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ON DIFFERENTIAL TRANSFORMATIONS BETWEEN CARTESIAN AND CURVILINEAR (GEODETIC) COORDINATES

by

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PREFACE

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1. INTRODUCTION

Depending on the nature of certain problems, geometers found it expedient to look beyond the strict Cartesian systems of coordinates.

Lamé's investigations mark a brilliant achievement in the history of curvilinear systems. In addition to his contributions to differential geometry, his now classical reference [Lamé, 1837] introduced for the first time the idea of curvilinear elliptic coordinates (today the name "ellipsoidal" is generally accepted). The same basic theory has been used by geodesists up to recent times [Molodenskii et al., 1960].

After Lamé, scores of mathematicians became interested in the subject, improving and generalizing the methods. Among them, [Darboux, 1898] some sixty years later, wrote several papers originating the concept of moving frames, which later was universalized by [Cartan, 1935].

In modern times, mathematics has progressed toward the maximum degree of generalization. With the advent of Absolute Differential Geometry, the study of such abstract topics as m-dimensional manifolds in n-space and the concepts of tensor and differential forms have drastically revolutionized the field.

Even in geodesy some pioneering steps in these areas have been taken [Marussi, 1949], [Hotine, 1969], [Grafarend, 1975], culminating in the periodic celebration of the Hotine Symposiums on Mathematical Geodesy.

The present report benefits from some of the above methods in order to develop differential transformations between Cartesian and curvilinear orthogonal coordinates. However, only matrix algebra is used for the presentation of the basic concepts. The fact that second order Cartesian tensors reduce to 3 x 3 matrices frequently is overlooked.

After defining in Chapter 2 the reference systems used in this work, Chapter 3 introduces the rotation (R), "metric" (H) and Jacobian (J) matrices of the transformations between Cartesian and curvilinear coordinate systems. A value of R as a function of H and J is presented. Likewise an analytical expression for J^{-1} as a function of H^{-1}

and R is obtained. Subsequently, in Chapter 4, emphasis is placed on showing that the differential equations published in the English translation of [Molodenskii et al., 1960] are equivalent to conventional similarity transformations. This dissipates the confusion created recently by some authors [Badekas, 1969], [Krakiwsky and Thompson, 1974] who credited [Molodenskii et al., 1962] with a model they never wrote. A discussion of scaling methods follows.

Chapter 5 introduces ellipsoidal coordinates, to which the general theory developed in Chapter 3 is applied. Finally, differential transformations between ellipsoidal and geodetic coordinates are established.

2. REFERENCE COORDINATE SYSTEMS

The principal problem of geodesy may be stated as follows [Hirvonen, 1960]: "Find the space coordinates of any point P at the physical surface S of the earth when a sufficient number of geodetic operations have been carried out along S."

Therefore, in order to know the position of P, the definition of an appropriate system of coordinates is of primary importance.

Due to the nature of the rotational motions of the earth and to other geodynamic problems, a rigorously defined system of the accuracy of our current observational capabilities, is not presently available. A recent colloquium organized by the IAU (International Astronomical Union) in Toruń, Poland, was the first attempt to coordinate the work of different groups in the international scientific community for the future definition and selection of reliable reference frames [Kołaczek and Weiffenbach, 1974].

In the present report, only those earth fixed coordinate systems (Terrestrial Systems) which are commonly used in geodesy will be described. The reader is assumed to be familiar with other celestial systems used frequently in astronomy and conveniently defined, for example, in [Mueller, 1969]. With regard to the dynamically defined coordinate systems, generally best suited for geophysical problems, see the description in [Munk and MacDonald, 1960].

In the first place, a broad division between Cartesian and curvilinear systems may be made. Due to the nature of the basic reference surface in geodetic problems, sometimes it is convenient to use curvilinear coordinates instead of spatial rectangular coordinates. This is especially true when the ellipsoid is used as the basic reference.

In the following sections the coordinate systems used in this report and their notation will be presented in order to avoid any possible confusion.

2.1 Quasi-Geocentric Cartesian Systems

$(x, y, z) \equiv$ "Geographic" or Mean Terrestrial System

Origin: Close to the geocenter (center of mass of the earth, including the atmosphere)

z axis: Directed toward the CIO (Conventional International Origin) as defined by the IPMS (International Polar Motion Service) and the BIH (Bureau International de l'Heure).

x axis: Passes through the point of zero longitude as defined by the 1968 BIH system [Guinot et al., 1971].

y axis: Forms a right-handed coordinate system with the x and z axes.

A redefinition of this system is plausible in the future [see Kołaczek and Weiffenbach, 1974, pp. 34-37].

World Systems. These are systems defined by particular satellite solutions accomplished by different organizations. Two wide categories can be mentioned:

Dynamic Solutions (Geocentric)

$(x, y, z)_{GSFC}$ Goddard Space Flight Center

$(x, y, z)_{SAO} \equiv$ Smithsonian Astrophysical Observatory

$(x, y, z)_{NWL} \equiv$ Naval Weapons Surface Center

Geometric Solutions (Non-Geocentric)

$(x, y, z)_{NOS} \equiv$ National Ocean Survey

$(x, y, z)_{OSU} \equiv$ Ohio State University

For a complete description of the different published solutions and their corresponding references, consult [Mueller, 1975].

Some as yet unexplained differences between the orientation of the world systems with respect to the geographic system are reported in [Mueller et al., 1973].

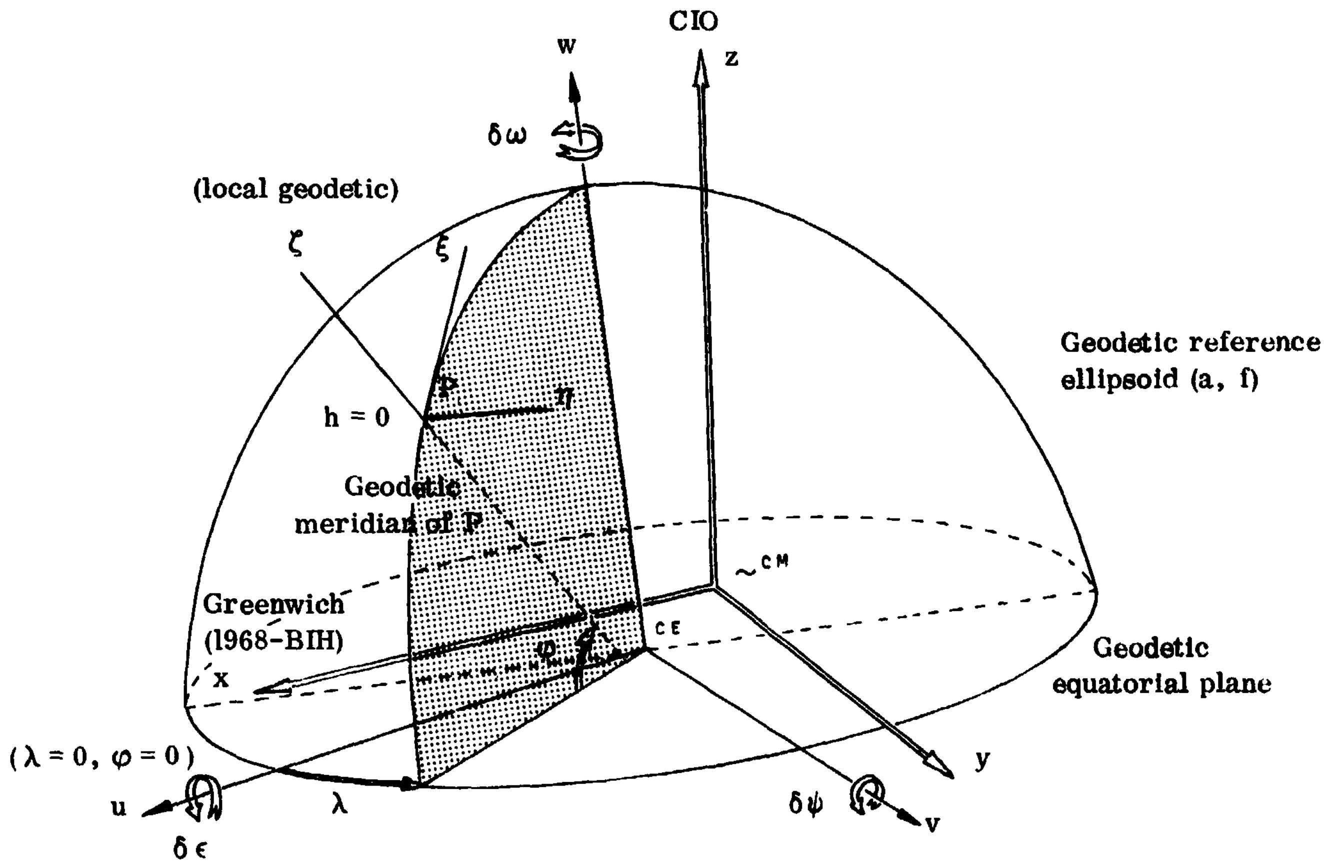


Fig. 2.1 "Geographic," Geodetic and Local Cartesian Systems

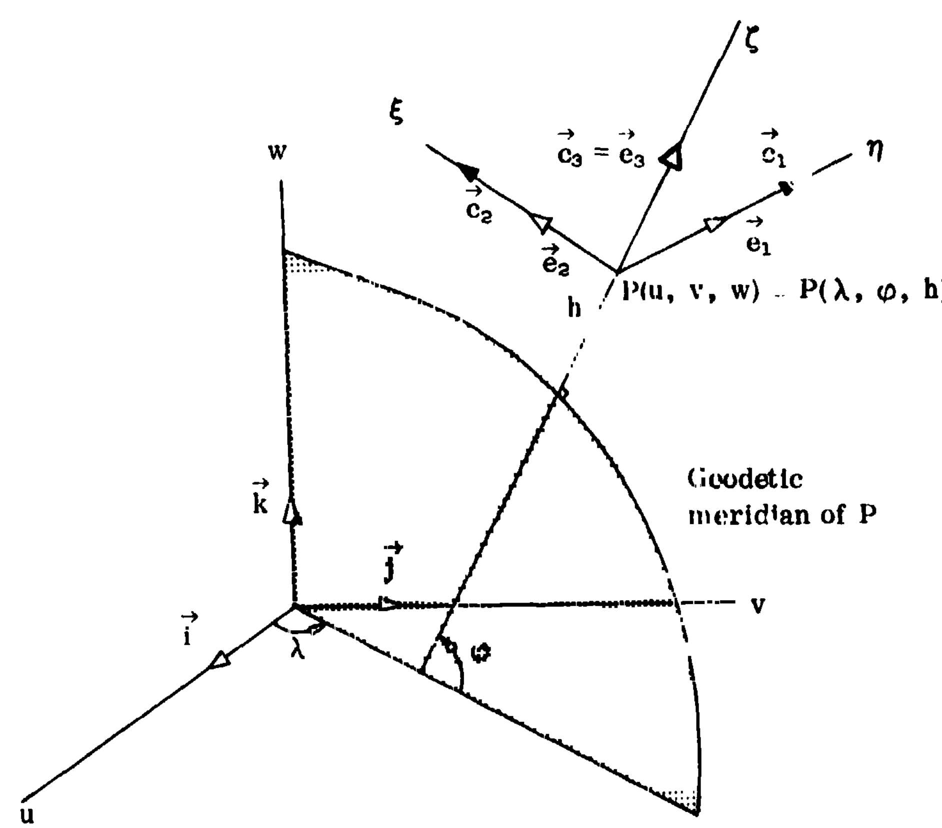


Fig. 2.2 Orthonormal Bases ($\vec{i}$, $\vec{j}$, $\vec{k}$) and ($\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$)

$(u, v, w) \equiv$ Geodetic Systems (one for each particular local datum)

Origin: The center of the reference ellipsoid used for defining the datum in question.

w axis: Coincides with the semiminor axis b of the reference ellipsoid.

u axis: Passes through the point ($\lambda = 0$, $\varphi = 0$)

v axis: Forms a right-handed frame with u and w axes.

Errors in the deflections of the vertical adopted at the datum origin, in the observed astronomic latitude and longitude, and the adoption of improper parameters of the referenced ellipsoid shift the origin of this system from the geocenter by amounts δu, δv, δw.

The improper application of the Laplace condition and errors in the astronomic azimuth introduce non-parallelism between the geodetic and geographic systems. The relationship is established through the rotations $\delta\epsilon$, $\delta\psi$, $\delta\omega$. See Fig. 2.1. Examples of this type of system defined through the datum coordinates are:

$(u, v, w)_{NAD} \equiv$ North American Datum

$(u, v, w)_{EUD} \equiv$ European Datum

2.2 Curvilinear Systems of Coordinates

$(\lambda, \varphi, h) \equiv$ Curvilinear Geodetic Coordinates

λ: Geodetic longitude. Angle between the plane uw and the geodetic meridian plane of the point P measured positive toward the east (see Fig. 2.1 and 2.2).

$$0 \leq \lambda \leq 2\pi$$

φ: Geodetic latitude. Angle between the normal to the ellipsoid at P and the plane uv.

$$-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$$

h: Geodetic height. Distance along the normal to the reference ellipsoid between P and the surface of this ellipsoid.

$(\lambda, \beta, \tilde{u}) \equiv$ Curvilinear Ellipsoidal (Rotational) Coordinates (as defined in [Heiskanen and Moritz, 1967])

λ: The same as above (i.e. geodetic longitude = ellipsoidal longitude).

β: Ellipsoidal or reduced latitude (see Fig. 5.1).

$\tilde{u}$: Semiminor axis of the confocal ellipsoid through P.

$(\Lambda, \Phi, H) \equiv$ Natural or Astronomical Coordinates

This curvilinear coordinate system refers to the instantaneous terrestrial system. In this report only the reduced astronomic coordinates will be used. Consult [Mueller, 1969] for the corresponding definitions.

λ^*: Reduced astronomic longitude

φ^*: Reduced astronomic latitude

2.3 Local Frames of Reference

$(\eta, \xi, \zeta) \equiv$ Local Geodetic Frame

Origin: The point $P(\lambda, \varphi, h)$. In the case when P is on the earth surface, the local coordinate system will be called topocentric.

ζ axis: Normal through P to the reference ellipsoid. The positive sign in the outward direction.

η axis: Normal to ζ and the geodetic meridian plane (when $h = 0$, tangent to the geodetic parallel of P.) Positive in the direction of increasing λ.

ξ axis: Perpendicular to η and ζ forming a right-handed system (when $h = 0$, tangent to the geodetic meridian of P.) Positive in the direction of increasing φ. See Figures 2.1 and 2.2.

$(\tilde{\eta}, \tilde{\xi}, \tilde{\zeta}) \equiv$ Local Ellipsoidal Frame

Origin: At the point $P(\lambda, \beta, \tilde{u})$. The above definition for the topocentric frame applies here also.

$\tilde{\zeta}$ axis: Normal to the confocal ellipsoid of semiminor axis $\tilde{u}$, which passes through P. Positive in the outward direction.

$\tilde{\eta}$ axis: Normal to $\tilde{\zeta}$ and the ellipsoidal meridian plane of P (i.e., tangent to the ellipsoidal parallel of P.) Positive in the direction of increasing λ.

$\tilde{\xi}$ axis: Normal to $\tilde{\eta}$ and the confocal hyperboloid passing through P (i.e., tangent to the confocal ellipsoid at P.) Positive in the direction of increasing β.

(η^*, ξ^*, ζ^*) ≡ <u>Local Astronomic</u> Frame

Origin: At point P.

ζ^* axis: Normal through P to the geop of P (i.e. tangent at P to the plumb line passing through P.) Positive outwards.

η^* axis: Normal to ζ^* and to the mean astronomical meridian of P (positive in the direction of increasing astronomic longitude.)

ξ^* axis: Normal to ζ^* and η^* forming a right-handed system. Positive in the direction of increasing astronomic longitude.

In the ideal case of parallelism between the (x, y, z) and (u, v, w) systems, the transformation between the astronomic and geodetic coordinates is done through the deflection of the vertical components η' and ξ'.

3. CURVILINEAR GEODETIC COORDINATES

3.1 General Comments

Consider three families of surfaces represented by the parametric equations:

$$\lambda = \lambda(u, v, w) \qquad \varphi = \varphi(u, v, w) \qquad h = h(u, v, w) \tag{3.1-1}$$

where (u, v, w) are Cartesian coordinates. This is really a transformation between points in the (u, v, w) Euclidean E^3 space lying in a certain domain and points in a certain domain in the (λ, φ, h) space, generally a "non-flat" space. These domains naturally will exclude all singular points of the transformation.

Assuming now that λ, φ, h are variable parameters, for each constant value of the parameters the family of surfaces will define three "coordinate surfaces" intersecting in "coordinate lines or curves". In general, one surface of each family passes through a chosen point and a neighboring point will be determined by neighboring values of the parameters, thus dividing the space into elementary cells which in general are not rectangular parallelepipedons. If to each value of (u, v, w) corresponds a unique value of (λ, φ, h), then any point P is uniquely determined by the three surfaces through the point.

The quantities λ, φ, h are called the "curvilinear coordinates" of the point P.

The most convenient system of curvilinear coordinates for geodetic applications are determined by families of surfaces which intersect each other everywhere at right angles. In such a case we have a "triply-orthogonal" family of surfaces or an "orthogonal curvilinear system".

Assuming that (3.1-1) represents any set of orthogonal curvilinear coordinates, in that which follows, the general theory is going to be particularized, first to the very well-known set of geodetic coordinates and later to some other curvilinear orthogonal systems used frequently in geodesy and geophysics. All the matrix relationships derived, even though deduced for a particular curvilinear geodetic system, may be applied to any set of orthogonal curvilinear coordinates.

3.2 The Local Base $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$

The coordinate transformation between the curvilinear geodetic coordinates and the Cartesian coordinates may be expressed symbolically by

$$(\lambda, \varphi, h) \xrightarrow[(a,f)]{} (u, v, w)$$

and is defined by the well-known matrix relation

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} (N+h)\cos\varphi\cos\lambda \\ (N+h)\cos\varphi\sin\lambda \\ [N(1-e^2)+h]\sin\varphi \end{bmatrix}_{(a,f)} \tag{3.2-1}$$

where N, the principal radius of curvature in the prime vertical plane is given by

$$N = \frac{a}{(1 - e^2 \sin^2\varphi)^{1/2}} \tag{3.2-2}$$

and

$$e^2 = 2f - f^2 \qquad \text{exactly.}$$

Relation (3.2-1) can also be expressed in general by the usual parametric form:

$$u = u(\lambda, \varphi, h) \qquad v = v(\lambda, \varphi, h) \qquad w = w(\lambda, \varphi, h) \tag{3.2-3}$$

In order to have a coordinate system of practical value, the following conditions will be satisfied everywhere except at isolated singular points (e.g., the poles):

a) Each point (u, v, w) has a unique set of curvilinear coordinates; that is, there is a one-to-one correspondence between the (λ, φ, h) and (u, v, w) coordinates. Therefore, the Jacobian determinant of transformation (3.2-3) is not zero.

b) Equation (3.2-3) can be solved for λ, φ, h giving the inverse transformation:

$$\lambda = \lambda(u, v, w) \qquad \varphi = \varphi(u, v, w) \qquad h = h(u, v, w) \tag{3.2-4}$$

This cannot be obtained by an explicit <u>simple</u> closed expression, but can be implemented through iteration [Heiskanen and Moritz, 1967], [Rapp, 1975], [Bartelme and Meissl, 1975] or directly [Paul, 1973] and [Benning, 1974].

d) In addition to these conditions, the tangents at a point P to the λ, φ, h "coordinate lines" through this point are perpendicular, so that the curvilinear system is orthogonal.

Equation (3.2-3) in vector notation may be written as:

$$\vec{r} = \vec{r}(\lambda, \varphi, h) \tag{3.2-5}$$

where

$$\vec{r} = u\vec{i} + v\vec{j} + w\vec{k} \tag{3.2-6}$$

and $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors along the u, v, w axes.

The tangent vectors to the (curvilinear) coordinate lines at P are defined by:

$$\vec{c}_1 = \frac{\partial \vec{r}}{\partial \lambda} = \frac{\partial u}{\partial \lambda}\vec{i} + \frac{\partial v}{\partial \lambda}\vec{j} + \frac{\partial w}{\partial \lambda}\vec{k} \tag{3.2-7a}$$

$$\vec{c}_2 = \frac{\partial \vec{r}}{\partial \varphi} = \frac{\partial u}{\partial \varphi}\vec{i} + \frac{\partial v}{\partial \varphi}\vec{j} + \frac{\partial w}{\partial \varphi}\vec{k} \tag{3.2-7b}$$

$$\vec{c}_3 = \frac{\partial \vec{r}}{\partial h} = \frac{\partial u}{\partial h}\vec{i} + \frac{\partial v}{\partial h}\vec{j} + \frac{\partial w}{\partial h}\vec{k} \tag{3.2-7c}$$

From (3.2–1) one can obtain

$$\begin{aligned}
&\frac{\partial u}{\partial \lambda} = -(N+h)\cos\varphi\sin\lambda &&\frac{\partial u}{\partial \varphi} = -(M+h)\sin\varphi\cos\lambda &&\frac{\partial u}{\partial h} = \cos\varphi\cos\lambda \\
&\frac{\partial v}{\partial \lambda} = (N+h)\cos\varphi\cos\lambda &&\frac{\partial v}{\partial \varphi} = -(M+h)\sin\varphi\sin\lambda &&\frac{\partial v}{\partial h} = \cos\varphi\sin\lambda \\
&\frac{\partial w}{\partial \lambda} = 0 &&\frac{\partial w}{\partial \varphi} = (M+h)\cos\varphi &&\frac{\partial w}{\partial h} = \sin\varphi
\end{aligned} \tag{3.2-8}$$

where M, the principal radius of curvature in the meridian plane is

$$M = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}} \tag{3.2-9}$$

Using (3.2-7) and (3.2-8) it can be shown that the $\vec{c}_i$ (i = 1, 2, 3) vectors are orthogonal, that is,

$$\vec{c}_i \cdot \vec{c}_j = 0 \quad \forall i \neq j \tag{3.2-10}$$

Computations involving curvilinear coordinates are greatly simplified if the coordinate curves and the vectors $\vec{c}_i$ are orthogonal, as in this case; otherwise the introduction of tensors will be required.

