

## CMPS 2200 Assignment 2

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In this assignment we'll work on applying the methods we've learned to analyze recurrences, and also see their behavior in practice. As with previous assignments, some of your answers will go in `main.py`. You should feel free to edit this file with your answers; for handwritten work please scan your work and submit a PDF titled `assignment-02.pdf` and push to your github repository.

- Derive asymptotic upper bounds of work for each recurrence below.

$$\begin{aligned} & \text{cost} \cdot W(n) = 2W(n/3) + 1. \quad \left. \begin{array}{l} \text{level } i \text{ has } 2^i \text{ nodes and the cost at each node is 1} \\ \text{each level costs } 2^i \end{array} \right\} \log_3 n \quad \text{cost} = \sum_{i=0}^{\log_3 n} 2^i < \frac{2}{1} * 2^{\log_3 n} = 2 * n^{\log_3 2} = O(n^{\log_3 2}) \end{aligned}$$

$$\begin{aligned} & \text{cost} \cdot W(n) = 5W(n/4) + n. \quad \left. \begin{array}{l} \text{level } i \text{ has } 5^i \text{ nodes} \\ \text{cost of each node is } \frac{n}{4^i} \end{array} \right\} \text{each level costs } 5^i * \left(\frac{n}{4^i}\right) = \left(\frac{5}{4}\right)^i n \\ & \sum_{i=0}^{\log_4 n} \left(\frac{5}{4}\right)^i n = \frac{5}{4-1} * \left(\frac{5}{4}\right)^{\log_4 n} n = 5 * \frac{5^{\log_4 n}}{4^{\log_4 n}} = \frac{5 n^{\log_4 5}}{4^{\log_4 n}} = O(n^{\log_4 5}) \end{aligned}$$

$$\begin{aligned} & W(n) = 7W(n/7) + n. \quad \left. \begin{array}{l} n \rightarrow \frac{n}{7} \text{ level } i \text{ has } 7^i \text{ nodes} \\ n \rightarrow \frac{n}{7^2} \text{ each level costs } 7^i * \frac{n}{7^i} = n \end{array} \right\} \log_7 n \quad \sum_{i=0}^{\log_7 n} n = n \sum_{i=0}^{\log_7 n} 1 = n \log_7 n = O(n \log n) \end{aligned}$$

$$\begin{aligned} & W(n) = 9W(n/3) + n^2. \quad \left. \begin{array}{l} n^2 \rightarrow \frac{n^2}{3} \text{ level } i \text{ has } 3^i \text{ nodes} \\ n^2 \rightarrow \frac{n^2}{3^2} \text{ each level costs } 3^i * \frac{n^2}{3^i} = n^2 \end{array} \right\} \log_3 n \quad \sum_{i=0}^{\log_3 n} n^2 = n^2 \sum_{i=0}^{\log_3 n} 1 = n^2 \log_3 n = O(n^2 \log n) \end{aligned}$$

$$\begin{aligned} & W(n) = 8W(n/2) + n^3. \quad \left. \begin{array}{l} 2^n \rightarrow \frac{2^n}{2} \text{ level } i \text{ has } 2^i \text{ nodes} \\ 2^n \rightarrow \frac{2^n}{2^2} \text{ each level costs } 2^i * \frac{2^n}{2^i} = n^3 \end{array} \right\} \log_2 n \quad \sum_{i=0}^{\log_2 n} n^3 = n^3 \sum_{i=0}^{\log_2 n} 1 = n^3 \log_2 n = O(n^3 \log n) \end{aligned}$$

$$W(n) = 49W(n/25) + n^{3/2} \log n.$$

$$\begin{aligned} & 49 * \left(\frac{n}{25}\right)^{3/2} \log \frac{n}{25} \rightarrow \frac{49n^{3/2}}{25} \log \frac{n}{25} \rightarrow \frac{n}{25} \rightarrow = \frac{49}{25} n^{3/2} \log \frac{n}{25} = \left(\frac{49}{25}\right) n^{3/2} (\log n - \log 25) \\ & 49^2 * \left(\frac{n}{25^2}\right)^{3/2} \log \frac{n}{25^2} = \frac{49^2 n^{3/2}}{25^3} \log \frac{n}{25^2} \rightarrow \frac{n}{25^2} \\ & 49^3 * \left(\frac{n}{25^3}\right)^{3/2} \log \frac{n}{25^3} = \frac{49^3 n^{3/2}}{25^6} \log \frac{n}{25^3} \rightarrow \frac{n}{25^3} \end{aligned}$$

cost of each level is  $\frac{49^i \cdot n^{3/2} \log \frac{n}{25^i}}{25^{3i}}$

1 Brick method  $\Rightarrow$  root dominated

Thus, we get  $O(n^{3/2} \log n)$

- $W(n) = W(n-1) + 2$   
 $= \omega(n-2) + 2 + 2 = \omega(n-2) + 4$   
 $= \omega(n-3) + 2 + 2 + 2 = \omega(n-3) + 6$   
 $\vdots$   
 $\omega(n) = \sum_{i=1}^n 2 = 2 \sum_{i=1}^n 1 = 2n = \boxed{O(n)}$

