(s) = \frac{A_1}{s+0.1-j3} + \frac{A_2}{s+0.1+j3} \qquad \text{... (4.8.28)}$$

Values of A_1 and A_2 can be obtained as follows :

$$A_1 = (s+0.1-j3)\cdot H_a(s)\Big|_{s=-0.1+j3} = \frac{s+0.1}{s+0.1+j3}\Bigg|_{s=-0.1+j3}$$

$$= \frac{-0.1 + j\,3 + 0.1}{-0.1 + j\,3 + 0.1 + j\,3} = \frac{1}{2}$$

$$A_2 = \left(s + 0.1 + j\,3\right) \cdot H_a\,(s)\Big|_{s=-0.1-j\,3} = \left.\frac{s + 0.1}{s + 0.1 - j\,3}\right|_{s=-0.1-j\,3}$$

$$= \frac{-0.1 - j\,3 + 0.1}{-0.1 - j\,3 + 0.1 - j\,3} = \frac{1}{2}$$

Hence equation (4.8.28) becomes,

$$H_a(s) = \frac{\frac{1}{2}}{s + 0.1 - j\,3} + \frac{\frac{1}{2}}{s + 0.1 + j\,3} \qquad \text{... (4.8.29)}$$

We know that impulse invariance transformation is given by equation (4.8.29) as,

$$\frac{1}{s - p_k} \to \frac{1}{1 - e^{p_k T} z^{-1}}$$

Using this relation we can obtain the system function for digital filter from equation (4.8.29) as,

$$H\,(z) = \frac{\frac{1}{2}}{1 - e^{-0.1T + j\,3T}\; z^{-1}} + \frac{\frac{1}{2}}{1 - e^{-0.1T - j\,3T}\; z^{-1}}$$

This system function can be simplified further as,

$$H\,(z) = \frac{1 - \left(e^{-0.1T}\;\cos 3T\right) z^{-1}}{1 - \left(2\,e^{-0.1T}\;\cos 3T\right) z^{-1} + e^{-0.2\,T}\; z^{-1}}$$

Observe that this equation can also be obtained using equation (4.8.21).

Example 4.8.3 *If* $H_a\,(s) = \dfrac{1}{(s+1)\,(s+2)}$ *, find the corresponding* $H\,(z)$ *using impulse invariance method for sampling frequency of 5 samples/sec.*

Solution : Let us first expand $H_a\,(s)$ in partial fractions. i.e.,

$$H_a\,(s) = \frac{c_1}{s+1} + \frac{c_2}{s+2}$$

$$\therefore \qquad c_1 = (s+1)\,H_a\,(s)\Big|_{s=-1} = \left.\frac{1}{s+2}\right|_{s=-1} = \frac{1}{-1+2} = 1$$

$$c_2 = (s+2)\;H_a\,(s)\Big|_{s=-2} = \left.\frac{1}{s+1}\right|_{s=-2} = \frac{1}{-2+1} = -1$$

Hence $H_a(s)$ becomes,

$$H_a(s) = \frac{1}{s+1} - \frac{1}{s+2} \qquad \text{... (4.8.30)}$$

It is given that sampling frequency $F_s = 5$ Hz

$\therefore$ Sampling period $T = \frac{1}{F_s} = \frac{1}{5} = 0.2$

From equation (4.8.19), impulse invariance transformation is given as,

$$\frac{1}{s-p_k} \rightarrow \frac{1}{1-e^{p_k T} z^{-1}}$$

Now let us apply this transformation to individual terms of $H_a(s)$ of equation (4.8.30) i.e.,

$$\frac{1}{s+1} \rightarrow \frac{1}{1-e^{-1\times 0.2} z^{-1}}, \qquad \text{Here} \quad p_1 = -1$$

and,

$$\frac{1}{s+2} \rightarrow \frac{1}{1-e^{-2\times 0.2} z^{-1}}, \qquad \text{Here} \quad p_2 = -2$$

Hence $H(z)$ of digital filter becomes,

$$H(z) = \frac{1}{1-e^{-1\times 0.2} z^{-1}} - \frac{1}{1-e^{-2\times 0.2} z^{-1}}$$

$$\therefore \qquad H(z) = \frac{1}{1-e^{-0.2} z^{-1}} - \frac{1}{1-e^{-0.4} z^{-1}} = \frac{1}{1-0.818\, z^{-1}} - \frac{1}{1-0.67\, z^{-1}}$$

On simplifying this equation we get,

$$H(z) = \frac{0.148\, z}{z^2 - 1.48\, z + 0.548}$$

Example 4.8.4 *Develop the digital transfer function G(z) from the causal analog transfer function* $H(s) = \frac{16(s+2)}{(s^2+2s+5)(s+3)}$ *using impulse invariance method. Assume T = 1 sec.*

AU : May-04, Marks 8

Solution :

$$H(s) = \frac{16(s+2)}{(s^2+2s+5)(s+3)} = \frac{1-j3}{(s+1-j2)} + \frac{1+j3}{(s+1+j2)} - \frac{2}{s+3}$$

Impulse invariance transformation is given as,

$$\frac{1}{s-p_k} \rightarrow \frac{1}{1-e^{p_k T} z^{-1}}$$

Here T = 1 and applying above equation to $H(s)$,

$$\therefore \quad H(z) = \frac{1-j3}{1-e^{(-1+j2)\times 1}z^{-1}} + \frac{1+j3}{1-e^{(-1-j2)\times 1}z^{-1}} - \frac{2}{1-e^{-3\times 1}z^{-1}}$$

After simplification of above equation we get,

$$H(z) = \frac{2+2.3z^{-1}}{1-0.3z^{-1}+0.134z^{-2}} - \frac{2}{1-0.05z^{-1}}$$

Example 4.8.5 *Convert $H(s) = (2s - 1)/(s^2 + 5s + 4)$ to $H(z)$, using impulse invariant method with sampling period = 0.5 sec.* **AU : Dec.-06, Marks 10**

Solution :

$$H(s) = \frac{2s-1}{s^2+5s+4}$$

$$= \frac{2s+1}{(s+1)(s+4)}$$

$$= \frac{\frac{7}{3}}{s+4} - \frac{\frac{1}{3}}{s+1}$$

Here let us apply impulse invariant transformation,

$$\frac{1}{s-p_k} \rightarrow \frac{1}{1-e^{p_k T}z^{-1}} \quad \text{with T = 0.5 sec}$$

$$\therefore \quad H(z) = \frac{\frac{7}{3}}{1-e^{-4\times 0.5}z^{-1}} - \frac{\frac{1}{3}}{1-e^{-1\times 0.5}z^{-1}}$$

$$= \frac{\frac{7}{3}}{1-0.135z^{-1}} - \frac{\frac{1}{3}}{1-0.6z^{-1}}$$

This is required digital filter transfer function.

Review Question

1. *Explain the impulse invariance method of IIR filter design. What is its draw back ?*

AU : May-05, Marks 10, Dec.-05, Marks 8, May-06, Marks 8, May-11, Marks 4, Dec.-12, Marks 6

4.8.3 IIR Filter Design by Bilinear Transformation

4.8.3.1 $s \rightarrow z$ Transformation

In the impulse invariance method the impulse response of analog filter is sampled. We know that whenever sampling takes place problems due to aliasing occur. This aliasing takes place in frequency domain. Hence to design higher frequency filters using impulse invariance, sampling frequencies should be high. This limits the use of impulse invariance method to only low pass (or lower frequencies) type of filters.

Hence we will discuss a different type of mapping from analog to digital domain which overcomes the limitations of impulse invariance method. It is called bilinear transformation and given as

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \quad \text{... (4.8.31)}$$

The above equation can also be written as,

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \quad \text{... (4.8.32)}$$

4.8.3.2 s-plane to z-plane Mapping

We know that $s = \sigma + j\,\Omega$ and polar from of 'z' is $z = r\,e^{j\,\omega}$. Putting this value of 'z' in above equation we get,

$$s = \frac{2}{T}\,\frac{r\,e^{j\,\omega}-1}{r\,e^{j\,\omega}+1}$$

We know that $e^{j\,\omega} = \cos\,\omega + j\,\sin\,\omega$, hence above equation becomes,

$$s = \frac{2}{T}\cdot\frac{r\,(\cos\omega + j\,\sin\omega)-1}{r\,(\cos\omega + j\,\sin\omega)+1}$$

Separating the real and imaginary parts of this equation we get,

$$s = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\,\cos\omega} + j\,\frac{2r\,\sin\omega}{1+r^2+2r\,\cos\omega}\right) \quad \text{... (4.8.33)}$$

We know that $s = \sigma + j\,\Omega$. Comparing with above equation, we get the values of σ and Ω as follows :

$$\sigma = \frac{2}{T}\cdot\frac{r^2-1}{1+r^2+2r\,\cos\omega} \quad \text{... (4.8.34)}$$

and $$\Omega = \frac{2}{T} \cdot \frac{2r \sin \omega}{1 + r^2 + 2r \cos \omega} \qquad \text{... (4.8.35)}$$

From equation (4.8.34) we have following :

i) If $r > 1$, then $\sigma > 0$

ii) If $r < 1$, then $\sigma < 0$

iii) If $r = 1$, then $\sigma = 0$

The first statement indicates that the right hand side of s-plane (i.e. $\sigma > 0$) maps outside of the unit circle (i.e. $r > 1$).

The second statement indicates that the left hand side of s-plane (i.e. $\sigma < 0$) maps inside of the unit circle (i.e. $r < 1$).

The third statement indicates that the $j\Omega$ axis in s-plane (i.e. $\sigma = 0$) maps on the unit circle (i.e. $r = 1$).

This mapping is similar to that of impulse invariance method shown in Fig. 4.8.2. But in impulse invariance the mapping was valid only for poles. But bilinear transform maps poles as well as zeros. The mapping shows that a stable analog filter is converted to stable digital filter.

4.8.3.3 Frequency Warping

Now let us see how the imaginary axis $j\Omega$ is mapped on unit circle. Equation (4.8.35) gives relationship between Ω and ω. i.e.,

$$\Omega = \frac{2}{T} \frac{2r \sin \omega}{1 + r^2 + 2r \cos \omega}$$

If we want the relationship of $j\Omega$ axis in s-plane (i.e. $\sigma = 0$) to unit circle in z-plane (i.e. $r = 1$) we have to put $r = 1$ in above equation. Then we get,

$$\Omega = \frac{2}{T} \frac{2 \sin \omega}{1 + 1 + 2 \cos \omega}$$

$$= \frac{2}{T} \frac{\sin \omega}{1 + \cos \omega} = \frac{2}{T} \frac{2 \sin \frac{\omega}{2} \cos \frac{\omega}{2}}{2 \cos^2 \frac{\omega}{2}}$$

$$= \frac{2}{T} \tan \frac{\omega}{2} \qquad \text{... (4.8.36)}$$

or $$\omega = 2 \tan^{-1} \frac{\Omega T}{2} \qquad \text{... (4.8.37)}$$

This equation shows that the entire range of 'Ω' maps only once in $-\pi \le \omega \le \pi$. This mapping is highly nonlinear. Fig. 4.8.3 shows this mapping.

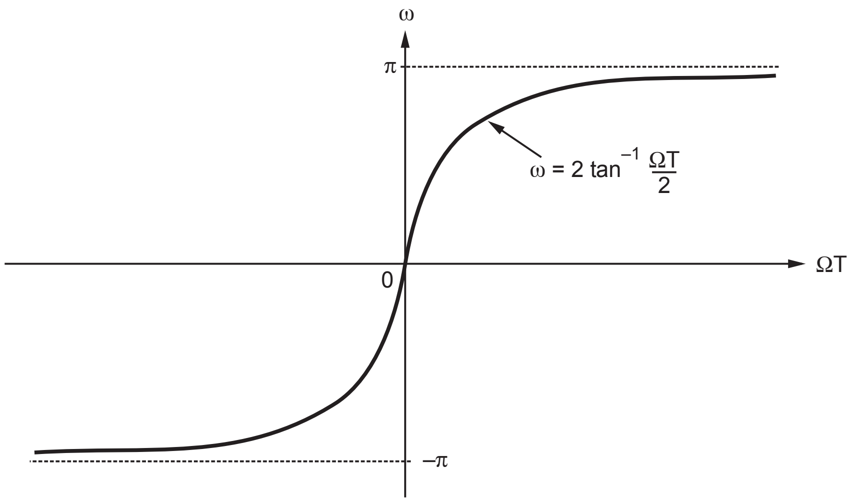


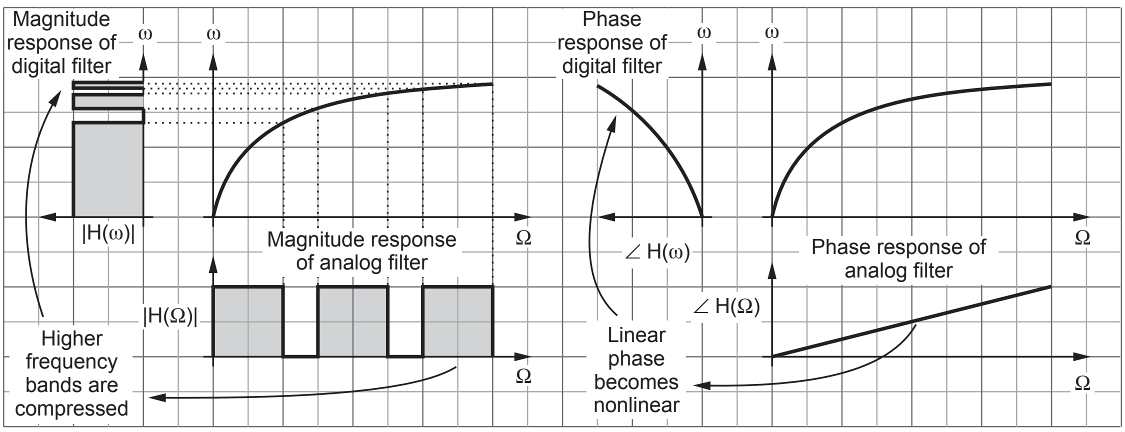
Fig. 4.8.3 Mapping between 'Ω' and 'ω' in bilinear transformation

Observe the nonlinearity in the relationship between 'ω' and 'Ω'. It is called *frequency warping*.

Effect of frequency warping on magnitude and phase response

1) Fig. 4.8.4 shows the effect of frequency warping on magnitude and phase response. In Fig. 4.8.4(a) observe that higher frequency bands are compressed while transformed to digital domain.

2) Similarly in Fig. 4.8.4 (b) observe that linear phase of the original analog filter becomes nonlinear when converted to digital domain.



(a) (b)

Fig. 4.8.4 Effect of frequency warping on magnitude and phase

No aliasing in bilinear transformation

The main difference between impulse invariance and bilinear transformation is that there is no aliasing effect in bilinear transformation. This is the major advantage of bilinear transformation. Observe that the complete $j\Omega$ axis is mapped on the unit circle only once. But in impulse invariance the segments $\frac{(2k-1)\pi}{T} \leq \Omega \leq \frac{(2k+1)\pi}{T}$ of $j\Omega$ axis are all mapped on unit circle repeatedly. Thus the transformation is many to one. Hence problem of aliasing takes place in impulse invariance method. The problem with bilinear transformation is that the frequency relationship is nonlinear.

4.8.4 Comparison between Impulse Invariance Method with Bilinear Transform Method

Sr. No.	Impulse invariance	Bilinear transform
1.	$\frac{1}{s-p_k} \rightarrow \frac{1}{1-e^{p_k T} z^{-1}}$	$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$
2.	$\frac{(2k-1)}{T}\pi \leq \Omega \leq \frac{(2k+1)}{T}\pi$ of $j\Omega$ axis are mapped on unit circle $-\pi \leq \omega \leq \pi$. Hence there is aliasing in frequency domain.	There is unique mapping of $j\Omega$ axis on unit circle. Hence there is no aliasing.
3.	$\omega = \Omega T$	$\omega = 2\tan^{-1}\frac{\Omega T}{2}$
4.	Frequency relationship is linear.	Nonlinear frequency relationship.
5.	Maps only poles.	Maps poles as well as zeros.
6.	Used for lowpass filters of lower cutoff frequency.	Used for all the types of filters.

Examples for Understanding

Example 4.8.6 *The system function of the analog filter is given as,*

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+16}$$

Obtain the system function of the digital filter using bilinear transformation which is resonant at $\omega_r = \frac{\pi}{2}$.

Solution : From the denominator of $H_a(s)$ we can write the poles of analog filter as,

$$(s+0.1)^2+16 = (s+0.1-j4)(s+0.1+j4)$$

$$\therefore \quad s = -0.1+j4 \quad \text{and} \quad s = -0.1-j4$$

There are the two complex conjugate poles. We know that $s = \sigma + j\,\Omega$. Thus the values of 'σ' and 'Ω' for these two poles are,

$$\sigma = -0.1 \quad \text{and} \quad \Omega = \pm 4$$

A function is said to be resonant at its poles. Hence $H_a(s)$ will be resonant at,

$$s = -0.1 \pm j4$$

In other words we can state that $H_a(s)$ will be resonant at $\Omega = 4$. It is required that the digital filter should be resonant at $\omega_r = \frac{\pi}{2}$. This means the bilinear transformation should map $\Omega = 4$ into $\omega_r = \frac{\pi}{2}$. From equation 4.8.36 the relationship between 'Ω' and 'ω' is given as,

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

$$\therefore \qquad T = \frac{2}{\Omega} \tan \frac{\omega}{2}$$

Putting for $\Omega = 4$ and $\omega_r = \omega = \frac{\pi}{2}$,

$$T = \frac{2}{4} \tan \frac{\pi}{4} = \frac{1}{2}$$

This means if we select $T = \frac{1}{2}$, then the resonant frequency $\Omega = 4$ of analog filter will map into $\omega_r = \frac{\pi}{2}$ of digital filter in bilinear transformation. The bilinear transformation is given by equation (4.8.31) as,

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

At $\qquad T = \frac{1}{2}, \qquad s = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

Putting for this value of 's' in $H_a(s)$ we get $H(z)$ i.e.,

$$H(z) = H_a(s)\Big|_{s=4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.1}{\left[4\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.1\right]^2 + 16}$$

On simplifying the above equation we get,

$$H(z) = \frac{0.128 + 0.006\,z^{-1} - 0.122\,z^{-2}}{1 + 0.0006\,z^{-1} + 0.975\,z^{-2}} = \frac{0.128\,z^2 + 0.006\,z - 0.122}{z^2 + 0.0006\,z + 0.975}$$

The roots of denominator of $H(z)$ are poles of $H(z)$. They are located at $z = -0.0003 + j\,0.9874208$ and $z = -0.0003 - j\,0.9874208$ converting these poles to their polar values,

$$z = 0.9874208\; e^{\pm j\,1.5711001}$$

We know that $z = r\,e^{j\omega}$, hence, $r = 0.9874208$ and $\omega = \pm 1.5711001 \approx \pm \frac{\pi}{2}$. Thus the two complex conjugate poles are located at $\omega = \pm \frac{\pi}{2}$. Hence $H(z)$ will be resonant at $\omega = \frac{\pi}{2}$.

Example 4.8.7 *Develop a transformation for the solution of a first order linear constant coefficient differential equation by using trapezoidal rule for the integral approximation.*

Solution : Consider the system function,

$$H_a(s) = \frac{b}{s+a} \quad \dots (4.8.38)$$

This system is also described by the first order differential equation,

$$\frac{d}{dt} y(t) + a y(t) = b\, x(t) \quad \dots (4.8.39)$$

Let $y'(t)$ be the derivative of $y(t)$. Then we can write,

$$y(t) = \int_{t_0}^{t} y'(\tau)\, d\tau + y(t_0) \quad \dots (4.8.40)$$

Here observe that $y(t)$ is obtained by integrating the derivative $y'(t)$. Heere $y(t_0)$ is the initial value of $y(t)$. Let us approximate the integration in above equation by trapezoidal rule. Let $t = nT$ and $t_0 = nT - T$. Then,

$$\int_{t_0}^{t} y'(\tau) d\tau = \frac{T}{2}\,[y'(nT) + y'(nT - T)]$$

Hence equation (4.8.40) can be written as,

$$y(nT) = \frac{T}{2}\,[y'(nT) + y'(nT - T)] + y(nT - T) \quad \dots (4.8.41)$$

Let the differential equation of equation (4.8.39) be evaluated at $t = nT$. Then we get,

$$y'(nT) + a y(nT) = b x(nT)$$

$$\therefore \quad y'(\mathrm{nT}) = -a y(nT) + b x\,(nT)$$

Delaying above equation by T, i.e.,

$$\therefore \quad y'(\mathrm{nT-T}) = -ay(nT-T)+bx(nT-T)$$

Putting for $y'(\mathrm{nT})$ and $y'(nT-T)$ from above two equations in equation (4.8.41) we get,

$$y(nT) = \frac{T}{2}[-ay(nT)+bx(nT)-ay(nT-T)+bx(nT-T)]+y(nT-T)$$

$$y(nT) = -\frac{aT}{2}y(nT)+\frac{bT}{2}x(nT)-\frac{aT}{2}y(nT-T)+\frac{bT}{2}x(nT-T)+y(nT-T)$$

$$\therefore \left(1+\frac{aT}{2}\right)y(nT)-\left(1-\frac{aT}{2}\right)y(nT-T) = \frac{bT}{2}[x(nT)+x(nT-T)]$$

For simplicity of notations,

Let $y(nT) \equiv y(n)$ and $x(nT) \equiv x(n)$. Then above equation becomes,

$$\left(1+\frac{aT}{2}\right)y(n)-\left(1-\frac{aT}{2}\right)y(n-1) = \frac{bT}{2}[x(n)+x(n-1)]$$

Taking z-transform of above equation,

$$\left(1+\frac{aT}{2}\right)Y(z)-\left(1-\frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}[X(z)+z^{-1}X(z)]$$

$$\therefore \quad Y(z)\left[1+\frac{aT}{2}-\left(1-\frac{aT}{2}\right)z^{-1}\right] = \frac{bT}{2}(1+z^{-1})X(z)$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{bT}{2}(1+z^{-1})}{1+\frac{aT}{2}-\left(1-\frac{aT}{2}\right)z^{-1}} = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}$$

On comparing above equation with equation (4.8.38) the mapping from s-plane to z-plane is given as,

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Examples with Solution

Example 4.8.8 *Convert the analog transfer function of the 2^{nd} order Butterworth filter into digital transfer function using bilinear transformation.*

AU : May-05, Marks 6

Solution : The system function of normalized second order butterworth filter is given as,

$$H_a(s) = \frac{1}{s^2 + \sqrt{2}\, s + 1}$$

Bilinear transformation is given as,

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Let us assume T = 1,

$$z = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\therefore \quad H(z) = H_a(s)\Big|_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{1}{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2\right] + \sqrt{2}\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] + 1}$$

$$= \frac{0.127\,(1 + 2\,z^{-1} + z^{-2})}{1 - 0.766 z^{-1} + 0.277\, z^{-2}}$$

Example 4.8.9 *Design a single pole lowpass digital filter with a 3 dB bandwidth of* 0.2π *, using the bilinear transformation applied to the analog filter,*

$$H(s) = \frac{\Omega_c}{s + \Omega_c}$$

Here Ω_c *is the 3-dB bandwidth of the analog filter.*

Solution : For bilinear transformation,

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2} = \frac{2}{T}\tan\frac{0.2\pi}{2}, \qquad \text{since } \omega_c = 0.2\pi \text{ given}$$

$$= \frac{0.65}{T}$$

Putting this value of Ω_c in $H(s)$,

$$H(s) = \frac{\Omega_c}{s + \Omega_c} = \frac{\frac{0.65}{T}}{\frac{s + 0.65}{T}}$$

Now $H(z) = H(s)\big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$ By bilinear transformation

$$= \frac{\frac{0.65}{T}}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+\frac{0.65}{T}} = \frac{\frac{0.65}{T}}{\frac{2}{T}\left(\frac{z-1}{z+1}\right)+\frac{0.65}{T}}$$

Simplifying the above equation,

$$= \frac{0.245(z+1)}{z-0.509}$$

This is the required single pole digital filter.

Example 4.8.10 *For the analog transfer fuction H(s) = 2/(s + 1)(s + 3) determine H(z) using bilinear transformation. With T = 0.1 sec.* **AU : April-10, Marks 8, May-12, Marks 16**

Solution : $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$

$$s = \frac{2}{0.1}\frac{1-z^{-1}}{1+z^{-1}}, \text{ Here } T = 0.1 \text{ sec}$$

$$s = 20\,\frac{1-z^{-1}}{1+z^{-1}}$$

$$H(z) = H(s)\Bigg|_{s=\frac{20(1-z^{-1})}{\left(1+z^{-1}\right)}}$$

$$= \frac{2}{(s+1)(s+3)}\Bigg|_{s=\frac{20\,(1-z^{-1})}{\left(1+z^{-1}\right)}}$$

$$= \frac{2}{\left[\frac{20(1-z^{-1})}{1+z^{-1}}+1\right]\left[\frac{20(1-z^{-1})}{1+z^{-1}}+3\right]} = \frac{0.00414\ (1+z^{-1})^2}{1-1.64\,z^{-1}+0.67\,z^{-2}}$$

Example 4.8.11 *Apply bilinear transformation to* $H(s) = \dfrac{2}{(s+1)\,(s+2)}$ *with T = 1 sec and find H(z).* **AU : May-16, Marks 8**

Solution : Here

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{1}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \text{ since T = 1 sec}$$

$$\therefore \quad s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\therefore \quad H(z) = H(s)\Big|_{s=2\left(\frac{1-z^{-1}}{1-z^{-1}}\right)} = \frac{2}{\left[\dfrac{2(1-z^{-1})}{(1+z^{-1})}+1\right]\left[\dfrac{2(1-z^{-1})}{1+z^{-1}}+2\right]}$$

$$= \frac{2(1+z^{-1})^2}{(2-2z^{-1}+1+z^{-1})\;(2-2z^{-1}+2+2z^{-1})}$$

$$\frac{(1+z^{-1})^2}{2(3-z^{-1})} = \frac{0.167\;(1+z^{-1})^2}{(1-0.33\;z^{-1})}$$

Example 4.8.12 *Obtain the system function of the digital filter if the analog filter is* $H_a(s) = 1/[(s+0.2)^2+2]$*. Using the impulse invariance method and the bilinear transformation method obtain the digital filter.* **AU : Dec.-16, Marks 16**

Solution : i) Impulse invariance method :

Let us rearrange the given equation as,

$$H_a(s) = \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{(s+0.2)^2+(\sqrt{2})^2}$$

Consider the standard impulse invariance transformation formula,

$$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

with b = $\sqrt{2}$, a = 0.2 and $\tau = 1$ in above equation,

$$H(z) = \frac{1}{\sqrt{2}}\cdot\frac{e^{-0.2}(\sin\sqrt{2})z^{-1}}{1-2e^{-0.2}(\cos\sqrt{2})z^{-1}+e^{-2\times 0.2}z^{-2}} = \frac{0.572z^{-1}}{1-0.255\;z^{-1}+0.67\;z^{-2}}$$

ii) Bilinear transformation

$$H_a(s) = \frac{1}{(s+0.2)^2+2} = \frac{1}{s^2+0.4s+0.04+2} = \frac{1}{s^2+0.4s+2.04}$$

Bilinear transformation is given as,

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = H_a(s)\Big|_{s=\frac{2(1-z^{-1})}{1+z^{-1}}} \text{ assuming T = 1}$$

$$= \frac{(1+z^{-1})^2}{2.44-4z^{-1}+1.6z^{-2}}$$

Examples for Practice

Example 4.8.13 : *An analog filter has the following system function. Convert this filter into a digital filter using bilinear transformation.* $H(s) = \frac{1}{(s+2)^2(s+1)}$

$$\left[\textbf{Ans. : } H(z) = \frac{(1+z^{-1})^3}{(6-2z^{-1}+6z^{-2})\ \ (3-z^{-1})}\right]$$

Example 4.8.14 : *Convert the Bessel filter* $H(s) = 3/(s^2+3s+3)$ *using bilinear transformation with* $T_s = 0.5$ *sec.* **AU : Dec.-06, Marks 8**

$$\left[\textbf{Ans. : } H(z) = \frac{0.157(1+2z^{-1}+z^{-2})}{1-0.1z^{-1}-0.263z^{-2}}\right]$$

Review Questions

1. *Explain the bilinear transform method of IIR filter design. What is warping effect ? Explain the poles and zeros mapping procedure clearly.* **AU : May-06, Marks 6, May-07, Marks 16**
2. *Compare the impulse invariance and bilinear transformation methods.*
3. *What is warping effect ? What is its effect on magnnitude and phase response ?*
4. *What are the requirements of converting a stable analog filter into a stable digital filter.*

4.9 Design of IIR Filters using Butterworth and Chebyshev Approximations

AU : Dec.-11, 15, May-14, 15, 17

Now let us study the design of digital filters using analog filter approximations.

4.9.1 Design Steps

Following steps must be adopted while designing the IIR filter.

Fig. 4.9.1 shows common minimum steps required for the design of IIR filter using filter approximations. In case of normalized filter and frequency transformations, two steps will be added. In this section we will study the design of IIR filters using analog filter approximations.

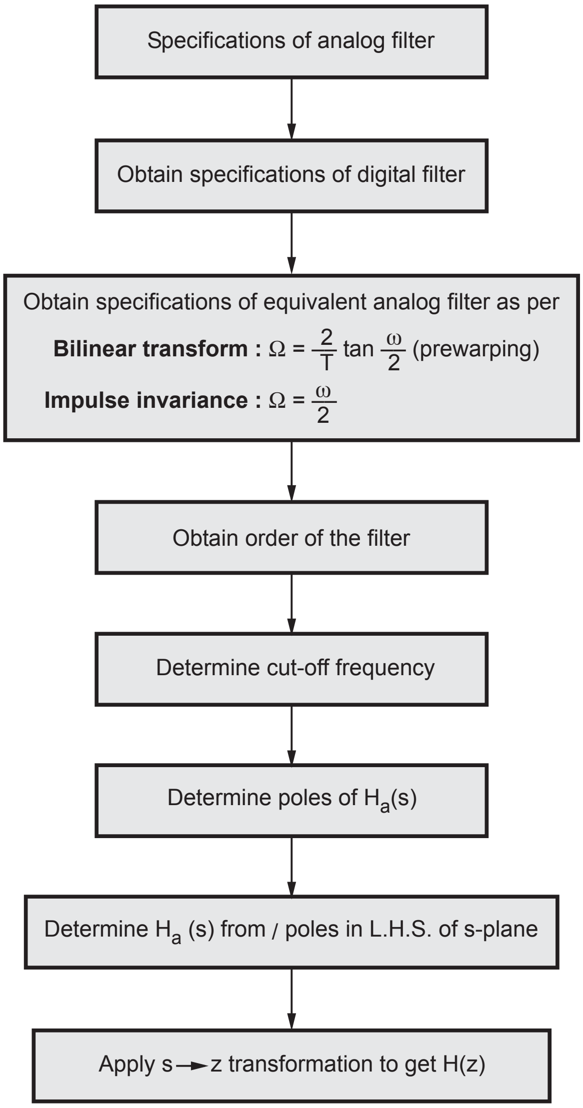


Fig. 4.9.1 Common steps required in IIR filter design

4.9.2 Solved Examples on Butterworth Approximation

Examples for Understanding

Example 4.9.1 *Design a second order lowpass DT butterworth filter with cut-off frequency of 1 kHz and sampling frequency of 10^4 samples/sec by bilinear transformation.*

Solution : Given data :

Order of the filter, $N = 2$

Analog filter cut-off frequency $F_c = 1000$ Hz

Sampling frequency $F_s = 10{,}000$ Hz

Defining specifications for digital filter :

Order of the filter, $N = 2$ (given) cut-off frequency f_c of the digital filter can be obtained by applying the formula for conversion of continuous to discrete time frequencies. $\left(\text{i.e. } f = \frac{F}{F_s}\right)$. Hence,

$$f_c = \frac{F_c}{F_s} = \frac{1000}{10000} = 0.1 \quad \text{cycles/sample}$$

We know that angular frequency of the discrete time signal is given as ($\omega = 2\pi f$) i.e.,

$$\omega_c = 2\pi f_c = 2\pi \times 0.1 = 0.2\,\pi \quad \text{radians/sample}$$

This is the angular discrete time cut-off frequency required.

To obtain specifications of equivalent analog filter for bilinear Transformation :

In this example we are using bilinear transformation. We have cut-off frequency of the digital filter as $\omega_c = 0.2\,\pi$. Now we should determine the equivalent cut-off frequency (Ω_c) of analog filter according to bilinear transformation. The frequency relationship in bilinear transformation is given as,

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

$$\Omega_c = \frac{2}{T} \tan \frac{\omega_c}{2}$$

Here 'T' is the sampling duration and it is given as $T = \frac{1}{F_s}$. Here $F_s = 10000$ Hz. Hence $T = \frac{1}{10000}$ sec. Putting these values above equation becomes,

$$\Omega_c = \frac{2}{\left(\frac{1}{10000}\right)} tan\left(\frac{0.2\,\pi}{2}\right) = 6498.4 \quad \text{radians/sec}$$

Thus we have the specifications of analog filter for bilinear transformation as,

$$\Omega_c = 6498.4 \quad \text{radians/sec} \quad \text{and}$$

$$N = 2$$

Important note :

Here observe that first we obtained digital filter cutoff frequency. Then we obtained analog filter cutoff frequency according to bilinear transformation frequency relationship. Sometimes this procedure is called *prewarping*. This prewarping removes the warping effect when we apply bilinear transformation to analog filter system function. Hence the cutoff frequencies are mapped properly. The same procedure was done in previous example also for impulse invariance method, eventhough there is no warping concept. Hence this is the standard procedure for impulse invariance as well as bilinear transformation.

To obtain poles of $H_a(s)$:

The poles of $H_a(s) \cdot H_a(-s)$ are given by equation (4.7.2) as,

$$p_k = \pm \Omega_c \; e^{j(N+2k+1)\pi/2N}, \quad k = 0, 1, 2, \ldots, N-1$$

For $\Omega_c = 6498.4$ and N = 2 above equation becomes,

$$p_k = \pm\, 6498.4 \; e^{j(3+2k)\pi/4}, \quad k = 0, 1 \qquad \text{... (4.9.1)}$$

With $k = 0$ in above equation we get,

$$\therefore \quad p_0 = \pm\, 6498.4 \; e^{j\,3\pi/4}$$

$$= -\,4595.0627 + j\,4595.0627 \;\; \text{and} \;\; 4595.0627 - j\,4595.0627$$

With $k = 1$ in equation (4.9.1) we get

$$p_1 = \pm\, 6498.4 \; e^{j\,5\pi/4}$$

$$= -\,4595.0627 - j\,4595.0627 \;\; \text{and} \;\; 4595.0627 + j\,4595.0627$$

The poles of $H_a(s)$ will be the poles lying in left half of the s-plane. i.e.,

$$s_1 = -\,4595 + j\,4595.0627 \quad \text{and}$$

$$s_1^* = -\,4595 - j\,4595.0627$$

Observe that the two poles lying in left half of s-plane are complex conjugate of each other. Since this is second order filter there are two poles. Note that the poles always occur in complex conjugate pairs if 'N' is even. If 'N' is odd, then $(N-1)$ poles occur in complex conjugate pairs and one pole lies on real axis. Hence the coefficients are not imaginary.

To obtain system function $H_a(s)$:

For butterworth approximation, the system function for second order filter is given as,

$$H_a(s) = \frac{\Omega_c^2}{(s-s_1)\left(s-s_1^*\right)}$$

Here the order is '2' hence numerator is equal to Ω_c^2. Putting values in above equation we get,

$$H_a(s) = \frac{(6498.4)^2}{(s+4595.0627-j\,4595.0627)\,(s+4595.0627+j\,4595.0627)}$$

$$= \frac{(6498.4)^2}{(s+4595.0627)^2+(4595.0627)^2}$$

$$= \frac{(6498.4)^2}{s^2+9190.125\,s+42.23\times 10^6} \quad \text{... (4.9.2)}$$

This is the system function of analog filter.

To obtain $H(z)$ using bilinear transformation :

Next step is to obtain system function of the digital filter i.e. $H(z)$ by applying bilinear transformation to $H_a(s)$. Bilinear transformation is given as,

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

We have $T = \frac{1}{10000}$, then above equation becomes,

$$s = \frac{2}{\left(\frac{1}{10000}\right)}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= 2\times 10^4\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \quad \text{... (4.9.3)}$$

Putting for 's' from above equation in equation (4.9.2) we get $H(z)$ i.e.,

$$H(z) = \frac{(6498.4)^2}{\left[2\times10^4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 9190.125\left[2\times10^4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] + 42.23\times10^6}$$

On simplifying above equation we get,

$$H(z) = \frac{0.0676\left(1+2z^{-1}+z^{-2}\right)}{1-1.143z^{-1}+0.4128z^{-2}} \qquad \text{... (4.9.4)}$$

This is the required system function of digital lowpass second order butterworth filter.

Example 4.9.2 *To show that $\frac{2}{T}$ can be assumed to be '1' while solving problems of filter design using bilinear transformation.*

Explain this concept with the help of redesigning second order lowpass filter of last example (i.e. example 4.9.1)

Solution : Important (What we are doing ?) :

We have already designed a second order lowpass filter of $F_c = 1$ kHz and $F_s = 10{,}000$ Hz in previous example. Here we will consider the same filter design but we will assume $\frac{2}{T} = 1$. This means we will show that $\frac{2}{T}$ cancels out.

Given data :

Order to filter $N = 2$

Analog filter cut-off frequency $F_c = 1000$ Hz sampling frequency $F_s = 10000$ Hz.