From (3.2-7) and (3.2-8) it can be seen that the $\vec{c}_i$ $(i = 1, 2, 3)$ vectors are not unit vectors; thus it will be practical to replace the $\vec{c}_i$ by unit vectors $\vec{e}_i$ $(i = 1, 2, 3)$ having the same directions.

Defining,

$$\vec{e}_1 = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial \lambda} \tag{3.2-11a}$$

$$\vec{e}_2 = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \varphi} \tag{3.2-11b}$$

$$\vec{e}_3 = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial h} \tag{3.2-11c}$$

where h_i represents the corresponding modulus of the vectors $\vec{c}_i$, or:

$$h_1 = \left|\frac{\partial \vec{r}}{\partial \lambda}\right| = \left[\left(\frac{\partial u}{\partial \lambda}\right)^2 + \left(\frac{\partial v}{\partial \lambda}\right)^2 + \left(\frac{\partial w}{\partial \lambda}\right)^2\right]^{1/2} = (N + h)\cos\varphi \tag{3.2-12a}$$

$$h_2 = \left|\frac{\partial \vec{r}}{\partial \varphi}\right| = \left[\left(\frac{\partial u}{\partial \varphi}\right)^2 + \left(\frac{\partial v}{\partial \varphi}\right)^2 + \left(\frac{\partial w}{\partial \varphi}\right)^2\right]^{1/2} = (M + h) \tag{3.2-12b}$$

$$h_3 = \left|\frac{\partial \vec{r}}{\partial h}\right| = \left[\left(\frac{\partial u}{\partial h}\right)^2 + \left(\frac{\partial v}{\partial h}\right)^2 + \left(\frac{\partial w}{\partial h}\right)^2\right]^{1/2} = 1 \tag{3.2-12c}$$

Thus using expressions (3.2-7), (3.2-8) and (3.2-12) in (3.2-11)

$$\vec{e}_1 = -\sin\lambda\,\vec{i} + \cos\lambda\,\vec{j} \tag{3.2-13a}$$

$$\vec{e}_2 = -\sin\varphi\cos\lambda\,\vec{i} - \sin\varphi\sin\lambda\,\vec{j} + \cos\varphi\,\vec{k} \tag{3.2-13b}$$

$$\vec{e}_3 = \cos\varphi\cos\lambda\,\vec{i} + \cos\varphi\sin\lambda\,\vec{j} + \sin\varphi\,\vec{k} \tag{3.2-13c}$$

It can be observed that the vectors $\vec{e}_i$ $(i = 1, 2, 3)$ are of unit length and mutually orthogonal, i.e.,

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} \qquad \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{3.2-14}$$

where δ_{ij} in the above formula is referred to as the "Kronecker delta".

Thus the vectors $\vec{e}_i$ form an orthonormal base in the Euclidean space E^3 just as do the vectors $\vec{i}$, $\vec{j}$, $\vec{k}$. There is, however, one fundamental difference between the two bases: whereas the $\vec{i}$, $\vec{j}$, $\vec{k}$ are fixed directions, the directions of $\vec{e}_i (i = 1,2,3)$ in general will vary from point to point because the coordinate lines are curved. This can be seen from (3.2-13) where the $\vec{e}_i$ vectors are functions of λ and φ. A frame like $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ which is parameter-dependent is called a "moving frame." The theory of the "moving frames of reference" was greatly extended and generalized by [Cartan, 1935] who referred to it in French as "repère mobil." For an introductory study of the matter and some bibliography, see [Grafarend, 1975].

The rectangular Cartesian reference frame (η, ξ, ζ) whose axes have the same direction as the unit vectors $\vec{e}_i$ is said to be the "local coordinate reference frame" which is attached to the point P. The coordinates (or components) of any vector $\vec{\nu}$ in this local moving frame (η, ξ, ζ) are termed the local coordinates of $\vec{\nu}$. Thus,

$$\vec{\nu} = \eta\vec{e}_1 + \xi\vec{e}_2 + \zeta\vec{e}_3 \tag{3.2-15}$$

It is important to note that because the vectors $\vec{e}_i$ are functions of λ and φ, in general the components of $\frac{\partial\vec{\nu}}{\partial\lambda}$ are not $\frac{\partial\eta}{\partial\lambda}, \frac{\partial\xi}{\partial\lambda}, \frac{\partial\zeta}{\partial\lambda}$ but

$$\frac{\partial\vec{\nu}}{\partial\lambda} = \frac{\partial\eta}{\partial\lambda}\vec{e}_1 + \frac{\partial\xi}{\partial\lambda}\vec{e}_2 + \frac{\partial\zeta}{\partial\lambda}\vec{e}_3 + \eta\frac{\partial\vec{e}_1}{\partial\lambda} + \xi\frac{\partial\vec{e}_2}{\partial\lambda} + \zeta\frac{\partial\vec{e}_3}{\partial\lambda} \tag{3.2-16}$$

Clearly the same logic will apply to $\frac{\partial\vec{\nu}}{\partial\varphi}$. This dependence of the vectors $\vec{e}_i$ on the geodetic coordinates will be implied always when a vector $\vec{\nu}$ is written in the form $\vec{\nu}(\eta, \xi, \zeta)$.

3.3 The Rotation Matrix R

Denoting by $[u \;\; v \;\; w]^T$ the 3 x 1 matrix whose elements are the coordinates of a free vector $\vec{\nu}$ in the fixed Cartesian frame and by $[\eta \;\; \xi \;\; \zeta]^T$ the column matrix whose elements are the coordinates of $\vec{\nu}$ in the local system, the transformation of components of the free vector $\vec{\nu}$ from a reference frame to the other is given by:

$$\begin{bmatrix}\eta\\ \xi\\ \zeta\end{bmatrix} \quad R \begin{bmatrix}u\\ v\\ w\end{bmatrix} \tag{3.3-1}$$

or symbolically may be represented by the mapping

$$(u, v, w) \xrightarrow{R} (\eta, \xi, \zeta)$$

where the rotation matrix R of the transformation can be deduced from Figure 2.2 by simple geometric considerations as follows:

$$R = R_1(90 - \varphi)\, R_3(\lambda + 90) = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \end{bmatrix} \qquad (3.3\text{-}2)$$

It must be pointed out here that in the following the interpretation of R will always be that of an orthogonal transformation from the geocentric to the local system. Nevertheless, knowing that R is an orthogonal matrix ($R R^T = I \Rightarrow R^{-1} = R^T$), the inverse transformation can also be written as

$$(\eta, \xi, \zeta) \xrightarrow{R^T} (u, v, w)$$

It should be noted that the rows of R are the components of the orthonormal vectors $\vec{e}_i$ (i = 1, 2, 3) given in (3.2-13). Therefore it follows immediately that

$$\begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} = R \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} \qquad (3.3\text{-}3)$$

which gives the transformation between the two orthonormal bases. It is known that the rows of R represent as well the direction cosines of the vectors $\vec{e}_i$ (i = 1, 2, 3), that is, the direction cosines of the normals with respect to the three coordinate lines or surfaces at P.

3.4 Element of Arc and Orthogonality

It was proved already that the vectors $\vec{e}_i$ (i = 1, 2, 3) tangent to the coordinate lines are orthogonal.

A different way to see that the curvilinear geodetic coordinates are orthogonal is computing the element of arc (element of distance) in these coordinates. From (3.2-3)

$$u = u(\lambda, \varphi, h) \qquad v = v(\lambda, \varphi, h) \qquad w = w(\lambda, \varphi, h)$$

The total differentials of the functions u, v, w are

$$du = \frac{\partial u}{\partial \lambda} d\lambda + \frac{\partial u}{\partial \varphi} d\varphi + \frac{\partial u}{\partial h} dh \tag{3.4-1a}$$

$$dv = \frac{\partial v}{\partial \lambda} d\lambda + \frac{\partial v}{\partial \varphi} d\varphi + \frac{\partial v}{\partial h} dh \tag{3.4-1b}$$

$$dw = \frac{\partial w}{\partial \lambda} d\lambda + \frac{\partial w}{\partial \varphi} d\varphi + \frac{\partial w}{\partial h} dh \tag{3.4-1c}$$

or in matrix notation

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = J \begin{bmatrix} d\lambda \\ d\varphi \\ dh \end{bmatrix} \tag{3.4-2}$$

where J is the Jacobian or functional matrix and may be expressed by:

$$J = \frac{\partial(u, v, w)}{\partial(\lambda, \varphi, h)} = \frac{(u, v, w)}{(\lambda, \varphi, h)} = \begin{bmatrix} \frac{\partial u}{\partial \lambda} & \frac{\partial u}{\partial \varphi} & \frac{\partial u}{\partial h} \\ \frac{\partial v}{\partial \lambda} & \frac{\partial v}{\partial \varphi} & \frac{\partial v}{\partial h} \\ \frac{\partial w}{\partial \lambda} & \frac{\partial w}{\partial \varphi} & \frac{\partial w}{\partial h} \end{bmatrix} \tag{3.4-3}$$

Thus the Jacobian matrix of a coordinate transformation can be interpreted as the matrix of a certain linear change of coordinates, namely:

$$(d\lambda, d\varphi, dh) \xrightarrow{J} (du, dv, dw)$$

As mentioned previously, the Jacobian determinant is not zero. Therefore

$$|J| \neq 0 \rightarrow J \text{ is not singular} \rightarrow J^{-1} \text{ exists.}$$

The square of the line element in the (u, v, w) system is

$$ds^2 = du^2 + dv^2 + dw^2 \tag{3.4-4}$$

Substituting above the values from (3.4-1) one can obtain the following equation

$$ds^2 = h_1^2\, d\lambda^2 + h_2^2\, d\varphi^2 + h_3^2\, dh^2 + 2h_4^2\, d\lambda\, d\varphi + 2h_5^2\, d\varphi\, dh + 2h_6^2\, dh\, d\lambda \tag{3.4-5}$$

where the values h_1, h_2, h_3 are given by (3.2-12) and

$$h_4 = \left[\frac{\partial u}{\partial \lambda}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial \lambda}\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial \lambda}\frac{\partial w}{\partial \varphi} \right]^{\frac{1}{2}} \tag{3.4-6a}$$

$$h_5 = \left[\frac{\partial u}{\partial \varphi}\frac{\partial u}{\partial h} + \frac{\partial v}{\partial \varphi}\frac{\partial v}{\partial h} + \frac{\partial w}{\partial \varphi}\frac{\partial w}{\partial h} \right]^{\frac{1}{2}} \tag{3.4-6b}$$

$$h_6 = \left[\frac{\partial u}{\partial h}\frac{\partial u}{\partial \lambda} + \frac{\partial v}{\partial h}\frac{\partial v}{\partial \lambda} + \frac{\partial w}{\partial h}\frac{\partial w}{\partial \lambda} \right]^{\frac{1}{2}} \tag{3.4-6c}$$

Replacing the values presented in (3.2-8) in equation (3.4-6), it is easy to find that the necessary and sufficient condition for orthogonality is

$$h_4 = h_5 = h_6 = 0 \tag{3.4-7}$$

Therefore the absence of the terms $d\lambda d\varphi$, $d\varphi dh$, and $dh d\lambda$ in (3.4-5) is the evidence that the curvilinear coordinates (λ, φ, h) are orthogonal. The transformation is called conformal when elements of arc in the neighborhood of a point in the (u, v, w) system are proportional to the elements of arc in the neighborhood of the corresponding point in the (λ, φ, h) curvilinear system. That is when

$$ds^2 = du^2 + dv^2 + dw^2 = k^2(d\lambda^2 + d\varphi^2 + dh^2) \tag{3.4-8}$$

Thus conformality requires

$$h_1 = h_2 = h_3 \tag{3.4-9}$$

The above is in agreement with the fact that conformality implies orthogonality but not viceversa.

It can be observed that equation (3.4-5) gives the linear element ds in three-dimensional space. Clearly when limited to surface transformations the so-called Gaussian fundamental quantities will be present. For example, in the case of an ellipsoid the following identities are established:

$$E = h_1 \qquad G = h_2 \qquad F = h_4 \tag{3.4-10}$$

When $F = 0$ the condition for orthogonality exists, and if simultaneously $E = G$ the transformation is conformal.

Equation (3.4-5) can easily be written in matrix notation as follows:

$$ds^2 = [du \quad dv \quad dw] \begin{bmatrix} du \\ dv \\ dw \end{bmatrix} \tag{3.4-11}$$

But recalling equation (3.4-2)

$$ds^2 = [d\lambda \quad d\varphi \quad dh]\, J^T J \begin{bmatrix} d\lambda \\ d\varphi \\ dh \end{bmatrix} \tag{3.4-12}$$

Thus the transformation between two sets of coordinates in E^3 will be orthogonal if the matrix product of the Jacobian transpose by the Jacobian is a diagonal matrix.

3.5 The "Metric Matrix" H

The differentials du, dv, dw may also be represented as a free vector. They behave under transformation of coordinates as do free vector components; thus from equation (3.3-1)

$$\begin{bmatrix} d\eta \\ d\xi \\ d\zeta \end{bmatrix} = R \begin{bmatrix} du \\ dv \\ dw \end{bmatrix} \tag{3.5-1}$$

Using (3.4-2) it is possible to express the relationship between the differential arc-length of the curvilinear coordinates along the "coordinate lines" and their projection on the (η, ξ, ζ) system as

$$\begin{bmatrix} d\eta \\ d\xi \\ d\zeta \end{bmatrix} = RJ \begin{bmatrix} d\lambda \\ d\varphi \\ dh \end{bmatrix} \qquad (3.5\text{-}2)$$

where the value of the matrix product RJ according to (3.3-2), (3.2-8) and (3.4-3) is

$$RJ = \begin{bmatrix} (N+h)\cos\varphi & 0 & 0 \\ 0 & (M+h) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix} = H \qquad (3.5\text{-}3)$$

Thus symbolically one may represent

$$(d\lambda,\ d\varphi,\ dh) \xrightarrow{H} (d\eta,\ d\xi,\ d\zeta)$$

where the matrix of the transformation H will be called the "metric matrix" by analogy with tensor terminology.

From (3.5-3) the following basic relation can be written:

$$RJ = H \qquad (3.5\text{-}4)$$

and using the orthogonality property of R,

$$J = R^T H \qquad (3.5\text{-}5)$$

Then

$$J^T J = H^T R R^T H$$

and H being diagonal the following results:

$$J^T J = H^2 \qquad (3.5\text{-}6)$$

as can easily be proved by simple multiplication of matrices.

Comparing (3.5-6) with (3.4-12) it may be deduced that the diagonality of the metric matrix H is a consequence of the orthogonality of the curvilinear coordinate system.

3.6 Jacobian Determinant and Its Applications

Equation (3.5-6) provides a simple way to obtain the value of the Jacobian determinant. Taking determinants in (3.5-6)

$$|J^T J| = |H^2| \Rightarrow |J^T|\,|J| = |H|^2 \quad \text{but } |J^T| = |J|$$

Thus

$$|J|^2 = |H|^2$$

and finally

$$|J| = |H| = h_1\, h_2\, h_3 \tag{3.6-1}$$

where h_i ($i = 1, 2, 3$) are given by equation (3.2-12).

It is important to mention here that while the determinants of J and H are always equal, the matrix J is equal to the matrix H only when $R = I$, as can be seen from (3.5-4). This will be equivalent to making the frame (u, v, w) parallel to the local (η, ξ, ζ) frame through the pertinent rotations.

The functional (or Jacobian) determinant in the case of geodetic coordinates is

$$|J| = (M + h)\,(N + h)\cos\varphi \tag{3.6-2}$$

Thus, aside from points where $\cos\varphi = 0 \Rightarrow \varphi = \pm\frac{\pi}{2} \Rightarrow |J| = 0$, the transformation is locally one-to-one, implying that any point on the w (polar) axis is a singular point in this specific transformation.

Equation (3.6-1) is also very convenient for the computation of elements of area along the different coordinate surfaces and the element of volume between them.

$$dA_1 = h_1 h_2\, d\lambda\, d\varphi \tag{3.6-3a}$$

$$dA_2 = h_1 h_3\, d\lambda\, dh \tag{3.6-3b}$$

$$dA_3 = h_2 h_3\, d\varphi\, dh \tag{3.6-3c}$$

$$dV = h_1 h_2 h_3\, d\lambda\, d\varphi\, dh \tag{3.6-3d}$$

For the particular case of geodetic coordinates on the reference ellipsoid

$$dA = MN\cos\varphi\, d\lambda\, d\varphi \tag{3.6-4a}$$

$$dV = (M + h)(N + h)\cos\varphi\, d\lambda\, d\varphi\, dh \tag{3.6-4b}$$

It should be noted that the Jacobian may be either positive or negative, the difference of sign being of the same nature as the consideration of an area or volume and their reflection. Due to the fact that the elements of area or volume are considered positive when the increments of the variables are positive, the absolute value of the Jacobian will always be taken.

3.7 Analytic Expressions for the Inverse of the Jacobian Matrix

From (3.5-5) it is evident that

$$J^{-1} = H^{-1}R \tag{3.7-1}$$

which provides an analytical way of obtaining the inverse of the Jacobian matrix. Using (3.5-3) and (3.3-2) one obtains

$$J^{-1} = \frac{\partial(\lambda, \varphi, h)}{\partial(u, v, w)} = \frac{(\lambda, \varphi, h)}{(u, v, w)} = \begin{bmatrix} \frac{\partial\lambda}{\partial u} & \frac{\partial\lambda}{\partial v} & \frac{\partial\lambda}{\partial w} \\ \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} & \frac{\partial\varphi}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{\sin\lambda}{(N+h)\cos\varphi} & \frac{\cos\lambda}{(N+h)\cos\varphi} & 0 \\ -\frac{\sin\varphi\cos\lambda}{M+h} & -\frac{\sin\varphi\sin\lambda}{M+h} & \frac{\cos\varphi}{M+h} \\ \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \end{bmatrix} \tag{3.7-2}$$

The classical way to obtain the elements of J^{-1} is to solve the following nine equations:

$$\frac{\partial u}{\partial\lambda}\frac{\partial\lambda}{\partial u} + \frac{\partial u}{\partial\varphi}\frac{\partial\varphi}{\partial u} + \frac{\partial u}{\partial h}\frac{\partial h}{\partial u} = 1 \tag{3.7-3a}$$

$$\frac{\partial v}{\partial\lambda}\frac{\partial\lambda}{\partial v} + \frac{\partial v}{\partial\varphi}\frac{\partial\varphi}{\partial v} + \frac{\partial v}{\partial h}\frac{\partial h}{\partial v} = 1 \tag{3.7-3b}$$

$$\frac{\partial w}{\partial \lambda}\frac{\partial \lambda}{\partial w} + \frac{\partial w}{\partial \varphi}\frac{\partial \varphi}{\partial w} + \frac{\partial w}{\partial h}\frac{\partial h}{\partial w} = 1 \tag{3.7-3c}$$

$$\frac{\partial u}{\partial \lambda}\frac{\partial \lambda}{\partial v} + \frac{\partial u}{\partial \varphi}\frac{\partial \varphi}{\partial v} + \frac{\partial u}{\partial h}\frac{\partial h}{\partial v} = 0 \tag{3.7-3d}$$

$$\frac{\partial u}{\partial \lambda}\frac{\partial \lambda}{\partial w} + \frac{\partial u}{\partial \varphi}\frac{\partial \varphi}{\partial w} + \frac{\partial u}{\partial h}\frac{\partial h}{\partial w} = 0 \tag{3.7-3e}$$

$$\frac{\partial v}{\partial \lambda}\frac{\partial \lambda}{\partial u} + \frac{\partial v}{\partial \varphi}\frac{\partial \varphi}{\partial u} + \frac{\partial v}{\partial h}\frac{\partial h}{\partial u} = 0 \tag{3.7-3f}$$

$$\frac{\partial v}{\partial \lambda}\frac{\partial \lambda}{\partial w} + \frac{\partial v}{\partial \varphi}\frac{\partial \varphi}{\partial w} + \frac{\partial v}{\partial h}\frac{\partial h}{\partial w} = 0 \tag{3.7-3g}$$

$$\frac{\partial w}{\partial \lambda}\frac{\partial \lambda}{\partial u} + \frac{\partial w}{\partial \varphi}\frac{\partial \varphi}{\partial u} + \frac{\partial w}{\partial h}\frac{\partial h}{\partial u} = 0 \tag{3.7-3h}$$

$$\frac{\partial w}{\partial \lambda}\frac{\partial \lambda}{\partial v} + \frac{\partial w}{\partial \varphi}\frac{\partial \varphi}{\partial v} + \frac{\partial w}{\partial h}\frac{\partial h}{\partial v} = 0 \tag{3.7-3i}$$

which reduce in matrix notation to

$$J\,J^{-1} = I \tag{3.7-4}$$

or

$$(J\,J^{-1})^{T} = I \Rightarrow (J^{-1})^{T} J^{T} = I \tag{3.7-5}$$

but from (3.7-1) $J^{-1} = H^{-1}R$, thus substituting this above, the following two equalities can be written:

$$R^{T} H^{-1} J^{T} = I \tag{3.7-6}$$

and

$$J\,H^{-1} R = I \tag{3.7-7}$$

Premultiplying both sides of (3.7-6) by J^{-1}, which is equal to $H^{-1}R$,

$$(H^{-1}R)\,(R^{T} H^{-1})\,J^{T} = J^{-1} \qquad \text{or}$$

$$J^{-1} = (H^{2})^{-1} J^{T} \tag{3.7-8}$$

where clearly $(H^{-1})^{2} = (H^{2})^{-1}$ due to the diagonality of H. This is another way of computing J^{-1} independent of R as a function of J and H. Substituting (3.7-8) in (3.7-4) the following equality is established:

$$J(H^2)^{-1}J^T = I$$

Equation (3.7-8) gives

$$J = [(H^2)^{-1}J^T]^{-1} = (J^T)^{-1}H^2 = (J^{-1})^T H^2$$

Therefore

$$J = (J^{-1})^T H^2 \qquad (3.7\text{-}9)$$

3.8 The Matrix R as a function of H and J

Finally it is possible to write the expressions for the rotation matrix R as a function of the metric and Jacobian matrices.

