

063-0605-00L : Computational Structural Design 1

# Computational Graphic Statics

Dr. Juney Lee & Dr. Lluis Enrique

Autumn Semester 2021

**ETH** zürich  DARCH

**BRG**

## Week 9

Friday, November 19th

Hour 1

**09:45 – 09:50** · Recap of Exercise 4

**09:50 – 10:30** · [Lecture 4 : Thrust Network Analysis](#)

**10:30 – 10:45** · Break

Hour 2

**10:45 – 11:35** · [Tutorial 4 : RhinoVAULT 2 \(RV2\) – Part 1](#)

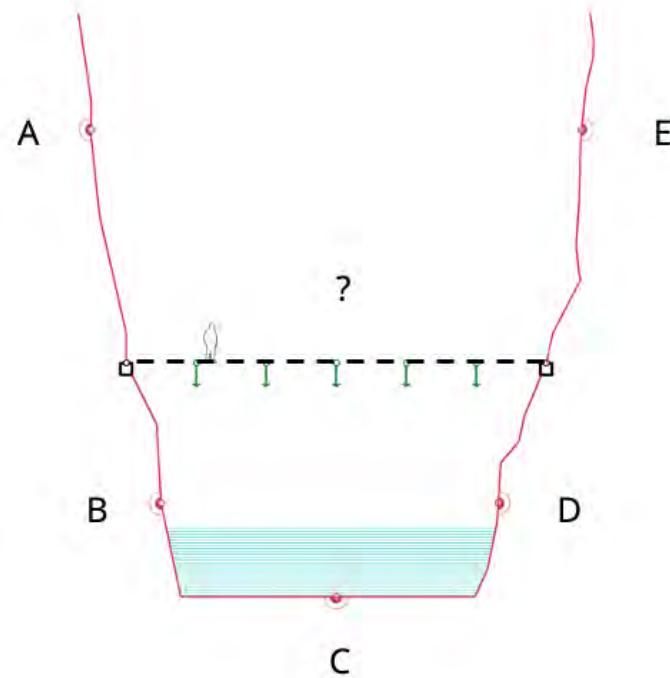
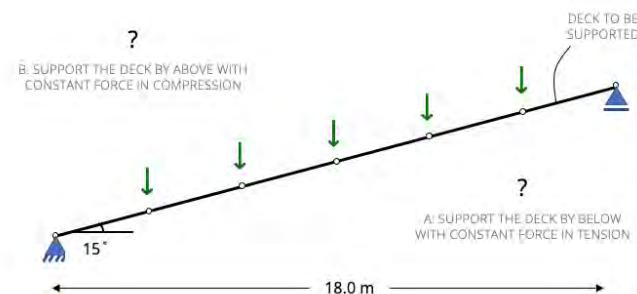
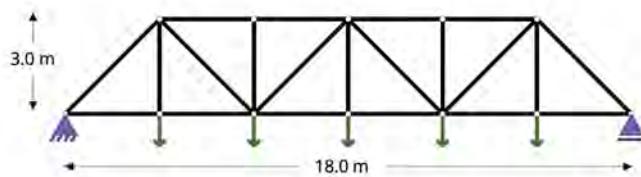
**11:30 – 11:45** · Break

Hour 3

**11:45 – 12:30** · [Tutorial 4 : RhinoVAULT 2 \(RV2\) – Part 2](#)

## Recap of exercise 4

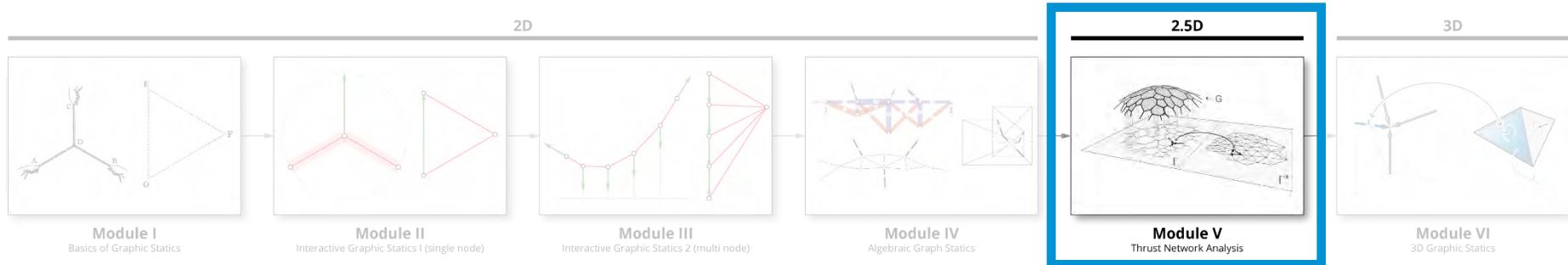
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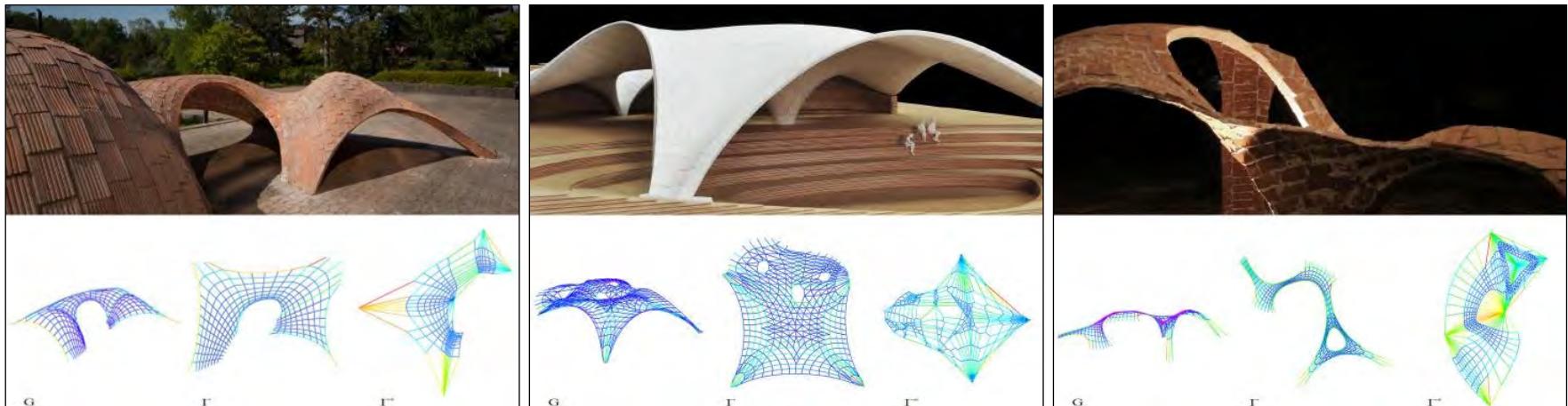
## Course schedule

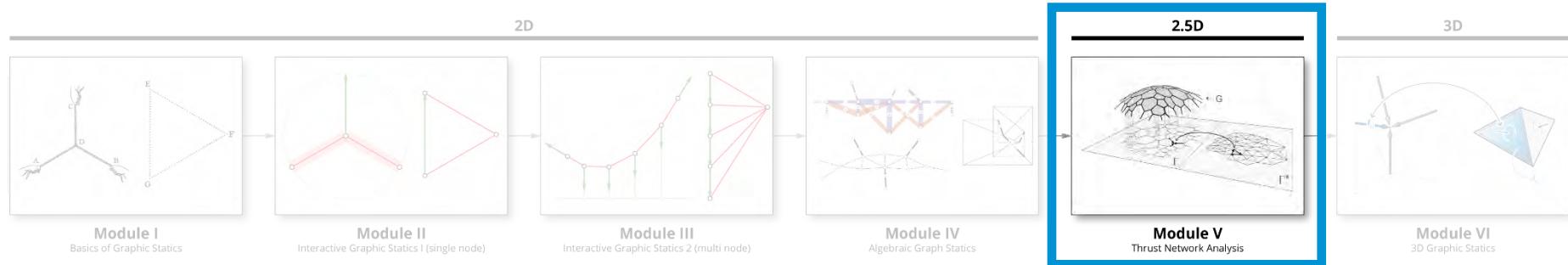
4

| Session Type/Week   | Topic              | Week            | Day | Time   | Lecturer                                 |
|---|--------------------|-----------------|-----|--|--|
| 1. planar structures<br><br>cables & arches<br><br>2D GS (procedural) | Introduction       | 1               |     | Lecture 1 <b>Introduction to graphic statics</b>   | Dr. Juney Lee                            |
|   |                    | Week 1 (9/24)   |     | 2 Tutorial 1 Basics of graphic statics on "blackboard"<br>Introduction to Rhino + Grasshopper (GH) | Dr. Lluís Enrique<br>Lotte Aldinger      |
|   |                    | Week 2 (10/1)   |     | 3 Work session 1 <b>Exercise 1: Reciprocal diagrams in Rhino + GH basics</b>                       | Instructors                              |
|   |                    |                 |     | Quick recap of Exercise 1  | Lotte Aldinger                           |
|   |                    | Week 2 (10/1)   | 1   | Lecture 2 <b>Algorithmic design &amp; thinking</b>   | Dr. Juney Lee                            |
|   |                    |                 | 2   | Tutorial 2 Single node bridge in GH (form to force)  | Lotte Aldinger                           |
|   |                    | Week 3 (10/8)   | 1   | Work session 2 <b>Exercise 2: Single node bridge with GH</b>                                       | Lotte Aldinger + Instructors             |
|   |                    |                 | 2   | Quick recap of Exercise 2  | Lotte Aldinger                           |
|   |                    | Week 4 (10/15)  | 1   | Lecture 3 <b>Computational graphic statics</b>   | Dr. Juney Lee                            |
|   |                    |                 | 2   | Tutorial 3 Multi-node bridge in Grasshopper  | Dr. Lluís Enrique                        |
|   |                    | Week 5 (10/22)  | 1   | Work session 3 <b>Exercise 3: Multi-node bridge with GH</b>  | Dr. Lluís Enrique + Instructors          |
| seminar week  |                    |                 |     |  |  |
| trusses   | 2D GS (AGS)        | Week 6          |     | Quick recap of Exercise 3  | Dr. Lluís Enrique                        |
|   |                    | Week 7 (11/5)   | 1   | Lecture 4 <b>Algebraic graph statics (AGS)</b>   | Dr. Juney Lee                            |
|   |                    |                 | 2   | compas_aggs + IGS (Interactive Graphic Statics) plugin   | Ricardo Avelino                          |
|   |                    | Week 8 (11/12)  | 1   | Work session 4 <b>Exercise 4: Truss problems with IGS</b>  | Ricardo Avelino + Instructors            |
| 2. surface structures<br><br>shells                                   | 2.5 GS (TNA)       | Week 9 (11/19)  |     | Quick recap of Exercise 4  | Ricardo Avelino                          |
|   |                    |                 | 1   | Lecture 5 <b>Thrust Network Analysis (TNA)</b>   | Dr. Juney Lee                            |
|   |                    |                 | 2   | compas_tna + RV2 (RhinoVIAU.T 2) plugin  | Dr. Juney Lee                            |
|   |                    | Week 10 (11/26) | 1   | Work session 5   |  |
|   |                    |                 | 2   | Quick recap of Exercise 5  | Dr. Juney Lee                            |
| 3. spatial structures<br><br>polyhedral structures                    | 3D GS (polyhedral) | Week 11 (12/3)  | 1   | Lecture 6 <b>3D graphic statics (3GS)</b>  | Dr. Juney Lee                            |
|   |                    |                 | 2   | compas_3gs + 3GS (3D graphic statics) plugin   | Dr. Juney Lee                            |
|   |                    | Week 12 (12/10) | 1   | Work session 6 <b>Exercise 6: Spatial structures with 3GS</b>                                      |  |
|   |                    |                 | 2   | Quick recap of Exercise 6  | Dr. Juney Lee                            |
|   |                    |                 | 1   | Pt I. Outlook on computational graphic statics   | Dr. Juney Lee                            |
|   |                    |                 | 2   | Pt II. Guest Lecture   | TBD                                      |
|   |                    |                 | 3   | Q/A  | Discussions, feedback, evaluations, etc. |



## Module V · Thrust Network Analysis & RhinoVAULT





## Module V · Thrust Network Analysis & RhinoVAULT



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Computational Graphic Statics

# Lecture 5

# Thrust Network Analysis

Friday, November 19th, 2021

Dr. Juney Lee

# **Form finding**



"As hangs the flexible line, so but inverted will stand the rigid arch."

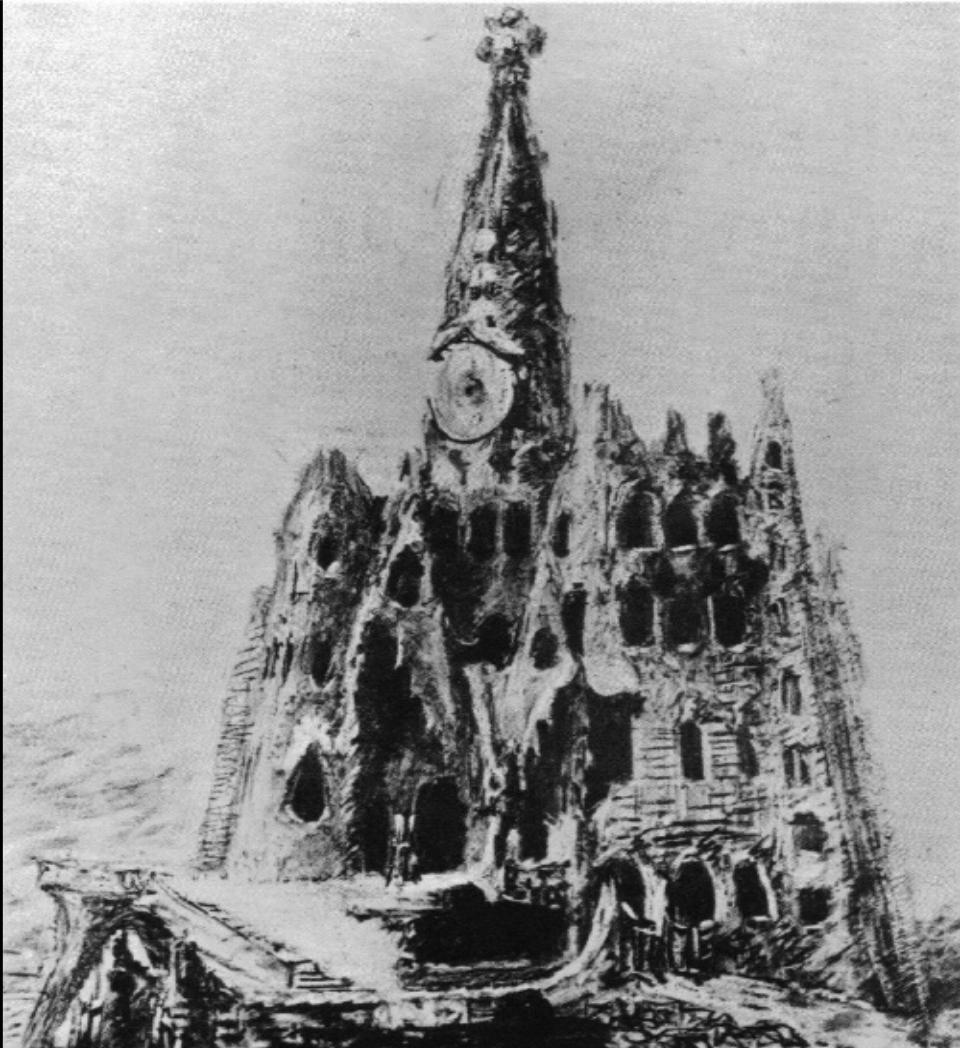
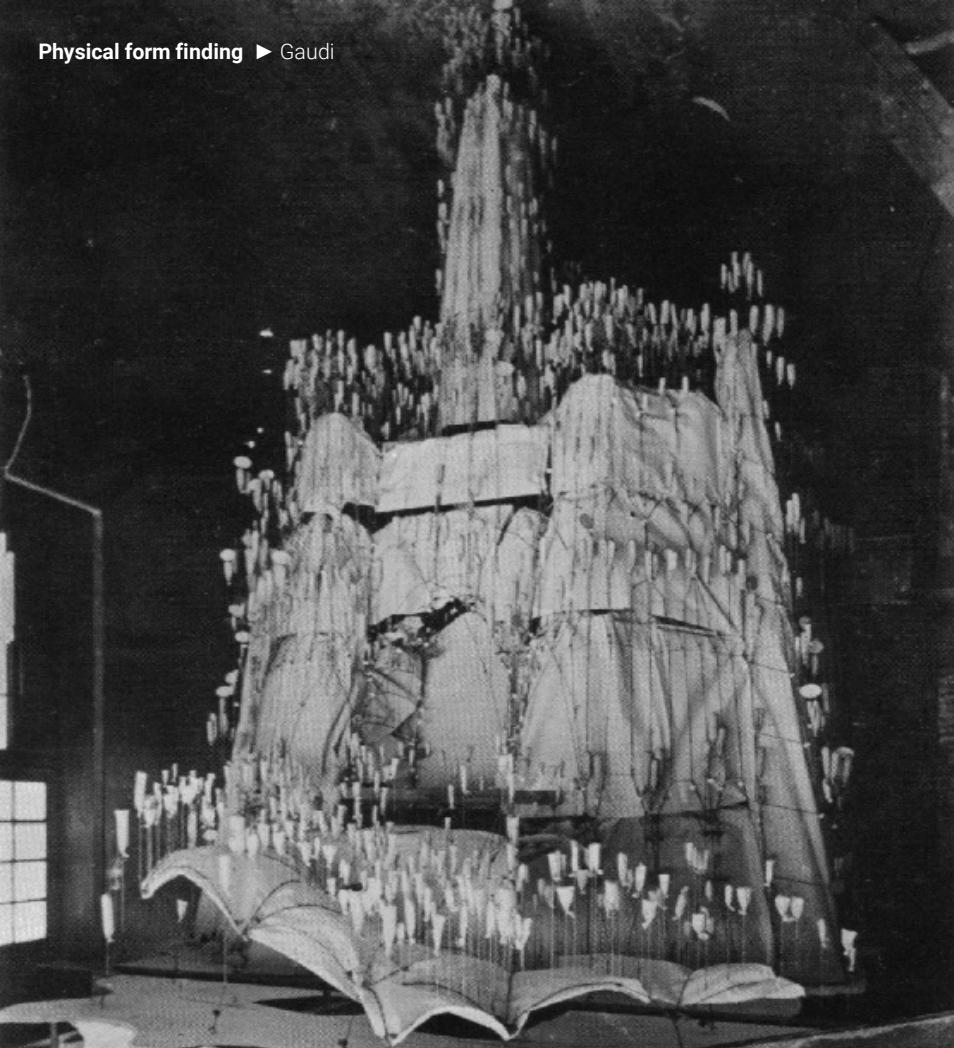
Hooke (1676)





Redesign of Gaudi's hanging model for the Colonia Güell | Images: Beotis (2007)

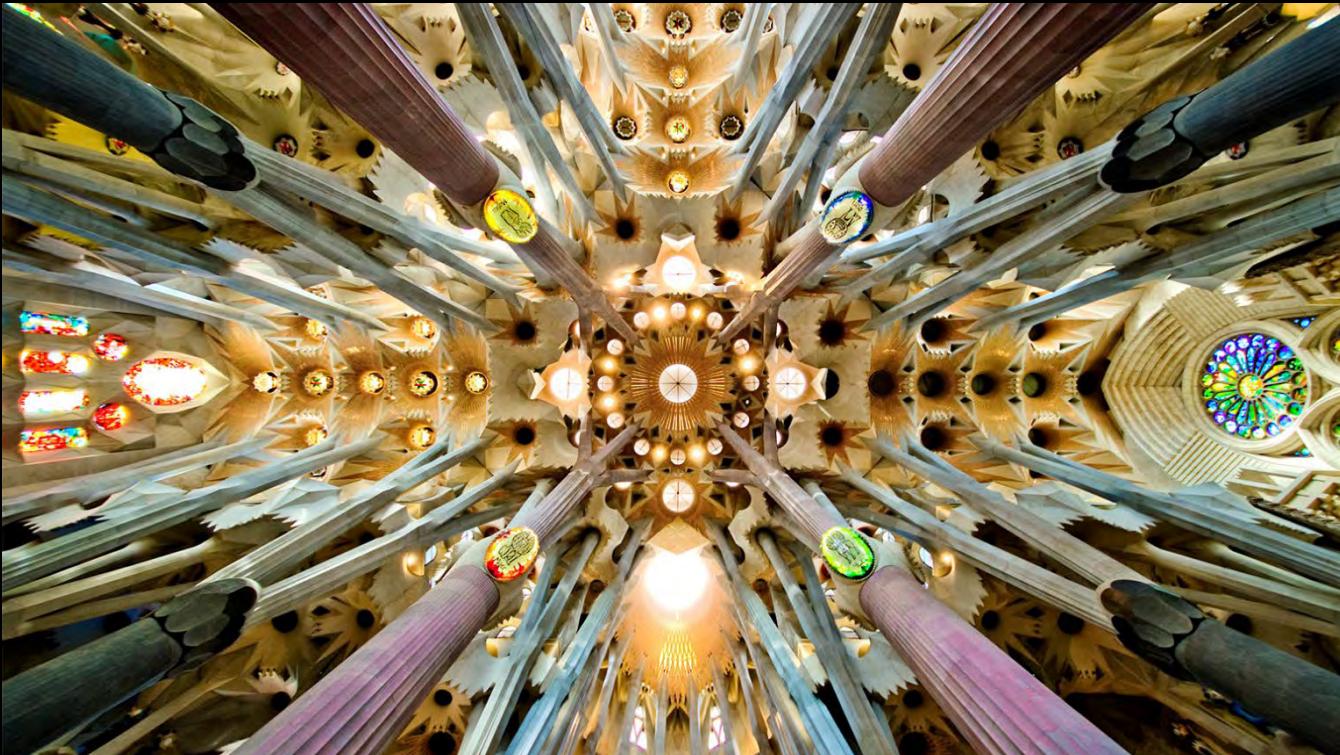
Physical form finding ▶ Gaudi



Gaudí's design of its exterior directly sketched on the inverted photograph of the hanging model | Images: Collins, 1963