Defining specifications for digital filter :

Order of the filter, $N = 2$ (given). Cut-off frequency f_c of the digital filter can be obtained by applying the formula for conversion of continuous to discrete time frequencies $\left(\text{i.e. } f = \frac{F}{F_s}\right)$. Hence,

$$f_c = \frac{F_c}{F_s} = \frac{1000}{10000} = 0.1 \text{ cycles/sample}$$

Hence $\omega_c = 2\pi f_c = 0.1\times 2\pi = 0.2\,\pi$ radius/sample

To obtain specifications of equivalent analog filter for bilinear transformation :

Here we are applying bilinear transformation hence the cut-off frequency of digital signal i.e. ω_c should be converted to its equivalent value of analog filter i.e. Ω_c according to bilinear transformation frequency relationship. This relationship is given as,

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

Now we will consider $\frac{2}{T} = 1$. Hence,

$$\Omega_c = \tan \frac{\omega_c}{2} = \tan \frac{0.2\,\pi}{2} = 0.325$$

Thus we have specifications of equivalent analog filter according to bilinear transformation are,

$$\Omega_c = 0.325$$

and $$N = 2$$

To obtain poles of $H_a(s)$:

The poles of $H_a(s) \cdot H_a(-s)$ are given by equation (4.7.2) as,

$$p_k = \pm \Omega_c \; e^{j(N+2k+1)\pi/2N}, \qquad k = 0, 1, 2 \ldots, N-1$$

For $N = 2$ and $\Omega_c = 0.325$ above equation becomes

$$p_k = \pm 0.325 \; e^{j(3+2k)\pi/4}, \quad k = 0, 1$$

With $k = 0$ in above equation,

$$p_0 = \pm 0.325 \; e^{j\,3\pi/4} = -0.229 + j\,0.229 \text{ and } 0.229 - j\,0.229$$

Similarly with $k = 1$ we get,

$$p_1 = \pm 0.325 \; e^{j5\pi/4} = -0.229 - j\,0.229 \text{ and } 0.229 + j\,0.229$$

For stable filter we have to consider poles lying in left half of s-plane. Hence poles of $H_a(s)$ will be,

$$s_1 = -0.229 + j\,0.229 \qquad \text{and} \qquad s_1^* = -0.229 - j\,0.229$$

To determine system function $H_a(s)$:

The system function of second order Butterworth lowpass filter is given as,

$$H_a(s) = \frac{\Omega_c^2}{(s - s_1)\left(s - s_1^*\right)}$$

Here numerator is Ω_c^2 since it is second order Butterworth filter. Putting the values of s_1 and s_1^* in above equation we get,

$$H_a(s) = \frac{(0.325)^2}{(s+0.229-j\,0.229)\,(s+0.229+j\,0.229)}$$

On simplifying this equation we get,

$$H_a(s) = \frac{(0.325)^2}{s^2+0.458\,s+0.1048}$$

To obtain $H(z)$ using bilinear transformation :

We know that bilinear transformation is given as,

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Since we have assumed $\frac{2}{T}=1$,

$$s = \frac{1-z^{-1}}{1+z^{-1}}$$

Putting for this value of 's' in $H_a(s)$ we get,

$$H(z) = \frac{(0.325)^2}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2+0.458\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1048}$$

On simplifying this equation we get,

$$H(z) = \frac{0.0675\left(1+2\,z^{-1}+z^{-2}\right)}{1-1.145\,z^{-1}+0.413\,z^{-2}}$$

Observe that this expression for $H(z)$ is same as that of equation (4.9.4) we obtained in the previous example. This shows that the factor $\frac{2}{T}$ cancels out, and hence we can assume $\frac{2}{T}=1$. This reduces the number of calculations.

Example 4.9.3 *Design a low pass Butterworth filter to satisfy passband cut-off = 0.2π, stopband cutoff = 0.3π, passband ripple = 7 dB, stopband ripple = 16 dB T = 1 sec. using impulse invariance method.*

Solution : Given data

$$A_p = 7 \text{ dB} \qquad \omega_p = 0.2\pi$$

$$A_s = 16 \text{ dB} \qquad \omega_s = 0.3\pi$$

Specifications of equivalent analog filter for impulse invariance transformation

Frequencies in analog and discrete domain are related as,

$$\omega = \Omega T$$

$$\therefore \qquad \Omega = \omega \qquad \text{since } T = 1 \text{ given.}$$

Hence analog filter specifications will be,

$$A_p = 7 \text{ dB} \qquad \Omega_p = \omega_p = 0.2\pi$$

$$A_s = 16 \text{ dB} \qquad \Omega_s = \omega_s = 0.3\pi$$

Order of the filter, *N*

$$N = \frac{1}{2}\frac{\log\left[\dfrac{10^{0.1A_s\,dB}-1}{10^{0.1A_p\,dB}-1}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{1}{2}\frac{\log\left[\dfrac{10^{0.1\times 16}-1}{10^{0.1\times 7}-1}\right]}{\log\left(\dfrac{0.3\pi}{0.2\pi}\right)}$$

$$= 2.79 \approx 3$$

Cut-off frequency, Ω_c

$$\Omega_c = \frac{1}{2}\left\{\frac{\Omega_p}{\left(10^{0.1A_p\,dB}-1\right)^{\frac{1}{2N}}}+\frac{\Omega_s}{\left(10^{0.1A_s\,dB}-1\right)^{\frac{1}{2N}}}\right\}$$

$$= \frac{1}{2}\left\{\frac{0.2\pi}{\left(10^{0.1\times 7}-1\right)^{\frac{1}{2\times 3}}}+\frac{0.3\pi}{\left(10^{0.1\times 16}-1\right)^{\frac{1}{2\times 3}}}\right\}$$

$$= 0.5$$

Poles of $H_a(s)$

$$p_k = \pm\Omega_c \, e^{j(N+2k+1)\pi/2N}, \quad k = 0,1,2,....N-1$$

For $N = 3$ and $\Omega_c = 0.5$, $p_k = \pm 0.5 e^{j(3+2k+1)\pi/6}$, $k = 0, 1, 2$

For $k=0$, $p_0 = \pm 0.5 \, e^{j4\pi/6} \Rightarrow -0.25 + j\,0.433$ and $0.25 - j\,0.433$

For $k=1$, $p_1 = \pm 0.5 e^{j\pi} \Rightarrow -0.5$ and 0.5

For $k=2$, $p_2 = \pm 0.5 e^{j} \Rightarrow -0.25 - j\,0.433$ and $0.25 + j\,0.433$

Thus the poles of $H_a(s)$ are,

$s_1 = -0.5$, $s_2 = -0.25 + j\,0.433$ and $s_2^* = -0.25 - j\,0.433$

System function $H_a(s)$

$$H_a(s) = \frac{\Omega_c^N}{(s-s_1)(s-s_2)\left(s-s_2^*\right)} = \frac{0.5^3}{(s+0.5)(s+0.25-j\,0.433)(s+0.25+j\,0.433)}$$

$$= \frac{0.125}{(s+0.5)\left(s^2+0.5s+0.25\right)} \quad \ldots(4.9.5)$$

$$= \frac{A}{s+0.5} + \frac{Bs+C}{s^2+0.5s+0.25} \quad \ldots(4.9.6)$$

Values of A, B, C can be obtained using standard partial fraction rules,

$$A = \left.\frac{0.125}{s^2+0.5s+0.25}\right|_{s=-0.5} = 0.5$$

From equation (4.9.6) we can write,

$$\frac{A\left(s^2+0.5s+0.25\right)+(Bs+C)(s+0.5)}{(s+0.5)\left(s^2+0.5\,s+0.25\right)}$$

Note that above equation must be same as equation (4.9.5). Hence equating numerators of above equation and equation (4.9.5),

$$A\left(s^2+0.5s+0.25\right)+(B\,s+C)(s+0.5) = 0.125$$

$$\therefore \quad (A+B)\,s^2+(0.5A+0.5B+C)\,s+(0.25A+0.5C) = 0.125$$

Equating the coefficients of s^2, s and constants,

$$\left.\begin{array}{l} A+B=0 \\ 0.5A+0.5B+C=0 \\ 0.25A+0.5C=0.125 \end{array}\right\} \begin{array}{l} \text{Putting } A=0.5 \text{ and solving these equations we get} \\ A=0.5, \quad B=-0.5 \text{ and } C=0 \end{array}$$

Hence equation (4.9.6) becomes

$$H_a(s) = \frac{0.5}{s+0.5} - \frac{0.5s}{s^2+0.5s+0.25} \qquad \ldots(4.9.7)$$

We have $s^2+0.5s+0.25 = (s+0.25-j\,0.433)(s+0.25+j\,0.433) = (s+0.25)^2+(0.433)^2$

Putting this expression in equation (4.9.7)

$$H_a(s) = \frac{0.5}{s+0.5} - \frac{0.5s}{(s+0.25)^2+(0.433)^2}$$

Let us rearrange this equation as follows :

$$H_a(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25-0.25}{(s+0.25)^2+(0.433)^2}\right]$$

$$= \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2+(0.433)^2} - \frac{0.25}{(s+0.25)^2+(0.433)^2}\right]$$

Let us rearrange second term inside the brackets as,

$$H_a(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2+(0.433)^2} - \frac{0.25\times\frac{0.433}{0.433}}{(s+0.25)^2+(0.433)^2}\right]$$

$$= \frac{0.5}{s+0.5} - 0.5\,\frac{s+0.25}{(s+0.25)^2+(0.433)^2} + 0.29.\frac{0.433}{(s+0.25)^2+(0.433)^2} \qquad \ldots(4.9.8)$$

System function of digital filter, *H(z)*

Here let us use following impulse invariant transformations for three terms in $H_a(s)$ of equation (4.9.8).

$$\frac{1}{s-p_k} \rightarrow \frac{1}{1-e^{p_k T}z^{-1}}$$

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}(\sin bT)\,z^{-1}}{1-2e^{-aT}(\cos bT)\,z^{-1}+e^{-2aT}z^{-2}}$$

With $T=1$, $p_k = -\ 0.5$, $a=0.25$ and $b=0.433$, $H(z)$ can be expressed as,

$$H(z) = \frac{0.5}{1-e^{-0.5}z^{-1}} - 0.5\frac{1-e^{-0.25}(\cos 0.433)\,z^{-1}}{1-2e^{-0.25}(\cos 0.433)\,z^{-1}+e^{-2(0.25)}z^{-2}}$$

$$+\ 0.29\frac{e^{-0.25}(\sin 0.433)\,z^{-1}}{1-2e^{-0.25}(\cos 0.433)\,z^{-1}+e^{-2(0.25)}z^{-2}}$$

$$= \frac{0.5}{1-0.6z^{-1}} - \frac{0.5\left(1-0.7z^{-1}\right)}{1-1.41z^{-1}+0.6z^{-2}} + \frac{0.1z^{-1}}{1-1.41z^{-1}+0.6z^{-2}}$$

$$\therefore \quad H(z) = \frac{0.5}{1-0.6z^{-1}} - \frac{0.5-0.45z^{-1}}{1-1.41z^{-1}+0.6z^{-2}}$$

This is the required digital filter system function.

Example 4.9.4 *Design high pass digital filter to meet the following specifications.*

Passband	2 - 4 *kHz*
Stopband	0 - 500 *Hz*
δ_p	3 *dB*
δ_s	20 *dB*

Assume butterworths approximation.

Solution :

i) Given data

$$A_p = 3 \text{ dB}, \quad F_p = 2 \text{ kHz}$$

$$A_s = 20 \text{ dB}, \quad F_s = 500 \text{ Hz}$$

Let us assume the sampling frequency to be 8000 Hz.

$$\therefore \quad f_p = \frac{F_p}{F_{samp}} = \frac{2000}{8000} = 0.25, \text{ hence } \omega_p = 2\pi f_p = 2\pi \times 0.25 = 0.5\pi$$

$$\text{and} \quad f_s = \frac{F_s}{F_{samp}} = \frac{500}{8000} = 0.0625, \text{ hence } \omega_s = 2\pi f_s = 2\pi \times 0.0625 = 0.125\pi$$

ii) To obtain the specification of equivalent analog filter for bilinear transformation (prewarping)

$$\text{Here} \quad \Omega = \frac{2}{T}\tan\frac{\omega}{2} = \tan\frac{\omega}{2} \quad \text{with } \frac{2}{T} = 1$$

$$\therefore \qquad \Omega_p = \tan\frac{0.5\pi}{2} = 1 \text{ rad/sec}$$

$$\Omega_s = \tan\frac{0.125\pi}{2} = 0.2 \text{ rad/sec}$$

Thus the specification of equivalent analog filter are,

$$A_p = 3\,\text{dB}, \quad \Omega_p = 1 \text{ rad/sec}$$

$$A_s = 20\,\text{dB}, \quad \Omega_s = 0.2 \text{ rad/sec}$$

iii) To obtain the specifications of lowpass filter

First we will design the lowpass filter for given specifications.

The new specifications will be,

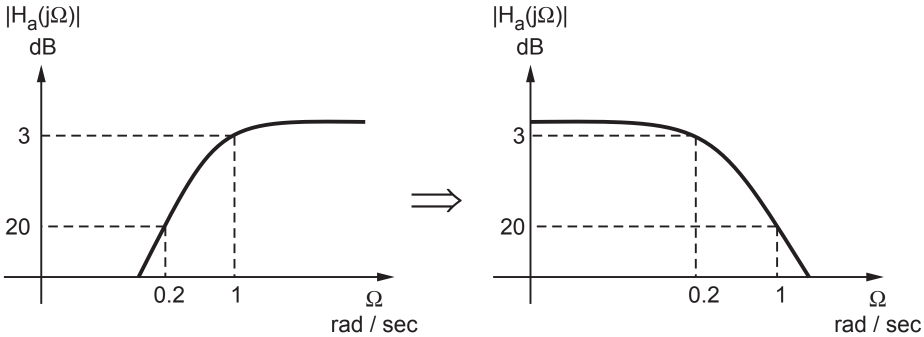


Fig. 4.9.2 Conversion from highpass to lowpass filter

$$\left.\begin{array}{ll} A_p = 3\,\text{dB} & \Omega_p = 0.2 \text{ rad / sec} \\ A_s = 20\,\text{dB} & \Omega_s = 1 \text{ rad / sec} \end{array}\right\}$$

Here note that the Ω_p and Ω_s are exchanged to convert highpass to lowpass.

Fig. 4.9.2 shows the characteristics of highpass filter and those of lowpass filter we are going to design.

iv) To obtain the order of the filter

$$N = \frac{1}{2}\frac{\log\left[\dfrac{10^{0.1A_s\,\text{dB}}-1}{10^{0.1A_p\,\text{dB}}-1}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{1}{2}\frac{\log\left[\dfrac{10^{0.1\times 20}-1}{10^{0.1\times 3}-1}\right]}{\log\left(\dfrac{1}{0.2}\right)} = 1.43$$

$$N \approx 2$$

v) To obtain system function of normalized Butterworth filter

The system function of normalized Butterworth filter of 2^{nd} order is given as,

$$H_{an(LP)}(s) = \frac{1}{s^2+b_1 s+1} = \frac{1}{s^2+\sqrt{2}\,s+1} \text{ from Table 4.7.1.}$$

vi) To obtain system function of required analog higpass filter by frequency transformation.

The lowpass to highpass frequency transformation is given as,

$$s \rightarrow \frac{\Omega_p\,\Omega_{HP}}{s}$$

Here Ω_p = 1 rad/sec the passband edge frequency of lowpass filter.

Ω_{HP} = 1 rad/sec the passband edge frequency of highpass filter.

$$\therefore \quad s \rightarrow \frac{1}{s}$$

$$\therefore \quad H_{a(HP)}(s) = H_{an(LP)}(s)\Big|_{s\rightarrow \frac{1}{s}} = \frac{1}{\left(\frac{1}{s}\right)^2+\sqrt{2}\left(\frac{1}{s}\right)+1} = \frac{s^2}{1+\sqrt{2}\;s+s^2} = \frac{s^2}{s^2+\sqrt{2}\;s+1}$$

vii) To obtain highpass digital filter by bilinear transformation.

Applying bilinear transformation to $H_{a(HP)}(s)$,

$$H(z) = H_{a(HP)}(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2+\sqrt{2}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1} = \frac{0.29(1-2z^{-1}+z^{-2})}{1+0.171z^{-2}}$$

Example 4.9.5 *Design a bandstop IIR filter to meet the following specifications :*

Lower passband :	*0 to 50 Hz*
Upper passband :	*450 to 500 Hz*
Stopband :	*200 to 300 Hz*
Passband ripple :	*3 dB*
Stopband attenuation :	*20 dB*
Sampling frequency :	*1 kHz*

Determine the following

i) Passband and stopband edge frequencies of a suitable prototype lowpass filter.

ii) Order, N of the prototype lowpass filter.

iii) Coefficients and transfer function using BLT method.

Solution : Given data

$$F_{l\,1} = 50 \text{ Hz}, \quad F_{l\,2} = 200 \text{ Hz}$$

$$F_{u\,1} = 300 \text{ Hz}, \quad F_{u\,2} = 450 \text{ Hz}$$

$$F_{SF} = 1000 \text{ Hz}$$

$$\therefore \quad f_{l\,1} = \frac{F_{l_1}}{F_{SF}} = \frac{50}{1000} = 0.05, \qquad f_{l\,2} = \frac{F_{l_2}}{F_{SF}} = \frac{200}{1000} = 0.2$$

$$\therefore \quad f_{u\,1} = \frac{F_{u_1}}{F_{SF}} = \frac{300}{1000} = 0.3, \qquad f_{u\,2} = \frac{F_{u_2}}{F_{SF}} = \frac{450}{1000} = 0.45$$

Hence

$$\omega_{l_1} = 2\pi f_{l_1} = 2\,\pi \times 0.05 = 0.3141592$$

$$\omega_{l_2} = 2\pi f_{l_2} = 2\,\pi \times 0.2 = 1.2566371$$

$$\omega_{u_1} = 2\pi f_{u_1} = 2\,\pi \times 0.3 = 1.8849556$$

$$\omega_{u_2} = 2\pi f_{u_2} = 2\,\pi \times 0.45 = 2.8274334$$

To obtain frequencies of analog filter as per bilinear transformation (prewarping)

$$\Omega_{l_1} = \tan\frac{\omega_{l_1}}{2} = \tan\frac{0.3141592}{2} = 0.1584 \text{ rad/sec}$$

$$\Omega_{l_2} = \tan\frac{\omega_{l_2}}{2} = \tan\frac{1.2566371}{2} = 0.7265 \text{ rad/sec}$$

$$\Omega_{u_1} = \tan\frac{\omega_{u_1}}{2} = \tan\frac{1.8849556}{2} = 1.37638 \text{ rad/sec}$$

$$\Omega_{u2} = \tan\frac{\omega_{u2}}{2} = \tan\frac{2.8274334}{2} = 6.3138 \text{ rad/sec}$$

i) To obtain passband and stopband edge frequencies of prototype lowpass filter

Lowpass to bandstop transformation is given as,

$$s \to \Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}$$

Here L.H.S. represents prototype lowpass filter and R..H.S. represents bandstop filter. Putting $s = j\Omega^p$ in L.H.S. and $s = j\Omega$ in R.H.S.,

$$j\Omega^p = \Omega_p \frac{j\Omega(\Omega_u - \Omega_l)}{(j\Omega)^2 + \Omega_u \Omega_l} = j\,\Omega_p \frac{\Omega(\Omega_u - \Omega_l)}{-\Omega^2 + \Omega_u \Omega_l}$$

Here Ω^p is the frequency of prototype lowpass filter and

Ω is the frequency of bandstop filter.

Ω_p is the cut-off frequency of prototype lowpass filter. It is 1 rad/sec.

From above equation we get,

$$\Omega^p = 1\,\frac{\Omega(\Omega_u - \Omega_l)}{-\Omega^2 + \Omega_u \Omega_l} \text{ with } \Omega_p = 1 \text{ rad/sec}$$

In this equation if we put $\Omega = \Omega_{l1} = 0.1584$ rad/sec we will get passband edge frequency of prototype lowpass filter. And putting $\Omega_u = \Omega_{u2} = 6.3138$, $\Omega_l = \Omega_{l1} = 0.1584$ we get,

$$\Omega_p^p = \frac{0.1584(6.3138 - 0.1584)}{-(0.1584)^2 + 6.3138 \times 0.1584} = 1 \text{ rad/sec}$$

Putting $\Omega = \Omega_{l2}$ we get stopband edge frequency i.e.,

$$\Omega_s^p = \frac{0.7265(6.3138 - 0.1584)}{-(0.7265)^2 + 6.3138 \times 0.1584} = 9.47 \text{ rad/sec}$$

Here $\Omega_p^p = 1$ rad/sec and $\Omega_s^p = 9.47$ rad/sec are passband edge and stopband edge frequencies of prototype lowpass filter respectively.

ii) To determine order '*N*' of prototype lowpass filter

The specifications are,

$$A_p = 3 \text{ dB} \qquad \Omega_p^p = 1 \text{ rad/sec}$$

$$A_s = 20 \text{ dB} \qquad \Omega_s^p = 9.47 \text{ rad/sec}$$

$$N = \frac{1}{2}\frac{\log\left[\dfrac{10^{0.1A_s dB}-1}{10^{0.1A_p dB}-1}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{1}{2}\frac{\log\left[\dfrac{10^{0.1\times 20}-1}{10^{0.1\times 3}-1}\right]}{\log\left(\dfrac{9.47}{1}\right)} = 1.023 \approx 1$$

iii) To obtain coefficients and transfer function

The system function of first order prototype lowpass filter is given as,

$$H_{an}(s) = \frac{1}{s+1}$$

The system function of desired digital filter can be obtained by applying lowpass to bandstop transformation. i.e.

$$s \to \Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}$$

$$\therefore \quad H_a(s) = \frac{1}{\Omega_p \dfrac{s(\Omega_u-\Omega_l)}{s^2+\Omega_u\Omega_l}+1}$$

Here $\Omega_p = \Omega_p^p = 1$ rad/sec and putting $\Omega_u = 6.3138$ and $\Omega_l = 0.1584$ we get,

$$H_a(s) = \frac{1}{\dfrac{s(6.3138-0.1584)}{s^2+6.3138\times 0.1584}+1} = \frac{1}{\dfrac{6.1554s}{s^2+1}+1} = \frac{s^2+1}{s^2+6.1554s+1}$$

Applying bilinear transform,

$$H(z) = H_a(s)\big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \quad \text{assuming } \frac{2}{T} = 1$$

$$= \frac{\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)^2+1}{\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)^2+6.1554\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+1} = \frac{0.24\,(1+z^{-2})}{1-0.5z^{-2}}$$

Example 4.9.6 *Design the second order lowpass digital filter of butterworth type using BLT for the specifications given below :*

Analog transfer function of the filter

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Cut-off frequency = 1 kHz, Sampling frequency = 10 kHz. **PU : Dec.-2000**

Solution : Given data :

$$H_{an}(s) = \frac{1}{s^2 + \sqrt{2}s + 1} \quad \text{(Normalized filter)}$$

$$F_c = 1 \text{ kHz}$$

$$F_{sF} = 10 \text{ kHz}$$

$$\therefore \quad f_c = \frac{F_c}{F_{sF}} = \frac{1 \text{ kHz}}{10 \text{ kHz}} = 0.1$$

$$\therefore \quad \omega_c = 2\pi f_c = 2\pi \times 0.1 = 0.2\pi$$

To obtain Ω_c as per BLT (prewarping)

$$\Omega_c = \frac{2}{T} \tan \frac{\omega_c}{2}$$

$$= \tan \frac{\omega_c}{2}, \qquad \text{Assuming } \frac{2}{T} = 1$$

$$= \tan \frac{0.2\pi}{2} = 0.1\pi$$

Lowpass to lowpass transformation analog filter

Lowpass to lowpass transformation is given as,

$$s \rightarrow \frac{\Omega_p}{\Omega_{Lp}} s$$

Here $\Omega_{Lp} = \Omega_c = 0.1\pi$, $\Omega_p = 1$ (For normalized filter)

$$\therefore \quad s \rightarrow \frac{1}{0.1\pi} s \Rightarrow s \rightarrow 3.183\, s$$

$$H_a(s) = H_{an}(s)\big|_{s \rightarrow 3.183s} = \left.\frac{1}{s^2 + \sqrt{2}s + 1}\right|_{s \rightarrow 3.183s} = \frac{1}{(3.183s)^2 + \sqrt{2} \times 3.183s + 1}$$

$$= \frac{1}{10.131s^2 + 4.5s + 1}$$

Bilinear transformation to get $H(z)$

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{1}{10.131s^2 + 4.5s + 1}\Bigg|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{1}{10.131\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)^2 + 4.5\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1} = \frac{0.064\left(1+2z^{-1}+z^{-2}\right)}{1-1.168z^{-1}+0.424z^{-2}}$$

Examples with Solutions

Example 4.9.7 *Design a butterworth filter using the impulse invariance method for the following specifications.*

$0.8 \le \left|H(e^{j\omega})\right| \le 1 \quad 0 \le \omega \le 0.2\pi$

$\left|H(e^{j\omega})\right| \le 0.2 \quad 0.6\pi \le \omega \le \pi$

AU : Dec.-11, Marks 10, Dec.-15, Marks 16

Solution : i) Specifications of equivalent analog filter

For impulse invariant transformation,

$$\omega = \Omega T$$

or $$\Omega = \frac{\omega}{T} = \omega \text{ for assuming } T = 1.$$

Hence specifications of analog filter are as follows :

$$A_p = 0.8 \quad , \Omega_p = 0.2\pi$$

$$A_s = 0.2 \quad , \Omega_s = 0.6\pi$$

ii) Order of the filter

$$N = \frac{1}{2}\frac{\log\left[\left(\dfrac{1}{A_s^2}-1\right)\Big/\left(\dfrac{1}{A_p^2}-1\right)\right]}{\log\dfrac{\Omega_s}{\Omega_p}} = \frac{1}{2}\frac{\log\left[\left(\dfrac{1}{0.2^2}-1\right)\Big/\left(\dfrac{1}{0.8^2}-1\right)\right]}{\log\left(\dfrac{0.6\pi}{0.2\pi}\right)}$$

$$= 1.7 \approx 2$$

iii) Cutoff frequency Ω_c

$$\Omega_c = \frac{1}{2}\left\{\frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)^{\frac{1}{2N}}}+\frac{\Omega_s}{\left(\frac{1}{A_s^2}-1\right)^{\frac{1}{2N}}}\right\}$$

$$= \frac{1}{2}\left\{\frac{0.2\pi}{\left(\frac{1}{0.8^2}-1\right)^{\frac{1}{2\times 2}}}+\frac{0.6\pi}{\left(\frac{1}{0.2^2}-1\right)^{\frac{1}{2\times 2}}}\right\} = 0.7885 \approx 0.79$$

iv) Poles of $H_a(s)$

$$p_k = \pm\, \Omega_c\, e^{j(N+2k+1)\pi/2N} \quad , \quad k = 0, 1, \ldots N-1$$

for $N = 2$, $$p_k = \pm\, 0.79\, e^{j(2+2k+1)\pi/(2\times 2)} \quad , \quad k = 0, 1$$

$$= \pm\, 0.79\, e^{j(3+2k)\pi/4} \quad , \quad k = 0, 1$$

$\therefore$ $$p_0 = \pm\, 0.79\, e^{j3\pi/4} = -0.56 + j0.56 \text{ and } 0.56 - j0.56$$

$$p_1 = \pm\, 0.79\, e^{j5\pi/4} = -0.56 - j0.56 \text{ and } 0.56 + j0.56$$

Thus $$s_1 = -0.56 + j0.56 \quad \text{and} \quad s_1^* = -0.56 - j0.56$$

v) System function $H_a(s)$

$$H_a(s) = \frac{\Omega_c^N}{(s-s_1)(s-s_1^*)} = \frac{0.79^2}{(s+0.56-j0.56)(s+0.56+j0.56)}$$

$$= \frac{0.79^2}{(s+0.56)^2+(0.56)^2}$$

Let us rearrange above equation as follows

$$H_a(s) = \frac{0.79^2}{0.56}\cdot\frac{0.56}{(s+0.56)^2+(0.56)^2} \qquad \ldots (4.9.9)$$

vi) Impulse invariant transformation

Here use $$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}\sin(bT)\,z^{-1}}{1-2e^{-aT}(\cos bT)\,z^{-1}+e^{-2aT}\,z^{-2}}$$

with a = 0.56 and b= 0.56 in equation (4.9.9) (with T = 1),

$$H(z) = \frac{(0.79)^2}{0.56} \cdot \frac{e^{-0.56} \sin(0.56)\, z^{-1}}{1-2e^{-0.56} \cos 0.56\, z^{-1} + e^{-2\times 0.56}\, z^{-2}}$$

$$= \frac{0.39\, z^{-1}}{1-0.97\, z^{-1}+0.32\, z^{-2}}$$

Example 4.9.8 *Design a third order Butterworth digital filter using impulse invariance method. Assume sampling period T = 1 sec.*

Solution : The system function of the normalized 3^{rd} order Butterworth filter is given as,

$$H_a(s) = \frac{1}{(s+1)\left(s^2+s+1\right)}$$

$$= \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1} = \frac{A\left(s^2+s+1\right)+(Bs+C)(s+1)}{(s+1)\left(s^2+s+1\right)} \quad \text{...(4.9.10)}$$

Numerators of above two equations must be same. i.e. ,

$$A\left(s^2+s+1\right)+(Bs+C)(s+1) = 1$$

$$\therefore \quad (A+B)\,s^2+(A+B+C)\,s+(A+C) = 1$$

Equating the coefficients of s^2, s and constants on both sides we get,

$$\left.\begin{aligned} A+B &= 0 \\ A+B+C &= 0 \\ A+C &= 1 \end{aligned}\right\} \begin{array}{l} \text{Solving these equations} \\ A = 1,\; B = -1 \;\text{ and }\; C = 0 \end{array}$$

Putting for A, B and C in equation (4.9.10),

$$H_a(s) = \frac{1}{s+1} - \frac{s}{s^2+s+1}$$

$$= \frac{1}{s+1} - \frac{s}{\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)} = \frac{1}{s+1} - \frac{s}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

Let us rearrange above equation as,

$$H_a(s) = \frac{1}{s+1} - \frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{s+1} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

Let us rearrange the last term of above equation as,

$$H_a(s) = \frac{1}{s+1} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\cdot\frac{\sqrt{3}/2}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{s+1} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + 0.577\cdot\frac{\sqrt{3}/2}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

Apply following impulse invariant formulae to above equation,

$$\frac{1}{s-p_k} \rightarrow \frac{1}{1-e^{p_kT}z^{-1}}$$

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos\ bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos\ bT)z^{-1}+e^{-2aT}z^{-2}}$$

with $p_k = -1$, $a = \frac{1}{2}$, $b = \frac{\sqrt{3}}{2}$ and $T = 1$,

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} - \frac{1-e^{-\frac{1}{2}}\left(\cos\frac{\sqrt{3}}{2}\right)z^{-1}}{1-2e^{-\frac{1}{2}}\left(\cos\frac{\sqrt{3}}{2}\right)z^{-1}+e^{-2\times\frac{1}{2}}z^{-2}} + \frac{0.577\cdot e^{-\frac{1}{2}}\left(\sin\frac{\sqrt{3}}{2}\right)z^{-1}}{1-2e^{-\frac{1}{2}}\left(\cos\frac{\sqrt{3}}{2}\right)z^{-1}+e^{-2\times\frac{1}{2}}z^{-2}}$$

Simplifying above equation,

$$H(z) = \frac{1}{1-0.367z^{-1}} - \frac{1-0.66\,z^{-1}}{1-0.785\,z^{-1}+0.367\,z^{-2}}$$

This is the system function of digital filter.

Example 4.9.9 *In the DSP system sampled at 1000 Hz, a notch bandpass filter at 200 Hz is to be added. Show the necessary extra poles and zeros to achieve this.*

Solution : $F_{BP} = 200\,\text{Hz}$ and $F_s = 1000$ Hz

$$\therefore \quad f_{BP} = \frac{F_{BP}}{F_s} = \frac{200}{1000} = 0.2$$

$$\therefore \quad \omega_{BP} = 2\pi f_{BP} = 2\pi \times 0.2 = 0.4\pi$$

- If we indicate $z = re^{j\omega_{BP}}$ with $r = 1$, i.e. the pole on unit circle,

$$z = e^{j(0.4\pi)} = 0.3 + j0.95$$

- Thus the pole must be located at $z = 0.3 + j0.95$. For the system to be realizable, the complex poles must occur in conjugate pairs. Hence one more pole must be located at $z = 0.3 - j0.95$.
- Since there are two poles, there must be two zeros located at z = 0 (origin). Thus,

Poles : $p_1 = 0.3 + j0.95$, $p_2 = 0.3 - j0.95$

Zeros : $z_1 = z_2 = 0$

Example 4.9.10 *Design a digital Butterworth filter that satisfies the following constraints, using bilinear transformation. Assume T = 1 sec.*

$$0.9 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le \omega \le \frac{\pi}{2}$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.2 \qquad \frac{3\pi}{4} \le \omega \le \pi$$

Solution : Given data :

$$A_p = 0.9 \qquad \omega_p = \frac{\pi}{2}$$

$$A_s = 0.2 \qquad \omega_s = \frac{3\pi}{4}$$

Specification of equivalent analog filter (prewarping)

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = \tan\frac{\omega_p}{2} \quad \text{Assuming } \frac{2}{T} = 1$$

$$= \tan\frac{\frac{\pi}{2}}{2} = 1$$

$$\Omega_s = \tan\frac{\omega_s}{2} = \tan\frac{\frac{3\pi}{4}}{2} = 2.4142$$

Now specifications are,

$$A_p = 0.9, \qquad \Omega_p = 1$$

$$A_s = 0.2 \qquad \Omega_s = 2.4142$$

Order of butterworth filter

$$N = \frac{1}{2}\frac{\log\left[\left(\frac{1}{A_s^2}-1\right)\Big/\left(\frac{1}{A_p^2}-1\right)\right]}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{1}{2}\frac{\log\left[\left(\frac{1}{(0.2)^2}-1\right)\Big/\left(\frac{1}{(0.9)^2}-1\right)\right]}{\log\left(\frac{2.4142}{1}\right)} = 2.625 \approx 3$$

Cut-off frequency

$$\Omega_c = \frac{1}{2}\left\{\frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)^{\frac{1}{2N}}} + \frac{\Omega_s}{\left(\frac{1}{A_s^2}-1\right)^{\frac{1}{2N}}}\right\} = \frac{1}{2}\left\{\frac{1}{\left(\frac{1}{(0.9)^2}-1\right)^{\frac{1}{6}}} + \frac{2.4142}{\left(\frac{1}{(0.2)^2}-1\right)^{\frac{1}{6}}}\right\} = 1.1$$

Poles of $H_a(s)$

$$p_k = \pm\,\Omega_c\, e^{j(N+2k+1)\pi/2N}, \qquad k = 0, 1, 2, \ldots\ldots N-1$$

$$\therefore \quad p_k = \pm 1.1\ e^{j(4+2K)\pi/6}, \qquad k = 0, 1, 2.$$

$$p_o = \pm 1.1\ e^{j4\pi/6} = -0.55 + j0.9526 \text{ and } 0.55 - j0.9526$$

$$p_1 = \pm 1.1\, e^{j\pi} = -1.1 \text{ or } 1.1$$

$$p_2 = \pm 1.1\, e^{j8\pi/6} = -0.55 - j0.9526 \text{ and } 0.55 + j0.9526$$

Here the pole pairs are,

$$s_1 = -0.55 + j0.9526 \text{ and } s_1^* - 0.55 - j0.9526$$

$$s_2 = -1.1$$

System function $H_a(s)$

$$H_a(s) = \frac{\Omega_c^3}{(s-s_1)\left(s-s_1^*\right)(s-s_2)} =$$

$$\frac{(1.1)^3}{(s+0.55-j0.9526)(s+0.55+j0.9526)(s+1.1)}$$

$$= \frac{(1.1)^3}{\left(s^2+1.1s+1.2\right)(s+1.1)}$$

System function of digital filter

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \qquad \text{since } \frac{2}{T} = 1\,(\text{assumed})$$

$$= \frac{(1.1)^3}{\left[\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1.1\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1.2\right]\left[\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1.1\right]}$$

$$= \frac{\left[1.1\left(1+z^{-1}\right)\right]^3}{\left(3.3+0.4z^{-1}+1.1z^{-2}\right)\left(2.1-0.1z^{-1}\right)}$$

$$= \frac{0.192\left(1+z^{-1}\right)^3}{\left(1+0.121\,z^{-1}+0.33\,z^{-2}\right)\left(1-0.047z^{-1}\right)}$$

Example 4.9.11 *Design digital highpass Butterworth filter for cut-off frequency = 30 Hz and sampling frequency = 150 Hz.* **PU : May-01**

Solution : Given data :

Highpass filter with $F_{HP} = 30$ Hz

and $F_s = 150$ Hz

Assume order of filter to be 1. i.e. $N = 1$

To obtain specifications of equivalent digital filter :

The cut-off frequency of digital highpass filter will be given as $\left(f = \frac{F}{F_s}\right)$ i.e.,

$$f_{HP} = \frac{F_{HP}}{F_s} = \frac{30}{150} = 0.2 \quad \text{cycles/sample}$$

$$\omega_{HP} = 2\pi f_{HP} = 2\pi \times 0.2 = 0.4\,\pi \quad \text{radians/sample}$$

Thus we have $\omega_{HP} = 0.4\,\pi$

and $N = 1$

These are specifications of equivalent digital filter.

To obtain the specifications of equivalent analog filter according to bilinear transformation :

Here we will redefine the frequencies of analog filter according to bilinear frequency relationship. This frequency relationship is given as,

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

$$\Omega_{HP} = \frac{2}{T} \tan \frac{\omega_{HP}}{2}$$

Here $T = \frac{1}{F_s} = \frac{1}{150}$ sec. and $\omega_{HP} = 0.4\,\pi$. Hence above equation gives,

$$\Omega_{HP} = \frac{2}{\left(\frac{1}{150}\right)} \tan\left(\frac{0.4\,\pi}{2}\right)$$

$$= 217.963 \quad \text{radians/sec}$$

Thus we have the specifications of analog filter according to bilinear transformation are,

$$\Omega_{HP} = 217.963$$

and $\quad N = 1$

Note that the procedure discussed here is sometimes called as prewarping. We are obtaining the prewarped frequency of analog filter. When we apply bilinear transformation to $H_a(s)$, the warping effect is compensated.