- $W(n) = W(n-1) + n^c$ , with  $c \geq 1$ .  
 $= \omega(n-2) + (n-1)^c + n^c$   
 $= \omega(n-3) + (n-2)^c + (n-1)^c + n^c$   
 $\omega(n) = n^c + (n-1)^c + (n-2)^c + \dots + 2^c + 1^c$   
 $< n^c + n^c + n^c + \dots + n^c + n^c$   
 $\Rightarrow n \cdot n^c = n^{c+1}$   
 $= \boxed{O(n^{c+1})}$

- $W(n) = W(\sqrt{n}) + 1$

2. Suppose that for a given task you are choosing between the following three algorithms:

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- Algorithm A solves problems by dividing them into five subproblems of half the size, recursively solving each subproblem, and then combining the solutions in linear time.  $O(n^{\log_2 5})$
- Algorithm B solves problems of size  $n$  by recursively solving two subproblems of size  $n-1$  and then combining the solutions in constant time.  $O(2^n)$
- Algorithm C solves problems of size  $n$  by dividing them into nine subproblems of size  $n/3$ , recursively solving each subproblem, and then combining the solutions in  $O(n^2)$  time.

$$W(n) = 5W\left(\frac{n}{2}\right) + n$$

$$W(n) = 2W(n-1) + 1$$

$$W(n) = 9W\left(\frac{n}{3}\right) + n^2$$

$$O(n^2 \log n)$$

What are the asymptotic running times of each of these algorithms? Which algorithm would you choose?

I would choose Algorithm C, because for non-massive  $n$  values the runtime is quicker, and this is because it grows slowly. Also, I likely will only need the algorithm for small values of  $n$ .

3. Now that you have some practice solving recurrences, let's work on implementing some algorithms. In lecture we discussed a divide and conquer algorithm for integer multiplication. This algorithm takes as input two  $n$ -bit strings  $x = \langle x_L, x_R \rangle$  and  $y = \langle y_L, y_R \rangle$  and computes the product  $xy$  by using the fact that  $xy = 2^{n/2}x_Ly_L + 2^{n/2}(x_Ly_R + x_Ry_L) + x_Ry_R$ . Use the stub functions in `main.py` to implement Karatsuba-Ofman algorithm for integer multiplication: a divide and conquer algorithm that runs in subquadratic time. Then test the empirical running times across a variety of inputs to test whether your code scales in the manner described by the asymptotic runtime. Please refer to Recitation 3 for some basic implementations, and Eqs (7) and (8) in the slides <https://github.com/allan-tulane/cmps2200-slides/blob/main/module-02-recurrences/recurrences-integer-multiplication.ipynb>

$$(x_L + x_R) \cdot (y_L + y_R) = (x_L \cdot y_L) + (x_L \cdot y_R) + (x_R \cdot y_L) + (x_R \cdot y_R)$$

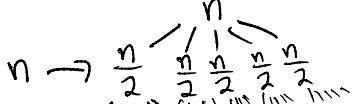
$$(x_L \cdot y_R) + (x_R \cdot y_L) = (x_L + x_R) \cdot (y_L + y_R) - (x_L \cdot y_L) - (x_R \cdot y_R)$$

$\equiv$

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The code's empirical running times line up.

$$W(n) = 5W\left(\frac{n}{2}\right) + n$$



$$5 \cdot \frac{n}{2} \rightarrow \frac{n}{2^2} \frac{n}{2^2} \frac{n}{2^2} \frac{n}{2^2} \frac{n}{2^2}$$

$$5^2 \cdot \frac{n}{2^2} \rightarrow \frac{n}{2^3} \frac{n}{2^3} \frac{n}{2^3} \frac{n}{2^3} \frac{n}{2^3}$$

$$5^3 \cdot \frac{n}{2^3} \Rightarrow \text{cost of each level is } \left(\frac{5}{2}\right)^i n$$

$$\sum_{i=0}^{\log_2 n} \left(\frac{5}{2}\right)^i n = n \sum_{i=0}^{\log_2 n} \left(\frac{5}{2}\right)^i = n \frac{\frac{5}{2}}{\frac{5}{2}-1} \cdot \frac{5^{\log_2 n}}{2^{\log_2 n}} \\ = n \frac{5}{3} \cdot \frac{n^{\log_2 5}}{n} = \frac{5}{3} n^{\log_2 5} = \boxed{O(n^{\log_2 5})}$$

$$W(n) = 2W(n-1) + 1$$

$$W(n-1) = 2W(n-2) + 1$$

$$W(n) = 2^2 W(n-2) + 2 + 1$$

$$W(n-2) = 2W(n-3) + 1$$

$$W(n) = 2^3 W(n-3) + 2^2 + 2 + 1$$

$$W(n) = 2^{n-1} W(n-(n-1)) + 2^{n-2} + \dots + 2^2 + 2 + 1$$

$$W(1) = 1$$

$$\Rightarrow W(n) = 2^{n-k} W(n-k) - 2^n \\ = \boxed{O(2^n)}$$

$$W(n) = 9W\left(\frac{n}{3}\right) + n^2 = \boxed{O(n^2 \log n)} \quad \leftarrow \text{From problem on first page}$$