Gaudí's Church of Colònia Güell, 1898





Physical Form Finding Models. | Isler Archive



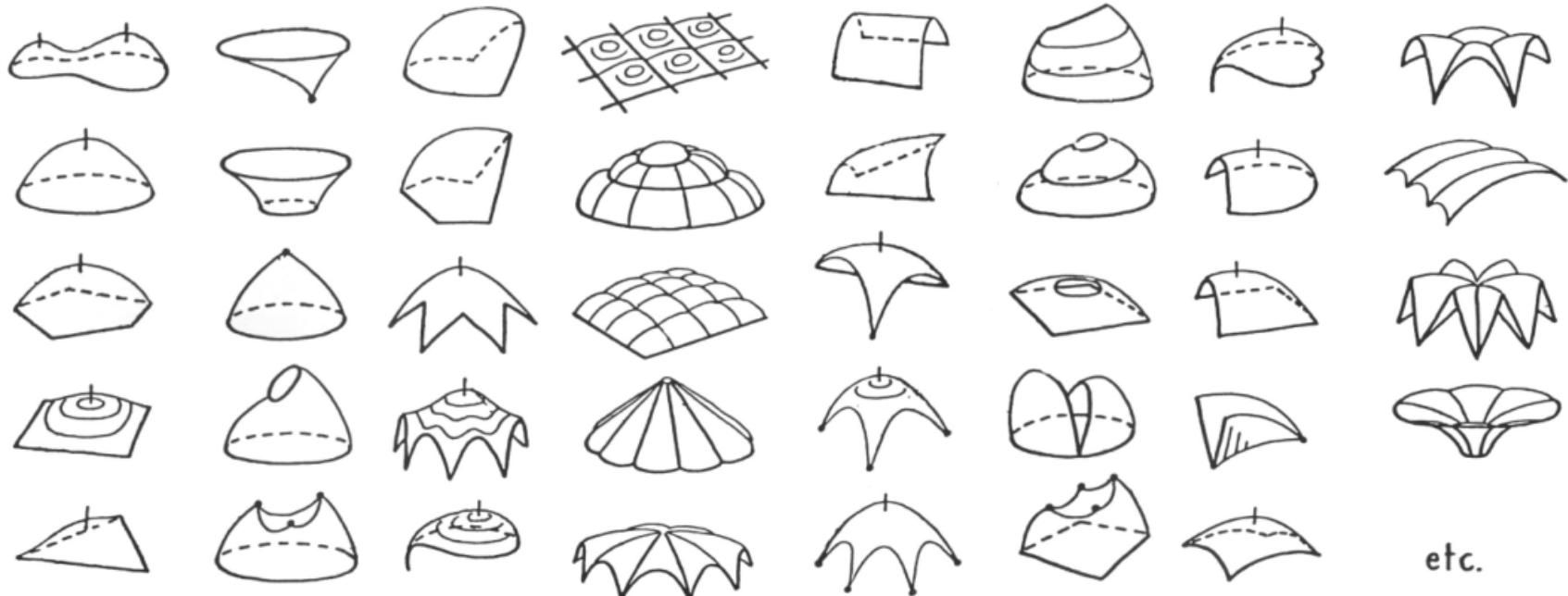
Tankstelle Deitingen Süd, Schweiz, 1968, Ing. Heinz Isler



Tankstelle Deitingen Süd, Schweiz, 1968, Ing.: Heinz Isler



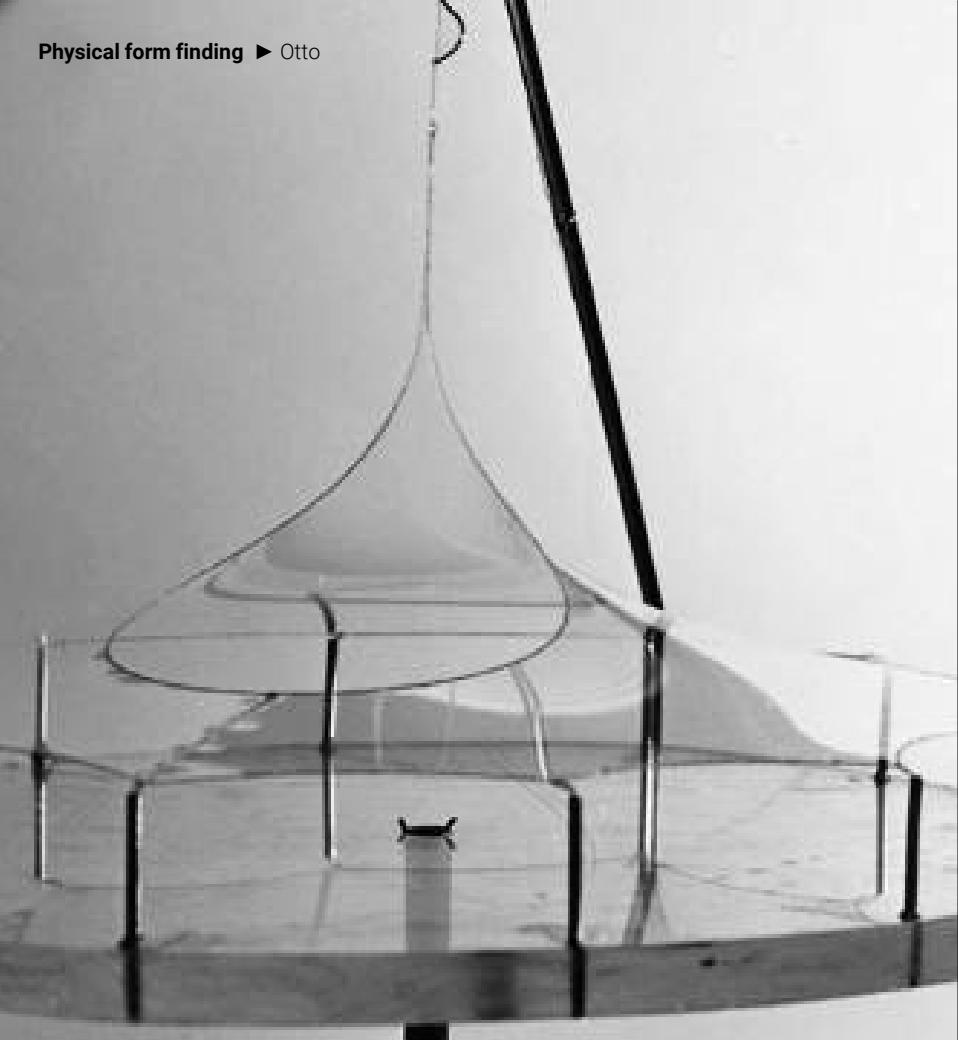
Tankstelle Deitingen Süd, Schweiz, 1968, Ing.: Heinz Isler



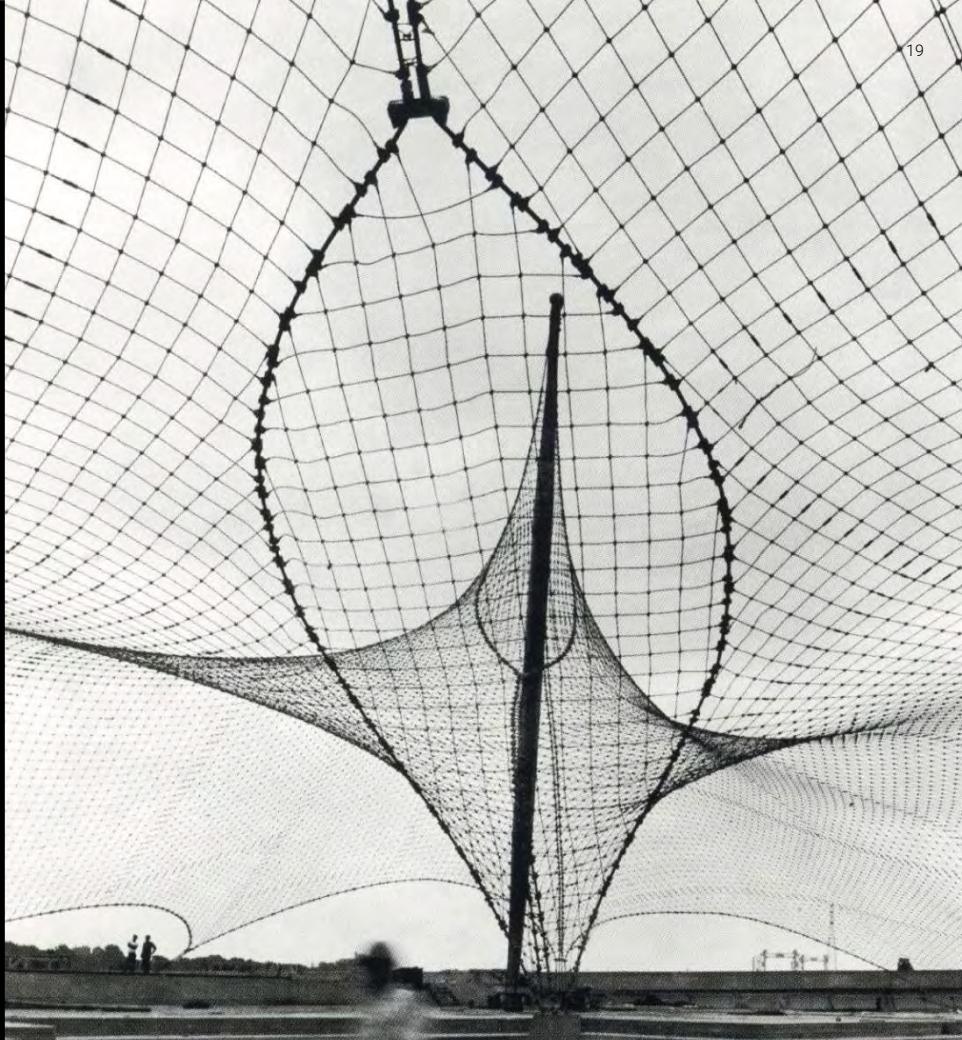
etc.

Physical form finding ► Otto

19



Soap bubble experiments | Frei Otto



German Pavilion at Expo 1967 | Frei Otto

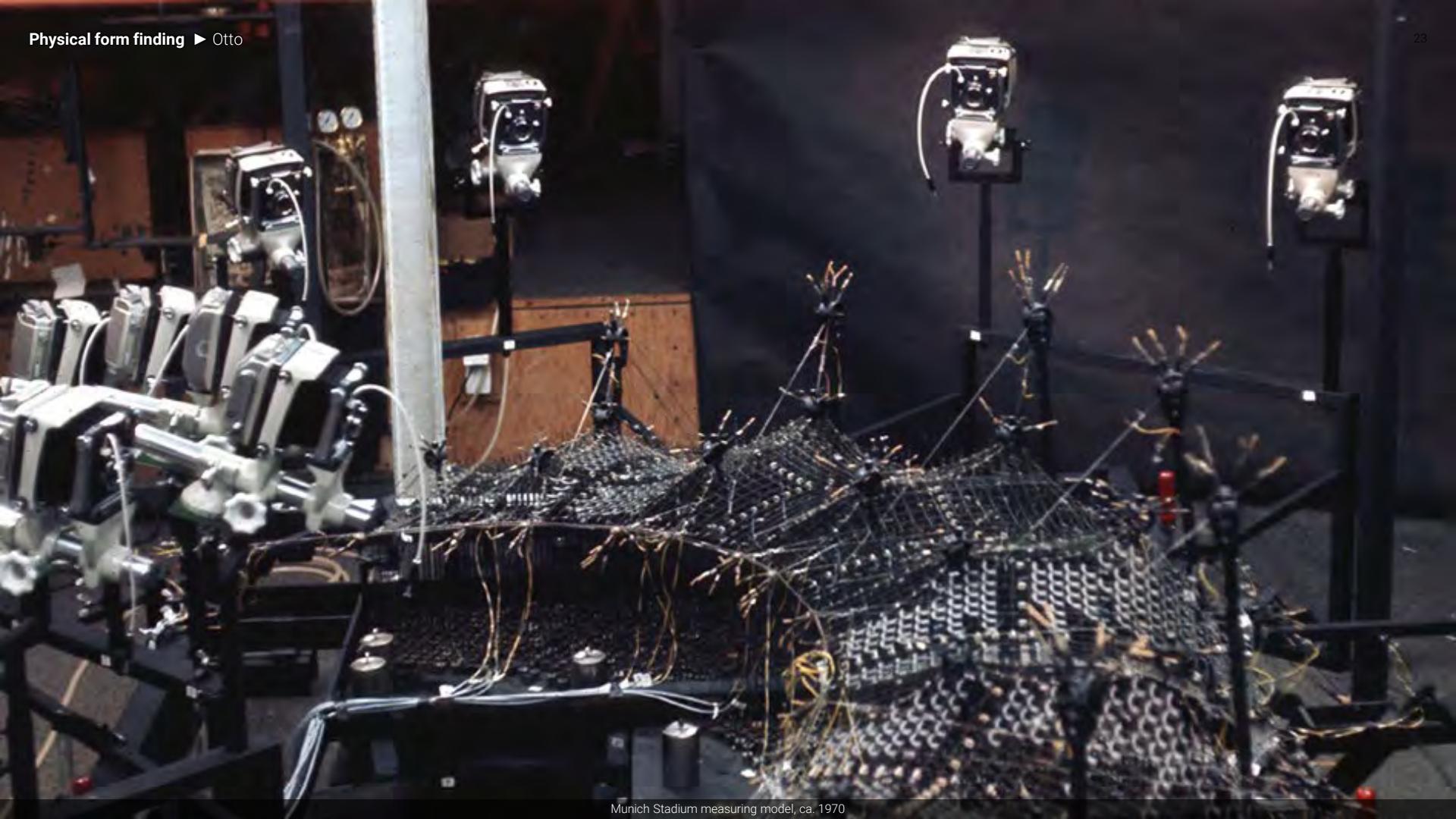
The Institute for Lightweight Structures (ILS)  
University of Stuttgart

Director; Frei Otto





Montreal Pavilion measuring model, 1966



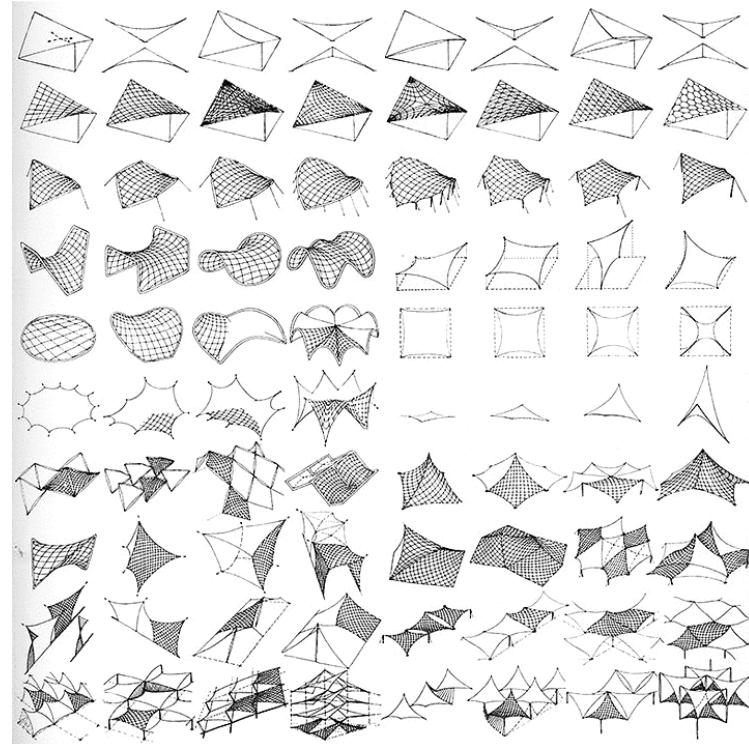
Munich Stadium measuring model, ca. 1970

# Computational **form finding**

## Form finding

“... forward process in which parameters are explicitly/directly controlled to find an ‘optimal’ geometry of a structure which is in static equilibrium with a design loading...”

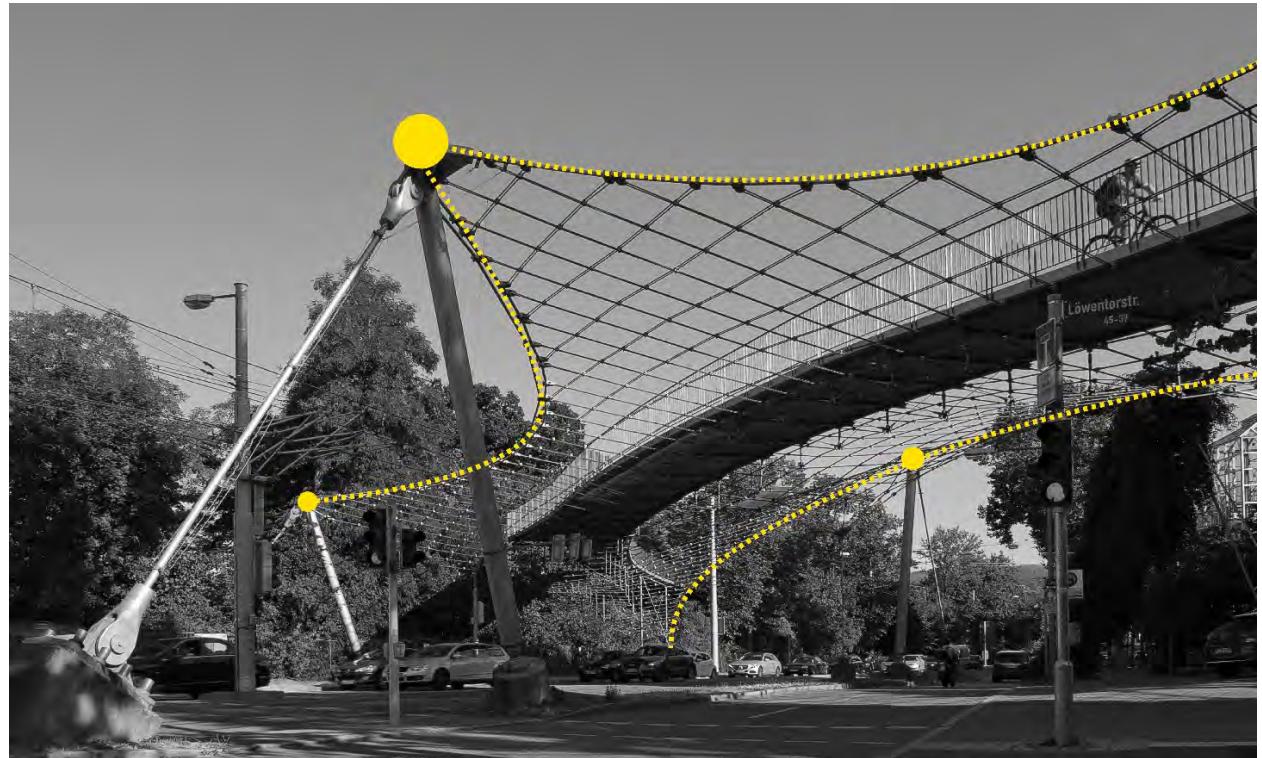
Adriaenssens et al. (2014)



