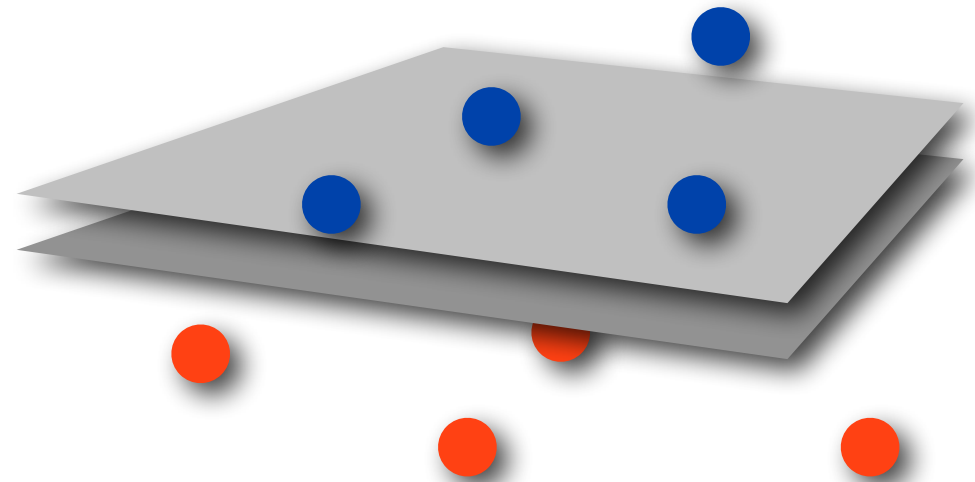
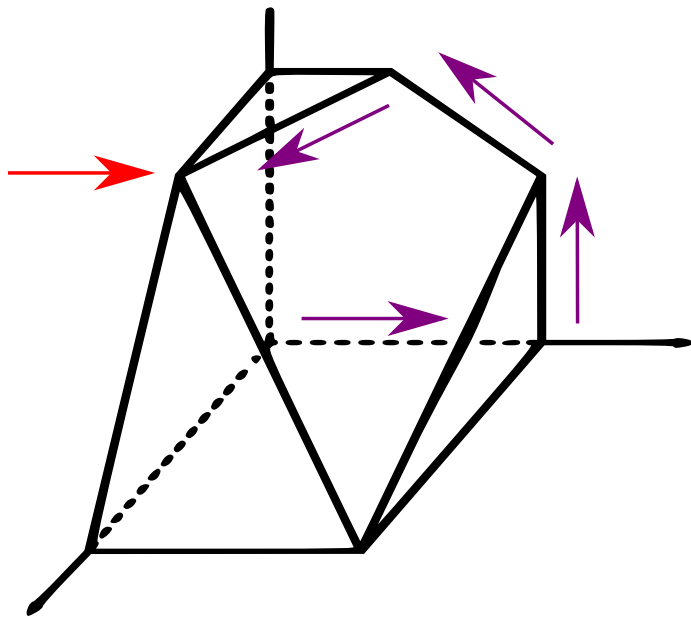


Linear Programming

with CCGAL



AlgoLab HS20 Tutorial #7 — Hung Hoang

Based on work by Luis Barba, Bernd Gärtner and Sebastian Stich.

Today

- ❖ Quick introduction to Linear Programming (formulation and representation)
- ❖ Linear Programming in CGAL
- ❖ Examples

Linear Programming (LP)

- ❖ Central topic in optimization
- ❖ Powerful modeling tool in many applications
- ❖ Attracted a lot of attention in optimization during the last six decades for two main reasons:
 - ❖ **Applicability**: Many real-world problems can be modeled using LP;
 - ❖ **Solvability**: Theoretically and practically efficient techniques for solving large-scale problems.

Linear Programming (LP)

An optimization problem subject to constraints.

- ❖ **Variables:** parameters whose values we can choose;
- ❖ **Objective function:** the criterion to minimize (e.g. cost) or maximize (e.g. profit);
- ❖ **Constraints:** limitations for choosing values for the variables.

Obs. Considering minimization is sufficient because minimizing $-f$ is equivalent to maximizing f .

LP—Mathematical Formulation

- ❖ **Problem:** Minimize a **linear** function in n variables subject to m **linear** (in)equality constraints.

- ❖ **Example** ($n=2, m=5$):

minimize

$-32y + 64$ subject to

$$x + y \leq 7$$

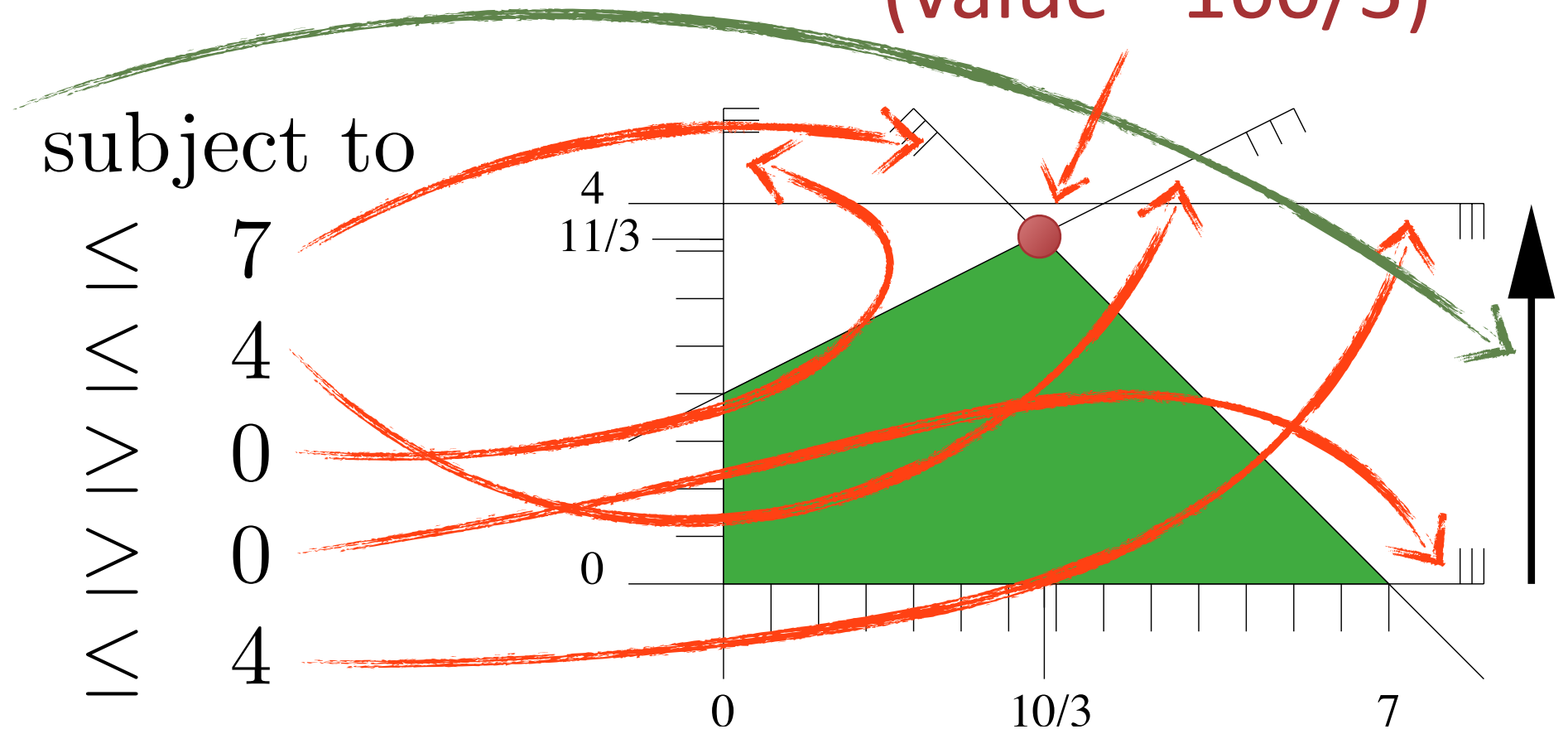
$$-x + 2y \leq 4$$

$$x \geq 0$$

$$y \geq 0$$

$$y \leq 4$$

Optimal solution
(value $-160/3$)



LP—Mathematical Formulation

- ❖ **Problem:** Minimize a linear function in n variables subject to m linear (in)equality constraints.

