Constant Coefficient Differential Equations

- DE with constants as the coefficients to the derivatives of y so of this form: $c_n y^{(n)} + c_{n-1}y^{(n-1)} + \cdots + c_1 y' + c_0 y = F(x)$

Homogenous

- just means $F(x) = 0$

How to Solve

- Re-write the derivatives of $y^{(n)}$ with a differential operator to the same power D^n
- treat as polynomial and factor
  - if degree > 2 then factoring may be difficult so us polynomial division
  - there's a thm that says a root must be evenly divisible by the constant at the end so have a finite pool of factors to try
- roots will return basis elements of the solution space

roots	Basis elements
$root = a, a$ is real	e^{ax}
$r = a \pm bi$	$e^{ax}\cos(bx), e^{ax}\sin(bx)$

- **NOTE**: roots with multiplicities result in multiple basis elements with increasing powers of x
- example sol: $y = c_1 e^{3x} + c_2 x e^{3x} + c_3 \cos(5x) + c_4 \sin(5x)$

Non-homogenous

- means $F(x)$ is a function
- requires annihilation

How to Solve

- solve the homogenous equation
- get a trial solution by finding the annihilator and applying it to both sides
- simplify y_T by omitting terms from y_H
- get y_P by plugging y_T into the original DE and back solving for the constants
- final answer is $y_H + y_P$ possibly can combine terms to further simplify

Term	Annihilator
e^{ax}	$D - a$
$x^k e^{ax}$	$(D-a)^{k+1}$
x^k	D^{k+1}
$e^{ax}\cos(bx), e^{ax}\sin(bx)$	$D^2 - 2aD + (a^2 + b^2)$
$x^k e^{ax}\cos(bx), x^k e^{ax}\sin(bx)$	$(D^2 - 2aD + (a^2 + b^2))^{k+1}$
$\cos(bx), \sin(bx)$	$D^2 + b^2$

- **NOTE**: if 2 terms are summed then the annihilator is the product of each term's annihilator

Cauchy Euler Differential Equations

- nth order Cauchy-Euler:
  - $x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1 x y' + a_0 y = 0$
- substitute: $x = e^z, z = \ln|x|$
- $xy' = D, x^2 y'' = D(D-1), \ldots, x^n y^{(n)} = D(D-1)(D-2)\ldots(D-(n-1))$
  - with D being a differential operator in terms of z so you must substitute at the end

Variation of Parameters

- when the thing to annihilate is not simple

How to solve

- solve homogenous
  - lets say you get basis elements $A, B, C, \ldots$ to solve the homogenous equation
- let $u_1 A, \ldots, u_n C$ be solutions

- set up a wronskian of the basis elements augmented with $\vec{0}$ but the last element is the thing to annihilate
- solve this system
  - **NOTE**: you're solving for $u_1', \ldots, u_n'$
- integrate and plug into the 'sol'
  - this is particular so still need: $y = y_H + y_P$

Eigenvalues

- to find eigenvalues simply solve for $\lambda : \det(A - \lambda I_n) = 0$
  - really means cofactor expand a matrix a whose diagonal look like $c - \lambda$ and find the roots
- might need the quadratic formula for this
  - this may mean finding complex roots
  - these come in conjugate pairs

Eigenvectors

- each eigenvalue has an eigenvector
- to find simply solve for $(A - \lambda I_n)\vec{v}_\lambda = 0$
  - means $A - \lambda I_n$ augmented with the zero matrix
- the resultant vector will have n parameters
- if $n \neq mul(\lambda)$
  - matrix is defective

Complex eigenvalues

- Now suppose we have $\lambda = a + bi$ with $\vec{v} = \vec{r} + i\vec{s}$
  - and complex conjugate pair
- yeilds sols:
  - $\vec{u}(t) = e^{at}(\cos(bt)\vec{r} - \sin(bt)\vec{s})$
  - $\vec{v}(t) = e^{at}(\cos(bt)\vec{s} + \sin(bt)\vec{r})$
  - **NOTE**: these are just the vectors so if asked to solve a system of DE make sure top multiply each vector by a constant

Solving Systems of DE

- find eigenvalues
- find eigenvectors
- solution: $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_{1P} + \cdots + c_n e^{\lambda_n t}\vec{v}_{nP}$
  - where $\vec{v}_{kP}$ is the corresponding eigenvector but particular
  - choose $\vec{v}_1 \ldots \vec{v}_n$ to be LI

Defective

- the 1st 2 steps are the same but finding eigenvectors has caveats
  - by the multiplicty, it will be $2, 1$ for $mul = 3$ or for $2, 2$ or $3, 1$ for $mul = 4$
  - try lowest order bu $(A - \lambda I_n)^n \vec{v}_n = 0$
    - note if $(A - \lambda I_n)^n = \vec{0}$ this is the highest order
  - if this is the highest order
    - choose and a vector of this form such that it's not order 1
    - choose another such that it's LI from the previous
    - find their corresponding order 1 by $\vec{v}_n = A \cdot \vec{u}_n$
  - otherwise you need order 3 so simple choose vec such that not order 2 or order 1
    - find the order 2 by $\vec{v}_2 = A \cdot \vec{v}_3$
    - find order 1 by $\vec{v}_1 = A \cdot \vec{v}_2$
- use the formula to write make sure to only chain where appropriate
- formula:
  - let $\vec{v}_k$ be the generalized kth order eigenvector
    - $e^{\lambda t}\vec{v}$
    - $e^{\lambda t}(\vec{v}_2 + t\vec{v})$
    - $e^{\lambda t}(\vec{v}_3 + t\vec{v}_2 + \frac{1}{2}t^2\vec{v})$
    - $\vdots$
    - $e^{\lambda t}(\vec{v}_k + t\vec{v}_{k-1} + \cdots + \frac{1}{(k-1)!}t^{k-1}\vec{v})$