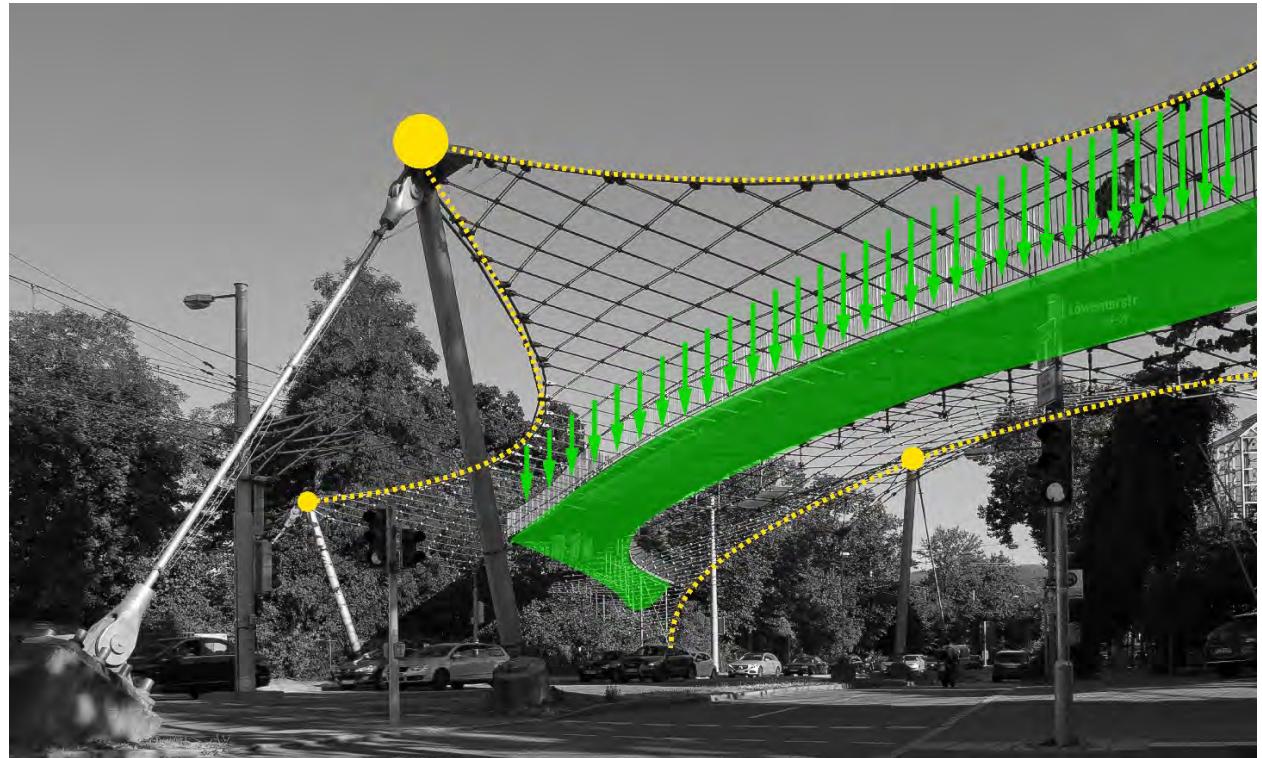


Schlaich Bergermann Partner, Lodzer Steg Stuttgart, 1992

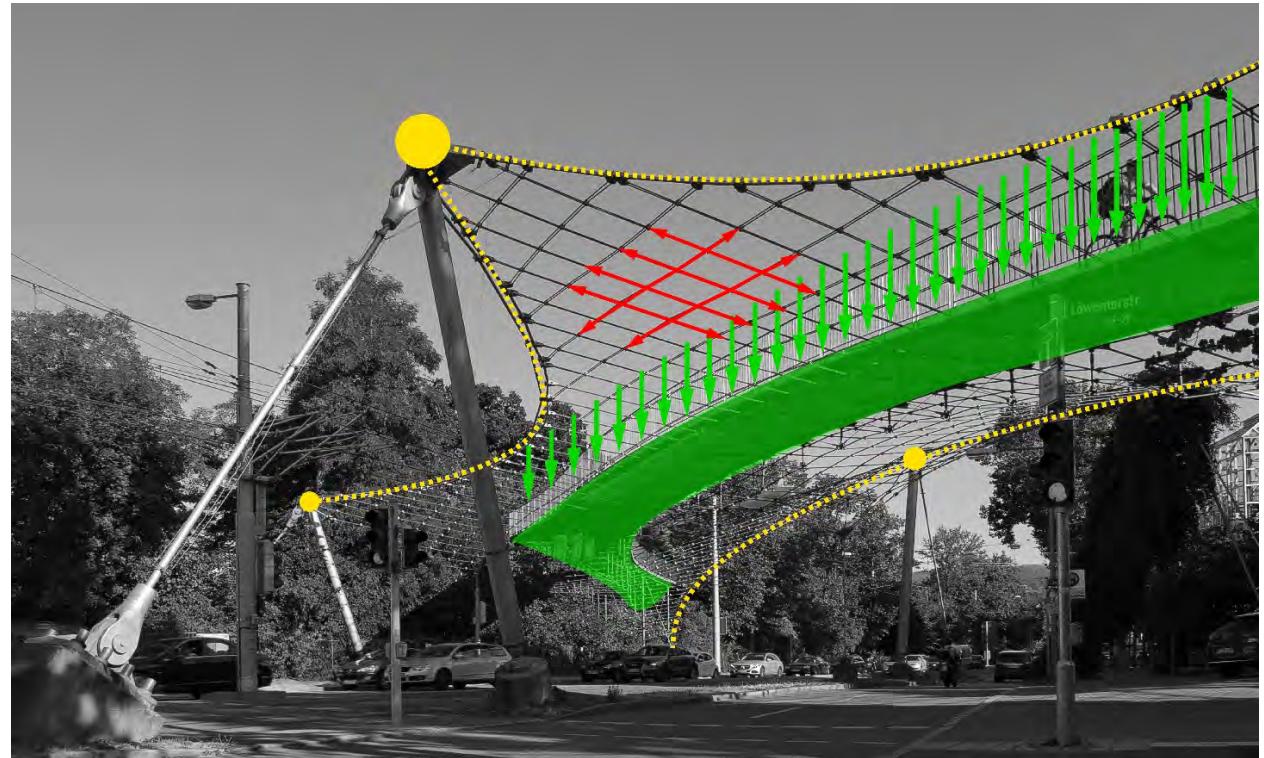
### 1. Defined boundary conditions



1. Defined boundary conditions
2. Defined loading case



1. Defined boundary conditions
2. Defined loading case
3. Defined state of self stress



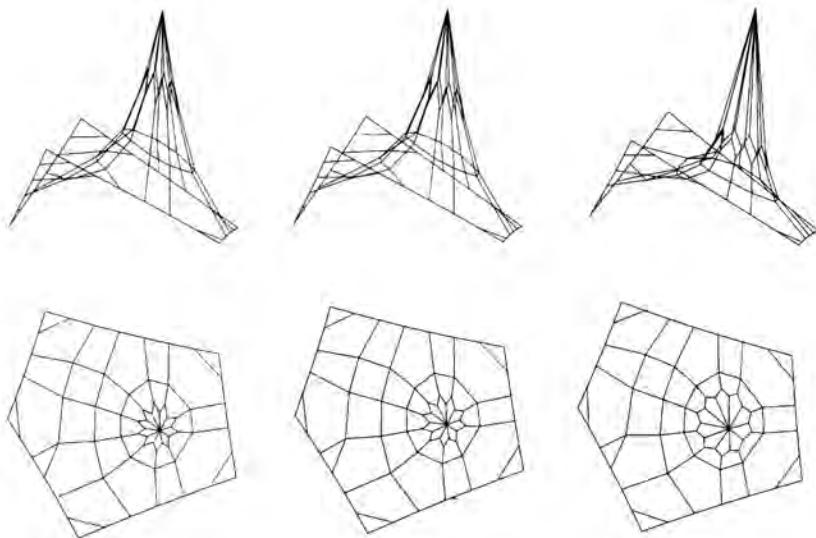


Fig. 7. Global changes in the force densities and lengthening of the radial branches.

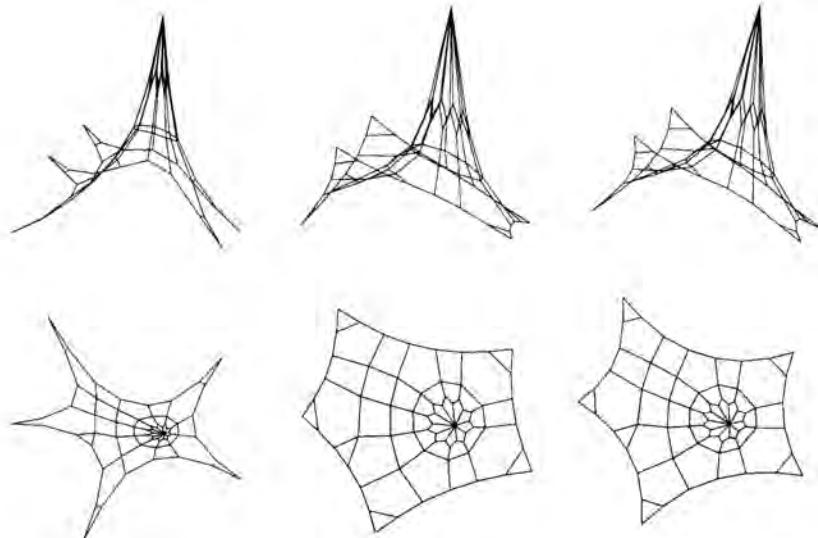


Fig. 8. Global changes in the force densities of the main cables.

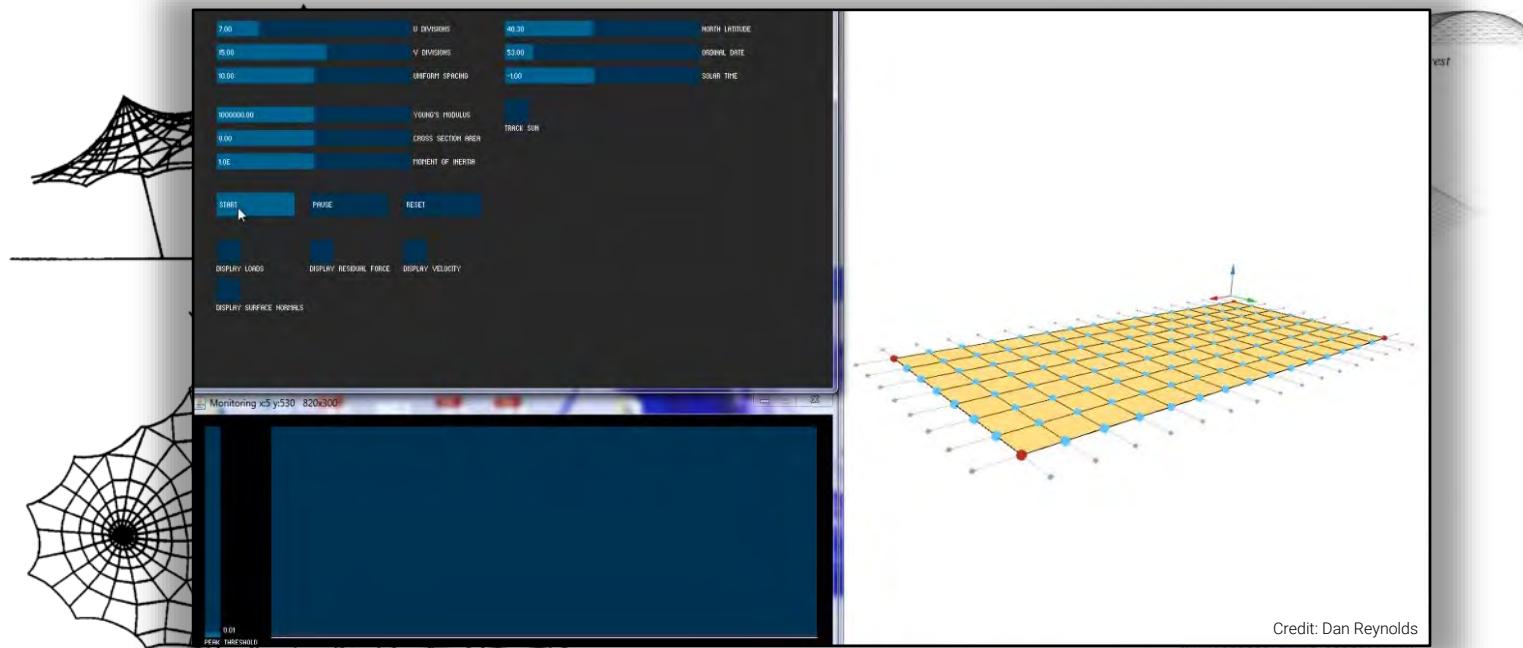


Figure 13. Outwards deflections due to loading

# Form finding with graphic statics

JOURNAL OF THE INTERNATIONAL ASSOCIATION FOR SHELL AND SPATIAL STRUCTURES J.IASS

## THRUST NETWORK ANALYSIS: A NEW METHODOLOGY FOR THREE-DIMENSIONAL EQUILIBRIUM

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**Editor's Note:** The first author of this paper is one of the four winners of the 2007 Hanga Prize, awarded for outstanding papers that are submitted for presentation and publication at the annual IASS Symposium by younger members of the Association (under 30 years old). It is re-published here with permission of the editors of the proceedings of the IASS 2007 Symposium: Structural Architecture: Toward the Future Looking to the Past, held in December 2007 in Venice, Italy.

### ABSTRACT

This paper presents a new methodology for generating compression-only vaulted surfaces and networks. The method finds possible funicular solutions under gravitational loading within a defined envelope. Using projective geometry, duality theory and linear optimization, it provides a graphical and intuitive method, adapted to both 2-D and 3-D analysis, that can be extended to fully three-dimensional problems. The proposed method is applicable for the analysis of vaulted historical structures, specifically in unreinforced masonry, as well as the design of new vaulted structures. This paper introduces the method and shows examples of applications in both fields.

**Keywords:** compression-only structures, unreinforced masonry vaults, funicular analysis, Thrust Network Analysis, reciprocal diagrams, form-finding, lower-bound analysis, structural optimization

### 1. INTRODUCTION

Medieval vault builders created complex forms carefully balanced in compression. The structural properties of these sophisticated forms are still poorly understood because of a lack of appropriate analysis methods, i.e. methods relating stability and form. Understanding the mechanics of these vaulted structures leads to new insights for both analysis and design.

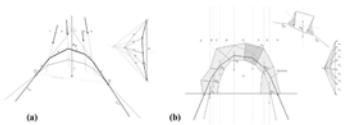


Figure 1. (a) Two possible compression-only equilibrium shapes for a random set of loads, and (b) an interactive thrust-line classification developed in [1]; the user can adapt the geometry by dragging control points and the structural feedback, in the form of a thrust-line, is updated in real-time. The magnitudes of the forces in the system are visualized in the accompanying funicular polygon (right).

1

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based on elastic solution can give one possible answer, they no longer suggest better form as was inherent to the more holistic graphical methods.

There is a real need for tools to better understand and visualize the stability of compression-only structures, such as historic unreinforced masonry structures, as well as design tools that can generate form. These problems are related to finding axial force structures in equilibrium acting only in compression or tension. Currently, graphic statics provides a holistic analysis and design tool for two-dimensional vaults [2]. With today's availability of powerful linear 3-D optimization software, the following question arises: can a fully three-dimensional version of thrust-line analysis provide the same freedom to explore the infinite equilibrium solutions for a certain loading condition?

### 2. METHODOLOGY

The *Thrust-Network Analysis* method presented in this paper is inspired by O'Dwyer's work on funicular analysis of vaulted masonry structures [2]. It is extended by adding the concept of duality between geometry and the in-plane internal forces of networks [3].

#### 2.1. Reciprocal figures

The proposed method produces funicular (compression-only) solutions for loading conditions where all loads are applied in the same direction, as is the case for a vault under its own weight. The solutions are compression-only that also means that the vaults can never curl back onto themselves, which would demand some elements to go into tension. The resulting three-dimensional networks can represent load paths throughout a structure. There is no constraint on the length of the branches

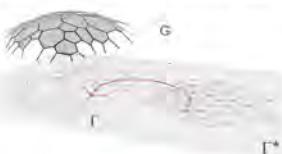


Figure 2. The primal grid  $\Gamma$  and dual grid  $\Gamma^*$  are related by a reciprocal relationship. The equilibrium of a node in one of them is guaranteed by a closed polygon in the other and vice versa. The labeling uses Routh's notation [6].

2

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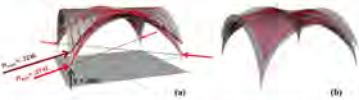


Figure 3. (a) Possible thrust values for this given vault range from 21% to 12% of its total weight. (b) L3 and rth action with the forces mostly spanning between the ribs as represented in the dual grid (c).

of lower-bound analysis. Put simply, if a compression-only network can be found that fits within the boundaries of a vault, then the vault will stand in compression. Note that if we want a solution without any tension (hinges) in the section, we want the solution to lie within the middle third of the section (defined by equation 6). This is a powerful concept for understanding the stability and proximity to collapse of such structures. Additional reading on this topic can be found in Heyman [10], O'Dwyer [2], Boothby [11] and Block et al. [1].

most useful characterization of the structural behavior of the vault. The minimum (or passive) thrust value is the least amount of thrust that will push sideways onto its neighboring elements, as a function of its self-weight and shape. The maximum (or active) state of thrust on the other hand represents the largest horizontal force this vault can provide. So, this value demonstrates how much horizontal force this vault can safely take from its neighboring elements.

Figure 7b shows a solution with three-dimensional web action. The distribution of the horizontal thrust along the perimeter regions in the dual grid (Fig. 7c) More detailed application of this methodology to masonry vaults can be found in [12].

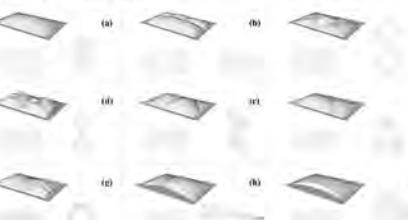
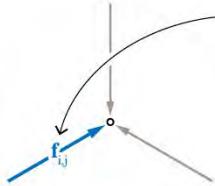


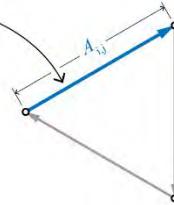
Figure 4. A series of snapshots, starting from a regular rectangular grid, showing the relationship between the primal grid and dual grid and the corresponding surfaces.