From (3.5-4)

$$R = H J^{-1} \qquad (3.8\text{-}1)$$

Substituting (3.7-8) above, the following is derived:

$$R = H^{-1} J^T \qquad (3.8\text{-}2)$$

and therefore the following matrix equality holds:

$$H J^{-1} = H^{-1} J^T \qquad (3.8\text{-}3)$$

Knowing, as was mentioned above (section 3.3) that the elements of the rows of R represent the direction cosines of the normals to the family of surfaces λ, φ, h that pass through a point P, equations (3.8-1) and (3.8-2) will provide general formulas for computing the nine direction cosines of these normals in two different ways.

For example, the direction cosines of the normal to the reference ellipsoid at a point (λ, φ, h) may be given by

$$\cos(u, h) = \frac{1}{h_3}\frac{\partial u}{\partial h} = h_3 \frac{\partial h}{\partial u} = \cos\varphi \cos\lambda$$

$$\cos(v, h) = \frac{1}{h_3}\frac{\partial v}{\partial h} = h_3 \frac{\partial h}{\partial v} = \cos\varphi \sin\lambda$$

$$\cos(w, h) = \frac{1}{h_3}\frac{\partial w}{\partial h} = h_3 \frac{\partial h}{\partial h} = \sin\varphi$$

This approach will be very helpful for other curvilinear orthogonal systems where the matrix R cannot be obtained by simple geometric considerations, as in the case of curvilinear ellipsoidal coordinates (see Section 5.2).

4. DIFFERENTIAL CHANGES BETWEEN CARTESIAN AND CURVILINEAR GEODETIC COORDINATES

4.1 Basic Equations

From (3.5-2) the following basic relation can be written:

$$\begin{bmatrix} d\eta \\ d\xi \\ d\zeta \end{bmatrix} = H \begin{bmatrix} d\lambda \\ d\varphi \\ dh \end{bmatrix} \tag{4.1-1}$$

which substituted in (3.5-1) gives the fundamental differential relations between the Cartesian (u, v, w) and curvilinear (λ, φ, h) coordinates.

$$\begin{bmatrix} d\lambda \\ d\varphi \\ dh \end{bmatrix} = H^{-1}R \begin{bmatrix} du \\ dv \\ dw \end{bmatrix} \tag{4.1-2}$$

This is usually written for the case of geodetic coordinates as

$$H \begin{bmatrix} d\lambda \\ d\varphi \\ dh \end{bmatrix} = R \begin{bmatrix} du \\ dv \\ dw \end{bmatrix} \tag{4.1-3}$$

or

$$\begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix}_{(a,f)} = R \begin{bmatrix} du \\ dv \\ dw \end{bmatrix} \tag{4.1-4}$$

where a particular ellipsoid is implied in the computation of N and M.

The above formula expresses the basic matrix equation relating the differential changes in the geodetic coordinates $(d\lambda, d\varphi, dh)$ of a point P referenced to a given ellipsoid (a, f), due to differential changes in the geodetic Cartesian coordinates of the point.

Considering the orthogonality of R,

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = R^T \begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix} \tag{4.1-5}$$

It is possible to show that this equation is equivalent to (3.4-2), namely,

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = \frac{\partial(u,\ v,\ w)}{\partial(\lambda,\ \varphi,\ h)} \begin{bmatrix} d\lambda \\ d\varphi \\ dh \end{bmatrix}$$

Clearly, if the point $P(\lambda, \varphi, h)$ is on the surface of the ellipsoid

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = R^T \begin{bmatrix} N\cos\varphi\, d\lambda \\ M\, d\varphi \\ 0 \end{bmatrix} \tag{4.1-6}$$

4.2 Differential Changes in (λ, φ, h) Due to Shifts, Rotations and Scaling of the (u, v, w) Cartesian System

As an illustration of the above theory, one can assume, for example, that it is desired to obtain the differential changes in the geodetic curvilinear coordinates (λ, φ, h) due to differential shift, rotation and scale changes of the Cartesian system. Then from (4.1-4) the total contribution may be expressed as:

$$\begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix} = R \left\{ \begin{bmatrix} du \\ dv \\ dw \end{bmatrix}_{\text{shift}} + \begin{bmatrix} du \\ dv \\ dw \end{bmatrix}_{\text{rotation}} + \begin{bmatrix} du \\ dv \\ dw \end{bmatrix}_{\text{scale}} \right\} \tag{4.2-1}$$

Each individual differential contribution will be studied separately in the following paragraphs.

4.2.1 Changes Due to Differential Shift of the Origin

If the geodetic system (u, v, w) is shifted by the amounts δu, δv, δw, then obviously one obtains

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix}_{shift} = \begin{bmatrix} \delta u \\ \delta v \\ \delta w \end{bmatrix} \tag{4.2-2}$$

Assuming the systems (u, v, w) and (x, y, z) to be parallel, the signs of the shift components may be given by one of the following conventions (see also Fig. 4.1):

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} \equiv \begin{cases} \text{(Geographic System)} - \text{(Geodetic System)} \\ \text{(Final } '' \text{)} - \text{(Initial } '' \text{)} \\ \text{(New } '' \text{)} - \text{(Old } '' \text{)} \\ \text{(Fixed } '' \text{)} - \text{(Moving } '' \text{)} \end{cases} \tag{4.2-3}$$

Thus for example the transformation of coordinates

$$\begin{array}{lcl} \text{Geodetic} & \longrightarrow & \text{Geographic} \quad \text{will be} \\ (u, v, w) & & (x, y, z) \end{array}$$

$$\text{Geographic} = \text{Geodetic} + (\text{Geographic} - \text{Geodetic})$$

and consequently

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} \delta u \\ \delta v \\ \delta w \end{bmatrix} \tag{4.2-4}$$

In general the above sign rules will be observed on the following pages, if not specified otherwise.

4.2.2 Changes Due to Rotation

As is known, it is possible to relate the coordinates of the two Cartesian systems having the same origin by the equation

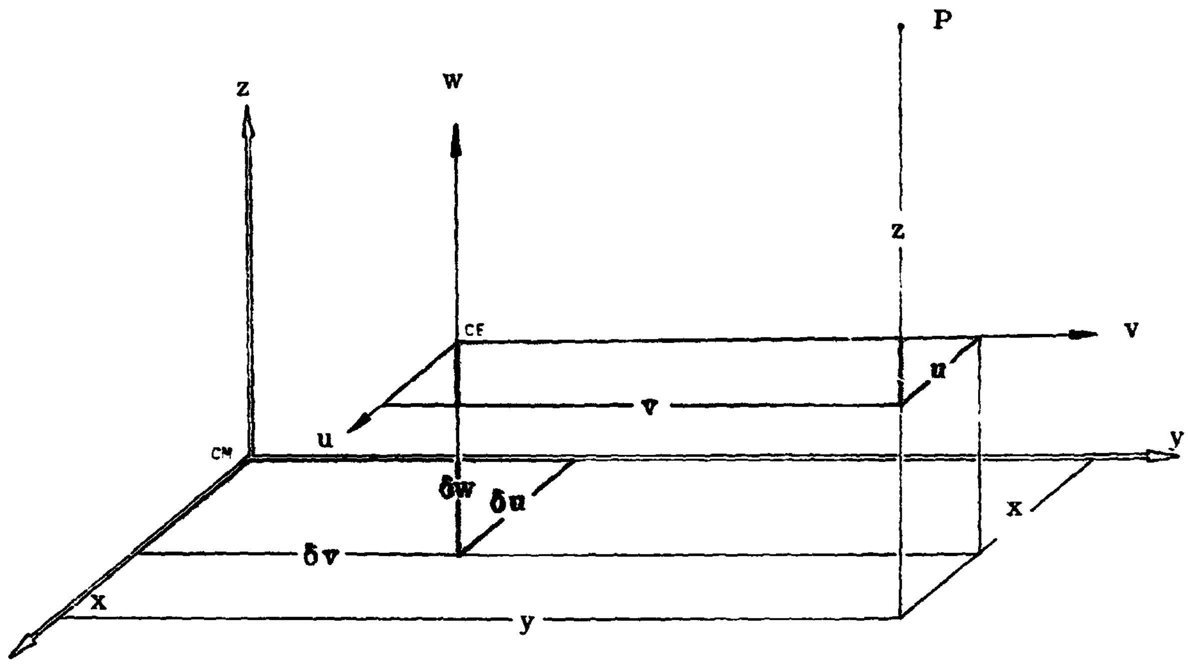


Fig. 4.1 Shifts Between the Geographic and Geodetic Systems

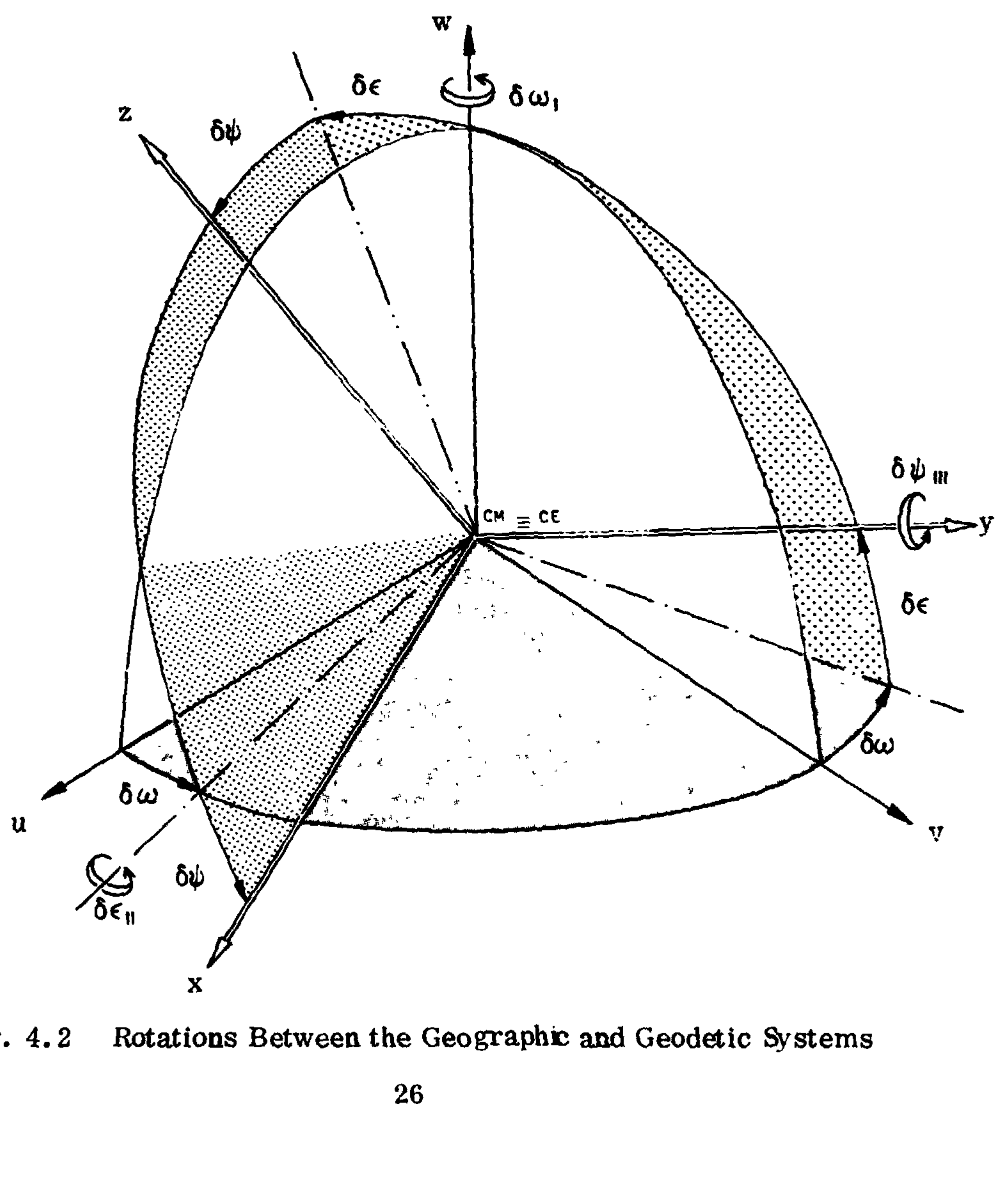


Fig. 4.2 Rotations Between the Geographic and Geodetic Systems

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underline{R} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad (4.2\text{-}5)$$

where the rotation matrix $\underline{R}$ can be written as follows

$$\underline{R} = R_3(\psi)\, R_1(\epsilon)\, R_3(\omega) =$$

$$= \begin{bmatrix} \cos\omega\cos\psi - \sin\omega\sin\epsilon\sin\psi & \sin\omega\cos\psi + \cos\omega\sin\epsilon\sin\psi & -\cos\epsilon\sin\psi \\ -\sin\omega\cos\epsilon & \cos\omega\cos\epsilon & \sin\epsilon \\ \cos\omega\sin\psi + \sin\omega\sin\epsilon\cos\psi & \sin\omega\sin\psi - \cos\omega\sin\epsilon\cos\psi & \cos\epsilon\cos\psi \end{bmatrix} \quad (4.2\text{-}6)$$

An introductory section on rotational matrix algebra may be consulted in [Goldstein, 1950].

In order to keep the sign convention established in the previous section, the differential changes in the coordinates (u, v, w) due to the rotations ω, ϵ, ψ are given by

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix}_{\text{rotation}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \underline{R} \begin{bmatrix} u \\ v \\ w \end{bmatrix} - \begin{bmatrix} u \\ v \\ w \end{bmatrix} = [\underline{R} - I] \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad (4.2\text{-}7)$$

Assuming now differentially small rotations $\delta\omega$, $\delta\epsilon$, $\delta\psi$ (see Fig. 4.2), it follows

$$\begin{aligned} \sin\omega &\approx \delta\omega \\ \sin\epsilon &\approx \delta\epsilon \\ \sin\psi &\approx \delta\psi \\ \cos\omega &= \cos\epsilon = \cos\psi \approx 1 \end{aligned} \quad (4.2\text{-}8)$$

and neglecting second-order terms, the rotation matrix becomes

$$\underline{R} \approx \begin{bmatrix} 1 & \delta\omega & -\delta\psi \\ -\delta\omega & 1 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 1 \end{bmatrix} = R_\delta \quad (4.2\text{-}9)$$

and by means of (4.2-7)

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix}_{\delta\omega,\,\delta\epsilon,\,\delta\psi} = \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \delta \underline{R} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad (4.2\text{-}10)$$

It can be observed that the matrix $\delta \underline{R}$ is a skew-symmetric (or antisymmetric) matrix

$$\delta \underline{R} + (\delta \underline{R})^T = 0 \qquad (4.2\text{-}11)$$

That is, if δr_{ij} (i, j = 1, 2, 3) are the elements of $\delta \underline{R}$

$$\begin{aligned} \delta r_{ij} &= 0 & \forall\, i = j \\ \delta r_{ji} &= -\delta r_{ij} & \forall\, i \neq j \end{aligned} \qquad (4.2\text{-}12)$$

This represents an important property for all differential rotation matrices. Notice that the R_δ matrix given by (4.2-9) is orthogonal only up to first-order terms. From (4.2-9) and (4.2-10)

$$R_\delta = I + \delta \underline{R} \qquad (4.2\text{-}13)$$

The condition for R_δ to be orthogonal is $R_\delta R_\delta^T = I$, but

$$R_\delta R_\delta^T = (I + \delta \underline{R})(I + \delta \underline{R})^T = (I + \delta \underline{R})(I - \delta \underline{R}) = I - (\delta \underline{R})^2 \neq I$$

4.2.3 Changes Due to Differential Scale Changes

The changes in u, v, w due to a differential scale change δL are

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix}_{\delta L} = \begin{bmatrix} u\,\delta L \\ v\,\delta L \\ w\,\delta L \end{bmatrix} = \delta L \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad (4.2\text{-}14)$$

where δL according to the sign convention mentioned in (4.2.1) is,

$$\delta L = (\text{Geographic scale}) - (\text{Geodetic scale}) \qquad (4.2\text{-}15)$$

Equation (4.2-14) can also be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \delta L \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{4.2-16}$$

From (4.2-16) it is obvious that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (1 + \delta L) \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{4.2-17}$$

which implies that the δL change in scale may be computed directly from the coordinates, but δL can also be obtained from the chord distances d_x and d_u in the respective systems.

$$d_x^2 = (x - x')^2 + (y - y')^2 + (z - z')^2 = [x - x' \quad y - y' \quad z - z'] \begin{bmatrix} x - x' \\ y - y' \\ z - z' \end{bmatrix}$$

$$= (1 + \delta L)^2 [u - u' \quad v - v' \quad w - w'] \begin{bmatrix} u - u' \\ v - v' \\ w - w' \end{bmatrix} = (1 + \delta L)^2 d_u^2 \tag{4.2-18}$$

Thus

$$d_x = (1 + \delta L)\, d_u \implies \delta L = \frac{d_x - d_u}{d_u} \tag{4.2-19}$$

Precautions should be taken, however, when in a least square adjustment, scale determination through chord distances is intended. Only an independent set of chords should be used [Leick and van Gelder, 1975].

4.2.4 Final Equation

Thus finally substituting in (4.2-1) the computed effects in the coordinates u, v, w due to shift, rotation and scale changes, the following equation is obtained:

$$\begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix} = R \left\{ \begin{bmatrix} \delta u \\ \delta v \\ \delta w \end{bmatrix} + \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \delta L \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right\} \tag{4.2-20}$$

which gives in matrix notation the changes in geodetic coordinates (λ, φ, h) due to shifts (δu, δv, δw), rotations $\delta\omega$, $\delta\epsilon$, $\delta\psi$ and scale δL

Clearly to express $d\lambda$, $d\varphi$, dh only in function of geodetic coordinates, equation (3.2-1) will be substituted in (4.2-20).

4.3 Similarity Transformations

It is proper to point out here that if one considers only changes in the Cartesian coordinates due to translations, rotations and scale change, after substituting (4.1-4) in the left side of (4.2-20) the following results:

$$R \begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = R \left\{ \begin{bmatrix} \delta u \\ \delta v \\ \delta w \end{bmatrix} + \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \delta L \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right\} \tag{4.3-1}$$

But according to the notation used in this report,

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{final} - \begin{bmatrix} u \\ v \\ w \end{bmatrix}_{initial} \tag{4.3-2}$$

Thus, finally after substitution of the above in (4.3-1) and omitting the rotation matrix R

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} \delta u \\ \delta v \\ \delta w \end{bmatrix} + \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \delta L \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{4.3-3}$$

But one can also write

$$\left.\begin{aligned} \delta u &= \Delta x \\ \delta v &= \Delta y \\ \delta w &= \Delta z \end{aligned}\right\} \equiv \begin{cases} \text{Shifts of the geodetic system with} \\ \text{respect to the geographic system} \end{cases} \tag{4.3-4a}$$

$$\delta L = \Delta \equiv \text{Geographic scale - Geodetic scale} \tag{4.3-4b}$$

Then one obtains the usual seven-parameter transformation between the coordinates of two Cartesian frames.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} + \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \Delta \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{4.3-5}$$

or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} + \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + (1+\Delta) \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{4.3-6}$$

or finally,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} + \begin{bmatrix} 1+\Delta & \delta\omega & -\delta\psi \\ -\delta\omega & 1+\Delta & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 1+\Delta \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{4.3-7}$$

Some authors use the notation $1 + \Delta = \Lambda$. Thus, recalling (4.3-7)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} + \begin{bmatrix} \Lambda & \delta\omega & -\delta\psi \\ -\delta\omega & \Lambda & \delta\epsilon \\ \delta\psi & -\delta\epsilon & \Lambda \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{4.3-8}$$

which is a form of the more general similarity transformation:

$$X = T + \Lambda \underline{R} U \tag{4.3-9}$$

Comparing (4.3-8) with (4.3-9) it is obvious that besides the assumption of small rotations $\underline{R} \approx R_\delta$ the products of Λ by the rotations were neglected in the differential transformations of the formula (4.3-8). For a complete development of the similarity transformations as used in geodesy, consult the discussion in [Leick and van Gelder, 1975].

Any one of the above type transformation models is generally referred to in geodetic literature as the "Bursa model" after [Bursa, 1966]. They have been popularized as mathematical models of the type $F(X, L) = 0$ in the least square solution for computing the seven parameters of the similarity transformation between world systems.

Nevertheless, it is unclear why several authors, among them [Badekas, 1969] and [Krakiwsky and Thompson, 1974] credited [Molodenskii et al., 1962] with a different model where the rotations and scale expansion are about some particular point (u_o, v_o, w_o) other than the origin of the Cartesian geodetic system.

