Pricing a European Put Option on a Zero-Coupon Bond

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In the following report, one can find a detailed explanation on how to price a European put option on a zero-coupon bond using three different methods : analytically, then using using a finite-different approach with the explicit method, and finally using the implicit method. All calculations are implemented with MATLAB.

1) The framework

We consider a Vasicek short rate model, such that:

$$dr_t = k(\theta - r(t))dt + \sigma_r dW_t$$

where the parameters k, θ and σ_r are given. Moreover, the maturity of the put option considered is $T_1 = 3/12$, while the maturity of the underlying (the zero-coupon bond) is $T_2 = 5 + 3/12$, i.e. 5 years and 3 months.

Before analyzing the different methods, it is important to highlight the fact that the boundary conditions will be the same independently of the method. Given the option considered is a put option, the boundary conditions are given by:

$$V(T_{max}, r_j) = max(0, K - P(T_1, T_2; r_j))$$
$$V(i, r_{min}) = 0$$
$$V(i, r_{max}) = K - P(T_1, T_2; r_{max})$$

2) The analytical method

Pricing the put option with the analytical method is straightforward. Indeed, we use the following analytical formula:

$$ZBP(0, T_1, T_2, K) = KP(0, T_1; r_0)N(-h + \tilde{\sigma_r}) - P(0, T_2; r_0)N(-h),$$

where the strike $K = 0.805$, the initial interest rate at t_0 is $r_0 = 0.042$, N is the cumulative distribution function of a standard gaussian random variable, and h and $\tilde{\sigma_r}$ have closed-form expressions given in the assignment. The zero-coupon bond price is considered to have an affine term structure and its closed-form expression is also given in the assignment.

Implementing this function in Matlab with the given parameters, one easily obtains:

$$ZBP(0, 3/12, 5 + 3/12, 0.805) = 0.075$$

In the following, we will use this analytical result as a benchmark to compare the results obtained with numerical methods.

3) The explicit method

Remember the option's PDE is given by:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial r} k(\theta - r) + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} \sigma_r^2 = rV$$

where the subscripts for time in front of r, the interest rate, and in front of V, the value of the option, have been omitted for clarity.

After discretization, this equation becomes:

$$\frac{V_{i,j} - V_{i,j-1}}{\delta t} + k(\theta - i\delta r) \frac{V_{i+1,j} - V_{i-1,j}}{\delta r} + \frac{1}{2} \sigma_r^2 \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{\delta r^2} = rV_{i,j}$$

Iterating this equation until the final time N, the equation can be written as:

$$\frac{V_{i,N} - V_{i,N-1}}{\delta t} + k(\theta - i\delta r) \frac{V_{i+1,N} - V_{i-1,N}}{\delta r} + \frac{1}{2} \sigma_r^2 \frac{V_{i+1,N} - 2V_{i,N} + V_{i-1,N}}{\delta r^2} = rV_{i,N}$$

Now, using the boundary conditions, the only unknown of this equation is V_{i,N_1}. Hence, we have:

$$V_{i,N-1} = a_i V_{i-1,N} + b_i V_{i,N} + c_i V_{i+1,N},$$

where

$$\begin{aligned}
a_i &= \frac{\delta t}{2}\left(\frac{\sigma_r^2}{\delta r^2} - \frac{k(\theta - i\delta r)}{\delta r}\right) \\
b_i &= 1 - \delta t\left(r + \frac{\sigma^2 r}{\delta r^2}\right) \\
c_i &= \frac{\delta t}{2}\left(\frac{\sigma_r^2}{\delta r^2} + \frac{k(\theta - i\delta r)}{\delta r}\right)
\end{aligned}$$

Simply iterating this equation backwards, we can obtain our put option price using linear interpolation.

The importance of this method, just like for the implicit method, is to choose a reasonable two-dimensional grid for interest rate and time intervals. In our case, we chose to experiment with different 'space' and time intervals: $I = \{100, 110, 120, ..., 980, 990, 1000\}$

The other crucial point is how we decided to bound our variable r. We were imposed to fix a minimal bound of 0, and we decided to fix a maximal bound of 1, as we considered the probability that the interest rate reaching 100% is negligible.

Finally, using this method, we obtain a vector zbp_{exp} with length L = $(1000/100) * 10 + 1 = 51$, with each row corresponding to the price of the put option depending on the interval over I.
This vector will be used later for a plot to compare our results with the analytical, the implicit and the explicit methods.

