Exploring the Cosmological Constant

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1 Introduction

Before Hubble's ground breaking discovery in 1929 where he identified a relationship between a Galaxy's' distance and its radial velocity, there was no reason to discredit a static universe. In fact around the when Albert Einstein first published his theory of General Relativity, the scientific society was not sure that galaxies outside our own existed. There appeared to be just as many stars and fuzzy patches moving towards us as there where moving away from us leading the community of the time to believe that we in fact lived in a static universe. The problem with this notion though is that Einstein's theory made such a universe science fiction.

In an effort to avoid the complexity of Einsteins field equations the concern is nicely illustrated through the use of Newtoniain theory. From Possion's equation for gravity.

$$\nabla^2\Phi = 4\pi G\rho \tag{1}$$

where Φ is the gravitational potential, G is the well known and appreciated gravitational constant, and rho is the mass density of the universe. Now noting that the force is the gradient of the potential implying that acceleration is proportional to the gradient ($\nabla\Phi = -\ddot{\vec{x}}$, here the acceleration is denoted by $\ddot{\vec{x}}$) we see that a static universe(i.e. $\dot{\vec{x}} = \ddot{\vec{x}} = 0$) only if,

Is there a source for this, or did you come up with this yourself? It's a somewhat forced extension of Lambda into Newtonian gravity...

$$\rho = \frac{1}{4\pi G}\nabla^2\Phi = 0$$

This is a universe completely void of matter [1], one merely needs to look up to see that thi the case!

However if one introduces a term, ad hoc, to equation 1 such that,

$$\nabla^2\Phi + \Lambda = 4\pi G\rho \tag{3}$$

a static universe is completely acceptable, since with this formulation it does not directly imply an empty one.

1.1 The Friedmann Equation

The Newtonian analysis is but an approximation to general relativity. In order to understand the full implications of the cosmological constant one must invoke the Friedmann equation, derived from the Einsteins field equations for gravity. With the inclusion of the cosmological constant it takes the following form

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{a^2(t)} + \frac{\Lambda}{3} \tag{4}$$

Here a is the scale factor, it represents the amount the universe has grown. Positions in the universe are defined by both a comoving coordinate and the scale factor ($r(t) = r_c a(t)$). We note

[1]From $E = mc^2$ also completely void of energy.

that all the time dependence is locked in a. The scale factor essentially encapsulates the dynamical history of the universe and predicts its future; it is normalized to be one now ($t = t_0$) and 0 at $t = 0$ for a big bang universe. k is the curvature constant which defines the geometrical topology of the universe. Table 1 introduces the relevance of this parameter. Λ is Einsteins Cosmological constant which, today, has been defined to be proportional to the density of Dark Energy in the universe. ρ is the density of the universe due to anything other than Λ.

k	**Physical Significance**
0	This is a flat or euclidian universe also known as an *Einstein de Sitter* universe. In such a universe the angles of a triangle add up to precisely 180°.
>0	This is commonly referred to as a closed universe. A common closed geometrical shape is a sphere. In such a universe the angles of a triangle add up to a value greater than 180°.
<0	This is commonly referred to as an open universe. A common open geometrical shape is the surface of a saddle. In such a universe the angles of a triangle add up to a value less than 180°.

Table 1: *A simple breakdown of the curvture constant*

It is essential to note that the examples referring to 2 dimensional surfaces are not directly applicable. The curvature is a measure of the full 3 dimensional space not some 2 dimensional subspace. The implications of the curvature constant along with the Cosmological constant will be further explored in the body of the text[2].

1.2 The Hubble Parameter

In 1927 Edwin Hubble declared to the world, through his publication, that all Galaxies not gravitationally influenced by our own or local group are moving away from us with some velocity proportional to their distance from us. The famous figure depicting this remarkable result is shown on figure **??**. From the plot the slope is approximately $500[\frac{km}{s \cdot Mpc}]$. The slope of this plot is referred to as the *Hubble Parameter*, $H(t)$, and it is time dependent. Hubble's preliminary measurement of his own constant was a grave overestimate due to biasing of galactic distances towards smaller distances.The accepted value of the Hubble parameter today is $H_0 \approx 72[\frac{km}{s \cdot Mpc}]$; we note the difference between $H(t)$ and H_0, where $H_0 = H(t_0) = H(t = today)$, here I hope it is apparent that t_0 denotes the time today or the time since the big bang. The nice thing about this parameter is that it is easily and directly measurable.