To obtain system function for prototype first order lowpass filter :

We want to design highpass filter. But we have to first obtain system function of normalized lowpass filter (i.e. also called prototype filter) and then frequency transformations are applied to get highpass filter system function.

Ths system function of the normalized lowpass filter is given by equation (4.4.34) as,

$$H_{an}(s) = \frac{1}{s^N + b_{N-1}s^{N-1} + \ldots + b_1 s + 1}$$

For $N = 1$,
$$H_{an}(s) = \frac{1}{s+1} \qquad \ldots (4.9.11)$$

This is the system function of the normalized lowpass Butterworth filter.

To obtain system function of analog highpass filter by frequency transformation :

Now we have to apply frequency transformation on system function of prototype lowpass filter to obtain system function of highpass filter. The lowpass to highpass transformation is given by equation (4.7.2) as,

$$s \rightarrow \frac{\Omega_p \, \Omega_{HP}}{s}$$

Here Ω_p is passband edge frequency of lowpass filter, which is equal to '1'. Hence $\Omega_p = 1$. Hence above transformation will be,

$$s \rightarrow \frac{\Omega_{HP}}{s}$$

We have obtained Ω_{HP} as 217.963 i.e. prewarped cut-off frequency of highpass filter. Hence,

$$s \rightarrow \frac{217.963}{s}$$

Hence system function of highpass filter is given as,

$$H_{aHP}(s) = H_{an}(s)\Big|_{s \rightarrow \frac{\Omega_{HP}}{s}} = \frac{1}{s+1}\Bigg|_{s \rightarrow \frac{217.963}{s}} = \frac{1}{\frac{217.963}{s} + 1}$$

$$= \frac{s}{s + 217.963} \qquad \ldots (4.9.12)$$

This is the system function of analog highpass filter.

To obtain $H(z)$ by bilinear transformation :

We know that bilinear transformation is given as,

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Here $T = \frac{1}{F_s} = \frac{1}{150}$. Hence above equation becomes,

$$s = \frac{2}{\left(\frac{1}{150}\right)}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = 300\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Applying this transformation to equation (4.9.12) we get,

$$H(z) = \frac{300\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{300\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+217.963} = \frac{0.5792\left(1-z^{-1}\right)}{1-0.1583\ z^{-1}}$$

This is the system function of the required digital highpass filter.

Example 4.9.12 *Determine H(z) for a butter worth filter satisfying the following constraints.*

$$\sqrt{0.5} \le |H(e^{j\omega})| \le 1\ ;\ 0 \le \omega \le \frac{\pi}{2}$$

$$|H(e^{j\omega})| \le 0.2;\ \frac{3\pi}{4} \le \omega \le \pi$$

with T = 1 s. Apply impulse invariant transformation. **AU : May-15, Marks 16**

Solution : Given data :

$$A_p = \sqrt{0.5}\ ,\quad \omega_p = \frac{\pi}{2}$$

$$A_s = 0.2\ ,\quad \omega_s = \frac{3\pi}{4}\ ,\ T = 1\ s$$

Specifications of equivalent analog filter :

For impulse invariant transformation,

$$\omega = \Omega T$$

$$\therefore \quad \Omega = \frac{\omega}{T} = \omega, \text{ since } T = 1$$

Hence specifications of equivalent analog filter will be,

$$A_p = \sqrt{0.5} = 0.707,\quad \Omega_p = \frac{\pi}{2} = 0.5\pi$$

$A_s = 0.2, \qquad \Omega_s = \frac{3\pi}{4} = 0.75\,\pi$

To determine order of butterworth filter :

$$N = \frac{1}{2}\frac{\log\left[\left(\frac{1}{A_s^2}-1\right)\Big/\left(\frac{1}{A_p^2}-1\right)\right]}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$= \frac{1}{2}\frac{\log\left[\left(\frac{1}{0.2^2}-1\right)\Big/\left(\frac{1}{0.70712^2}-1\right)\right]}{\log\left(\frac{0.75\pi}{0.5\pi}\right)} = 3.918 \approx 4$$

To determine cut-off frequency Ω_c :

$$\Omega_c = \frac{1}{2}\left\{\frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)^{\frac{1}{2N}}}+\frac{\Omega_s}{\left(\frac{1}{A_s^2}-1\right)^{\frac{1}{2N}}}\right\}$$

$$= \frac{1}{2}\left\{\frac{0.5\pi}{\left(\frac{1}{0.707^2}-1\right)^{\frac{1}{8}}}+\frac{0.75\pi}{\left(\frac{1}{0.2^2}-1\right)^{\frac{1}{8}}}\right\} = 1.577 \text{ rad/sec}$$

To determine poles of $H_a(s)$:

Poles of $H_a(s)$ are given as,

$$p_k = \pm\Omega_c\, e^{j(N+2k+1)\,\pi/2N}, \quad k = 0, 1, 2, \ldots\ldots N-1$$

$$\therefore\ p_k = \pm 1.577 e^{j(5+2k)\pi/8}, \quad k = 0, 1, 2, 3.$$

$$p_0 = \pm 1.577\, e^{j5\pi/8} = -0.6 + j1.457 \text{ and } 0.6 - j1.457$$

$$p_1 = \pm 1.577\, e^{j7\pi/8} = -1.457 + j\,0.6 \text{ and } 1.457 - j\,0.6$$

$$p_2 = \pm 1.577 e^{j9\pi/8} = -1.457 - j\,0.6 \text{ and } 1.457 + j\,0.6$$

$$p_3 = \pm 1.577\, e^{j11\pi/8} = -0.6 - j\,1.457 \text{ and } 0.6 + j\,1.457$$

The poles in LHS of s-plane in complex conjugate pairs must be selected for stable filter,

$$\therefore \quad s_1 = -0.6 + j\,1.457 \text{ and } s_1^* = -0.6 - j1.457$$

$$s_2 = -1.457 + j0.6 \text{ and } s_2^* = -1.457 - j0.6$$

To determine system function $H_a(s)$:

$$H_a(s) = \frac{\Omega_c^4}{(s-s_1)(s-s_1^*)\,(s-s_2)\,(s-s_2^*)}$$

$$= \frac{(1.577)^4}{(s+0.6-j1.457)(s+0.6+j1.457)\,(s+1.457-j\,0.6)(s+1.457+j\,0.6)}$$

$$= \frac{6.185}{(s^2+1.2s+2.483)(s^2+2.914\,s+2.483)} \qquad \ldots(4.9.13)$$

Convert above equation into partial fractions.

$$H_a(s) = \frac{As+B}{s^2+1.2\,s+2.483} + \frac{Cs+D}{s^2+2.914\,s+2.483} \qquad \ldots(4.9.14)$$

Comparing equation (4.9.13) and above equation,

$$(s^2+2.914s+2.483)\;(As+B)+(s^2+1.2s+2.483)\,(Cs+D) = 6.185$$

Compairing the coefficients of s^3 , s^2, s and constant terms,

$$A + C = 0$$

$$2.914\,A + B + 1.2\,C + D = 0$$

$$2.47\,A + 2.914\,B + 2.47\,C + 1.2\,D = 0$$

$$B + D = 2.483$$

Solving above equations,

$A = -1.451$, $B = -1.744$, $C = 1.451$ and $D = 4.211$

Hence equation (4.9.14) can be written as,

$$H_a(s) = -\frac{1.451\,s+1.744}{s^2+1.2s+2.483} + \frac{1.451\,s+4.211}{s^2+2.914s+2.467}$$

Rearranging above equation,

$$H_a(s) = -1.451 \frac{s+0.6+0.6}{(s+0.6)^2+(1.457)^2} + 1.451 \frac{s+1.457+1.457}{(s+1.457)^2+0.6^2}$$

$$= -1.451 \frac{s+0.6}{(s+0.6)^2+(1.457)^2} - 0.6 \frac{1.457}{(s+0.6)^2+(1.457)^2} + 1.457 \frac{s+1.457}{(s+1.457)^2+0.6^2}$$

$$+ 3.49 \frac{0.6}{(s+1.457)^2+(0.6)^2} \quad \ldots (4.9.15)$$

Applying impulse invariant transformation :

Here let us use following equations of impulse invariant transformation :

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT)\, z^{-1}}{1-2e^{-aT}\ (\cos bT)\ z^{-1}+e^{-2aT}\ z^{-2}}$$

and
$$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}\ (\cos bT)\ z^{-1}}{1-2e^{-aT}\ (\cos bT)\ z^{-1}+e^{-2aT}\ z^{-2}}$$

with $T = 1$ and putting appropriate values of a and b in equation (4.9.15),

$$H(z) = \frac{-1.451 - 0.232\, z^{-1}}{1 - 0.131\, z^{-1} + 0.3\, z^{-2}} + \frac{1.451 + 0.185\, z^{-1}}{1 - 0.386\, z^{-1} + 0.055\, z^{-2}}$$

Example 4.9.13 *Using the bilinear transform design a highpass filter, monotonic in passband with cut-off frequency 1000 Hz and down 10 dB at 350 Hz. The sampling frequency is 5000 Hz.* **AU : (ECE) Dec.-05, Marks 10, May-17, Marks 13**

Solution : i) Given data

$$F_c = 1000 \text{ Hz}$$

An attenuation of 3 dB is achieved at cutoff frequency F_c. Hence we can call it as pass band edge frequency F_p and attenuation of 3 dB can be called as passband attenuation A_p. i.e.,

$$A_p = 3 \text{ dB} \qquad F_b = 1000 \text{ Hz}$$

$$A_s = 10 \text{ dB} \qquad F_s = 350 \text{ Hz}$$

Sampling frequency $F_{SF} = 5000$ Hz.

The discrete time frequencies will be,

$$f_p = \frac{F_p}{F_{SF}} = \frac{1000}{5000} = 0.2$$

$$\therefore \omega_p = 2\pi f_p = 2\pi \times 0.2 = 0.4\,\pi$$

$$f_s = \frac{F_s}{F_{SF}} = \frac{350}{5000} = 0.07$$

$$\therefore \omega_p = 2\pi f_s = 2\pi \times 0.7 = 0.14\ \pi$$

Thus the specifications are,

$$A_p = 3 \text{ dB} \qquad \omega_p = 0.4\ \pi$$

$$A_s = 10 \text{ dB} \qquad \omega_s = 0.14\ \pi$$

ii) To obtain the specifications of equivalent analog filter for bilinear transformation (Prewarping)

Prewarping is given by $\Omega = \tan\frac{\omega}{2}$, assuming $\frac{2}{T} = 1$.

$$\therefore \qquad \Omega_p = \tan\frac{\omega_p}{2} = \tan\frac{0.4\ \pi}{2} = 0.7265$$

and

$$\Omega_s = \tan\frac{\omega_s}{2} = \tan\frac{0.14\ \pi}{2} = 0.2235$$

Thus, the specifications of equivalent analog filter are,

$$A_p = 3 \text{ dB} \qquad \Omega_p = 0.7265$$

$$A_s = 10 \text{ dB} \qquad \Omega_s = 0.2235$$

iii) To convert the specifications of analog filter to normalized specifications.

Here let us consider a normalized lowpass filter with $\Omega_p = 1$ rad/sec. Let us convert the specifications obtained in (ii) to lowpass filter. For this purpose Ω_s and Ω_p will be exchanged. i.e.,

$$\left.\begin{array}{l} A_p = 3\ dB, \quad \Omega_p = 0.2235 \\ A_s = 10\,d\text{B}, \quad \Omega_s = 0.7265 \end{array}\right\} \text{Specifications of lowpass filter}$$

If $\Omega_p = 1$ rad/sec normalized, then,

$$\Omega_s(\text{normalized}) = \frac{\Omega_s}{\Omega_p} = \frac{0.7265}{0.2235} = 3.25 \text{ rad/sec}$$

Thus new specifications of normalized lowpass filter are,

$$A_p = 3 \text{ dB} \qquad \Omega_p = 1$$

$$A_s = 10 \text{ dB} \qquad \Omega_s = 3.25$$

iv) To obtain order of normalized lowpass filter

$$N = \frac{1}{2}\frac{\log\left[\dfrac{10^{0.1A_s dB}-1}{10^{0.1A_p dB}-1}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

$$= \frac{1}{2}\frac{\log\left[\dfrac{10^{0.1\times 10}-1}{10^{0.1\times 3}-1}\right]}{\log\left(\dfrac{3.25}{1}\right)} = 0.934 \approx 1$$

Thus the filter of order $N = 1$ will be enough.

v) To obtain the system function of normalized lowpass filter of order 1

$$H_{an}(s) = \frac{1}{s+1} \quad \text{from the Butterworth polynomial table.}$$

vi) To apply frequency transformation for lowpass to highpass

Lowpass to highpass transformation is given as,

$$\therefore \quad s \rightarrow \frac{\Omega_p \Omega_{HP}}{s}$$

Here $\Omega_p = 1$ rad/sec, $\Omega_{HP} = 0.7265$

$$s \rightarrow \frac{0.7265}{s}$$

$$\therefore \quad H_a(s) = \frac{1}{\dfrac{0.7265}{s}+1} = \frac{s}{s+0.7265}$$

vii) To apply bilinear transformation

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{\dfrac{1-z^{-1}}{1+z^{-1}}}{\dfrac{1-z^{-1}}{1+z^{-1}}+0.7265} = \frac{0.579(1-z^{-1})}{1-0.16z^{-1}}$$

This is the required digital highpass filter transfer function.

Examples for Practice

Example 4.9.14 : *Design digital Butterworth transformation filter using bilinear transformation*

$\omega_p = 0.23\pi$, $\omega_s = 0.43\pi$, $A_p = 2\,dB$, $A_s = 11\,dB$, $T = 1\,sec$.

[**Hint and Ans. :** $N = 2$, $\Omega_c = 0.433$ rad/sec., $H_a(s) = \dfrac{0.187}{s^2 + 0.6s + 0.18}$,

$$H(z) = \frac{0.1\left(1 + 2z^{-1} + z^{-2}\right)}{1 - 0.92z^{-1} + 0.326z^{-2}}]$$

Example 4.9.15 : *Design a second order bandpass digital butterworth filter with passband of 200 Hz to 300 Hz and sampling frequency of 2000 Hz using bilinear transformation.*

$$[\text{ Ans. : } H(z) = \frac{0.1367\left(1 - z^{-1}\right)}{1 - 1.237z^{-1} + 0.726z^{-2}}]$$

4.9.3 Solved Examples on Chebyshev Approximation

Examples for Understanding

Example 4.9.16 *Design the digital filter using Chebyshev approximation and bilinear transformation to meet the following specifications :*

Passband ripple = 1 dB for $0 \leq \omega \leq 0.15\pi$

Stopband attenuation ≥ 20 dB for $0.45\pi \leq \omega \leq \pi$

Solution : Given data :

$$A_p = 1 \text{ dB} \qquad 0 \leq \omega_p \leq 0.15\pi$$

$$A_s = 20 \text{ dB} \qquad \omega_s \leq 0.45\pi$$

To obtain specification of equivalent analog filter for bilinear transformation (Prewarping)

Here we have to design the filter using bilinear transformation. Hence it is necessary to apply prewarping. The frequency relationship is given as,

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} \quad \text{for bilinear transformation}$$

$$= \tan\frac{\omega}{2} \quad \text{assuming } \frac{2}{T} = 1.$$

$$\therefore \qquad \Omega_p = \tan\frac{\omega_p}{2} = \tan\frac{0.15\pi}{2} = 0.24 \text{ rad/sec.}$$

$$\text{and} \qquad \Omega_s = \tan\frac{\omega_s}{2} = \tan\frac{0.45\pi}{2} = 0.85 \text{ rad/sec.}$$

Thus the prewarped specifications of euivalent analog filter are,

$$A_p = 1 \text{ dB} \qquad \Omega_p = 0.24 \text{ rad/sec.}$$

$$A_p = 20 \text{ dB} \qquad \Omega_s = 0.85 \text{ rad/sec.}$$

To obtain order of Chebyshev filter

Here A_p and A_s are in dB. Hence order of Chebyshev filter is given by equation (4.7.31) as,

$$N = \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1A_s}-1}{10^{0.1A_p}-1}}}{\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\times 20}-1}{10^{0.1\times 1}-1}}}{\cosh^{-1}\left(\dfrac{0.85}{0.24}\right)} = 1.892 \approx 2$$

The next higher integer of 1.982 is 2. Hence we will required a chebyshev filter of 2nd order.

To obtain poles of Chebyshev filter

Since $A_p = 1$ dB is in dB use equation (4.7.29) to calculate ε .

i.e. $$\varepsilon = \sqrt{10^{0.1\,A_p}-1} = \sqrt{10^{0.1\times 1}-1} = 0.508$$

$\therefore$ $$\mu = \frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon} \quad \text{from equation (4.7.34 (b))}$$

$$= \frac{1+\sqrt{1+0.508^2}}{0.508} = 4.176$$

$\therefore$ $$a = \Omega_p \left(\frac{\mu^{\frac{1}{N}} - \mu^{-\frac{1}{N}}}{2}\right) \quad \text{By equation (4.7.34 (b))}$$

$$= 0.24\left(\frac{4.176^{\frac{1}{2}} - 4.176^{-\frac{1}{2}}}{2}\right) = 0.186$$

and $$b = \Omega_p \left(\frac{\mu^{\frac{1}{N}} + \mu^{-\frac{1}{N}}}{2} \right) \quad \text{By equation (4.7.34 (b))}$$

$$= 0.24 \left(\frac{4.176^{\frac{1}{2}} - 4.176^{-\frac{1}{2}}}{2} \right) = 0.304$$

Following table calculates the poles as per equation (4.7.34 (a)) and (4.7.34 (b)). Here $k = 0, 1, 2 \ldots N-1$. Since $N = 2$, $k = 0, 1$

k	$\phi_k = \frac{(2k+N+1)\pi}{2N}$	$\sigma_k = a \cos \phi_k$	$\Omega_k = b \sin \phi_k$	$p_k = \sigma_k + j\Omega_k$
0	$\phi_0 = \frac{(2\times 0+2+1)\pi}{2\times 2}$ $= \frac{3\pi}{4}$	$\sigma_0 = 0.186 \cos \frac{3\pi}{4}$ $= -0.131$	$\Omega_k = 0.304 \sin \frac{3\pi}{4}$ $= 0.215$	$p_0 = -0.131 + j\,0.215$
1	$\phi_1 = \frac{(2\times 1+2+1)\pi}{2\times 2}$ $= \frac{5\pi}{4}$	$\sigma_1 = 0.186 \cos \frac{5\pi}{4}$ $= -0.131$	$\Omega_k = 0.304 \sin \frac{5\pi}{4}$ $= -0.215$	$p_1 = -0.131 - j\,0.215$

From the above table the poles are complex conjugates i.e.,

$$s_1 = -0.131 + \mathrm{j}\,0.215 \text{ and } s_1^* = -0.131 - \mathrm{j}\,0.215 \qquad \ldots (4.9.16)$$

To obtain system function $H_a(s)$ of Chebyshev filter

By equation (4.7.35) system function is given as,

$$H_a(s) = \frac{k}{(s-s_1)(s-s_1^*)} \quad \text{for } N = 2$$

$$= \frac{k}{(s+0.131-j\,0.215)(s+0.131+j\,0.215)}$$

Putting values from equation (4.9.16)

$$= \frac{k}{(s+0.131)^2+(0.215)^2} = \frac{k}{s^2+0.262s+0.063} \qquad \ldots (4.9.17)$$

Here note the the last term in denominator of above equation. It is b_0. Thus $b_0 = 0.063$. For $N = 2$ i.e. even order,

$$k = \frac{b_0}{\sqrt{1+\varepsilon^2}} \quad \text{By equation (4.7.37)}$$

$$= \frac{0.063}{\sqrt{1+0.508^2}} = 0.056$$

Putting this value of k in equation (4.9.17),

$$H_a(s) = \frac{0.056}{s^2+0.262\,s+0.063}$$

vi) To obtain *H*(*z*) using bilinear transformation

System function of digital filter is obtained using bilinear transformation as,

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \left.\frac{0.056}{s^2+0.262\,s+0.063}\right|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \quad \text{assuming } \frac{2}{T} = 1$$

$$= \frac{0.056}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2+0.262\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.063} = \frac{0.042\,(1+z^{-1})^2}{1-1.414\,z^{-1}+0.604\,z^{-2}}$$

Example 4.9.17 *The specification of the lowpass filter are given as,*

$0.8 \le |H(\omega)| \le 1 \quad$ *for* $\quad 0 \le \omega \le 0.2\,\pi$

$|H(\omega)| \le 0.2 \quad$ *for* $\quad 0.32\,\pi \le \omega \le \pi$

Design the Chebyshev filter using bilinear transformation.

AU : May-14, Marks 16

Solution : Given data

$$A_p = 0.8\,, \quad \omega_p = 0.2\,\pi$$

$$A_s = 0.2\,, \quad \omega_s = 0.32\,\pi$$

Prewarping

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} = \tan\frac{\omega}{2} \quad \text{assuming } \frac{2}{T} = 1$$

$$\therefore \quad \Omega_p = \tan\frac{\omega_p}{2} = \tan\frac{0.2\,\pi}{2} = 0.325 \text{ rad/sec.}$$

$$\text{and} \quad \Omega_s = \tan\frac{\omega_s}{2} = \tan\frac{0.32\,\pi}{2} = 0.55 \text{ rad/sec.}$$

Thus the prewarped specifications of equivalent analog filter are,

$$A_p = 0.8\,, \quad \Omega_p = 0.325 \text{ rad/sec.}$$

$$A_s = 0.2\,, \quad \Omega_s = 0.55 \text{ rad/sec.}$$

Order of Chebyshev filter

Let us first calculate,

$$\varepsilon = \sqrt{\frac{1}{A_p^2}-1} \quad \text{By equation (4.7.28)}$$

$$= \sqrt{\frac{1}{0.8^2}-1} = 0.75$$

and
$$\delta = \sqrt{\frac{1}{A_s^2}-1} \quad \text{By equation (4.7.28)}$$

$$= \sqrt{\frac{1}{0.2^2}-1} = 4.9$$

The order of Chebyshev filter is given in terms of ε and δ as,

$$N = \frac{\cosh^{-1}(\delta/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} = \frac{\cosh^{-1}\left(\frac{4.9}{0.75}\right)}{\cosh^{-1}\left(\frac{0.55}{0.325}\right)} = 2.29 \approx 3$$

Poles of $H_a(s)$

Let us calculate following values defined in equation (4.7.34 (b))

$$\mu = \frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon} = \frac{1+\sqrt{1+0.75^2}}{0.75} = 3$$

$$a = \Omega_p\left(\frac{\mu^{\frac{1}{N}}-\mu^{-\frac{1}{N}}}{2}\right) = 0.325\left(\frac{3^{\frac{1}{3}}-3^{-\frac{1}{3}}}{2}\right) = 0.12$$

$$b = \Omega_p\left(\frac{\mu^{\frac{1}{N}}+\mu^{-\frac{1}{N}}}{2}\right) = 0.325\left(\frac{3^{\frac{1}{3}}+3^{-\frac{1}{3}}}{2}\right) = 0.347$$

$$\phi_k = \frac{(2k+N+1)\pi}{2N}, \; k = 0, 1, \ldots N-1 \quad \text{By equation (4.7.34 (b))}$$

For $N = 3$
$$\phi_k = \frac{(2k+3+1)\pi}{2\times 3}, \quad k = 0, 1, 2$$

or
$$\phi_k = \frac{(2k+4)\pi}{6}, \quad k = 0, 1, 2$$

Following table illustrates the calculations of poles

k	$\phi_k = \frac{(2k+4)\pi}{6}$	$\sigma_k = a \cos \phi_k$	$\Omega_k = b \sin \phi_k$	$p_k = \sigma_k + j\Omega_k$
0	$\phi_0 = \frac{(2\times 0+4)\pi}{6}$ $= \frac{4\pi}{6}$	$\sigma_0 = 0.12 \cos \frac{4\pi}{6}$ $= -0.06$	$\Omega_0 = 0.347 \sin \frac{4\pi}{6}$ $= 0.3$	$p_0 = -0.06 + j\,0.3$
1	$\phi_1 = \frac{(2\times 1+4)\pi}{6}$ $= \pi$	$\sigma_1 = 0.12 \cos \pi$ $= -0.12$	$\Omega_1 = 0.347 \sin \pi$ $= 0$	$p_1 = -0.12$
2	$\phi_2 = \frac{(2\times 2+4)\pi}{6}$ $= \frac{8\pi}{6}$	$\sigma_2 = 0.12 \cos \frac{8\pi}{6}$ $= -0.06$	$\Omega_2 = 0.347 \sin \frac{8\pi}{6}$ $= -0.3$	$p_2 = -0.06 - j\,0.3$

From above table complex conjugate poles are combined as follows :

$$s_1 = p_1 = -0.12$$

$$s_2 = p_0 = -0.06 + j\,0.3 \quad \text{and} \quad s_2^* = p_2 = -0.06 - j\,0.3$$

System function $H_a(s)$

$$H_a(s) = \frac{k}{(s-s_1)(s-s_1^*)(s-s_2^*)} \quad \text{By equation (4.7.35)}$$

$$= \frac{k}{(s+0.12)(s+0.06-j\,0.3)(s+0.06+j\,0.3)} = \frac{k}{(s+0.12)[(s+0.6)^2+(0.3)^2]}$$

$$= \frac{k}{(s+0.12)(s^2+0.12\,s+0.0936)}$$

Here $\quad b_0 = 0.12 \times 0.0936 = 0.011232$

From equation (4.7.37), $k = b_0 = 0.011232$ since order N is odd. Then above system function will be,

$$H_a(s) = \frac{0.011232}{(s+0.12)(s^2+0.12\,s+0.0936)}$$

***H*(*z*) using bilinear transformation**

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \left.\frac{0.011232}{(s+0.12)(s^2+0.12\,s+0.0936)}\right|_{s=\frac{1-z^{-1}}{1+z^{-1}}}, \qquad \text{since } \frac{2}{T} = 1$$

$$= \frac{0.011232}{\left[\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.12\right]\left[\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2+0.12\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.0936\right]}$$

$$= \frac{0.0083\,(1+z^{-1})^3}{(1-0.785\,z^{-1})(1-1.493\,z^{-1}+0.802\,z^{-2})}$$

This is the system function of required digital filter.

Example 4.9.18 *Determine the order of a Chebyshev digital lowpass filter to meet the following specifications : In the passband extending from 0 to 0.25π, a ripple of not more than 2 dB is allowed. In the stopband extending from 0.4π to π, attenuation can be more than 40 dB. Use bilinear transformation method.*

Solution : Given data

$$A_p = 2 \text{ dB} \qquad \omega_p = 0.25\pi$$

$$A_s = 40 \text{ dB} \qquad \omega_s = 0.4\pi$$

Equivalent analog filter specifications for bilinear transformation

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = \tan\frac{0.25\pi}{2} = 0.414, \text{ (assuming } \frac{2}{T} = 1)$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = \tan\frac{0.4\pi}{2} = 0.727$$

Order of Chebyshev filter

$$N = \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1A_s}-1}{10^{0.1A_p}-1}}}{\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\times 40}-1}{10^{0.1\times 2}-1}}}{\cosh^{-1}\left(\dfrac{0.727}{0.414}\right)} = 4.786 \approx 5$$

Order of Butterworth filter

$$N = \frac{\log\sqrt{\dfrac{10^{0.1A_s}-1}{10^{0.1A_p}-1}}}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{\log\sqrt{\dfrac{10^{0.1\times 40}-1}{10^{0.1\times 2}-1}}}{\log\left(\dfrac{0.727}{0.414}\right)} = 8.65 \approx 9$$

Thus the order of Chebyshev filter is lower than that of Butterworth filter.

Examples for Practice

Example 4.9.19 *Design a second order notch (bandstop) filter with following characteristics :*
Notch frequencies 400 Hz to 600 Hz

Sampling frequency 2.4 kHz
Ripple in the passband and monotonic decay in the notch. Use bilinear transformation method. **(Ans. :** $H(z) = \dfrac{1.577 - 2z^{-1} + z^{-2}}{1.792 - 0.846\, z^{-1} + 1.362\, z^{-2}}$ **)**

Example 4.9.20 *The digital lowpass filter is to be designed that has a passband cut-off frequency $\omega_p = 0.375\pi$ with $\delta_p = 0.01$ and a stopband cut-off frequency $\omega_s = 0.5\pi$ with $\delta_s = 0.01$. The filter is to be designed using bilinear transformation. What orders of the Butterworth, Chebyshev filters are necessary to meet design specifications?*

[Hint and Ans. : $\Omega_p = 0.668$, $\Omega_s = 1$, $N_{Butterworth} = 17$, $N_{Chebyshev} = 8$**]**

Example 4.9.21 *Design a digital LPF to satisfy the following passband ripple $1 \leq |H(j\Omega)| \leq 0$ for $0 \leq \Omega \leq 1404\pi$ rad/sec and stopband attenuation $|H(j\Omega)|$ dB > 60 for $\Omega \geq 8268\pi$ rad/sec. The sampling interval $T_s = 10^{-4}$ sec. Use bilinear transformation technique for designing.*

[Hints and Ans. : $N = 3$, $\Omega_c = 0.3193$, $H_a(s) = \dfrac{(0.3193)^3}{(s+0.3193)\,(s^2+0.318s+0.101)}$

$H(z) = \dfrac{0.0173(1+z^{-1})^3}{(1-0.515z^{-1})(1-1.267z^{-1}+0.552z^{-2})}$**]**

Review Question

1. *Explain the steps in designing the IIR filters.*

4.10 Comparison of FIR and IIR Filters

FIR and IIR filters can be compared on the basis of number of parameters. Table 4.10.1 shows the comparison of FIR and IIR filters. This comparison can also be considered as advantages and disadvantages of FIR and IIR filters.

Sr. No.	Parameter or characteristic	IIR filters	FIR filters
1	Unit sample response	$h(n)$ is infinite in duration.	$h(n)$ is finite in duration.
2	Poles and zeros	Poles as well as zeros are present. Sometimes all pole filters are also designed.	These are all zero filters.
3	Recursive/Nonrecursive and feedback from output	These filters use feedback from output. They are recursive filters.	These filters do not use feedback. They are nonrecursive.
4	Phase characteristic	Nonlinear phase response. Linear phase is obtained if $H(z) = \pm z^{-1} H(z^{-1})$	Linear phase response for h(n) $= \pm\, h(m-1-n)$

5	Stability of the filter	These filters are to be designed for stability.	These are inherently stable filters.
6	Number of multiplications required	Less.	More.
7	Complexity of implementation	More.	Less.
8	Memory requirement	Less memory is required.	More memory is required.
9	Can simulate prototype analog filters	Yes.	No.
10	Order of filter for similar specifications	Requires lower order.	Requires higher order.
11	Availability of design softwares	Good.	Very good.
12	Design procedure	Complicated.	Less complicated.
13	Processing time	Less time is required.	More time is required.
14	Design methods	(i) Bilinear transform. (ii) Impulse invariance.	(i) Windowing. (ii) Frequency sampling.
15	Applications	Can be used where sharp cutoff characteristics with minimum order are required.	Used where linear phase characteristic is essential.

Table 4.10.1 Comparison or Advantages / Disadvantages of FIR and IIR filters

The selection between FIR and IIR filter is based on following criteria :

i) If linear phase requirement is critical then FIR filters are used.

ii) If sharp cutoff characteristic with minimum order is required, then IIR filters are required.

Thus IIR filters are used in maximum number of applications when phase characteristic is not very important. For the design of FIR as well as IIR filters good number of standard softwares are available.

Till now we have not discussed about finite wordlength effects. IIR and FIR filters can be compared on the basis of these effects also. FIR filters are least affected due to finite wordlength effects, whereas IIR filters are affected more.

4.11 Short Answered Questions [2 Marks Each]

Q.1 State the conditions for FIR filters to have linear phase.

Madras Univ. : Oct. -2000; April-01; May-08, 11

Ans. : A FIR filter will have linear phase if, $h(n) = h(M-1-n)$ for symmetric response. $h(n) = -h(M-1-n)$ for anitsymmetric response.

Here 'M' is the length of unit sample response of the filter.

Q.2 State Gibb's phenomenon.

Madras Univ., : Oct. -2000, April-01, Dec.-08, May-09, 12, 16

Ans. : While designing FIR filters, unit sample response is obtained from desired frequency response $H_d(\omega)$. The desired unit sample response can be obtained by taking inverse Fourier transform or Fourier series. Normally $h_d(n)$ is infinite in length. Hence it is truncated and only finite number of samples are taken. This truncation introduces oscillations or ringing of $H(\omega)$ in passband and stopband. This effect is called Gibb's phenomenon. It can be reduced by using suitable windows.

Q.3 What is frequency warping ?

Madras Univ. : Oct. -2000; April-01, Dec.-07, May-08, 09, Dec.-10, May-10, Dec.-12, May-12

Ans. : The frequency relationship in bilinear transformation is given as,

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

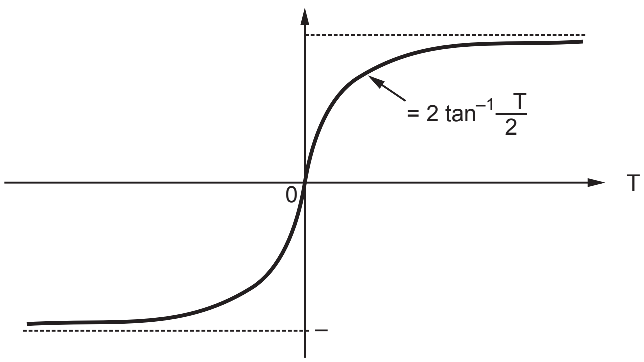


Fig. 4.11.1 Mapping between 'Ω' and 'ω' in bilinear transformation

or $$\omega = 2\tan^{-1} \frac{\Omega T}{2}$$

Fig. 4.11.1 shows this relationship. In this figure observe that the relationship is highly nonlinear at higher values of 'ω'. The entire frequency range $(-\infty \ to + \infty)$ is mapped into $+\pi$ to $-\pi$. This nonlinearity is called warping effect.

Q.4 Mention the disadvantages of bilinear transformation technique.

Madras Univ. : Oct.-2000

Ans. :

1. There is nonlinearity in the frequency relationship. This nonlinearity is large at high frequencies.
2. Impulse response and phase response of the analog filter is not preserved during bilinear mapping.

Q.5 Why impulse invariant method is not preferred in the design of highpass IIR filters ? **Madras Univ. : Oct.-2000**

Ans. : In impulse invariant method, the segments of $\frac{(2k-1)\pi}{T} \le \Omega \le \frac{(2k+1)\pi}{T}$ are mapped on the unit circle repeatedly. Hence first set, i.e. $-\frac{\pi}{T} \le \Omega \le \frac{\pi}{T}$ is mapped correctly. Then $\frac{\pi}{T} \le \Omega \le \frac{3\pi}{T}$ is mapped on the same circle. Thus one point on the circle represents multiple analog frequencies. Hence high frequencies are actually mapped as low frequencies. Therefore all high frequencies are aliased frequencies. Hence impulse invariant technique is not much suitable for high pass filters. But for lowpass filters it is better, since actual mapping takes place.

Q.6 What is windowing and why it is necessary ? **Madras Univ. : April-2000, Dec.-12, 16**

Ans. : Unit sample response of the desired filter is obtained from frequency response $H_d(w)$. This unit sample response is normally infinite in length. Hence it is truncated to some finite length. This truncation creates oscillations in passband and stopband of the filter. This problem can be avoided with windowing . The desired unit sample response is multiplied with suitable window. The length of the window can be selected to desired value. Due to windowing, the unit sample response of the filter is reshaped such that ringing (oscillations) are reduced.

Q.7 What are the attractive aspects of frequency sampling design ? **Madras Univ. : April-2000**

Ans. : In frequency sampling method, the desired magnitude response is sampled. These samples of frequency response are DFT coefficients. These frequency samples can be set as per the requirement. When the number of samples are limited, then desired frequency response is implemented effectively. Narrowband frequency selective filters can be implemented better with the help of frequency sampling.

Q.8 Why do we go for analog approximation to design a digital filter ? **Madras Univ. : April-2000**

Ans. : These are effective filter approximation techniques available in analog domain. Using transformation methods a stable analog filter can be converted to stable digital

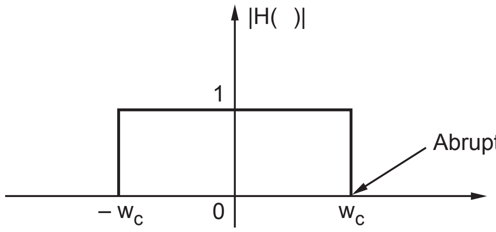


Fig. 4.11.2 Frequency response of ideal lowpass filter

filters. Hence it becomes easier to design IIR filters from analog filters. But such effective approximations are not available in discrete domain.

Q.9 What is the effect of having abrupt discontinuity in frequency response of FIR filters? **Madras Univ., Oct.-99**

Ans. : Consider a low pass filter having frequency response as shown in Fig. 4.11.2.

Observe that there is abrupt discontinuity in the frequency response at ω_c. Due to this discontinuity, the impulse response becomes infinite in length i.e.,

$$h(n) = \begin{cases} \dfrac{\sin \omega_c n}{\pi n} & for\, n \neq 0 \\ \dfrac{\omega_c}{n} & for\, n = 0 \end{cases}$$

It is clear from above equation that $h(n)$ is infinite in length. Hence it is necessary to truncate this response.

Q.10 What are the characteristic features of FIR filters ?