- ❖ **Example** ($n=2$, $m=5$):

minimize

$-32y + 64$ subject to

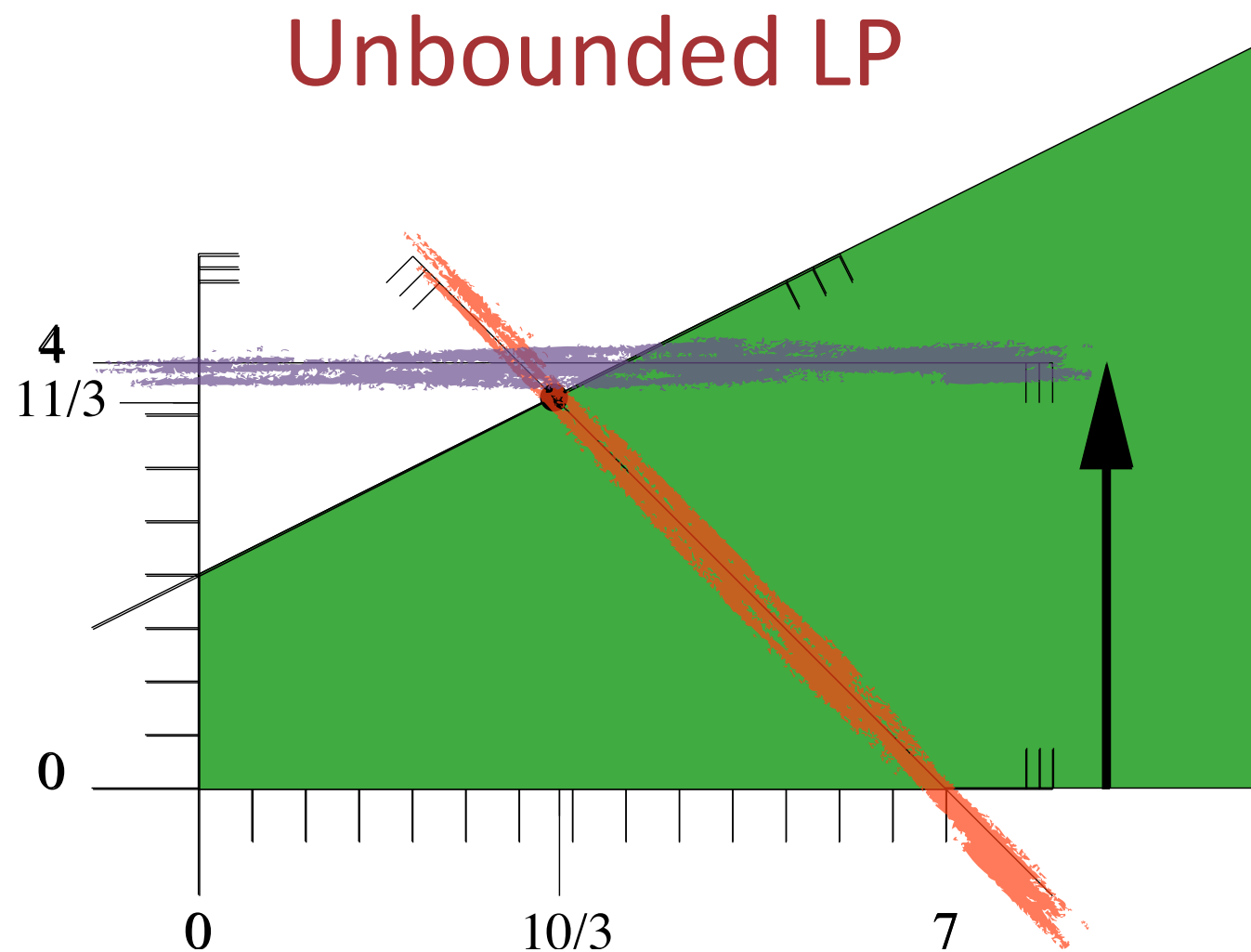
$$x + y \leq 7$$

$$-x + 2y \leq 4$$

$$x \geq 0$$

$$y \geq 0$$

$$y \leq 4$$



LP—Mathematical Formulation

- ❖ **Problem:** Minimize a linear function in n variables subject to m linear (in)equality constraints.

- ❖ **Example** ($n=2$, $m=5$):

minimize

$-32y + 64$ subject to

$$x + y \leq 7$$

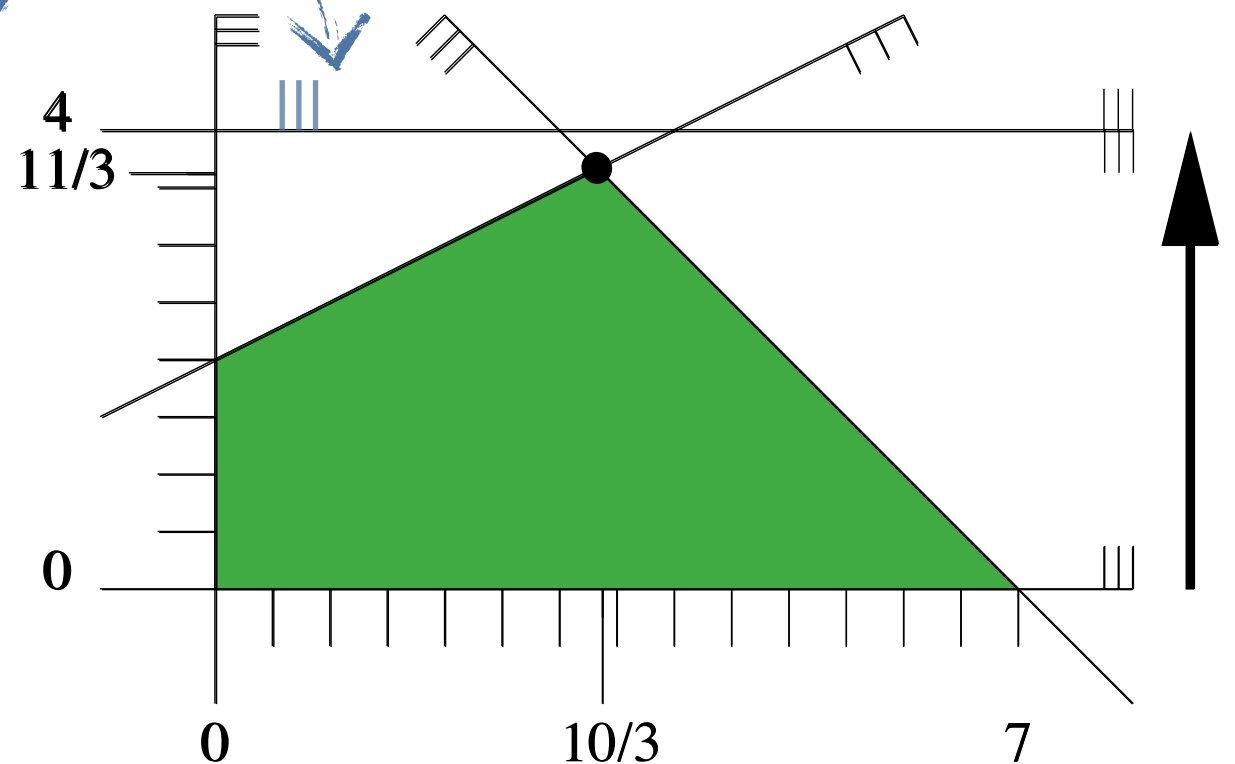
$$-x + 2y \leq 4$$

$$x \geq 0$$

$$y \geq 0$$

$$y \geq 4 \quad y \leq 4$$

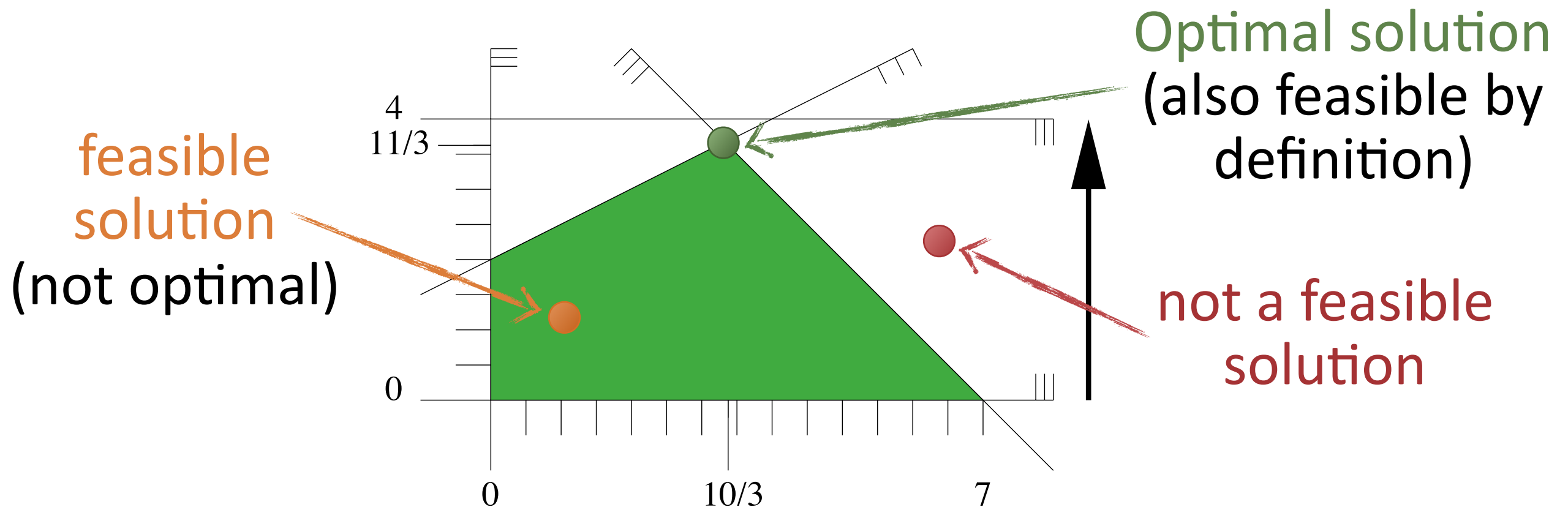
Infeasible LP



Feasible and Optimal Solutions

A **feasible solution** is an assignment of values to the variables of an LP that satisfies all constraints.