We mentioned that the Hubble Parameter is time dependent. In fact it is a measure of how quickly the universe is changing, this is also true of the time derivative of the scale factor and it is not difficult to show (see equation 5.4 in Liddle) that,

$$H(t) = \frac{\dot{a}(t)}{a(t)} \tag{5}$$

This relation becomes useful when trying to simplify the Friedmann equation and express it in totality by observable quantities.

[2] We will be neglecting contributions from matter and readiation to the total density ρ.

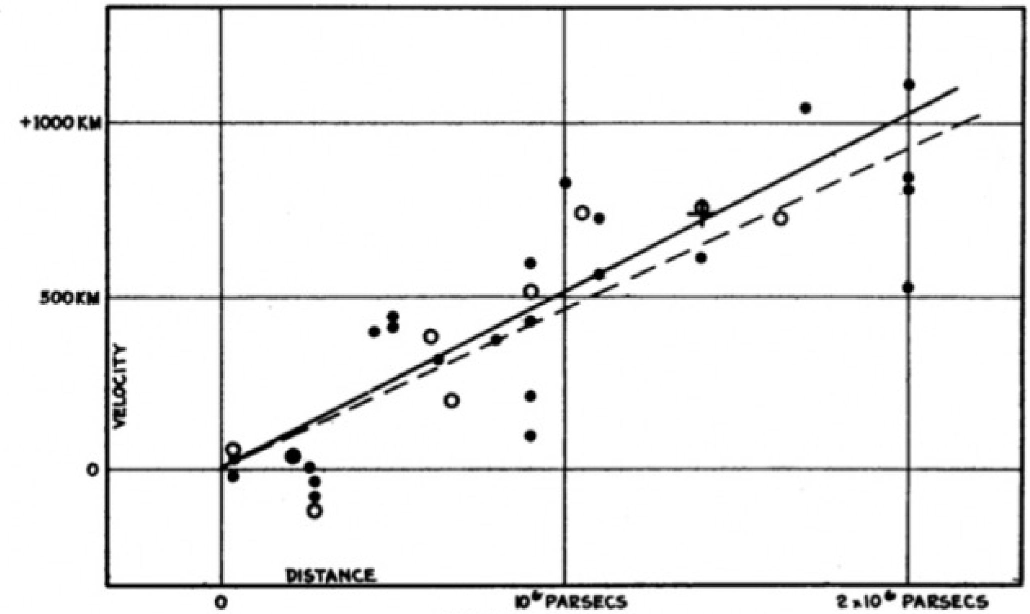

FIGURE 1

Velocity-Distance Relation among Extra-Galactic Nebulae.

Figure 1: *Hubbles origional plot depicting the relation between a galaxies resesional velocity and its distance from us. One notes that he has mislable his y axis to have units of* $[km]$ *when they should have units of veocity not distance i.e.* $[km/s]$

2 Simplifying Friedmann's Equation

The Friedmann equation (eq. 4) takes a particularly useful form when one considers the **Density** which is the density for a flat (i.e. $k = 0$). Substituting zero for k,

$$H^2(t) = \left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{\Lambda}{3} \tag{6}$$

$$\text{Defining: } \rho_\Lambda = \frac{\Lambda}{8\pi G} \quad \& \quad \rho_c = \rho_\Lambda + \rho \tag{7}$$

$$\text{We see that:} \qquad \rho_c = \frac{3H^2}{8\pi G} \tag{8}$$

ρ_c is the *Critical Density*.

With the critical density in hand we can proceed to express the Friedmann equation in terms of dimensionless parameters. Because we are only concerned with the cosmological parameter (Λ)

and curvature we may express the Friedmann equation as,

$$H^2(t) = \frac{8\pi G}{3}\rho_\Lambda - \frac{kc^2}{a^2(t)}, \quad \text{Dividing both sides by } \rho_c \tag{9}$$

$$\Longrightarrow 1 = \frac{\rho_\Lambda}{\rho_c} - \frac{kc^2}{a^2(t)\rho_c}, \quad \text{Making the following definitions,} \tag{10}$$

$$\Omega_\Lambda(t) = \frac{\rho_\Lambda}{\rho_c} = \rho_\Lambda \frac{8\pi G}{3H^2} \quad \& \quad \Omega_k(t) = -\frac{kc^2}{a^2\rho_c} = -\frac{kc^2}{a^2}\frac{8\pi G}{3H^2} \tag{11}$$

We find that the Friedmann equation takes the following form,

$$1 = \Omega_\Lambda(t) + \Omega_k(t) \tag{12}$$

There are a few important points to note about the Friedmann equation in this form.