Madras Univ. : April-99, May-11

Ans. :

1. FIR filters are all zero filters.
2. FIR filters are inherently stable filters.
3. FIR filters can have linear phase.

Q.11 : What is Butterworth filter approximation ?

Ans. : In butterworth filter approximation, the magnitude function is monotonically reducing. All poles of the Butterworth filter lie on the circle of radius Ω_c (cut-off frequency).

Q.12 : What is Chebyshev approximation ?

OR

Mention the significance of Chebyshev approximation. **AU : Dec.-10**

Ans. :

- In Chebyshev approximation, there are ripples in either passband or stopband.
- Poles of the Chebyshev filter lie on the ellipse.

- Chebyshev filters are more effective than butterworth filters for the same order.

Q.13 : What is normalized filter ?

Ans. : Normalized filter or prototype filter is the lowpass filter with cutoff frequency of 1 rad/sec.

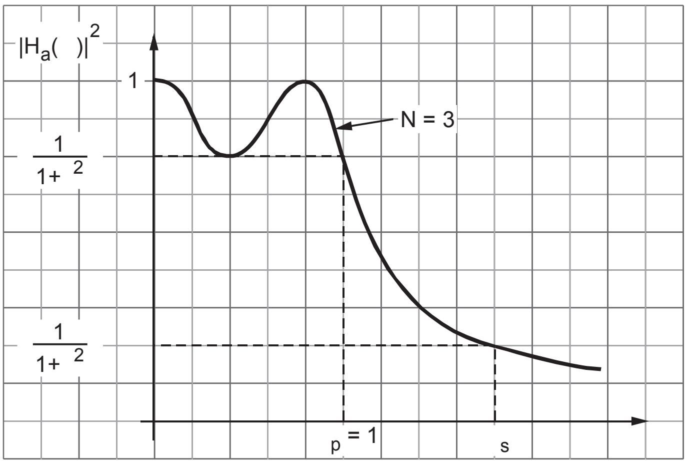


Fig. 4.11.3 Magnitude response of 3rd order Chebyshev filter

Q.14 What is canonic structure ? **AU : May-04**

Ans. : If the number of delays in the structure is equal to order of the difference equation or order of the transfer function, then it is called canonic form realization.

Q.15 Draw the magnitude response of 3rd order Chebyshev lowpass filter. **AU : May-04**

Ans. :

Q.16 What are the advantages and disadvantages of FIR filter ? **AU : May-05, Dec.-10, May-16**

Ans. : Advantages

1) FIR filters are inherently stable.

2) FIR filter implementation is simple.

3) FIR filters are capable of providing linear phase.

Disadvantages

1) FIR filters require higher order compared to IIR filters for providing similar magnitude response.

2) More memory and processing time is required by FIR filters.

Q.17 Name the different structures to implement IIR system. **AU : May-05**

Ans. : IIR systems can be implemented with the help of

1. Direct form structures

   i) Direct form - I ii) Direct form - II

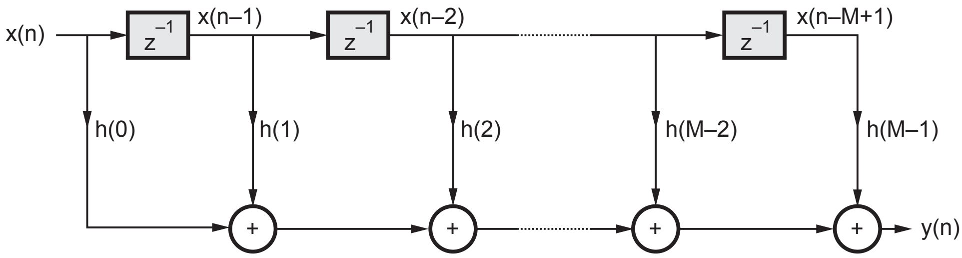


Fig. 4.11.4 Direct form realization of FIR filter

2. Cascade form structure
3. Parallel form structure
4. Lattice structure

Q.18 Draw a basic FIR filter structure. **AU : Dec.-05, 06**

Ans. : Following figure shows the direct form realization of FIR filter.

Q.19 What is bilinear transformation ? **AU : Dec.-05**

Ans. : The bilinear transformation is given as

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

- Right hand side of s-plane ($\sigma > 0$) maps outside of the unit circle.
- Left hand side of s-plane ($\sigma < 0$) maps inside of the unit circle.
- The j Ω axis in s-plane maps on unit circle.

Q.20 Draw the direct form-I structure of IIR filter. **AU : May-06**

Ans. : Fig. 4.11.5 shows direct form - I structure. (See Fig. 4.11.5 on next page)

Q.21 What is linear phase filter ? What is impulse response of such filter ? **AU : May-06**

Ans. : When phase becomes linear function of frequency, it is called linear phase filter. The impulse response satisfies following condition for linear phase,

$$h(n) = \pm h(M-1-n)$$

Q.22 Is bilinear transformation linear or not ? What is the merit and demerit of bilinear transformation ? **AU : Dec.-06**

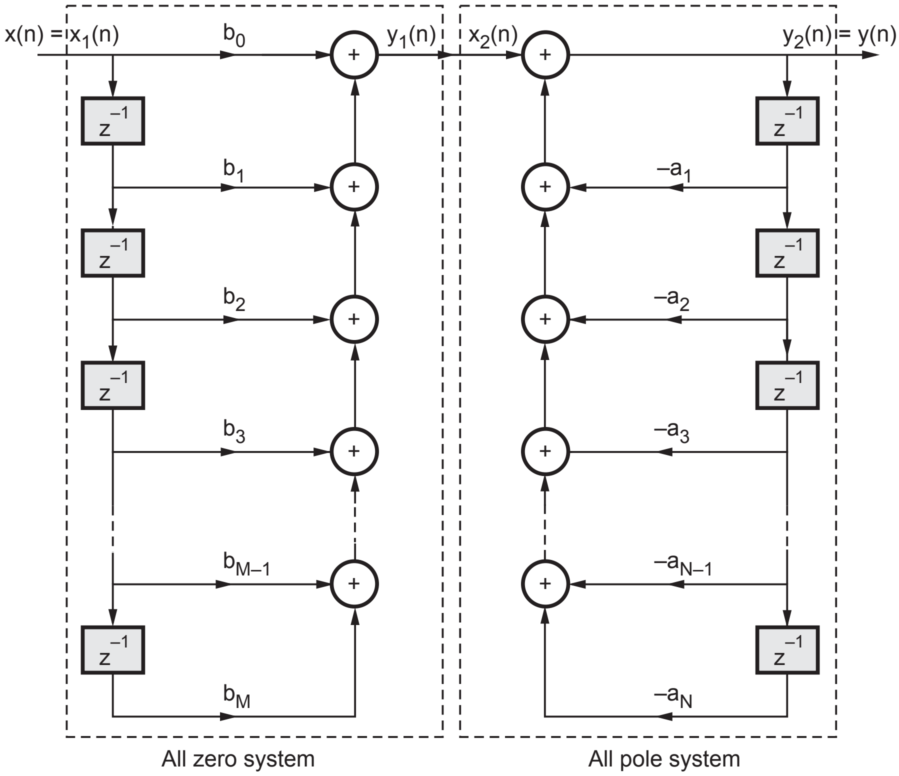


Fig. 4.11.5 Direct form-I of filters

Ans. :

- Bilinear transformation is nonlinear.
- **Merit** : Bilinear transformation avoids frequency aliasing while mapping.
- **Demerit** : Mapping is highly nonlinear, hence frequency warping effect takes place.

Q.23 Write the equation of Bartlett and Hamming window. **AU : May-07, 14**

Ans. : **Bartlett window :** $W_T(n) = 1 - \dfrac{2\left|n - \dfrac{M-1}{2}\right|}{M-1}$

Hamming window : $W_H(n) = 0.54 - 0.46\cos\dfrac{2\pi n}{M-1}$

For n = 0, 1, … M − 1

Q.24 Write the expression for location of poles of normalized Butterworth filter. **AU : May-07**

Ans. : $p_k = \pm\, \Omega_c\, e^{j(N+2k+1)\pi/2N}$, k = 0, 1, 2, … N − 1

Here Ω_c is cutoff frequency and

N is order of the filter.

For normalized Butterwort filter $\Omega_c = 1$,

$$p_k = e^{j(N+2k+1)\pi/2N}$$

Q.25 Why is FIR filter always stable ? AU : May-10

Ans. : Output of FIR filter is given as,

$$y(n) = \sum_{k=0}^{M-1} b_k\, x(n-k)$$

Here 'b_k' are coefficients of FIR filter. They are finite and bounded. Hence output y(n) will be bounded as long as input x(n – k) is bounded. Thus above equation of FIR filter represents a *stable* system.

Q.26 State the conditions for a digital filter to be causal and stable. AU : May-10, 12

Ans. : The system function of the digital filter should satisfy following conditions for the filter to be causal and stable.

i) ROC of the system function is exterior of some circle of radius 'r', i.e $|z| > r$.

ii) ROC must include unit circle, i.e. $r < 1$.

Thus, $|z| > r < 1$ for causal and stable filter.

Q.27 Mention the methods to convert analog filters into digital filters. AU : May-10, Dec.-16

Ans. : 1) Approximation of derivatives

2) Impulse invariance

3) Bilinear transformation

4) Matched z-transformation

Q.28 Differentiate between FIR and IIR filters. AU : May-12

Ans. : Refer table 4.10.1.

Q.29 Is the given transfer function H(z) = $\dfrac{1+0.8z^{-1}}{1-0.9z^{-1}}$ represent low pass filter or high pass filter ? AU : Dec.-13

Ans. :

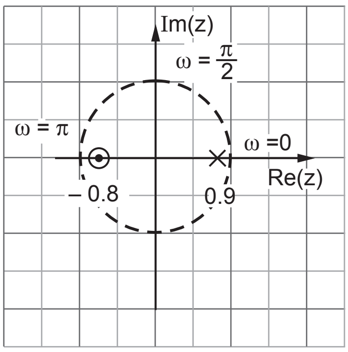


Fig. 4.11.6 Pole-zero plot

$H(z) = \frac{1+0.8z^{-1}}{1-0.9z^{-1}} = \frac{z+0.8}{z-0.9}$. The pole is located at p = 0.9 and zero is located at

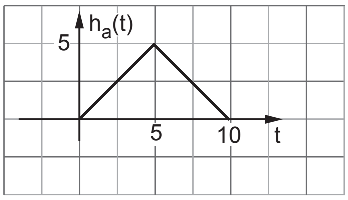


Fig. 4.11.7

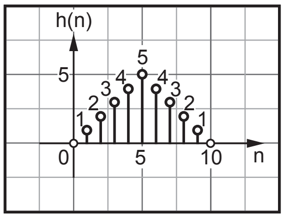


Fig. 4.11.8

z = –0.8. Fig. 4.11.6 shows the pole-zero plot. Observe that low frequencies will be boosted due to pole at z = 0.9 and high frequencies will be attenuated due to zero at z = –0.8. Hence this is a low pass filter.

Q.30 The impulse response of an analog filter is given in Fig. 4.11.7. Let $h(n) = h_a(nT)$, where T = 1. Determine the system function. **AU : Dec.-13, 15**

Ans. : After sampling $h_a(t)$, the given triangular function will be as shown in Fig. 4.11.8 since T = 1,

$$h(n) = \{\underset{\uparrow}{0}, 1, 2, 3, 4, 5, 4, 3, 2, 1, 0\}$$

$$\therefore \quad H(z) = \sum_{n=0}^{10} h(n) z^{-n} = z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + 5z^{-5} + 4z^{-6} + 3z^{-7} + 2z^{-8} + z^{-9}$$

$$= (z^{-1} + z^{-9}) + 2(z^{-2} + z^{-8}) + 3(z^{-3} + z^{-7}) + 4(z^{-4} + z^{-6}) + 5z^{-5}$$

Second method

The given triangular pulse can be expressed with the help of ramp functions as,

$$h_a(t) = t\,u(t) - 2\,(t-5)\,u\,(t-5) + (t-10)\,u\,(t-10)$$

$$h(n) = h_a(nT)\big|_{t=nT} = h_a(n)\big|_{t=n} \text{ since T = 1}$$

$$\therefore \quad h(n) = n\,u(n) - 2\,(n-5)\,u\,(n-5) + (n-10)\,u\,(n-10)$$

Taking z - transform using time shift property,

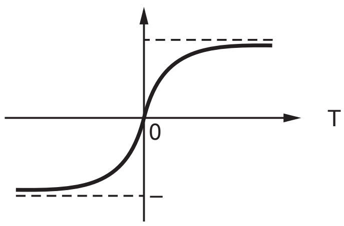


Fig. 4.11.9 Relationship between ω and Ω

$$H(z) = \frac{z^{-1}}{(1-z^{-1})^2} - 2z^{-5} \cdot \frac{z^{-1}}{(1-z^{-1})^2} + z^{-10} \cdot \frac{z^{-1}}{(1-z^{-1})^2} = \frac{z^{-1}(1-2z^{-5}-z^{-10})}{(1-z^{-1})^2}$$

Q.31 What is meant by prewarping ? **AU : May-11, 17, Dec.-15**

Ans. : In bilinear transformation discrete time frequency is given as,

$$\omega = 2\tan^{-1}\frac{\omega T}{2}$$

This relationship is highly nonlinear as shown in Fig. 4.11.9. It is called *frequency warping*. The problem of frequency warping can be corrected by applying reverse characteristics before applying bilinear transformation. It is given as,

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

This effect is called *prewarping*. It removes the nonlinear relationship in frequencies of analog signal and discrete time signal.

Q.32 Compare bilinear transformation and impulse invariant method of IIR filter design. **AU : Dec.-11**

Ans :

Sr. No.	Impulse invariant method	Bilinear transform method
1.	Only poles of H(s) are mapped.	Both poles and zeros of H(s) are mapped.
2.	Aliasing of frequencies takes place.	No aliasing since mapping is one to one.
3.	Linear frequency relationship.	Nonlinear frequency relationship.

Q.33 What is meant by linear phase response of a filter ? **AU : Dec.-11**

Ans : When the phase shift is directly proportional to frequency, i.e.,

$$\angle H(\omega) = k\,\omega \;,\; \text{k is constant}$$

This is called linear phase shift.

Q.34 Comment on the passband and stop band characteristics of butter worth filter. **AU : May-15**

Ans. : i) Response is monotonically reducing.

ii) $|H_a(\Omega)|^2 = 0.5$ for $\Omega = \Omega_c$ for all N.

iii) As the order is increased, the characteristic is close to ideal response.

Q.35 Realize the following causal linear phase FIR system function $H(z) = \frac{2}{3} + z^{-1} + \frac{2}{3}z^{-2}$. [Refer example 4.2.2] **AU : May-15**

Q.36 Obtain the cascade realization for the system function,

$$H(z) = \frac{\left(1+\frac{1}{4}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right)}$$ [Refer example 4.3.3]

AU ; May-16

Q.37 Write the advantages and disadvanages of digital filters.

OR

Why digital filters are more useful than analog filters ? **AU : Dec.-16**

Ans. : Advantages :

i) Phase response can be made linear.

ii) Digital filters can be made adaptive.

iii) Digital filters are stable.

iv) There are no environmental effects and component aging problems.

v) Data storage is possible. Hence offline filtering can be done.

Disadvantages :

i) Digital filters are not economical for simple filtering.

ii) Finite wordlength problems are present.

❑❑❑

5 Digital Signal Processors

Syllabus

Introduction - Architecture - Features - Addressing formats - Functional modes - Introduction to commercial DS Processors.

Contents

5.1 Introduction . . . . **May-11** · · · · · · · · · · · · · · · · Marks 16

5.2 Computer Architectures for Signal Processing

. . . . **May-11, 12, 14,**

. . . . **Dec.-10, 12, 13** · · · · · · · · · · · Marks 8

5.3 Addressing Formats . . . . **May- 11, 12, 15, 16, 17,**

. . . . **Dec.-11, 12, 13, 15, 16,** · · · Marks 16

5.4 On-Chip Peripherals

5.5 Introduction to Commercial Digital Signal Processors

. . . . **May-12, 14, 15, 16, 17,**

. . . . **Dec.-10, 12, 15, 16,** · · · · · · Marks 16

5.6 Selecting Digital Signal Processors . . . . **May-15** · · · · · · · · · · · · · · · · · Marks 2

5.7 Applications of DSP . . . . **May-16, Dec.-16,** · · · · · · · · · Marks 8

5.8 Short Answered Questions [2 Marks Each]

5.1 Introduction

AU : May-11

In this chapter we will briefly introduce DSP processor architecture and their features. There are few things common to all DSP algorithms such as,

i) Processing on arrays is involved.

ii) Majority of operations are multiply and accumulate.

iii) Linear and circular shifting of arrays is required.

These operations require large time when they are implemented on general purpose processors. This is because the hardware of general purpose processors is not optimized to perform such operations fast. Hence general purpose processors are not suitable for DSP operations. Particularly, real time DSP operations are very difficult on general purpose processors. Hence DSP processors having architecture suitable for DSP operations are developed.

5.1.1 Desirable Features of DSP Processors

Now let us see what features DSP processors should have so that DSP operations will be performed fast.

i) DSP processors should have multiple registers so that data (i.e. arrays) exchange from register to register is fast.

ii) DSP operations require multiple operands simultaneously. Hence DSP processor should have multiple operand fetch capacity.

iii) DSP processors should have circular buffers to support circular shift operations.

iv) The DSP processor should be able to perform multiply and accumulate operations very fast.

v) DSP processors should have multiple pointers to support multiple operands, jumps and shifts.

vi) Since DSP processors can be used with general processors, they should have multi processing ability.

vii) To support DSP operations fast, the DSP processors should have on chip memory.

viii) For real time applications interrupts and timers are required. Hence DSP processors should have powerful interrupt structure and timers.

The architectures of DSP processors are designed to have these features. The DSP processors from Analog Devices, Texas Instruments, Motorola etc. are commonly used.

Review Question

1. *State the desirable features of DSP processors.* **AU : May-11, Marks 16**

5.2 Computer Architectures for Signal Processing

AU : May-11, 12, 14, Dec.-10, 12, 13

Although signal processing operations can be performed on all the types of processors, they are not computationally efficient. The specific operations in DSP, such as multiply/accumulate, shifting, bit reversing, convolution needs special provisions in architectures for efficient computation. Different types of architectures are as follows :

1. Von-Neumann architecture.
2. Harvard architecture.
3. Modified Hardvard architecture.

5.2.1 Von-Neumann Architecture

All the general purpose processors normally have this type of architecture. Fig. 5.2.1 shows the Von-Neumann architecture.

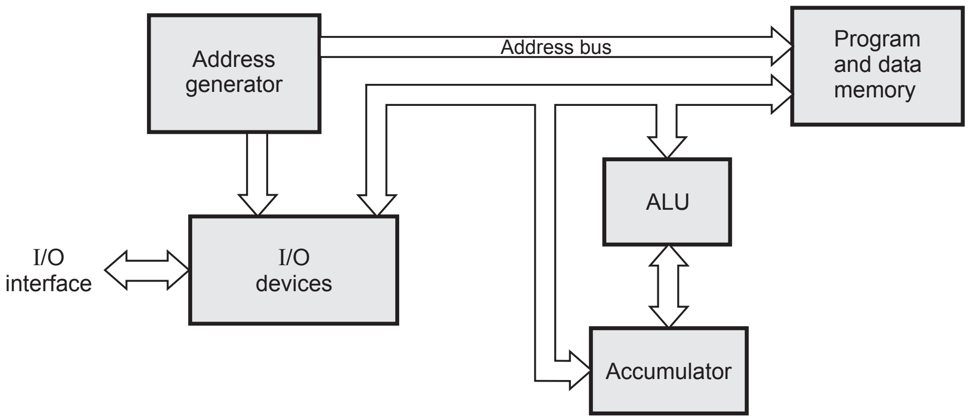


Fig. 5.2.1 Von-Neumann architecture

The architecture shares same memory for program and data. The processors perform instruction fetch, decode and execute operations sequentially. In such architecture, the speed can be increased by pipelining. This type of architecture contains common interval address and data bus, ALU, accumulator, I/O devices and common memory for program and data. This type of architecture is not suitable for DSP processors.

5.2.2 Harvard Architecture

- The Harvard architecture has separate memories for program and data. There are also separate, address and data buses for program and data. Because of these separate on chip memories and internal buses, the speed of execution in Harvard architecture is high.

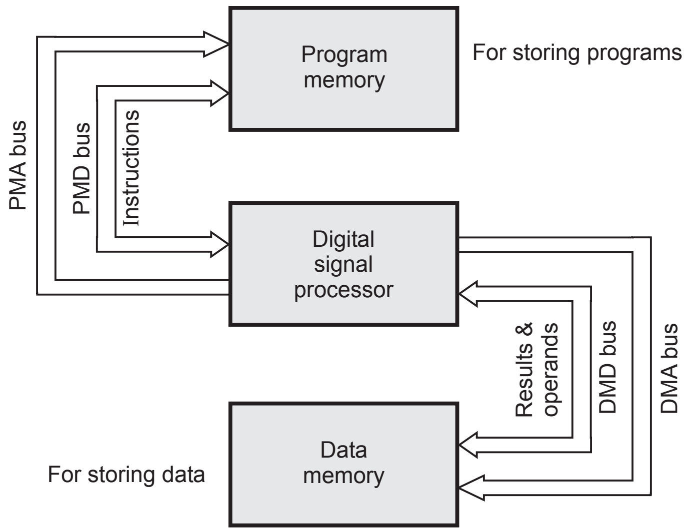


Fig. 5.2.2 Harvard architecture showing separate program and data memories

- In the above figure observe that there is Program Memory Address (PMA) bus and Program Memory Data (PMD) bus separate for program memory. Similarly there is separate Data Memory Data (DMD) bus and Data Memory Address (DMA) bus for data memory. This is all on chip. The digital signal processor includes various registers, address generators, ALUs etc.
- The PMD bus is used to get instructions from the program memory and DMD bus is used to exchange operands and results from data memory.
- The instruction code from program memory and operands from data memory can be fetched simultaneously. This parallel operation increases the speed.
- It is possible to fetch next instruction when current instruction is executed. That is, the fetch, decode and execute operations are done parallelly.

5.2.3 Modified Harvard Architecture

- Fig. 5.2.3 shows the block diagram for modified Harvard architecture. One set of bus is used to access program as well as data memories.
- The DMD bus can be used to transfer the data from program memory to data memory and vice-versa.
- Normally the program memory and data memory addresses are generated by separate address generators.

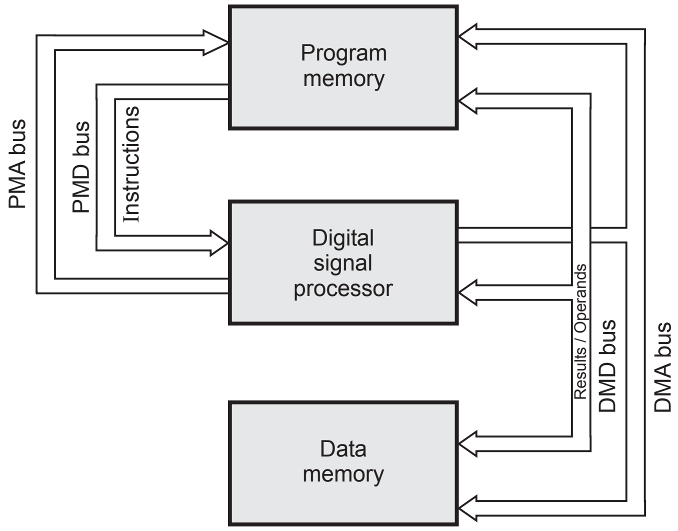


Fig. 5.2.3 Modified Harvard architecture data memory shared by programs

- This modified Harvard architecture is used in several P-DSPs such as DSP processors from Texas Instruments and Analog Devices.

5.2.4 Hardware Multiplier-Accumulator (MAC) Unit

- Most of the operations in DSP involve array multiplication. The operations such as convolution, correlation require multiply and accumulate operations. In real time applications, the array multiplication and accumulation must be completed before next sample of input comes. This requires very fast implementation of multiplication and accumulation.
- The dedicated hardware unit called MAC is used. It is called multiplier-accumulator (MAC). It is one of the computational unit in processor. The complete MAC operation is executed in one clock cycle.
- In Texas Instruments DSP processor 320C5X, the output of multiplier is stored into the product register. This product register contents are added to accumulator register ACC in central ALU.
- Fig. 5.2.4 shows the block diagram of typical MAC unit.

The MAC accepts two 16-bit 2's compliment fractional numbers and computes 32-bit product in a single cycle only. The X-register and Y-register holds the inputs to be multiplied.

- The DSP processors have a special instruction called MACD. This means multiply accumulate with data shift. The MACD instruction performs multiply, accumulate with accesses are required :

i) Fetch MACD instruction from program memory

ii) Fetch one of the operands from program memory

iii) Fetch second operand from data memory

iv) Data memory write.

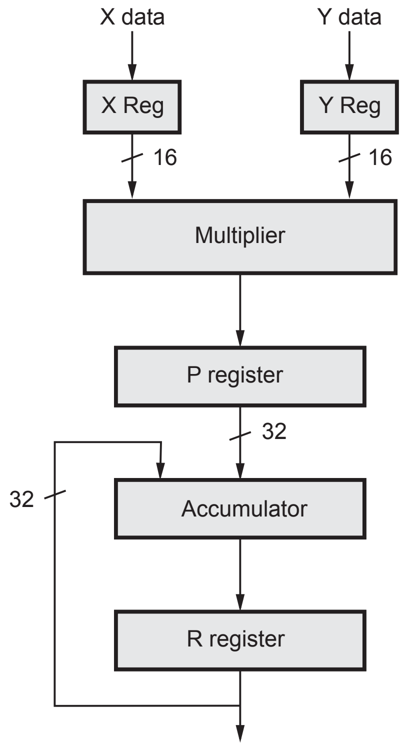


Fig. 5.2.4 Block diagram of MAC unit

- If this instruction is executed with conventional architecture (Von-Neumann architecture), then it requires four clock cycles. But Harvard and Modified Harvard architecture require less number of clock cycles.

5.2.5 Multiple Access Memory

The multiple access memory allows more than one memory access in a single clock cycle. The dual access RAM (DA-RAM) allows two memory access in a single clock cycle. The DARAM is connected to the DSP processor with two address and two data buses independently. This gives four memory accesses in a single clock period. The Harvard architecture allows multiple access memories to be interfaced to DSP processes.

5.2.6 Multiported Memory

- The multiported memory has the facility of interfacing multiple address and data buses. Fig. 5.2.5 shows the block diagram of a dual ported memory. This memory has two address buses and two data buses separately interfaced.

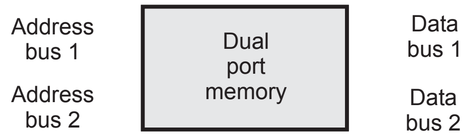


Fig. 5.2.5 Dual port memory

- The dual ported memory can allows two memory accesses in a single clock period.
- With the help of dual port memory, the program and data can be stored in a single memory chip and they can be accessed simultaneously.
- The motorola DSPs use single ported program memory and dual ported data memory. This allows one program memory access and two data memory access per clock cycle.
- The multiported memories increased number of pins and larger chip area which makes it more expensive and large in size.

5.2.7 Pipelining

Any instruction cycle can be split in following micro instructions :

i) Fetch : In this phase, an instruction is fetched from the memory.

ii) Decode : In this phase, an instruction is decoded.

iii) Read : An operand required for the instruction is fetched from the data memory.

iv) Execute : The operation is executed and results are stored at appropriate place.

Each of the above operations can be separately executed in different functional units. Fig. 5.2.6 shows how the instruction is executed without pipeline.

Value of T	Fetch	Decode	Read	Execute
1	I 1			
2		I 1		
3			I 1	
4				I 1
5	I 2			
6		I 2		
7			I 2	
8				I 2

Fig. 5.2.6 Instruction execution without pipeline

In the above figure observe that when instruction I 1 is in fetch phase, other units such as decode, read and execute are idle. Similarly when I 1 is in decode phase, other three units are idle. This means each functional unit is busy only for 25% of the total time.

- Fig. 5.2.7 shows the instruction execution with pipeline. Here observe that when I 1 is in decode phase, next instruction I 2 is fetched. Similarly when I 2 goes to decode phase, next instruction I 3 is fetched. Thus observe that all the functional units are executing four successive instructions at any time. On comparing Fig. 5.2.6 and Fig. 5.2.7 we observe that five instructions are executed in the same time if pipelining is used.

Value of T	Fetch	Decode	Read	Execute
1	I 1			
2	I 2	I 1		
3	I 3	I 2	I 1	
4	I 4	I 3	I 2	I 1
5	I 5	I 4	I 3	I 2
6		I 5	I 4	I 3
7			I 5	I 4
8				I 5

Fig. 5.2.7 Instruction execution with pipeline

5.2.8 Extended Parallelism

In DSP processor architecture parallel processing techniques are used extensively to increase computational efficiency. Very Long Instruction Word (VLIW), Single Instruction Multiple Data (SIMD) and superscalar processing are used to achieve this.

5.2.8.1 VLIW Architecture and Multiple ALUs

- Some of the DSP processors use Very Long Instruction Word (VLIW) architecture. Such architecture consists of multiple number of ALUs, MAC units, shifters etc.
- Fig. 5.2.8 shows the block diagram of VLIW architecture.
- The above architecture consists of multiported register file. It is used for fetching the operands and storing the results.
- The Read/Write cross bar provides parallel random access by functional units to the multiported register file. The functional units work concurrently with the load/store operation of data between a RAM and the register file.

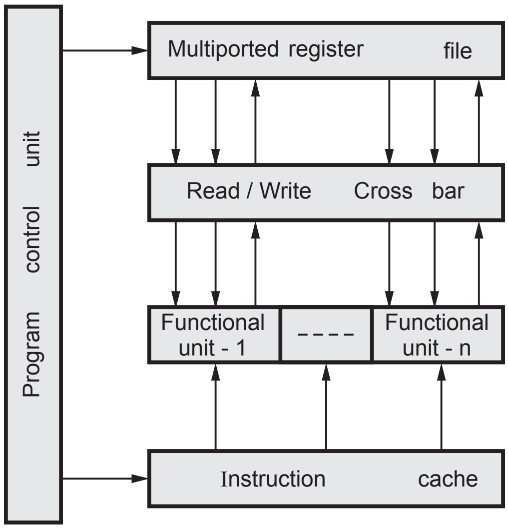


Fig. 5.2.8 VLIW architecture

- The program control unit provides the algorithm that executes independent parallel operations. The performance of VLIW architecture depends upon degree of parallelism.
- Normally 8 functional units are preferred. This number is limited by hardware cost of the multiported register file and crossbar switch.

5.2.8.2 Single Instruction Multiple Data (SIMD)

The SIMD is the technique used to data level parallelism. The data is distributed across different parallel computing units. These computing units are called processing units (PU) as shown in Fig. 5.2.9 Each PU is some functional unit, that performs some task on different pieces of distributed data. The single execution thread controls operations on all the pieces of the data. The SIMD handles data manipulations.

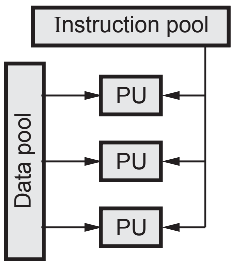


Fig. 5.2.9 SIMD

Advantages

- The same value is being added (or subtracted) to a large number of data points (which is most common operation in many multimedia applications. SIMD processor will do this work with a single instruction very efficiently on complete data.
- SIMD include mainly those instructions that can be applied to all of the data in one operation.

Disadvantages

- All algorithms cannot take advantage of SIMD.
- Most of the compilers do not generate SIMD instructions.
- Programming with SIMD instructions involves various low level challenges because of architecture, data, number representation mismatch.

5.2.8.3 Superscalar Processing

The superscalar processing employs instruction-level parallelism. Therefore multiple instructions are executed in one cycle. The power PC and pentium processors use superscalar architectures. Analog Devices Tiger SHARC processor also have superscalar architecture. Fig. 5.2.10 shows superscalar architecture.

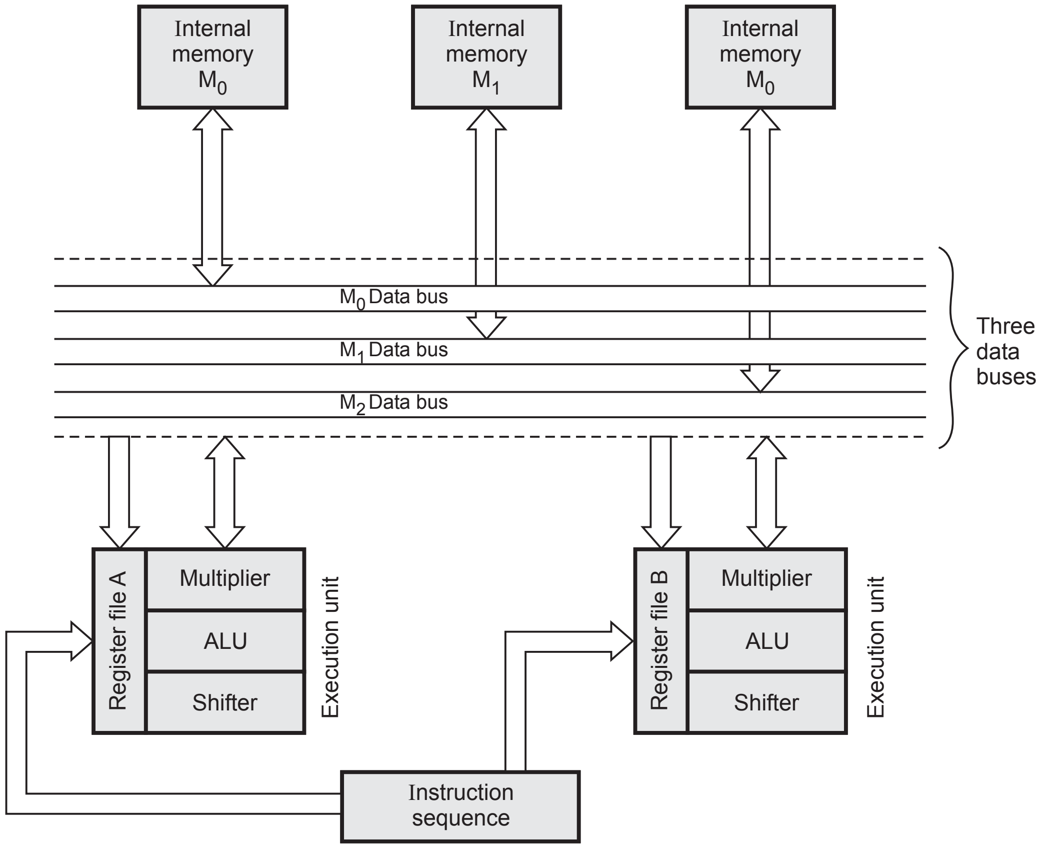


Fig. 5.2.10 Superscalar architecture

In this figure observe that there are multiple memories with separate data buses. Multiple instructions can be executed simultaneously with separate set of execution units consisting of ALU, multiplier and shifters. The execution units take inputs from register file and returns results in the same.

5.2.9 Special Instructions

DSPs perform variety of special operations and hence they have special instructions to serve these operations. These special instructions can be classified as,

1. Instructions that support basic DSP operations.
2. Instructions which reduce overhead in loops.
3. Application specific instructions.

Normally following type of DSP operations are performed with the help of special instructions :

1. Shift-multiply-accumulate operations in filtering and correlation.
2. Repeat operations in loop and nested loops.
3. Adaptive filtering and operations in LMS algorithms.
4. Bit-reversed addressing in FFT algorithms.
5. Code-book search in speech coding.
6. Surround sound operations in digital audio.
7. Viterbi decoding operations.

There are large number of similar other operations those can be performed with the help of special instructions. Most of the above operations are completely implemented in single cycle.

Advantages of Special Instructions

1. Special instructions provide compact code which occupies less memory.
2. Execution speed is increased that provides better computational efficiency.

5.2.10 Replication

Replication means using multiple arithmetic units. Such arithmetic units are ALU, multiplier or memories. There is only one CPU. The arithmetic units operate inparallel.

5.2.11 On-chip Memory/Cache

On-chip memory is provided in almost every DSP processor to enhance the operating speed. The program code is loaded from external memories to on-chip memories at the time of initialization. This increases the speed of operations. Some times the repeated sections of the program are loaded in on-chip memories. This reduces the time of fetching instructions and increases the speed of execution.

Review Questions

1. *Explain the following architectures with the help of block diagrams :*
   i) *Von-Neumann architecture* **AU : Dec.-10, 13, Marks 8**
   ii) *Harvard architecture* **AU : Dec.-12, Marks 8**
   iii) *Modified Harvard architecture.* **AU : Dec.-13, Marks 8**
2. *Draw the block diagram of hardware multiplier-accumulator unit and explain the same.* **AU : Dec.-10, May-14, Marks 8**
3. *Write a short notes on the following*
   i) *Multiported memory* ii) *VLIW architecture*
   iii) *SIMD* iv) *Superscalar processing*
   v) *Special instructions* vi) *On-chip memory/cache.*
4. *State advantages of VLIW architecture.* **AU : Dec.-12, Marks 8**
5. *Explain the functional modes present in DSP processor.* **AU : May-11, 12, Marks 8**
6. *Explain about pipelining in DSP.* **AU : May-12, 14, Marks 8**

5.3 Addressing Formats **AU : May- 11, 12, 15, 16, 17, Dec.-11, 12, 13, 15, 16**

Conventional microprocessors have addressing modes such as direct, indirect, immediate etc. The DSP processors have additional addressing modes because of which execution is fast. These addressing modes are discussed next.

5.3.1 Short Immediate Addressing

The operand is specified using a short constant. This short constant becomes the part of a single word instruction. In TMS320C5X series of DSP processors 8-bit operand can be specified as one of the operand in single word instructions such as add, subtract, AND, OR, XOR etc.