A feasible solution is **optimal** if it minimizes the objective function (among all feasible solutions).



Types of Linear Programs

A linear program is of one of the following three types:

- ❖ **Optimal:** there exists an optimal solution (possibly many) that attains the unique minimum of the objective function;
- ❖ **Unbounded:** there exist feasible solutions that attain arbitrarily small values of the objective function;
- ❖ **Infeasible:** There is no feasible solution.

LP—Matrix Formulation

- ❖ **Problem:** Minimize a linear function in n variables subject to m linear (in)equality constraints!

- ❖ **Example** ($n=2, m=5$):

convert “ \geq ” to “ \leq ” by
inverting coefficients

minimize

$-32y + 64$ subject to

$$x + y \leq 7$$

$$-x + 2y \leq 4$$

$$-x \leq 0 \quad x \geq 0$$

$$-y \leq 0 \quad y \geq 0$$

$$y \leq 4$$



A			b
1	1	x	7
-1	2		4
-1	0	y	0
0	-1		0
0	1		4

LP—Matrix Formulation

- ❖ **Problem:** Minimize a linear function in n variables subject to m linear (in)equality constraints!

- ❖ **Example** ($n=2, m=5$):

minimize

$-32y + 64$ subject to

$$\begin{array}{rcl} x + y & \leq & 7 \\ -x + 2y & \leq & 4 \\ -x & \leq & 0 \\ -y & \leq & 0 \\ y & \leq & 4 \end{array}$$

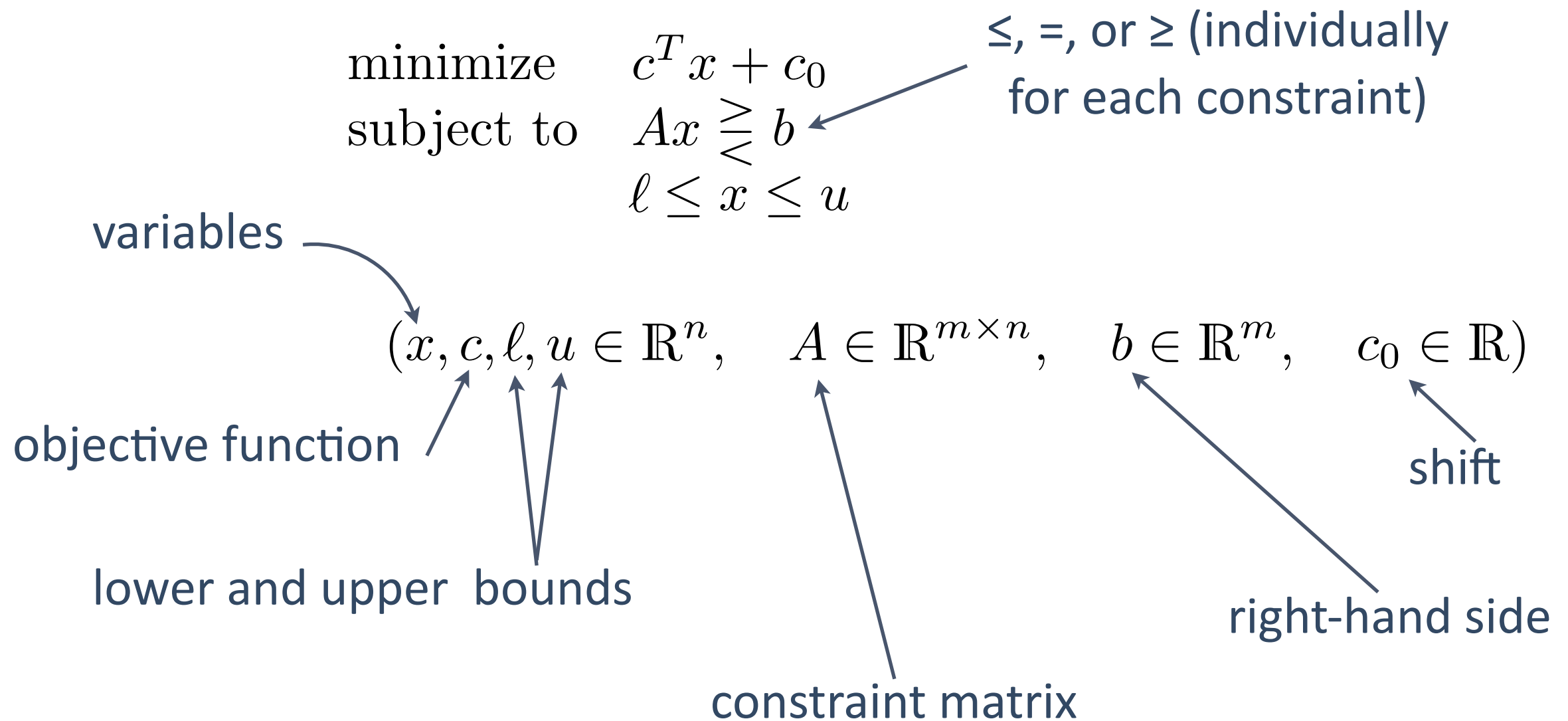
$$\begin{array}{c} c^T \\ 0 \\ -32 \end{array} \begin{array}{cc} x & y \end{array} + \begin{array}{c} c_0 \\ 64 \end{array}$$



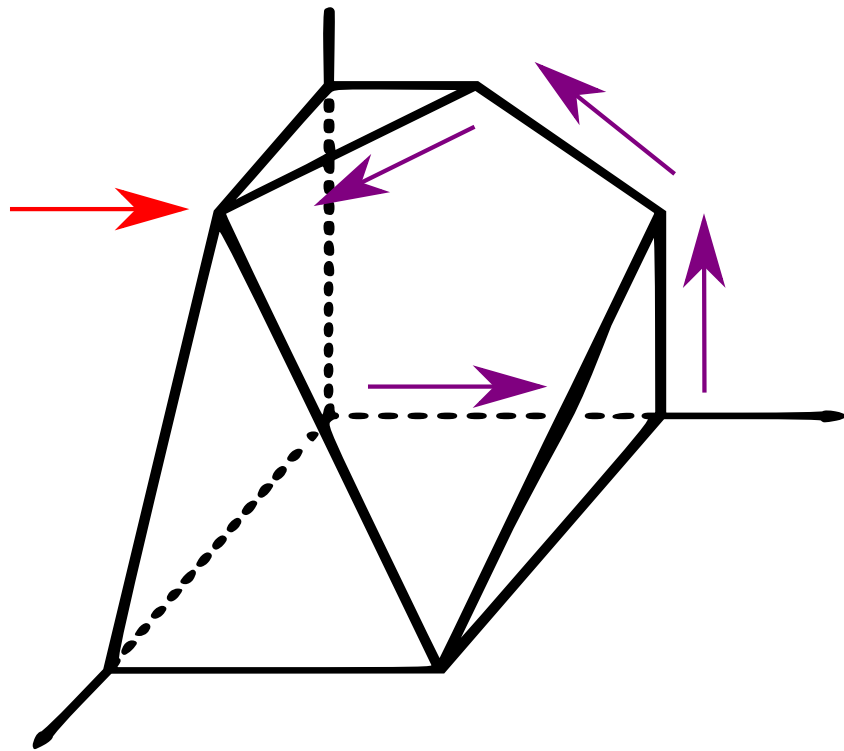
$$\begin{array}{cc} 1 & 1 \\ -1 & 2 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{array} \begin{array}{c} x \\ y \end{array} \leq \begin{array}{c} 7 \\ 4 \\ 0 \\ 0 \\ 4 \end{array}$$

A b

LP—General Form in CGAL



LP—Geometric Interpretation

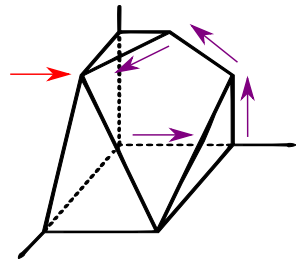


$$\begin{array}{ll}\text{minimize} & c^T x + c_0 \\ \text{subject to} & Ax \leq b \\ & \ell \leq x \leq u\end{array}$$

$$P = \{x \mid Ax \leq b\}$$

- ❖ The constraints define an n -dimensional convex polyhedron P with $\leq m$ faces.
- ❖ Optimum objective value is attained at a vertex of P .
- ❖ But $\# \text{vertices}(P)$ can be exponential in m and n .
=> Trying all vertices not feasible.

