

$$\log_b n$$

$$\sum_{i=0}^{\log_b n} c$$

$$\log_b n = \log_b r$$

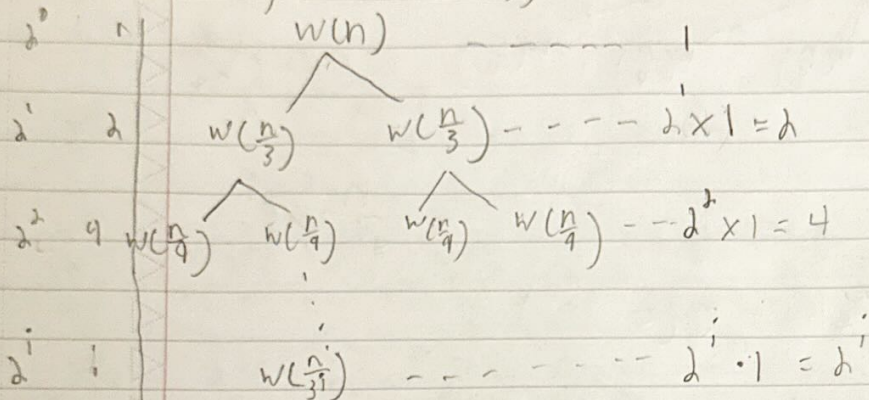
$$\sum_{i=0}^{\log_b n} a^i \cdot c$$

$$\text{depth} = \log_3 n$$

Part 1. Asymptotic Analysis

$$1. \quad w(n) = 2w(n/3) + 1$$

$$w(n) = a w(n/b) + c$$



$$\frac{n}{3^i} \quad n = 3^i \quad i = \log_3 n$$

$$\sum_{i=0}^{\log_3 n} 2^i \cdot 1$$

$$2^{\log_3 n}$$

$$w(n) = O(n^{\log_3 2})$$

$$\log 2^n$$

helped by TA.

2. $w(n) = 5w(\frac{n}{4}) + n$

Recursion tree for $w(n) = 5w(\frac{n}{4}) + n$:

- Level 0: $w(n)$ (cost: n)
- Level 1: $5w(\frac{n}{4})$ (cost: $5 \cdot \frac{n}{4}$)
- Level 2: $25w(\frac{n}{16})$ (cost: $25 \cdot \frac{n}{16}$)
- ...
- Level i : $5^i w(\frac{n}{4^i})$ (cost: $5^i \cdot \frac{n}{4^i}$)
- Level $\log_4 n$: $5^{\log_4 n} w(1)$ (cost: $5^{\log_4 n}$)

Sum of costs:

$$n \sum_{i=0}^{\log_4 n} \frac{n}{4^i} \times 5^i$$

$$n \sum_{i=0}^{\log_4 n} \left(\frac{5}{4}\right)^i$$

Since $\frac{5}{4} > 1$, the sum is dominated by the last term:

$$w(n) = O(n \log n)$$

3. $w(n) = 7w(\frac{n}{7}) + n$

Recursion tree for $w(n) = 7w(\frac{n}{7}) + n$:

- Level 0: $w(n)$ (cost: n)
- Level 1: $7w(\frac{n}{7})$ (cost: $7 \cdot \frac{n}{7} = n$)
- Level 2: $49w(\frac{n}{49})$ (cost: $49 \cdot \frac{n}{49} = n$)
- ...
- Level i : $7^i w(\frac{n}{7^i})$ (cost: $7^i \times \frac{n}{7^i} = n$)
- Level $\log_7 n$: $7^{\log_7 n} w(1)$ (cost: $7^{\log_7 n}$)

Sum of costs:

$$n \sum_{i=0}^{\log_7 n} \frac{n}{7^i} \times 7^i$$

Since the cost is constant at each level, the total cost is:

$$n \sum_{i=0}^{\log_7 n} 1 = n \log_7 n$$

balanced search tree

$$w(n) = O(n \log n)$$

$$4. \quad W(n) = 9W\left(\frac{n}{3}\right) + n^2$$

$$\begin{array}{l} 9^0 \quad 1 \\ 9^1 \quad 9 \\ 9^i \quad \frac{n}{3^i} \end{array} \quad \begin{array}{c} W(n) \\ \swarrow \quad \downarrow \quad \searrow \\ W\left(\frac{n}{3}\right)^2 \quad W\left(\frac{n}{3}\right)^2 \quad W\left(\frac{n}{3}\right)^2 \quad W\left(\frac{n}{3}\right)^2 \quad W\left(\frac{n}{3}\right)^2 \quad W\left(\frac{n}{3}\right)^2 \quad W\left(\frac{n}{3}\right)^2 \quad W\left(\frac{n}{3}\right)^2 \end{array} \quad \begin{array}{l} 9^0 \cdot n^2 = n^2 \\ \dots 9^i \left(\frac{n}{3}\right)^2 = n^2 \end{array}$$