5.3.2 Short Direct Addressing

The lower order address of the operand is specified in the single word instruction. In TMS320CXX DSP processors, lower 7 bits of the address are specified as the part of the instruction. Higher 9 bits of the address are stored in the data page pointer. Each such data page consists of 128 words.

5.3.3 Memory-Mapped Addressing

The CPU and I/O registors are accessed as memory location. These registers are mapped in the starting page or final page of the memory space. In TMS320C5X, page 0 corresponds to CPU and I/O registers.

5.3.4 Indirect Addressing

The addresses of operands are stored in the indirect address registers. In TMS320CXX processors such registers are called auxiliary registers (ARs). Any of these ARs can be updated when operands fetched by these registers are being executed. The ARs are incremented or decremented automatically by the value specified in offset register. For TMS320CXX processors the offset register is called INDEX register.

5.3.5 Bit Reversed Addressing Mode

For the computation of FFT, the input data is required in bit reversed format. There is no need to actually reshuffle the data in bit reversed sequence. The serially arranged data in the memory or buffer can be given to the processor in bit reversed mode with the help of bit reversed addressing mode. With this addressing mode an address is incremented/decremented by the number represented in bit reversed form.

5.3.6 Circular Addressing

With this mode, the data stored in the memory can be read/written in circular fashion. This increases the utility of the memory. The memory is organized as a circular buffer. The beginning and ending addresses of the circular buffer are continuously monitored. If the address exceeds ending address of the memory, then it is set at the beginning address of the memory. This is nothing but circular addressing.

Review Questions

1. *Write a short note on memory mapped register addressing.* **AU : Dec.-12, Marks 4**
2. *Explain circular addressing mode.* **AU : Dec.-12, Marks 4**
3. *Explain addressing formats of DSP processors.*
   AU : Dec.-11, 15, Marks 16, May-11, 12, 13, 16, Marks 8, May-15, Marks 10 Dec.-16, Marks 3, May-17, Marks 13
4. *Write the architectural details of DSP processor.* **AU : Dec.-11, Marks 16**

5.4 On-Chip Peripherals

The DSPs have many peripherals on the chip to support its operation. These on-chip peripherals reduce the DSP system around P-DSP.

5.4.1 On-Chip Timer

The timer generate single pulse or periodic train of pulses. The period of the single pulse or frequency of the pulse train can be programmed. The on-chip timer can be used for

i) Generation of periodic interrupts to P-DSPs.

ii) Generation of sampling clocks for A/D converters.

iii) Timing signals

5.4.2 Serial Port

The serial ports have input and output buffers. These ports also have serial to parallel and parallel to serial converters. The serial ports can operate in asynchronous mode or in synchronous mode. The serial port allows following operations :

i) Communication between P-DSP and external peripherals such as A/D converter, D/A converter or RS 232 device.

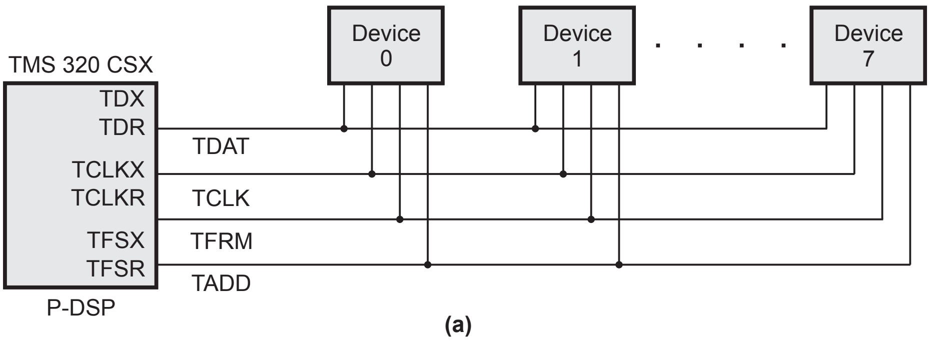
ii) Parallel writes from P-DSP and serial transmission. Receives serially from external peripherals and gives parallel data to P-DSPs.

iii) Generate interupts when serial port output buffer is empty or when input buffer is full.

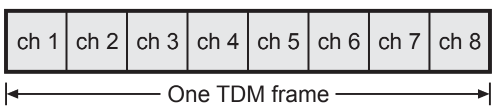
iv) Allows synchronous and asynchronous transmission.

5.4.3 TDM Serial Port

The TDM serial port allows the P-DSP and peripherals to communicate using time division multiplexing. Fig. 5.4.1 (a) shows how eight peripherals can be interfaced to P-DSP.



(a)



(b)

Fig. 5.4.1 a) Commutation through TDM serial port
b) One TDM frame with eight slots

As shown in Fig. 5.4.1 (a), the TDM serial port uses following four lines for serial communication.

TDAT : Used for data transmission into TDM channel by authorized device and used for data reception by authorized device.

TCLK : Bit clock used by transmiting and receiving devices.

TFRM : Frame sync signal is transmitted on this line.

TADD : Address of the serial device that is authorized to output data in a particular time slot.

The TFRM signal indicates the begining of a TDM frame. The TDM frame allots the TDM channel to eight-devices as shown in Fig. 5.4.1. (b).

5.4.4 Parallel Port

The parallel port allows faster data transmission compared to serial port. The parallel port includes data lines as well as additional lines for strobing and handshaking. Some times data bus itself is used for parallel port. The parallel port is then addressed by using I/O instructions. The parallel port is asigned a fixed address space.

5.4.5 Bit I/O Ports

These I/O ports have single bit lines. These bit I/O ports can be individually operated. These I/O ports do not have handshaking signals. Bit I/O ports are used for,

i) Control purposes as well as data transfer.

ii) Conditional branching or calls.

5.4.6 Host Port

The host port is a parallel port that is 8 or 16-bits wide. All P-DSPs normally have host port. It is used for following purposes :

i) P-DSPs communicate with host processors such as microprocessor or PC through host port.

ii) The host processor generates interrupts to P-DSP and load data on reset through host port.

iii) Host port is also used for data communication with host processor.

5.4.7 Comm Ports

Comm ports are parallel ports. P-DSPs from Texas Instruments as well as Analog Devices both have comm ports which vary from 6 to 8. Normally each comm port have 8 bits. comm ports are used for communication between P-DSPs when they are operating in multiprocessor system. The 23 bit wide data can be split into four 8 bit words. Then the data is exchanged amog P-DSPs over four different comm ports.

5.4.8 A/D and D/A Converter

The on-chip A/D and D/A converters are useful for P-DSPs which are used for voice applications such as mobiles and answering machines.

5.5 Introduction to Commercial Digital Signal Processors

AU : May-12, 14, 15, 16, 17, Dec.-10, 12, 15, 16

The general purpose DSP are the high speed microprocessors with specific architectural modifications and instructions to suit DSP operations. The DSP processors employ parallelism, Harvard architecture pipelining and dedicated hardware to improve the speed of operation. The general purpose DSPs have on-chip memories, peripherals, SIMD, VLIW and superscalar architectures to improve the performance further. The DSP processors are of two types : fixed point and floating point DSP processors.

5.5.1 TMS 320C5x Processors

Fig. 5.5.1 shows the functional block diagram or architecture of TMS 320C5x family of DSP processors.

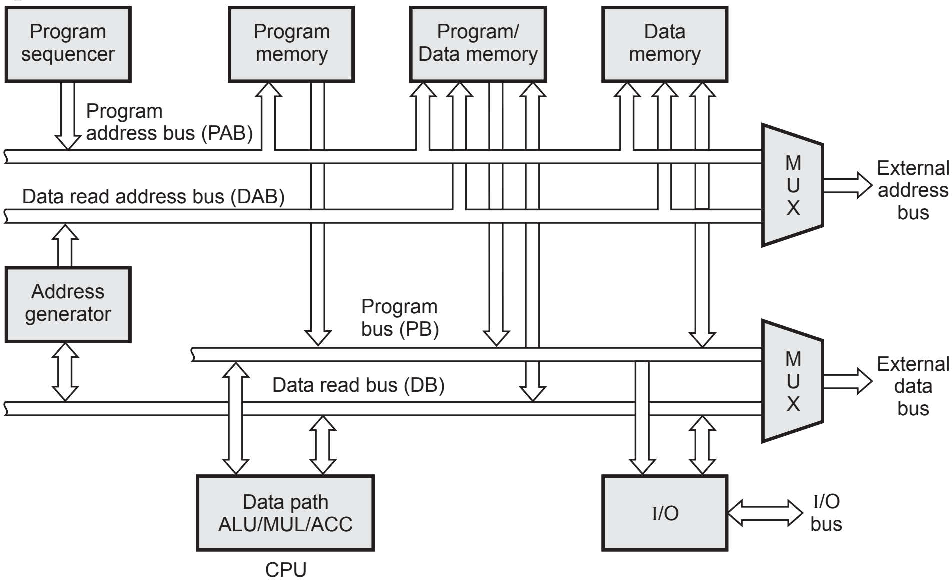


Fig. 5.5.1 Architecture of TMS 320C5x processors

A. Bus Structure

Observe that there are separate program and data buses. The program address bus (PAB) addresses to the program memory. The data read address bus (DAB) addresses to the program as well as data memory. The program bus (PB) carries the instructions from program memory. These instructions are given further to CPU for execution. The data read bus (DB) carries the data required for execution. It gets the data from the I/O ports, CPU or data memory. Because of the four types of buses for program and data, high degree of parallelism is obtained. Therefore multiple operations are performed in single instruction cycle.

B. Data Path Units

1. Central Processing Unit

The central processing unit consists of 32-bit ALU/accumulator, scaling shifter, parallel logic unit (PLU), parallel multiplier and auxiliary register arithmetic unit (ARAU).

2. 32-bit ALU/accumulator

The 32-bit ALU and accumulator performs arithmetic and logical functions. Almost all these functions are executed in single cycle. ALU can also perform boolean operations. The ALU takes its operands from accumulator, shifter and multiplier.

3. Scaling shifter

The scaling shifter has a 16-bit input connected to the data bus and 32-bit output connected to the ALU. The scaling shifter produces a left shift of 0 to 16-bits on the input data. The other shifters perform numerical scaling, bit extraction, extended precision arithmetic and overflow prevention.

4. Parallel logic unit (PLU)

The parallel logic unit (PLU) is the second logic unit. It executes logic operations on the data without affecting the contents of ACC. PLU provides bit manipulation which can be used to set, clear, test or toggle bits in data memory control or status registers.

5. 16 $\times$ 16 bit parallel multiplier

This is 16 $\times$ 16 bit hardware multiplier is capable of multiplying signed or unsigned 32-bit product in a single machine cycle.

The two number being multiplied are treated as 2's complement number and the result is also a 32-bit 2's complement number.

6. Auxiliary registers and auxiliary register arithmetic unit (ARAU)

There is a register file of eight auxiliary registers. These registers are used for temporary data storage. The auxiliary register file (AR0-AR7) is connected to the auxiliary register arithmetic unit (ARAU). The contents of the auxiliary registers can be stored in data memory or used as inputs to central arithmetic logic unit (CALU). The ARAU helps to speed up the operation of CALU.

7. Program Controller

The program controller decodes the instructions, manages the CPU pipeline, stores the status of CPU operations and decodes conditional operations. The program controller consists of program sequencer, address generator, program counter, instruction register, status and control registers and hardware stack.

C. I/O Ports

There are total 64 I/O ports. Out of these, there are 16 ports memory mapped in data memory space.

5.5.2 TMS 320C54X Processors

- This processor series contain all the features of the basic architecture. It has number of additional features for improved speed and performance.
- This series of processors have advanced modified Harvard architecture.
- The TMS 320C54X is upward compatible to earlier fixed point processors such as 'C2X, 'C2XX and 'C5X processors.
- It is 16-bit fixed-point DSP processor family.

Advantages of 'C54X Devices

1. Enhanced Harvard architecture, which include one program bus, three data buses and four address buses.
2. CPU has high degree of parallelism and application specific hardware logic.
3. It has highly specialized instruction set for faster algorithms.
4. Modular architecture design for fast development of spinoff devices.
5. It has increased performance and low power consumption.

5.5.2.1 Features of 'C54X

A. CPU

1. One program bus, three data buses and four address buses.
2. 40-bit ALU, including 40-bit barrel shifter and two independent 40-bit accumulators.
3. 17-bit × 17-bit parallel multiplier coupled to 40-bit dedicated adder for nonpipelined single cycle multiply / accumulate (MAC) operation.

4. Compare, select, store unit (CSSU) for the add / compare selection of viterbi operator.
5. Exponent encoder to compute the exponent of 40-bit accumulator value in single cycle.
6. Two address generators, including eight auxiliary registers and two auxiliary register arithmetic units.
7. Multiple-CPU / core architecture on some devices.

B. Memory

1. 192 K words × 16-bit addressable memory space.
2. Extended program memory in some devices.

C. Instruction Set

1. Single instruction repeat and block repeat operations.
2. Block memory move operations.
3. 32-bit long operand instructions.
4. Instructions with 2 or 3 operand simultaneous reads.
5. Parallel load and parallel store instructions.
6. Conditional store instructions.
7. Fast return from interrupt.

D. On-chip peripherals

1. Software programmable wait state generator.
2. Programmable bank switching logic.
3. On-chip PLL generator with internal generator with internal oscillator.
4. External bus-off control to disable the external data bus, address bus, and control signals.
5. Programmable timer.
6. Bus hold feature for data bus.

5.5.2.2 Architecture of 'C54X

Fig. 5.5.2 shows an internal architecture of TMS 320CS4X DSP processor.

See Fig. 5.5.2 on next page.

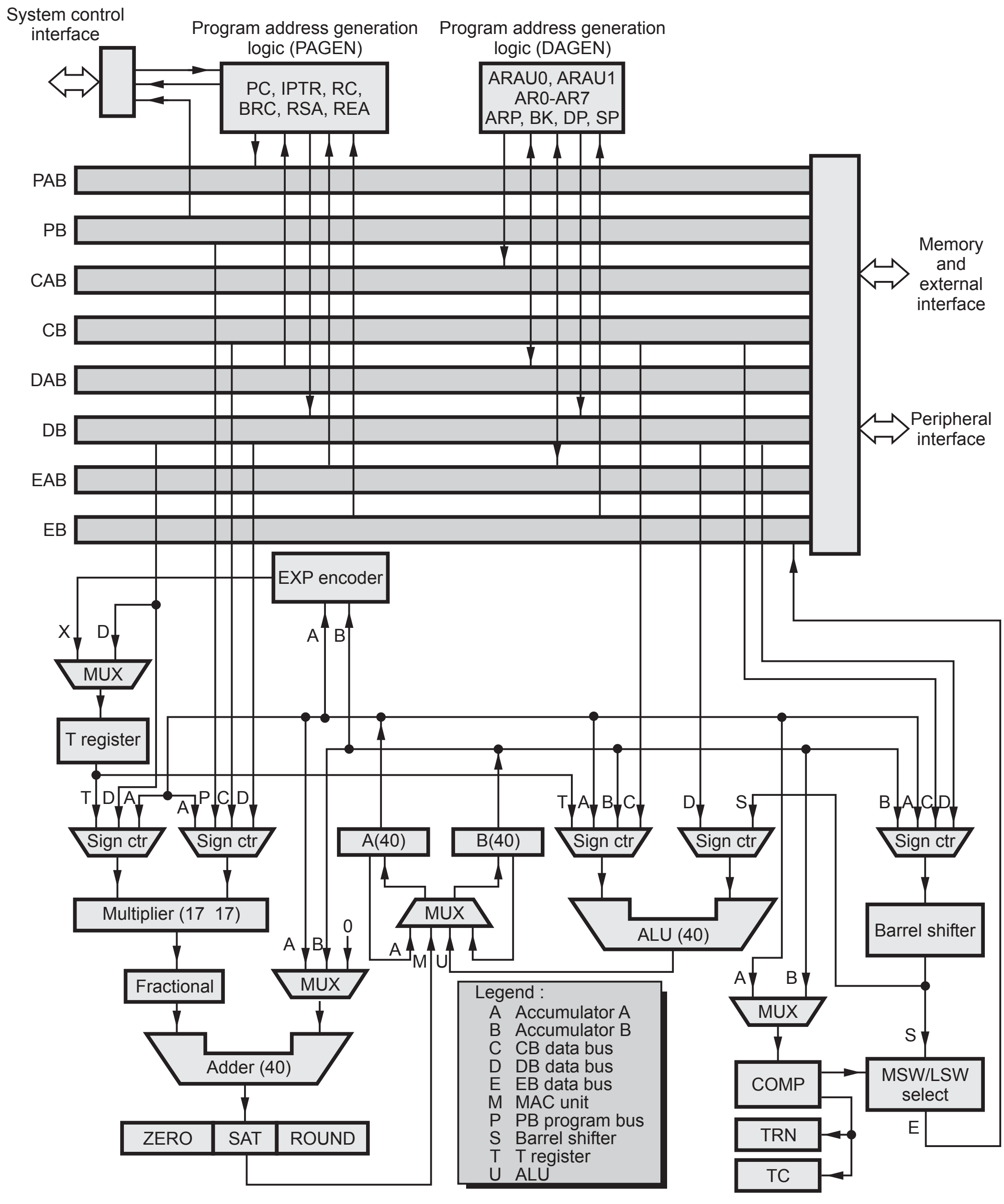


Fig. 5.5.2 Architecture of TMS 320C54X DSP processor

Bus Architecture

- There are eight major 16-bit buses (four program / data bus and four address buses).
- Program bus (PB) carries instruction code and immediate operands from program memory.
- Three address buses (CB, DB and EB) interconnect CPU, data address generation logic, program address generation logic, on-chip peripherals and data memory.
- Four address buses (PAB, CAB, DAB and EAB) carry the addresses needed for instruction execution.

Internal Memory Organization

- There are three individually selectable spaces : program, data and I/O space.
- There are 26 CPU registers plus peripheral registers that are mapped in data memory space.
- The 'C54X devices can contain RAM as well as ROM.
- On-chip ROM is part of program memory space, and in some cases part of data memory space.
- There can be DARAM, SARAM, Two way shared RAM on the chip.
- On-chip memory can be protected from being manipulated externally.

CPU

The CPU contains :

- **40 - bit ALU :** Performs 2's complement arithmetic with 40-bit ALU and two 40 bit accumulator. The ALU can also perform boolean operations.
- **Accumulators :** There are two accumulators A and B. They store the output from the ALU or the multiplier / adder block.
- **Barrel shifter :** It's 40-bit input can come from accumulators or data memory. It's 40-bit output is connected to ALU or data memory. It can produce left shift of 0 to 31 bits and a right shift of 0 to 16 bits.
- **Multiplier / Adder Unit :** It performs 17×17 - bit 2's complement multiplication with a 40-bit addition in a single instruction cycle.
- **Compare, Select and Store Unit (CSSU) :** It performs comparisons between the accumulators high and low word.

Data Addressing

The 'C54X DSP processors have seven basic data addressing modes :

1. Immediate addressing

2. Absolute addressing
3. Accumulator addressing
4. Direct addressing
5. Indirect addressing
6. Memory-mapped register addressing
7. Stack addressing

Program Memory Addressing

- The program memory is addressed with program counter (PC). The PC is used to fetch individual instructions.
- PC is loaded by program address generator (PAGEN). PAGEN increments PC.

Pipeline Operation

- The 'C54X DSP has six levels : prefetch, fetch, decode, access, read and execute.
- One to six instructions can be active in a single cycle.

On-Chip Peripherals

- **General purpose I/O pins :** These pins can be read or written through software control. These pins are $\overline{\text{BIO}}$ and XF.
- **Software programmable Wait - state Generator :** It extends external bus cycles upto seven machine cycles to interface with slower off-chip memory and I/O devices.
- **Programmable Bank-Switching Logic :** It can automatically insert one cycle when an access crosses memory bank boundaries inside program memory or data memory space.
- **Hardware timer :** It provides 16-bit timing circuit with 4-bit prescaler. The timer can be stopped, restarted, reset or disabled by specific status bits.
- **Clock generator :** The clock can be generated by two options (a) internal oscillator or (b) PLL circuit.
- **DMA controller :** It transfers data between points in the memory map without intervention by the CPU. The data can be moved to and from program data memory, on-chip peripherals or external memory devices.
- **Host Port Interface (HPI) :** It is parallel port. It provides an interface to a host processor. The information is exchanged between 'C54X and host processor through on-chip memory.
- **Serial ports :** There are four types of serial ports
  (i) Synchronous, (ii) Buffered,

(iii) Multi-channel buffered and (iv) Time division multiplexed.

Review Questions

1. *State the difference between fixed point and floating point DSP processors.*
2. *Explain the architecture of TMS 320C5X processor with the help of block diagram.* **AU : Dec.-10, Marks 10, May-12, Marks 8**
3. *Explain auxiliary registers in TMS320C5X DSP processors.* **AU : Dec.-12, Marks 6**
4. *Draw and explain the architecture of TMS320C54X.* **AU : May-14, Marks 16**
5. *Draw the architecture of a DSP processor for implementing a DSP algorithm. Explain its features.* **AU : May-15, Marks 16**
6. *Write a note on commerical DSP processor.* **AU : May-15, Marks 6**
7. *With Suitable block diagram explain in detail about TMS320C54 DSP Processor and of its memory architecture.* **AU : Dec.-15, Marks 16**
8. *Explain the datapath architecture and the bus structure in a DSP processor with suitable diagram.* **AU : May-16, Marks 8**
9. *With neat figure explain the architecture of any one type of a DS processor.* **AU : Dec.-16, Marks 8**
10. *Discuss the features and architecture of TMS 320C50 processor.* **AU : May-17, Marks 13**

5.6 Selecting Digital Signal Processors **AU : May-15**

The DSP processors are selected for particular application depending upon four factors :

1. Architectural features

The architectural features should match to the specific application. Following features are to be considered :

i) Size of on-chip memory.

ii) DMA capability and multiprocessor support.

iii) Special instructions to upport DSP operations.

iv) I/O capabilities.

2. Execution speed

The speed of execution is critical especially in real time applications. Following features are examined for execution speed :

i) Clock speed of the processor.

ii) Million Instructions Per Second (MIPS).

iii) Speed of benchmark algorithms such as FFT, FIR and IIR filters.

3. Type of arithmetic

The arithmetic types are :

i) Fixed point and ii) Floating point

The arithmetic decides cost of the processor. It is also selected depending upon volume of production.

4. Wordlength

The wordlength has direct relation with

i) Noise and errors.

ii) Precision directly related to wordlength.

iii) Signal quality.

High precision is required in audio applications. The telecommunication applications can have low precision. Normally 16-bit (telecommunications) and 24-bit (audio) wordlengths are most common. Cost is also one of the factors in deciding precision.

5.6.1 Comparison between DSP Processors and General Purpose Processors

We have seen that the architecture of DSP processors is developed so that array processing operations are implemented fast. General purpose processors have many features but their architecture do not support fast processing of arrays. Table 5.6.1 presents the comparison between DSP processors and general purpose processors.

Sr. No.	Parameter	DSP Processors	General Purpose Processors
1.	Instruction cycle.	Instructions are executed in single cycle of the clock i.e. true instruction cycle.	Multiple clock cycles are required for execution of one instructions.
2.	Instruction execution.	Parallel execution is possible.	Execution of instructions is always sequential.
3.	Operand fetch from memory.	Multiple operands are fetched simultaneously.	Operands are fetched sequentially.
4.	Memories.	Separate program and data memories.	Normally no such separate memories are present.
5.	On-chip/off-chip memories.	Program and data memories are present on-chip and extendable off-chip.	Normally on-chip cache memory is present. Main memory is off-chip.

6.	Program flow control.	Program sequencer and instruction register takes care of program flow.	Program counter maintains the flow of execution.
7.	Queuing/pipelining.	Queuing is implicite through instruction register and instruction cache.	Queue is performed explicitely by queue registers for pipelining of instructions.
8.	Address generation.	Addresses are generated combinely by DAGs and program sequencer.	Program counter is incremented sequentially to generate addresses.
9.	Address/data bus multiplexing.	Address and data buses are not multiplexed. They are separate on-chip as well as off-chip.	Address/data buses can be separate on the chip but usually multiplexed off-chip.
10.	Computational units.	Three separate computational units : ALU, MAC and shifter.	ALU is the main computational unit.
11.	On-chip address and data buses.	Separate address and data buses for program memory and data memories and result bus. i.e. PMA, DMA, PMD, DMD and R-bus.	Address and data buses are the two buses on the chip.
12.	Addressing modes.	Direct and indirect addressing is supported.	Direct, indirect, register, register indirect, immediate, etc addressing modes are supported.
13.	Suitable for	Array processing operations	General purpose processing.

Table 5.6.1 Comparison between DSP processors and general purpose processors

In addition to above points, DSP processors have on-chip DMA control logic, host interface ports, timers, serial ports, boot address generators, program memory and data memory bus multiplexers etc. Whereas the general purpose processors have bus arbitration logic, interrupt controllers etc on the chip. Both the DSP processors as well as general purpose processors support multiprocessing and parallel processing environment. But this is not true for all DSP processors. Peripherals are interfaced in memory mapped I/O as well as I/O mapped I/O in general purpose processors. The DSP processors communicate to their peripherals by memory mapped I/O in data memory. In general purpose processors monitor programs (i.e. boot programs) are normally present in the external EPROM. The processor executes these programs. Whereas in DSP processors, boot programs are loaded from external boot memory into internal program memory to increase the speed of operation.

Review Questions

1. *How the DSP processors are selected ?*
2. *Compare general purpose processors and DSP processors.*

5.7 Applications of DSP

AU : May-16, Dec.-16

5.7.1 DSP based System For Audio Recording

Fig. 5.7.1 shows the block diagram of DSP based audio recording system. Compact discs process digital audio data. CDs are used on the large scale for audio storage and circulation. The CD audito is portable onl all digital machines.

Recording on CD

Fig. 5.7.1 shows the block diagram of audio signal processing and recording in CD system.

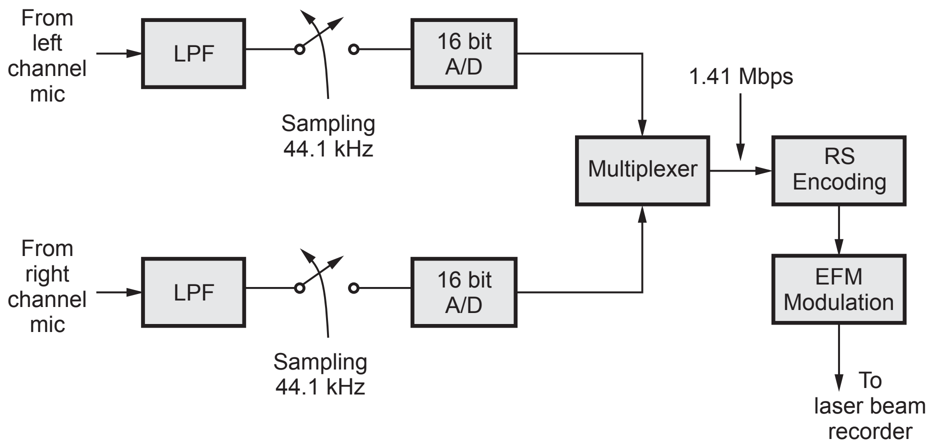


Fig. 5.7.1 Audio processing and CD recording system

The microphone signals collected from left and right channels are passed through bandlimiting low pass filters. These are the analog filters used to bandlimit the analog audio signals before sampling. This bandlimiting is used to avoid aliasing. The bandlimited analog audio signals are then sampled at the rate of 44.1 kHz. The sampled signal is converted to analog form by 16 bit A/D converter. The multiplexer combines the left and right channel digitized audio. Thus the output bit rate of multiplexer will be $44.1 \times 10^3 \times 2 \times 16$ = 1.41 Mbps. For error detection and correction RS coding is employed on this multiplexed digital data. Additional bits are added for control and display of information. The eight to fourteen modulation (EFM) translates each byte in the data stream to 14 bits code. This is particularly done to record data on the CD. After due synchronization, the digital data is given to the laser assembly for recording on the CD.

The digital data is recorded on the CD in form of a spiral track of bits. Each bit recorded on the CD occupies an area of 1 μm^2.

Reproduction from the CD

Fig. 5.7.2 shows the block diagram of reproduction and audio processing from CD.

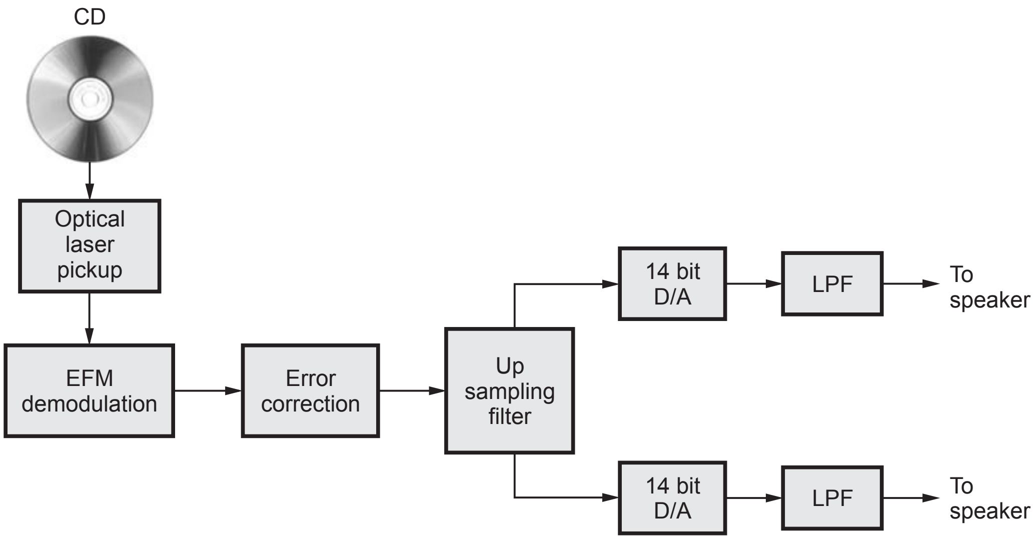


Fig. 5.7.2 Audio processing and reproduction form CD

The spiral tracks on the CD are read by laser optical pickup assembly. The disc rotates at 1.2 m/s speed. The equivalent electric signal from laser pick up is given to eight-to-fourteen demodulator. This removes the additional bits which are included by EFM modulator at the time of CD recording. The digital data is then given for error detection and correction. Errors are introduced due to manufacturing defects, damage, dust or fingure prints on the disc. If errors are not correctable, then they are replaced with zero sample values (i.e. concealed) or interpolated with nearby sample values. The digital data is then given to upsampling digital filter after time base correction. The upsampling digital filter operates at the rate of 4×44.1 kHz, i.e. 176.4 kHz sampling rate. This increased sampling rate makes output of D/A converter smoother. Hence analog filtering requirement is reduced. This upsampling provides 16-bit signal to noise ratio for 14-bit D/A converter. The signal is then passed through 14-bit D/A converter and low pass filtered. Note that the upsampling filter also isolates left and right audio channels. Finally the signal is given to loud speakers.

Features of CD audio

Frequency response	:	20 Hz to 20 kHz (+ 0.5 to – 1 dB)
Dynamic range	:	> 90 dB
Signal to noise ratio	:	> 90 dB
Separation between two channels	:	> 90 dB
Harmonic distortion	:	0.004 %

5.7.2 Radar Signal Processing using DSP

5.7.2.1 Radar Principles

- Fig. 5.7.3 shows the basic principle of radar operation. The radar consists of common or separate transmitting / receiving antenna, transmitter, receiver, display and control unit.
- The transmitting antenna sends RF signals across the target. The signal is intercepted and reflected back by the target. The receiving antenna collects this reflected signal.
- The reflected signal is the echo of transmitted signal. It is very weak signal. The receiver analyses the reflected signal and locates the distance and speed of the target.

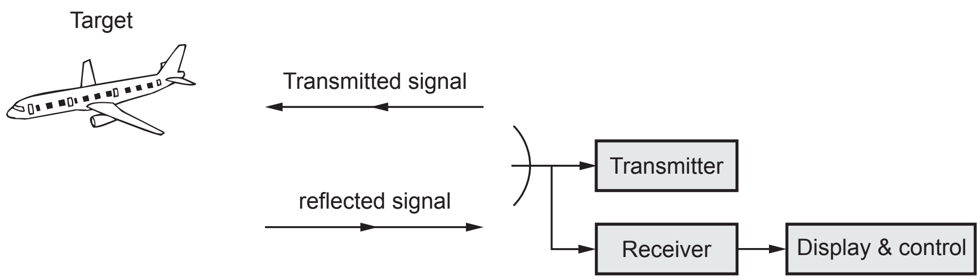


Fig. 5.7.3 Radar principle

5.7.2.2 Types of Radars

1. **Pulsed radar :** It sends pulses at periodic intervels towards target. The reflected train of pulses is analyzed at the receiver.
2. **CW doppler radar :** It sends continuous wave RF signal towards the target. The speed and location of the target is then established with the help of doppler effect.

3. **Moving Target Indication (MTI) radar :** It uses the doppler effect to minimize the clutter effects to locate the target that is moving. MTI senses the target movement by comparing the phase shift of the received signal with respect to transmitted signal.

5.7.2.3 Radar System and Parameter Considerations

Various parameters are to be considered while designing the DSP based analysis of radar signals.

1. **Pulse repetition frequency :** The number of radar pulses transmitted per second is called Pulse Repetition Frequency (PRF)
2. **Pulse Width (PW) :** The duration of transmitted pulse is called Pulse Width (PW).
3. **Maximum range of radar :** It is the distance beyond which the target cannot be detected. It depends upon transmitted signal power, tramitting antenna gain, effective area of receiving antenna, cross section of the target etc.
4. **Receiver noise and SNR :** The SNR puts a limit on maximum range of the radar. The SNR depends upon climate and location of the radar.

5.7.2.4 DSP based Radar System

- Fig. 5.7.4 shows the radar signal processing using DSP. It uses common transmitting and receiving antenna. The reflected signal from the target is obtained from antenna by receiver.

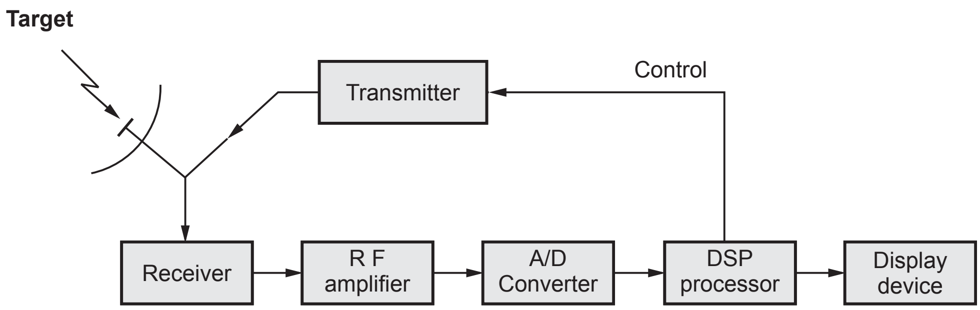


Fig. 5.7.4 DSP based radar system

- The recieved signal is very weak. It is amplified by RF amplifier with enough bandwidth. The received signal is then converted to digital form by A/D converter.
- The digital signal is then given to DSP processor or DSP based system. The DSP processor performs various DSP operations such as FFT, cross-correlation, digital filtering to establish the location and speed of the target.
- The DSP controls transmitter to send pulses towards target as well as sends necessary information to the display device.

Advantages

i) Efficient noise suppression

ii) Accurate tracking of the target.

iii) Efficient calculations.

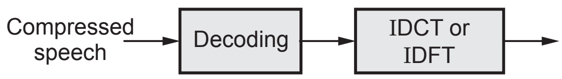
5.7.3 Speech Coding

- **Principle :** Speech coding is used for compression/decompression. It reduces the transmission data rate of the speech so that more channels can be accomodated in the given bandwidth.
- Speech coding can be done in time domain or frequency domain. Time domain techniques include pulse code modulation and delta modulation. Frequency domain coding include transform coding and subband coding.

5.7.3.1 Transform Coding

- Fig. 5.7.5 shows the block diagram of transform coding of speech. The speech signal is first converted to frequency domain by DCT or DFT. Then quantization is performed. It removes insignificant frequnecy components.

(a) Speech coder



(b) Speech decoder

Fig. 5.7.5 Transform coding

- The encoder encodes only significant frequency components that forms compressed speech signal.
- The compressed speech is given to decoder. The decoder obtains frequency components back form encoded data. Such decoded spectrum is inverted by IDCT or IDFT to get time domain speech signal.
- Note that some data is lost at the time of quantization. Such data is redundant and does not affect speech quality.
- **Application :** MPEG audio coding uses transform coding.

5.7.3.2 Subband Coding of Audio Signals

- **Necessity of subband coding** : Speech energy is not uniformly distributed over the complete frequency range. More energy is contained in lower frequency bands and less energy is contained in higher frequency bands. Hence lower frequency bands are coded with more number of bits and high frequency bands are coded with less number of bits.
- **Definition of subband coding** : The speech signal is divided into several frequency bands. These frequency band do not overlap each other. They are called subbands. Each subband is encoded separately.
- **Block diagram of subband coding** :

  Fig. 5.7.6 shows the block diagram of subband coding

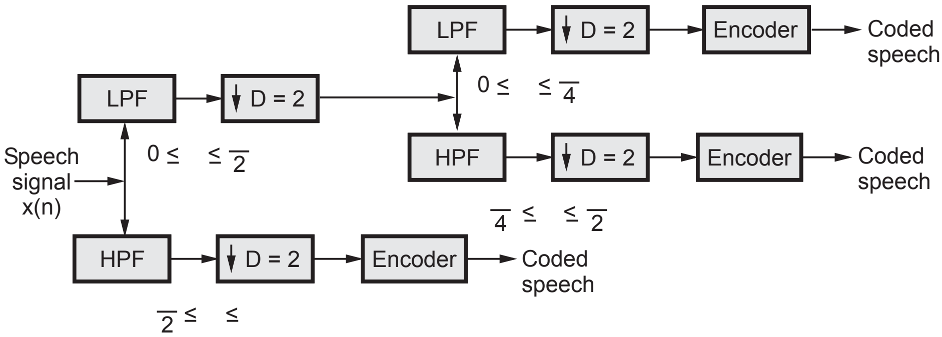


Fig. 5.7.6 Block diagram of subband coding

- The speech signal containing the frequencies from 0 to π radians is passed through a pair of lowpass filter and high pass filter.
- These lowpass and high pass filters split the signal into two non overlapping frequency bands. These bands are $0 \le \omega \le \frac{\pi}{2}$ and $\frac{\pi}{2} \le \omega \le \pi$. Fig. 5.7.7 shows these bands.