LP—Complexity of CGAL Solver



$$\begin{aligned} &\text{minimize} && c^T x + c_0 \\ &\text{subject to} && Ax \geq b \\ & && \ell \leq x \leq u \\ & && P = \{x \mid Ax \leq b\} \end{aligned}$$

- ❖ The constraints define an n -dimensional convex polyhedron P with $\leq m$ faces.
- ❖ CGAL solver uses a simplex-type algorithm: Walk from vertex to vertex along edges of P while improving the objective value.
- ❖ Worst-case complexity is exponential in n and m .

For $\min \{n, m\}$ small, the complexity is $O(\max \{n, m\})$.

$\min \{m, n\}$: 0

20

200

Easy!

Avoid!

LP in CGAL—Preamble

- ❖ Choose input type (in this order of preference):
 - ❖ `int, long`
 - ❖ `CGAL::Gmpz` (arbitrary precision integer)
 - ❖ `CGAL::Gmpq` (arbitrary precision rational) } use the GNU Multi-Precision Library (GMP)
- ❖ Choose exact type for solver:
 - ❖ `CGAL::Gmpq` (if this is the input type)
 - ❖ `CGAL::Gmpz` (in all other cases)

```
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpz.h>

// choose input type (input coefficients must fit)
typedef int IT;
// choose exact type for solver (CGAL::Gmpz or CGAL::Gmpq)
typedef CGAL::Gmpz ET;
```

LP in CGAL—Preamble (2)

- ❖ Define program and solution type:

```
// program and solution types  
typedef CGAL::Quadratic_program<IT> Program;  
typedef CGAL::Quadratic_program_solution<ET> Solution;
```

- ❖ Yep, it is called Quadratic... (more general concept) but also used for linear programs.
- ❖ Internally, the solver uses CGAL::Quotient<ET>.

LP in CGAL—Setup

✧ Enter the program data:

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \left\{ \begin{array}{ll} x + y & \leq 7 \\ -x + 2y & \leq 4 \\ x & \geq 0 \\ y & \geq 0 \\ y & \leq 4 \end{array} \right. \\ & Ax \leq b \end{array}$$

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

// set the coefficients of A and b
const int X = 0;
const int Y = 1;
lp.set_a(X, 0, 1); lp.set_a(Y, 0, 1); lp.set_b(0, 7); // x + y <= 7
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4); // -x + 2y <= 4
// set upper bound
lp.set_u(Y, true, 4); // y <= 4
// objective function
lp.set_c(Y, -32); // -32y
lp.set_c0(64); // +64
```

LP in CGAL—Setup

❖ Enter the program data:

Default inequality is \leq .

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \left\{ \begin{array}{ll} x + y & \leq 7 \\ -x + 2y & \leq 4 \\ x & \geq 0 \\ y & \geq 0 \\ y & \leq 4 \end{array} \right. \end{array}$$

$Ax \leq b$

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// set upper bound
lp.set_u(Y, true, 4); // y <= 4
// objective function
lp.set_c(Y, -32); // -32y
lp.set_c0(64); // +64
```

LP in CGAL—Setup

❖ Enter the program data:

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \begin{cases} x + y \leq 7 \\ -x + 2y \leq 4 \\ x \geq 0 \\ y \geq 0 \\ y \leq 4 \end{cases} \end{array}$$

$Ax \leq b$

```
// create an LP with  $Ax \leq b$ , lower bound 0 and no upper bounds
```

```
Program lp (CGAL::SMALLER, true, 0, false, 0);
```

```
// set the coefficients of A and b
```

```
const int X = 0;
```

```
const int Y = 1;
```

```
lp.set_a(X, 0, 1); lp.set_a(Y, 0, 1); lp.set_b(0, 7); //  $x + y \leq 7$ 
```

```
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4); //  $-x + 2y \leq 4$ 
```

```
// set upper bound
```

```
lp.set_u(Y, true, 4); //  $y \leq 4$ 
```

```
// objective function
```

```
lp.set_c(Y, -32); //  $-32y$ 
```

```
lp.set_c0(64); //  $+64$ 
```

Lower bounds: All variables must be \geq zero.

LP in CGAL—Setup

- ✧ Enter the program data:

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \left\{ \begin{array}{ll} x + y & \leq 7 \\ -x + 2y & \leq 4 \\ x & \geq 0 \\ y & \geq 0 \\ y & \leq 4 \end{array} \right. \\ & Ax \leq b \end{array}$$

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// create an LP with Ax <= b, lower bound 0 and no upper bounds
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lp.set_a(X, 0, 1); lp.set_a(Y, 0, 1); lp.set_b(0, 7); // x + y <= 7
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4); // -x + 2y <= 4
// set upper bound
lp.set_u(Y, true, 4); // y <= 4
// objective function
lp.set_c(Y, -32); // -32y
lp.set_c0(64); // +64
```

No upper bounds.

LP in CGAL—Setup

- ✦ Enter the program data:

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \begin{cases} x + y \leq 7 \\ -x + 2y \leq 4 \\ x \geq 0 \\ y \geq 0 \\ y \leq 4 \end{cases} \\ & Ax \leq b \end{array}$$

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);
```

```
// set the coefficients of A and b
```

```
const int X = 0;
const int Y = 1;
```

Convenient alias
(index) for variables.

```
lp.set_a(X, 0, 1); lp.set_a(Y, 0, 1); lp.set_b(0, 7); // x + y <= 7
```

```
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4); // -x + 2y <= 4
```

```
// set upper bound
```

```
lp.set_u(Y, true, 4); // y <= 4
```

```
// objective function
```

```
lp.set_c(Y, -32); // -32y
```

```
lp.set_c0(64); // +64
```

LP in CGAL—Setup

- ✦ Enter the program data:

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \left\{ \begin{array}{ll} x + y \leq 7 \\ -x + 2y \leq 4 \\ x \geq 0 \\ y \geq 0 \\ y \leq 4 \end{array} \right. \end{array}$$

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
```

```
Program lp (CGAL::SMALLER, true, 0, false, 0);
```

```
// set the coefficients of A and b
```

```
const int X = 0;
```

```
const int Y = 1;
```

```
lp.set_a(X, 0, 1); lp.set_a(Y, 0, 1); lp.set_b(0, 7); // x + y <= 7
```

```
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4); // -x + 2y <= 4
```

```
// set upper bound
```

```
lp.set_u(Y, true, 4); // y <= 4
```

```
// objective function
```

```
lp.set_c(Y, -32); // -32y
```

```
lp.set_c0(64); // +64
```

Constraint/inequality #0.

LP in CGAL—Setup

✦ Enter the program data:

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \left\{ \begin{array}{ll} x + y & \leq 7 \\ -x + 2y & \leq 4 \\ x & \geq 0 \\ y & \geq 0 \\ y & \leq 4 \end{array} \right. \end{array}$$

$Ax \leq b$

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
```

```
Program lp (CGAL::SMALLER, true, 0, false, 0);
```

```
// set the coefficients of A and b
```

```
const int X = 0;
```

```
const int Y = 1;
```

```
lp.set_a(X, 0, 1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);
```

```
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);
```

```
// set upper bound
```

```
lp.set_u(Y, true, 4);
```

```
// objective function
```

```
lp.set_c(Y, -32);
```

```
lp.set_c0(64);
```

Constraint/inequality #1.

// x + y <= 7

// -x + 2y <= 4

// y <= 4

// -32y

// +64

LP in CGAL—Setup

❖ Enter the program data:

You can use `set_a()` and `set_b()` for the upper bound, too. But we recommend `set_u()` for convenience and efficiency.