- The Ω's are dimensionless.
- Although ρ_Λ is a constant (i.e. independent of time) $\Omega_\Lambda(t)$ is not. This is because the critical density, ρ_c, is dependent on time.
- The sum of the Ω's is always 1, FOR ALL TIME!

It would behove us to express the Friedmann equation in terms of quantities we can observe to today (i.e. $t = t_0$). First noting that $H_0 = H(t_0)$ and $a(t_0) = 1$, It is not difficult to see from equation 11 that if we multiply the top and bottom of both Ω_Λ and Ω_k by H_0 we arrive to the conclusion that,

$$\Omega_\Lambda = \Omega_\Lambda(t_0)\frac{H_0^2}{H^2} = \Omega_{\Lambda,0}\frac{H_0^2}{H^2} \tag{13}$$

$$\Omega_k = \frac{\Omega_{k,0}}{a^2}\frac{H_0}{H^2} \tag{14}$$

Further by exploiting the third bullet point above we see that $\Omega_{k,0} = 1 - \Omega_{\Lambda,0}$, allowing us to express the Friedmann equation (eq 2) as,

$$H^2(t) = H_0^2\left[\Omega_{\Lambda,0} + \frac{1}{a^2(t)}(1 - \Omega_{\Lambda,0})\right] \tag{15}$$

3 The Curvature Constant

In section 1.1 we introduced the curvature constant k and attempted to provide some conceptional backing in table 1. As a constant it is time independent and its sign indicates the type of universe we live in. By reinserting Ω_k into equation 15 we can solve for k explicitly arriving to,

$$k = \frac{3a^2}{8\pi Gc^2}(\Omega_{\Lambda,0}H_0^2 - H^2) \tag{16}$$

Because k must remain a constant we can evaluate the expression above for $t = t_0$ to arrive to the value of k for all of time.

$$k = \frac{3H_0^2}{8\pi Gc^2}(\Omega_{\Lambda,0} - 1) \tag{17}$$

$3H_0^2/(8\pi Gc^2)$ must be positive, all the information regarding the sign of k is held in the second term within the parentheses. The following table, table 2, outlines the relationship between $\Omega_{\Lambda,0}$ and k.

$\Omega_{\Lambda,0}$	k	Type of Universe
$= 1$	0	Flat
> 1	>0	Closed
< 1	<0	Open

Table 2: *Describing the relationship between $\Omega_{\Lambda,0}$ and k. Equation 17 mathematically outlines this relationship.*

4 The Scale Factor

The scale factor is truly central to cosmology since it describes how the distance between different points in space evolves given the other parameters in Friedmann equation. Of particular interest is whether our universe (i.e. our Λ and k universe) will expand forever, eventually contract and return to a singularity, or contract to later expand. These possible outcomes all depend on particular combinations of the density parameters and initial conditions. For the universes that change direction, it is of particular interest to identify the value of the scale factor at the **turn-around point** $t = t_{turn}$.

At t_{turn}, $\dot{a}$ must be zero, i.e. $\dot{a}(t_{turn}) = 0$. From equation 15 we see that this occurs when,

$$a(t_{turn}) = a_{turn} = \sqrt{1 - \frac{1}{\Omega_{\Lambda,0}}} \tag{18}$$

One should note that for $0 < \Omega_{\Lambda,0} < 1$, a_{turn} is imaginary and thus there is no turn-around point.To further investigate this turn-around point we must consider the second derivative of a or in other words the acceleration equation. The second derivative is an indicator of of the concavity of a and will shed light on the evolution of the scale factor after the turn-around point, i.e. decreasing, increasing, or constant. To this end we can exploit the acceleration equation (see Liddle equation 3.18). For a pure lambda universe it takes the following form,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_\Lambda + \frac{3p}{c^2}\right), \quad \text{For this universe } p = p_\Lambda = -\rho_\Lambda c^2 \text{ (Liddle eq 7.9)} \tag{19}$$

$$\frac{\ddot{a}}{a} = \frac{8\pi G}{3}\rho_\Lambda = \Omega_{\Lambda,0} H_0^2 \tag{20}$$

Because $a > 0$ for all time and so is H_0^2 all the sign dependence dependence is in $\Omega_{\Lambda,0}$, who,from equation 11 depends, in turn depends on the sign of ρ_Λ. The following table, table 3, illustrates the regimes of interest.