3



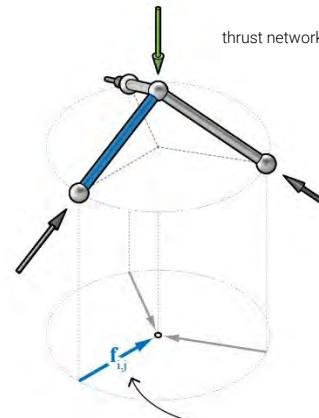


form diagram

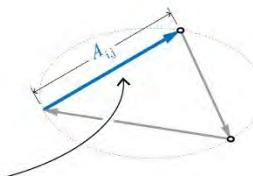


force diagram

2 D

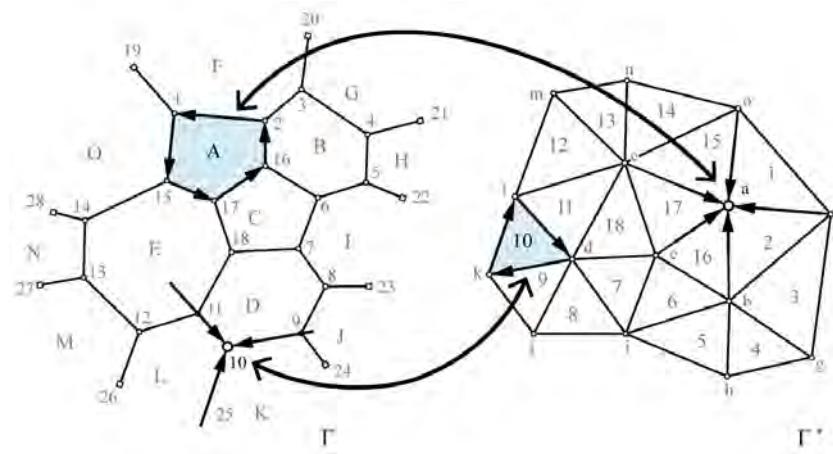


form diagram



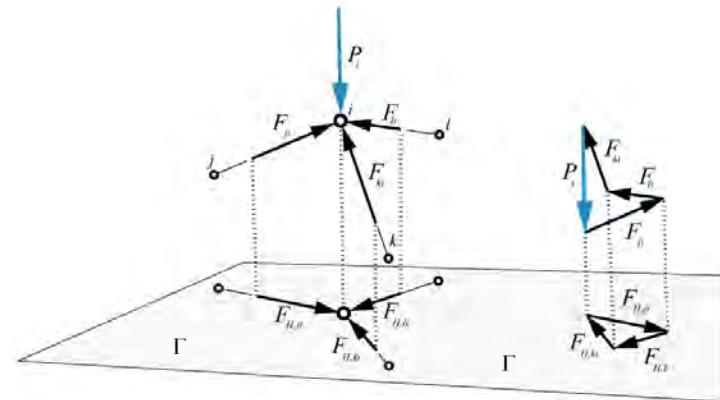
force diagram

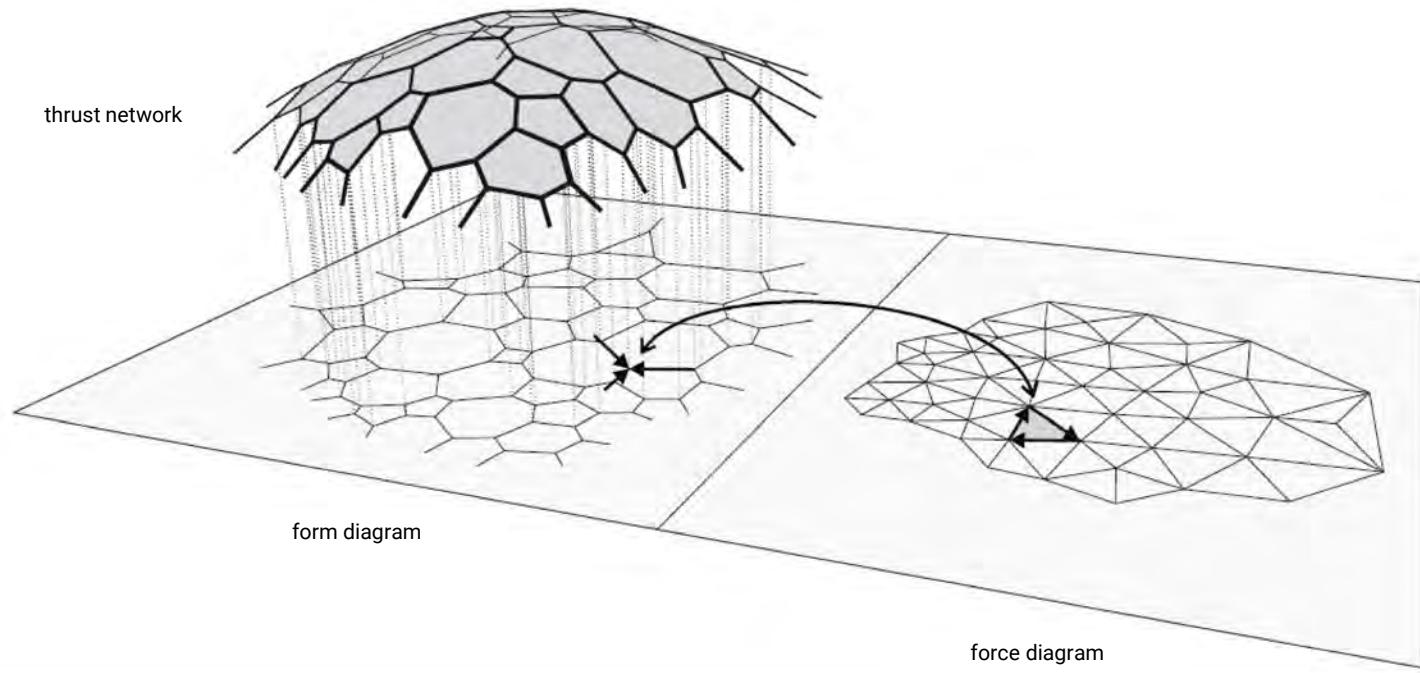
2.5 D

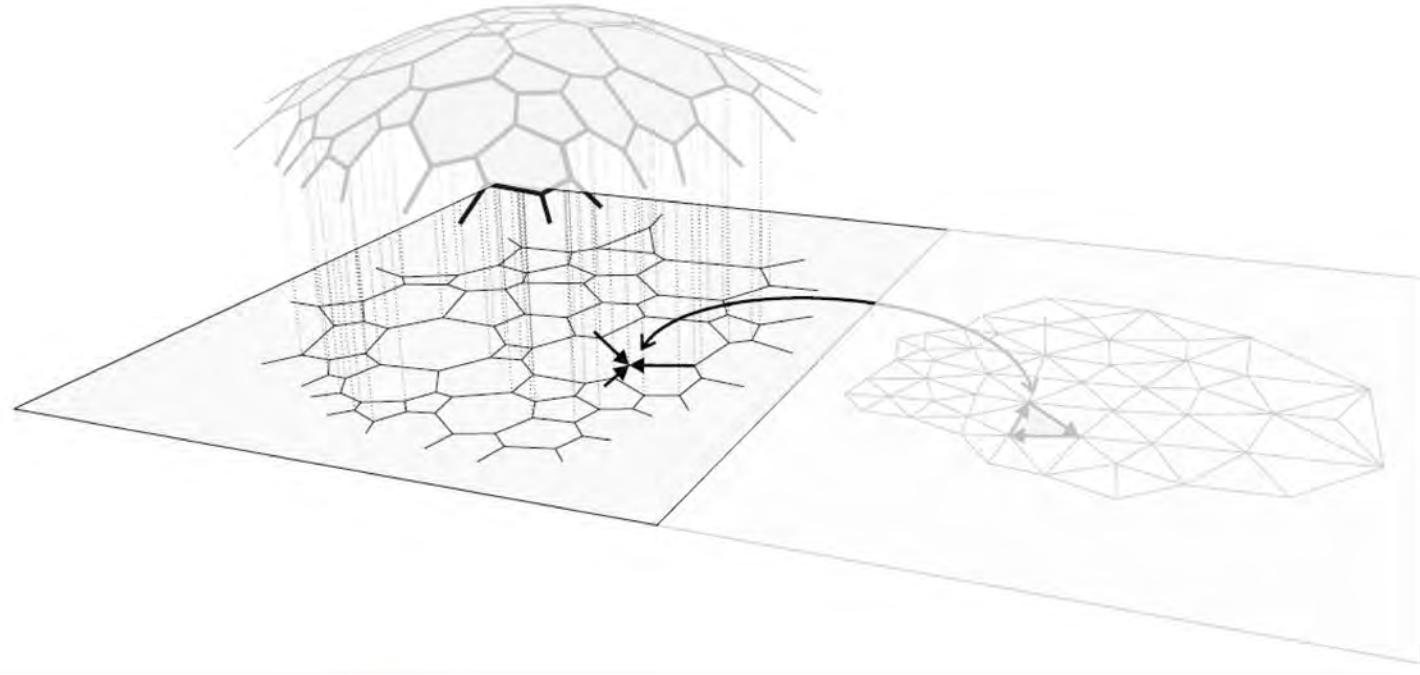


Form Diagram

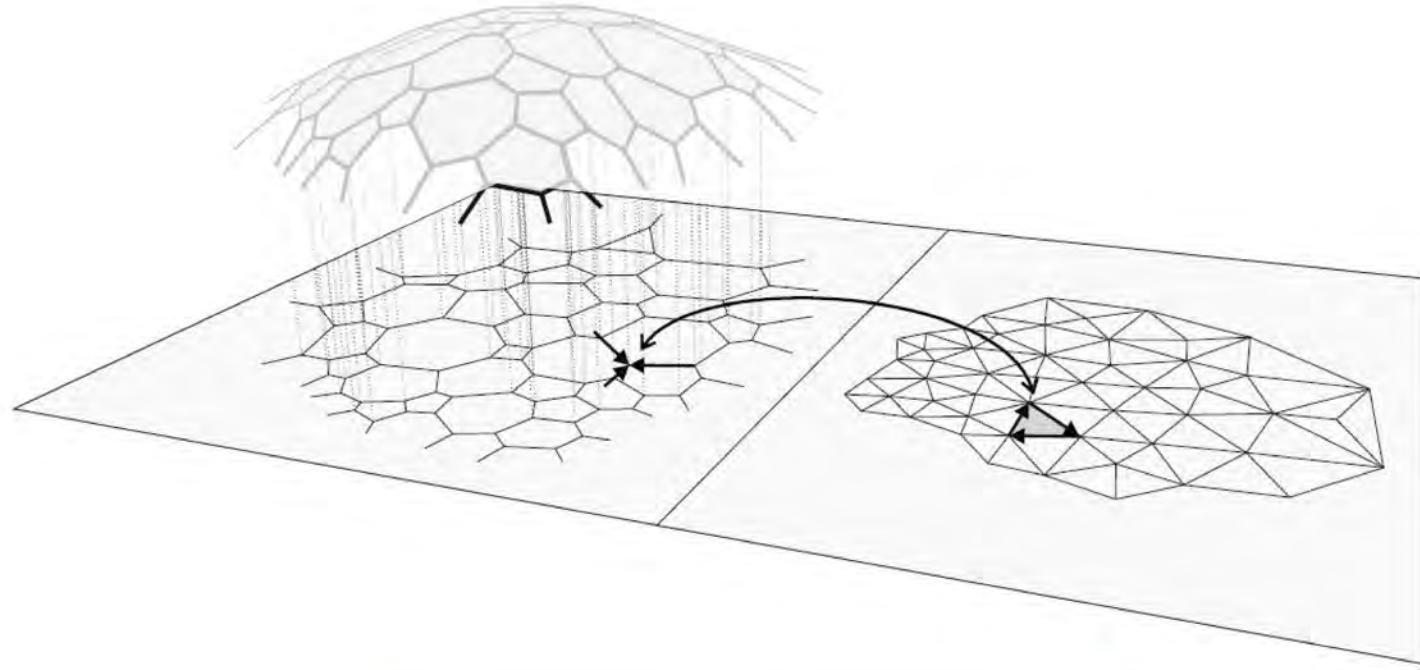
Force Diagram

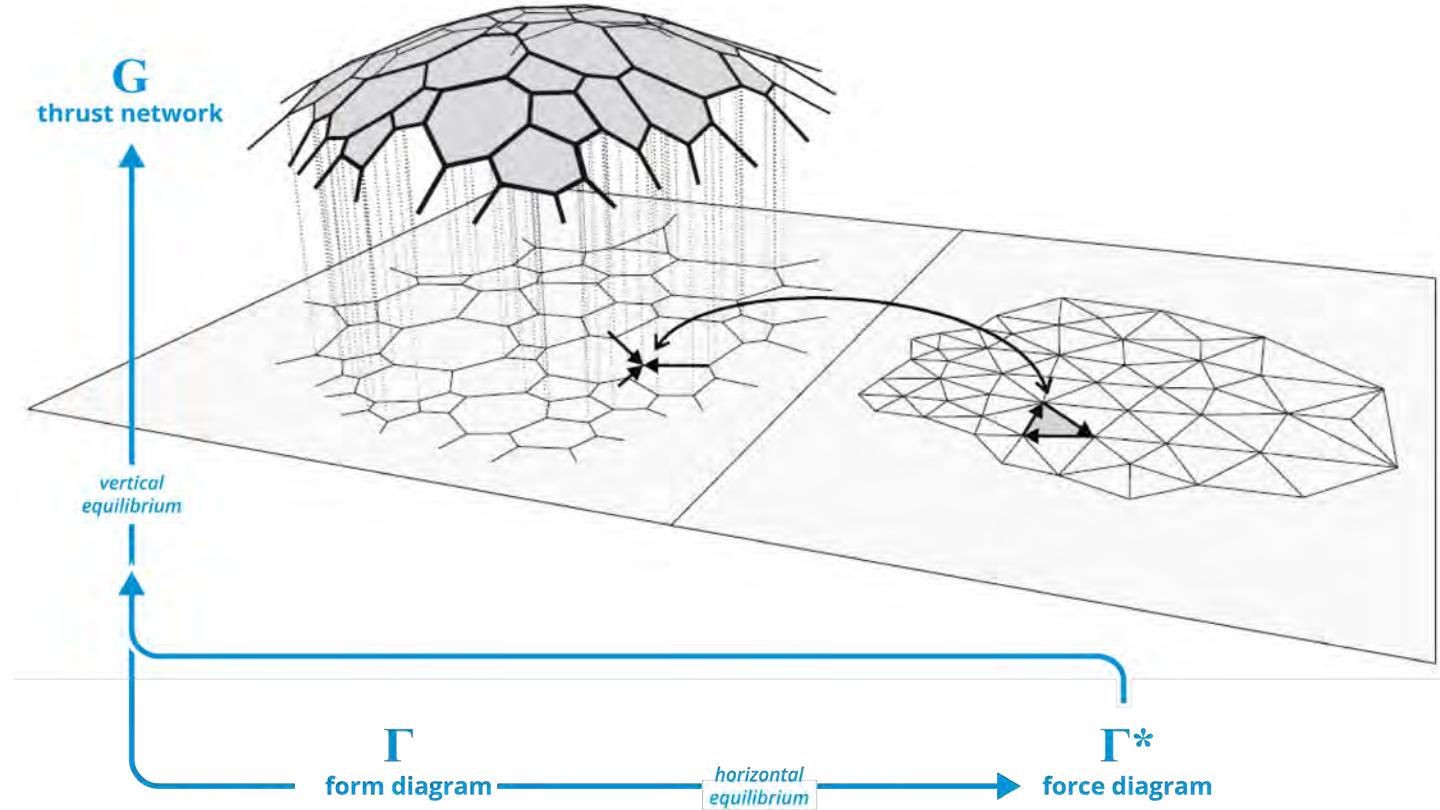


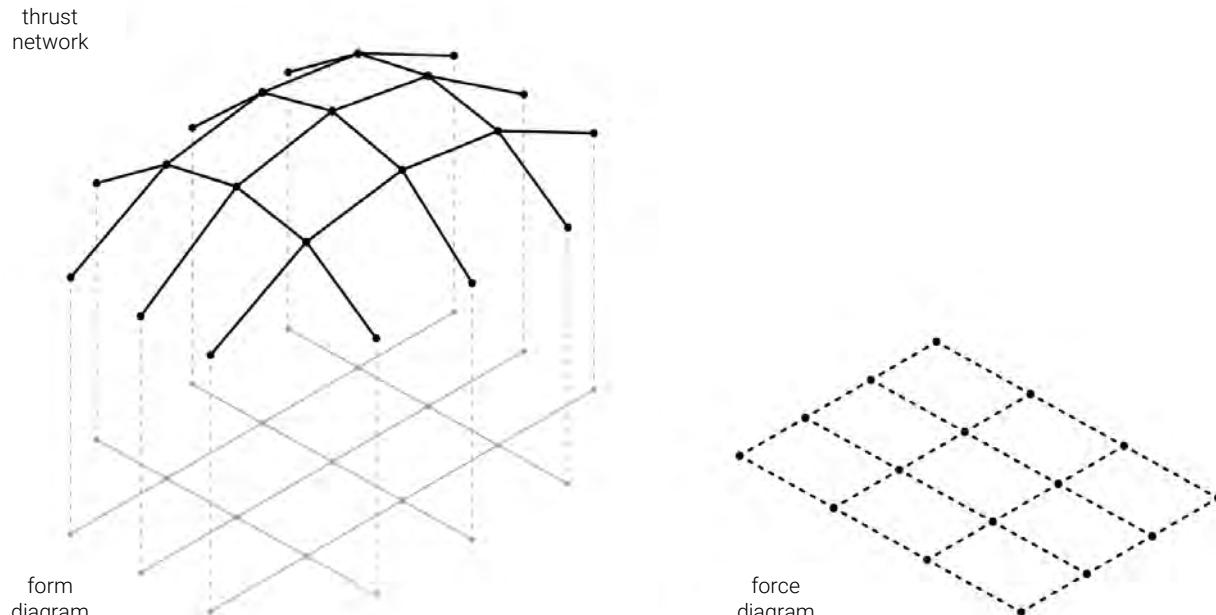




**Γ**  
form diagram







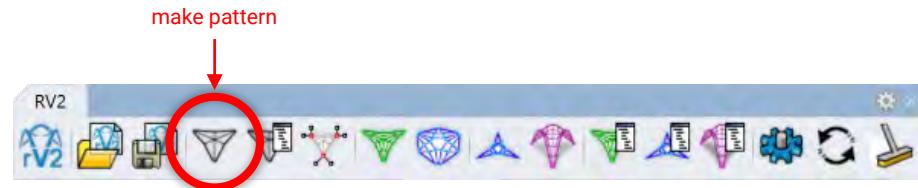
## RV2 workflow



## TNA steps

1. Pattern
2. Boundary conditions
3. Form diagram
4. Dual diagram
5. Horizontal equilibrium
6. Force diagram
7. Vertical equilibrium
8. Thrust diagram