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \begin{cases} x + y \leq 7 \\ -x + 2y \leq 4 \\ x \geq 0 \\ y \geq 0 \\ y \leq 4 \end{cases} \end{array}$$

$Ax \leq b$

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

// set the coefficients of A and b
const int X = 0;
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lp.set_a(X, 0, 1); lp.set_a(Y, 0, 1); lp.set_b(0, 7); // x + y <= 7
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// set upper bound
lp.set_u(Y, true, 4); // y <= 4
// objective function
lp.set_c(Y, -32); // -32y
lp.set_c0(64); // +64
```

Upper bound for y.

LP in CGAL—Setup

- ✦ Enter the program data:

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \left\{ \begin{array}{ll} x + y & \leq 7 \\ -x + 2y & \leq 4 \\ x & \geq 0 \\ y & \geq 0 \\ y & \leq 4 \end{array} \right. \\ & Ax \leq b \end{array}$$

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

// set the coefficients of A and b
const int X = 0;
const int Y = 1;
lp.set_a(X, 0, 1); lp.set_a(Y, 0, 1); lp.set_b(0, 7); // x + y <= 7
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4); // -x + 2y <= 4
// set upper bound
lp.set_u(Y, true, 4); // y <= 4
// objective function
lp.set_c(Y, -32); // -32y
lp.set_c0(64); // +64
```

LP in CGAL—Setup

- ✦ Enter the program data:

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \left\{ \begin{array}{ll} x + y & \leq 7 \\ -x + 2y & \leq 4 \\ x & \geq 0 \\ y & \geq 0 \\ y & \leq 4 \end{array} \right. \\ & Ax \leq b \end{array}$$

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

// set the coefficients of A and b
const int X = 0;
const int Y = 1;
lp.set_a(X, 0, 1); lp.set_a(Y, 0, 1); lp.set_b(0, 7); // x + y <= 7
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4); // -x + 2y <= 4
// set upper bound
lp.set_u(Y, true, 4); // y <= 4
// objective function
lp.set_c(Y, -32); // -32y
lp.set_c0(64); // +64
```

LP in CGAL—Solve

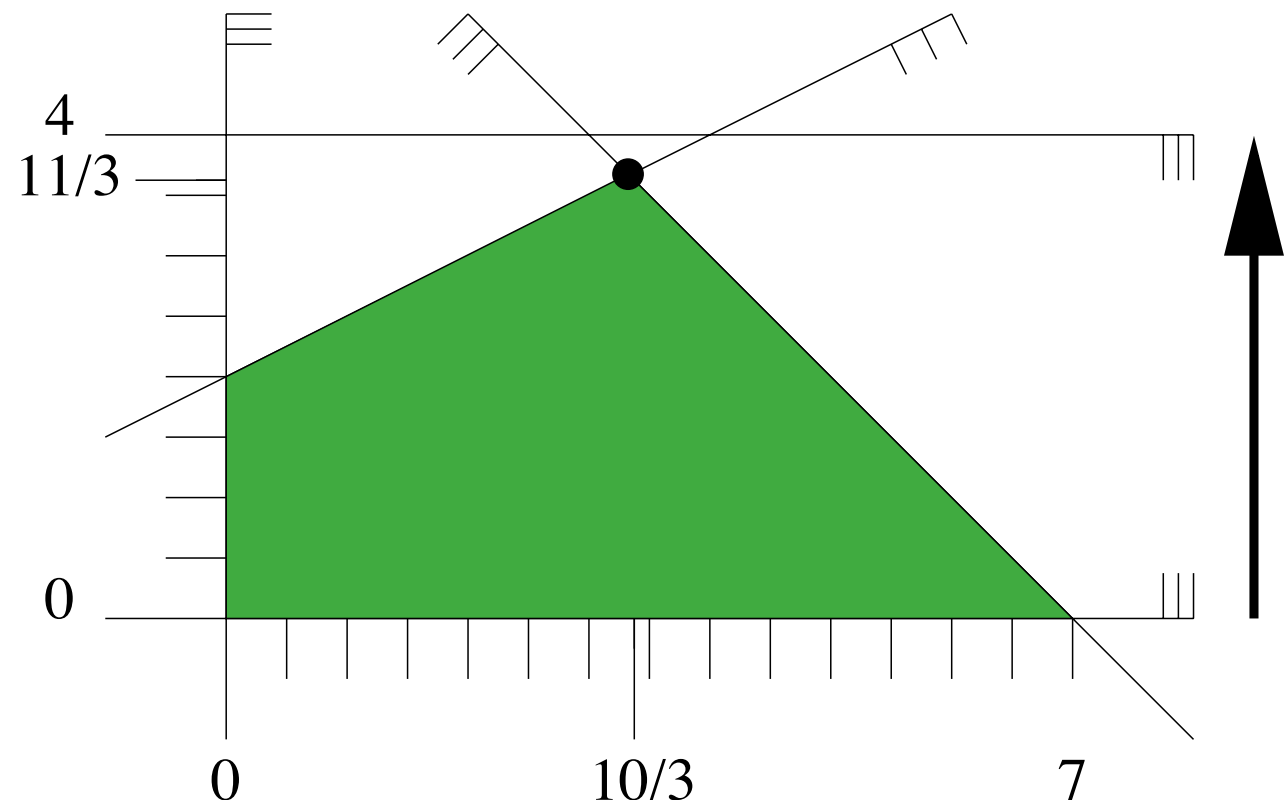
- ❖ Call the LP solver and output the solution:

```
// solve the program, using ET as the exact type
Solution s = CGAL::solve_linear_program(lp, ET());

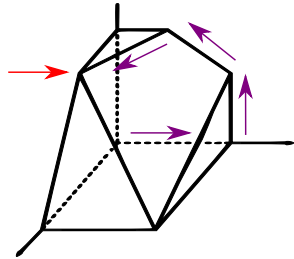
// output solution
std::cout << s;
```

- ❖ Output:

