Subspaces

- to prove S is a subspace for V, S must be
  - closed under $\odot$
  - closed under $\oplus$
- **NOTE**: a quick way to disprove is if the $\vec{0} \notin S$
- also, closed just means preforming the operation results in an object inside the same space

Basis and Dimension

- finding a basis means finding the general element and iterating through parameters setting one to 1 and the rest to 0
- dimension is the number of vectors is the basis

Linear Dependence

- if the only sol to $c_1 v_1 + \cdots + c_n v_n = 0$ is $c_1 = \cdots = c_n = 0$
- set up a homogenous system of equations and if infinite solutions LD, unique sol LI
- any set that includes $\vec{0}$ is LD
- let $v_1 ... v_k$ be a set of vectors in V
- let $\dim(V) = n$
- Case 1. $k < n$
  - Not a spanning set. Not a basis.
- Case 2. $k > n$
  - LD. Not a basis.
- Case 3. $k = n$
  - if spanning, LI, basis
  - if not span, LD, not a basis
  - so have to check span or LI
- wronskian: if you can find one value for which the wronskian is non-zero it's LI

Span

- to check is $v \in$ span of V unravel the vectors into matrices and if it's consistent (ie. has solution(s)) it's in the span
- if you have less vectors than dimension automatically not a aspanning set

Row/Col Space

- Let A be and $m \times n$ matrix
- `def`:
  - $rowspace(A)$ is span of rows, subspace of $\mathbb{R}^n$
  - $colspace(A)$ is span of columns, subspace of $\mathbb{R}^m$
  - $nullspace(A)$ is sols to $Ax = 0$, subspace of $\mathbb{R}^n$
  - note: $\dim(nullspace(A))$ is called the $nullity(A)$
  - $\dim(rowspace(A)) = \dim(colspace(A)) = rank(A)$
- How to find basis:
  - row: put A into ref use non-zero rows
  - col: put A into ref, use **ORIGINAL** cols corresponding to pivot position
  - null: put A into ref and solve the homogenous system
- `rank nullity thm`: let A be $m \times n$. Then $rank(A) + nullity(A) = n$
  - note: $\dim(nullspace(A))$ is called the $nullity(A)$

Inner Product

- define an `inner product` to be a mapping $< u, v > \Rightarrow \mathbb{R}$. The input is 2 vectors and the output is a scalar that satisfies 4 properties:
  - $< u, u > \geq 0$ **AND** $< u, u >= 0 \Leftrightarrow u = 0_V$ (the subscript indicates $\vec{0} \in V$)
  - $< u, v >=< v, u >$

- $< ku, v >= k < u, v >$
- $< u + v, w >=< u, w > + < v, w >$

- def: let V be an inner product space. Then the norm of a vector u is $\|u\| = \sqrt{< u, u >}$
- `orthogonal`: let v, w be vectors in IP space. if $< v, w >= 0$ then v, w are orthogonal ($\perp$)

Gram Schmidt Process

- `def`: projection of v onto u: $P(v, u) = \frac{<v,u>}{<u,u>} u$
- let $v_1 \dots v_n$ be a basis for IP space V
- construct $u_1 \dots u_n$ tp be an orthogonal basis for the IP space V
  - $u_1 = v_1$
  - $u_2 = v_2 - P(v_2, u_1)$
  - $u_3 = v_3 - P(v_3, u_2) - P(v_3, u_1)$
  - $\vdots$
  - $u_n = v_n - P(v_n, u_{n-1}) - \dots - P(v_n, u_1)$
- `noramlize`: divide each basis element by its magnitude

Linear Transformations

- `def`: let V, W be vector spaces. Then a mapping T is called a linear transformation written as $T : V \to W$ if
  - $T(v_1 + v_2) = T(v_1) + T(v_2)$
  - $T(kv) = kT(v)$

Kernel, Range

- `kernel`: to find the kernel, set $T(v) = 0_w$
  - unravel into matrix and solve
  - a non trivial solution means the kernel has something other than $\vec{0}$
- $T : V \to W$. T is a LT
  - V is the domain
  - W is the co-domain
- $kr(T) \subseteq V$
- $kr(T)\{v \in V : T(v) = 0_w\}$
  - everything that gets mapped to 0
  - subspace of V always contains $\vec{0}$
- $Rng(T) \subseteq W$
- $Rng(T) = \{w \in W : \exists v \in V w/T(v) = w\}$
  - everything in the co-domain which is covered by the transformation
  - solving for Rng means first solving Ker, use ranl nullity thm to solve for dim, is general $T(v)$ to get Rng, and get the basis by inputting 0 and 1 for parameters in Rng
  - if gen elem has more params then dimension, rewrite
- `general rank nullity thm`: $\dim(ker(T)) + \dim(Rng(T)) = \dim(V)$
  - **NOTE**: $\dim(ker(T)) \leq \dim(V)$
  - $\dim(Rng(T)) \leq \dim(W)$
- `1-to-1`: $T : V \to W$ is 1-to-1 if $T(v_1) = T(v_2) \Rightarrow v_1 = v_2$
  - each element of the domain, when transformed, maps to a single distinct element of the co-domain. this only happens if the kernel is just 0
  - $ker(T) = \{0_V\}$
  - $\dim(ker(T)) = 0$
- `onto`: $T : V \to W$ is ontoi if $\forall w \in W \exists v \in V : T(v) = w$ (ae. domain when transformed covers the entire co-domain)
  - $Rng(T) = W$
  - $\dim(Rng(T)) = \dim(W)$