$\Omega_{\Lambda,0}$	a_{turn}	Concavity
> 0 & < 1	Imaginary, No turn around	N/A
> 1	Real	up
< 0	Real	Down

Table 3: *Describing the relationship between $\Omega_{\Lambda,0}$, a_{turn}, and the concavity of $a(t)$.*

You need more extended discussion here of what's going on... specifically, you didn't say anything about whether and when each universe will contract... all of this can be deduced without fully solving the Friedmann equatiom

4.1 Solutions For the Scale Factor $a(t)$

Equation 2 is a first order linear differential equation for $a(t)$. To make this clear it may be written in the following way,

$$\dot{a} = \frac{da}{dt} = H_0\sqrt{a^2\Omega_{\Lambda,0} + (1-\Omega_{\Lambda,0})} \tag{21}$$

or

$$\frac{da}{\sqrt{a^2\Omega_{\Lambda,0} + (1-\Omega_{\Lambda,0})}} = H_0 dt \tag{22}$$

For $\Omega_{\Lambda,0} \neq 0$ the solution for $a(t)$ takes the form,

$$a(t) = \frac{1}{2\Omega_{\Lambda,0}} e^{-\sqrt{\Omega_{\Lambda,0}}(H_0t+C)} \left(e^{2\sqrt{\Omega_{\Lambda,0}}(H_0t+C)} + \Omega_{\Lambda,0}(\Omega_{\Lambda,0}-1)\right) \tag{23}$$

The constant C is defined by the boundary condition of interest. When $\Omega_{\Lambda,0}$ is zero we are dealing with a pure curvature universe and must rewrite equation 21 by replacing $\Omega_{\Lambda,0}$ with $\Omega_{k,0}$. The differential equation becomes,

$$\dot{a} = H_0\sqrt{\Omega_{k,0}} \tag{24}$$

Which is easily solved,

$$a(t) = H_0\sqrt{\Omega_{k,0}}t + C \qquad \textbf{for} \quad \Omega_{\Lambda,0} = 0 \tag{25}$$

4.1.1 Big Bang Universe

For a big bang universe the boundary condition of interest is that a be zero when t is zero. Enforcing this condition on equation 23,

$$0 = \frac{1}{2\Omega_{\Lambda,0}} e^{-\sqrt{\Omega_{\Lambda,0}}C} \left(e^{2\sqrt{\Omega_{\Lambda,0}}C} + \Omega_{\Lambda,0}(\Omega_{\Lambda,0}-1)\right) \tag{26}$$

$$C = \frac{1}{2\sqrt{\Omega_{\Lambda,0}}} \ln\left(\Omega_{\Lambda,0}(1-\Omega_{\Lambda,0})\right) \tag{27}$$

Plugging C back into eq 23,

$$a(t) = \frac{1}{2\Omega_{\Lambda,0}\sqrt{\Omega_{\Lambda}(1-\Omega_{\Lambda,0})}} e^{-\sqrt{\Omega_{\Lambda,0}}H_0t} \left(e^{2\sqrt{\Omega_{\Lambda,0}}H_0t} + \Omega_{\Lambda,0}(\Omega_{\Lambda,0}-1)\right) \tag{28}$$

$$= \Omega_+ e^{\sqrt{\Omega_{\Lambda,0}}H_0t} + \Omega_- e^{-\sqrt{\Omega_{\Lambda,0}}H_0t} \tag{29}$$

Where,

$$\Omega_+ = \frac{1}{2\Omega_{\Lambda,0}\sqrt{\Omega_{\Lambda}(1-\Omega_{\Lambda,0})}} \quad \& \quad \Omega_- = -\frac{\sqrt{\Omega_{\Lambda}(1-\Omega_{\Lambda,0})}}{2\Omega_{\Lambda,0}} \tag{30}$$

This has the from of a hyperbolic function if $1 > \Omega_{\Lambda,0} > 0$, since in this regime the quantities within the square roots remain positive and real. One can note that in this case the a diverges as t goes to infinity; implying that the universe has not true limit or limiting age. With some algebraic manipulation (recall Euler's formula) $a(t)$ becomes,

$$a(t) = \sqrt{\frac{1-\Omega_{\Lambda,0}}{\Omega_{\Lambda,0}}} \sinh\left(H_0\sqrt{\Omega_{\Lambda,0}}t\right) \qquad \textbf{for} \qquad 0 < \Omega_{\Lambda,0} < 1 \tag{31}$$