```
status:          OPTIMAL
objective value: -160/3
variable values:
  0: 10/3
  1: 11/3
```



When to apply LP?

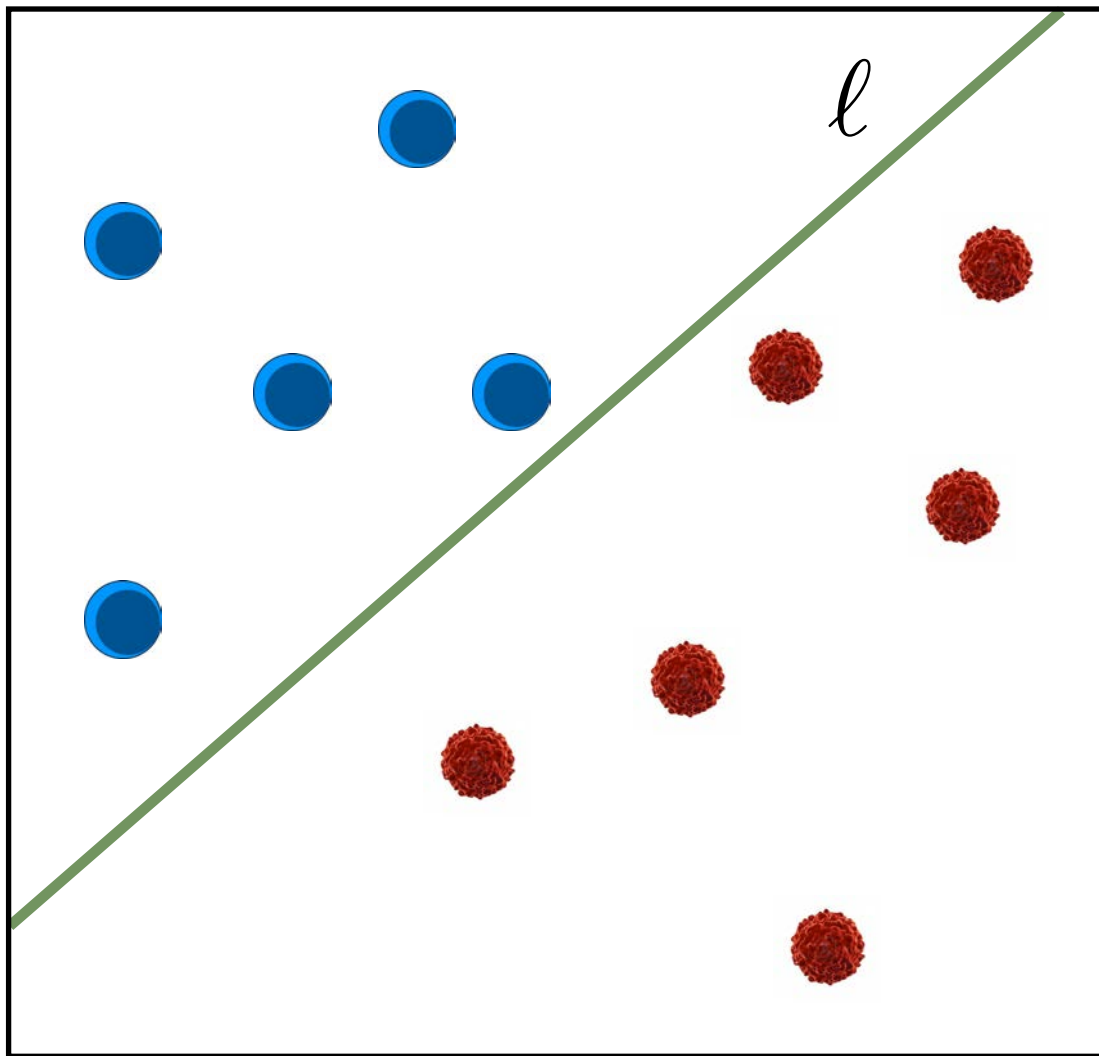


$$\begin{aligned} &\text{minimize} && c^T x + c_0 \\ &\text{subject to} && Ax \geq b \\ &&& \ell \leq x \leq u \\ &&& P = \{x \mid Ax \leq b\} \end{aligned}$$

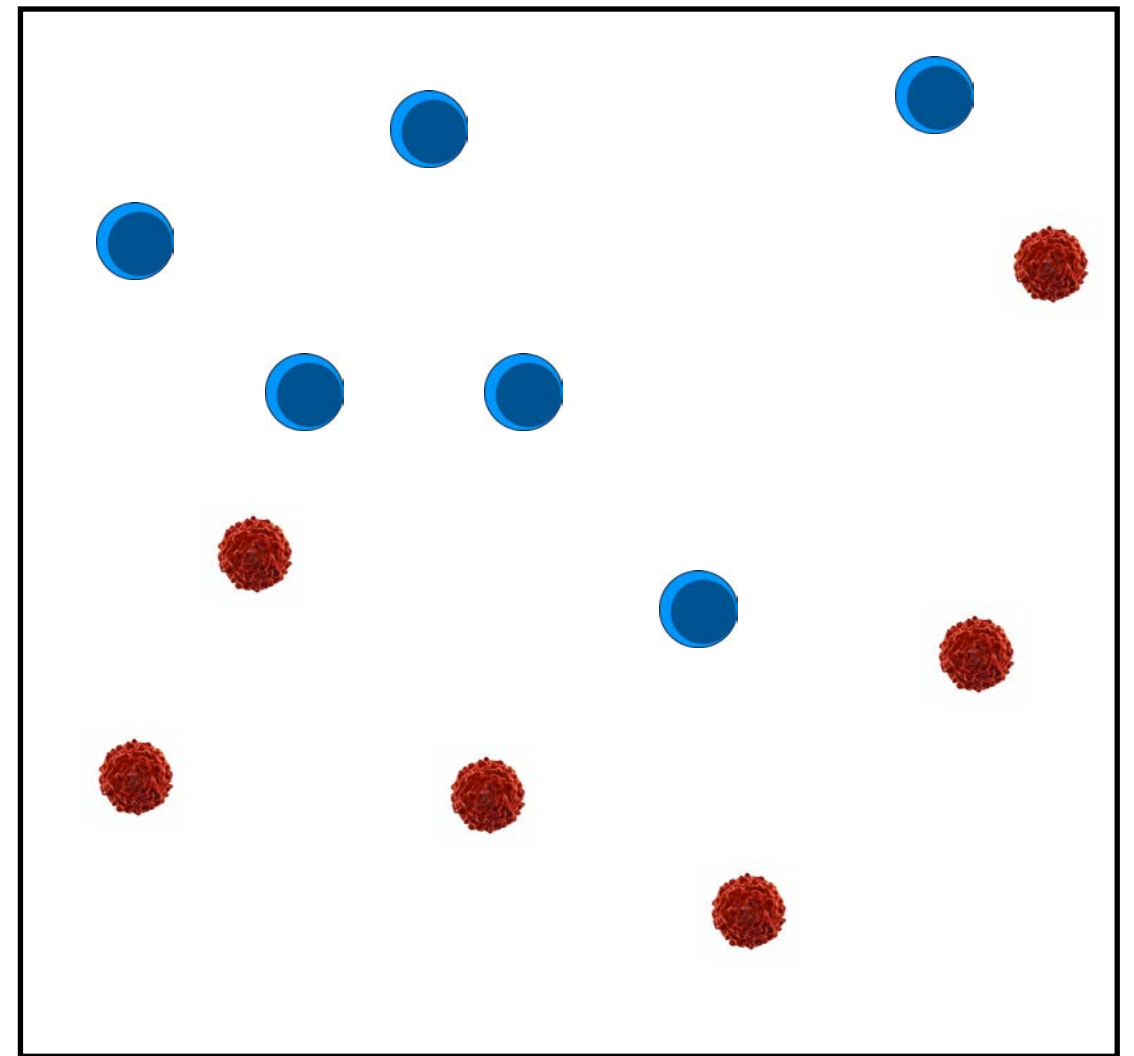
- ✦ If a problem can be modelled using n variables and m constraints s.t.
 - ✦ constraints are **linear** (in)equalities in the variables
 - ✦ at least **one of** n or m is **small** (s.a.).
- ✦ Agenda/Checklist:
 - ✦ What are variables, what are constraints? Output
 - ✦ What is n , what is m ? Is one of n or m small? Input
 - ✦ Are all constraints linear in the variables?

Ex. I: Linear Separation

- ❖ **Given:** A set **R** of red points and a set **B** of blue points.
- ❖ **Want:** A line ℓ that separates **R** and **B**.



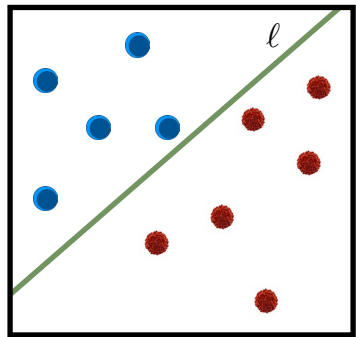
Linearly separable



No solution 🙄

Ex. I: Linear Separation

- ❖ **Given:** A set **R** of red points and a set **B** of blue points.
- ❖ **Want:** A line ℓ that separates **R** and **B**.



- ❖ **Agenda/Checklist:**

- ❖ What are variables, what are constraints?
- ❖ What is n , what is m ? Is one of n or m small?
- ❖ Are all constraints linear in the variables?

$n=3 \Rightarrow$ small
 $m=|\mathbf{R}|+|\mathbf{B}|$

Yes!

- ❖ **Variables/Output:** We are looking for a line ℓ .

$\ell : \{(x,y) : ax + by + c = 0\}$, for a, b, c unknown.

- ❖ **Constraints/Input:** **R** and **B** on different sides of ℓ .

$ap_x + bp_y \leq c$, for $(p_x, p_y) \in \mathbf{R}$ and

$aq_x + bq_y \geq c$, for $(q_x, q_y) \in \mathbf{B}$.

Linear Separation

```
// example: decide whether two point sets R and B can be
// separated by a line

#include <iostream>
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpz.h>

// choose input type (input coefficients must fit)
typedef int IT;
// choose exact type for solver (CGAL::Gmpz or CGAL::Gmpq)
typedef CGAL::Gmpz ET;

// program and solution types
typedef CGAL::Quadratic_program<IT> Program;
typedef CGAL::Quadratic_program_solution<ET> Solution;

int main()
{
    // create an LP with  $Ax \leq b$  and no lower/upper bounds
    Program lp (CGAL::SMALLER, false, 0, false, 0);
    const int a = 0;
    const int b = 1;
    const int c = 2;

    // number of red and blue points
    int m; std::cin >> m;
    int n; std::cin >> n;
```

```
// read the red points
for (int i = 0; i < m; ++i) {
    int x; std::cin >> x;
    int y; std::cin >> y;
    // set up constraint  $a x + b y + c \leq 0$ 
    lp.set_a(a, i, x);
    lp.set_a(b, i, y);
    lp.set_a(c, i, 1);
}
// read the blue points
for (int i = 0; i < n; ++i) {
    int x; std::cin >> x;
    int y; std::cin >> y;
    // set up constraint  $a x + b y + c \geq 0$ 
    lp.set_a(a, m+i, -x);
    lp.set_a(b, m+i, -y);
    lp.set_a(c, m+i, -1);
}
// no objective function, just feasibility

// solve the program, using ET as the exact type
Solution s = CGAL::solve_linear_program(lp, ET());
assert(s.solves_linear_program(lp));

// output solution
std::cout << s;
}
```

Output for
 $R=\{(0,0)\}$ and
 $B=\{(1,1)\}$:

status: OPTIMAL
objective value: 0/1
variable values:
0: 0/1
1: 0/1
2: 0/1

Oops, not a line...

Linear Separation

```
// example: decide whether two point sets R and B can be
// separated by a nonvertical line

#include <iostream>
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpz.h>

// choose input type (input coefficients must fit)
typedef int IT;
// choose exact type for solver (CGAL::Gmpz or CGAL::Gmpq)
typedef CGAL::Gmpz ET;