## 1. Pattern

2. Boundary conditions

3. Form diagram

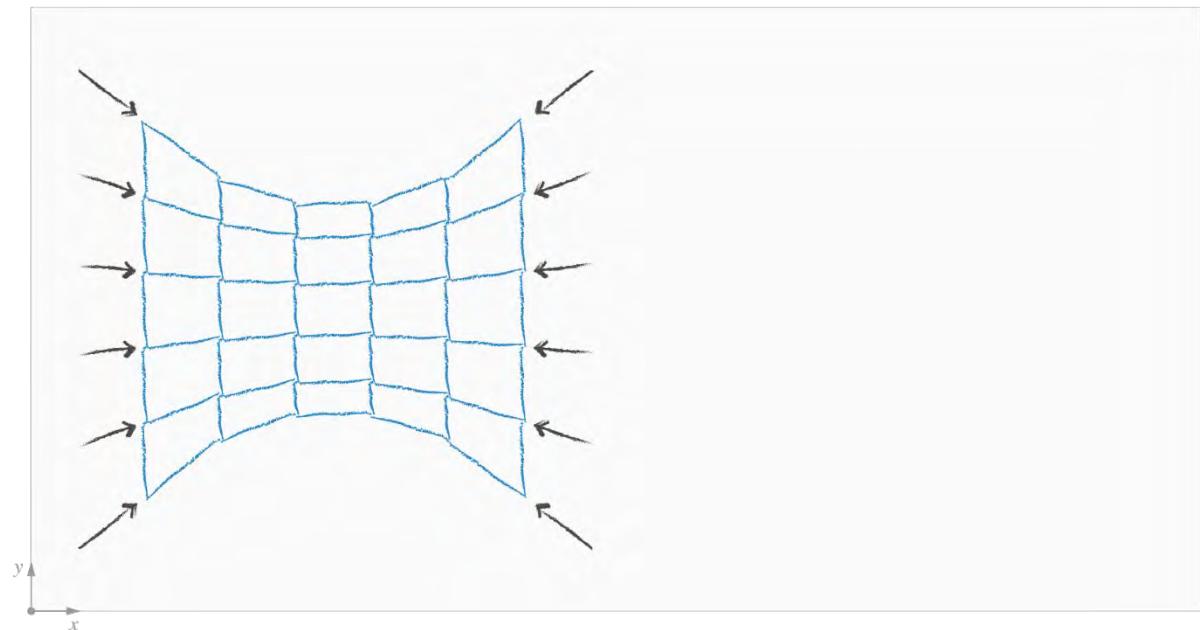
4. Dual diagram

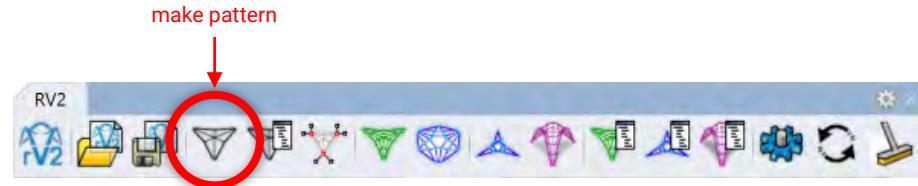
5. Horizontal equilibrium

6. Force diagram

7. Vertical equilibrium

8. Thrust diagram





## 1. Pattern

## 2. Boundary conditions

3. Form diagram

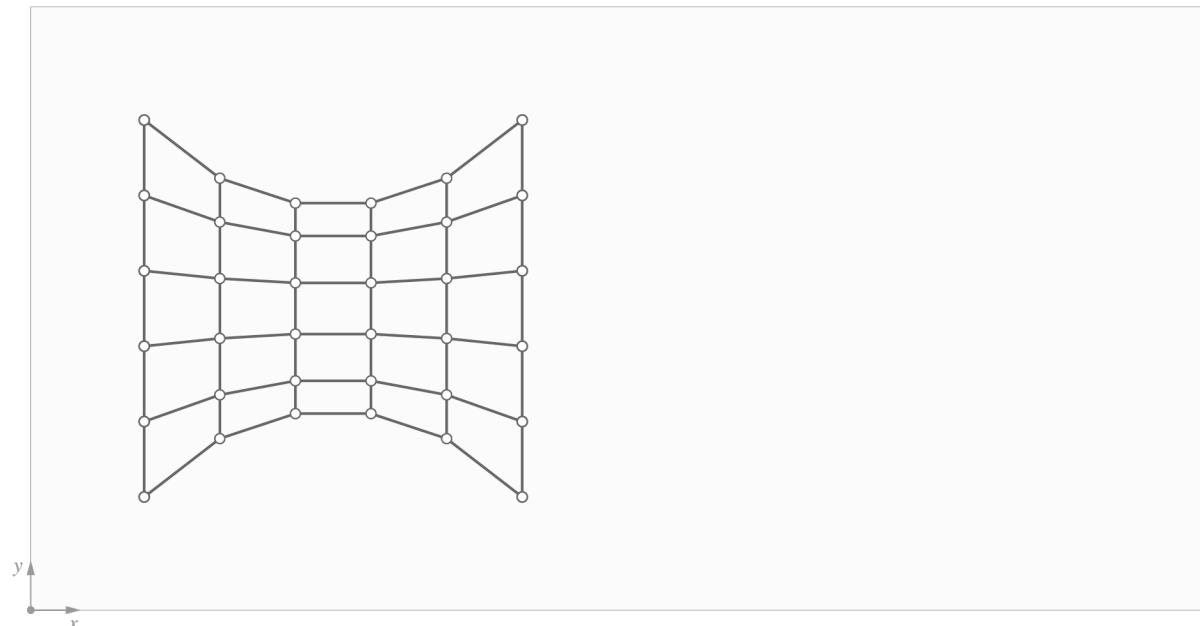
4. Dual diagram

5. Horizontal equilibrium

6. Force diagram

7. Vertical equilibrium

8. Thrust diagram



define boundary conditions



## 1. Pattern

## 2. Boundary conditions

3. Form diagram

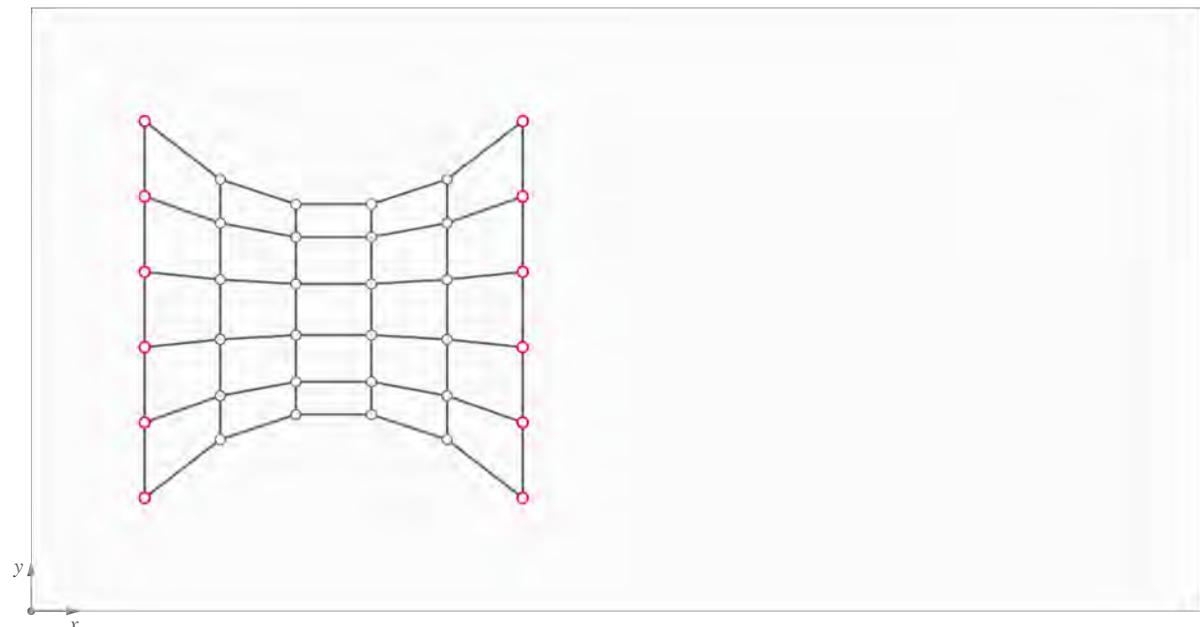
4. Dual diagram

5. Horizontal equilibrium

6. Force diagram

7. Vertical equilibrium

8. Thrust diagram



define boundary conditions



## 1. Pattern

## 2. Boundary conditions

3. Form diagram

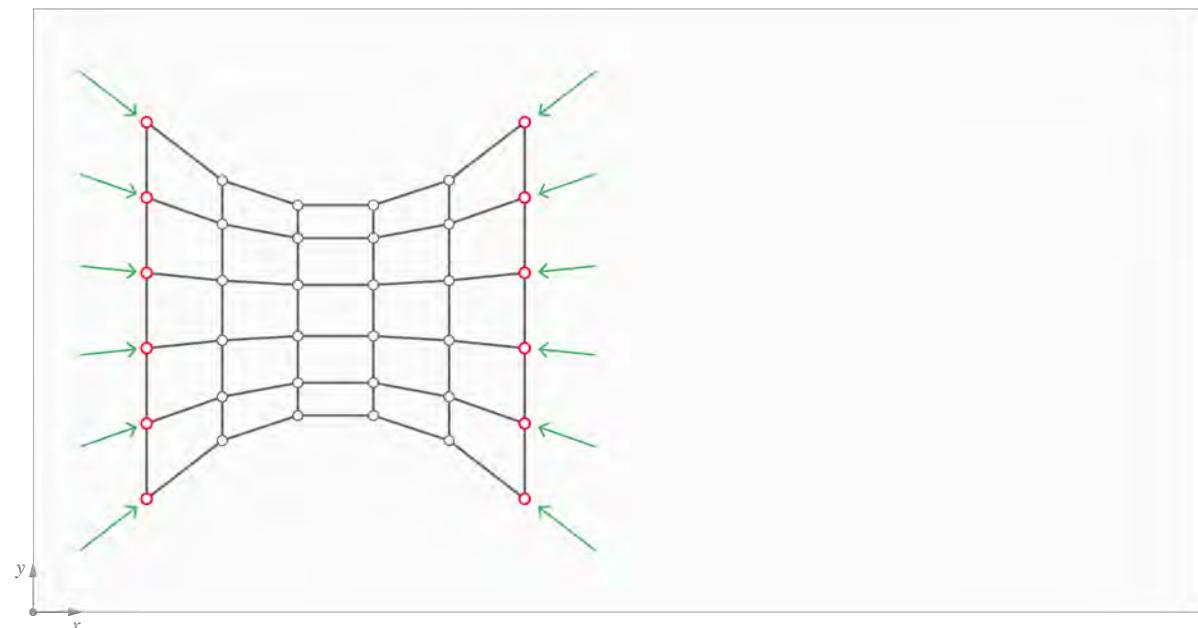
4. Dual diagram

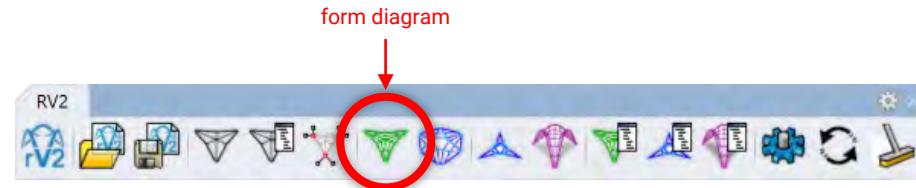
5. Horizontal equilibrium

6. Force diagram

7. Vertical equilibrium

8. Thrust diagram





1. Pattern

2. Boundary conditions

**3. Form diagram**

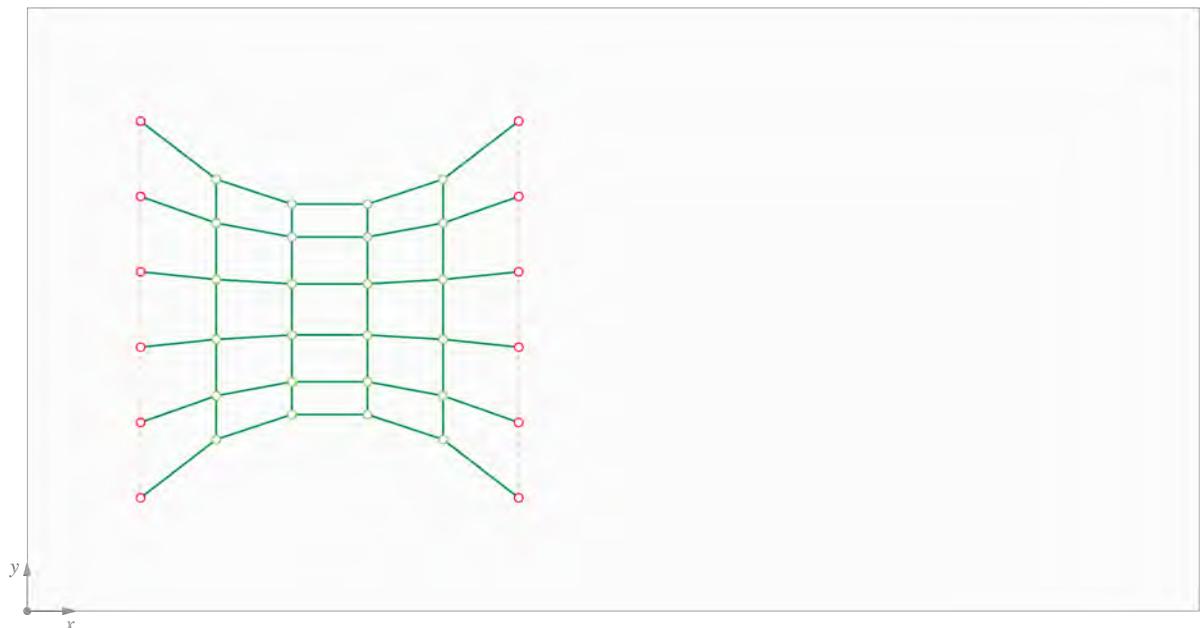
4. Dual diagram

5. Horizontal equilibrium

6. Force diagram

7. Vertical equilibrium

8. Thrust diagram



dual diagram



1. Pattern

2. Boundary conditions

3. Form diagram

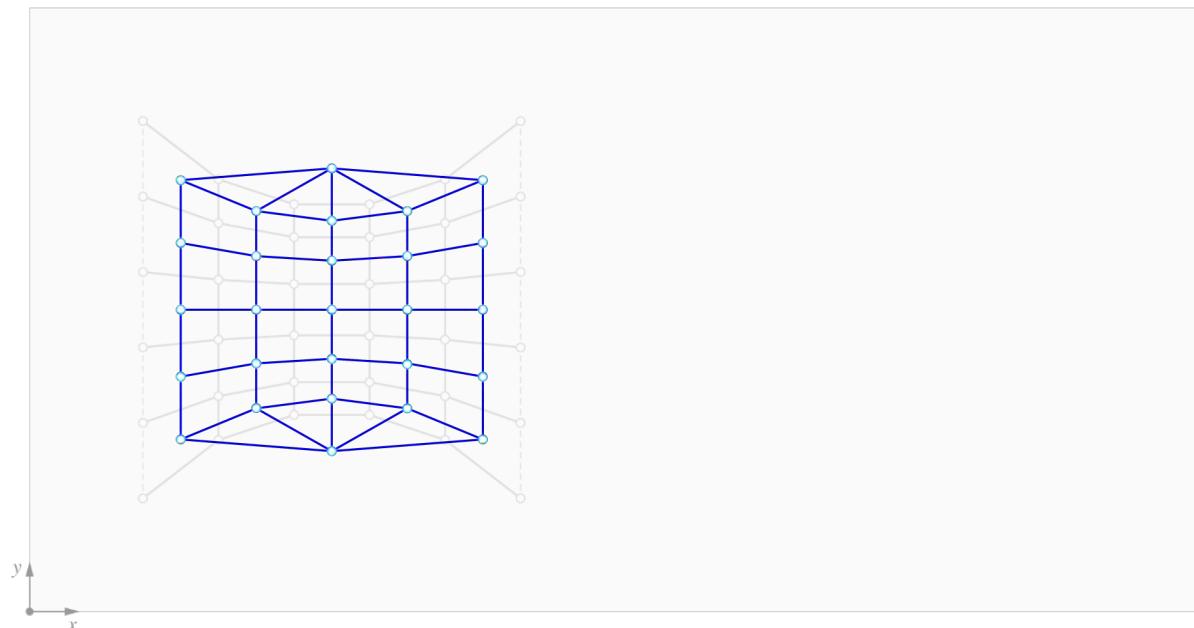
4. Dual diagram

5. Horizontal equilibrium

6. Force diagram

7. Vertical equilibrium

8. Thrust diagram



dual diagram



1. Pattern

2. Boundary conditions

3. Form diagram

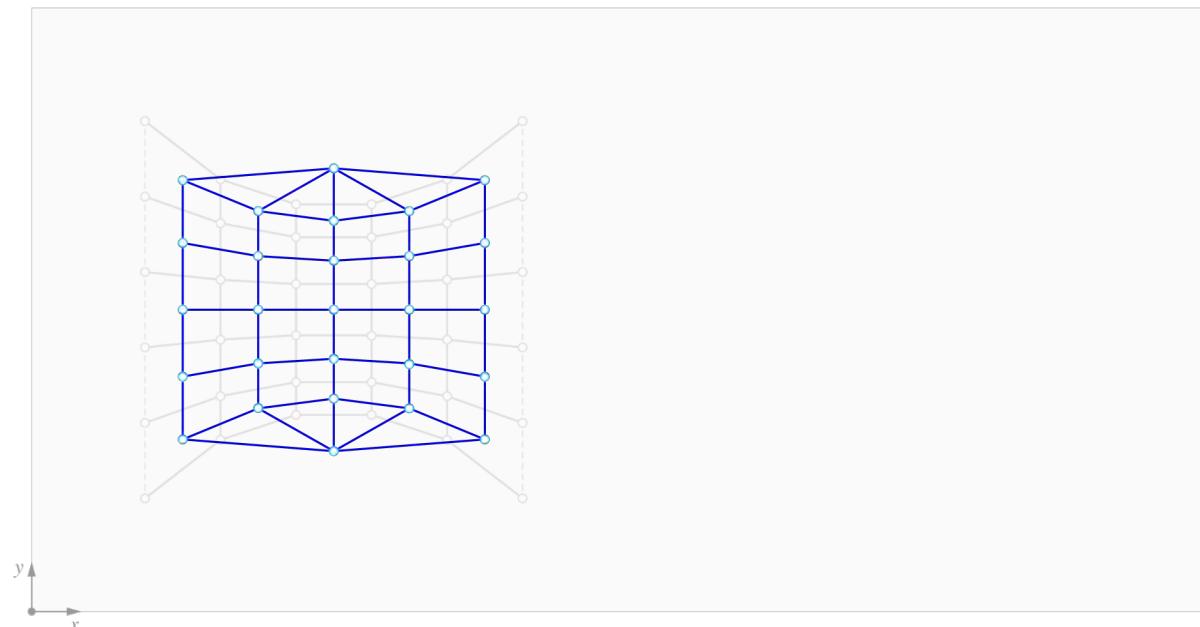
4. Dual diagram

5. Horizontal equilibrium

6. Force diagram

7. Vertical equilibrium

8. Thrust diagram



dual diagram



1. Pattern

2. Boundary conditions

3. Form diagram

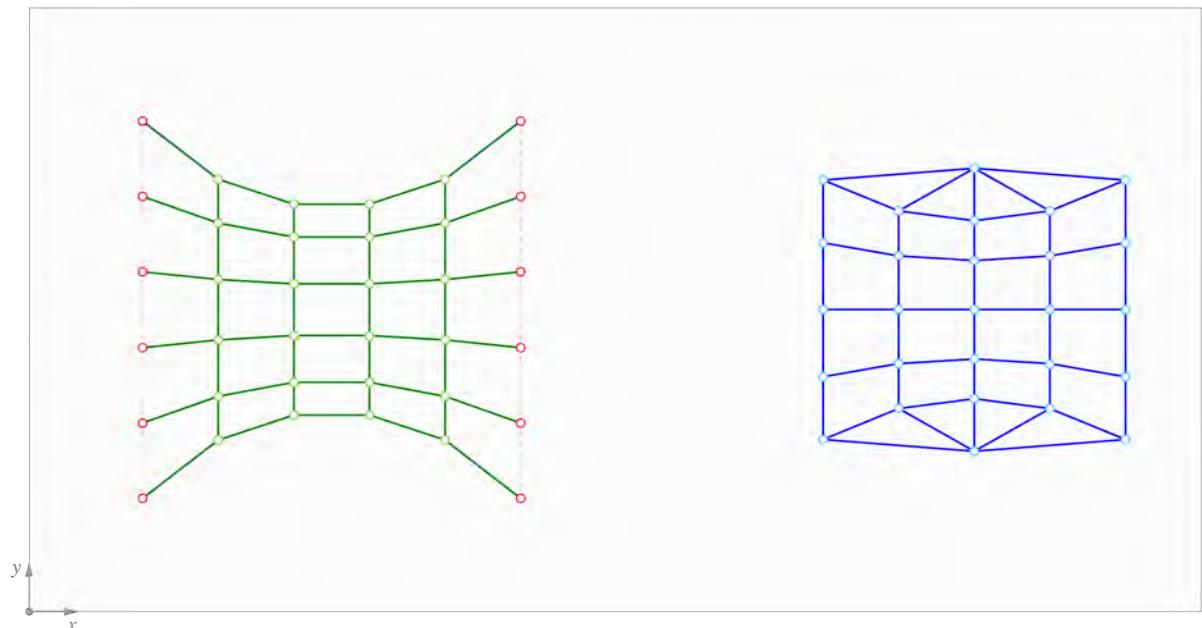
4. Dual diagram

5. Horizontal equilibrium

6. Force diagram

7. Vertical equilibrium

8. Thrust diagram



dual diagram



1. Pattern

2. Boundary conditions

3. Form diagram

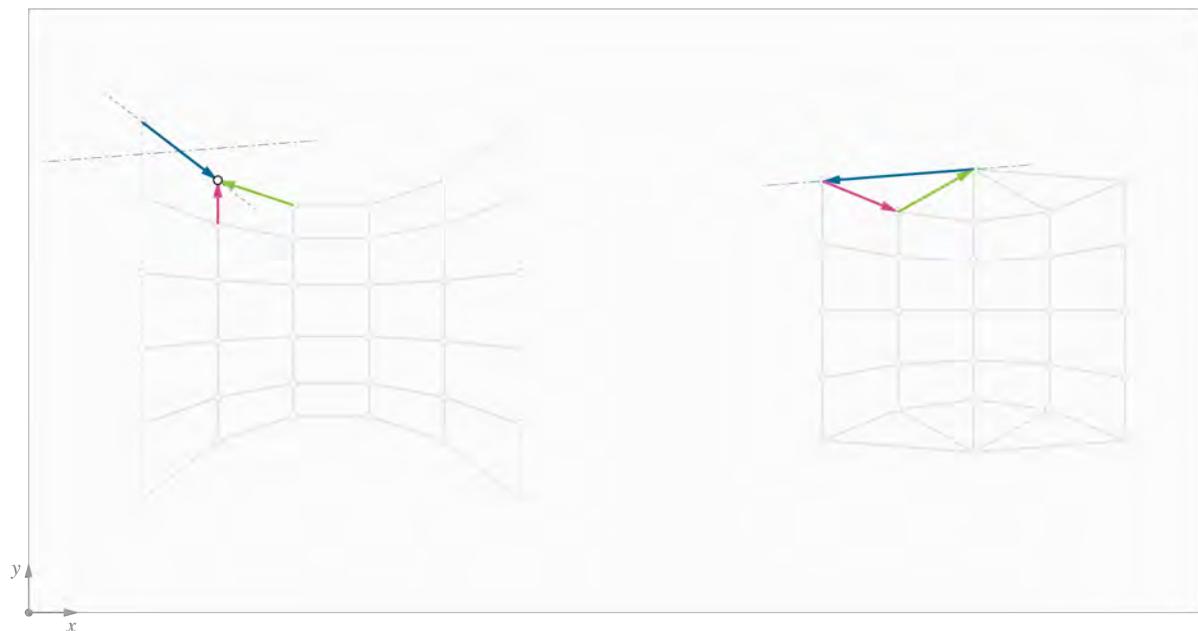
4. Dual diagram

5. Horizontal equilibrium

6. Force diagram

7. Vertical equilibrium

8. Thrust diagram



horizontal equilibrium



1. Pattern

2. Boundary conditions

3. Form diagram

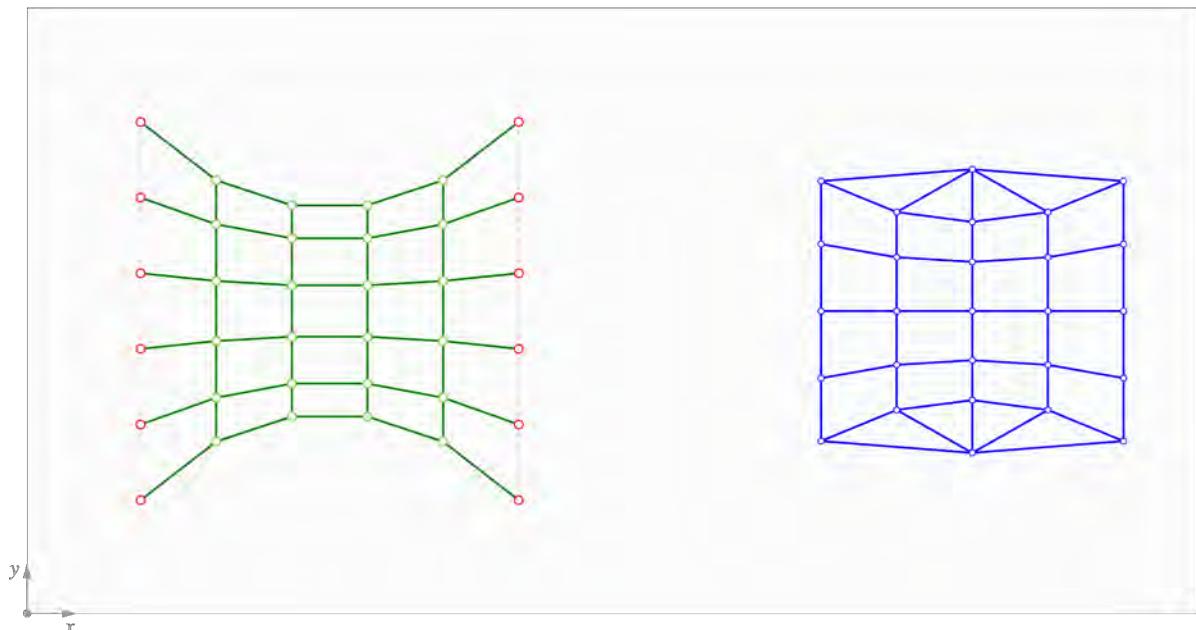
4. Dual diagram

5. Horizontal equilibrium

6. Force diagram

7. Vertical equilibrium

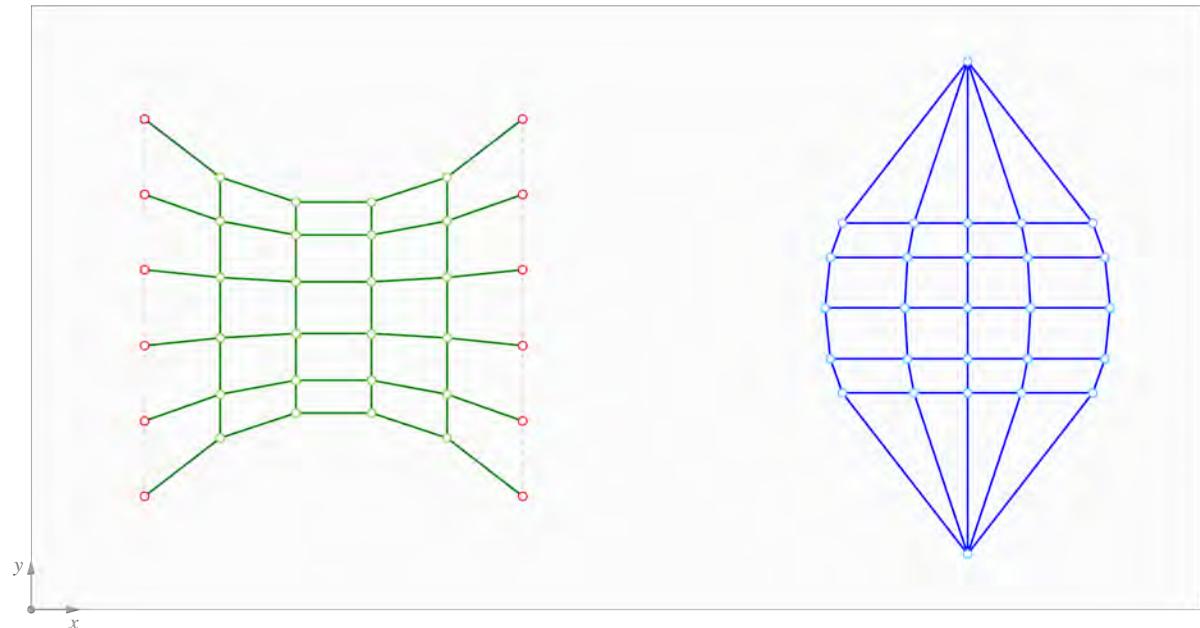
8. Thrust diagram



horizontal equilibrium



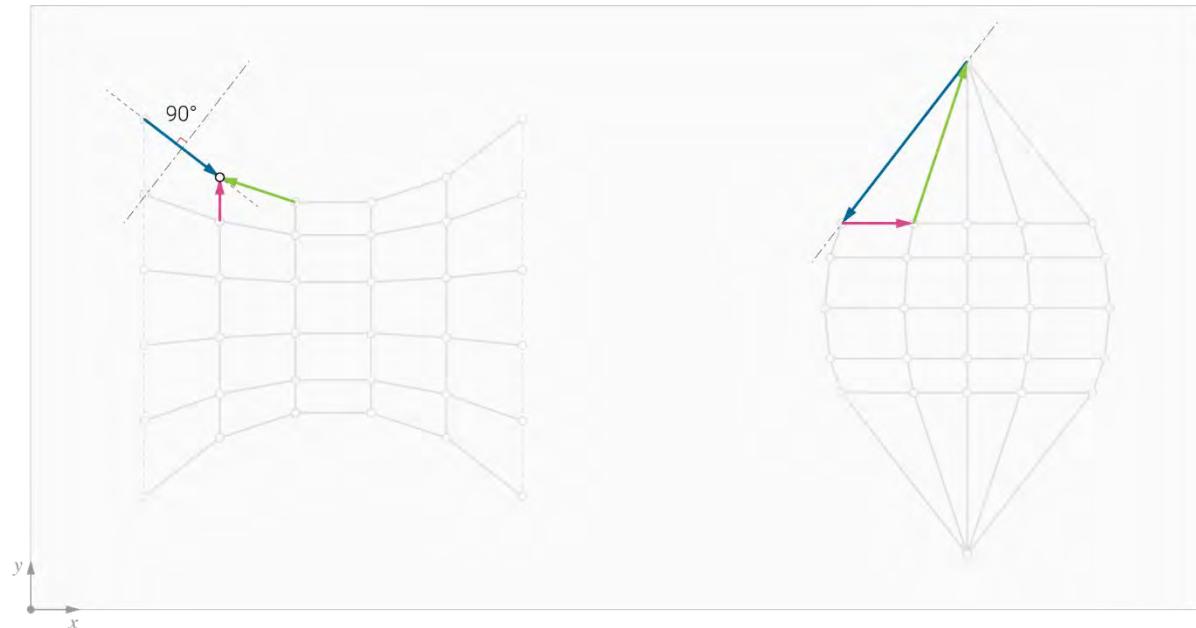
1. Pattern
2. Boundary conditions
3. Form diagram
4. Dual diagram
5. Horizontal equilibrium
- 6. Force diagram**
7. Vertical equilibrium
8. Thrust diagram



horizontal equilibrium



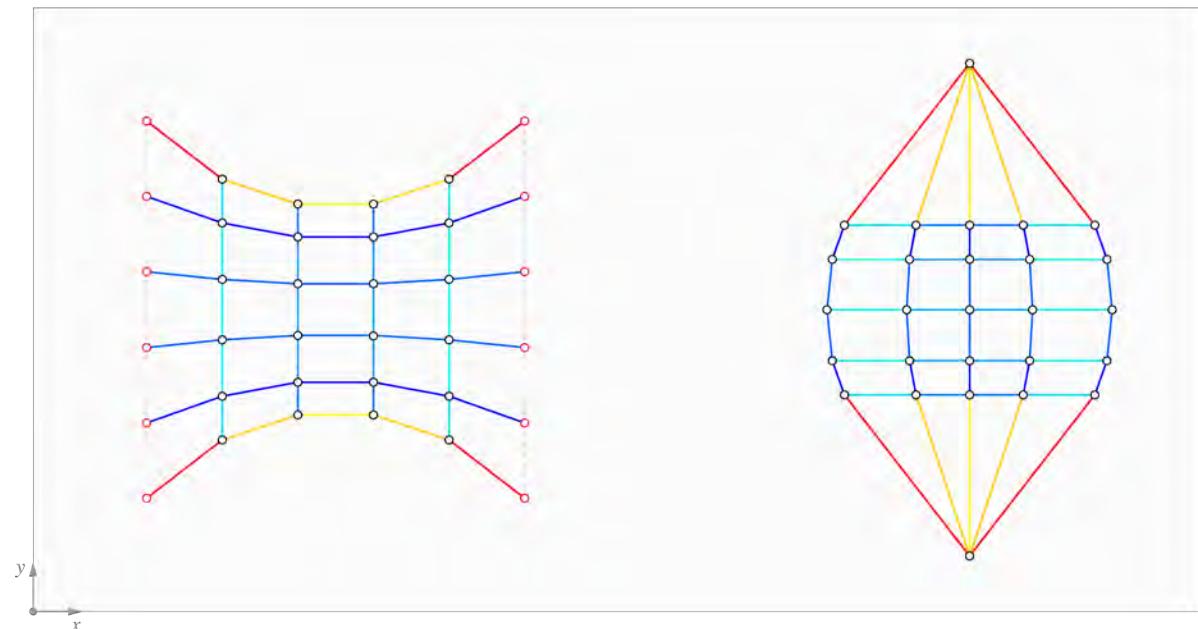
1. Pattern
2. Boundary conditions
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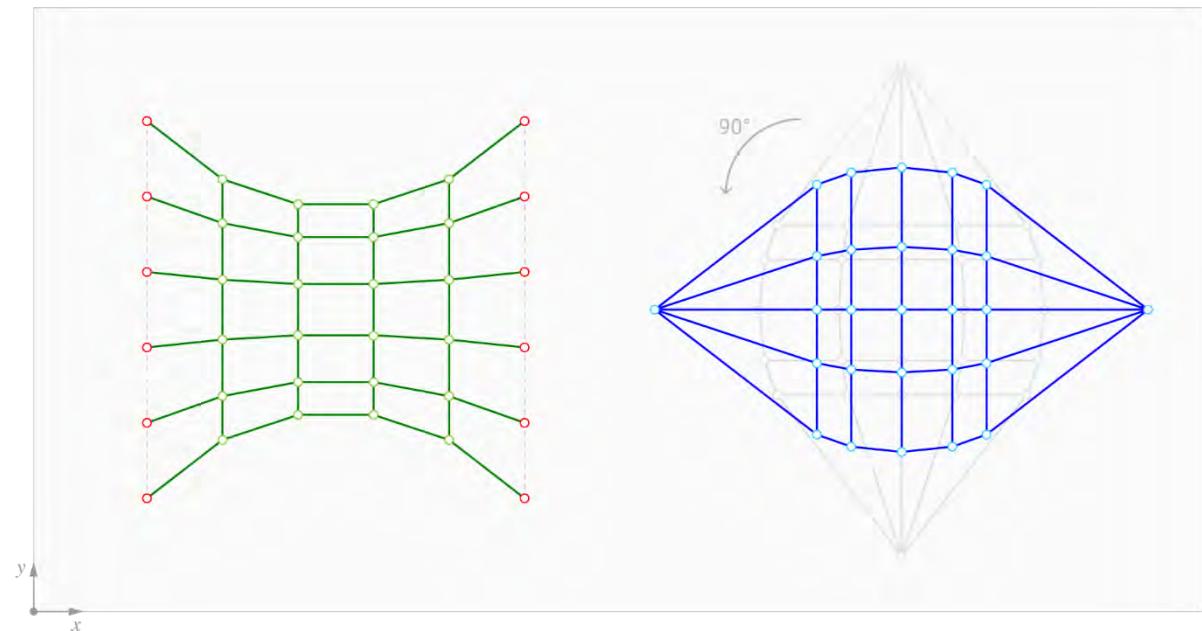
horizontal equilibrium