If on the other hand $\Omega_{\Lambda,0} < 0$ or > 1, everywhere $\sqrt{\Omega_{\Lambda,0}}$ appears in the previous equation, we may write, without any loss of generality, $i\sqrt{|\Omega_{\Lambda,0}|}$ such that,

$$a(t) = \Omega_+ e^{i\sqrt{|\Omega_{\Lambda,0}|}H_0 t} + \Omega_- e^{-i\sqrt{|\Omega_{\Lambda,0}|}H_0 t} \tag{32}$$

$$= \sqrt{\frac{1+|\Omega_{\Lambda,0}|}{|\Omega_{\Lambda,0}|}} \sin\left(H_0\sqrt{|\Omega_{\Lambda,0}|t}\right) \qquad \textbf{for} \quad \Omega_{\Lambda,0} < 0 \quad \& \quad \Omega_{\Lambda,0}{>}1 \tag{33}$$

Which changes the solution from a hyperbolic one to an oscillating trigonometric one. Thus the universe is bound to recollapse to $a = 0$ within some finite time, t_P. From equation 32 it is evident that the angular period is $2\pi/(\sqrt{|\Omega_{\Lambda,0}|}H_0)$ meaning that the next time a is zero is half that, $t_P = \pi/(\sqrt{|\Omega_{\Lambda,0}|}H_0)$. Because $a > 0$ for all time the concavity of this solution must be down, in other word a reaches a maximum. Any values of $\Omega_{\Lambda,0}$ that make it such that this is not the case are unphysical.

Figure 2, depicts the evolution of a as function of both $\Omega_{\Lambda,0}$ and t in the form of three dimensional plot.s In particular one should notice that figure (b) clearly shows the linear behavior of a as $\Omega_{\Lambda,0}$ approaches 0 as predicted by equation 25 although the solution is oscillatory. One can see this by taylor expanding the solution and keeping the leading term, from equation 33

$$a(t) \approx \sqrt{1+|\Omega_{\Lambda,0}|}H_0 t = \sqrt{1-\Omega_{\Lambda,0}}H_0 t = \sqrt{\Omega_{k,0}}H_0 t \qquad \text{as} \qquad \Omega_{\Lambda,0} \to 0^- \tag{34}$$

Exactly[3] as equation 33.

It is important to note that although I have plotted (b) and (c) out to large t. these universes are of finite age due to the restriction that a must always be greater than 0. Thus for any particular configuration a starts at 0 and some time later reaches 0 again indicating the end.

4.1.2 No Big Bang

For universes without a Big Bang the only boundary condition available to us for determining the value of C is the normalization of the scale factor. Recall that $a(t_0) = 1$. Enforcing that condition,

$$1 = \frac{1}{2\Omega_{\Lambda,0}} e^{-\sqrt{\Omega_{\Lambda,0}}(H_0 t_0 + C)} \left(e^{2\sqrt{\Omega_{\Lambda,0}}(H_0 t_0 + C)} + \Omega_{\Lambda,0}(\Omega_{\Lambda,0} - 1)\right) \tag{35}$$

$$C_\pm = -H_0 t_0 + \frac{\ln\left(\Omega_{\Lambda,0} \pm \sqrt{\Omega_{\Lambda,0}}\right)}{\sqrt{\Omega_{\Lambda,0}}} \tag{36}$$

Inserting these values for C,

$$a(t) = \frac{e^{-H_0\sqrt{\Omega_{\Lambda,0}}(t-t_0)}}{2\sqrt{\Omega_{\Lambda,0}}} \left[e^{2H_0\sqrt{\Omega_{\Lambda,0}}(t-t_0)}\left(\sqrt{\Omega_{\Lambda,0}} \pm 1\right) + \sqrt{\Omega_{\Lambda,0}} \mp 1\right] \tag{37}$$

$$= \Omega_\pm e^{H_0\sqrt{\Omega_{\Lambda,0}}(t-t_0)} + \Omega_\mp e^{-H_0\sqrt{\Omega_{\Lambda,0}}(t-t_0)} \tag{38}$$

[3] Approcahing 0 from the left is of unpmost importance scice that is what allowed us to chage $+|\Omega_{\Lambda,0}|$ to $-\Omega_{\Lambda,0}$.