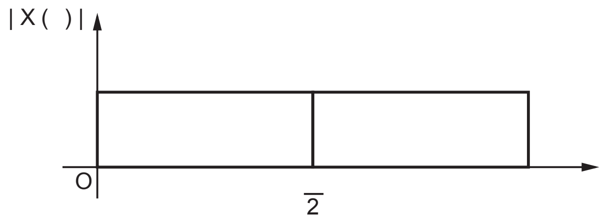


Fig. 5.7.7 Frequency bands

- The lower frequency band is further split into two frequency bands i.e. $0 \le \omega \le \frac{\pi}{4}$ and $\frac{\pi}{4} \le \omega \le \frac{\pi}{2}$. This is shown in Fig. 5.7.8. Note that higher frequency band of $\frac{\pi}{2} \le \omega \le \pi$ is not split further. It is directly encoded since it contains less energy.

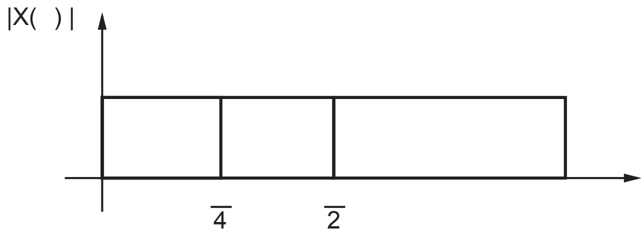


Fig. 5.7.8

- The subbands are decimated immediately after they are filtered. This is because, the frequency range is reduced after filtering. Hence the sampling frequency can also be reduced. In Fig. 5.7.6 observe that decimation is performated after enery stage of splitting. This requires less number of bits for encoding.
- **Filters used for subband coding**
- Fig. 5.7.9 shows the characteristic of low pass and high pass filters used for subband coding. It is the characteristic of Quadrature Mirror Filter (QMF). These are physically realizable filters. They eliminate the aliasing resulting due to decimation.

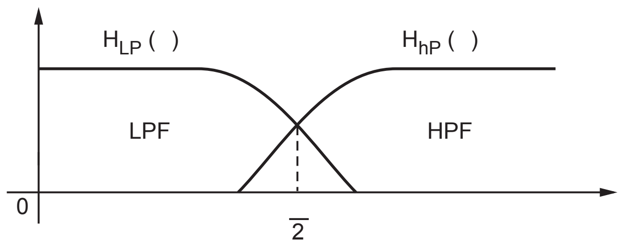


Fig. 5.7.9 QMF characteristic

Applications of Subband Coding

i) Coding and compression of speech signals.

ii) Data compression of image signals.

iii) It provides bandwidth compression, when the energy of the signal is unevenly spread over the spectrum.

Synthesis of Subband Coded Signal

Fig. 5.7.10 shows the block diagram of synthesis of subband coded signal.

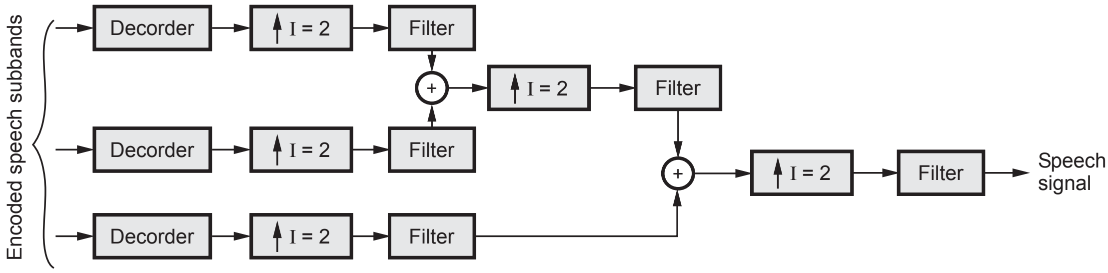


Fig. 5.7.10 Block diagram of synthesis of subband coded signal

Observe that the above diagram performs exactly the reverse function of that of Fig. 5.7.6.

Interpolation of two subband makes their sampling rates equal. Then the two signals are filtered and combined.

5.7.4 Digital Mobile Telephony

Fig. 5.7.11 shows the block diagram of third generation (3G) GSM mobile. The speaker and microphone are interfaced to DSP system through A/D and D/A converter. The GSM vocoder compresses voice signal at 13 kbps. It uses rectangular pulse excited linear predictive coding with long term prediction (RPE - LTP). The coding algorithm has been implemented on motorola DSP 56000 or Texas TMS 320C50 processors. DSP performs other operations such as multipath equalization, signal strength and quality measurements, voice messaging, error control coding, modulation and demodulation. The effects of multipath propagation are reduced by use of digital equalizer at the receiver. A known 26 bits long sequence is transmitted at regular intervals. At the receiver, the equalizer uses this framing sequence to adjust the coefficients of digital filter. This digital filter reduces the effects of multipath propagation.

The DSP algorithms are designed to modulate the carrier in GSM system. The demodulating algorithms are also implemented. Error detection and correction is obtained by channel interleaving/deinterleaving, burst formating algorithms in DSP. The data is encrypted/decrypted using ciphering/deciphering algorithms implemented in DSP.

The interface with display, keyboard and SIM card is done through layers 1, 2, 3 protocols. The layer 1 protocol is implemented in DSP. The signal level is also monitored with the help of DSP. This helps to adjust the power levels of base stations. The signals from surrounding base stations are also monitored. This helps to analyze co-channel interference.

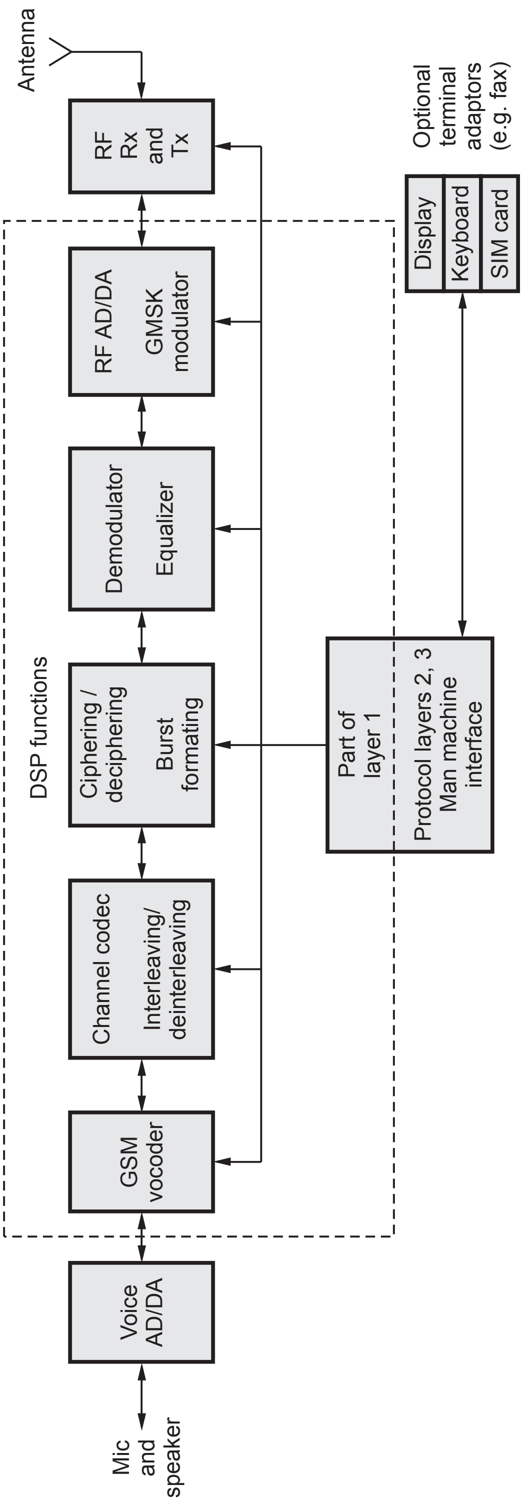


Fig. 5.7.11 Functional block diagram of a GSM phone employing DSP

5.7.5 Adaptive Telephone Echo Cancellation

In the telephone network, the subscribers are connected to telephone exchange by two wire circuit. The exchanges are connected by four wire circuits. The two wire circuit is bidirectional and carries signals in both the directions. The four wire circuit has separate paths for transmission and reception. The hybrid coil at the exchange provides the interface between two wire and four wire circuit as shown in Fig. 5.7.12 (a).

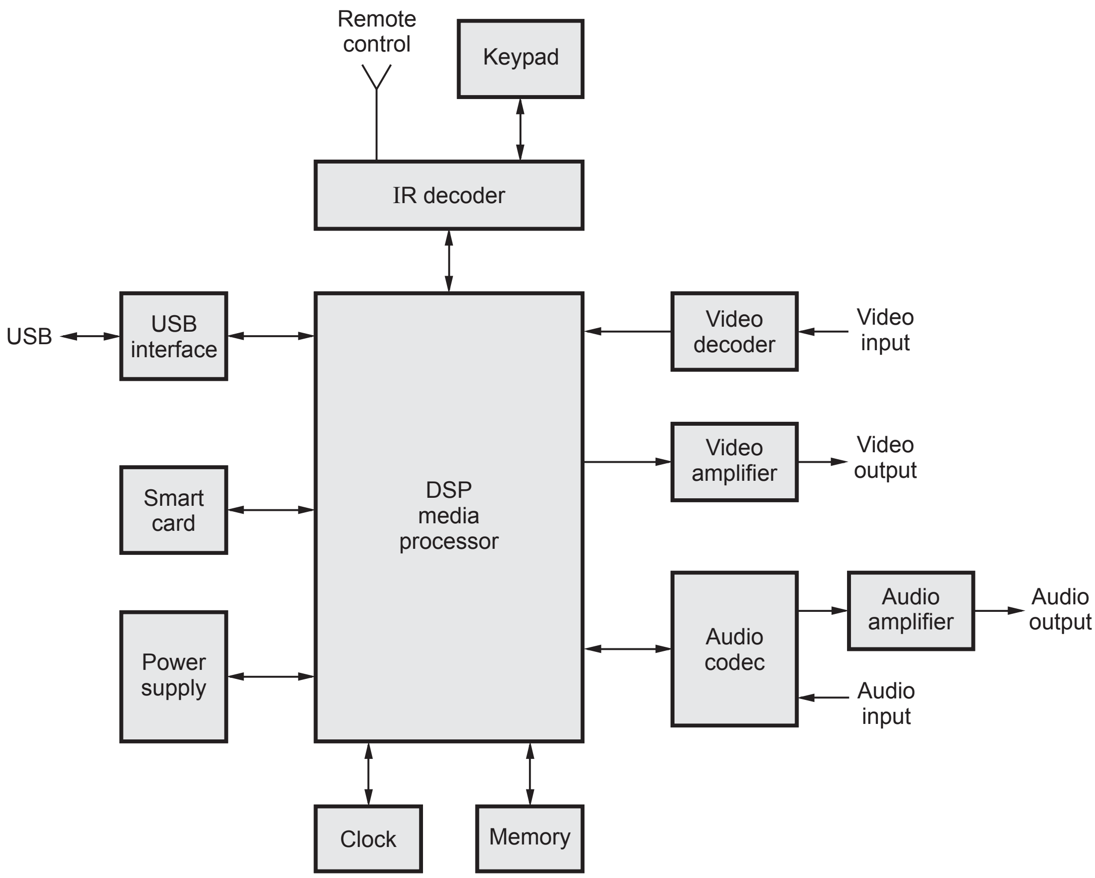


Fig. 5.7.12 Block diagram of DSP based set-top-box

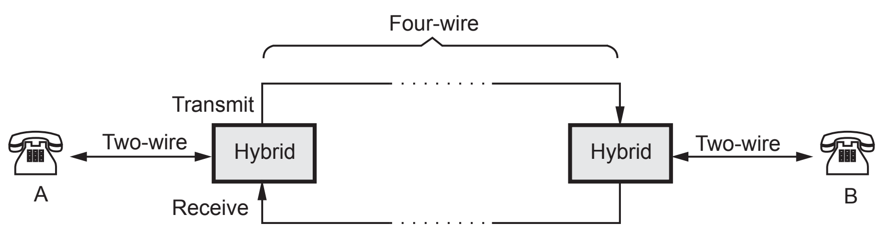


Fig. 5.7.12 (a) Two/Four wire interface in telephone network

The hybrid circuit provides impedance matching between two and four wire circuits. Because of this perfect impedance matching, there are no reflections (also called as echos) on the lines. Actually the hybrid coil is shared among several subscribers. The impedance matching is length dependent. The subscribers have different lengths of their two wire circuits. Hence the impedance matching between two wire circuit with four wire circuit at the hybrid coil is not perfect. Because of this impedance unbalance, reflections take place. These reflections are called echos. As the delay and amplitude of the echo increases it becomes annoying to the talker, hence it should be removed.

DSP techniques can be used for echo cancellation. Fig. 5.7.13 shows the echo cancellation scheme. This echo cancellation scheme uses adaptive filtering. The circuit generates replica of the echo using the signal in the receive path and subtracts it from the signal in the transmit path. The echo canceller in Fig. 5.7.13 is a digital adaptive filter. The coefficients of this filter are adjusted (adopted) such that echo signals are attenuated below 40 dB.

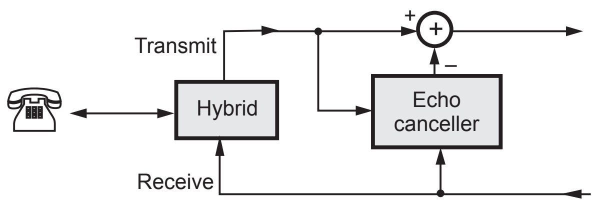


Fig. 5.7.13 Echo cancellation scheme

5.7.6 Fetal ECG Monitoring

- Fig 5.7.14 shows the block diagram of Fetal ECG monitoring since direct ECG of the fetus cannot be taken, signal from scalp is taken with reference to thigh of the mother. At the same time urine pressure of the mother is also taken. These three signals combinely form fetal ECA.

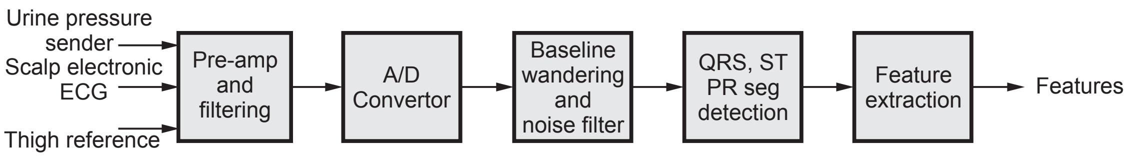


Fig. 5.7.14 Fetal ECG monitoring using DSP

- **Pre-amplifier and filtering :** The pre-amplifier and filter is analog circuit that is located near the sensor itself. It amplifies the signals and removes noise.
- **A/D converter :** The analog ECG signal is converted to its digital equivalent. Normally 8 bit to 16 bit A/D conversion is used. The sampling frequency is upto 500 Hz.

- **Baseline wandering and noise removal :** The baseline of the ECG signal shifts significantly. Also muscle noise is added. This distortion in ECG signal is removed with the help of low frequency filters. Baseline shift frequency is very low compared to ECG frequency.
- **Waveform detection :** DSP algorithms are developed to detect QRS peak, ST segment and PR segment. The R to R peaks interval gives heart rate.

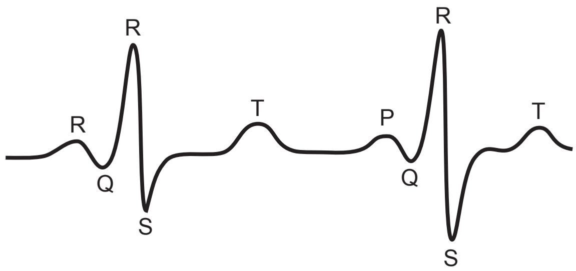


Fig. 5.7.15 ECG signal

- Stress/Distress is represented by shift of ST segment. The width of QRS wave also indicates blood pumping activity of heart.
- **Feature extraction :** The significant features are extracted based on above observations. Heart-rate, stress/distress etc. are detected as mentioned above.

Review Questions

1. *Design a DSP based system for the process of Audio signals in an audio recorder system.* **AU : May-16, Marks 8**
2. *Elaborate on Radar signal processing using a DSP processor.* **AU : May-16, Marks 8**
3. *Explain the use of DSP in telephone echo cancellation.*
4. *How DSP is used in ECG monitoring ?*
5. *Elaborate one application of digital signal processing with a DS processor.* **AU : Dec.-16, Marks 8**

5.8 Short Answered Questions [2 Marks Each]

Q.1 What is BSAR instruction ? Give an example. **AU : Dec.-10**

Ans. : This is TMS320C5 assembly language instruction,

Syntax : BSAR *shift*

Operands : $1 \leq \text{shift} \leq 16$

Description : The contents of the accumulator are arithmetically right barrel shifted by 1 to 16 bits. This shift is defined in the *shift* operand of the instruction. This instruction is executed in single cycle. If SXM bit is cleared, the high order bits of the ACC are zero filled. If SXM bit is set, the high order bits of ACC are sign extended.

Example : BSAR 16; (SXM = 0)

$$\underbrace{ACC : 0010000}_{\text{Before execlution}} \quad \underbrace{ACC : 0000\,0001}_{\text{After execution}}$$

This instruction is an accumulator memory reference instruction.

Q.2 Give special features of DSP processors. AU : May-11, Dec.-11

Ans. : i) Multiply - accumulate operations are implemented fast.

ii) Multiple operands and arrays are handled simultaneously.

iii)Circular buffers are provided.

iv)Multiple pointers for jumps and shifts.

Q.3 What is pipelining ? AU : Dec.-11, May-17

Ans : In pipelining , the functional units simultaneously execute instruction fetch, decode, read and execute operations. When one instruction is being decoded, next instruction is fetched. This concept of pipelining increases computational efficiency of the processor.

Q.4 What is the function of parallel logic unit in DSP processor ? AU : Dec.-12

Ans : The Parallel Logic Unit or PLU is the unit that executes logic operations on the data without affecting the contents of ACC. PLU provides the bit manipulation which can be used to set, clear, test or toggle bits in data memory, control or status registers.

Q.5 What is meant by bit reversed addressing mode ? What is the applications for which this addressing mode is preferred ? (Refer section 5.3.5) AU : Dec.-13

Q.6 Compare RISC and CISC processors. AU : Dec.-13

Ans. :

Sr. No.	RISC	CISC
1.	Simple instruction taking one cycle.	Complex instruction taking multiple cycle.
2.	Multiple register sets.	Single register set.
3.	Highly pipelined.	Not pipelined.
4.	Instruction executed by hardware.	Instructions executed by microprogram.

Q.7 List various registers used with ARAU. AU : May-14

Ans. : Auxiliary registers file AR0 - AR7. They are used for indirect addressing. Index (INDX) and auxiliary register compare register (ARCR) are used to calculate indirect address.

Q.8 What are different buses of TMS320C54x processor and list their functions. AU : May-14

Ans. :

i) 4 Program buses (PB) carries instruction code and immediate operands from program memory.

ii) Three address buses (CD, DB and EB) interconnect CPU, data address generation logic, program address generation logic, on-chip peripheral and data memory.

iii) Four address buses (PAB, CAB, DAB and EAB) carry address needed for instruction execution.

Q.9 How do a digital signal processor differ from other processors. AU : May-15

Ans. : Refer table 5.6.1.

Q.10 What is the advantage of Harvard Architecture in a DS Processor ? AU : Dec.-15

Ans. : The Harvard architecture has seperate memories for program and data. There are also seperate address and data buses for program and data. Because of these seperate on chip memories and internal buses, the speed of execution in Harvard architecture is high.

Q.11 How is a DS Processor applicable for motor control applications ? AU : Dec.-15

Ans. : Motor control applications require firing of devices in the converters. It also requires sensing of speed, current and voltage of the motor. There inputs are fed to DSP processor and control algorithm is written to generate fining pulses. DSP processors architecture allows an intelligent and efficient approach to rednce system cost to provide compact and cheaper motor control drives.

Q.12 What are the merits and demierits of VLIW architecture ? AU : May-16

Ans. : **Merits :** (i) No complicated logic is required to check for dependencies.

ii) Complicated instruction scheduling is eliminated.

iii) Performance is improved.

Demerits : i) Code size is increased due to empty slots.

ii) Memory bandwidth is increased.

iii) All pipelines can be stalled due to cache miss in one of them.

Q.13 What are the factors that influence the selection of DSP processor for an application ? AU : May-16

Ans. : Following factors influence selection of a DSP.

i) Architechiral features. ii) Execution speed iii) Type of arthmetic iv) Wordlength

Q.14 Write some commercial DSP processors. **AU : May-17**

Ans. : i) TMS 320 C 54X DSP processors.

ii) ARM DSP processors

iii) ADSP 21XX DSP processors.

Q.15 State how spectrum meter application can be designed with DS Processor. **AU : Dec.-16**

Ans. : The block diagram of spectrum meter is given as,

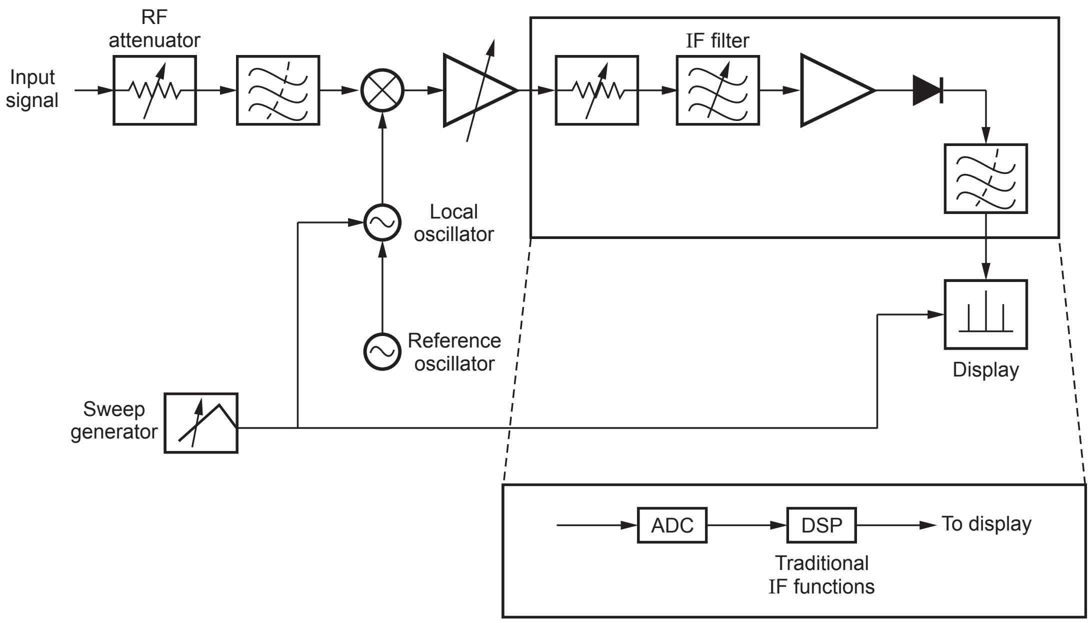


Fig. 5.8.1 Spectrum meter

- Spectrum meters have a heterodyne receiver principle to generate display of signal frequency components. Modern instruments use digital processor to handle several of traditional IF functions.

❑❑❑

May - 2015 **Discrete Time Systems and Signal Processing** **Semester - IV (EEE)** **(Regulation 2013)**	**AU** **Solved Paper**

Time : Three Hours] **[Maximum Marks : 100**

Answer ALL questions

PART A - (10 × 2 = 20 Marks)

Q.1 *Check if the system described by the difference equation y (n) = ay (n – 1) + x (n) with y (0) = 1 is stable.*
[Refer Two Marks Q.33 of Chapter - 2]

Q.2 *Differentiate between energy and power singals.*
[Refer Two Marks Q.25 of Chapter - 1]

Q.3 *Determine the z-transform of* $x(n) = a^n$.
[Refer Two Marks Q.34 of Chapter - 2]

Q.4 *Find the DFT of the sequence x(n) = {1,1,0,0}.*
[Refer Two Marks Q.18 of Chapter - 3]

Q.5 *Determine the Fourier Transform of the signal* $x(t) = \sin \omega_0 t$.
[Refer Two Marks Q.35 of Chapter - 2]

Q.6 *Draw the basic butterfly flow graph for the computation in the DIT FFT algorithm.*
[Refer Two Marks Q.9 of Chapter - 3]

Q.7 *Comment on the passband and stop band characteristics of butter worth filter.*
[Refer Two Marks Q.34 of Chapter - 4]

Q.8 *Realize the following causal linear phase FIR system function*
$H(z) = \frac{2}{3} + z^{-1} + \frac{2}{3} z^{-2}$.
[Refer Two Marks Q.35 of Chapter - 4]

Q.9 *How do a digital signal processor differ from other processors.*
[Refer Two Marks Q.9 of Chapter - 5]

Q.10 *State any two application of DSP.*
[Refer Two Marks Q.26 of Chapter - 1]

PART B - (5 × 16 = 80 Marks)

Q.11 a) *i) Find the impulse response of a discrete time invarient system whose difference equation is given by : y (n) = y (n – 1) + 0.5 y (n – 2) + x (n) + x (n – 1).*
[Refer Example 2.9.17] **[12]**

ii) Explain the properties of discrete time system. **[4]**
[Refer section 1.4]

OR

b) *i) A discrete time system is represented by the following difference equation in which x (n) is input and y (n) is output. y (n) = 3y (n – 1) – nx (n) + 4x (n – 1) + 2x (n + 1) and n ≥ 0. Is this system linear ? Shift invariant ? Causal ? In each case, justify your answer.* **[12]**
[Refer example 1.4.9]

ii) What is meant by quantization and quantization error ? **[4]**
[Refer section 1.5.1]

Q.12 a) *i) Find the z-transform of* $x(n) = n^2 u(n)$. **[8]**
[Refer example 2.4.6]

ii) Find the inverse z-transform of $X(z) = \dfrac{z}{3z^2 - 4z + 1}$, *for ROC*

(i) $|z| > 1$, *(ii)* $|z| < \dfrac{1}{3}$, *(iii)* $\dfrac{1}{3} < |z| < 1$. **[Refer example 2.5.7]** **[8]**

OR

b) *i) Convolute the following two sequences* $x_1(n) = \{0, 1, 4, -2\}$ *and* $x_2(n) = \{1, 2, 2, 2\}$. **[Refer example 2.4.8]** **[8]**

ii) Find the frequency response of the LTI system governed by the equation $y(n) = a_1\, y(n-1) - a_2\, y(n-2) - x(n)$. **[Refer example 2.10.12]** **[8]**

Q.13 a) *i) Determine the DFT of the sequence* $x(n) = \begin{cases} \dfrac{1}{4}, & \text{for } 0 \le n \le 2 \\ 0, & \text{otherwise} \end{cases}$ **[8]**
[Refer example 3.1.7]

ii) Draw the flow graph of an 8-point DIF-FFT algorithm and explain.
[Refer section 3.6.2 (Fig. 3.6.19)] **[8]**

OR

b) *i) Given x (n) = n + 1, and N = 8, find X (K) using DIT, FFT algorithm.*
[Refer example 3.6.2] **[8]**

ii) Use 4 – point inverse FFT for the DFT result {6, –2 + j2 –2, –2 – j2} and determine the input sequence. **[Refer example 3.7.2]** **[8]**

Q.14 a) *i) A low pass filter is to be designed with the following desired frequency response.*

$$H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega}, & \dfrac{-\pi}{4} \leq |\omega| \leq \dfrac{\pi}{4} \\ 0, & \dfrac{\pi}{4} < |\omega| \leq \pi \end{cases}$$

Determine the filter coefficients $h_d(n)$ *if the window function is defined as*

$$\omega(n) = \begin{cases} 1 & 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

[Refer example 4.6.9] **[16]**

OR

b) *Determine H (z) for a Butter worth filter satisfying the following constraints.*

$$\sqrt{0.5} \leq |H(e^{j\omega})| \leq 1 \; ; \; 0 \leq \omega \leq \frac{\pi}{2}$$

$$|H(e^{j\omega})| \leq 0.2; \; \frac{3\pi}{4} \leq \omega \leq \pi$$

with T = 1 s. Apply impulse invariant transformation.
[Refer example 4.9.12] **[16]**

Q.15 a) *Draw the architecture of a DSP processor for implementing a DSP algorithm. explain its features.* **[Refer section 5.5]** **[16]**

OR

b) *i) Name the different addressing modes of a DSP processor. Explain them with an example.* **[Refer section 5.3]** **[10]**

ii) Write a note on commercial DSP processor. **[Refer section 5.5.2]** **[6]**

❑❑❑

December - 2015 Discrete Time Systems and Signal Processing Semester - IV (EEE) (27216) (Regulation 2013)	AU Solved Paper

Time : Three Hours] **[Maximum Marks : 100**

Answer ALL Questions.

PART A - (10 $\times$ 2 = 20 Marks)

Q.1 *Given a continuous signal x(t) = 2 cos 300 πt. What is the nyquist rate and fundamental frequency of the signal.* **(Refer Q.27, section 1.6)**

Q.2 *Determine x(n) = u(n) is a power signal or an energy signal* **(Refer Q.28, section 1.6)**

Q.3 *What is ROC of Z transform ? State its properties.* **(Refer Q.12 and Q.16, section 2.11)**

Q.4 *State initial and final value theorem of Z transform* **(Refer Q.32, section 2.11)**

Q.5 *Calculate the percentage saving in calculation in a 256 point radix–2 FFT when Compared to direct FFT.* **(Refer Q.21, section 3.8)**

Q.6 *State circular frequency shift property of DFT.* **(Refer Q.20, section 3.8)**

Q.7 *Define pre–wraping effect? Why it is employed ?* **(Refer Q.31, section 4.11)**

Q.8 *The impulse response of analog filter is given in Fig. 1. Let $h(n) = h_a(nT)$ where T = 1. Determine the system function.* **(Refer Q.30, section 4.11)**

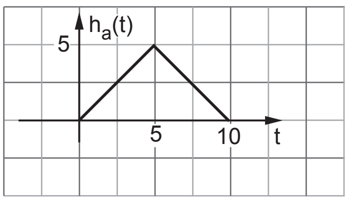


Fig. 1

Q.9 *What is the advantage of Harvard Architecture in a DS Processor ?* **(Refer Q.10, section 5.7)**

Q.10 *How is a DS Processor applicable for motor control applications ?* **(Refer Q.11, section 5.7)**

PART B - (5 × 16 = 80 Marks)

Q.11 a) i) *Check the causality and stability of the systems*
$y(n) = x(-n) + x(n-2) + x(2n-1)$ **[8]**

ii) *Check the system for linearity and time variance* $y(n) = (n-1)x(n) + C$. **[8]**

(Refer Example 1.4.7) **[8]**

OR

b) *i) What is meant by energy and power signal ? Determine whether the following signal are energy or power or neither energy nor power signals.*

1) $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$ **[4]**

2) $x_2(n) = \sin\left(\frac{\pi}{6}n\right)$ **(Refer Example 1.2.3 (i) and (iii))** **[4]**

ii) State and prove the Sampling theorem **(Refer section 1.5.2)** **[8]**

Q.12 a) *i) Find the Z transform and ROC of* $x(n) = r^n \cos(n\theta)u(n)$
(Refer Example 2.4.4 (i)) **[8]**

ii) Find the inverse Z transform of $X(z) = \frac{z}{3z^2 - 4z + 1}$ $ROC|Z|>1$
(Refer Example 2.5.7) **[8]**

OR

b) *Using z–transform determine the response y(n) for* $n \geq 0$ *if*
$y(n) = \left(\frac{1}{2}\right)y(n-1) + x(n), x(n) = \left(\frac{1}{3}\right)^n u(n)$ *and* $y(-1) = 0$
(Refer Example 2.9.6) **[16]**

Q.13 a) *i) The first five points of the eight point DFT of a real valued sequence are (0.25, 0,125 – j 0.3018, 0, 0.125 – j0.0518). Determine the remaining three points.*
(Refer Example 3.3.5) **[4]**

ii) Compute the eight point DFT of the sequence x = {0, 1, 2, 3, 4, 5, 6, 7} using DIF FFT algorithm. **(Refer Example 3.6.9)** **[12]**

OR

b) *i) Find the inverse DFT of*
$X(k) = \{7, -\sqrt{2} - j\sqrt{2}, -j, \sqrt{2} - j\sqrt{2}, 1, \sqrt{2} + j\sqrt{2}, j, -\sqrt{2} + j\sqrt{2}\}$
(Refer Example 3.1.5) **[12]**

ii) Using FFT algorithm compute the DFT of x(n) = {2, 2, 2, 2}
(Refer Example 3.6.5) **[4]**

Q.14 a) *Design a Butterworth filter using the Impulse invariance method for the following specifications.* **(Refer Example 4.9.7)** **[16]**

$$0.8 \leq \left| H(e^{j\omega}) \right| \leq 1 \qquad 0 \leq \omega \leq 0.2\pi$$

$$\left| H(e^{j\omega}) \right| \leq 0.2 \qquad 0.6\pi \leq \omega \leq \pi$$

OR

b) *Design a filter with desired frequency response.*

$$H_d(e^{j\omega}) = e^{-j3\omega} \quad \text{for} \quad \frac{-3\pi}{4} \leq \omega \leq \frac{3\pi}{4}$$

$$= 0 \quad \text{for} \quad \frac{3\pi}{4} \leq |\omega| \leq \pi$$ **(Refer Example 4.6.8)**

Using a Hanning window for N = 7 **[16]**

Q.15 a) *Explain the various addressing modes of a commercial DSP processor.* **(Refer section 5.3)** **[16]**

OR

b) *With Suitable block diagram explain in detail about TMS320C54 DSP Processor and of its memory architecture.* **(Refer section 5.5)** **[16]**

❑❑❑

May - 2016 Discrete Time Systems and Signal Processing Semester - IV (EEE) (57318) Regulation 2013	AU Solved Paper

Time : Three Hours] [Maximum Marks : 100

Answer ALL Questions.

PART - A (10 $\times$ 2 = 20 Marks)

Q.1 *Determine if the system described by the equation* $y(n) = x(n) + \frac{1}{x(n-1)}$ *is causal or non causal.* **(Refer Q.29, section 1.6)**

Q.2 *What is an Anti-Aliasing filter ?* **(Refer Q.1, section 1.6)**

Q.3 *Determine the Z-transform and ROC of the following finite duration signals*

i) $x(n) = \{3, 2, 2, 3, 5, 0, 1\}$ **(Refer Q.17, section 2.11)**

ii) $x(n) = \delta(n-k)$ **(Refer Q.30, section 2.11)**

Q.4 *Compute the convolution of the two sequences*

$x(n) = \{2, 1, 0, 0.5\}$ *and* $h(n) = \{2, 2, 1, 1\}$ **(Refer Q.26, section 2.11)**

Q.5 *Draw the flow graph of a 4 point radix-2 DIT-FFT butterfly structure for DFT.* **(Refer Q.22, section 3.8)**

Q.6 *What are the applications of FFT algorithm ?* **(Refer Q.23, section 3.8)**

Q.7 *Obtain the cascade realization for the system function,*

$$H(z) = \frac{\left(1+\frac{1}{4}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right)}$$ **(Refer Q.36, section 4.11)**

Q.8 *Mention the advantages of FIR filters over IIR filters.* **(Refer Q.16, section 4.11)**

Q.9 *What are the merits and demierits of VLIW architecture ?* **(Refer Q.12, section 5.7)**

Q.10 *What are the factors that influence the selection of DSP processor for an application ?* **(Refer Q.13, section 5.7)**

PART - B (5 × 16 = 80 Marks)

Q.11 a) i) *Determine if the signals, $x_1(n)$ and $x_2(n)$ are power, energy or neither energy nor power signals.*

$x_1(n) = \left(\frac{1}{3}\right)^n u(n)$ and $x_2(n) = e^{2n}u(n)$. **(Refer Example 1.2.3)** **[8]**

ii) *What is the input signal x(n) that will generate the output sequence*

y(n) = {1, 5, 10, 11, 8, 4, 1} for a system with impulse response

h(n) = {1, 2, 1} **(Refer Example 2.9.11)** **[8]**

OR

b) *i) A signal x(t) = $\text{sinc}(50\pi t)$ is sampled at a rate of (1) 20 Hz (2) 50 Hz and (3) 75 Hz. For each of these cases, explain if you can recover the signal x(t) from the samples signal.* **(Refer Example 1.5.2)** **[6]**

ii) Determine whether or not each of the following signals is periodic. If the signal is periodic, specify its fundamental period.