$$i = \log_3 n \quad n^2 \sum_{i=0}^{\log_3 n} \left(\frac{n}{3^i}\right)^2 \cdot 9^i$$

$$\left(\frac{n}{3^i}\right)^2 \cdot 9^i = n^2$$

$$W(n) = O(n^2 \log n) \quad \text{balanced}$$

$$5. \quad 8W\left(\frac{n}{2}\right) + n^3$$

$$\begin{array}{l} 8^0 \\ 8^1 \\ 8^i \end{array} \quad \begin{array}{c} W(n) \\ \swarrow \quad \downarrow \quad \searrow \\ W\left(\frac{n}{2}\right)^2 \quad W\left(\frac{n}{2}\right)^2 \quad W\left(\frac{n}{2}\right)^2 \quad W\left(\frac{n}{2}\right)^2 \quad W\left(\frac{n}{2}\right)^2 \quad W\left(\frac{n}{2}\right)^2 \quad W\left(\frac{n}{2}\right)^2 \quad W\left(\frac{n}{2}\right)^2 \end{array} \quad \begin{array}{l} 8^0 \cdot n^3 = n^3 \\ 8^1 \left(\frac{n}{2}\right)^3 = n^3 \\ 8^i \left(\frac{n}{2^i}\right)^3 = n^3 \end{array}$$

$$i = \log_2 n \quad n^3 \sum_{i=0}^{\log_2 n} 8^i \left(\frac{n}{2^i}\right)^3$$

$$W(n) = O(n^3 \log n)$$

balanced

$$6. \quad 49W\left(\frac{n}{25}\right) + n^{3/2} \log n$$

$$\begin{array}{l} 49^0 \\ 49^1 \\ \frac{n}{25^i} \end{array} \quad \begin{array}{c} W(n) \\ \swarrow \quad \downarrow \quad \searrow \\ W\left(\frac{n}{25}\right)^{3/2} \log\left(\frac{n}{25}\right) \quad \dots \quad W\left(\frac{n}{25}\right)^{3/2} \log\left(\frac{n}{25}\right) \quad W\left(\frac{n}{25}\right)^{3/2} \log\left(\frac{n}{25}\right) \quad W\left(\frac{n}{25}\right)^{3/2} \log\left(\frac{n}{25}\right) \end{array} \quad \begin{array}{l} n^{3/2} \log n \\ 49 \left(\frac{n}{25}\right)^{3/2} \log\left(\frac{n}{25}\right) = n^{3/2} \log n \\ 49^i \left(\frac{n}{25^i}\right)^{3/2} \log\left(\frac{n}{25^i}\right) = n^{3/2} \log\left(\frac{n}{25^i}\right) \end{array}$$

$$i = \log_{25} n \quad n^{3/2} \log n \sum_{i=0}^{\log_{25} n} 49^i \left(\frac{n}{25^i}\right)^{3/2} \log\left(\frac{n}{25^i}\right)$$

$$W(n) = O(n^{3/2} \log n)$$

$$7. w(h) = w(h-1) + 2$$

$$\begin{array}{lcl} 1^0 & 0 & w(h) \dots \dots \dots 2 \\ 1^1 & 1 & w(h-1) \dots \dots \dots 2 \\ & \vdots & \vdots \\ 1^i & i & w(h-i) \dots \dots \dots 2 \\ & \vdots & \vdots \\ 1^{n-1} & n-1 & w(h-1) \dots \dots \dots 2 \end{array}$$

$$\sum_{i=0}^{n-1} 2 = 2(n-1)$$

$$w(h) = 2n - 2$$

$$w(h) = O(h)$$

$$8. w(h) = w(h-1) + n^l \quad \text{with } l \geq 1$$

$$\begin{array}{lcl} h & 0 & w(h) \dots \dots \dots n^l \\ h-1 & 1 & (h-1)^l \dots \dots \dots (n-1)^l \\ & \vdots & \vdots \\ h-i & i & (h-i)^l \dots \dots \dots (n-i)^l \end{array}$$