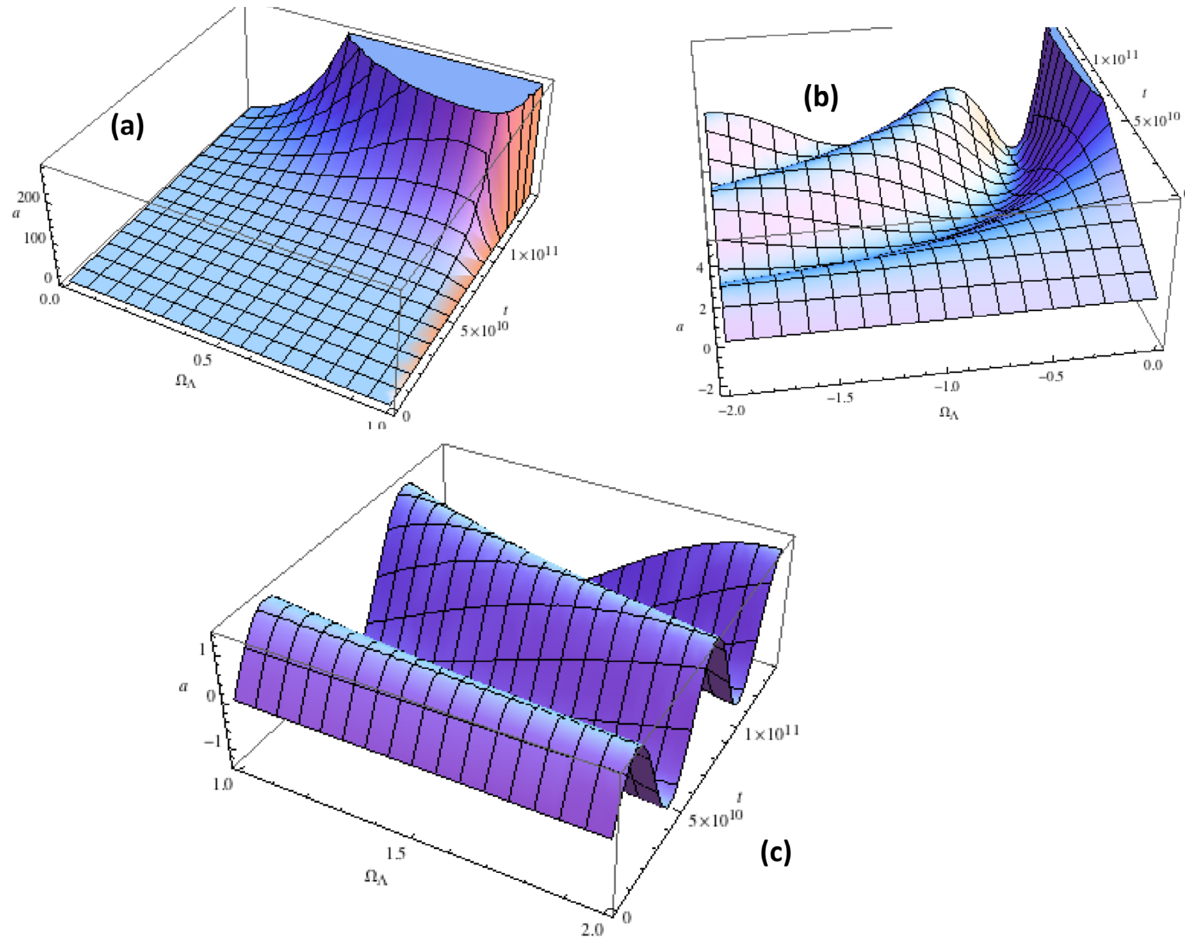

Figure 2: *The surfaces depict the behavior of the scale factor as a function of both* t *and* $\Omega_{\Lambda,0}$. *(a), (b), (c) are representative of the two solution, presented in equations 31 and* **??** *for the 3 regions of interest:* $0 < \Omega_{\Lambda,0} < 1$, $\Omega_{\Lambda,0} < 0$, *and* $1 < \Omega_{\Lambda,0}$ *respectively.*

Where we have defined,

$$\Omega_\pm = \frac{\sqrt{\Omega_{\Lambda,0}} \pm 1}{2\sqrt{\Omega_{\Lambda,0}}} \quad \& \quad \Omega_\mp = \frac{\sqrt{\Omega_{\Lambda,0}} \mp 1}{2\sqrt{\Omega_{\Lambda,0}}} \tag{39}$$

For $\Omega_{\Lambda,0} < 0$ the solution is oscillatory and similar to the discussion of equation 32,

$$a(t) = \Omega_\pm e^{iH_0\sqrt{|\Omega_{\Lambda,0}|}(t-t_0)} + \Omega_\mp e^{-iH_0\sqrt{|\Omega_{\Lambda,0}|}(t-t_0)} \qquad \textbf{for} \quad \Omega_{\Lambda,0} < 0 \tag{40}$$

Note that we must also substitute $i\sqrt{|\Omega_{\Lambda,0}|}$ for $\sqrt{\Omega_{\Lambda,0}}$ in $\Omega_\pm$ and $\Omega_\mp$.