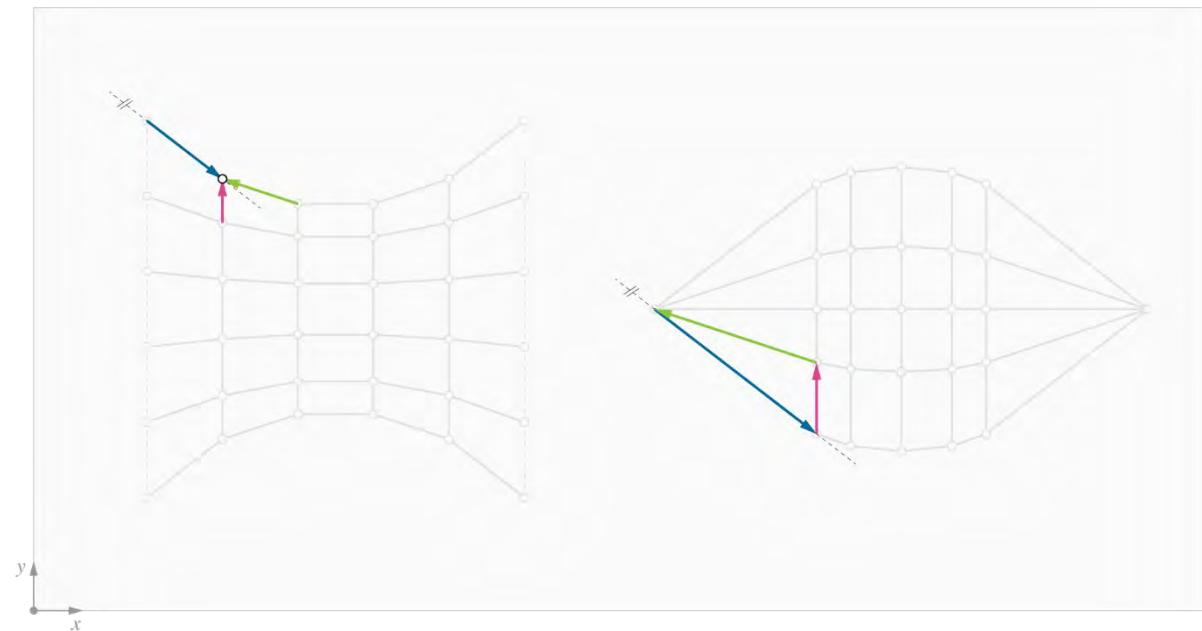
1. Pattern
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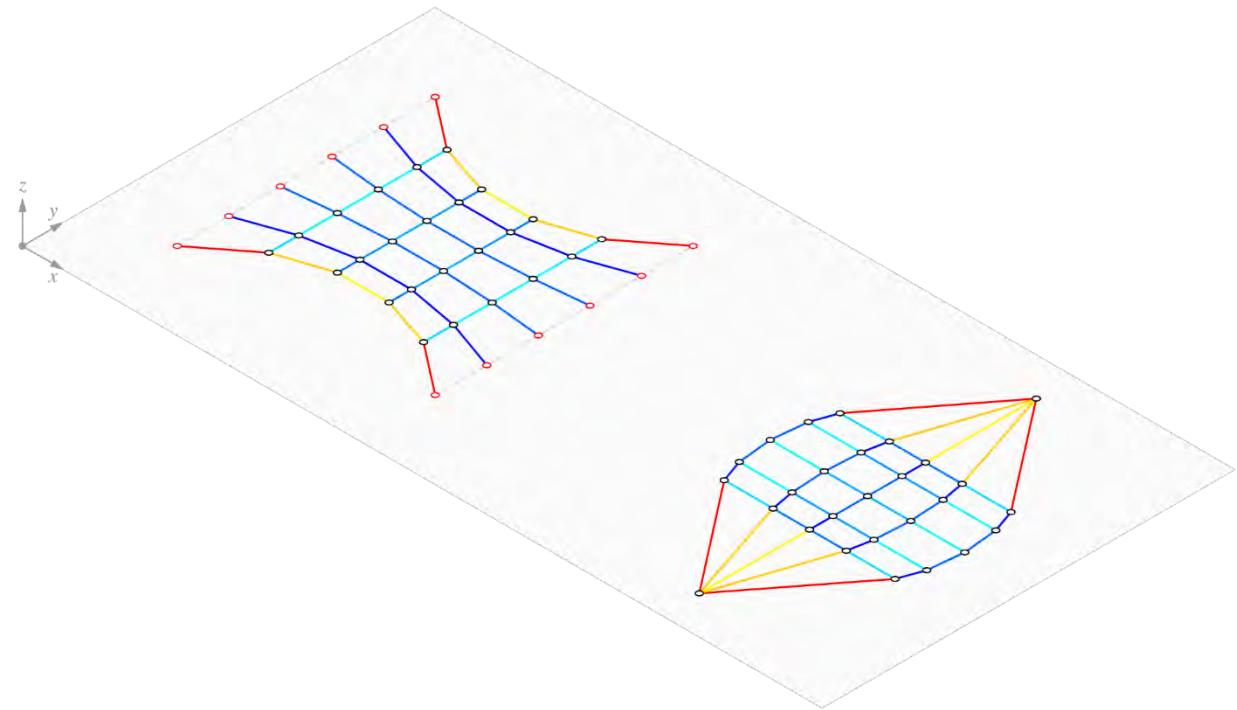
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3. Form diagram
4. Dual diagram
5. Horizontal equilibrium
- 6. Force diagram**
7. Vertical equilibrium
8. Thrust diagram



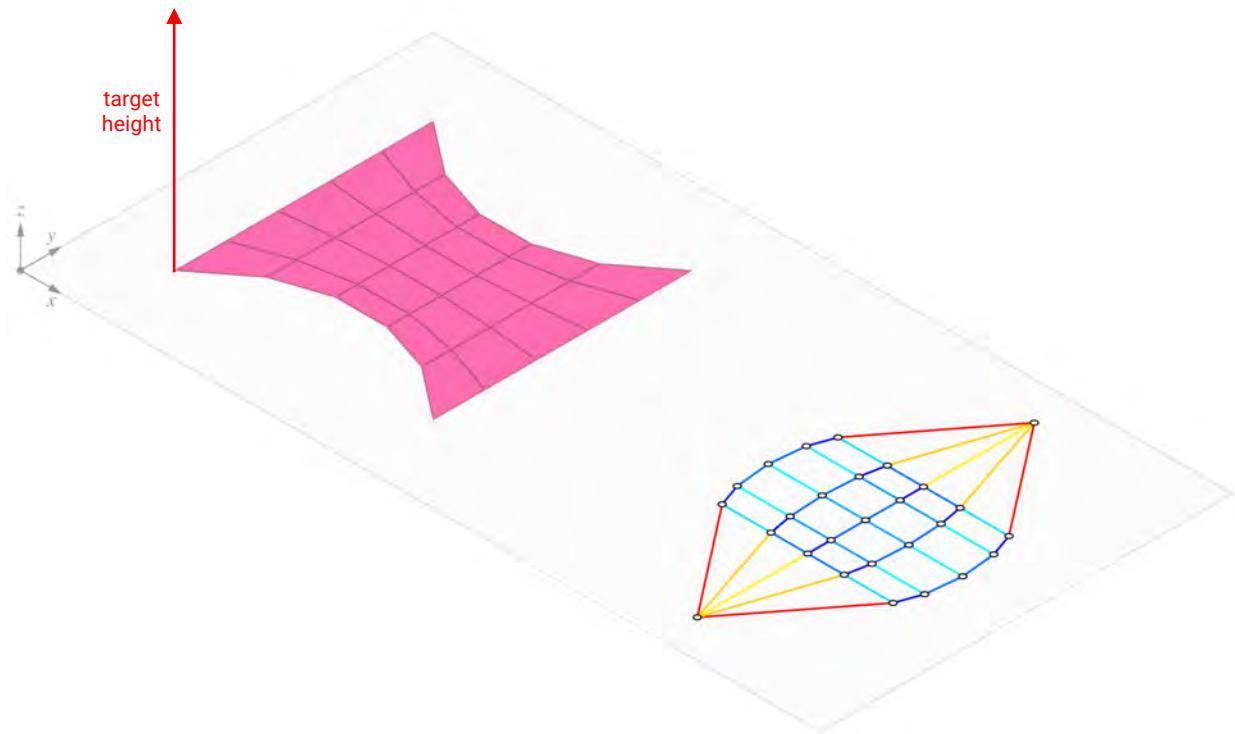
vertical equilibrium



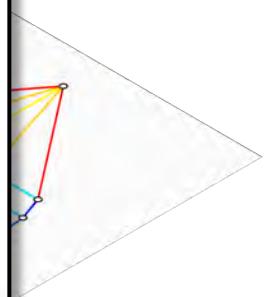
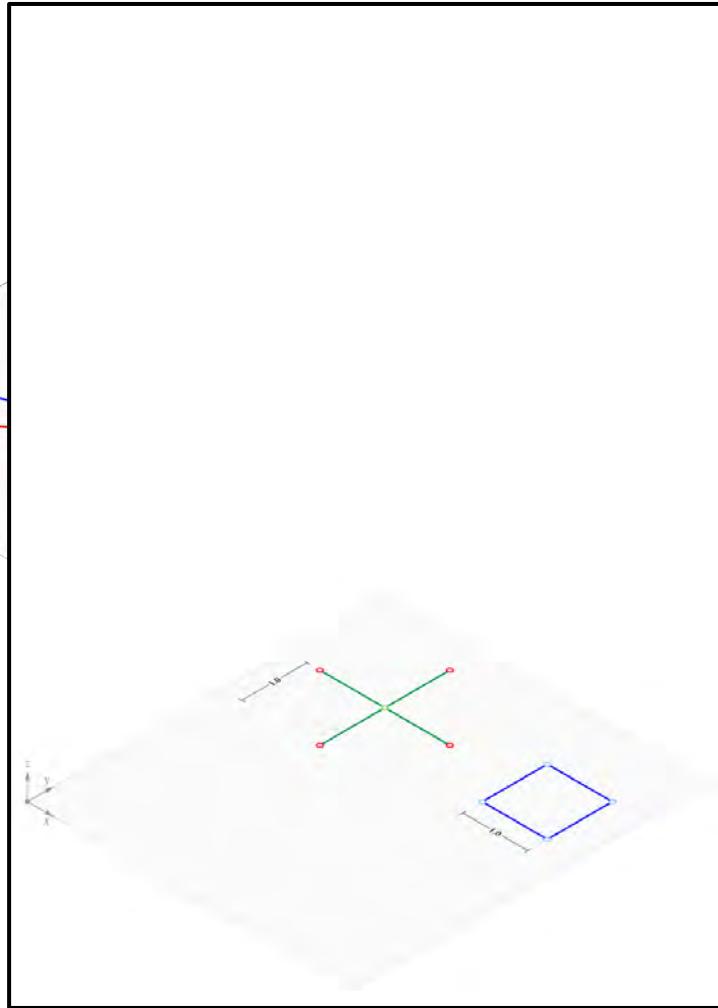
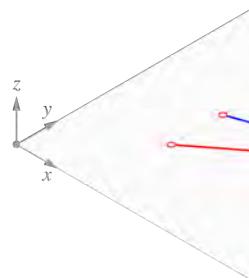
1. Pattern
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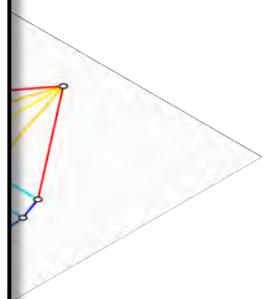
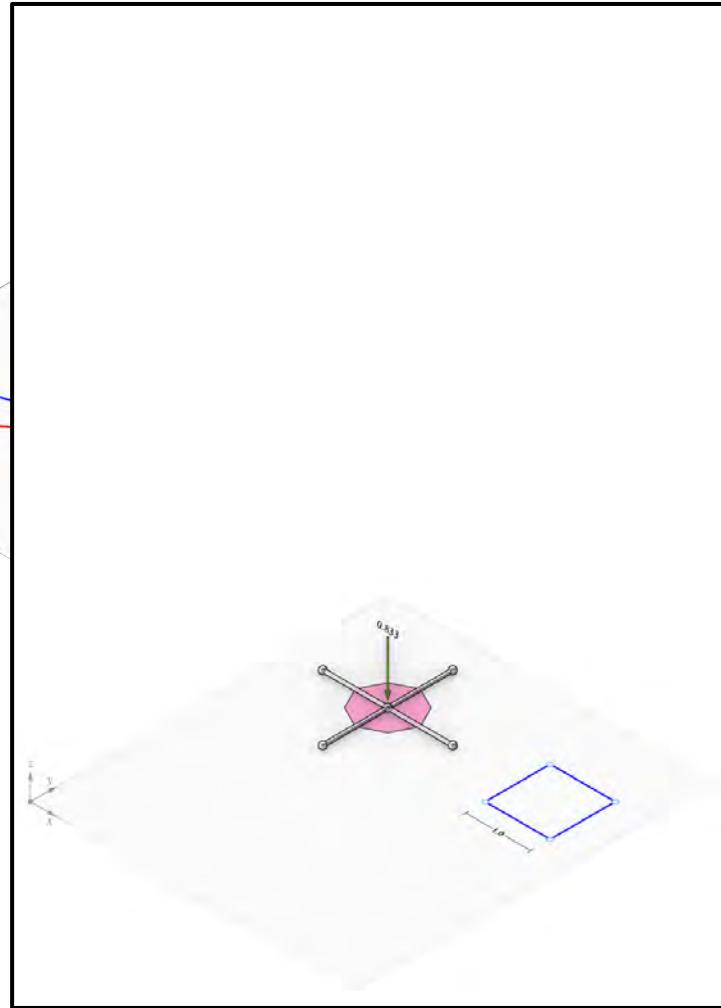
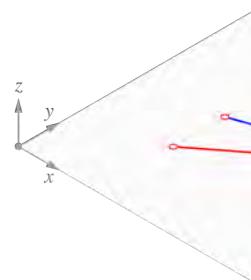
1. Pattern
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8. Thrust diagram



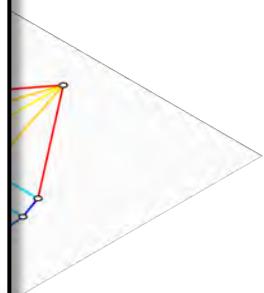
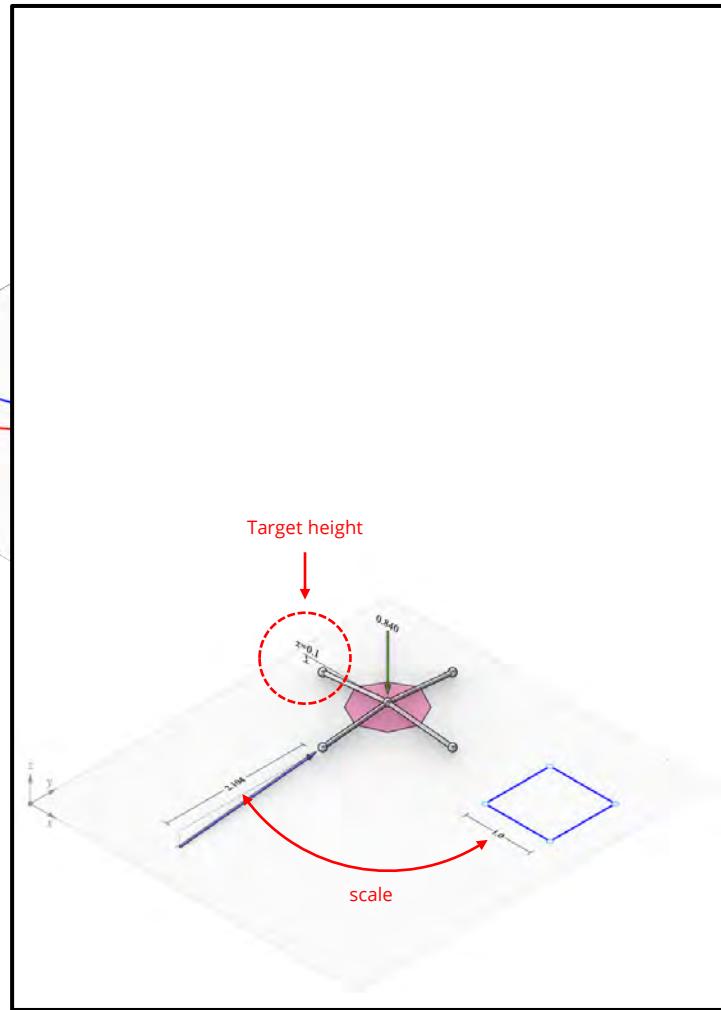
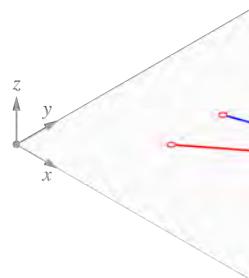
1. Pattern
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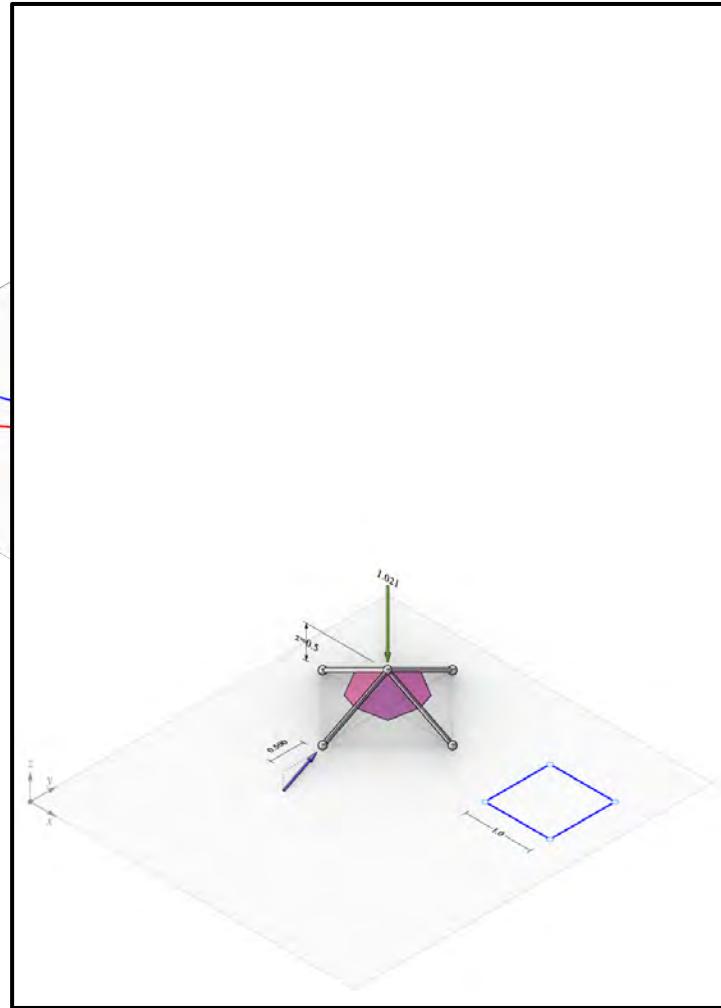
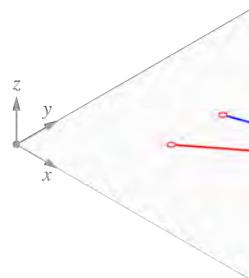
1. Pattern
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8. Thrust diagram