depth = n-1

$$w(h) = \sum_{(h-i)=1}^{n-1} (h-i)^l$$

$$w(h) = \sum_{(h-i)=1}^h (h-i)^l = O(h^{l+1})$$

$$w(h) = O(h^{l+1})$$

$$9. w(n) = w(\sqrt{n}) + 1$$

$$\text{depth } 1 \log \log n$$

$$1^0 \quad 1 \quad w(h) \dots \dots \dots 1^0 \cdot 1 \approx 1$$

$$1^1 \quad 2 \quad w(h^{1/2}) \dots \dots \dots 1^1 \cdot h^{1/2} = h^{1/2} \approx 1$$

$$1^2 \quad w(h^{1/4}) \dots \dots \dots 1^2 \cdot h^{1/4} = h^{1/4} \approx 1$$

$$1^3 \quad w(h^{1/8}) \dots \dots \dots (h^{1/8})^{1/8} \cdot 1 \approx 1$$

$$1^4 \quad w(h^{1/16}) \dots \dots \dots (h^{1/16})^{1/16} \cdot 1 \approx 1$$

$$1^5 \quad w(h^{1/32}) \dots \dots \dots (h^{1/32})^{1/32} \cdot 1 \approx 1$$

$$1^6 \quad w(h^{1/64}) \dots \dots \dots (h^{1/64})^{1/64} \cdot 1 \approx 1$$

$$1^7 \quad w(h^{1/128}) \dots \dots \dots (h^{1/128})^{1/128} \cdot 1 \approx 1$$

$$\text{depth } \log_2 n$$

$$\sum_{i=0}^{\log_2 n} (n^{1/2^i})^{1/2^i}$$

$$w(h) = O(\log \log h)$$

Part 2, Algorithm comparison

A. $w(n) = 5w(\frac{n}{5}) + n$

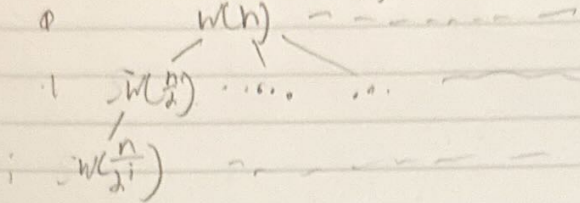
5^0

5^1

5^i

$2^{\frac{n}{5^i}}$

$n=2^i$



$$5^0 \cdot n = n$$

$$5^1 (\frac{n}{5}) = \frac{5n}{5}$$

$$5^i (\frac{n}{5^i})$$

$$i = \log_5 n$$

$$n \sum_{i=0}^{\log_5 n} 5^i (\frac{n}{5^i})$$

$$w(n) (5^{\log_5 n})$$

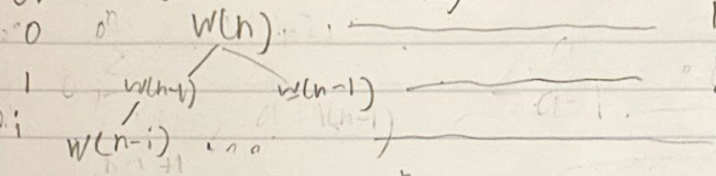
leaf dominates

B. $w(n) = 2w(n-1) + 1$

2^0

2^1

2^i



$$\sum_{i=0}^n a^i = \frac{a^{n+1} - 1}{a - 1} < \frac{a}{a-1} \cdot a^n$$

$$\sum_{i=0}^{n-1} 2^i = \frac{2^{n+1} - 1}{2 - 1} = 2^n - 1$$

$$w(n) = O(2^n)$$

C. $w(n) = 9w(\frac{n}{3}) + n^2$

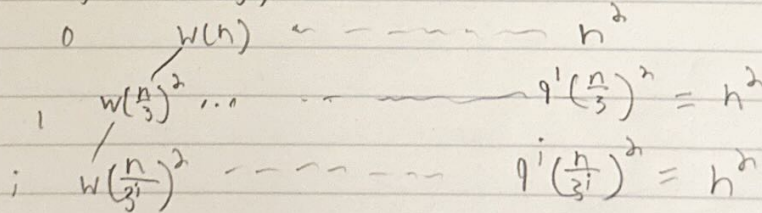
9^0

9^1

9^i

$$i = \log_3 n$$

$n=$



$$9^1 (\frac{n}{3})^2 = n^2$$

$$9^i (\frac{n}{3^i})^2 = n^2$$

$$n^2 \sum_{i=0}^{\log_3 n} 9^i (\frac{n}{3^i})^2$$

$$w(n) = O(n^2 \log n)$$

The third algorithm is better than the first
C is the best