In the following sections the equations given by [Molodenskii et al., 1962] will be presented adopting the general criterion of this work. It will be shown that they are not different from a similarity transformation of the form (4.3-5) except in the way the scale is applied. In th connection the similarities of eliminating the variations of scale through changes in the semimajor axis of the reference ellipsoid will be explained.

4.4 Differential Transformations According to [Molodenskii et al., 1962]

Before fully developing the formulas given by [Molodenskii et al., 1962], the effect of differential changes in the Cartesian geodetic coordinates due to changes in the size (a) and flattening (f) of the ellipsoid will be treated.

4.4.1 Changes Due to Variations δa and δf

If the original reference ellipsoid parameters are changed, their effect on the (u, v, w) coordinates can be expressed in matrix notation as follows:

$$
\begin{bmatrix} du \\ dv \\ dw \end{bmatrix}_{\delta a, \delta f} = \begin{bmatrix} \frac{\partial u}{\partial a} & \frac{\partial u}{\partial f} \\ \frac{\partial v}{\partial a} & \frac{\partial v}{\partial f} \\ \frac{\partial w}{\partial a} & \frac{\partial w}{\partial f} \end{bmatrix} \begin{bmatrix} \delta a \\ \delta f \end{bmatrix} \tag{4.4-1}
$$

where

$$
\delta a = a_{new} - a_{old} \tag{4.4-2a}
$$

$$
\delta f = f_{new} - f_{old} \tag{4.4-2b}
$$

and from (3.2-1) it is possible to compute [Rapp, 1975a]:

$$
\begin{aligned}
\frac{\partial u}{\partial a} &= \frac{\cos\varphi \cos\lambda}{W} & \frac{\partial u}{\partial f} &= \frac{a(1-f)\sin^2\varphi\cos\varphi\cos\lambda}{W^3} \\
\frac{\partial v}{\partial a} &= \frac{\cos\varphi \sin\lambda}{W} & \frac{\partial v}{\partial f} &= \frac{a(1-f)\sin^2\varphi\cos\varphi\sin\lambda}{W^3} \\
\frac{\partial w}{\partial a} &= \frac{(1-e^2)\sin\varphi}{W} & \frac{\partial w}{\partial f} &= (M\sin^2\varphi - 2N)(1-f)\sin\varphi
\end{aligned} \tag{4.4-3}
$$

where

$$
W^2 = 1 - e^2\sin^2\varphi \qquad M = \frac{a(1-e^2)}{W^3} \qquad N = \frac{a}{W} \tag{4.4-4}
$$

4.4.2 General Equations

Literally following Molodenskii et al., [1962], their equation (I.3.2) using the differential matrix approach and the notation of this paper may be written as

$$
\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = \begin{bmatrix} \delta u_o \\ \delta v_o \\ \delta w_o \end{bmatrix} + \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} u - u_o \\ v - v_o \\ w - w_o \end{bmatrix}
$$

$$
+ R^T \begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial a} & \frac{\partial u}{\partial f} \\ \frac{\partial v}{\partial a} & \frac{\partial v}{\partial f} \\ \frac{\partial w}{\partial a} & \frac{\partial w}{\partial f} \end{bmatrix} \begin{bmatrix} \delta a \\ \delta f \end{bmatrix} \tag{4.4-5}
$$

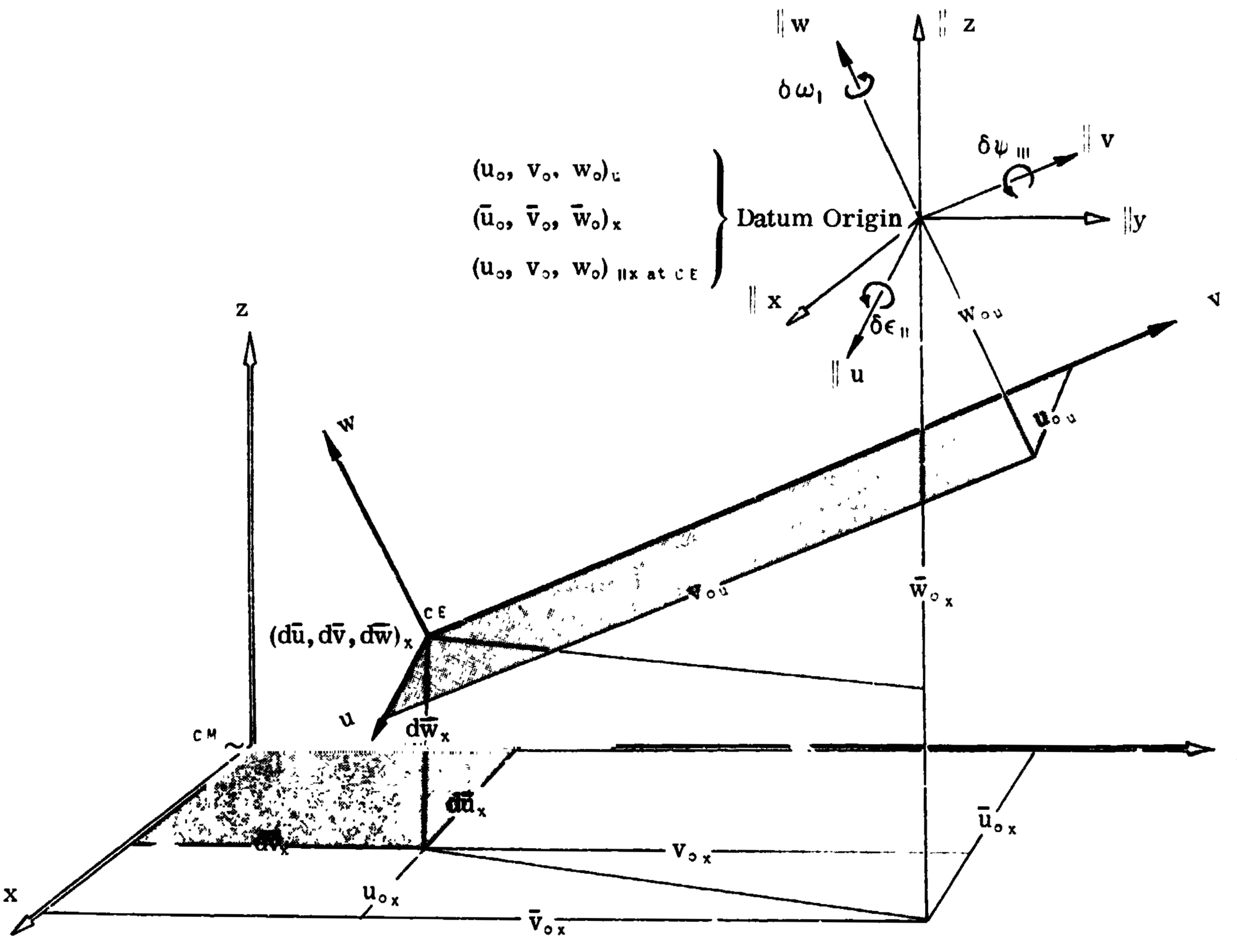


Fig. 4.3 Datum Origin Coordinates in the Geographic and Geodetic Systems

Clearly in the above equation differential changes ($d\lambda$, $d\varphi$, dh) in the geodetic coordinates at any particular point are considered, in addition to the changes δa and δf.

Before going further, it is important to understand what $[\delta u_o \quad \delta v_o \quad \delta w_o]^T$ is. This is not very clear in the original text and probably originated the confusion when equation (4.4-5) was used as a model. (At this moment it is proper to mention that in the often quoted work by Molodenskii et al., [1962] a local system different from the one used in this paper is assumed. Appendix A contains the relationship between the matrix notation used here and the equations in the original English translation from the Russian).

Following [Molodenskii et al., 1962] the shifts between the origins of the geographic and geodetic systems become (see Fig. 4.3)

$$\begin{bmatrix} d\bar{u} \\ d\bar{v} \\ d\bar{w} \end{bmatrix}_x = \begin{bmatrix} \bar{u}_o \\ \bar{v}_o \\ \bar{w}_o \end{bmatrix}_x - \begin{bmatrix} u_o \\ v_o \\ w_o \end{bmatrix}_x = \begin{bmatrix} \bar{u}_o \\ \bar{v}_o \\ \bar{w}_o \end{bmatrix}_x - \left\{ \begin{bmatrix} u_o \\ v_o \\ w_o \end{bmatrix}_u + \begin{bmatrix} du_o \\ dv_o \\ dw_o \end{bmatrix}_{rot} \right\}_x =$$

$$= \begin{bmatrix} \bar{u}_o \\ \bar{v}_o \\ \bar{w}_o \end{bmatrix}_x - \left\{ \begin{bmatrix} u_o \\ v_o \\ w_o \end{bmatrix}_u + \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} u_o \\ v_o \\ w_o \end{bmatrix}_u \right\}_x \qquad (4.4\text{-}6)$$

where for clarity, small subindices are used to represent the Cartesian system to which the components of the column vectors are referred. Consequently, it is possible to write the following relation equivalent to equation (I.3.4) in [Molodenskii et al., 1962]

$$\begin{bmatrix} d\bar{u} \\ d\bar{v} \\ d\bar{w} \end{bmatrix} = \begin{bmatrix} \delta u_o \\ \delta v_o \\ \delta w_o \end{bmatrix} - \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} u_o \\ v_o \\ w_o \end{bmatrix} \qquad (4.4\text{-}7)$$

and therefore from (4.4-6) and (4.4-7) one may conclude

$$\begin{bmatrix} \delta u_o \\ \delta v_o \\ \delta w_o \end{bmatrix} = \begin{bmatrix} \bar{u}_o \\ \bar{v}_o \\ \bar{w}_o \end{bmatrix}_x - \begin{bmatrix} u_o \\ v_o \\ w_o \end{bmatrix}_u \qquad (4.4\text{-}8)$$

Probably the main reason why in [Molodenskii et al., 1962] the value of (4.4-8) is not given explicitly is that what they called "progressive translations of the ellipsoid" are the difference between the two sets of coordinates of a particular point (in this case the origin of the datum) which are in different Cartesian systems (x, y, z and u, v, w). This difference, although rigorously correct, is somewhat difficult to visualize. Nevertheless it is perfectly clear that equation (4.4-8) does not represent the shifts between the origins of the (x, y, z) and (u, v, w) systems, as interpreted by some authors when the Molodenskii equations were used as a model. It is obvious that the second term on the right hand side of (4.4-7), equivalent to a rotation about (u_o, v_o, w_o) of a system parallel to (u, v, w), is neglected when the vector $[\delta u_o \;\; \delta v_o \;\; \delta w_o]^T$ is thought to be the shifts between the geographic and geodetic Cartesian systems. Substituting the value of $[\delta u_o \;\; \delta v_o \;\; \delta w_o]^T$ from (4.4-7) in (4.4-5) it may be shown that

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = \begin{bmatrix} d\bar{u} \\ d\bar{v} \\ d\bar{w} \end{bmatrix} + \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + R^T \begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial a} & \frac{\partial u}{\partial f} \\ \frac{\partial v}{\partial a} & \frac{\partial v}{\partial f} \\ \frac{\partial w}{\partial a} & \frac{\partial w}{\partial f} \end{bmatrix} \begin{bmatrix} \delta a \\ \delta f \end{bmatrix} \qquad (4.4\text{-}9)$$

and finally with the assumption ($d\lambda = d\varphi = dh = 0$), it follows that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} d\bar{u} \\ d\bar{v} \\ d\bar{w} \end{bmatrix} + \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial a} & \frac{\partial u}{\partial f} \\ \frac{\partial v}{\partial a} & \frac{\partial v}{\partial f} \\ \frac{\partial w}{\partial a} & \frac{\partial w}{\partial f} \end{bmatrix} \begin{bmatrix} \delta a \\ \delta f \end{bmatrix} \qquad (4.4\text{-}10)$$

which is exactly the similarity transformation given by (4.3-5) with the notation

$$\left.\begin{aligned} d\bar{u} &= \Delta x \\ d\bar{v} &= \Delta y \\ d\bar{w} &= \Delta z \end{aligned}\right\} = \left\{\begin{array}{l}\text{shifts between the origins of the two coordinate systems} \\ \text{with the sign convention of (4.3-4a)}\end{array}\right.$$

and the replacement of the effect of a scale change Δ by the change in the semi-major axis of the ellipsoid. In the above equation it may be assumed $\delta f = 0$.

Equation (4.4-10) is not given in the above form in [Molodenskii et al., 1962] and was derived here only with the intention of showing that the model obtained is strictly a similarity transformation of the " Bursa type " without anything special introduced besides the scaling variation mentioned above. Later the difference between these two scaling approaches will be fully explained.

What Molodenskii presented as expression (1.3.3) is nothing else but equation (4.4-9) premultiplied on both sides by the rotation matrix R, namely

$$R\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = R\begin{bmatrix} d\bar{u} \\ d\bar{v} \\ d\bar{w} \end{bmatrix} + R\begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$+\begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix} + R\begin{bmatrix} \dfrac{\partial u}{\partial a} & \dfrac{\partial u}{\partial f} \\ \dfrac{\partial v}{\partial a} & \dfrac{\partial v}{\partial f} \\ \dfrac{\partial w}{\partial a} & \dfrac{\partial w}{\partial f} \end{bmatrix}\begin{bmatrix} \delta a \\ \delta f \end{bmatrix} \qquad (4.4\text{-}11)$$

4.5 Comparison of Scaling Methods by Means of δL or δa

Before giving the individual formulation for each method, it must be understood that a change δL is always applied to the unit length of the Cartesian coordinate system involved. Therefore it may be considered as scaling the space or a change in its metric. It can be applied either to the geodetic or to the geographic system, depending on the adopted sign convention.

Nevertheless, a change δa is always applied to the geodetic system and practically represents a "network scale."

4.5.1 Changes $d\lambda$, $d\varphi$, dh Due to a Change δL in the Scale

These changes may be obtained very easily by making use of the relations (4.1-4) and (4.2-14), namely

$$\begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix}_{\delta L} = R \begin{bmatrix} u\,\delta L \\ v\,\delta L \\ w\,\delta L \end{bmatrix} = R \begin{bmatrix} (N+h)\cos\varphi\cos\lambda\,\delta L \\ (N+h)\cos\varphi\sin\lambda\,\delta L \\ [N(1-e^2)+h]\sin\varphi\,\delta L \end{bmatrix} \qquad (4.5\text{-}1)$$

Finally, it can be proved that (see Appendix B)

$$d\lambda_{\delta L} = 0 \qquad (4.5\text{-}2a)$$

$$d\varphi_{\delta L} = -\frac{N e^2 \sin\varphi\cos\varphi}{M+h}\,\delta L \qquad (4.5\text{-}2b)$$

$$dh_{\delta L} = (aW + h)\,\delta L \qquad (4.5\text{-}2c)$$

Consequently, as expected, for a reference rotational ellipsoid, there is not any influence in the geodetic longitude due to a change of scale in the length unit of the Cartesian geodetic system.

4.5.2 Changes $d\lambda$, $d\varphi$, dh Due to a Change δa in the Semimajor Axis of the Ellipsoid

The changes (du, dv, dw) in the Cartesian coordinates in function of changes of the geodetic coordinates $(d\lambda, d\varphi, dh)$ and differential changes in ellipsoidal parameters δa, δf may be expressed by

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial a} & \frac{\partial u}{\partial f} \\ \frac{\partial v}{\partial a} & \frac{\partial v}{\partial f} \\ \frac{\partial w}{\partial a} & \frac{\partial w}{\partial f} \end{bmatrix} \begin{bmatrix} \delta a \\ \delta f \end{bmatrix} + R^T \begin{bmatrix} (M+h)\cos\varphi\, d\lambda \\ (N+h)\, d\varphi \\ dh \end{bmatrix} \tag{4.5-3}$$

Thus, assuming that the point (u, v, w) remains fixed in space, the effect of differential changes δa, δf on (λ, φ, h) is:

$$\begin{bmatrix} (M+h)\cos\varphi\, d\lambda \\ (N+h)\, d\varphi \\ dh \end{bmatrix} = -R \begin{bmatrix} \frac{\partial u}{\partial a} & \frac{\partial u}{\partial f} \\ \frac{\partial v}{\partial a} & \frac{\partial v}{\partial f} \\ \frac{\partial w}{\partial a} & \frac{\partial w}{\partial f} \end{bmatrix} \begin{bmatrix} \delta a \\ \delta f \end{bmatrix} \tag{4.5-4}$$

Therefore, recalling (4.4-1) and (4.4-3)

$$\begin{bmatrix} (M+h)\cos\varphi\, d\lambda \\ (N+h)\, d\varphi \\ dh \end{bmatrix}_{\delta a} = -R \begin{bmatrix} du \\ dv \\ dw \end{bmatrix}_{\delta a} = -R \begin{bmatrix} \frac{\cos\varphi\cos\lambda}{W}\delta a \\ \frac{\cos\varphi\sin\lambda}{W}\delta a \\ \frac{(1-e^2)\sin\varphi}{W}\delta a \end{bmatrix} \tag{4.5-5}$$

The above matrix operations yield (see Appendix B)

$$d\lambda_{\delta a} = 0 \tag{4.5-6a}$$

$$d\varphi_{\delta a} = \frac{N e^2 \sin\varphi\cos\varphi}{(M+h)\, a}\,\delta a \tag{4.5-6b}$$

$$dh_{\delta a} = -W\,\delta a \tag{4.5-6c}$$

As in the previous case, changing the semimajor axis of the reference ellipsoid does not affect the geodetic longitude. Nevertheless, there are some differences between the two methods of scaling, as will be explained in the next section.

4.5.3 Comparison of Scaling Methods

Assume that it is desired to absorb the metric scale change in the coordinate system by a change δa in the semimajor axis of the ellipsoid. A positive change δL will not alter the ellipsoid; nevertheless, the new value of the semimajor axis (using the new yardstick due to a positive δL) will fulfill the inequality,

$$a_{new} < a_{old} \qquad \forall \delta L > 0$$

Thus, $$\delta a_L = a_{new} - a_{old} < 0,$$

therefore, $$\delta L > 0 \Longrightarrow \delta a_L < 0,$$

and finally, $$\delta a_L = -a\,\delta L \tag{4.5-7}$$

Therefore, substituting the above formula in (4.5-4b) and (4.5-4c),

$$d\varphi_{\delta a_L} = -\frac{N \cdots \cos\varphi}{\cdots}\,\delta L = d\varphi_{\delta L} \tag{4.5-8}$$

and

$$dh_{\delta a_L} = aW\delta L \neq dh_{\delta L} = (aW + h)\delta L \tag{4.5-9}$$

Thus when the semimajor axis of the ellipsoid changes by the amount δa_L due only to scale variation, namely (4.5-7), the differential changes in the latitude are equal to the differential changes produced by a scaling of the coordinate system. However, the differential changes in the heights are different, as can be seen from (4.5-9). This is in accordance with the remarks made by Hotine [1969, p. 264]:

> "Most of the systematic error in scale of a network could be eliminated by altering the size of the base spheroid in the geodetic coordinate system... However, this procedure would vitiate the height dimension and would result in some inaccuracy even in a two-dimensional adjustment which ignores geodetic heights, especially if the network covers a considerable area."

The difference between the two methods of scaling is

$$\Delta h = dh_{\delta L} - dh_{\delta a_L} = h\,\delta L \tag{4.5-10}$$

which means that for points on the ellipsoid ($h = 0$) the two scaling methods are identical.

It can be observed that in any case the value of Δh may be neglected, as can be shown by a simple example. Taking the values of δL and h excessively, e.g.

$$\delta L = 6 \times 10^{-6} \quad \text{and} \quad h = 5 \text{ km}$$

$$\Delta h = 30 \times 10^{-6} \text{km.} = 30 \text{ mm.}$$

Thus for all practical cases, the method of scaling followed by Molodenskii is not different from scaling the system through changes in the unit length of the Cartesian axes.

However, rigorously speaking, in general it is possible to assume changes in the semimajor axis of the ellipsoid δa and at the same time, scale changes δL.

This, in fact, signifies that one can change the metric scale δL of the Cartesian system without changing the size of the ellipsoid, although every length measured with the new scale unit will be different. This is thus equivalent to changing the metric of the space, that is, its unit of length.

On the other hand, it is also completely valid to assume a change δa in the size of the ellipsoid independent of any scale change. This may be considered as a "network scale change" but clearly the units along the axis of the Cartesian coordinate system will not undergo any variation. That is, the unit of length (i.e. the scale of the three-dimensional space) remains the same.

4.6 Effects of Other Differential Changes on the Geodetic Curvilinear Coordinates

The theory given in the previous sections is general and may be used in any case desired. The basic equation is (4.1-4), in which the individual changes $[du \;\; dv \;\; dw]^T$ should be replaced by values corresponding to the particular problem at hand. Since the number of possibilities is unlimited (note that this general approach may also be applied to "spatial or three-dimensional geodesy"), in this section only two examples will be shown.