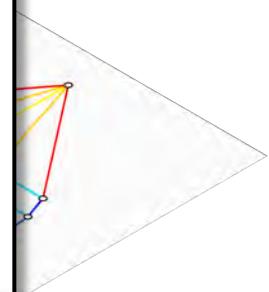
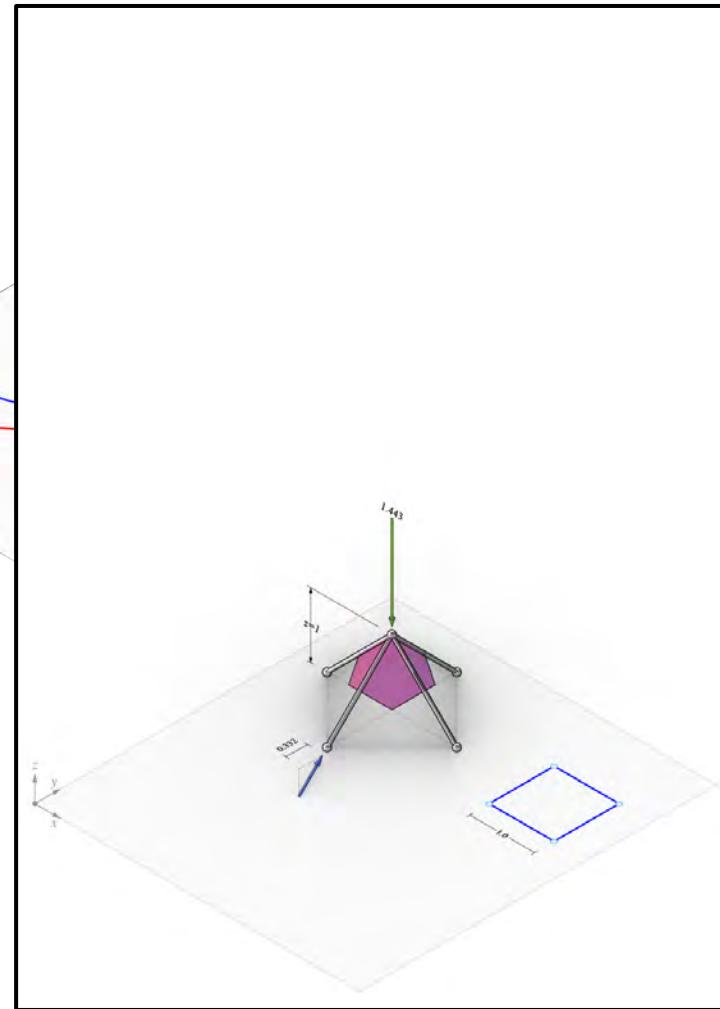
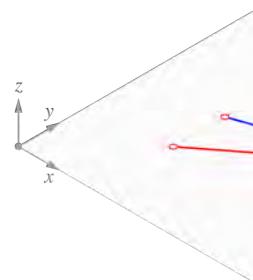
1. Pattern
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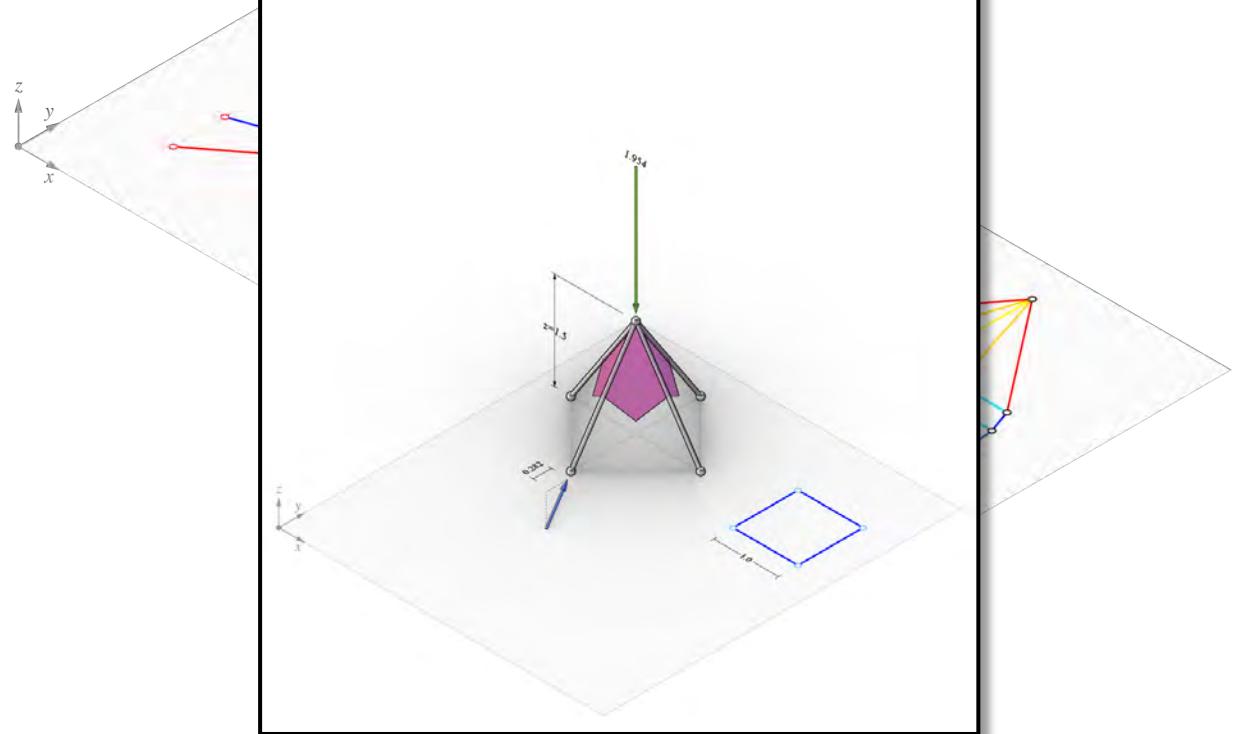
1. Pattern
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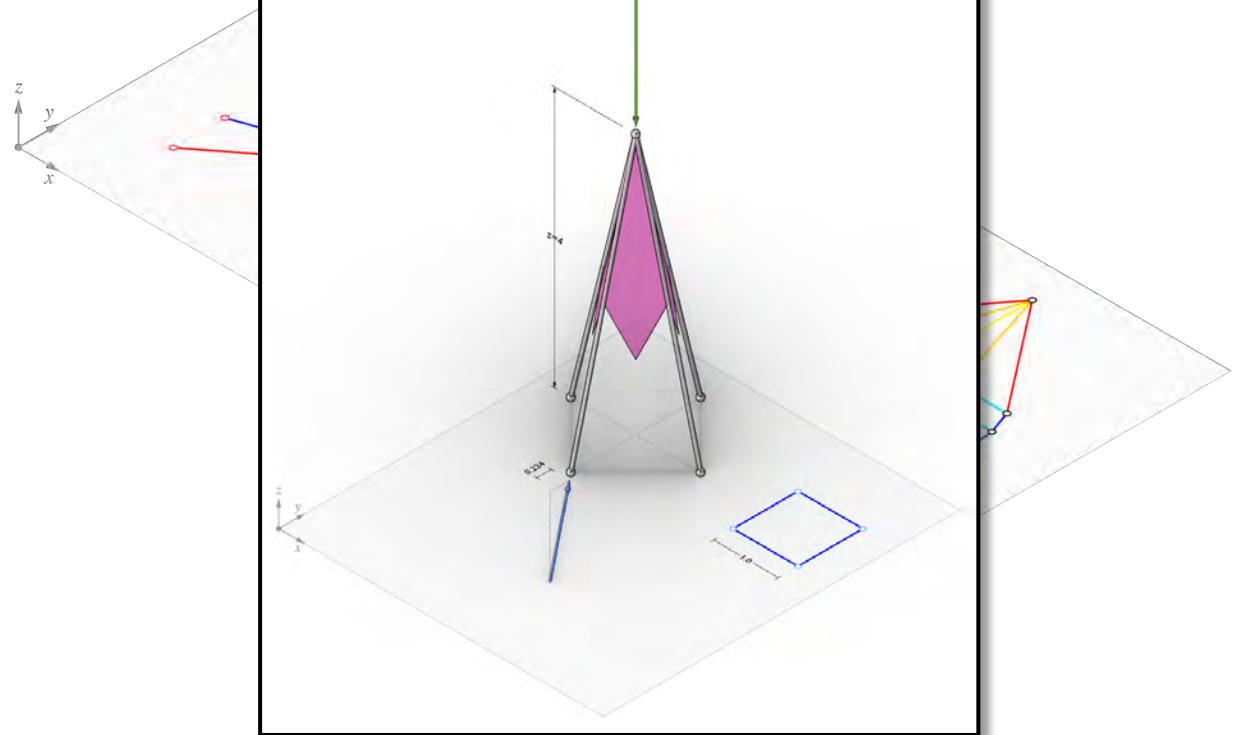
1. Pattern
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1. Pattern
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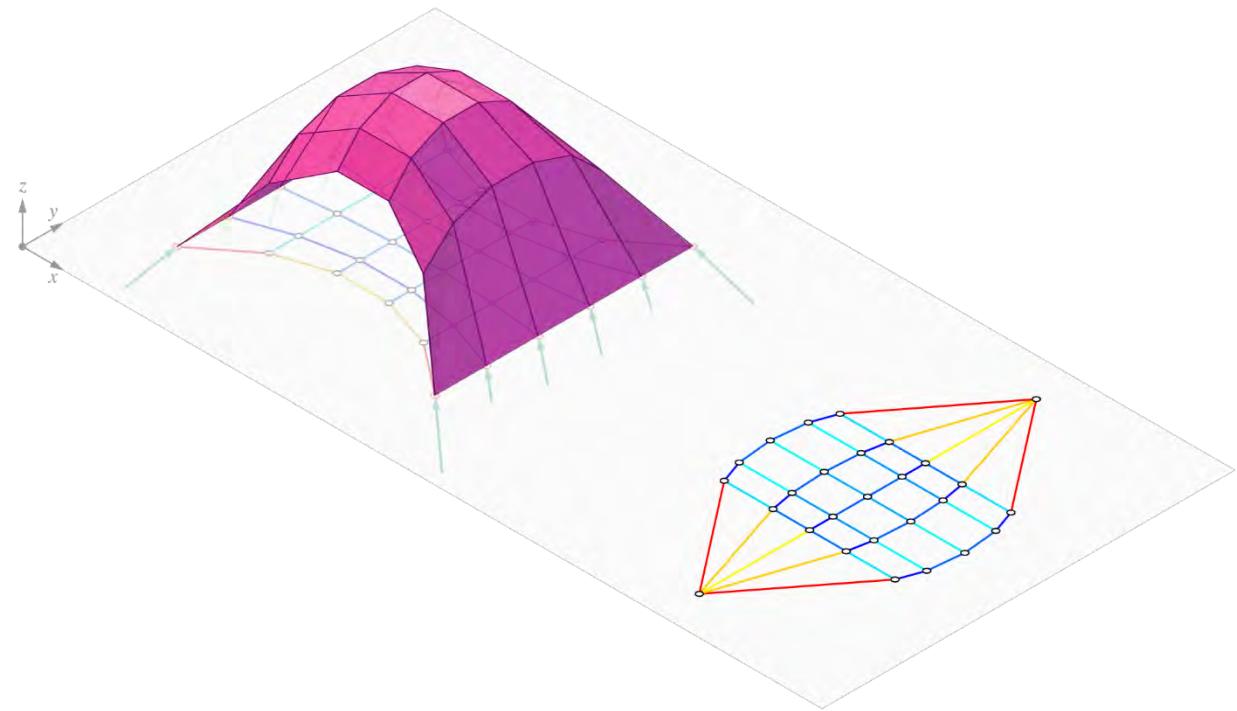
1. Pattern
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modify diagrams



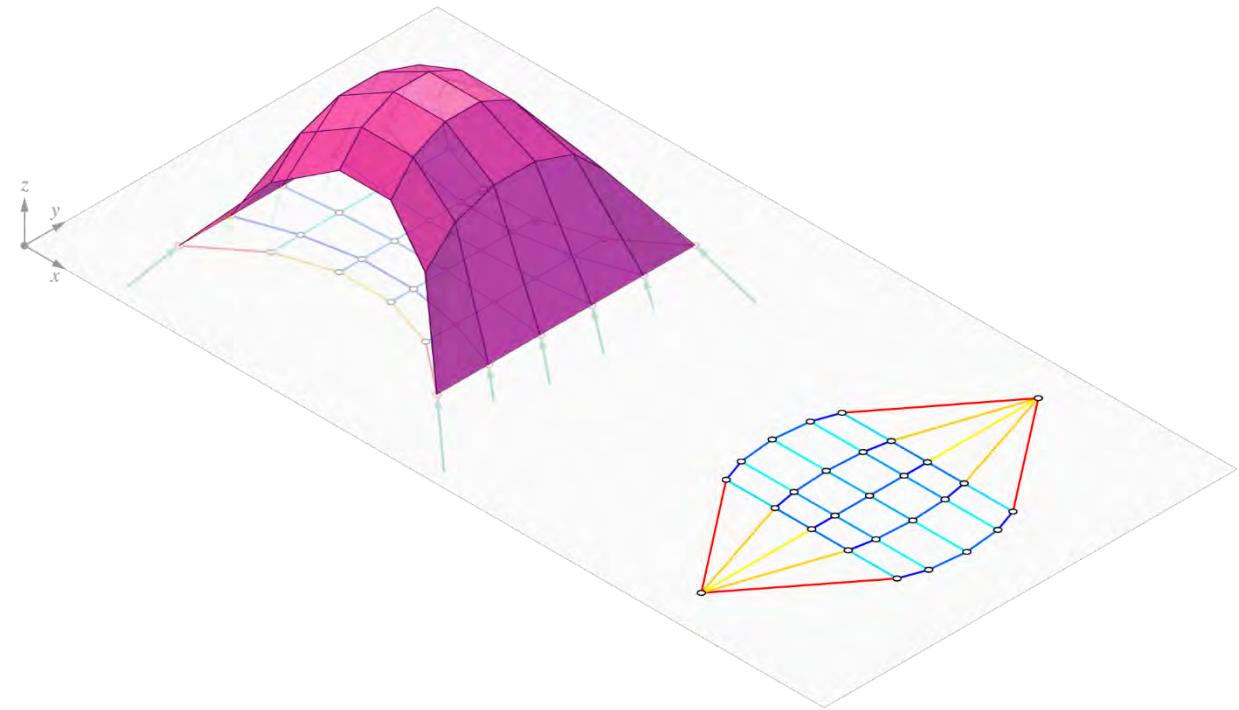
1. Pattern
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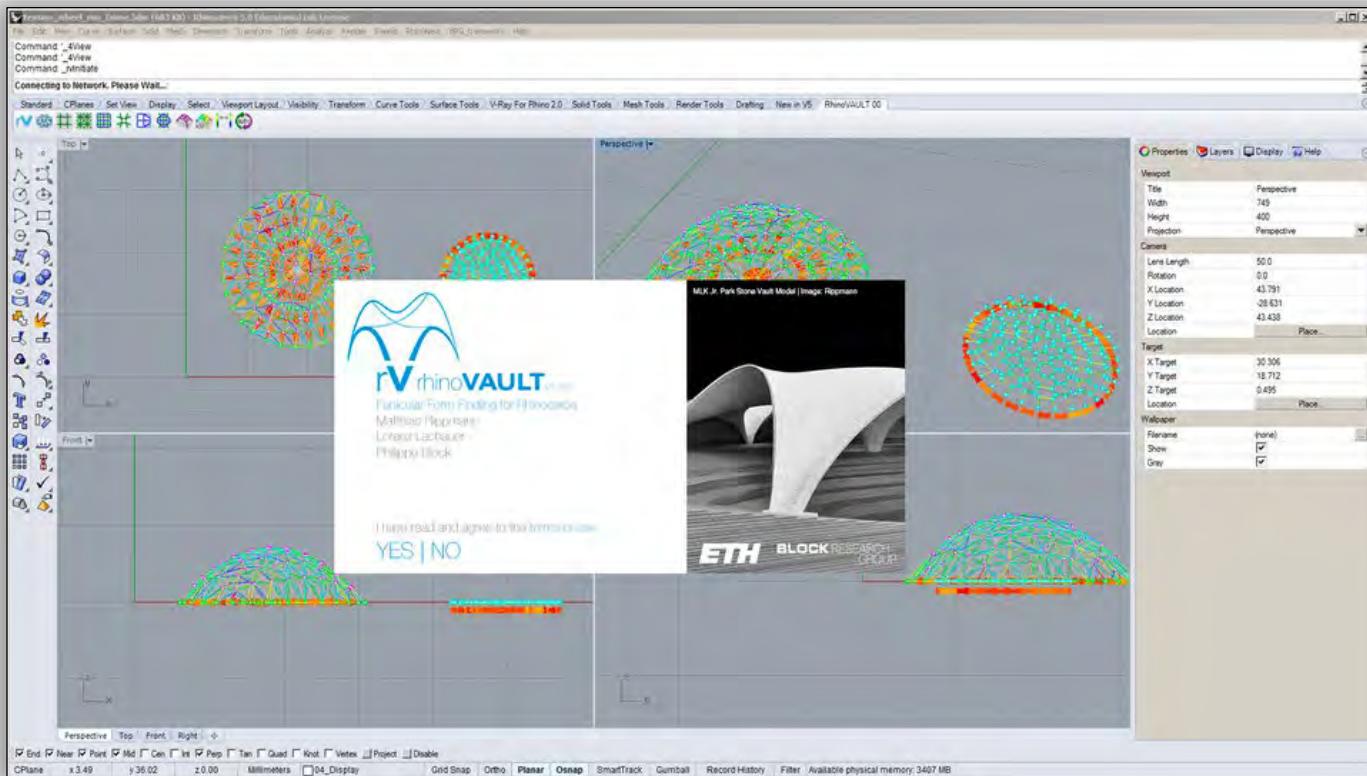
utility functions



1. Pattern
2. Boundary conditions
3. Form diagram
4. Dual diagram
5. Horizontal equilibrium
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7. Vertical equilibrium
8. Thrust diagram



## Implementation ► rhinoVAULT



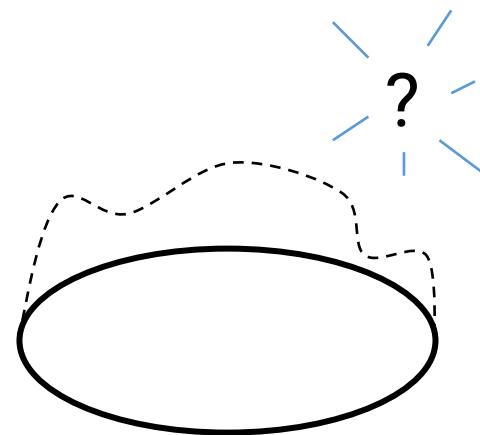
# rhino**VAULT**

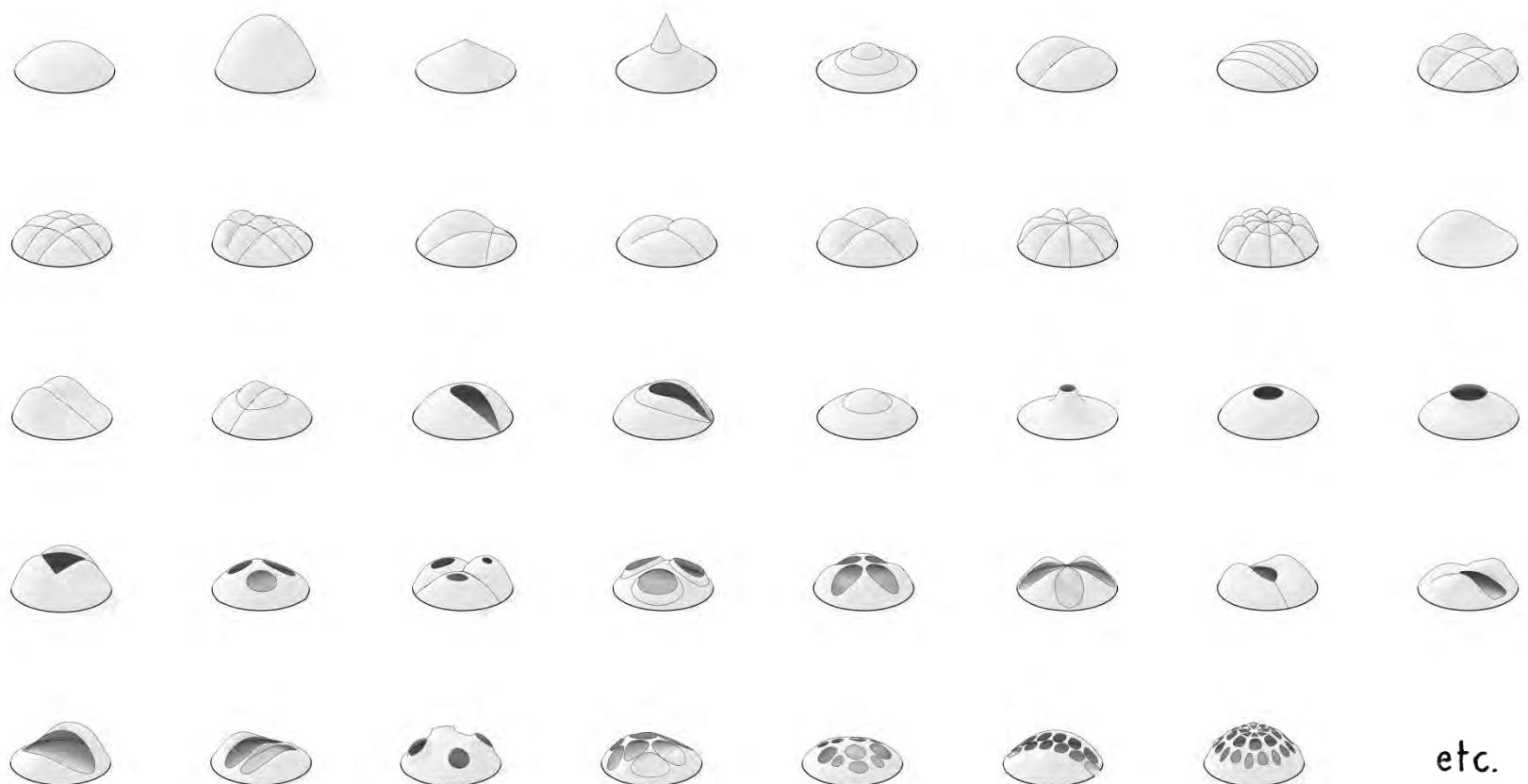
MATTHIAS RIPPmann  
LORENZ LACHAUER  
PHILIPPE BLOCK