As in section 4.1.1 figure 3 depicts the relationship between a and both t and $\Omega_{\Lambda,0}$. Note that t_0 and H_0 have both been normalized to one and we have only chosen to depict the second solution of equations 38 and 40 (i.e. $\Omega_{\pm} \rightarrow \Omega_{-}$ and $\Omega_{\mp} \rightarrow \Omega_{+}$). One can see from both figure (a) that under particular conditions one can generate a bouncing universe, one that reaches a minimum then continues to expand forever. Figure (b) is very similar to figure 2 (c) except that it starts at a maximum or close to a maximum then plummets down until it reaches zero. Figure (c) is a cross-section of figure (a), I chose $\Omega_{\Lambda,0} = 2$ in order to depict an example of a "Bouncing Universe."

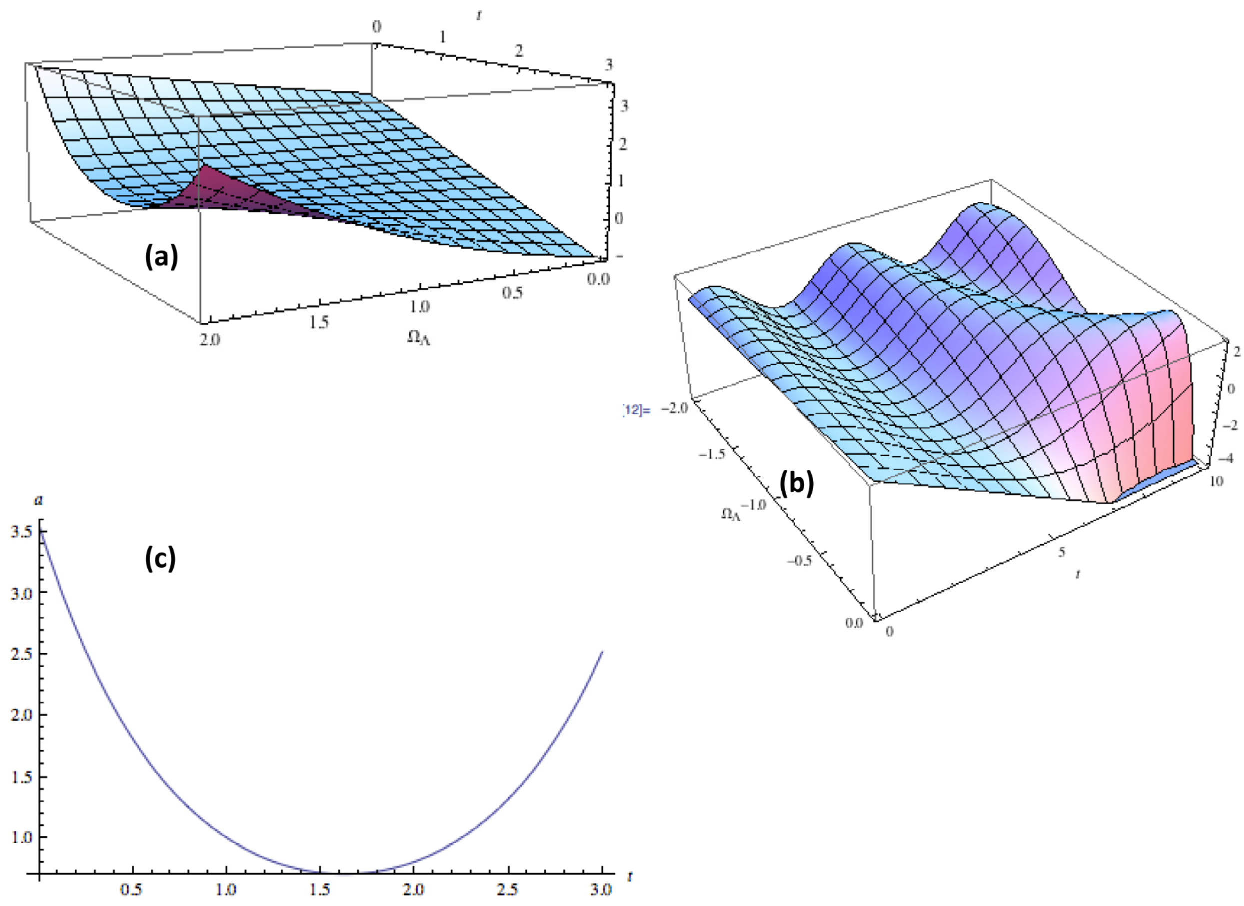

Figure 3: *The surfaces depict the behavior of the scale factor as a function of both t and $\Omega_{\Lambda,0}$. Note that t_0 and H_0 have both been normalized to one and we have only chosen to depict the second solution of equations 38 and 40 (i.e. $\Omega_{\pm} \rightarrow \Omega_{-}$ and $\Omega_{\mp} \rightarrow \Omega_{+}$). (a) and (c) correspond to solution 38, (b) to the other. Figure (c) is a cross-section of figure (a), I chose $\Omega_{\Lambda,0} = 2$ in order to depict an example of a "Bouncing Universe."*

A Constant Ω_Λ

We mentioned in section 2 that although ρ_Λ is constant Ω_Λ is not because the critical density, ρ_c (see equation 8), is time dependent. Yet there is a value of $\Omega_{\Lambda,0}$ that causes Ω_Λ to remain constant.

By definition in order for a quantity to remain constant with respect to time its time derivative must be equal to zero for all time. With this in mind,

$$\Omega_\Lambda = \Omega_{\Lambda,0}\frac{H_0^2}{H^2}, \quad \text{one can show:} \tag{41}$$

$$\dot{\Omega}_\Lambda = \frac{2H_0^2}{H^3 a}(H^2 a - \ddot{a}) \tag{42}$$