4.6.1 Differential Changes in (λ, φ, h) Due to a Change $(d\lambda_o, d\varphi_o, dh_o)$ at the Datum Origin $(\lambda_o, \varphi_o, h_o)$

Recalling (4.1-4), this may be expressed as

$$\begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix} = R \begin{bmatrix} du \\ dv \\ dw \end{bmatrix}_o \tag{4.6-1}$$

but from (4.5-5)

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix}_o = R_o^T \begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix}_o \tag{4.6-2}$$

where

$$R_o = R_1(90 - \varphi_o)\, R_3(90 + \lambda_o) \tag{4.6-3}$$

Thus:

$$\begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix} = R\, R_o^T \begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix}_o \tag{4.6-4}$$

and

$$R\,R_o^T = \begin{bmatrix} \cos(\lambda-\lambda_o) & \sin\varphi_o\sin(\lambda-\lambda_o) & -\cos\varphi_o\sin(\lambda-\lambda_o) \\ -\sin\varphi\sin(\lambda-\lambda_o) & \cos\varphi_o\cos\varphi + \sin\varphi_o\sin\varphi\cos(\lambda-\lambda_o) & \sin\varphi_o\cos\varphi - \cos\varphi_o\sin\varphi\cos(\lambda-\lambda_o) \\ \cos\varphi\sin(\lambda-\lambda_o) & \cos\varphi_o\sin\varphi - \sin\varphi_o\cos\varphi\cos(\lambda-\lambda_o) & \sin\varphi_o\sin\varphi + \cos\varphi_o\cos\varphi\cos(\lambda-\lambda_o) \end{bmatrix} \tag{4.6-5}$$

This transformation really consists first in making the local system at the datum origin parallel to the geodetic system through R_o^T. A new rotation R will finally make the rotated curvilinear frame parallel to the particular local system at any point P.

The complete operation may be expressed symbolically by the following commutative diagram:

$$\begin{tikzcd}
(\eta, \xi, \zeta)_o \arrow[rr, "R_1 R_o^T"] \arrow[dr, "R_o^T"'] & & (\eta, \xi, \zeta)_1 \\
& (u, v, w)_o \arrow[ur, "R_1"'] &
\end{tikzcd}$$

4.6.2 Effect of Rotations on the Curvilinear Geodetic Coordinates

From (4.1-4) and (4.2-10)

$$\begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix}_{\delta R} = R \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} (N+h)\cos\varphi\cos\lambda \\ (N+h)\cos\varphi\sin\lambda \\ [N(1-e^2)+h]\sin\varphi \end{bmatrix} \tag{4.6-6}$$

After the above matrix multiplications are performed and after simplification (see Appendix C), the following three equations are obtained:

$$d\lambda_{\delta R} = -\delta\omega + \delta\epsilon\left(1 - \frac{Ne^2}{N+h}\right)\tan\varphi\cos\lambda + \delta\psi\left(1 - \frac{Ne^2}{N+h}\right)\tan\varphi\sin\lambda \tag{4.6-7a}$$

$$d\varphi_{\delta R} = -\delta\epsilon\sin\lambda\,\frac{aW+h}{M+h} - \delta\psi\cos\lambda\,\frac{aW+h}{M+h} \tag{4.6-7b}$$

$$dh_{\delta R} = -\delta\epsilon N e^2\sin\varphi\cos\varphi\sin\lambda + \delta\psi N e^2\sin\varphi\cos\varphi\cos\lambda \tag{4.6-7c}$$

The above equations in similar form are also given in [Hotine, 1969, p. 263]. In Appendix A the reader may find in equations (A.1-12) the effect of rotations as given in [Molodenskii et al., 1962]. Note that equations (4.6-7) are not completely rigorous expressions because the assumption of small rotations is implicit in the matrix $\delta\underline{R}$. See equation (4.2-10).

The complete rigorous expressions in matrix notation may be obtained if in place of $\delta\underline{R}$ in (4.6-5), the matrix $[\underline{R} - I]$ is substituted, where $\underline{R}$ is given by (4.2-6). The utility of equations of the type (4.6-6) in the application of minimal constraints to curvilinear coordinates is discussed in Appendix D.

5. APPLICATIONS TO OTHER CURVILINEAR SYSTEMS

5.1 Ellipsoidal Coordinates

The theory developed in Chapters 3 and 4 was applied exclusively to a case of orthogonal curvilinear coordinates, the geodetic coordinates (λ, φ, h). It will be shown at this time that the above theory may also be used in other orthogonal curvilinear systems of common application in geodesy and geophysics, such as ellipsoidal and spherical coordinates.

Although in both instances, the family of surfaces is triply orthogonal, one basic difference, however, should not be overlooked. It is simply that while the coordinate lines and surfaces generated by geodetic coordinates are orthogonal, they are nevertheless not confocal. In this chapter only confocal families of surfaces will be treated. The following implications hold:

$$\text{confocal} \implies \text{orthogonal}$$

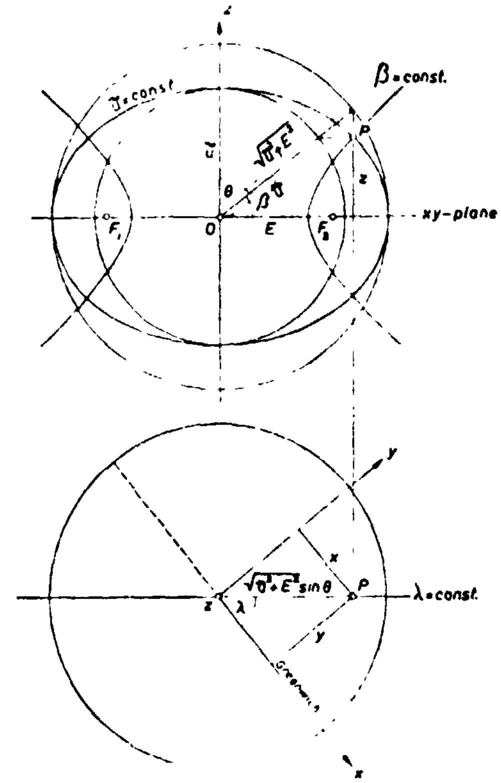
$$\text{orthogonal} \not\Rightarrow \text{confocal}$$

Appendix E reviews the properties of some common families of ellipsoids.

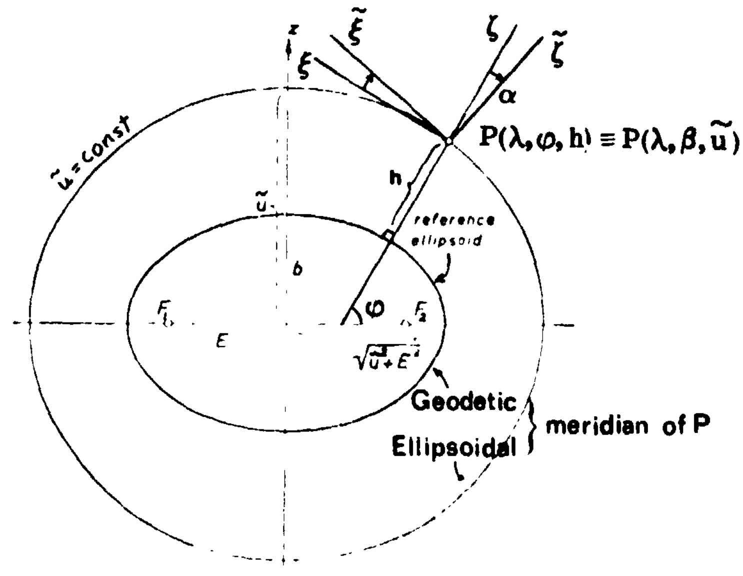
For the sake of generality, it will be convenient to give first the transformation equations between the Cartesian and general ellipsoidal coordinates (reference ellipsoid of three parameters) from which, as is known, two degenerated cases can be obtained: rotational ellipsoidal (reference ellipsoid of two parameters), also called spheroidal by some authors, and spherical coordinates.

On the following pages, it will be assumed that the reference ellipsoid for the ellipsoidal as well as the curvilinear geodetic coordinates is geocentric. Therefore the notation (x, y, z) will be used for the Cartesian system.

The transformation equations between the different curvilinear and Cartesian coordinates are given below. It is assumed that the reader is familiar with the basic definitions. Otherwise [Hobson, 1931] or [Heiskanen and Moritz, 1967] can be consulted.



a) **Ellipsoidal (Rotational) Coordinates**



b) **Local Ellipsoidal and Geodetic Frames**

Fig. 5.1 Ellipsoidal Coordinates (after [Heiskanen and Moritz, 1967.])

$(\tilde{\lambda}, \tilde{\beta}, \tilde{u})$

General ellipsoidal

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (\tilde{u}^2 + E^2)^{\frac{1}{2}} \cos\tilde{\beta} \cos\tilde{\lambda} \\ (\tilde{u}^2 + E_1^2)^{\frac{1}{2}} \cos\tilde{\beta} \sin\tilde{\lambda} \\ \tilde{u} \sin\tilde{\beta} \end{bmatrix}_{(\tilde{u}, E, E_1)} \qquad (5.1\text{-}1)$$

$$-\frac{\pi}{2} \le \tilde{\beta} \le \frac{\pi}{2} \quad ; \quad 0 \le \tilde{\lambda} \le 2\pi \quad ; \quad E_1^2 \le \tilde{u}^2 < \infty \quad ; \quad E^2 > E_1^2 > 0$$

$$E_1 \to E \implies \begin{cases} \tilde{\beta} \to \beta \\ \tilde{\lambda} \to \lambda \end{cases}$$

$(\lambda, \beta, \tilde{u})$; $E = E_1$

Rotational ellipsoidal
Spheroidal

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (\tilde{u}^2 + E^2)^{\frac{1}{2}} \cos\beta \cos\lambda \\ (\tilde{u}^2 + E^2)^{\frac{1}{2}} \cos\beta \sin\lambda \\ \tilde{u} \sin\beta \end{bmatrix}_{(\tilde{u}, E)} \qquad (5.1\text{-}2)$$

$$E \to 0 \implies \begin{cases} \tilde{u} \to r \\ \beta \to \psi \end{cases}$$

(λ, ψ, r) ; $E = 0$

Spherical

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \cos\psi \cos\lambda \\ r \cos\psi \sin\lambda \\ r \sin\psi \end{bmatrix}_{r} \qquad (5.1\text{-}3)$$

The elements of the Jacobian matrix for the transformation between general ellipsoidal and Cartesian coordinates may be computed easily:

$$\frac{\partial x}{\partial \tilde{\lambda}} = -(\tilde{u}^2 + E^2)^{\frac{1}{2}} \cos\tilde{\beta} \sin\tilde{\lambda} \qquad \frac{\partial x}{\partial \tilde{\beta}} = -(\tilde{u}^2 + E^2)^{\frac{1}{2}} \sin\tilde{\beta} \cos\tilde{\lambda} \qquad \frac{\partial x}{\partial \tilde{u}} = \frac{\tilde{u}}{(\tilde{u}^2 + E^2)^{\frac{1}{2}}} \cos\tilde{\beta} \cos\tilde{\lambda}$$

$$\frac{\partial y}{\partial \tilde{\lambda}} = (\tilde{u}^2 + E_1^2)^{\frac{1}{2}} \cos\tilde{\beta} \cos\tilde{\lambda} \qquad \frac{\partial y}{\partial \tilde{\beta}} = -(\tilde{u}^2 + E_1^2)^{\frac{1}{2}} \sin\tilde{\beta} \sin\tilde{\lambda} \qquad \frac{\partial y}{\partial \tilde{u}} = \frac{\tilde{u}}{(\tilde{u}^2 + E_1^2)^{\frac{1}{2}}} \cos\tilde{\beta} \sin\tilde{\lambda}$$

$$\frac{\partial z}{\partial \tilde{\lambda}} = 0 \qquad \frac{\partial z}{\partial \tilde{\beta}} = \tilde{u} \cos\tilde{\beta} \qquad \frac{\partial z}{\partial \tilde{u}} = \sin\tilde{\beta}$$

(5.1-4)

and then considering equations (3.2-12) and the above results, the elements of the metric matrix when the reference ellipsoid has the parameters ($\tilde{u}$, E, E_1) are obtained as follows:

$$h_1 = \cos\tilde{\beta}[\tilde{u}^2 + E^2\sin^2\tilde{\lambda} + E_1^2\cos^2\tilde{\lambda}]^{\frac{1}{2}} \tag{5.1-5a}$$

$$h_2 = [\tilde{u}^2 + \sin^2\tilde{\beta}\,(E^2\cos^2\tilde{\lambda} + E_1^2\sin^2\tilde{\lambda})]^{\frac{1}{2}} \tag{5.1-5b}$$

$$h_3 = \left[\tilde{u}^2\cos^2\tilde{\beta}\left(\frac{\cos^2\tilde{\lambda}}{\tilde{u}^2 + E^2} + \frac{\sin^2\tilde{\lambda}}{\tilde{u}^2 + E_1^2}\right) + \sin^2\tilde{\beta}\right]^{\frac{1}{2}} \tag{5.1-5c}$$

Assuming now $E = E_1$, the elements of the Jacobian and metric matrices for the transformation between ellipsoidal (rotational) and Cartesian coordinates are immediately obtained

$$\frac{\partial x}{\partial\lambda} = -(\tilde{u}^2 + E^2)^{\frac{1}{2}}\cos\beta\sin\lambda \qquad \frac{\partial x}{\partial\beta} = -(\tilde{u}^2 + E^2)^{\frac{1}{2}}\sin\beta\cos\lambda \qquad \frac{\partial x}{\partial\tilde{u}} = \frac{\tilde{u}}{(\tilde{u}^2 + E^2)^{\frac{1}{2}}}\cos\beta\cos\lambda$$

$$\frac{\partial y}{\partial\lambda} = (\tilde{u}^2 + E^2)^{\frac{1}{2}}\cos\beta\cos\lambda \qquad \frac{\partial y}{\partial\beta} = -(\tilde{u}^2 + E^2)^{\frac{1}{2}}\sin\beta\sin\lambda \qquad \frac{\partial y}{\partial\tilde{u}} = \frac{\tilde{u}}{(\tilde{u}^2 + E^2)^{\frac{1}{2}}}\cos\beta\sin\lambda$$

$$\frac{\partial z}{\partial\lambda} = 0 \qquad \frac{\partial z}{\partial\beta} = \tilde{u}\cos\beta \qquad \frac{\partial z}{\partial\tilde{u}} = \sin\beta \tag{5.1-6}$$

and

$$h_1 = \cos\beta\,(\tilde{u}^2 + E^2)^{\frac{1}{2}} \tag{5.1-7a}$$

$$h_2 = (\tilde{u}^2 + E^2\sin^2\beta)^{\frac{1}{2}} \tag{5.1-7b}$$

$$h_3 = [\frac{\tilde{u}^2}{\tilde{u}^2 + E^2}\cos^2\beta + \sin^2\beta]^{\frac{1}{2}} = [\frac{u^2 + E^2\sin^2\beta}{\tilde{u}^2 + E^2}]^{\frac{1}{2}} = w \tag{5.1-7c}$$

Finally, for the simple case of spherical coordinates (E = 0), the following known relations are obtained:

$$\frac{\partial x}{\partial \lambda} = -r\cos\psi\sin\lambda \qquad \frac{\partial x}{\partial \psi} = -r\sin\psi\cos\lambda \qquad \frac{\partial x}{\partial r} = \cos\psi\cos\lambda$$

$$\frac{\partial y}{\partial \lambda} = r\cos\psi\cos\lambda \qquad \frac{\partial y}{\partial \psi} = -r\sin\psi\sin\lambda \qquad \frac{\partial y}{\partial r} = \cos\psi\sin\lambda \qquad (5.1\text{-}8)$$

$$\frac{\partial z}{\partial \lambda} = 0 \qquad \frac{\partial z}{\partial \psi} = r\cos\psi \qquad \frac{\partial z}{\partial r} = \sin\psi$$

and

$$h_1 = r\cos\psi \qquad (5.1\text{-}9a)$$

$$h_2 = r \qquad (5.1\text{-}9b)$$

$$h_3 = 1 \qquad (5.1\text{-}9c)$$

5.2 Transformations Between the Normal Gravity Vector Components

Because of the nature of the reference body, the normal gravity field is generally represented in function of ellipsoidal coordinates. The function $U(\tilde{u}, \beta)$ is given for example by equation (2-62) in [Heiskanen and Moritz, 1967] for a particular rotational ellipsoid with semimajor axis a, ' ~ear excentricity E, gravitational constant kM and rotational velocity ω.

Assume now that the following transformation is desired:

$$(\gamma_x, \gamma_y, \gamma_z) \longrightarrow (\gamma_{\tilde{\eta}}, \gamma_{\tilde{\xi}}, \gamma_{\tilde{\zeta}}) \qquad (5.2\text{-}1)$$

where

$$\vec{\gamma} = \text{grad } U \equiv (\gamma_x, \gamma_y, \gamma_z) = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right)$$

That is, given the components of the vector $\vec{\gamma}$ (treated as a free vector) in the geocentric system (x, y, z), obtain the components of the normal gravity vector in a local ellipsoidal frame $(\tilde{\eta}, \tilde{\xi}, \tilde{\zeta})$ at the point $P(\lambda, \beta, \tilde{u})$.

The above transformation will be performed through an orthogonal rotation matrix. Therefore the inverse transformation (which is really the one practically used) is immediately available.

To obtain the rotational matrix in the transformation (5.2-1) is not as simple by geometrical deductions as in the case of geodetic coordinates. The main reason being that it is very difficult to visualize the connection between the angle β and the local ellipsoidal frame (see Fig. 5.1).

In cases like this, as mentioned in Section 3.8, equations (3.8-1) and (3.8-2) may be applied. For this particular curvilinear system, the following relation holds:

$$R_e = H_e^{-1} J_e^{T} \tag{5.2-2}$$

where the elements in the matrices J_e and H_e are given by equations (5.1-6) and (5.1-7) respectively. Thus

$$\begin{bmatrix} \gamma_{\tilde{\eta}} \\ \gamma_{\tilde{\xi}} \\ \gamma_{\tilde{\zeta}} \end{bmatrix} = R_e \begin{bmatrix} \gamma_x \\ \gamma_y \\ \gamma_z \end{bmatrix} = H_e^{-1} J_e^{T} \begin{bmatrix} \gamma_x \\ \gamma_y \\ \gamma_z \end{bmatrix} \tag{5.2-3}$$

$$\begin{bmatrix} \gamma_{\tilde{\eta}} \\ \gamma_{\tilde{\xi}} \\ \gamma_{\tilde{\zeta}} \end{bmatrix} = \begin{bmatrix} 1/(\tilde{u}^2 + E^2)^{\frac{1}{2}}\cos\beta & & \\ & 1/w(\tilde{u}^2 + E^2)^{\frac{1}{2}} & \\ & & 1/w \end{bmatrix} \times$$

$$\begin{bmatrix} -(\tilde{u}^2 + E^2)^{\frac{1}{2}}\cos\beta\sin\lambda & (\tilde{u}^2 + E^2)^{\frac{1}{2}}\cos\beta\cos\lambda & 0 \\ -(\tilde{u}^2 + E^2)^{\frac{1}{2}}\sin\beta\cos\lambda & -(\tilde{u}^2 + E^2)^{\frac{1}{2}}\sin\beta\sin\lambda & \tilde{u}\cos\beta \\ \tilde{u}\cos\beta\cos\lambda / (\tilde{u}^2 + E^2)^{\frac{1}{2}} & \tilde{u}\cos\beta\sin\lambda / (\tilde{u}^2 + E^2)^{\frac{1}{2}} & \sin\beta \end{bmatrix} \begin{bmatrix} \gamma_x \\ \gamma_y \\ \gamma_z \end{bmatrix} \tag{5.2-4}$$

The above is also given in equation (6-11) of [Heiskanen and Moritz, 1967], although the final result is obtained by a different approach. Observe that the notation of the equations presented here is in accordance with the general criteria of this report.

After the matrix multiplication in (5.2-4) is performed, the rotational matrix of the transformation (5.2-1) is as follows:

$$R_e = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\frac{1}{w}\sin\beta\cos\lambda & -\frac{1}{w}\sin\beta\sin\lambda & \frac{\tilde{u}}{w(\tilde{u}^2+E^2)^{\frac{1}{2}}}\cos\beta \\ \frac{\tilde{u}}{w(\tilde{u}^2+E^2)^{\frac{1}{2}}}\cos\beta\cos\lambda & \frac{\tilde{u}}{w(\tilde{u}^2+E^2)^{\frac{1}{2}}}\cos\beta\sin\lambda & \frac{\sin\beta}{w} \end{bmatrix} \qquad (5.2\text{-}5)$$

5.3 Differential Transformations Between Cartesian and Ellipsoidal Coordinates

The transformation between differential changes in Cartesian and ellipsoidal coordinates can immediately be obtained treating the basic equations of Section 4.1 in a more general way.

Applying (4.1-3) to this specific case,

$$H_e \begin{bmatrix} d\lambda \\ d\beta \\ d\tilde{u} \end{bmatrix} = R_e \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \qquad (5.3\text{-}1)$$

but recalling (3.8-1) and (3.8-2)

$$R_e = H_e^{-1} J_e^{T} = H_e J_e^{-1} \qquad (5.3\text{-}2)$$

one obtains

$$
\begin{bmatrix} d\lambda \\ d\beta \\ d\tilde{u} \end{bmatrix} = H_e^{-1} R_e \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \quad \text{and} \quad \begin{cases} \begin{bmatrix} d\lambda \\ d\beta \\ d\tilde{u} \end{bmatrix} = (H_e^{-1})^2 J_e^T \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} & (5.3\text{-}3) \\[2em] \begin{bmatrix} d\lambda \\ d\beta \\ d\tilde{u} \end{bmatrix} = J_e^{-1} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} & (5.3\text{-}4) \end{cases}
$$

Observe that equation (5.3-4) was expected, if the total differential approach is recalled.