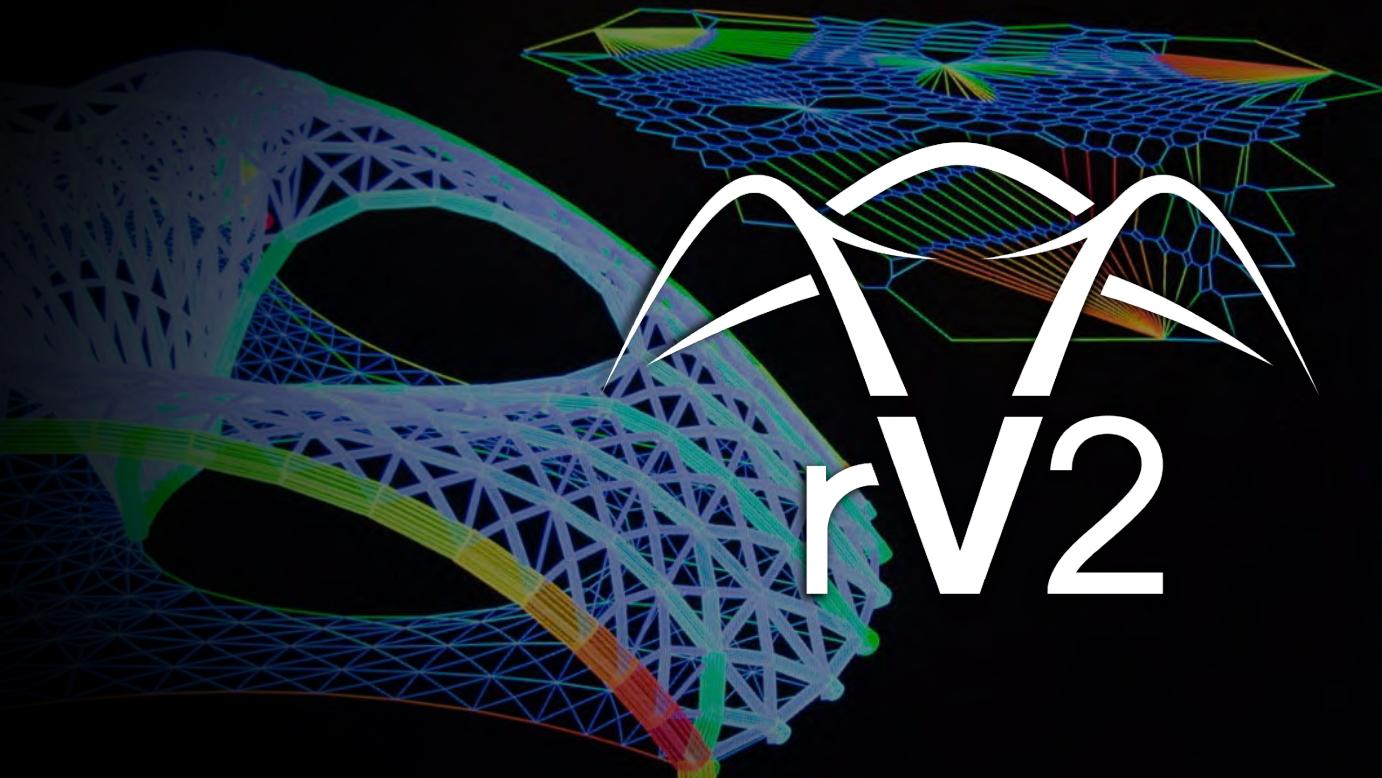


**ETH** Zürich

**BLOCK** RESEARCH  
GROUP







```
if len(neighbors) == 1:
    u = sorted([(key, network.edges_iter(key))])
else:
    u = sorted(network.edges_iter(key))
v = _find_first_neighbour(u, network)
_find_edge(v, v, network)
for u, v in network.edges_iter():
    if network.halfedge[v][v] is None:
        _find_edge_face(v, v, network)
    if network.halfedge[v][u] is None:
        _find_edge_face(v, u, network)
_break_faces(network, breakpoints)
return network.face
```

```
def _find_first_neighbour(key, network):
    angles = []
    nbrs = network.halfedge[key].keys()
    if len(nbrs) == 1:
        return nbrs[0]
    vu = [-1, -1, 0]
    for nbr in nbrs:
        w = [network.vertex[nbr][i] for i in 'XZY']
        v = [network.vertex[key][i] for i in 'XZY']
        vw = [w[0] - v[0], w[1] - v[1], 0]
        angles.append(angle_smallest_vectors(vu, vw))
    return nbrs[angles.index(min(angles))]
```

```
def _sort_neighbours(network, ccw=True):
    sorted_neighbours = {}
```

```
c
```

```
def convex_hull():
    pts, faces = [], []
    for pt in points:
        if pt not in pts:
            pts.append(pt)
            face = []
            for n in network.halfedge[pt].keys():
                if not dup:
                    face.append(n)
                    dup = True
                if not dup:
                    pts.append(pt)
                    if face:
                        faces.append(face)
                    faces.append(face)
    return pts, faces
```

```
if __name__ == "__main__":
    import math
    import random

    from compas.geometry import distance_point_point
    from compas_rhino.helpers.mesh import draw_mesh
    from compas.datastructures import Mesh
```

```
radius = 5
origin = (0., 0., 0.)
count = 0
points = []
while count < 1000:
    x = (random.random() - 0.5) * radius
    y = (random.random() - 0.5) * radius
    z = (random.random() - 0.5) * radius
    pt = x, y, z
    if distance_point_point(origin, pt) > radius:
        points.append(pt)
        count += 1
faces = convex_hull(points)
```

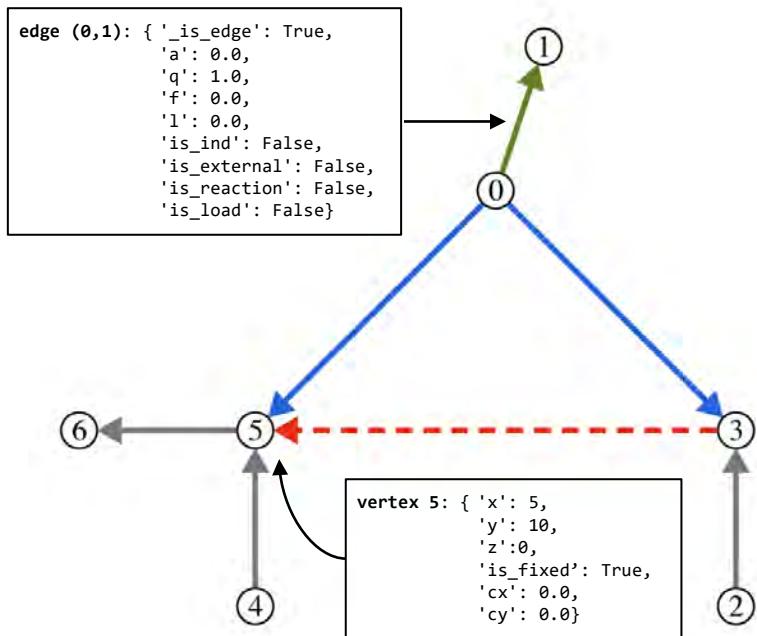


COMPASS

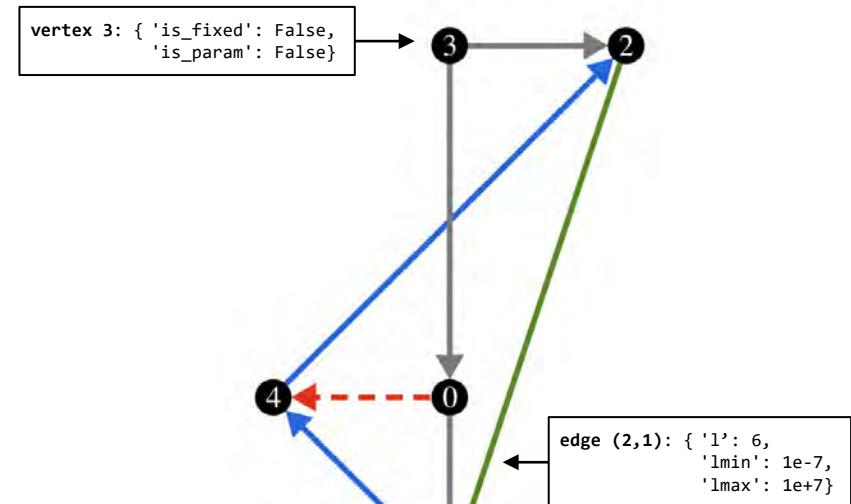
Open-source, Python-based framework for computational research and collaboration  
in architecture, engineering and digital fabrication

```
    if key in mesh.vertices_coordinates:
        mesh.vertices_coordinates[key] = vertex[key]
    else:
        mesh.vertices_coordinates[key] = vertex
    mesh.vertices_neighbours[key] = nbrs
    center_of_mass_polygon([key_xy[nbr] for nbr in nbrs])
    attr['x'] = p[0]
    attr['y'] = d * (c[1] - p[1])
    attr['z'] = d * (c[2] - p[2])
    if callback:
        callback(mesh, k, callback_args)
    if smooth_mesh_length:
        smooth_mesh_length(mesh, lmin, lmax, fixed)
    if callback:
        if not callable(callback):
            raise Exception('Callback is not callable')
    fixed = fixed or []
    fixed = set(fixed)
```

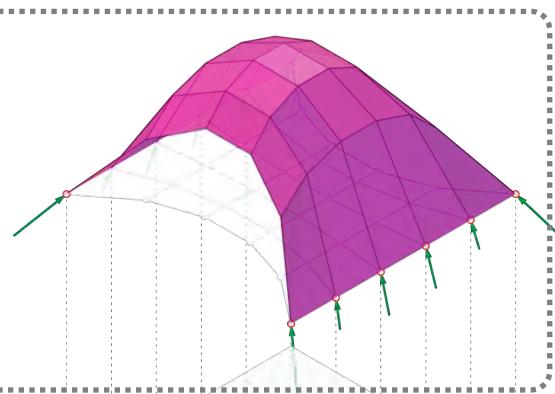
form diagram data



force diagram data

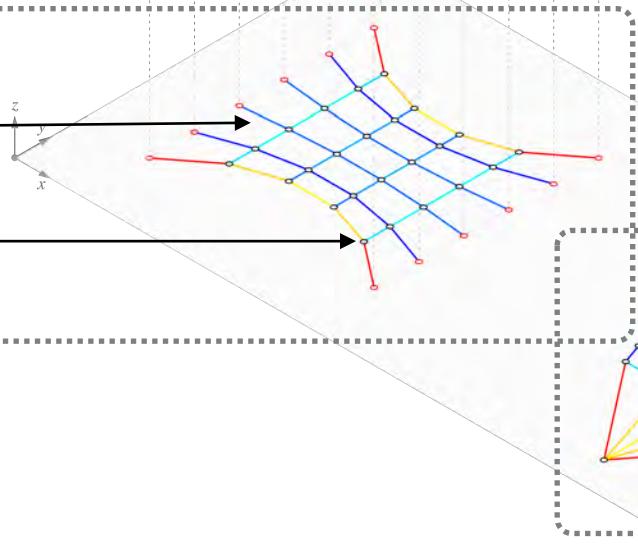


### thrust diagram data



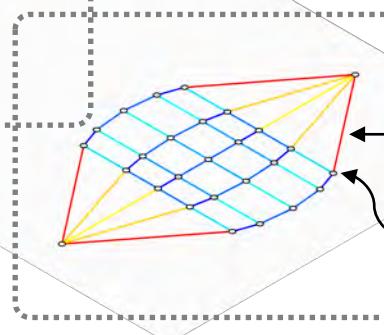
### form diagram data

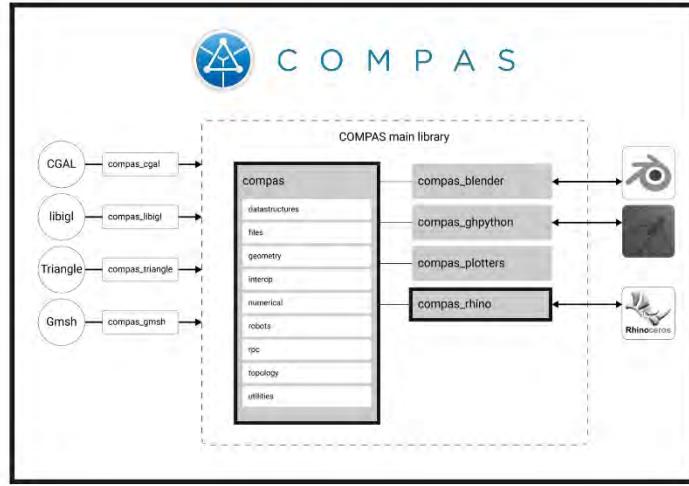
```
edge (u, v): {'q': 1.0,  
               'lmin': 0.0,  
               'lmax': 1e+7,  
               ...}  
  
vertex: {'is_fixed': False,  
        'is_anchor': False,  
        'x': 0.0,  
        'y': 0.0,  
        'z': 0.0,  
        ...}
```

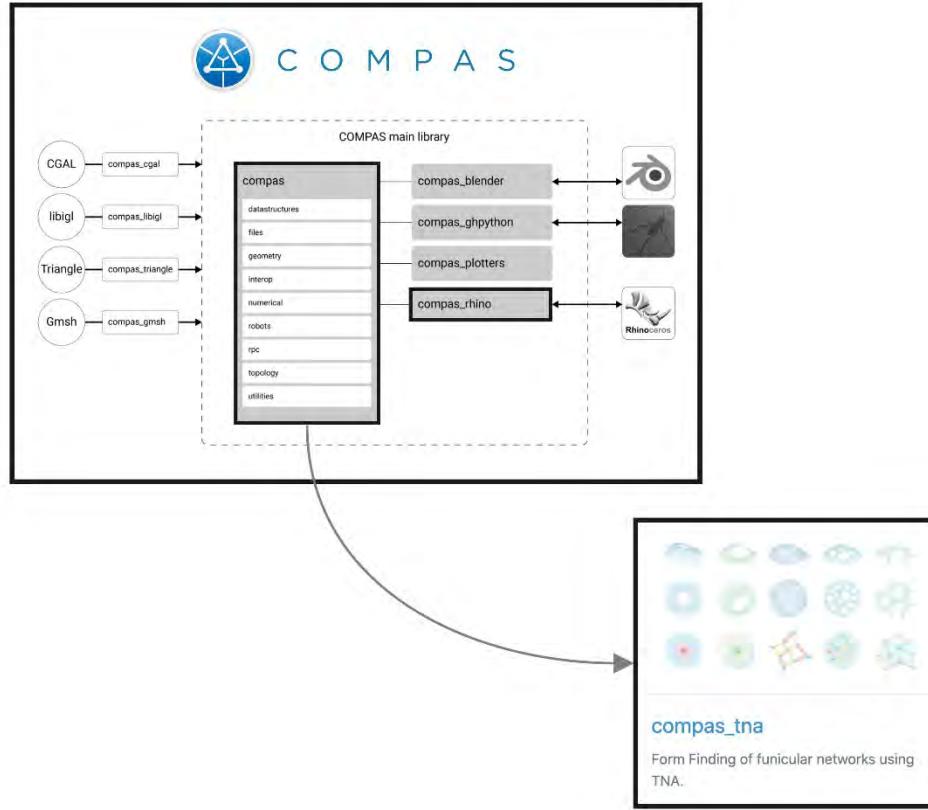


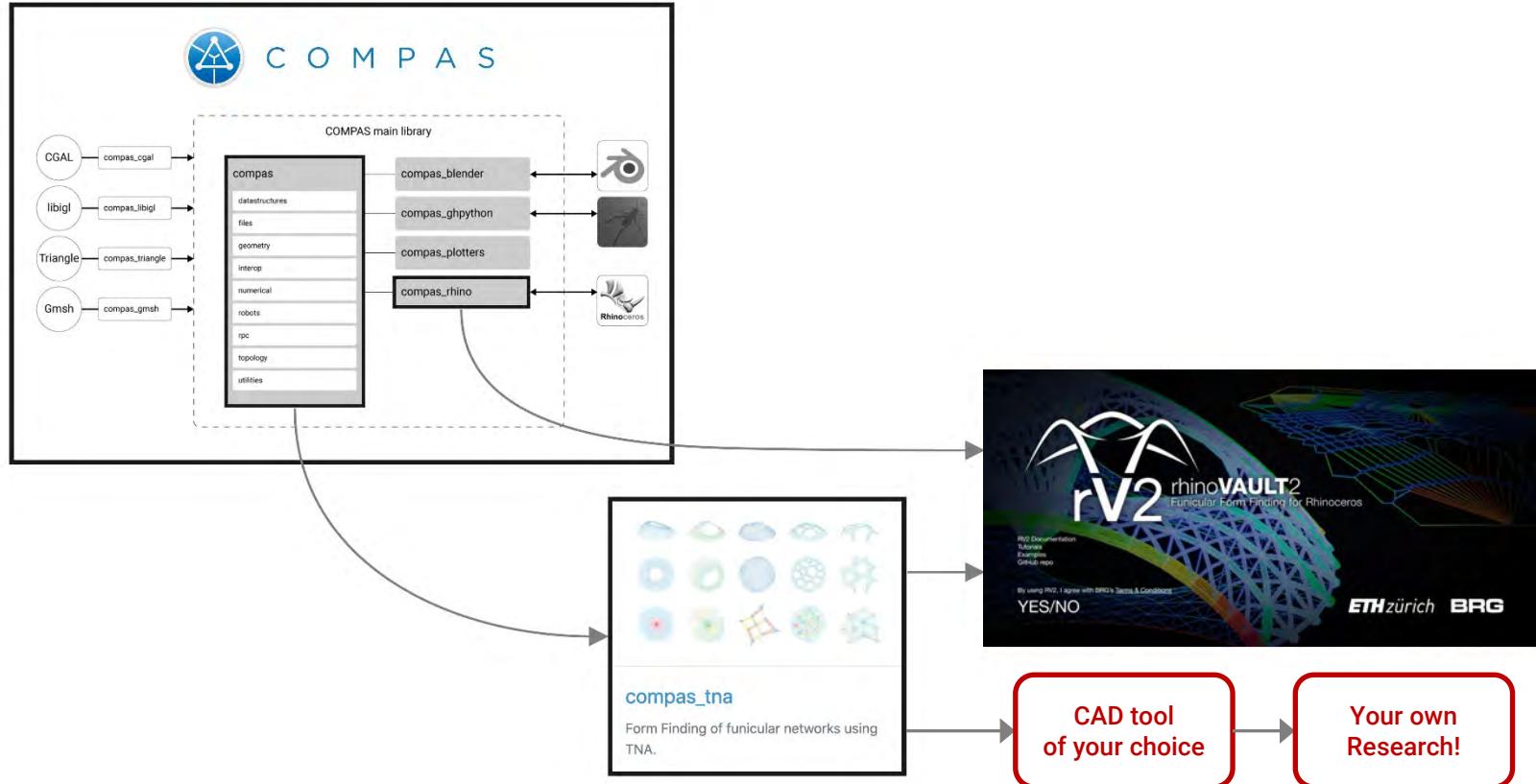
### force diagram data

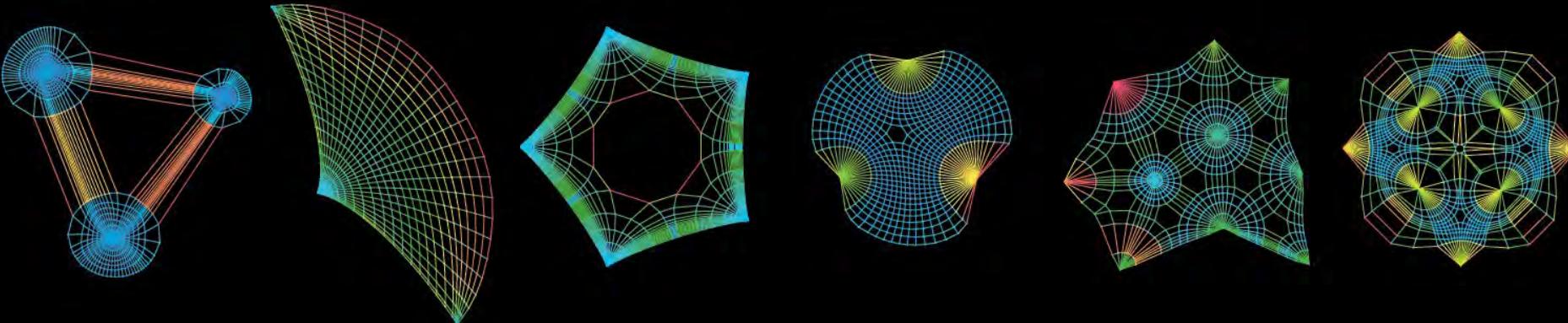
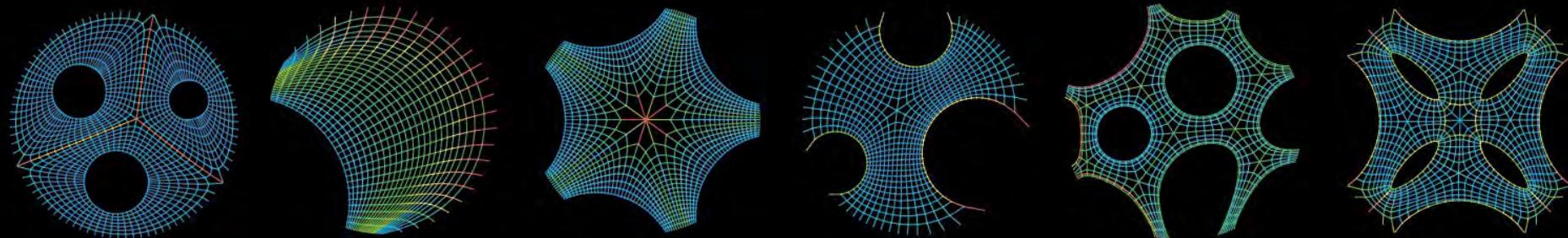
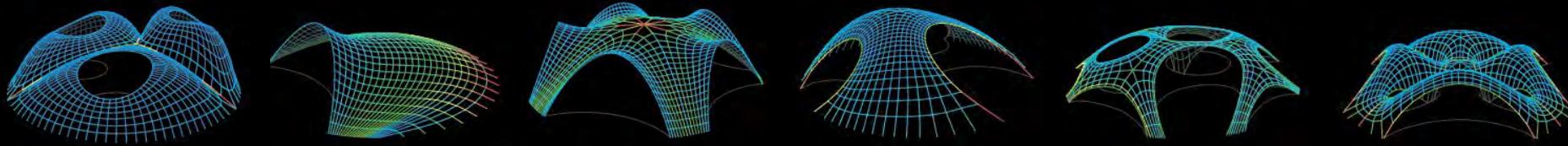
```
edge (u, v): {'_a': 0.0,  
               'lmin': 0.0,  
               'lmax': 1e+7}  
  
vertex: {'is_fixed': False,  
        'x': 0.0,  
        'y': 0.0,  
        'z': 0.0}
```











063-0605-00L : Computational Structural Design 1  
Computational Graphic Statics

<http://block.arch.ethz.ch>



**ETH** zürich

**DARCH**



**BRG**



compas dashboard

**v0.7.1**

+



RhinoVAULT 2

**V1.4.3**