// program and solution types
typedef CGAL::Quadratic_program<IT> Program;
typedef CGAL::Quadratic_program_solution<ET> Solution;

int main()
{
    // create an LP with  $Ax \leq b$  and no lower/upper bounds
    Program lp (CGAL::SMALLER, false, 0, false, 0);
    const int a = 0;
    const int b = 1;
    const int c = 2;

    // number of red and blue points
    int m; std::cin >> m;
    int n; std::cin >> n;
```

```
// read the red points
for (int i = 0; i < m; ++i) {
    int x; std::cin >> x;
    int y; std::cin >> y;
    // set up constraint  $a x + b y + c \leq 0$ 
    lp.set_a(a, i, x);
    lp.set_a(b, i, y);
    lp.set_a(c, i, 1);
}

// read the blue points
for (int i = 0; i < n; ++i) {
    int x; std::cin >> x;
    int y; std::cin >> y;
    // set up constraint  $a x + b y + c \geq 0$ 
    lp.set_a(a, m+i, -x);
    lp.set_a(b, m+i, -y);
    lp.set_a(c, m+i, -1);
}

// enforce that b is positive
lp.set_l(b, true, 1);

// solve the program, using ET as the exact type
Solution s = CGAL::solve_linear_program(lp, ET());
assert(s.solves_linear_program(lp));

// output solution
std::cout << s;
}
```

Output for
 $R=\{(0,0)\}$ and
 $B=\{(1,1)\}$:

status: OPTIMAL
objective value: 0/1
variable values:
0: 0/1
1: 1/1
2: 0/1

Fix: Set $b \neq 0$.

But this is not a valid inequality.

Linear Separation

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```

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// read the red points
for (int i = 0; i < m; ++i) {
    int x; std::cin >> x;
    int y; std::cin >> y;
    // set up constraint  $a x + b y + c \leq 0$ 
    lp.set_a(a, i, x);
    lp.set_a(b, i, y);
    lp.set_a(c, i, 1);
}

// read the blue points
for (int i = 0; i < n; ++i) {
    int x; std::cin >> x;
    int y; std::cin >> y;
    // set up constraint  $a x + b y + c \geq 0$ 
    lp.set_a(a, m+i, -x);
    lp.set_a(b, m+i, -y);
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Solution s = CGAL::solve_linear_program(lp, ET());
assert(s.solves_linear_program(lp));

// output solution
std::cout << s;
}
```

Output for

$R = \{(0,1), (1,2)\}$ and

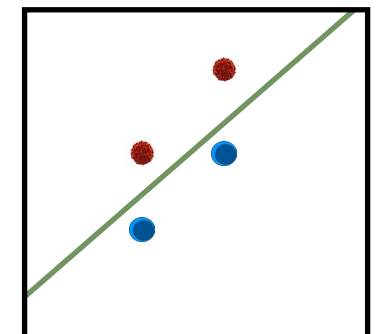
$B = \{(0,0), (1,1)\}$:

status:

INFEASIBLE

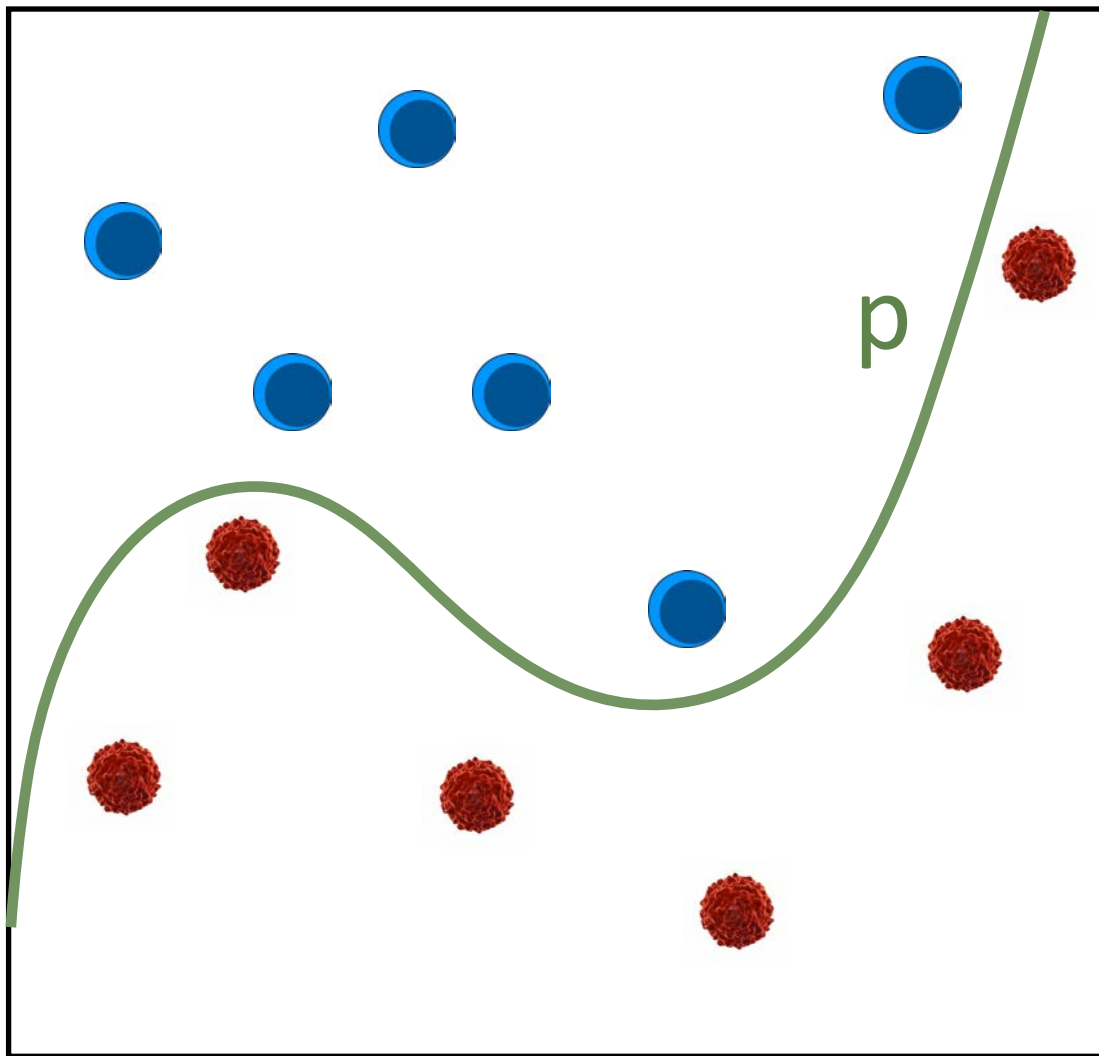
But R and B are linearly separable...

Fix: If the LP is infeasible, change the constraint $b \geq 1$ into $b \leq -1$ and re-solve.