From (5.3-3) the analytical form of the inverse of the Jacobian may be computed, giving

$$
J_e^{-1} = \frac{1}{w^2(\tilde{u}^2 + E^2)^{\frac{1}{2}}} \begin{bmatrix} -w^2 \sin\lambda/\cos\beta & w^2 \cos\lambda/\cos\beta & 0 \\ -\sin\beta\cos\lambda & -\sin\beta\sin\lambda & \dfrac{\tilde{u}\cos\beta}{(\tilde{u}^2 + E^2)^{\frac{1}{2}}} \\ \tilde{u}\cos\beta\cos\lambda & \tilde{u}\cos\beta\sin\lambda & \sin\beta(\tilde{u}^2 + E^2)^{\frac{1}{2}} \end{bmatrix} \tag{5.3-5}
$$

5.4 General Commutative Diagram for the Transformation of Free Vector Components

Figure 5.2 represents all possible transformations or mappings between free vector components in spatial rectangular coordinate systems. Emphasis has been placed on the coordinate systems discussed in this report (mainly geodetic and rotational ellipsoidal) but the same logic may be applied to any other orthogonal curvilinear coordinate system.

For an easy recall, some of the basic matrices represented in the diagram are given according to equation number in the following table:

Table 5.1 Equation Numbers of Commutative Diagram Matrices

Matrix Type	Geodetic System		Rot. Ellipsoidal System	
	Symbol	Equation #	Symbol	Equation #
Jacobian	J	(3.2-8)	J_e	(5.1-6)
Jacobian Inverse	J^{-1}	(3.7-2)	J_e^{-1}	(5.3-5)
Metric (Diagonal)	H	(3.2-12)	H_e	(5.1-7)
Rotation (Orthogonal)	R	(3.3-2)	R_e	(5.2-5)

As a simple application from the diagram, let's assume that the components $(\gamma_{\tilde{\eta}}, \gamma_{\tilde{\xi}}, \gamma_{\tilde{\zeta}})$ are desired. The diagram gives immediately

$$\begin{bmatrix} \partial F / \partial \tilde{\eta} \\ \partial F / \partial \tilde{\xi} \\ \partial F / \partial \tilde{\zeta} \end{bmatrix} = H_e^{-1} \begin{bmatrix} \partial F / \partial \lambda \\ \partial F / \partial \beta \\ \partial F / \partial \tilde{u} \end{bmatrix} \qquad (5.4\text{-}1)$$

where F is any scalar function.

Assuming F to be the spheropotential function U, one can obtain the components of the normal gravity $\vec{\gamma}$ along the local ellipsoidal system at the point $P(\lambda, \beta, \tilde{u})$

$$\begin{bmatrix} \gamma_{\tilde{\eta}} \\ \gamma_{\tilde{\xi}} \\ \gamma_{\tilde{\zeta}} \end{bmatrix} = H_e^{-1} \begin{bmatrix} \partial U / \partial \lambda \\ \partial U / \partial \beta \\ \partial U / \partial \tilde{u} \end{bmatrix} \qquad (5.4\text{-}2)$$

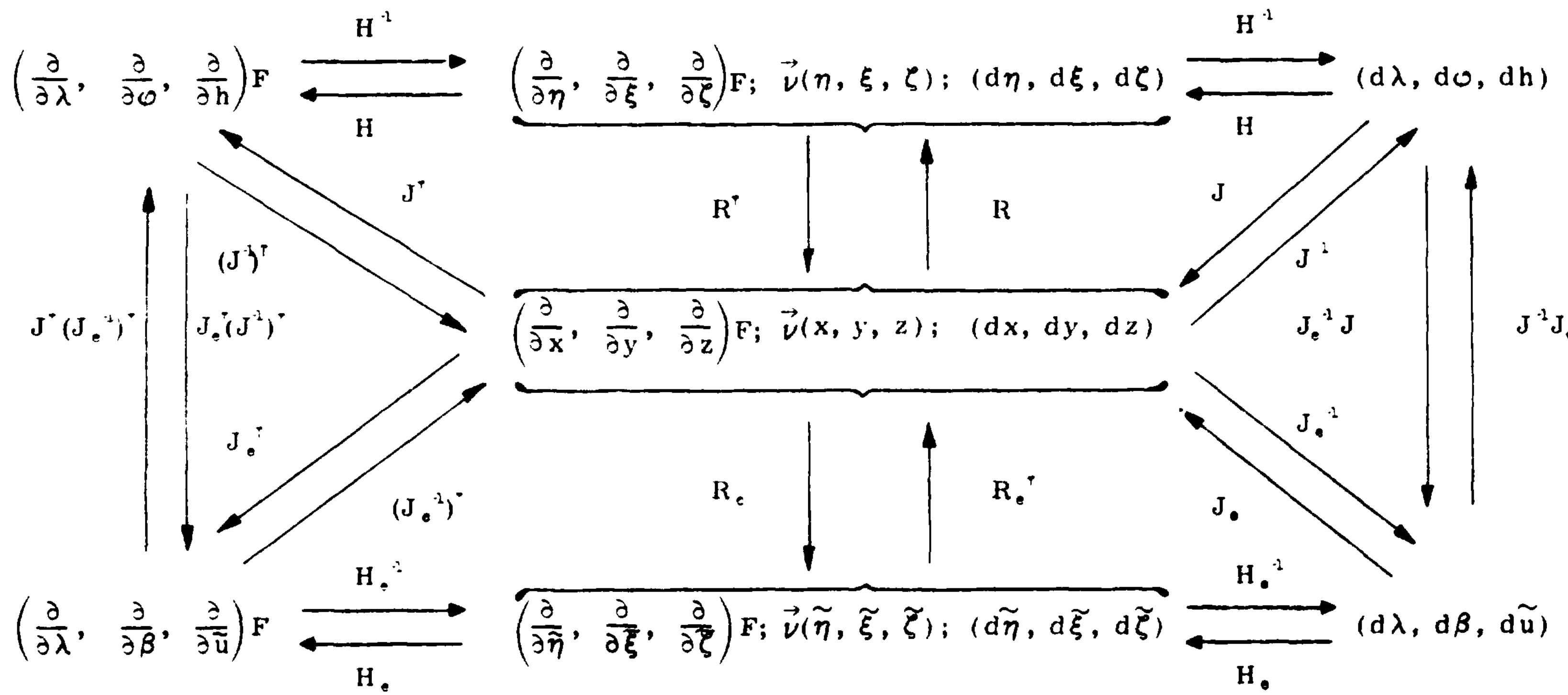


Fig. 5.2 Commutative Diagram for Transforming Free Vector Components Between Cartesian, Geodetic and Ellipsoidal Curvilinear Coordinates

Thus the values of $(\gamma_{\tilde{\eta}}, \gamma_{\tilde{\xi}}, \gamma_{\tilde{\zeta}})$ can be computed readily from the metric matrix of the transformation between ellipsoidal and Cartesian coordinates and the partials of the known function $U(\tilde{u}, \beta)$ respect to the ellipsoidal coordinates. Clearly, for a rotational ellipsoid $\partial U / \partial \lambda = 0$.

Once $\vec{\gamma}\,(\gamma_{\tilde{\eta}}, \gamma_{\tilde{\xi}}, \gamma_{\tilde{\zeta}})$ is computed, it is immediate to obtain $(\gamma_x, \gamma_y, \gamma_z)$ as explained in Section 3.2, through the transformation

$$(\gamma_{\tilde{\eta}}, \gamma_{\tilde{\xi}}, \gamma_{\tilde{\zeta}}) \xrightarrow{R_e^T} (\gamma_x, \gamma_y, \gamma_z)$$

which is also implied in Fig. 5.2.

Observe that the diagram of Fig. 5.2 is commutative. This means that there are several ways to obtain the transformation between two sets of free vector components. The selection of the approach will depend on the type of data readily available. For example, the differential transformation between ellipsoidal and geodetic coordinates may be performed according to the following possibilities.

$$(d\lambda, d\beta, d\tilde{u}) \xrightarrow{H^{-1} R R_e^T H_e} (d\lambda, d\varphi, dh)$$

$$(d\lambda, d\beta, d\tilde{u}) \xrightarrow{J^{-1} R_e^T H_e} (d\lambda, d\varphi, dh)$$

$$(d\lambda, d\beta, d\tilde{u}) \xrightarrow{H^{-1} R J_e} (d\lambda, d\varphi, dh)$$

$$(d\lambda, d\beta, d\tilde{u}) \xrightarrow{J^{-1} J_e} (d\lambda, d\varphi, dh)$$

where the matrix of the transformation is given by:

$$J^{-1} J_e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \begin{array}{l} [(\tilde{u}^2 + E^2)^{\frac{1}{2}} \sin\beta \sin\varphi \\ \quad + \tilde{u} \cos\beta \cos\varphi] \,/\, (M + h) \end{array} & \begin{array}{l} [-\tilde{u}(\tilde{u}^2 + E^2)^{-\frac{1}{2}} \cos\beta \sin\varphi \\ \quad + \sin\beta \cos\varphi] \,/\, (M + h) \end{array} \\ 0 & \begin{array}{l} -(\tilde{u}^2 + E^2)^{\frac{1}{2}} \sin\beta \cos\varphi \\ \quad + \tilde{u} \cos\beta \sin\varphi \end{array} & \begin{array}{l} \tilde{u}(\tilde{u}^2 + E^2)^{-\frac{1}{2}} \cos\beta \sin\varphi \\ \quad + \sin\beta \sin\varphi \end{array} \end{bmatrix} \tag{5.4-3}$$

Notice that there is no change in the transformation of $d\lambda$ between ellipsoidal and geodetic coordinates, as expected.

5.5 The Rotation Matrix $\mathfrak{R}$ Between the Geodetic and Ellipsoidal Local Systems

At this point it will be interesting to study the ·nsformation between the local geodetic and ellipsoidal systems. That is, we are interested in obtaining the rotation matrix $\mathfrak{R}$ of the transformation

$$(\tilde{\eta}, \tilde{\xi}, \tilde{\zeta}) \xrightarrow{\mathfrak{R}} (\eta, \xi, \zeta)$$

From the diagram o: Fig. 5.2,

$$\begin{bmatrix} \eta \\ \xi \\ \zeta \end{bmatrix} = R R_o^T \begin{bmatrix} \tilde{\eta} \\ \tilde{\xi} \\ \tilde{\zeta} \end{bmatrix} \tag{5.5-1}$$

Thus, after matrix multiplication,

$$\mathfrak{R} = R R_o^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & m & n \\ 0 & -n & m \end{bmatrix} \tag{5.5-2a}$$

where

$$m = \frac{1}{w} [\tilde{u} (\tilde{u}^2 + E^2)^{-\frac{1}{2}} \cos\beta \cos\varphi + \sin\beta \sin\varphi] \tag{5.5-2b}$$

$$n = \frac{1}{w} [-\tilde{u} (\tilde{u}^2 + E^2)^{-\frac{1}{2}} \cos\beta \sin\varphi + \sin\beta \cos\varphi] \tag{5.5-2c}$$

The matrix $\mathfrak{R}$ will transform components of the normal gravity vector $\vec{\gamma}$ from the local ellipsoidal to the local geodetic system, or viceversa. The orthogonality of $\mathfrak{R}$ may be proved easily.

Notice that $\mathcal{R}$ represents a counterclockwise rotation α along the $\tilde{\eta}$ axis (Fig. 5.1b) where

$$\cos\alpha = m$$

and

$$\sin\alpha = n$$

This can also be shown by an independent approach considering the orthonormal bases $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ and $(\vec{\tilde{e}}_1, \vec{\tilde{e}}_2, \vec{\tilde{e}}_3)$ along the respective local frames.

Clearly,

$$\cos\alpha = \frac{\vec{e}_2 \cdot \vec{\tilde{e}}_2}{|\vec{e}_2| \, . \, |\vec{\tilde{e}}_2|} = \frac{\vec{e}_3 \cdot \vec{\tilde{e}}_3}{|\vec{e}_3| \, . \, |\vec{\tilde{e}}_3|}$$

The components of the vectors of the base $\vec{e}_i$ $(i = 1, 2, 3)$ may be found in equations (3.2-13). They are also given by the row elements of the matrix R in equation (3.3-2); the components of the vectors in the base $\vec{\tilde{e}}_i$ $(i = 1, 2, 3)$ are the elements of the rows of R_e given by (5.2-5).

Considering that these bases are orthonormal, that is, all the vectors are unit vectors, it follows that

$$\cos\alpha = \vec{e}_2 \cdot \vec{\tilde{e}}_2 = \vec{e}_3 \cdot \vec{\tilde{e}}_3 = \frac{1}{w}\left[\tilde{u}(\tilde{u}^2 + E^2)^{-\frac{1}{2}}\cos\beta\cos\varphi + \sin\beta\sin\varphi\right]$$

Therefore, after algebraic manipulation and simplification,

$$\sin^2\alpha = 1 - \cos^2\alpha = \frac{1}{\tilde{u}^2 + E^2\sin^2\beta}\left[(\tilde{u}^2 + E^2)^{\frac{1}{2}}\sin\beta\cos\varphi - \tilde{u}\cos\beta\sin\varphi\right]^2$$

Thus, finally

$$\sin\alpha = \frac{1}{w}\left[\sin\beta\cos\varphi - \tilde{u}(\tilde{u}^2 + E^2)^{-\frac{1}{2}}\cos\beta\sin\varphi\right]$$

5.6 Transformation between Local Geodetic and Local Astronomic Systems

The same criterion of the above section may also be applied to obtaining the transformation of vector components between the local geodetic (η, ξ, ζ) and local astronomic (η^*, ξ^*, ζ^*) systems.

Clearly in this case, the mapping

$$(\eta, \xi, \zeta) \longrightarrow (\eta^*, \xi^*, \zeta^*)$$

will be obtained as follows:

$$\begin{bmatrix} \eta^* \\ \xi^* \\ \zeta^* \end{bmatrix} = R^* R^T \begin{bmatrix} \eta \\ \xi \\ \zeta \end{bmatrix} \qquad (5.6\text{-}1)$$

where

$$\begin{aligned} R^* &= R_1(90 - \varphi^*)\, R_3(90 + \lambda^*) \\ \varphi^* &= \text{reduced astronomic latitude} \\ \lambda^* &= \text{reduced astronomic longitude} \end{aligned} \qquad (5.6\text{-}2)$$

and

$$R^* R^T = \begin{bmatrix} \cos(\lambda^* - \lambda) & \sin\varphi \sin(\lambda^* - \lambda) & -\cos\varphi \sin(\lambda^* - \lambda) \\ -\sin\varphi^* \sin(\lambda^* - \lambda) & \begin{matrix} \cos\varphi \cos\varphi^* \\ + \sin\varphi \sin\varphi^* \cos(\lambda^* - \lambda) \end{matrix} & \begin{matrix} \sin\varphi \cos\varphi^* \\ - \cos\varphi \sin\varphi^* \cos(\lambda^* - \lambda) \end{matrix} \\ \cos\varphi^* \sin(\lambda^* - \lambda) & \begin{matrix} \cos\varphi \sin\varphi^* \\ - \sin\varphi \cos\varphi^* \cos(\lambda^* - \lambda) \end{matrix} & \begin{matrix} \sin\varphi \sin\varphi^* \\ + \cos\varphi \cos\varphi^* \cos(\lambda^* - \lambda) \end{matrix} \end{bmatrix}$$

(5.6-3)

Assuming now small differences between the geodetic and astronomic coordinates:

$$\varphi^* = \varphi + \delta\varphi \qquad (5.6\text{-}3a)$$

$$\lambda^* = \lambda + \delta\lambda \qquad (5.6\text{-}3b)$$

and with the simplifications

$$\begin{aligned} \sin(\lambda^* - \lambda) &\approx \delta\lambda & \sin(\varphi^* - \varphi) &\approx \delta\varphi \\ \cos(\lambda^* - \lambda) &\approx 1 & \cos(\varphi^* - \varphi) &\approx 1 \\ \sin\varphi^* &\approx \sin\varphi & \cos\varphi^* &\approx \cos\varphi \end{aligned} \tag{5.6-4}$$

it follows

$$RR^{*T} \approx \begin{bmatrix} 1 & \sin\varphi\delta\lambda & -\cos\varphi\delta\lambda \\ -\sin\varphi\delta\lambda & 1 & -\delta\varphi \\ \cos\varphi\delta\lambda & \delta\varphi & 1 \end{bmatrix} = R_\delta^* \tag{5.6-5}$$

This is really the case when one wants to transform between the local geodetic (η, ξ, ζ) and local astronomic (η^*, ξ^*, ζ^*) systems at some particular station P where the deflections of the vertical η' and ξ' are known.

Then,

$$\xi' = \varphi^* - \varphi = \delta\varphi \tag{5.6-6a}$$

$$\eta' = (\lambda^* - \lambda)\cos\varphi = \delta\lambda\cos\varphi \Longrightarrow \sin\varphi\delta\lambda = \eta'\tan\varphi \tag{5.6-6b}$$

Therefore, substituting in (5.6-5)

$$RR^{*T} \approx \begin{bmatrix} 1 & \eta'\tan\varphi & -\eta' \\ -\eta'\tan\varphi & 1 & -\xi' \\ \eta' & \xi' & 1 \end{bmatrix} = R_\delta' \tag{5.6-7}$$

Finally, if differential changes in (η, ξ, ζ) due to small rotations of the local geodetic system by amounts $(\delta\lambda, \delta\varphi)$ at the point (λ, φ) are sought,

$$\begin{bmatrix} d\eta \\ d\xi \\ d\zeta \end{bmatrix} = [R_\delta^* - I] \begin{bmatrix} \eta \\ \xi \\ \zeta \end{bmatrix} \tag{5.6-8}$$

where R_δ^* is given by (5.6-5), thus

$$\begin{bmatrix} d\eta \\ d\xi \\ d\zeta \end{bmatrix} = \begin{bmatrix} 0 & \sin\varphi\delta\lambda & -\cos\varphi\delta\lambda \\ -\sin\varphi\delta\lambda & 0 & -\delta\varphi \\ \cos\varphi\delta\lambda & \delta\varphi & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \xi \\ \zeta \end{bmatrix} = \delta R^* \begin{bmatrix} \eta \\ \xi \\ \zeta \end{bmatrix} \tag{5.6-9}$$

6. SUMMARY

The present work uses a general matrix approach in reviewing some basic differential transformations between Cartesian and curvilinear coordinate systems.

The methods discussed here are applicable to any type of orthogonal curvilinear coordinates. Nevertheless in this report only geodetic and ellipsoidal (rotational) coordinates are examined.

As an application of the theory, differential changes in geodetic coordinates due to shift, rotation and scale of the geodetic system are found. The same results may be obtained employing other methods, such as the total differential approach or tensor calculus.

This study also tries to clarify the confusion in recent geodetic literature regarding the so-called "Molodenskii model," which is used in the least squares solution of the seven transformation parameters between world and geodetic (datum) systems. Careful consideration of the differential equations given in [Molodenskii et al., 1962] shows that the model attributed to them is not implied in their work.

After defining three basic transformation matrices (rotation R, metric H and Jacobian J), mappings between differential changes in the Cartesian and orthogonal curvilinear coordinates are established. This is illustrated by a general commutative diagram (Fig. 5.2). As an example, the following differential transformation is presented:

$$(d\lambda,\ d\beta,\ d\tilde{u}) \rightleftarrows (d\lambda,\ d\varphi,\ dh)$$

(To the best knowledge of the author, the transformation matrix of this mapping is given here for the first time.) Thus, it is possible to obtain differential changes in the geodetic coordinates of a point $P(\lambda,\ \varphi,\ h)$ $P(\lambda,\ \beta,\ \tilde{u})$, when the ellipsoidal coordinates change differentially or viceversa.

As an immediate result of the above-mentioned diagram, it is possible to obtain the components of the normal gravity vector in the local geodetic system; thus the component of attraction along the geodetic normal can be found.

Finally, the rotation matrix $\Re$ between the local geodetic and local ellipsoidal frames is examined.

REFERENCES

Badekas, J., 1969. "Investigations Related to the Establishment of a World Geodetic System," Reports of the Department of Geodetic Science, No. 124, The Ohio State University, Columbus, Ohio.

Bartelme, N. and Meissl, P., 1975. "Ein einfaches, rasches und numerisch stabiles Verfahren zur Bestimmung des Kurzesten Abstandes eines Punktes von einem sphäroidischen Rotationsellipsoid," Allgemeine Vermessungs-Nachrichten, 12, 436-439.

Benning, W., 1974. "Der Kurzeste Abstand eines in rechtwinkligen Koordinaten gegebenen Aussenpunktes vom Ellipsoid," Allgemeine Vermessungs-Nachrichten, 11, 429-433.

Blaha, G., 1971. "Inner Adjustment Constraints with Emphasis on Range Observations," Reports of the Department of Geodetic Science, No. 148, The Ohio State University, Columbus, Ohio.

Bursa, M., 1966. "Fundamentals of the Theory of Geometric Satellite Geodesy," Travaux de l'Institute Géophysique de l'Académie Tchécoslovaque des Sciences, No. 241.

Cartan, E., 1935. "La Méthode du Repère Mobile, La Théorie des Groupes Continus et les Espaces Généralisés," Actualités Scientifiques et Industrielles, No. 194, Paris.

Darboux, G., 1898. Leçons sur les systèmes orthogonaux et les cordonnés curvilignes. Gauthier-Villars, Paris.

Goldstein, H., 1950. Classical Mechanics. Addison-Wesley Publishing Company, Reading, Massachusetts.

Grafarend, E., 1975. "Cartan Frames and the Foundation of Physical Geodesy," in Methoden und Verfahren der Mathematischen Physik/Band 12. Mathematical Geodesy, Part I. Brosowski, B. and Martensen, E. (Editors) Bibliographisches Institut - Wissenschaftsverlag, Zurich.

Guinot, B., M. Feissel and M. Granveaud, 1971. Annual Report for 1970. Bureau International de l'Heure, Paris.

Heiskanen, W.A. and H. Moritz, 1967. Physical Geodesy. W. H. Freeman and Company, San Francisco.