4) The implicit method

Similarly to the explicit method, the implicit method relies on a discretisation of the PDE. The difference lies in the fact that the forward difference is used for $\frac{\partial V}{\partial t}$.

Hence, after discretisation and using the forward difference, we obtain:

$$V_{i+1,j} = a_j V_{i,j-1} + b_j V_{i,j} + c_j V_{i,j+1}$$

The parameters a_j, b_j and c_j are just the same as for the explicit method, but with inverted signs.

Hence, this equation can be represented in the following matrix form:

$$MV_i = V_{i+1} + b$$

$$\begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & ... & ... & ... & \\ & & a_{M-2} & b_{M-2} & c_{M-2} \\ & & & a_{M-1} & b_{M-1} \end{bmatrix} \begin{bmatrix} v_{i,1} \\ v_{i,2} \\ v_{i,3} \\ ... \\ v_{i,M-2} \\ v_{i,M-1} \end{bmatrix} = \begin{bmatrix} v_{i+1,1} \\ v_{i+1,2} \\ v_{i+1,3} \\ ... \\ v_{i+1,M-2} \\ v_{i+1,M-1} \end{bmatrix} - \begin{bmatrix} a_1 v_{i,0} \\ 0 \\ 0 \\ ... \\ 0 \\ c_{M-1} v_{i,M} \end{bmatrix}$$

$$\iff V_i = M^{-1}(V_{i+1} + b)$$

Again, the option price is obtained using linear interpolation and we obtain an $L \times 1$ vector zbp_{imp} containing different values for the put option price depending on the length of the interval for time and interest rates grids.

5) Comparing results

We can now compare all the results we obtained using the different methods. Below is a plot joining all our results, where we have plotted on the x-axis the intervals I, from 100 to 1000, and on the y-axis the different put-option price depending on the length of the grids. The dashed lines correspond to our confidence interval of 99% (any deviation above the upper line and below the lower line corresponds to a deviation of more than 1% in absolute value of the analytical price).

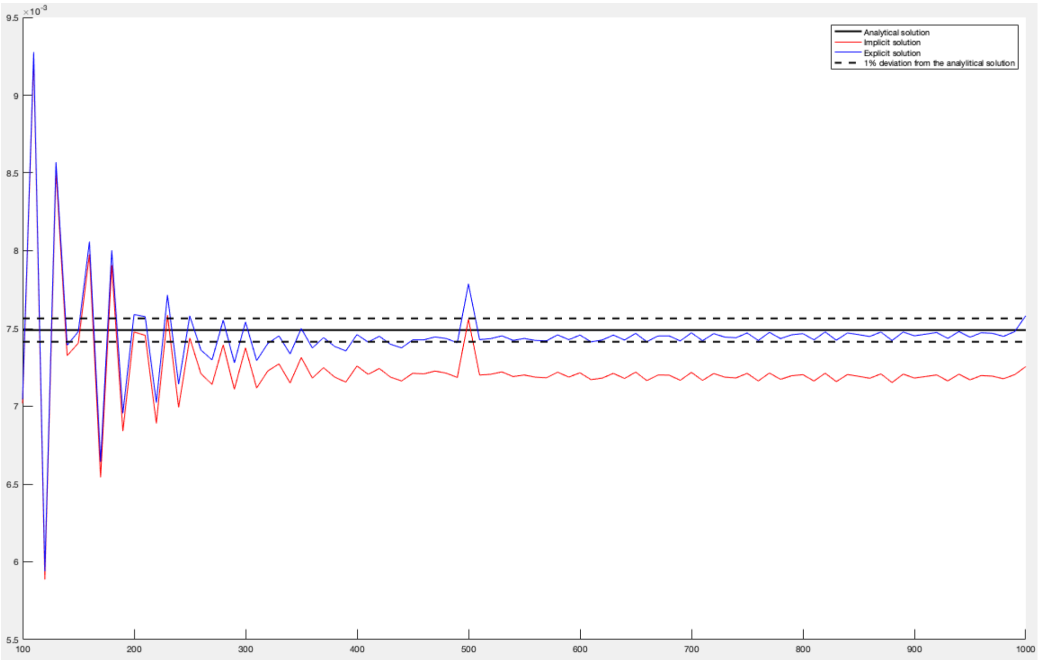

Figure 1: Convergence of the finite-difference numerical methods towards the analytical method

One can see that while the explicit method seems to converge to the analytical solution as the length of grids increases, it is not the case for the implicit method.