1) *x(n)* = $e^{j6\pi n}$ **[5]**

2) *x(n)* = $\cos\frac{\pi}{3}n + \cos\frac{3\pi}{4}n$ **(Refer Example 1.2.4)** **[5]**

Q.12 a) i) *Find x(n) if X(z) =* $\dfrac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}$ **(Refer Example 2.5.6)** **[6]**

ii) *Find the response of the causal system y(n) – y(n – 1) = x(n)+x(n – 1) to the input x(n) = u(n). Test its stability.* **(Refer Example 2.9.8)** **[10]**

OR

b) *Find the impulse response, frequency response, magnitude response and phase response of the second order system.*

$y(n) - y(n-1) + \frac{3}{16}y(n-2) = x(n) - \frac{1}{2}x(n-1)$ **(Refer Similar Example 2.10.11)** **[16]**

Q.13 a) i) *Summarize the steps of radix – 2 DIT–FFT algorithm* **(Refer section 3.6.1)[8]**

ii) *Compute the 4 point DFT of the sequence x(n) = {0, 1, 2, 3} using DIT and DIF algorithm.* **(Refer Example 3.6.10)** **[8]**

OR

b) *Find the IDFT of the sequence*

$X(k) = \{4, 1 - j\ 2.414, 0, 1 - j\ 0.414, 0, 1 + j\ 0.414, 0, 1 + j\ 2.414\}$ *Using DIF algorithm* **(Refer Example 3.7.3)** **[16]**

Q.14 a) *Design an ideal low pass filter with a frquency response.*

$$H_d(e^{j\omega}) = 1 \text{ for } \frac{-\pi}{2} \leq \omega \leq \frac{\pi}{2}$$

$$= 0 \text{ for } \frac{\pi}{2} \leq |\omega| \leq \pi$$ **(Refer Example 4.6.10)**

Find the values of h(n) for N = 11. Find H(z) and the filter coefficients. **[16]**

OR

b) *i) Given the specifications* $\alpha_p = 3$ *dB,* $\alpha_s = 10$ *dB,* $f_p = 1$ *kHz and* $f_s = 2$ *kHz. Determine the order of the filter using Chebyshev approximation. Find H(s).* **(Refer Example 4.7.8)** **[8]**

ii) Apply bilinear transformation to **[8]**

$$H(s) = \frac{2}{(s+1)\,(s+2)}$$ *with T = 1 sec and find H(z)* **(Refer Example 4.8.11)**

Q.15 a) *i) Discuss on the addressing modes supported by a DSP processor.* **(Refer section 5.3)** **[8]**

ii) Design a DSP based system for the process of Audio signals in an audio recorder system. **(Refer section 5.7.1)** **[8]**

OR

b) i) *Explain the datapath architecture and the bus structure in a DSP processor with suitable diagram.* **(Refer section 5.5.1)** **[8]**

ii) *Elaborate on Radar signal processing using a DSP processor.* **(Refer section 5.7.2)**

❑❑❑

December - 2016 Discrete Time Systems and Signal Processing Semester - IV (EEE) (80375) Regulation 2013	AU Solved Paper

Time : Three Hours] **[Maximum Marks : 100**

Answer ALL Questions.

PART - A (10 × 2 = 20 Marks)

Q.1 *Distinguish between discrete signal and digital signal representations.* **(Refer Two Marks Q.6 of section 1.6)**

Q.2 *If y(n) = x(n + 1) + x(n – 2), is the system causal ?* **(Refer Two Marks Q.30 of section 1.6)**

Q.3 *Find the system transfer function H(z) if y(n) = x(n) + y(n –1).* **(Refer Two Marks Q.36 of section 2.11)**

Q.4 *Explain the relationship between s-plane and z-plane* **(Refer Two Marks Q.37 of section 2.11)**

Q.5 *Why is it required to do zero padding in DFT analysis ?* **(Refer Two Marks Q.26 of section 3.8)**

Q.6 *What is need for windowing techniques on Fourier transformed signals ?* **(Refer Two Marks Q.6 of section 4.11)**

Q.7 *Why are digital filters more useful than analog filters ?* **(Refer Two Marks Q.37 of section 4.11)**

Q.8 *Name one method that convert the transfer function of a analog into the digital filter.* **(Refer Two Marks Q.27 of section 4.11)**

Q.9 *What is Gibbs phenomena ?* **(Refer Two Marks Q.2 of section 4.11)**

Q.10 *State how spectrum meter application can be designed with DS processor.* **(Refer Two Marks Q.15 of section 5.8)**

PART - B (5 × 16 = 80 Marks)

Q.11 a) *With neat figure explain block diagram of digital signal processing system. State the advantages of convolution technique.* **(Refer section 1.1)** **[14 + 2]**

OR

b) *Distinguish the following with examples and formulae.*

i) *Energy vs power signal.* **(Refer table 1.2.1)**

ii) *Time variant vs time invariant system.* **(Refer section 1.4.2)**

Q.12 a) i) *Explain the role of windowing to design a FIR filter.* **(Refer section 4.6)**

ii) *Compare and explain on the choice and type of windows selection for signal analysis.* **(Refer section 4.5.6)**

iii) *Compute numerically the effect of Hamming windows and design the filter if*

Cut-off frequency = 100 Hz

Sampling frequency = 1000 Hz

Order of filter = 2

Filter length required = 5 **(Refer section 4.6)** **[6 + 6 + 4]**

OR

b) *Evaluate the following :*

i) *The impulse response h(n) for*
y(n) = x(n) + 2x(n – 1) – 4x(n – 2) + x(n – 3)
(Refer example 2.9.9)

ii) *The ROC of a finite duration signal x(n) = {2, –1, –2, –3, 0, –1}.*
(Refer example 2.2.1)

iii) *Inverse z-transform for* $X(z) = 1/(z - 1.5)^4$; *ROC :* $|z| > 1/4$.
(Refer example 2.5.4)

Q.13 a) *What is the need for frequency response analysis ? Determine the frequency response and plot the magnitude response and phase response for the system.*
y(n) = 2x(n) + x(n – 1) + y(n – 2) **(Refer example 2.10.11)** **[6 + 10]**

OR

b) *Describe the need for Bit reversal and the Butterfly structure. For a sequence x(n) = (4, 3, 2, 1 –1, 2, 3, 4) obtain the 8 pt FFT computation using DIT method.*
(Refer examples 3.6.1 and 3.6.6) **[4 + 12]**

Q.14 a) *Write briefly on any two of the following :* **[8 + 8]**

i) *Comparison of Butterworth and Chebyshev filter.* **(Refer section 4.7.5)**

ii) *Elaborate one application of digital signal processing with a DS processor.*
(Refer section 5.7)

iii) *A difference equation describing a filter is given by*
y(n) – 2y(n – 1) + y(n + 2) = x(n) + 1/2 x(n – 1) obtain direct form II structure. **(Refer example 4.3.7)**

OR

b) *Obtain the system function of the digital filter if the analog filter is* $H_a(s) = 1/[(s+0.2)^2 + 2]$. *Using the impulse invariance method and the Bilinear transformation method obtain the digital filter.* **(Refer Example 4.8.12)** **[8 + 8]**

Q.15 a) *Compute the following if :* x_1 *= [–1, –1, –1, 2];* x_2 *= [–2 –1, –1, –2]* **[10 + 6]**

i) Linear and circular convolution of a sequence.

ii) $x_1 ; x_2$ *subject to addition and multiplication.* **(Refer example 3.3.8)**

OR

b) *Write briefly an any two of the following ;* **[8 + 8]**

i) *Quantization and errors in DS processor.* **(Not in syllabus)**

ii) *With neat figure explain the architecture of any one type of a DS processor.* **(Refer section 5.5)**

iii) *The addressing modes of one type of DS processor.* **(Refer section 5.3)**

❑❑❑

# May - 2017 **Discrete Time Systems and Signal Processing** **Semester - IV (EEE) (71774) Regulation 2013**	# AU **Solved Paper**

Time : Three Hours] **[Maximum Marks : 100**

Answer ALL Questions.

PART - A ($10 \times 2 = 20$ Marks)

Q.1 *What are the energy and power of discrete signal ?* **(Refer Two Marks Q.31 of section 1.6)**

Q.2 *State sampling theorem.* **(Refer Two Marks Q.2 of section 1.6)**

Q.3 *Write the properties of region of convergence.* **(Refer Two Marks Q.16 of section 2.11)**

Q.4 *Find the convolution of the input signal (1, 2, 1) and the impulse response (1, 1, 1) using z-transform.* **(Refer Two Marks Q.24 of section 2.11)**

Q.5 *Define twiddle factor. Write its magnitude and phase angle.* **(Refer Two Marks Q.24 of section 3.8)**

Q.6 *Compute the number of multiplications and additions for 32 point DFT and FFT.* **(Refer Two Marks Q.25 of section 3.8)**

Q.7 *Write the advantages and disadvantages of digital filters.* **(Refer Two Marks Q.37 of section 4.11)**

Q.8 *Define prewarping effect.* **(Refer Two Marks Q.31 of section 4.11)**

Q.9 *What is pipelining and how to define its depth ?* **(Refer Two Marks Q.3, section 5.8)**

Q.10 *Write some commercial DSP processors.* **(Refer Two Marks Q.14 section 5.8)**

PART - B ($5 \times 16 = 80$ Marks)

Q.11 a) *Determine the following systems are linear, stability and time invariance of the system i) $y(n) = x(2n)$ ii) $y(n) = \cos x(n)$ iii) $y(n) = x(n) + nx(n+1)$.*
(Refer examples 1.4.5 (viii), (i), (vii)) **[13]**

OR

b) i) *Explain the process of quantization and its error types.*
(Refer section 1.5.1) **[10]**

ii) *Compute the Nyquist sampling frequency of the signal $x(t) = 4 \sin c(3t/\pi)$.*
(Refer example 1.5.3) **[3]**

Q.12 a) i) *State and prove convolution and Parseval's theorem using z-transform.*
(Refer sections 2.4.6 and 2.4.12) **[6]**

ii) *Find the z-transform of the system $x(n) = \cos(n\theta)\, u(n)$*
(Refer example 2.4.3(1)) **[7]**

OR

b) *Find the inverse z-transform of $X(z) = (x+1)/(x + 0.2)(x - 1)$, $|z| > 1$ using residue method.* **(Refer example 2.5.8)** **[13]**

Q.13 a) *Determine the 8 point DFT of the sequence $x(n) = \{1, 1, 1, 1, 1, 1, 0, 0\}$,*
(Refer example 3.1.3) **[13]**

OR

b) *Compute 8 point DFT of the given sequence using DIT algorithm*
$x(n) = \begin{Bmatrix} \pi & n \le 7 \\ 0 & otherwise \end{Bmatrix}$ **(Refer example 3.6.3)**

Q.14 a) *Design a 15 tap linear phase filter using frequency sampling method to the following discrete frequeny response* $H\left(\frac{2\pi k}{15}\right) = \begin{cases} 1 & 0 \le k \le 3 \\ 0.4 & k = 4 \\ 0 & k = 5, 6, 7 \end{cases}$

(Refer example 4.6.11) **[13]**

OR

b) *Using bilinear transformation, design a high pass filter, monotonic in passband with cut off frequency of 1000 Hz and down 10 dB at 330 Hz. The sampling frequency is 5000 Hz.* **(Refer example 4.9.13)** **[13]**

Q.15 a) *Discuss the features and architecture of TMS 320C50 processor.*
(Refer example 5.5.2) **[13]**

OR

b) *Explain the addressing modes and registers of DSP processors.*
(Refer section 5.3) **[13]**

PART - C (1 × 15 = 15 Marks)

Q.16 a) *The analog signal has a bandwidth of 4 kHz. If we use N point DFT with $N = 2^m$ (m is an integer) to compute the spectrum of the signal with resolution less than or equal to 25 Hz. Determine the minimum sampling rate, minimum number of required examples and minimum lengh of the analog signal. What is the step size required for quantize this signal.* **(Refer example 3.1.8)** **[15]**

OR

b) *Convert the single pole low pass filter with system function* $H(z) = \dfrac{0.5(1 + Z^{-1})}{1 - 0.302 Z^{-3}}$ *into band pass filter with upper and lower cut-off frequencies ωu and ωl respectively. The lowpass filter has 3 dB bandwidth and $\omega p = \pi/6$ and $\omega u = 3\pi/4$, $\omega l = \pi/4$ and draw its realization in direct form II.* **(Refer example 4.7.9)** **[15]**

❑❑❑

December - 2017 **Discrete Time Systems and Signal Processing** Semester - IV (EEE) (50483) Regulation 2013	AU Solved Paper

Time : Three Hours] **[Maximum Marks : 100**

Answer ALL Questions

PART - A (10 × 2 = 20 Marks)

Q.1 *State the Parseval's theorem for discrete time signal.*

Ans. : If $x(n)$ is a discrete signal and $x(n) \xleftrightarrow{\text{DTFT}} X(\omega)$ then Parseval's the states that

Energy of the signal is given as,

$$E = \frac{1}{2\pi} \int_{\pi}^{\pi} |X(\omega)|^2 \, d\omega$$

Q.2 *What is meant by aliasing effect ?* **(Refer Two Marks Q.3 of section 1.6)**

Q.3 *List the methods to find inverse Z transform.*

Ans. : i) Power series expansion

ii) Partial fraction expansion

iii) Long division method

iv) Contour integration

Q.4 *Write the conditions to define stability in ROC.*
(Refer Two Marks Q.6 of section 2.11)

Q.5 *Find the DFT of the signal $x(n) = a^n$.*

Ans. : DFT is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1$$

$$= \sum_{n=0}^{N-1} a^n \, e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} \left(a \, e^{-j2\pi k/N}\right)^n$$

using standard series summation we get,

$$\frac{1 - a^N e^{-j2\pi k}}{1 - a\, e^{-j2\pi k/N}}$$

Q.6 *Draw the butterfly structure for 2 point DFT using DIT-FFT algorithm.*
(Refer Two Marks Q.9 of section 3.8)

Q.7 *Draw the direct form I structure for 3^{rd} order system.*

Ans. :

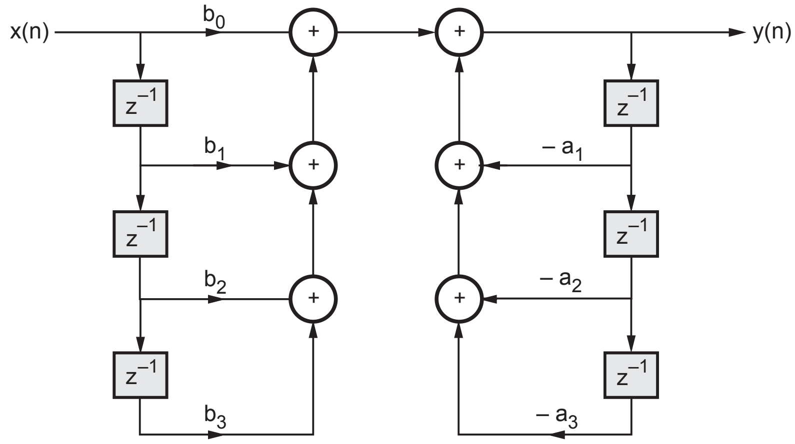


Fig. 1 Direct form I realisation

Q.8 *What is prewarping effect ?* **(Refer Two Marks Q.31 of section 4.11)**

Q.9 *Write the features of DSP processor.* **(Refer Two Marks Q.8 of section 5.8)**

Q.10 *List some example of commercial digital signal processor.*

Ans. : i) TMS 320 $C54X$

ii) Arm⊕DSP OMAP L138

iii) DSP C674X, C665X

iv) AM 335X

v) DSP C665X C6657 EVM.

PART - B (5 ×13 = 65 Marks)

Q.11 a) *i) Determine the power and energy of the given signal. State the signal is power or energy $x(n) = \sin\left(\frac{\pi n}{4}\right)$* **[4]**

Ans. : Energy is given by,

$$E = \sum_{n=-\infty}^{\infty} \left| \sin^2\left(\frac{\pi}{4} n\right) \right| = \sum_{n=-\infty}^{\infty} \left[\frac{1-\cos(\frac{\pi}{2} n)}{2} \right] = \infty$$

Power is given by,

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \sin^2\left(\frac{\pi}{4} n\right) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1-\cos\frac{\pi}{2} n}{2}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1}{2} - \lim_{N\to\infty} \frac{1}{2N+1} \sum_{N=-N}^{N} \frac{1}{2} \cos\frac{\pi}{2} n$$

The second term is summation of the cosine signal over complete two cycles. Hence its value is zero.

$$\therefore \qquad P = \lim_{N\to\infty} \frac{1}{2N+1} \frac{1}{2} \cdot (2N+1) = \frac{1}{2}$$

Since power is finite and non-zero. This signal is *power* signal.

ii) Determine the given signal is periodic or not $x(n) = \cos\left(\frac{2\pi n}{3}\right)$ **[3]**

Ans. : Compare the given signal with $x(n) = \cos(2\pi fn)$

$$\therefore \qquad 2\pi\, fn = \frac{2\pi}{3} n \Rightarrow f = \frac{k}{N} = \frac{2}{3} \text{ i.e. ratio of two integers.}$$

Hence, this signal is **periodic** $\boxed{N = 2}$

iii) Discuss the mathematical representation of signal. Write the difference between continuous and discrete time signal. **(Refer section 1.2)** **[6]**

OR

b) *i) Determine whether the system is linear or not* $y(n) = ax(n) + bx(n-1)$. **[3]**

Ans. : For two seperate inputs the system produces the response of

$$y_1(n) = \text{T}\,[x_1(n)] = a\,x_1(n) + b\,x_1(n-1)$$

$$y_2(n) = \text{T}\,[x_2(n)] = a\,x_2(n) + b\,x_2(n-1) \qquad \ldots(1)$$

The response of the system to linear combination of two inputs will be,

$$\begin{aligned} y_3(n) &= T\{a_1 x_1(n) + a_2 x_2(n)] \\ &= a[a_1 x_1(n) + a_2 x_2(n)] + b[a_1 x_1(n-1) + a_2 x_2(n-1)] \\ &= a\,a_1 x_1(n) + a\,a_2 x_2(n) + a_1 bx_1(n-1) + a_2 bx_2(n-1) \qquad \ldots(2) \end{aligned}$$

The linear combination of two outputs will be,

$$\begin{aligned} y'_3(n) &= a_1 y_1(n) + a_2 y_2(n) \\ &= a\,a_1 x_1(n) + a\,a_2 x_2(n) + a_1 bx_1(n-1) + a_2 bx_2(n-1) \qquad \ldots(3) \end{aligned}$$

$$y_3(n) = y'_3(n)$$

Hence system is linear.

ii) Determine whether the given system is causal or not $y(n) = x(n) + x^2(n-1)$ **[4]**

Ans. : $y(n) = x(n) + x^2(n-1)$

For $n = 0$; $y(0) = x(0) + x^2(-1)$

$n = 1$, $y(1) = x(1) + x^2(0)$

$n = -1$, $y(-1) = x(-1) + x^2(-2)$

From above values, the output $y(n)$ depends on present and past inputs and hence system is *causal*.

iii) Determine whether the system is time invariant and stability : $y(n) = e^{x(n)}$. **[6]**

Ans. : $y(n) = f[x(n)] = e^{x(n)}$

Time invariance :

Step 1 : The response to the delayed input $x(n-k)$ will be,

$$y(n,k) = e^{x(n-k)} \quad \ldots(1)$$

Step 2 : Now let us delay the output $y(n)$ by 'k' samples,

$$y(n-k) = e^{x(n-k)} \quad \ldots(2)$$

From (1) and (2), $y(n,k) = y(n-k)$

Hence the system is *time-invarient.*

Stability : As long as $x(n)$ is bounded $e^{x(n)}$ is also bounded. Hence this system is *stable*.

Q.12 a) *i) State and prove any three properties of Z transform.* **(Refer section 2.4)** **[8]**

ii) Find the Z transform of $x(n) = r^n \cos(n\theta) u(n)$. **(Refer Example 2.4.4(i))** **[5]**

OR

b) *i) A descrete system has a unit sample response.*
$h(n) = \frac{1}{2}\delta(n) + \delta(n-1) + \frac{1}{2}\delta(n-2)$. *Find the system frequency response.* **[7]**

Ans. : DTFT is given by

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2}\delta(n) + \delta(n-1) + \frac{1}{2}\delta(n-2)\right] e^{-j\omega n}$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty}\delta(n)\,e^{-j\omega n} + \sum_{n=-\infty}^{\infty}\delta(n-1)\,e^{-j\omega n} + \frac{1}{2}\sum_{n=-\infty}^{\infty}\delta(n-2)\,e^{-j\omega n}$$

$$= \frac{1}{2} + e^{-j\omega} + \frac{1}{2}e^{-j2\omega}$$

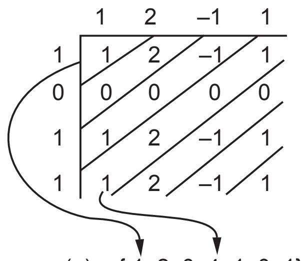
$$= e^{-j\omega}\left[\frac{1}{2}e^{j\omega} + 1 + \frac{1}{2}e^{-j\omega}\right]$$

$$H(\omega) = e^{-j\omega}\,[1+\cos\omega)]$$

$\therefore$ $|H(\omega)| = 1 + \cos\omega$ gives magnitude reponse and $\angle H(\omega) = -\omega$ gives frequency response.

ii) Find the convolution of the two sequences x(n) = (1, 2, –1, 1) and h(n) = (1, 0, 1, 1) using graphical method. **[6]**

Ans. : $y(n) = x(n) * h(n)$ by graphical method.



y(n) = { 1, 2, 0, 4, 1, 0, 1}

Q.13 a) *i) State and prove any two properties of DFT.* **(Refer section 3.3)** **[6]**

ii) Determine the DFT of the following sequence x(n) = (5,–1, 1, –1, 2) **[7]**

Ans. : $X_5 = [W_5]x_5$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0.309-j0.951 & -0.809-j0.587 & -0.809+j\,0.587 & 0.309+j\,0.951 \\ 1 & -0.809-j0.587 & 0.309+j\,0.951 & 0.309-j\,0.951 & -0.809+j\,0.587 \\ 1 & -0.809+j\,0.587 & 0.309-j\,0.951 & 0.309+j\,0.951 & -0.809-j\,0.587 \\ 1 & 0.309+j\,0.951 & -0.809+j0.587 & -0.809-j\,0.587 & 0.309-j\,0.951 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 5.3090+j\,1.6776 \\ 4.1910+j\,3.66551 \\ 4.1910-j\,3.66551 \\ 5.3090-j\,1.6776 \end{bmatrix}$$

OR

b) *Find the DFT of a sequence x(n) = (1, 2, 3, 4 ,4, 3, 2, 1) using DIT-FFT algorithm.* **[13]**

Ans. : Fig. 2 shows the signal flow graph along with stage wise results of above DFT.

Refer Fig. 2 on next page.

x(0) = 1, x(1) = 2, x(2) = 3, x(3) = 4, x(4) = 4, x(5) = 3, x(6) = 2, x(7) = 1

From Fig. 2, the DFT is,

$$X(0) = 20$$

$$X(1) = -5.828 - j\,2.414$$

$$X(2) = 0$$

$$X(3) = -0.172 - j\,0.414$$

$$X(4) = 0$$

$$X(5) = -0.172 + j\,0.414$$

$$X(6) = 0$$

$$X(7) = -0.528 + j\,0.414$$

Q.14 a) *Obtain an analog Chebyshev filter transfer function that satisfies the given contraints*

$$\frac{1}{\sqrt{2}} \leq |H(j\Omega)| \leq 1; \; 0 \leq \Omega \leq 2$$

$$|H(j\Omega)| < 0.1; \; \Omega \geq 4$$ **[13]**

Ans. : **Given :**

$$A_p = \frac{1}{\sqrt{2}} \quad \Omega_p = 2$$

$$A_s = 0.1 \quad \Omega_s = 4$$

Step 1 : Value of ε and δ

We know, $A_p = \dfrac{1}{\sqrt{1+\varepsilon^2}} = \dfrac{1}{\sqrt{2}} \Rightarrow \varepsilon = 1$

And, $A_s = \dfrac{1}{\sqrt{1+\delta^2}} = 0.1 \Rightarrow \delta = 9.95$

Step 2 : Order of the filter

The order of the filter is given by,

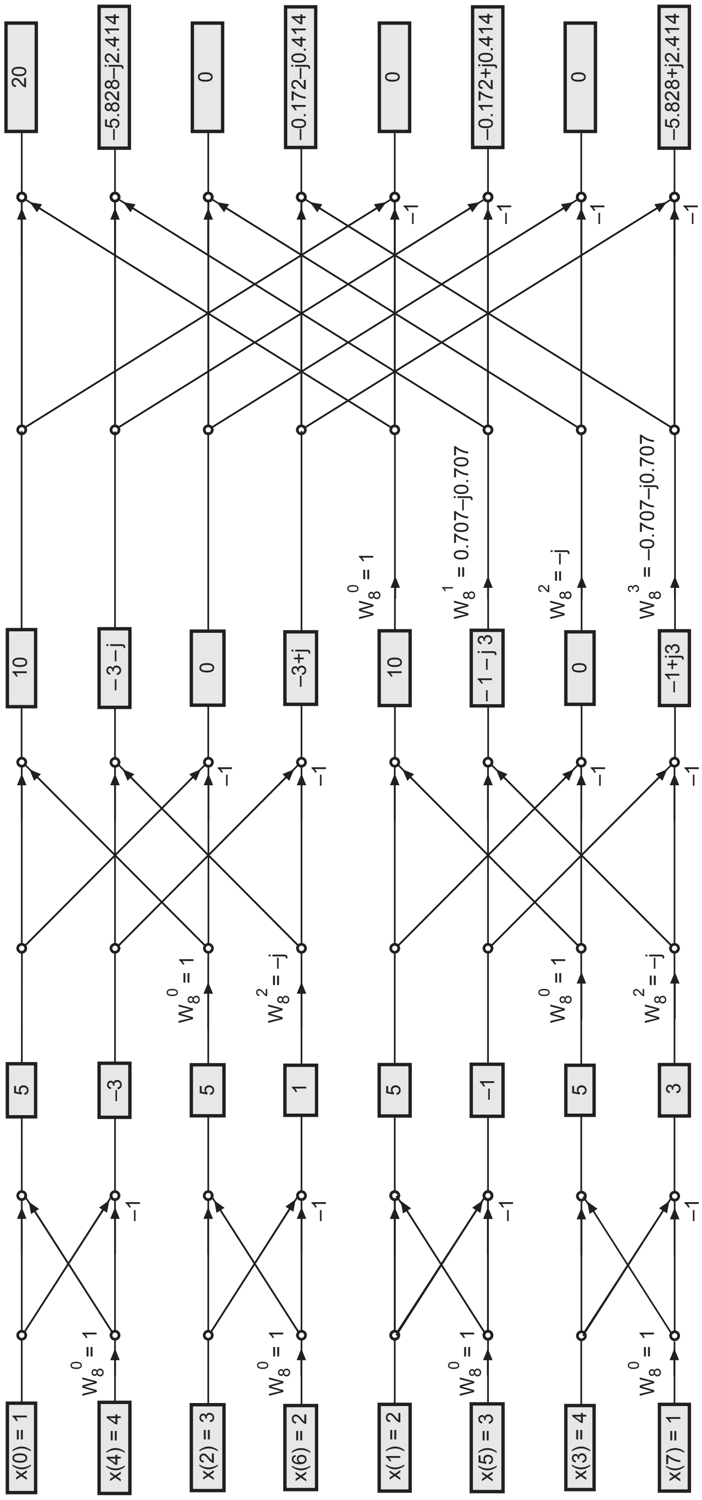


Fig. 2 8-point DFT using radix-2 DIT-FFT algorithm

$$N = \frac{\cosh^{-1}\left(\frac{\delta}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\cosh^{-1}(9.95)}{\cosh^{-1} 2} = 2.269 \cong 3$$

Step 3 : Values μ, a, b and poles

$$\mu = \frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon} = 2.414$$

$$a = \Omega_p \left[\frac{\mu^{\frac{1}{N}} - \mu^{-\frac{1}{N}}}{2}\right] = 2\left[\frac{(2.414)^{\frac{1}{3}} - (2.414)^{-\frac{1}{3}}}{2}\right] = 0.596$$

$$b = \Omega_p \left[\frac{\mu^{\frac{1}{N}} + \mu^{-\frac{1}{N}}}{2}\right] = 2\left[\frac{(2.414)^{\frac{1}{3}} + (2.414)^{-\frac{1}{3}}}{2}\right] = 2.087$$

Now, $\phi_k = \dfrac{(2k+N+1)\pi}{2N} \quad k = 0, 1, 2$

$$\phi_0 = \frac{2\pi}{3} = 120° \; ; \; \phi_1 = \pi = 180° \text{ and } \phi_2 = \frac{4\pi}{3} = 240°$$

k	ϕ_k	$\sigma_k = a\cos\phi_k$	$\Omega = b\sin\phi_k$	$P_k = \sigma_k + \Omega_k$
0	120°	−0.298	1.807	P_0 = −0.298 + j1.807
1	180°	−0.596	0	P_1 = −0.596
2	240°	−0.298	−1.807	P_2 = −0.298 − j1.807

Step 4 : To obtain $H_a(s)$

$$H_a(s) = \frac{k}{(s-s_1)(s-s_1^*)(s-s_2)} = \frac{k}{(s+0.596)(s+0.298-j1.807)(s+0.298+j1.807)}$$

Denominator of $H_a(s)$ will be,

$$D[H_a(s)] = (s+0.596)\,[s^2 + 0.596s + 3.354]$$

Now, $b_0 = 0.596 \times 3.354 = 1.998$

Since N is odd, $k = b_0 = 1.998$

$$\therefore \quad H_a(s) = \frac{1.998}{(s+0.596)(s^2+0.596\,s+3.354)}$$

OR

b) *Design an ideal low pass FIR filter with a frequency response.*

$$H_d(e^{j\omega}) = \begin{cases} 1 & \text{for } -\dfrac{\pi}{2} \le \omega \le \dfrac{\pi}{2} \\ 0 & \text{for } \dfrac{\pi}{2} \le \omega \le \pi \end{cases}$$

Find the values of h(n) for N = 11. Find H(z). Assume rectangular window.
(Refer Example 4.6.10) **[13]**

Q.15 a) *Draw the architecture of TMS320C50 and explain its functional units.*
(Refer section 5.5) **[13]**

OR

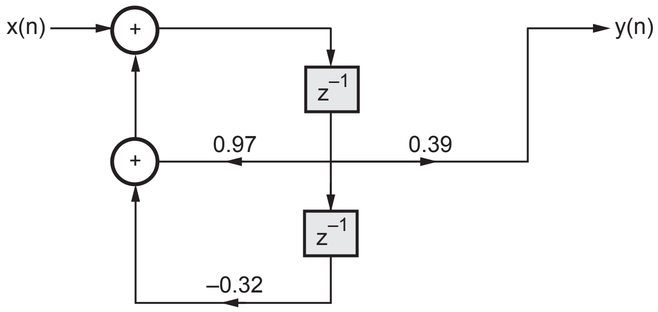
b) *Explain the classification of the instructions in DSP processor with suitable examples.* **(Refer section 5.3)** **[13]**

PART - C (1 × 15 = 15 Marks)

Q.16 a) *Design Butterworth filter using the impulse invariance method for the following specifications :* $0.8 \le \left|H(e^{j\omega})\right| \le 1,\ 0 \le \omega \le 0.2\pi$ *and* $\left|H(e^{j\omega})\right| \le 0.2,\ 0.6\pi \le \omega \le \pi$

Realize the designed filter using diect form II structure.
[Refer Example 4.9.7] **[15]**

Here $H(z) = \dfrac{0.39\,z^{-1}}{1 - 0.97\,z^{-1} + 0.32\,z^{-2}}$



OR

b) *i) How mapping from S-domain to Z-domain is achieved in bilinear transformation ?*
[Refer section 4.8.3] **[8]**

ii) Apply bilinear transformation to $H(s) = \dfrac{2}{(s+1)(s+2)}$.

[Refer Example 4.8.11] **[7]**

❑❑❑

May - 2018 Discrete Time Systems and Signal Processing Semester - IV (EEE) (41001) Regulation 2013	AU Solved Paper

Time : Three Hours] **[Maximum Marks : 100**

Answer ALL Questions

PART A - (10 $\times$ 2 = 20 Marks)

Q.1 *What is aliasing effect ?* **(Refer Two Marks Q.3 of section 1.6)**

Q.2 *List the sampling techniques.*

Ans. : There are three types of sampling techniques,

i) Flat top sampling

ii) Natural sampling

iii) Impulse sampling

Q.3 *What is the inverse z transform of* $H(z) = \frac{2z}{z-1/2}$?

Ans. :
$$H(z) = \frac{2z}{z-1/2} = \frac{2z}{z(1-\frac{1}{2}z^{-1})} = \frac{2}{1-\frac{1}{2}z^{-1}}$$

$$\therefore \quad H(z) = \frac{2}{1-\frac{1}{2}z^{-1}}$$

We have a standard z-transform pair,

$$z\{a^n u(n)\} = \frac{1}{1-az^{-1}} \text{ where } |z| > |a|$$

Here $a = 1/2$

$$\therefore \quad h(n) = (1/2)^n u(n)$$

Q.4 *What is zero padding ?*

Ans. : Consider we have a sequence *y(n)* with its length *M*. Now we want to find *N*-point DFT where $N > M$ of the sequence *x(n)*. In this case we need to add $(N - M)$ zeros to the sequence *y(n)* at its end, and this is called zero padding.

Q.5 *Find the DFT sequence of x(n) = {1, 1, 0, 0}*

Ans. :

$$x_4 = [W_4]x_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

Q.6 *What is meant by radix-4 FFT ?*

Ans. : In radix-4 FFT, the length of the DFT is a power of '4'. The sequence is decimated by the order of 4. This ends up in 4-point decimated sequences of x(n). The DFTs of 4-point sequences are calculated directly. Four 4-point DFTs are combined to have one 16-point DFT.