Hirvonen, R. A., 1960. "New Theory of the Gravimetric Geodesy," Annales Academiae Scientiarum Fennicae, Series A, III, Geologica-Geographica, No. 56, Helsinki.

Hobson, E.W., 1931. The Theory of Spherical and Ellipsoidal Harmonics. Cambridge University Press, London.

Hotine, M., 1969. Mathematical Geodesy, ESSA Monograph 2, U. S. Department of Commerce, Government Printing Office, Washington, D.C.

Kołaczek, B. and G. Weiffenbach, Editors, 1975. On Reference Coordinate Systems for Earth Dynamics, International Astronomical Union, Colloquium No. 26, Toruń, Poland. Institute of Higher Geodesy and Geodetic Astronomy, Warsaw Technical University, Warsaw, and The Smithsonian Astrophysical Observatory, Cambridge, Massachusetts.

Krakiwsky, E.J. and D.B. Thompson, 1974. "Mathematical Models for the Combination of Terrestrial and Satellite Networks," The Canadian Surveyor, 28, 606-615.

Lamé, G., 1837. "Sur les Surfaces Isothermes," Liouville Journal (1), 2.

Leick, A. and B. H. W. van Gelder, 1975. "Similarity Transformations and Geodetic Network Distortions Based on Doppler Satellite Observations," Reports of the Department of Geodetic Science, No. 235, The Ohio State University, Columbus, Ohio.

Marussi, A., 1949. "Fondements de Géométrie Differentielle Absolue du Champ Potential Terrestre," Bulletin Géodésique, 14, 411.

Molodenskii, M. S., V.F. Eremeev and M. I. Yurkina, 1960. Methods for Study of the External Gravitational Field and Figure of the Earth. Translation from Russian by Israel Program for Scientific Translations (1962), National Science Foundation, Washington, D.C.

Mueller, I. I., 1969. Spherical and Practical Astronomy as Applied to Geodesy, Frederick Ungar Publishing Co., New York.

Mueller, I.I., M. Kumar, J. P. Reilly, N. Saxena and T. Soler, 1973. "Global Satellite Triangulation and Trilateration for the National Geodetic Satellite Program," Reports of the Department of Geodetic Science, No. 199, The Ohio State University, Columbus, Ohio.

Mueller, I. I., 1975. "Tracking Station Positioning from Artificial Satellite Observations," Geophysical Surveys, 2, 243-276.

Munk, W. H. and G. J. F. MacDonald, 1960. The Rotation of the Earth, a Geophysical Discussion, Cambridge University Press, London.

Paul, M. K., 1973. "A Note on Computation of Geodetic Coordinates from Geocentric (Cartesian) Coordinates," Bulletin Géodésique, No. 108, 135-139.

Pope, A. J., 1971. "Transformation of Covariance Matrices Due to Changes in Minimal Control," paper distributed at the A.G.U. Fall Annual Meeting, December, San Francisco.

Rapp, R. H., 1975. "Geometric Geodesy. Vol. I (Basic Principles)," Lecture Notes, Department of Geodetic Science, The Ohio State University, Columbus, Ohio.

Rapp, R. H., 1975a. "Geometric Geodesy. Vol. II(Advanced Techniques)," Lecture Notes, Department of Geodetic Science, The Ohio State University, Columbus, Ohio.

Uotila, U.A., 1967. "Introduction to Adjustment Computations with Matrices," Lecture Notes, Department of Geodetic Science, The Ohio State University, Columbus, Ohio.

APPENDIX A (Referenced in Section 4.4.2)

A.1 Matrix Form of the Equations Given in [Molodenskii et al., 1962]

Those acquainted with the English translation of the work [Molodenskii et al., 1960] know that the approach followed there uses the strict procedure of differentiation of curvilinear coordinates on practically the same line as was introduced in the classical work by Lamé [1837]. Therefore, no mention of local or moving frames is evident in the Russian translation. Thus, in order to change over to the matrix notation of this report, a correspondence between frames and their rotation matrix must be established. This is shown in Fig. A.1.

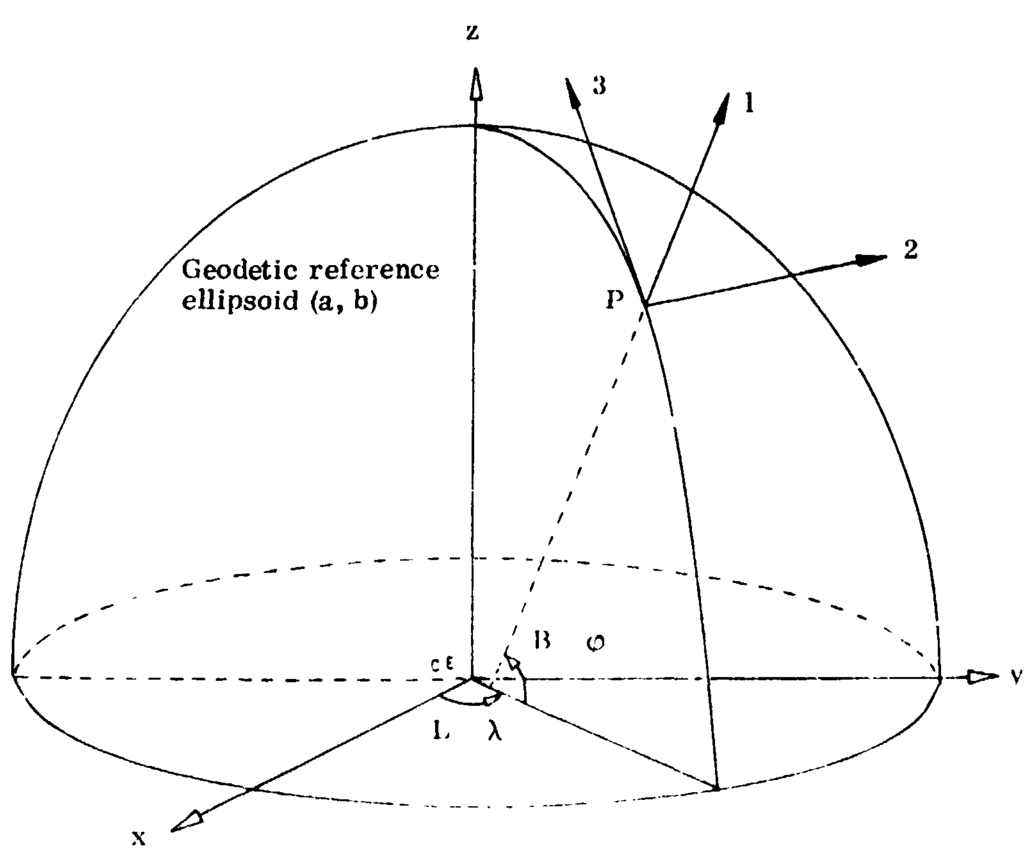


Fig. A. 1 Local Geodetic Frame in Molodenskii

Thus the rotation matrix $\mathbb{R}$ may be written:

$$\mathbb{R} = R_2(-\varphi)\, R_3(\lambda) = \begin{bmatrix} \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \\ -\sin\lambda & \cos\lambda & 0 \\ -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \end{bmatrix} \tag{A.1-1}$$

In the equations given in [Molodenskii et al., 1962] the differential changes (da, db) in the semi-major and semi-minor axes of the reference ellipsoid are introduced. However, in Section 4.4.2 of this report the flattening was used instead of b, thus the following substitutions must be taken into consideration

$$f = \frac{a-b}{a} \quad \Rightarrow \quad b = a - af \tag{A.1-2}$$

and differentiating above

$$db = da - a\,df - f\,da \quad \Rightarrow \quad db = da(1-f) - a\,df \tag{A.1-3}$$

Therefore

$$df = \frac{(1-f)\,da}{a} - \frac{db}{a} \quad \Rightarrow \quad df = \frac{b}{a^2}\,da - \frac{db}{a}$$

and finally

$$df = \frac{b}{a}\left(\frac{da}{a} - \frac{db}{b}\right) \tag{A.1-4}$$

The exact correspondence between equation (I.3.2) in [Molodenskii et al., 1962] and the matrix notation used in this report is as follows (see also Eq. 4.4.5):

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} dx_o \\ dy_o \\ dz_o \end{bmatrix} + \begin{bmatrix} 0 & \epsilon_z & -\epsilon_y \\ -\epsilon_z & 0 & \epsilon_x \\ \epsilon_y & -\epsilon_x & 0 \end{bmatrix} \begin{bmatrix} x - x_o \\ y - y_o \\ z - z_o \end{bmatrix} + \mathbb{R}^T \begin{bmatrix} dh \\ (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial f} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial f} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial f} \end{bmatrix} \begin{bmatrix} da \\ \frac{b}{a}\left(\frac{da}{a} - \frac{db}{b}\right) \end{bmatrix} \tag{A.1-5}$$

where now from (4.4-3), making use of (4.4-4), (A.1-2) and

$$1 - e^2 = \frac{b^2}{a^2}$$

it is possible to write

$$\frac{\partial x}{\partial f} = \frac{a}{b}\, M \sin^2\varphi \cos\varphi \sin\lambda \tag{A.1-6a}$$

$$\frac{\partial y}{\partial f} = \frac{a}{b}\, M \sin^2\varphi \cos\varphi \sin\lambda \tag{A.1-6b}$$

$$\frac{\partial z}{\partial f} = \frac{b}{a}\, (M - a N - M \cos^2\varphi) \sin\varphi \tag{A.1-6c}$$

In accordance with the notation in [Molodenskii et al., 1962], the following equalities are established.

$$p = a W \tag{A.1-7}$$

$$N = \frac{a^2}{p} \tag{A.1-8a}$$

$$M = \frac{a^3 b^2}{p^3} \tag{A.1-8b}$$

Premultiplying both sides of equation (A.1-5) by the rotation matrix $\mathbb{R}$ and recalling (4.4-8), equation (I.3.3) in [Molodenskii et al., 1962] follows immediately,

$$\mathbb{R}\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \mathbb{R}\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} + \mathbb{R}\begin{bmatrix} 0 & \epsilon_z & -\epsilon_y \\ -\epsilon_z & 0 & \epsilon_x \\ \epsilon_y & -\epsilon_x & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} dh \\ (N + h)\cos\varphi\, d\lambda \\ (M + h)\, d\varphi \end{bmatrix}$$

$$+ \mathbb{R}\begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial f} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial f} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial f} \end{bmatrix}\begin{bmatrix} da \\ \frac{b}{a}\left(\frac{da}{a} - \frac{db}{b}\right) \end{bmatrix} \tag{A.1-9}$$

It can be proved that Equation (A.1-9) corresponds to (I.3.3) given in [Molodenskii et al.,1962]. For example, it is immediate to show that the terms corresponding to the rotations in (A.1-9) are equal to the ones presented by Molodenskii et al., [1962].

$$\begin{bmatrix} (1) \\ (2) \\ (3) \end{bmatrix} = \mathbb{R} \begin{bmatrix} 0 & \epsilon_z & -\epsilon_y \\ -\epsilon_z & 0 & \epsilon_x \\ \epsilon_y & -\epsilon_x & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{(A.1-10)}$$

$$= \begin{bmatrix} \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \\ -\sin\lambda & \cos\lambda & 0 \\ -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \end{bmatrix} \begin{bmatrix} 0 & \epsilon_z & -\epsilon_y \\ -\epsilon_z & 0 & \epsilon_x \\ \epsilon_y & -\epsilon_x & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} -\cos\varphi\sin\lambda\,\epsilon_z + \sin\varphi\,\epsilon_y & \cos\varphi\cos\lambda\,\epsilon_z - \sin\varphi\,\epsilon_x & -\cos\varphi\cos\lambda\,\epsilon_y + \cos\varphi\sin\lambda\,\epsilon_x \\ -\cos\lambda\,\epsilon_z & -\sin\lambda\,\epsilon_z & \sin\lambda\,\epsilon_y + \cos\lambda\,\epsilon_x \\ \sin\varphi\sin\lambda\,\epsilon_z + \cos\varphi\,\epsilon_y & -\sin\varphi\cos\lambda\,\epsilon_z - \cos\varphi\,\epsilon_x & \sin\varphi\cos\lambda\,\epsilon_y - \sin\varphi\sin\lambda\,\epsilon_x \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

but

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (N+h)\cos\varphi\cos\lambda \\ (N+h)\cos\varphi\sin\lambda \\ [N(1-e^2)+h]\sin\varphi \end{bmatrix} = \begin{bmatrix} (N+h)\cos\varphi\cos\lambda \\ (N+h)\cos\varphi\sin\lambda \\ \left(N\frac{b^2}{a^2}+h\right)\sin\varphi \end{bmatrix}$$

thus

$$\begin{aligned} (1) &= -(N+h)\cos^2\varphi\sin\lambda\cos\lambda\,\epsilon_z + (N+h)\sin\varphi\cos\varphi\cos\lambda\,\epsilon_y + (N+h)\cos^2\varphi\sin\lambda\cos\lambda\,\epsilon_z \\ &\quad -(N+h)\sin\varphi\cos\varphi\sin\lambda\,\epsilon_x - \left(\frac{b^2}{a^2}N+h\right)\sin\varphi\cos\varphi\cos\lambda\,\epsilon_y \\ &\quad + \left(\frac{b^2}{a^2}N+h\right)\sin\varphi\cos\varphi\sin\lambda\,\epsilon_x \\ &= A\sin\varphi\cos\varphi(\epsilon_y\cos\lambda - \epsilon_x\sin\lambda) \end{aligned}$$

where

$$A = (N+h) - \left(\frac{b^2}{a^2}N+h\right) = N\left(1-\frac{b^2}{a^2}\right) = N\,\frac{a^2-b^2}{a^2} = \frac{a^2-b^2}{p}$$

therefore

$$(1) = \frac{a^2-b^2}{p}\sin\varphi\cos\varphi\,(\epsilon_y\cos\lambda - \epsilon_x\sin\lambda) \qquad \text{(A.1-11 a)}$$

Now

$$(2) = -(N+h)\cos\varphi\cos^2\lambda\,\epsilon_z - (N+h)\cos\varphi\sin^2\lambda\,\epsilon_z + \left(\frac{b^2}{a^2}N+h\right)\sin\varphi\sin\lambda\,\epsilon_y$$

$$+\left(\frac{b^2}{a^2}N+h\right)\sin\varphi\cos\lambda\,\epsilon_x$$

$$= -(N+h)\cos\varphi\,\epsilon_z + \left(\frac{b^2}{a^2}N+h\right)\sin\varphi(\epsilon_x\cos\lambda + \epsilon_y\sin\lambda)$$

and

$$(2) = -(N+h)\cos\varphi\,\epsilon_z + z(\epsilon_x\cos\lambda + \epsilon_y\sin\lambda) \qquad \text{(A.1-11b)}$$

Finally,

$$(3) = (N+h)\sin\varphi\cos\varphi\sin\lambda\cos\lambda\,\epsilon_z + (N+h)\cos^2\varphi\cos\lambda\,\epsilon_y$$

$$-(N+h)\sin\varphi\cos\varphi\sin\lambda\cos\lambda\,\epsilon_z - (N+h)\cos^2\varphi\sin\lambda\,\epsilon_x$$

$$+[N(1-e^2)+h]\sin^2\varphi\cos\lambda\,\epsilon_y - [N(1-e^2)+h]\sin^2\varphi\sin\lambda\,\epsilon_x$$

$$= B(\epsilon_y\cos\lambda - \epsilon_x\sin\lambda)$$

where

$$B = (N+h)\cos^2\varphi + [N(1-e^2)+h]\sin^2\varphi = N + h - Ne^2\sin^2\varphi = N\,W^2 + h = p + h$$

Thus,

$$(3) = (p+h)(\epsilon_y\cos\lambda - \epsilon_x\sin\lambda) \qquad \text{(A.1-11c)}$$

To conclude, one may write the effect of differential rotations ϵ_x, ϵ_y, ϵ_z on the geodetic coordinates as given by Molodenskii et al., [1962]

$$(N+h)\cos\varphi\,d\lambda = -(N+h)\cos\varphi\,\epsilon_z + z(\epsilon_x\cos\lambda + \epsilon_y\sin\lambda) \qquad \text{(A.1-12a)}$$

$$(M+h)\,d\varphi = (p+h)(\epsilon_y\cos\lambda - \epsilon_x\sin\lambda) \qquad \text{(A.1-12b)}$$

$$dh = \frac{a^2-b^2}{p}\sin\varphi\cos\varphi(\epsilon_y\cos\lambda - \epsilon_x\sin\lambda) \qquad \text{(A.1-12c)}$$

These equations are equivalent to (4.6-7) with

$$\epsilon_x \quad \delta\epsilon \qquad \epsilon_y \quad \delta\psi \qquad \epsilon_z \quad \delta\omega$$

and taking into account (A.1-7) and (A.1-8)

APPENDIX B (Referenced in Section 4.5)

B.1 Differential Changes in (λ, φ, h) Due to a Change δL in Scale

From (4.5-1)

$$\begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix}_{\delta L} = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \end{bmatrix} \begin{bmatrix} (N+h)\cos\varphi\cos\lambda\,\delta L \\ (N+h)\cos\varphi\sin\lambda\,\delta L \\ [N(1-e^2)+h]\sin\varphi\,\delta L \end{bmatrix}$$

After matrix multiplication

$$(N+h)\cos\varphi\, d\lambda_{\delta L} = -(N+h)\cos\varphi\sin\lambda\cos\lambda\,\delta L + (N+h)\cos\varphi\sin\lambda\cos\lambda\,\delta L = 0$$

Thus,

$$d\lambda_{\delta L} = 0 \tag{4.5-2a}$$

$$\begin{aligned} (M+h)\, d\varphi_{\delta L} &= -(N+h)\sin\varphi\cos\varphi\cos^2\lambda\,\delta L - (N+h)\sin\varphi\cos\varphi\sin^2\lambda\,\delta L \\ &\qquad + [N(1-e^2)+h]\sin\varphi\cos\varphi\,\delta L \\ &= -(N+h)\sin\varphi\cos\varphi\,\delta L + (N+h)\sin\varphi\cos\varphi\,\delta L - Ne^2\sin\varphi\cos\varphi\,\delta L \\ &= -Ne^2\sin\varphi\cos\varphi\,\delta L \end{aligned}$$

Therefore,

$$d\varphi_{\delta L} = -\frac{Ne^2\sin\varphi\cos\varphi}{M+h}\,\delta L \tag{4.5-2b}$$

Finally,

$$\begin{aligned} dh &= (N+h)\cos^2\varphi\cos^2\lambda\,\delta L + (N+h)\cos^2\varphi\sin^2\lambda\,\delta L + [N(1-e^2)+h]\sin^2\varphi\,\delta L \\ &= (N+h)\cos^2\varphi\,\delta L + (N+h)\sin^2\varphi\,\delta L - Ne^2\sin^2\varphi\,\delta L \\ &= (N+h)\,\delta L - Ne^2\sin^2\varphi\,\delta L \\ &= [N(1-e^2\sin^2\varphi)+h]\,\delta L \end{aligned}$$

but making use of equations (4.4-4) it follows that

$$dh_{\delta L} = (aW+h)\,\delta L \tag{4.5-2c}$$

B.2 <u>Differential Changes in (λ, φ, h) Due to a Change δa in the Semimajor Axis of the Ellipsoid</u>

From (4.5-3)

$$\begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\, d\varphi \\ dh \end{bmatrix}_{\delta a} = - \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \end{bmatrix} \begin{bmatrix} \dfrac{\cos\varphi\cos\lambda}{W}\,\delta a \\ \dfrac{\cos\varphi\sin\lambda}{W}\,\delta a \\ \dfrac{(1-e^2)\sin\varphi}{W}\,\delta a \end{bmatrix}$$

$$(N+h)\cos\varphi\, d\lambda_{\delta a} = \frac{\cos\varphi\sin\lambda\cos\lambda}{W}\,\delta a - \frac{\cos\varphi\sin\lambda\cos\lambda}{W}\,\delta a = 0$$

Thus,

$$d\lambda_{\delta a} = 0 \qquad (4.5\text{-}6a)$$

$$\begin{aligned}(M+h)\, d\varphi_{\delta a} &= \frac{\sin\varphi\cos\varphi\cos^2\lambda}{W}\,\delta a + \frac{\sin\varphi\cos\varphi\sin^2\lambda}{W}\,\delta a - \frac{(1-e^2)\sin\varphi\cos\varphi}{W}\,\delta a \\ &= \frac{\sin\varphi\cos\varphi}{W}\,\delta a - \frac{(1-e^2)\sin\varphi\cos\varphi}{W}\,\delta a \\ &= \frac{e^2\sin\varphi\cos\varphi}{W}\,\delta a = \frac{N e^2\sin\varphi\cos\varphi}{a}\,\delta a\end{aligned}$$

Therefore,

$$d\varphi_{\delta a} = \frac{N e^2\sin\varphi\cos\varphi}{(M+h)\,a}\,\delta a \qquad (4.5\text{-}6b)$$

Finally,

$$\begin{aligned}dh_{\delta a} &= -\frac{\cos^2\varphi\cos^2\lambda}{W}\,\delta a - \frac{\cos^2\varphi\sin^2\lambda}{W}\,\delta a - \frac{(1-e^2)\sin^2\varphi}{W}\,\delta a \\ &= -\frac{\cos^2\varphi}{W}\,\delta a - \frac{(1-e^2)\sin^2\varphi}{W}\,\delta a \\ &= -\frac{1-e^2\sin^2\varphi}{W}\,\delta a\end{aligned}$$

Thus,

$$dh_{\delta a} = -W\,\delta a \qquad (4.5\text{-}6c)$$

APPENDIX C (Referenced in Section 4.6)

