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# Parameter Estimation and Inverse Problems

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# Preface

This textbook evolved from a course in geophysical inverse methods taught during the past decade at New Mexico Tech, first by Rick Aster and, for the last five years, jointly between Rick Aster and Brian Borchers. The audience for the course has included a broad range of first- or second-year graduate students (and occasionally advanced undergraduates) from geophysics, hydrology, mathematics, astronomy, and other disciplines. Cliff Thurber joined this collaboration during the past three years and has taught a similar course at the University of Wisconsin.

Our principal goal for this text is to promote fundamental understanding of parameter estimation and inverse problem philosophy and methodology, specifically regarding such key issues as uncertainty, ill-posedness, regularization, bias, and resolution. We emphasize theoretical points with illustrative examples, and MATLAB codes that implement these examples are provided on a companion CD. Throughout the examples and exercises, a CD icon indicates that there is additional material on the CD. Exercises include a mix of programming and theoretical problems.

This book has necessarily had to distill a tremendous body of mathematics and science going back to (at least) Newton and Gauss. We hope that it will find a broad audience of students and professionals interested in the general problem of estimating physical models from data. Because this is an introductory text surveying a very broad field, we have not been able to go into great depth. However, each chapter has a “notes and further reading” section to help guide the reader to further exploration of specific topics. Where appropriate, we have also directly referenced research contributions to the field.

Some advanced topics have been deliberately omitted from the book because of space limitations and/or because we expect that many readers would not be sufficiently familiar with the required mathematics. For example, readers with a strong mathematical background may be surprised that we consider only inverse problems with discrete data and discretized models. By doing this we avoid the much of the technical complexity of functional analysis. Some advanced applications and topics that we have omitted include inverse scattering problems, seismic diffraction tomography, wavelets, data assimilation, and expectation maximization (EM) methods.

We expect that readers of this book will have prior familiarity with calculus, differential equations, linear algebra, probability, and statistics at the

undergraduate level. In our experience, many students are in need of at least a review of these topics, and we typically spend the first two to three weeks of the course reviewing this material from Appendices A, B, and C.

Chapters 1 through 5 form the heart of the book, and should be covered in sequence. Chapters 6, 7, and 8 are independent of each other, but depend strongly on the material in Chapters 1 through 5. As such, they may be covered in any order. Chapters 9 and 10 are independent of Chapters 6 through 8, but are most appropriately covered in sequence. Chapter 11 is independent of the specifics of Chapters 6 through 10, and provides an alternate view on, and summary of, key statistical and inverse theory issues.

If significant time is allotted for review of linear algebra, vector calculus, probability, and statistics in the appendices, there will probably not be time to cover the entire book in one semester. However, it should be possible for teachers to cover the majority of the material by selectively using material in the chapters following Chapter 5.

We especially wish to acknowledge our own professors and mentors in this field, including Kei Aki, Robert Parker, and Peter Shearer. We thank our many colleagues, including our own students, who provided sustained encouragement and feedback, particularly James Beck, Elena Resmerita, Charlotte Rowe, Tyson Strand, and Suzan van der Lee. Stuart Anderson, Greg Beroza, Ken Creager, Ken Dueker, Eliza Michalopoulou, Paul Segall, Anne Sheehan, and Kristy Tiampo deserve special mention for their classroom testing of early versions of this text. Robert Nowack, Gary Pavlis, Randall Richardson, and Steve Roecker provided thorough reviews that substantially improved the final manuscript. We offer special thanks to Per Christian Hansen of the Technical University of Denmark for collaboration in the incorporation of his Regularization Tools, which we highly recommend as an adjunct to this text. We also thank the editorial staff at Academic Press, especially Frank Cynar and Jennifer Hele, for essential advice and direction. Suzanne Borchers and Susan Delap provided valuable proofreading and graphics expertise. Brian Borchers was a visiting fellow at the Institute for Pure and Applied Mathematics (IPAM) at UCLA, and Rick Aster was partially supported by the New Mexico Tech Geophysical Research Center during the preparation of the text.

Rick Aster, Brian Borchers, and Cliff Thurber  
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