Q.7 *Obtain the direct form-I realization for the given difference equation*
$y(n) = 0.5y(n-1) - 0.25\, y(n-2) + x(n) + 0.4\, x(n-1)$

Ans. : Fig. 1 shows the direct form-I realization below

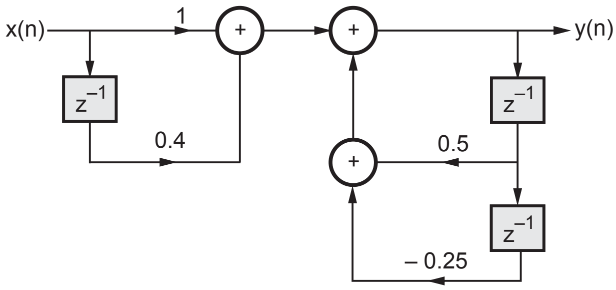


Fig. 1 Direct form-I realization

Q.8 *Distinguish the IIR and FIR filter.* **(Refer Two Marks Q.28 of section 4.11)**

Q.9 *What are the stages in pipelining process ?* **(Refer Two Marks Q.3 of section 5.8)**

Q.10 *Write the applications of commercial digital signal processor.*

Ans. : Below are few of the commercial applications of digital signal processors :

i) Speech recognition ii) Digital communications

iii) Radars and Sonars iv) Seismology and biomedicine

PART B - (5 × 13 = 65 Marks)

Q.11 a) *Explain the classification of continuous time signals with its mathematical representation.* **(Refer section 1.2)** **[13]**

OR

b) *Describe the different types of system and write the condition to state the system with its types.* **(Refer section 1.4)** **[13]**

Q.12 a) *i) Find the Z transform of $x(n) = r^n \cos(n\theta)u(n)$.* **(Refer Example 2.4.4 (i))** **[9]**

ii) State and proof the Parseval's theorem. **(Refer section 2.10.2.10)** **[4]**

OR

b) *i) Find the circular convolution of the two sequences* $x_1(n) = \{1, 2, 2, 1\}$ *and* $x_2(n) = \{1, 2, 3, 1\}$ **[8]**

Ans. : Circular convolution using matrix method :

$$y(n) = x_1(n) \; (N) \; x_2(n) = x_1(n) \; (4) \; x_2(n)$$

$$\therefore \quad y(n) = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \\ 10 \\ 12 \end{bmatrix}$$

$$\therefore \quad y(n) = \{11, 9, 10, 12\}$$

ii) How do you obtain the magnitude and phase response of DTFT ?
(Refer section 2.10.4) **[5]**

Q.13 a) *State and proof any four properties of DFT.* **(Refer section 3.3)** **[13]**

OR

b) *Determine the DFT of the given sequence x(n) = {1, –1, –1, –1, 1, 1, 1, –1} using DIT FFT algorithm.* **[13]**

Ans. :

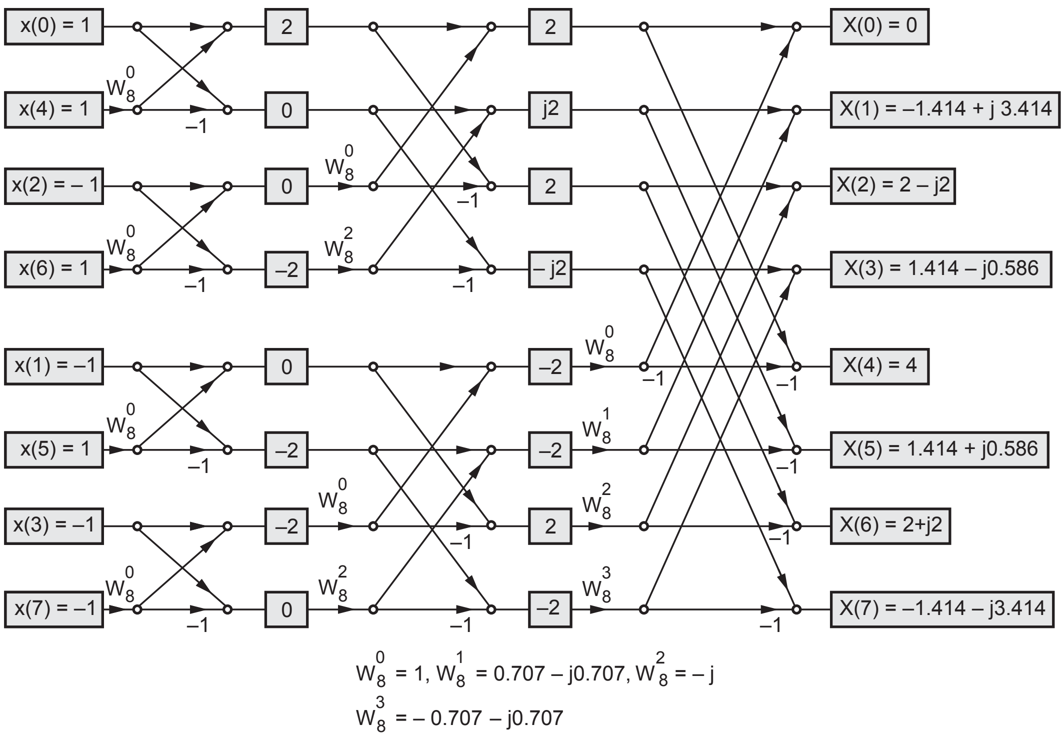


Fig. 2

Q.14 a) *Design a Chebyshev filter for the following specification using bilinear transformation.*

$0.8 \leq |He^{(j\omega)}| \leq 1, \quad 0 \leq \omega \leq 0.2\pi$

$|He^{(j\omega)}| \leq 0.2, 0.6\pi \leq \omega \leq \pi$ **[13]**

Ans. :

i) Given data :

$$A_p = 0.8,\ \omega_p = 0.2\ \pi$$

$$A_s = 0.2,\ \omega_s = 0.6\ \pi$$

ii) Prewarping

$$\Omega = \frac{2}{T} \tan \omega/2 = \tan \omega/2 \quad \text{assuming } \frac{2}{T} = 1$$

$$\therefore \quad \Omega_p = \tan(\omega_p/2) = \tan\left(\frac{0.2\pi}{2}\right) = 0.325 \text{ rad/sec}$$

$$\therefore \quad \Omega_s = \tan(\omega_s/2) = \tan\left(\frac{0.6\pi}{2}\right) = 1.376 \text{ rad/sec}$$

Therefore prewarped specifications are as follows :

$$A_p = 0.8, \quad \Omega_p = 0.325 \text{ rad/sec}$$

$$A_s = 0.2, \quad \Omega_s = 1.376 \text{ rad/sec}$$

iii) Order of the filter :

$$\varepsilon = \sqrt{\frac{1}{A_p^2} - 1} = \sqrt{\frac{1}{(0.8)^2} - 1} = 0.75$$

and

$$\delta = \sqrt{\frac{1}{A_s^2} - 1} = \sqrt{\frac{1}{(0.2)^2} - 1} = 4.9$$

Order is given by,

$$N = \frac{\cosh^{-1}(\delta/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} = \frac{\cosh^{-1}(4.9/0.75)}{\cosh^{-1}\left(\frac{1.376}{0.325}\right)} = \frac{2.564}{2.122}$$

$$= 1.20 \cong 2$$

iv) Poles of $H_a(s)$:

$$\mu = \frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon} = \frac{1+\sqrt{1+(0.75)^2}}{0.75} = 3$$

$$a = \Omega_p\left(\frac{\mu^{1/N} - \mu^{1/N}}{2}\right) = 0.325\left(\frac{3^{1/3} - 3^{-1/3}}{2}\right) = 0.12$$

$$b = \Omega_p\left(\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right) = 0.325\left(\frac{3^{1/3} + 3^{-1/3}}{2}\right) = 0.347$$

Now $\phi_k = \left(\frac{2k+N+1}{2N}\right)\pi,\ k = 0, 1, \ldots, N-1$

For $N = 2$, $\phi_k = \left(\frac{2k+3}{4}\right)\pi,\ k = 0, 1$

k	ϕ_k	$\sigma_k = a\cos\phi_k$	$\Omega_k = b\sin\phi_k$	$p_k = \sigma_k + j\Omega_k$
0	$\phi_0 = 3\pi/4$	$\sigma_0 = -0.084$	$\Omega_0 = 0.245$	$p_0 = -0.084 + j\,0.245$
1	$\phi_1 = 5\pi/4$	$\sigma_1 = -0.084$	$\Omega_1 = -0.245$	$p_1 = -0.084 - j\,0.245$

v) System function

$$H_s(s) = \frac{k}{(s-s_1)(s-s_1^*)} = \frac{k}{(s+0.084-\mathrm{j}0.245)(s+0.084+j\,0.245)}$$

Denominator of

$$H_s(s) = (s+0.084)^2 + (0.245)^2$$

$$= s^2 + 0.0070 + 0.168s + 0.0600$$

$$= s^2 + 0.168\,s + 0.0670$$

Hence, $b_0 = 0.0670$

Here $N = 2$, $k = \frac{b_0}{\sqrt{1+\varepsilon^2}} = \frac{0.670}{\sqrt{1+(0.75)^2}} = 0.536$

$\therefore$ $$H_a(s) = \frac{0.536}{s^2 + 0.168\,s + 0.670}$$

vi) Bilinear transformation

$$H(z) = H_a(s)\Big|_{s=\left[\frac{1-z^{-1}}{1+z^{-1}}\right]} \quad \text{[Here } 2/T = 1\text{]}$$

$$= \frac{0.536}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.168\left[\frac{1-z^{-1}}{1+z^{-1}}\right] + 0.670}$$

On evaluating it further

$$H(z) = \frac{0.052(1+z^{-1})^2}{1-1.3480\,z^{-1}+0.608\,z^{-2}}$$

OR

b) *Design a filter using hamming window with the specification N = 7 of the system*

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & \frac{-\pi}{4} \le \omega \le \frac{\pi}{4}, \\ 0, & \frac{-\pi}{4} \le \omega \le \pi \end{cases}$$

[13]

Ans. :

Step 1 : To obtain $h_d(n)$

$$h_d(n) = \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{-j3\omega} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{j\omega(n-3)} d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-3)}}{j(n-3)}\right]_{-\pi/4}^{\pi/4}$$

On evaluating it further, we get

$$h_d(n) = \frac{\sin\frac{\pi}{4}(n-3)}{\pi(n-3)} \text{ for } n \ne 3$$

for n = 3,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\omega(0)} d\omega = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} d\omega = 1/4$$

$$\therefore \qquad h_d(n) = \begin{cases} \frac{\sin\frac{\pi}{4}(n-3)}{\pi(n-3)} & \text{for } n \ne 3 \\ 1/4 & n = 3 \end{cases}$$

Step 2 : To apply windowing to $h_d(n)$ to get h(n)

Here Hamming window is to be used, It is given as,

$$W_H(n) = \begin{cases} 0.54 - 0.46\cos\dfrac{2\pi n}{M-1} & \text{for } n = 0, 1, \ldots, M-1 \\ 0 & \text{otherwise} \end{cases}$$

for $M = 7$,

$$W_H(n) = 0.54 - 0.46\cos\frac{2\pi n}{6} \text{ for n = 0, 1, 2, ..., 6}$$

$$= 0.54 - 0.46\cos\frac{\pi n}{3} \text{ for n = 0, 1, 2, ..., 6}$$

Following table shows $h_d(n)$, $W_H(n)$ and $h(n)$ as per above equations.

n	$h_d(n)$	$W_{Hamm}(n)$	$h(n) = h_d(n) \cdot W_{Hamm}(n)$
0	0.075	0.08	0.006
1	0.159	0.31	0.04929
2	0.225	0.77	0.17325
3	0.25	1	0.25
4	0.225	0.77	0.17325
5	0.159	0.31	0.4929
6	0.075	0.08	0.006

Q.15 a) *Explain the various types of addressing modes of digital signal processor with suitable example.* **(Refer section 5.3)** **[13]**

OR

b) *Draw the structure of central processing unit and explain each unit with its function.* **(Refer section 5.5.2.2)** **[13]**

PART C - (1 × 15 = 15 Marks)

Q.16 a) *Determine the frequency response $H(e^{j\omega})$ for the given system and plot magnitude and phase response, $y(n) + \frac{1}{4}y(n-1) = x(n) + x(n-1)$* **[15]**

Ans. : i) Take z-transform on both sides,

$$Y(z) + \frac{1}{4}z^{-1}Y(z) = X(z) + z^{-1}X(z)$$

$$\therefore Y(z)\left[1 + \frac{1}{4}z^{-1}\right] = X(z)[1 + z^{-1}]$$

$$\therefore \qquad H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1+\frac{1}{4}z^{-1}}$$

Now $\qquad z = e^{j\omega}$

$\therefore$ frequency response,

$$H(e^{j\omega}) = \frac{1+e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega}}$$

ii) Magnitude response,

$$\left|H(e^{j\omega})\right| = \frac{1+\cos\omega - j\sin\omega}{1+\frac{1}{4}\cos\omega - \frac{j}{4}\sin\omega}$$

$$= \frac{[(1+\cos\omega)^2 + \sin^2\omega]^{1/2}}{\left[\left(1+\frac{1}{4}\cos\omega\right)^2 + \frac{1}{16}\sin^2\omega\right]^{1/2}}$$

$$= \frac{[1+\cos^2\omega + 2\cos\omega + \sin^2\omega]^{1/2}}{\left[1+\frac{1}{16}\cos\omega^2 + \frac{1}{2}\cos\omega + \frac{1}{16}\sin^2\omega\right]}$$

$$= \frac{[2(1+\cos\omega)]^{1/2}}{\left[\frac{17}{16}+\frac{1}{2}\cos\omega\right]^{1/2}}$$

$$= \frac{2\cos\omega/2}{[1.0625+0.5\cos\omega]^{1/2}}$$

iii) Phase response,

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{\sin\omega}{1+\cos\omega}\right) - \tan^{-1}\left[\frac{(-0.25\sin\omega)}{(1+0.25\cos\omega)}\right]$$

Now

$$\tan^{-1}\left(\frac{\sin\omega}{1+\cos\omega}\right) = \tan^{-1}\left(\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\cos^2\left(\frac{\omega}{2}\right)}\right)$$

$$= \tan^{-1}\left[\tan\frac{\omega}{2}\right] = \frac{\omega}{2}$$

And $$\angle H(e^{j\omega}) = \frac{\omega}{2} - \tan^{-1}\left[\frac{-0.25\sin\omega}{1+0.25\cos\omega}\right]$$

Following table gives the values of magnitude and phase for various angles.

ω	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$\lvert H(e^{j\omega})\rvert$	1.6	1.55	1.371	0.91	0
$\angle H(e^{j\omega})$	0	0.542	1.03	1.389	1.570

Fig. 3 shows the magnitude and phase response as calculated above

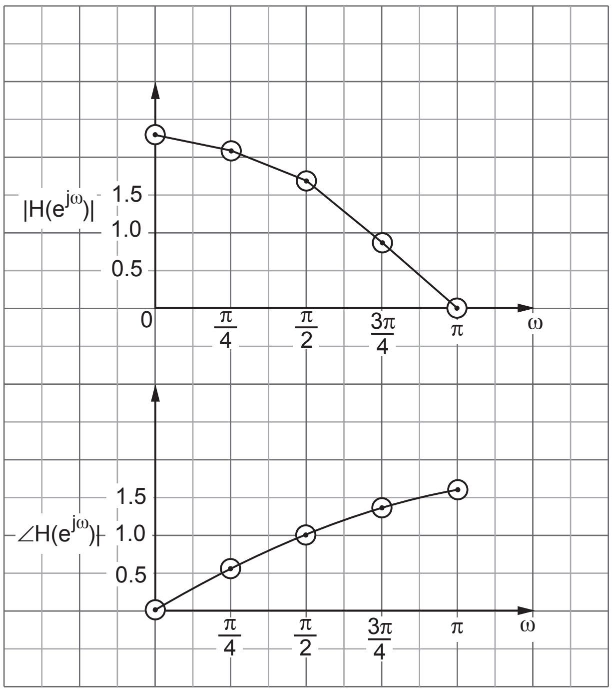


Fig. 3 Magnitude and Phase response

OR

b) *Determine the impulse response of the given difference equation $y(n) = y(n-1) + 0.25\,y(n-2) + x(n) + x(n-1)$. Plot the pole zero pattern and check its stability.* **[15]**

Ans. :

i) To obtain impulse response

Take z-transform of given difference equation,

$$Y(z) = z^{-1}Y(z) + 0.25\, z^{-2}Y(z) + X(z) + z^{-1}X(z)$$

$$\therefore\ Y(z)[1 - z^{-1} - 0.252^{-2}] = X(z)[1 + z^{-1}]$$

$$\therefore\quad H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 - z^{-1} - 0.25\, z^{-2}}$$

$$\therefore\quad H(z) = \frac{z(z+1)}{z^2 - z - 0.25}$$

$$\therefore\quad \frac{H(z)}{z} = \frac{(z+1)}{z^2 - z - 0.25} = \frac{(z+1)}{(z-1.207)(z+0.207)}$$

$$\frac{H(z)}{z} = \frac{A_1}{(z-1.207)} + \frac{A_2}{(z+0.207)}$$

values for A_1 and A_2 arc obtained as follows :

$$A_1 = (z-1.207)\frac{H(z)}{z}\bigg|_{z=1.207} = \frac{z+1}{z+0.207}\bigg|_{z=1.207}$$

$$= \frac{2.207}{1.414} = 1.56$$

$$A_2 = (z+0.207)\frac{H(z)}{z}\bigg|_{z=-0.207} = \frac{z+1}{z-1.207}\bigg|_{z=-0.207}$$

$$= \frac{0.793}{-1.414} = -\,0.56$$

$$\therefore\quad \frac{H(z)}{z} = \frac{1.56}{(z-1.207)} - \frac{0.56}{(z+0.207)}$$

$$H(z) = \frac{1.56\, z}{z-1.207} - \frac{0.56\, z}{z+0.207}$$

$$H(z) = \frac{1.56}{1-1.207\, z^{-1}} - \frac{0.56}{1-0.207\, z^{-1}}$$

Taking inverse z-transform,

$$h(n) = 1.56(1.207)^n u(n) - 0.56(0.207)^n u(n)$$

ii) To obtain the pole-zero pattern and stability

We know that,

$$H(z) = \frac{z(z+1)}{(z-1.207)(z+0.207)}$$

zeros are at, $z_1 = 0$, $z_2 = -1$ and poles are at, $p_1 = 1.207$, $p_2 = -0.207$.

Fig. 4 shows the pole-zero plot.

Stability : The system is stable, if all its pole lie inside the unit circle. Here observe that one pole $p_1 = 1.207$ lies outside the unit circle. Hence this system is *unstable*.

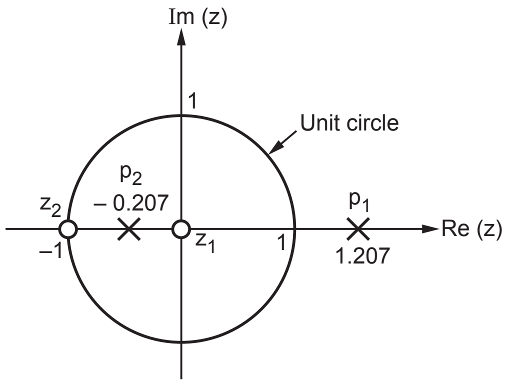


Fig. 4

❑❑❑

December - 2018 **Discrete Time Systems and Signal Processing** **Semester - IV (EEE) (20456) Regulation 2013**	**AU** **Solved Paper**

Time : Three Hours] **[Maximum Marks : 100**

Answer ALL Questions

PART - A (10 × 2 = 20 Marks)

Q.1 *Define spectral density.*

Ans. : The power or energy content of the signal with respect to frequency, is called spectral density. It's unit is watts per Hz. The spectral density can be Power Spectral Density (PSD) or Energy Specral Density (ESD).

Q.2 *What is Nyquist rate ?* **(Refer Two Marks Q.22 of section 1.6)**

Q.3 *Find the stability of the system whose impulse response $h(n) = 2^n u(n)$.*

Ans. : Consider,

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} [(2)^2 u(n)] = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 \ldots\ldots + 2^{\infty} = \infty$$

Since $\sum_{n=-\infty}^{\infty} |h(n)|$ is not absolutely summable, system is **unstable**.

Q.4 *What is relation between z-transform and DTFT ?*
(Refer Two Marks Q.20 of section 2.11)

Q.5 *Find the DFT sequence of x(n) = (1, 1, 0, 0).* **(Refer Q.5 of May-2018)**

Q.6 *State and proof the circular frequency shifting property of DFT.*
(Refer section 3.3.8)

Q.7 *Draw the direct from I realization for the given system*
$y(n) = -0.1\, y(n-1) + 0.2y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$

Ans. : Fig. 1 shows direct form I.

Q.8 *Define warping effect.* **(Refer Two Marks Q.3 of section 4.11)**

Q.9 *What are buses used in DSP processor ?* **(Refer Two Marks Q.8 of section 5.8)**

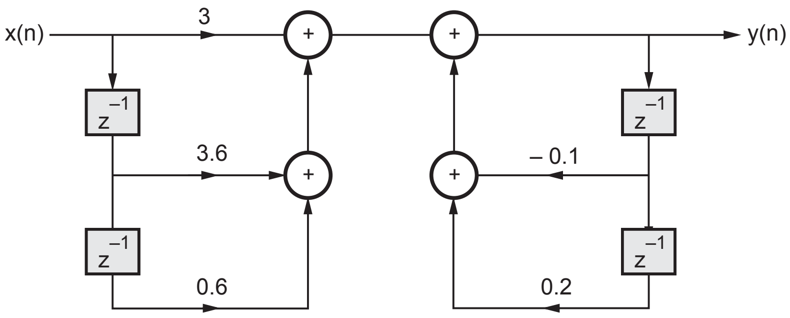


Fig. 1 Direct form I

Q.10 *List the features to select the digital signal processor.*
(Refer Two Marks Q.13 of section 5.8)

PART - B (5 × 13 = 65 Marks)

Q.11 a) *i) Illustrate the condition of the system to be causal and lineartity. Check the same for the given system* $y(n) = x(n) + \frac{1}{x(n-1)}$. **[7]**

Ans. : Please refer section 1.4.3 and 1.4.4 for conditions of causality and linearity.

Causality : Here

$$y(n) = x(n) + \frac{1}{x(n-1)}$$

The output at n^{th} time instant depends upon n^{th} or $(n-1)^{th}$ input. This means present output depands upon present or past inputs. Hence the system is *Causal.*

Linearity : Outputs due to two inputs $x_1(n)$ and $x_2(n)$ will be,

$$y_1(n) = T\{x_1(n)\} = x_1(n) + \frac{1}{x_1(n-1)} \text{ and}$$

$$y_2(n) = T\{x_2(n)\} = x_2(n) + \frac{1}{x_2(n-1)}$$

Linear combination of above two outputs will be,

$$y'_3(n) = a_1 y_1(n) + a_2 y_2(n)$$

$$= a_1 x_1(n) + a_1 \frac{1}{x_1(n-1)} + a_2 x_2(n) + a_2 \frac{1}{x_2(n-1)}$$

Response of the system to linear combination of two inputs will be,

$$y_3(n) = T\{a_1 x_1(n) + a_2 x_2(n-1)\}$$

$$= a_1 x_1(n) + a_2 x_2(n-1) + \frac{1}{a_1 x_1(n-1) + a_2 x_2(n-1)}$$

Since $y_3(n) \neq y'_3(n)$, the system is *nonliear.*

ii) Check the time invariant and stability of the given system y(n) = cos x(n).
(Refer Example 1.4.5 (i)) **[6]**

OR

b) *i) Determine the values of power and energy of the given signal*
$x(n) = \sin\left(\frac{\pi}{4} - n\right)$

[5]

Ans. :

$$\mathbf{P} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} [x(n)]^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left[\sin\left(\frac{\pi}{4} - n\right)\right]^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1-\cos 2\left(\frac{\pi}{4} - n\right)}{2}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1}{2} - \underbrace{\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1}{2}\cos\left[2\left(\frac{\pi}{4} - n\right)\right]}_{\text{Summation of cosine wave over complete cycle which will be zero}}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \cdot \frac{1}{2} \cdot (2N+1) = \frac{1}{2}$$

Since power is finite, it will be power signal.

ii) Explain the types of signals with its mathematical expression and neat diagram.
(Refer section 1.2)

[8]

Q.12 a) *Find the inverse z-transform of*
$X(z) = \dfrac{z^3 + z^2}{(z-1)(z-3)} ROC|z| > 3$ **[13]**

Ans. :

Step 1 : $$X(z) = \frac{z^3 + z^2}{(z-1)(z-3)}$$

$$\therefore \quad \frac{X(z)}{z} = \frac{z^2 + z}{(z-1)(z-3)}$$

On evaluating it further,

$$\frac{X(z)}{z} = 1 + \frac{5z-3}{(z-1)(z-3)} = 1 + \text{A(z)}$$

Step 2 : Partial fraction for A(z),

$$\frac{5z-3}{(z-1)(z-3)} = \frac{A_1}{z-1} + \frac{A_2}{z-3}$$

$$\therefore \quad A_1 = (z-1)\frac{(5z-3)}{(z-1)(z-3)}\bigg|_{z=1} = -1$$

$$A_2 = (z-3)\frac{(5z-3)}{(z-1)(z-3)}\bigg|_{z=3} = 6$$

$$\therefore \quad A(z) = \frac{-1}{z-1} + \frac{6}{z-3}$$

Step 3 : $$\frac{X(z)}{z} = 1 - \frac{1}{z-1} + \frac{6}{z-3}$$

$$\therefore \quad \text{X(z)} = z - \frac{z}{z-1} + \frac{6z}{z-3}$$

Taking inverse z-transform,

$$x(n) = \delta(n+1) - u(n) + 6(3)^n u(n)$$

OR

b) *Find the frequency response for the given sequence and plot its magnitude response and phase response x(n) =* $\begin{cases} 1 & \text{for } n = -2-1,0,1,2 \\ 0 & \text{otherwise} \end{cases}$ **[13]**

Ans. : i) The **frequency response** is given by,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=-2}^{\infty} x(n)e^{-j\omega n}$$

$$= x(-2)e^{j2\omega} + x(-1)e^{j\omega} + x(0) + x(1)e^{-j\omega} + x(2)e^{-2j\omega}$$

$$= e^{2j\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-2j\omega}$$

$$= 1 + 2\cos\omega + 2\cos 2\omega \qquad \because [\frac{e^{j\theta} + e^{-j\theta}}{2} - \cos\theta \text{ Euler's identity}]$$

ii) Magnitude response,

$$|X(\omega)| = |1 + 2\cos\omega + 2\cos 2\omega|$$

iii) Phase response,

$$\angle X(\omega) = 0 \text{ for } X(\omega) > 0$$

$$= \pm\pi \text{ for } X(\omega) < 0$$

Now, calculating the value of $X(\omega)$, $|X(\omega)|$ and $\angle X(\omega)$,

for $\omega = 0$ $\quad X(\omega) = 1 + 2 + 2 = 5$

$|X(\omega)| = 5$ and $\angle X(\omega) = 0$

for $\omega = \frac{\pi}{6} = 30°$

$\therefore \quad X(\omega) = 1 + 2\cos 30° + 2\cos 60° = 3.73$

$|X(\omega)| = 3.73$ and $\angle X(\omega) = 0°$

Similarly on calculating for the different value of ω, we get the values as shown in table.

ω	0	$\pi/6$	$\pi/4$	$\pi/2$	$3\pi/4$	$5\pi/6$	π
$X(\omega)$	5	3.73	2.414	–1	–0.414	0.268	1
$\|X(\omega)\|$	5	3.73	2.414	1	0.414	0.268	1
$\angle X(\omega)$	0	0	0	$-\pi$	$-\pi$	0	0

The magnitude and phase response is as shown below.

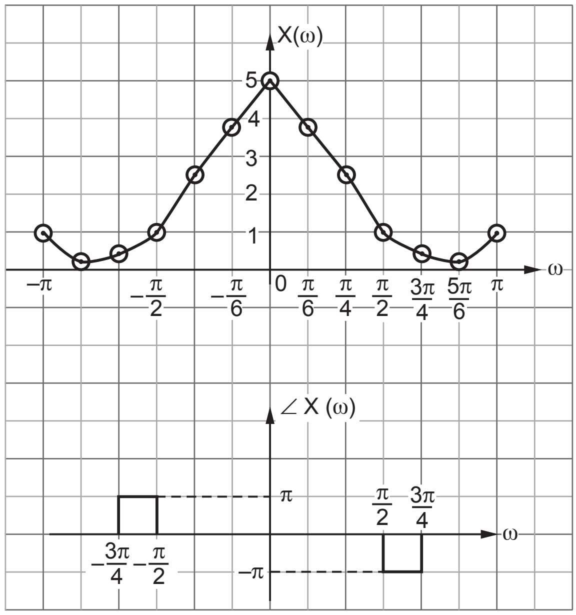
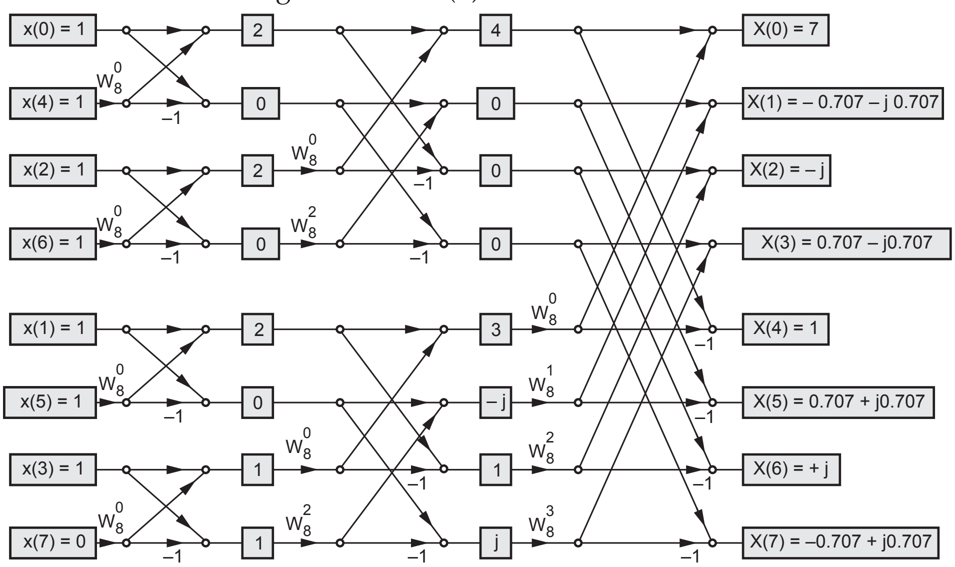


Fig. 2 Magnitude and Phase plot

Q.13 a) *Determine the DFT of a sequence x(n) = {1,1,1,1,1,1,1,0} using DIT algorithm.* **[13]**

Ans. : Below is DIT-FFT algorithm for $x(n)$



Here, $W_8^0 = 1$, $W_8^1 = 0.707 - j0.707$

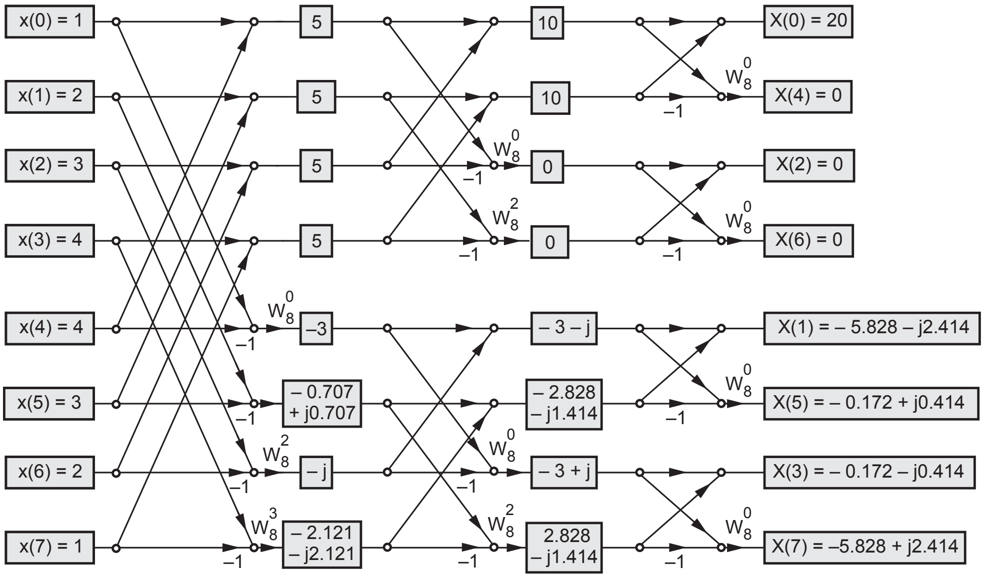
$W_8^2 = -j$, $W_8^3 = -0.707 - j0.707$

Fig. 3 DIT-FFT algorithm

OR

b) *Determine the DFT of the given sequence x(n) = {1, 2, 3, 4, 4, 3, 2, 1} using DIF FFT algorithm.* **[13]**

Ans. :



$W_8^0 = 1$, $W_8^1 = 0.707 - j0.707$

$W_8^2 = -j$, $W_8^3 = -0.707 - j0.707$

Fig. 4 DIF-FFT algorithm

Q.14 a) *Determine the order of the filter using Chebyshev approximation for the given specification* $\alpha_p = 3\,dB$, $\alpha_s = 16\,dB$, $f_p = 1\,kHz$ *and* $f_s = 2\,kHz$. *Find H(s).* **[13]**

Ans. : Step 1 : Specifications

Here $f_p = 1000$ Hz Hence $\Omega_p = 2\pi f_p = 2000\,\pi$ rad/sec

and $f_s = 2000$ Hz hence $\Omega_s = 4000\,\pi$ rad/sec

Thus, $A_p = 3$dB, $\Omega_p = 2000\,\pi$ rad/sec

$A_s = 16$ dB, $\Omega_s = 4000\,\pi$ rad/sec

Step 2 : Order of the filter

$$N = \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1A_s}-1}{10^{0.1A_p}-1}}}{\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\times 16}-1}{10^{0.1\times 3}-1}}}{\cosh^{-1}\left(\dfrac{4000\pi}{2000\pi}\right)} = 1.91 \cong 2$$

Step 3 : Values of ε, μ, a, b

$$\varepsilon = \sqrt{10^{0.1A_p} - 1} = \sqrt{10^{0.1\times 3} - 1} = 0.998$$

$$\mu = \frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon} = \frac{1+\sqrt{1+0.998^2}}{0.998} = 2.417$$

$$a = \Omega_p\left(\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right) = 2000\pi\left[\frac{2.417^{1/2} - 2.417^{-1/2}}{2}\right] = 2863.4$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 2000\pi\left[\frac{2.417^{1/2} + 2.417^{-1/2}}{2}\right] = 6904.8$$

Step 4 : Poles

It is given by $P_k = \sigma_k + j\Omega k$, **k** = 0, 1

Following table calculates the poles fork = 0 and 1.

k	$\phi_k = \frac{(2K+N+1)\pi}{2N}$	$\sigma_k = a\cos\phi_k$	$\Omega_k = b\sin\phi_k$	$p_k = \sigma_k + j\Omega k$
0	$\phi_0 = \frac{(0+2+1)\pi}{4}$ $= 3\pi/4$	σ_0 = 2863.4 cos $3\pi/4$ = –2024.7	Ω_0 = 6904 sin $3\pi/4$ = 4882.4	p_0 = –2024.7 + j 4882.4
1	$5\pi/4$	σ_1 = – 2024.7	Ω_1 = –4882.4	p_1 = –2024.7 –j 4882.4

$$H_a(s) = \frac{k}{(s-p_0)(s-p_1)} = \frac{k}{(s+2024.7-j\,4882.4)\,(s+2024.7+j\,4882.4)}$$

$$= \frac{k}{(s+2024.7)^2 + (4882.4)^2}$$

Here $\quad b_0 = (2024.7)^2 + (4882.4)^2 = 27937239.85$

and $\quad \boldsymbol{k} = \dfrac{b_0}{\sqrt{1+\varepsilon^2}}$ for N even $= \dfrac{27937239.85}{\sqrt{1+0.998^2}} = 19.77\times 10^6$

$$\therefore \quad H_a(s) = \frac{19.77\times 10^6}{(s+2024.7)^2 + (4882.4)^2}$$

is equation for analog Chebyshev filter.

OR

b) *Design an ideal high pass filter using Hanning window with the specification N = 11 of the system* $H_d(e^{j\omega}) = 1$ *for* $\frac{\pi}{4} \leq |\omega| \leq \pi$ *; otherwise zero* $|\omega| \leq \frac{\pi}{4}$ **[13]**

Ans. : i) To obtain $h_d(n)$

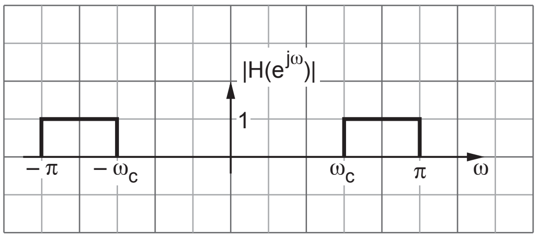


Fig. 5 High pass filter

The frequency response is given by,

$$H_d(e^{j\omega}) = \begin{cases} e^{j\omega\tau} & \text{for } \omega_c \leq |\omega| \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \qquad \text{...By inverse Fourier transform}$$

$$= \frac{1}{2\pi} \left\{ \int_{-\pi}^{-\omega_c} e^{-j\omega\tau} e^{j\omega n} d\omega + \int_{\omega_c}^{\pi} e^{-j\omega\tau} e^{j\omega n} d\omega \right\}$$

On simplifying, we get

$$h_d(n) = \begin{cases} \dfrac{\sin \pi (n-\tau) - \sin \omega_c (n-\tau)}{\pi(n-\tau)} & \text{for } n \neq \tau \\ 1 - \dfrac{\omega_c}{\pi} & \text{for } n = \tau \end{cases}$$

Here $\boldsymbol{M} = \boldsymbol{N} = 11$ hence $\tau = \frac{M-1}{2} = \frac{11-1}{2} = 5$

$$\therefore \quad h_d(n) = \begin{cases} \dfrac{\sin \pi (n-5) - \sin \frac{\pi}{4}(n-5)}{\pi (n-5)}, & n \neq 5 \\ 1 - \dfrac{\pi/4}{\pi}, & n = 5 \end{cases}$$

ii) To obtain ( ), ( ) ( )

Hanning window is given as

$$w(n) = \frac{1}{2}\left(1-\cos\frac{2\pi n}{M-1}\right) = \frac{1}{2}\left[1-\cos\left(\frac{2\pi n}{10}\right)\right] \qquad n_1 = 0, 1, 2, ...,10$$

$$w(n) = \frac{1}{2}\left(1-\cos\frac{\pi n}{5}\right) \qquad n = 0, 1, 2, ...,10$$

Following table shows the calculations of $h_d(n)$, $w_H(n)$ and $h(n) = h_d(n) \cdot w_H(n)$

n	$h_d(n)$	$w_{Han}(n)$	$h(n)$
0, 10	0.045	0	0
1, 9	0	0.095	0
2, 8	– 0.075	0.345	– 0.02591
3, 7	– 0.159	0.654	– 0.1040
4, 6	– 0.225	0.904	– 0.203
5	0.75	1	0.75

Q.15 a) *Explain the vaarious types of addressing modes of digital signal processor with suitable example.* **(Refer section 5.3)** **[13]**

OR

b) *Draw the structure of central processing unit and explain each unit with its function.* **(Refer section 5.5.2.2)** **[13]**

PART - C (1 × 15 = 15 Marks)

Q.16 a) *Design an ideal bandpass filter with a frequency response*

$$H_d(e^{j\omega}) = \begin{cases} 1 & \text{for } \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$ *find the values of h(n) for N = 11 and plot the frequency response.* **[15]**

Ans. : **i) To obtain $h_d(n)$**

The given bandpass filter has a passband from $\omega_{c_1} = \frac{\pi}{4}$ to $\omega_{c_2} = \frac{3\pi}{4}$ rad/sample.

The desired unit sample response of the ideal bandpass filter is given by equation (4.6.20) as,

$$h_d(n) = \begin{cases} \dfrac{\sin \omega_{c_2}(n-\tau) - \sin \omega_{c_1}(n-\tau)}{\pi(n-\tau)} & \text{for } n \neq \tau \\ \dfrac{\omega_{c_2} - \omega_{c_1}}{\pi} & \text{for } n = \tau \end{cases}$$

Here $\tau = \dfrac{M-1}{2} = \dfrac{11-1}{2} = 5$, Putting values in above equation,

$$h_d(n) = \begin{cases} \dfrac{\sin\left[\dfrac{3\pi(n-5)}{4}\right] - \sin\left[\dfrac{\pi(n-5)}{4}\right]}{\pi(n-5)} & \text{for } n \neq 5 \\ \dfrac{\dfrac{3\pi}{4} - \dfrac{\pi}{4}}{\pi} = \dfrac{1}{5} & \text{for } n = 5 \end{cases}$$

ii) To obtain h(n) by windowing

Here since rectangular window is given,

$$h(n) = h_d(n) \quad \text{for} \quad 0 \leq n \leq M-1$$

i.e. $$h(n) = h_d(n) \quad \text{for} \quad 0 \leq n \leq 10$$

Following are the values calculated as per above equation :

$h(0) = 0$ $h(6) = 0$

$h(1) = 0$ $h(7) = -0.3183$

$h(2) = 0$ $h(8) = 0$

$h(3) = -0.3183$ $h(9) = 0$

$h(4) = 0$ $h(10) = 0$

$h(5) = \dfrac{1}{5}$

OR

b) *Compute the response of the system $y(n) = 0.7y(n-1) - 0.12\,y(n-2) + x(n-1) + x(n-2)$ to input $x(n) = nu(n)$. Is the system stable ?*
(Refer Example 2.9.7) **[15]**

❑❑❑

Notes