C.1 Differential Changes in (λ, φ, h) Due to Small Rotations $\delta\epsilon, \delta\psi, \delta\omega$

From (4.6-5):

$$\begin{bmatrix} (N+h)\cos\varphi\, d\lambda \\ (M+h)\,d\varphi \\ dh \end{bmatrix}_{\delta R} = R \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} (N+h)\cos\varphi\cos\lambda \\ (N+h)\cos\varphi\sin\lambda \\ [N(1-e^2)+h]\sin\varphi \end{bmatrix}$$

$$= \begin{bmatrix} -\cos\lambda\,\delta\omega & -\sin\lambda\,\delta\omega & \sin\lambda\,\delta\psi + \cos\lambda\,\delta\epsilon \\ \sin\varphi\sin\lambda\,\delta\omega + \cos\varphi\,\delta\psi & -\sin\varphi\cos\lambda\,\delta\omega - \cos\varphi\,\delta\epsilon & \sin\varphi\cos\lambda\,\delta\psi - \sin\varphi\sin\lambda\,\delta\epsilon \\ -\cos\varphi\sin\lambda\,\delta\omega + \sin\varphi\,\delta\psi & \cos\varphi\cos\lambda\,\delta\omega - \sin\varphi\,\delta\epsilon & -\cos\varphi\cos\lambda\,\delta\psi + \cos\varphi\sin\lambda\,\delta\epsilon \end{bmatrix} \begin{bmatrix} (N+h)\cos\varphi\cos\lambda \\ (N+h)\cos\varphi\sin\lambda \\ [N(1-e^2)+h]\sin\varphi \end{bmatrix}$$

$$\begin{aligned} (N+h)\cos\varphi\, d\lambda_{\delta R} &= -(N+h)\cos\varphi\cos^2\lambda\,\delta\omega - (N+h)\cos\varphi\sin^2\lambda\,\delta\omega \\ &\quad + \sin\lambda\sin\varphi\,\delta\psi\,[N(1-e^2)+h] + \sin\varphi\cos\lambda\,\delta\epsilon\,[N(1-e^2)+h] \\ &= -(N+h)\cos\varphi\,\delta\omega + [(N+h) - Ne^2]\sin\varphi(\sin\lambda\,\delta\psi + \cos\lambda\,\delta\epsilon) \end{aligned}$$

Then

$$\begin{aligned} d\lambda_{\delta R} &= -\delta\omega + \frac{(N+h)\sin\varphi - Ne^2\sin\varphi}{(N+h)\cos\varphi}(\sin\lambda\,\delta\psi + \cos\lambda\,\delta\epsilon) \\ &= -\delta\omega + \tan\varphi\left(1 - \frac{Ne^2}{N+h}\right)(\sin\lambda\,\delta\psi + \cos\lambda\,\delta\epsilon) \end{aligned}$$

Thus, finally:

$$d\lambda_{\delta R} = -\delta\omega + \delta\epsilon\left(1 - \frac{Ne^2}{N+h}\right)\tan\varphi\cos\lambda + \delta\psi\left(1 - \frac{Ne^2}{N+h}\right)\tan\varphi\sin\lambda \qquad (4.6\text{-}7a)$$

$$\begin{aligned} (M+h)\,d\varphi_{\delta R} = {} & (N+h)\cos\varphi\cos\lambda\sin\varphi\sin\lambda\,\delta\omega + (N+h)\cos^2\varphi\cos\lambda\,\delta\psi \\ & - (N+h)\cos\varphi\sin\lambda\sin\varphi\cos\lambda\,\delta\omega - (N+h)\cos^2\varphi\sin\lambda\,\delta\epsilon \\ & + [N(1-e^2)+h]\sin^2\varphi\cos\lambda\,\delta\psi - [N(1-e^2)+h]\sin^2\varphi\sin\lambda\,\delta\epsilon \end{aligned}$$

$$(M+h)\,d\varphi_{\delta R} = (N+h)\cos^2\varphi\cos\lambda\,\delta\psi + (N+h)\sin^2\varphi\cos\lambda\,\delta\psi - Ne^2\sin^2\varphi\cos\lambda\,\delta\psi$$
$$- (N+h)\cos^2\varphi\sin\lambda\,\delta\epsilon - (N+h)\sin^2\varphi\sin\lambda\,\delta\epsilon + Ne^2\sin^2\varphi\sin\lambda\,\delta\epsilon$$

$$= \delta\psi\cos\lambda[(N+h) - Ne^2\sin^2\varphi] - \delta\epsilon\sin\lambda[(N+h) - Ne^2\sin^2\varphi]$$

But,

$$(N+h) - Ne^2\sin^2\varphi = N(1 - e^2\sin^2\varphi) + h = NW^2 + h = \frac{a}{W}W^2 + h = aW + h$$

Therefore,

$$(M+h)\,d\varphi = \delta\psi\cos\lambda(aW+h) - \delta\epsilon\sin\lambda(aW+h)$$

and finally,

$$d\varphi_{\delta R} = -\delta\epsilon\sin\lambda\,\frac{aW+h}{M+h} + \delta\psi\cos\lambda\,\frac{aW+h}{M+h} \qquad (4.6\text{-}7b)$$

$$dh_{\delta R} = -(N+h)\cos^2\varphi\cos\lambda\sin\lambda\,\delta\omega + (N+h)\cos\varphi\cos\lambda\sin\varphi\,\delta\psi$$
$$+ (N+h)\cos^2\varphi\sin\lambda\cos\lambda\,\delta\omega - (N+h)\cos\varphi\sin\lambda\sin\varphi\,\delta\epsilon$$
$$- [N(1-e^2)+h]\sin\varphi\cos\varphi\cos\lambda\,\delta\psi + [N(1-e^2)+h]\sin\varphi\cos\varphi\sin\lambda\,\delta\epsilon$$

$$= -\delta\psi\sin\varphi\cos\varphi\cos\lambda[(N+h) - Ne^2 - (N+h)]$$
$$+ \delta\epsilon\sin\varphi\cos\varphi\sin\lambda[(N+h) - Ne^2 - (N+h)]$$

Thus:

$$dh_{\delta R} = -\delta\epsilon Ne^2\sin\varphi\cos\varphi\sin\lambda + \delta\psi Ne^2\sin\varphi\cos\varphi\cos\lambda \qquad (4.6\text{-}7c)$$

APPENDIX D (Referenced in Section 4.6.2)

D.1 Inner (Minimal) Constraints in Curvilinear Coordinates

D.1.1 Introduction

Papers on minimal constraints and their application to geodesy are abundant in the literature. The basic principles introduced here follow [Pope, 1971] where the interested reader can consult the fundamental references on this topic.

It is well known that in most geodetic problems the set of normal equations

$$N X + U = 0 \qquad \text{(D.1-1)}$$

is a singular system when the original observation equations $F(X, L) = 0$ do not contain some peculiar constraints.

In the specific case of a spatial network the following relations hold

$$_{n}N_{n} \implies \text{Rank}(N) = r \qquad \text{(D.1-2)}$$

where $\qquad r < n \quad \text{and} \quad e = n - r \leq 7$

The value e is generally called the rank deficiency of N or the degrees of freedom of the network (not to be confused with the concept of degrees of freedom a least squares adjustment [see Uotila, 1967]).

As a consequence of (D.1-2)

$$| N | = 0 \implies N \text{ is singular}$$

One way to solve equation (D.1-1) in this case is by bordering the normal matrix N and solving the system:

$$\begin{bmatrix} N & E \\ E^T & 0 \end{bmatrix} \begin{bmatrix} X \\ -K \end{bmatrix} = \begin{bmatrix} -U \\ 0 \end{bmatrix} \qquad \text{(D.1-3)}$$

where $_{n}E_{e}$ is the basis for the solution space (null space of N) of the homogeneous equation. Therefore

$$N E = 0 \qquad \text{(D.1-4)}$$

Obviously property (D.1-4) also implies

$$A E = 0 \qquad \text{(D.1-5)}$$

The complete solution of the normal equation (D.1-1) in the case of singularity is:

$$X = X_o + E\beta \tag{D.1-6}$$

where

$X_o \equiv$ any particular solution of $NX_o = -U$

and β, $E\beta \equiv$ complete solution of the corresponding homogeneous equation $NX = 0$.

Minimal constraints are the smallest number of constraints e that produce a nonsingular matrix M, and minimizes $X^T X$,

$$M = \begin{bmatrix} N & E \\ E^T & 0 \end{bmatrix} \tag{D.1-7}$$

Of all the minimal constraints possible, some have simple geometric interpretations; this subset of minimal constraints is called "inner constraints " [Blaha, 1971].

As an illustration, assume that in E^3 only angles are measured in order to establish a network of points. Clearly, the degrees of freedom of this network will be seven, if one considers that translations, rotations and scale variations will change the coordinates of the points, although without affecting the values of the measured angles. In other words, one may say that coordinates are not estimable quantities. Thus, in this example

$$e = T + R + S = 7$$

where

$T \equiv$ number of constraints required for origin $= 3$

$R \equiv$ number of constraints required for orientation $= 3$

$S \equiv$ number of constraints required for scale $= 1$

Therefore,

$$_nE_e = (\underset{n\times 3}{E_T} \quad \underset{n\times 3}{E_R} \quad \underset{n\times 1}{E_S})_{n\times 7} \tag{D.1-8}$$

D.1.2 Inner Constraints in Rectangular Coordinates

Still following [Pope, 1971] the set of inner constraints when rectangular coordinates are used, may be obtained through the differential changes in the Cartesian coordinates due to translations (shifts), rotations and scale, that is,

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}_T = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{D.1-9a}$$

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}_R = \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{D.1-9b}$$

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}_S = \delta L \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{D.1-9c}$$

Thus, one can write:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_t = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}_t + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}_t = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}_t + \underset{3 \times 3}{I} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{D.1-10}$$

which is in the form of (D.1-6) and gives,

$$E_{T_t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_t \quad ; \quad \beta = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{D.1-11}$$

In the same way, the inner orientation constraints can be found,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_t = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}_t + \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}_t \begin{bmatrix} \delta\epsilon \\ \delta\psi \\ \delta\omega \end{bmatrix} \tag{D.1-12}$$

Therefore,

$$E_{R_i} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}_i \quad ; \quad \beta = \begin{bmatrix} \delta\epsilon \\ \delta\psi \\ \delta\omega \end{bmatrix} \tag{D.1-13}$$

For the inner scale constraint one has

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_i = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}_i + \begin{bmatrix} x \\ y \\ z \end{bmatrix}_i \delta L \tag{D.1-14}$$

Thus,

$$E_{S_i} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_i \quad \text{and} \quad \beta = \delta L \tag{D.1-15}$$

Finally the matrix (D.1-8) is given by

$$\underset{3 \times 7}{E_i} = \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 0 & -z & y & x \\ 0 & 1 & 0 & z & 0 & -x & y \\ 0 & 0 & 1 & -y & x & 0 & z \end{array}\right]_i \tag{D.1-16}$$

D.1.3 Inner Constraints in Curvilinear Coordinates (Spherical Case)

Curvilinear coordinates are always referred to some basic surface which introduces restrictions in the number of degrees of freedom needed for solving the network singularity.

For example, in the case of a flat surface (plane), the degrees of freedom of an angular network are only four,

$$e_p = T + R + S = 2 + 1 + 1 = 4$$

This is also true for spherical networks,

$$e_s = 4$$

Assuming that the scale of the spherical network is fixed through the radius r of the sphere, only three constraints are needed to resolve the singularity. Clearly the degrees of freedom in this instance are three rotations along the x, y, z axis. Thus, applying (D.1-9b) to the case of spherical coordinates, one can write the following from (4.6-6) with

$h = 0, \quad M = N = r, \quad u = x, \quad v = y$ and $w = z$:

$$\begin{bmatrix} r\cos\varphi \, d\lambda \\ r\,d\lambda \\ dh \end{bmatrix} = R \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{(D.1-17)}$$

or

$$\begin{bmatrix} r\cos\varphi \, d\lambda \\ r\,d\varphi \\ dh \end{bmatrix} = R \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} \delta\epsilon \\ \delta\psi \\ \delta\omega \end{bmatrix} \qquad \text{(D.1-18)}$$

Therefore,

$$E_R = R \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \text{ and } \beta = \begin{bmatrix} \delta\epsilon \\ \delta\psi \\ \delta\omega \end{bmatrix} \qquad \text{(D.1-19)}$$

Knowing that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\varphi\cos\lambda \\ r\cos\varphi\sin\lambda \\ r\sin\varphi \end{bmatrix} \qquad \text{(D.1-20)}$$

Thus finally,

$$E_R = \begin{bmatrix} r\cos\lambda\sin\varphi & r\sin\lambda\sin\varphi & -r\cos\varphi \\ -r\sin\lambda & r\cos\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (D.1-21)$$

Assuming that the parameters in the normal equations are given by the vector matrix

$$X = [d\lambda_1 \quad d\varphi_1 \quad d\lambda_2 \quad d\varphi_2 \ldots d\lambda_i \quad d\varphi_i \ldots d\lambda_n \quad d\varphi_n]^T \qquad (D.1-22)$$

The set of submatrices E_{R_i} required in order to avoid singularity are given in the following form:

$$\underset{2\times 3}{E_{R_i}} = \begin{bmatrix} \cos\lambda\tan\varphi & \sin\lambda\tan\varphi & -1 \\ -\sin\lambda & \cos\lambda & 0 \end{bmatrix}_i \qquad (D.1-23)$$

The use of the submatrices $\underset{2\times 3}{E_{R_i}}$ in the bordering of the normal matrix N for the solution of the singular system can be interpreted geometrically as in the rectangular case. It will give the "best" orientation to the spherical triangulation with points $(\lambda, \varphi)_i$.

When only local networks on a sphere are involved, it will be more appropriate to rotate about a geocentric Cartesian system parallel to the local frame $(\eta, \xi, \zeta)_o$ at the center of the network. In this case the following transformation applies,

$$\begin{bmatrix} \delta\epsilon \\ \delta\psi \\ \delta\omega \end{bmatrix} = R_o^T \begin{bmatrix} \delta\epsilon_o \\ \delta\psi_o \\ \delta\omega_o \end{bmatrix} \qquad (D.1-24)$$

and after substitution of the above in (D.1-18) one has the matrix equation,

$$\begin{bmatrix} r\cos\varphi\, d\lambda \\ r\, d\varphi \\ dh \end{bmatrix} = R \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} R_o^T \begin{bmatrix} \delta\epsilon_o \\ \delta\psi_o \\ \delta\omega_o \end{bmatrix} \qquad (D.1-25)$$

or

$$E_{R_0} = R\begin{bmatrix} 0 & -x & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} R_0^T = E_R \;\; R_0^T \;\; ; \;\; \beta = \begin{bmatrix} \delta\epsilon_0 \\ \delta\psi_0 \\ \delta\omega_0 \end{bmatrix} \qquad \text{(D. 1-26)}$$

That is,

$$E_{R_0} = \begin{bmatrix} r\sin\varphi\sin(\lambda-\lambda_0) & \begin{matrix}-r(\sin\varphi\sin\varphi_0\cos(\lambda-\lambda_0) \\ +\cos\varphi\cos\varphi_0)\end{matrix} & \begin{matrix} r(\sin\varphi\cos\varphi_0\cos(\lambda-\lambda_0) \\ -\sin\varphi_0\cos\varphi)\end{matrix} \\ r\cos(\lambda-\lambda_0) & r\sin\varphi_0\sin(\lambda-\lambda_0) & -r\cos\varphi_0\sin(\lambda-\lambda_0) \\ 0 & 0 & 0 \end{bmatrix}$$

(D.1-27)

and finally, when the parameters are given in the form of (D.1-22) the matrix $_{n}E_{e}$ (e = 3) is composed of the following submatrices:

$$\underset{2\times3}{E_{R_{01}}} = \begin{bmatrix} \sin(\lambda-\lambda_0)\tan\varphi & \begin{matrix}-\tan\varphi\sin\varphi_0\cos(\lambda-\lambda_0) \\ +\cos\varphi_0\end{matrix} & \begin{matrix}\tan\varphi\cos\varphi_0\cos(\lambda-\lambda_0) \\ -\sin\varphi_0\end{matrix} \\ \cos(\lambda-\lambda_0) & \sin\varphi_0\sin(\lambda-\lambda_0) & -\cos\varphi_0\sin(\lambda-\lambda_0) \end{bmatrix},$$

(D.1-28)

APPENDIX E (Referenced in Section 5.1)

E.1 Families of Rotational Ellipsoids

E.1.1 Confocal, Similar and Quasi-Parallel Ellipsoids

The following relations are immediately obtainable:

$$E = \sqrt{a^2 - b^2} \implies dE = \frac{a\,da - b\,db}{\sqrt{a^2 - b^2}} \qquad \text{(E.1-1)}$$

$$f = \frac{a - b}{a} \implies df = \frac{b}{a}\left(\frac{da}{a} - \frac{db}{b}\right) \qquad \text{(E.1-2)}$$

$$e = \frac{E}{a} \implies de = \frac{b(b\,da - a\,db)}{a^2\sqrt{a^2 - b^2}} \qquad \text{(E.1-3)}$$

From the above basic relations, it is possible to define the following types of ellipsoids:

Confocal Ellipsoids. A family of ellipsoids is called confocal if

$$E = \text{constant} \implies dE = 0 \qquad \text{(E.1-4)}$$

From (E.1-1) the condition for confocality is found immediately,

$$a\,da = b\,db \implies \frac{da}{b} = \frac{db}{a} \qquad \text{(E.1-5)}$$

or

$$db = \frac{a}{b}\,da \qquad \text{(E.1-6)}$$

Substituting property (E.1-5) in (E.1-2) and (E.1-3) one has,

$$df_c = \frac{b^2 - a^2}{a^2 b}\,da = -\frac{E^2}{a^2 b}\,da \qquad \text{(E.1-7)}$$

and

$$de_c = -\frac{E}{a^2}\, da \tag{E.1-8}$$

which is also obvious from the differentiation of $e = \frac{E}{a}$ for E = constant.

Similar Ellipsoids. The name "similar" is applied to a family of ellipsoids when

$$\left.\begin{matrix} f \\ \text{or} \\ e \end{matrix}\right\} = \text{constant} \implies \left.\begin{matrix} df \\ \text{or} \\ de \end{matrix}\right\} = 0 \qquad \begin{matrix} \text{(E.1-9)} \\ \\ \text{(E.1-10)} \end{matrix}$$

From (E.1-2) or (E.1-3) the similarity condition follows immediately:

$$b\,da = a\,db \implies \frac{da}{a} = \frac{db}{b} \tag{E.1-11}$$

and therefore,

$$db = \frac{b}{a}\, da \tag{E.1-12}$$

Substituting the above equation in (E.1-2) and (E.1-3) one has

$$dE_s = e\,da = \sqrt{2f - f^2}\, da \tag{E.1-13}$$

Obviously, the same result is obtained by differentiation $E = ea$ with e = const.

Quasi-Parallel Ellipsoids. A family of ellipsoids is called "quasi-parallel" (the author was unable to find anywhere in mathematical literature a name for this family) if the following property holds:

$$da = db \tag{E.1-14}$$

This implies:

$$f_p a_p = a - b = \text{constant} \tag{E.1-15}$$

That is, for any family of quasi-parallel ellipsoids, the product of its flattening by its semimajor axis is constant. The value of the constant is the difference between the semimajor and semiminor axes of any ellipsoid in the family.

From (E.1-1), (E.1-2) and (E.1-3) it is possible to obtain

$$dE_p = \frac{f}{e}\,da = \frac{a-b}{E}\,da \tag{E.1-16}$$

$$df_p = \frac{b-a}{a^2}\,da = -\frac{f}{a}\,da \implies \frac{df}{f} = -\frac{da}{a} \tag{E.1-17}$$

$$de_p = -\frac{b}{a^2}\sqrt{\frac{a-b}{a+b}}\,da \tag{E.1.-18}$$

The following inequalities hold:

$$| db_s | < | db_p | < | db_c | \tag{E.1-19}$$

$$| dE_p | < | dE_s |$$

$$| df_p | < | df_c |$$

$$| de_p | < | de_c |$$

E.1.2 The Variation dh of the Geodetic Height

After matrix multiplication equation (4.5-4) gives:

$$dh = -W\,da + \frac{a(1-f)}{W}\sin^2\varphi\,df \tag{E.1-20}$$

Therefore, the variation dh of the geodetic height, according to the different cases mentioned in the previous section, may be obtained.

a) Confocal case : $dE = 0$

Substituting the value of df given by equation (E.1-7) in (E.1-20), after simplication one has:

$$dh_c = -\frac{da}{W} \tag{E.1-21}$$

b) Similar case : $df = 0$

Obviously, from (E.1-20):

$$dh_s = -W\,da \qquad (E.1\text{-}22)$$

c) Quasi-parallel case : $da = db$

This case, although more involved, is also easy to obtain:

$$dh_p = -W\,da + \frac{(b-a)b}{a^2 W}\sin^2\varphi\,da$$

or

$$dh_p = -\frac{a-(a-b)\sin^2\varphi}{\sqrt{a^2-(a^2-b^2)\sin^2\varphi}}\,da \qquad (E.1\text{-}23)$$

As a consequence of (E.1-23) one concludes that the variation of h in the case $da = db$ is not constant. This is the primary reason for the name "quasi-parallel" for this family of ellipsoids.

From (E.1-23) the maximum value of dh_p is obtained at

$$\varphi = \text{arc}\sin\sqrt{\frac{a}{a+b}} \qquad (E.1 - 24)$$

where

$$dh_{pM} = \frac{2\sqrt{ab}}{a+b}\,da \qquad (E.1\text{-}25)$$