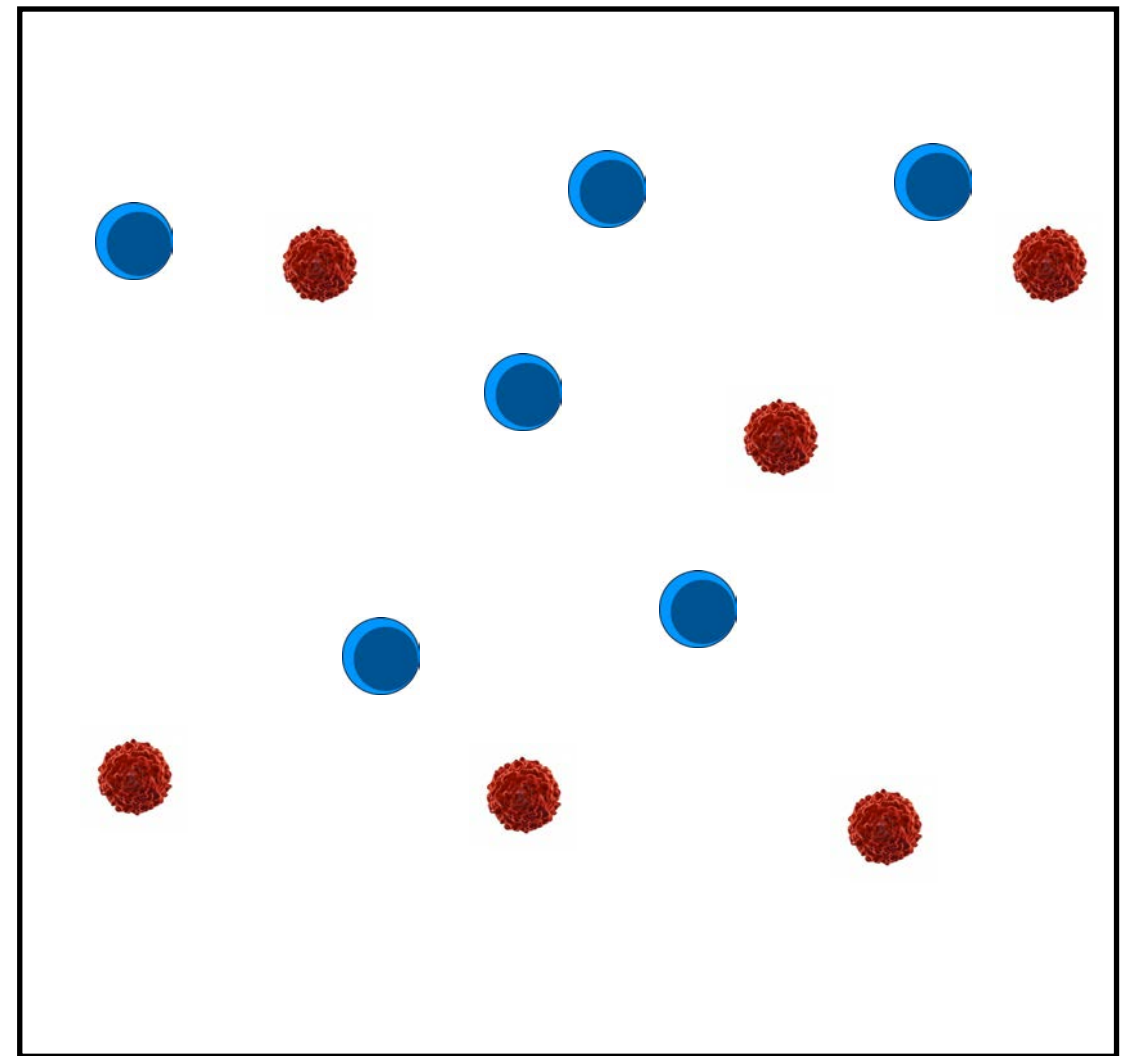


Ex. 2: Cubic Separation

- ❖ **Given:** A set R of red points and a set B of blue points.
- ❖ **Want:** A cubic polynomial p that separates R and B .



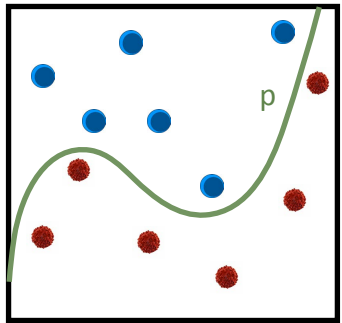
Separable



No solution 😞

Ex. 2: Cubic Separation

- ❖ **Given:** A set **R** of red points and a set **B** of blue points.
- ❖ **Want:** A cubic polynomial **p** that separates **R** and **B**.



- ❖ **Agenda/Checklist:**

- ❖ What are variables, what are constraints?
- ❖ What is n , what is m ? Is one of n or m small?
- ❖ Are all constraints linear in the variables?

$n=10 \Rightarrow$ small
 $m=|R|+|B|$

Yes!

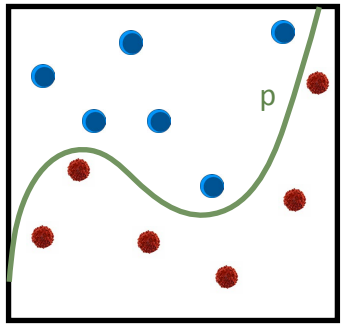
- ❖ **Variables/Output:** We want a cubic polynomial **p** :
 $\{(x,y) : a x^3 + b x^2 y + c x y^2 + d y^3 + e x^2 + f x y + g y^2 + h x + i y + j\}$,
for a, \dots, j unknown.

- ❖ **Constraints/Input:** **R** and **B** on different sides of **p**.

$$a p_x^3 + b p_x^2 p_y + c p_x p_y^2 + d p_y^3 + e p_x^2 + f p_x p_y + g p_y^2 + h p_x + i p_y \leq j, \quad \forall (p_x, p_y) \in R,$$
$$a q_x^3 + b q_x^2 q_y + c q_x q_y^2 + d q_y^3 + e q_x^2 + f q_x q_y + g q_y^2 + h q_x + i q_y \geq j, \quad \forall (q_x, q_y) \in B.$$

Ex. 2: Cubic Separation

- ❖ **Given:** A set **R** of red points and a set **B** of blue points.
- ❖ **Want:** A cubic polynomial **p** that separates **R** and **B**.



❖ Agenda/Checklist:

- ❖ What are variables, what are constraints?
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$n=10 \Rightarrow$ small
 $m=|R|+|B|$

➔ **Why?** There are powers here...

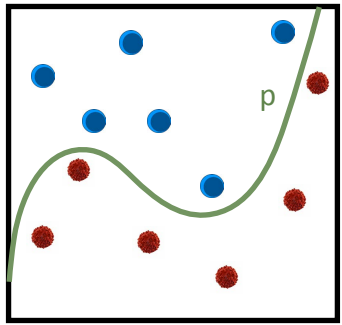
- ❖ **Variables:** $a, b, c, d, e, f, g, h, i, j$.
- ❖ **Constraints:** **R** and **B** on different sides of **p**.

$$a p_x^3 + b p_x^2 p_y + c p_x p_y^2 + d p_y^3 + e p_x^2 + f p_x p_y + g p_y^2 + h p_x + i p_y \leq j, \quad \forall (p_x, p_y) \in R,$$
$$a q_x^3 + b q_x^2 q_y + c q_x q_y^2 + d q_y^3 + e q_x^2 + f q_x q_y + g q_y^2 + h q_x + i q_y \geq j, \quad \forall (q_x, q_y) \in B.$$

- ❖ Constraints are linear in $a, b, c, d, e, f, g, h, i, j$.

Ex. 2: Cubic Separation

- ❖ **Given:** A set **R** of red points and a set **B** of blue points.
- ❖ **Want:** A cubic polynomial **p** that separates **R** and **B**.



- ❖ **Agenda/Checklist:**
- ❖ What are variables, what are constraints?
- ❖ What is n , what is m ? Is one of n or m small?
- ❖ Are all constraints linear in the variables?

→ **Why?** There are powers here...

- ❖ **Ex.** For $(2,3) \in R$ the constraint

$$a p_x^3 + b p_x^2 p_y + c p_x p_y^2 + d p_y^3 + e p_x^2 + f p_x p_y + g p_y^2 + h p_x + i p_y \leq j$$

reads

$$8a + 12b + 18c + 27d + 4e + 6f + 9g + 2h + 3i \leq j.$$

Careful with Indices

- ❖ The matrix A is adjusted dynamically to the entries and indices you provide.

```
// create an LP with  $Ax \leq b$ , lower bound 0 and no upper bounds  
Program lp (CGAL::SMALLER, true, 0, false, 0);  
  
const int X = 10000;  
lp.set_a(X, 0, 1);  
// now A has  $\geq 10000$  columns...
```

- ❖ Even if all these coefficients are zero, this affects efficiency (and space usage).

Nonnegative Solver

- ❖ There is a special solver for the case where all variables are nonnegative.

```
// solve the program, using ET as the exact type  
Solution s = CGAL::solve_nonnegative_linear_program(lp, ET());
```

- ❖ It may (or may not) be more efficient than the general solver on specific instances.
- ❖ **Attention!** The nonnegative solver ignores all other lower and upper bounds (specified with the constructor or with `set_l` or `set_u`).

Processing Solutions

- ❖ There are various functions to access the solution.

```
// solve the program, using ET as the exact type
Solution s = CGAL::solve_linear_program(lp, ET());
if (s.is_optimal()) { ... }
if (s.objective_value() < 0) { ... }
if (s.is_unbounded()) { ... }
if (s.is_infeasible()) { ... }
Solution::Variable_value_iterator opt = s.variable_values_begin();
// keep in mind that the values are of type Quotient<ET>
CGAL::Quotient<ET> res = *opt;
std::cout << res.numerator() << "/" << res.denominator() << "\n";
```

- ❖ In a maximization problem you probably want to invert the objective value...

How to avoid cycling

- ❖ The solver is driven by a pricing strategy.
- ❖ By default this strategy is deterministic and **may cycle** => run into an infinite loop.
- ❖ To avoid cycling, use Bland's Rule (see code below).

```
CGAL::Quadratic_program_options options;  
options.set_pricing_strategy(CGAL::QP_BLAND);  
Solution s = CGAL::solve_linear_program(lp, ET(), options);
```

- ❖ Bland's Rule is slower => only use where needed.
- ❖ Usually you will notice on the public test sets.

Known Bug :=(

- ❖ Do not copy or assign objects of type **CGAL::Quadratic_program_solution<ET>!**
- ❖ **Workaround 1:** If you want to pass or return such an instance to/from a function, pass a pointer to the instance instead!
- ❖ **Workaround 2:** Just don't do it!