Thus if we are interested in a static Ω_Λ we must enforce $H^2 = \ddot{a}/a$. Noting that we are particularly interested in the case where $\Omega_\Lambda = \Omega_{\Lambda,0}$. From eq 41, we see that this happens when $H^2 = H_0^2$. From the condition of interest this implies that $\ddot{a}/a = H^2 = H_0^2$, This relation is only satisfied if $H_0 = 0$ and $a =$ constant, meaning that the universe is and always has been static.

We also come to the conclusion,through the use of equation 20 ,that $\Omega_{\Lambda,0} = 1$. This also implies that the universe must be flat, $k = 0$. This can be seen from table 2 and equation 2, with $\Omega_\Lambda = \Omega_{\Lambda,0} = 1$, Ω_k must equal 0.

B The Age of the Universe

Because the scale factor has been normalized to be one "today" (t_0), in accordance to cosmological formalism which we have followed throughout, for universes with a big bang it is relatively straight forward to compute this time. One merely needs to invert equations 31 and 33 to find,

$$t_0 = \frac{1}{H_0\sqrt{\Omega_{\Lambda,0}}}\sinh^{-1}\left(\sqrt{\frac{\Omega_{\Lambda,0}}{1-\Omega_{\Lambda,0}}}\right) \quad \textbf{for} \quad 0 < \Omega_{\Lambda,0} < 1 \tag{43}$$

$$t_0 = \frac{1}{H_0\sqrt{|\Omega_{\Lambda,0}|}}\sin^{-1}\left(\sqrt{\frac{|\Omega_{\Lambda,0}|}{1+|\Omega_{\Lambda,0}|}}\right) \quad \textbf{for} \quad \Omega_{\Lambda,0} < 0 \quad \& \quad \Omega_{\Lambda,0} > 1 \tag{44}$$

t_0 vs $\Omega_{\Lambda,0}$ for both cases is plotted on figure 4. The discontinuity at $\Omega_{\Lambda,0} = 1$ can be explained by the fact that as $\Omega_\Lambda \to 1$ the universe becomes static (see appendix A) and thus the age infinite; t_0 drops of quickly as $|\Omega_{\Lambda,0}| \to \infty$. A large value of the cosmological constant indicates a large magnitude of acceleration that will reduce the time necessary to observe the values of current parameters such as H_0

For universes without a Big Bang, the word "age" no longer has a clear meaning. What marks the "beginning"? t_0 is now only conventionally relevant as the normalization of the scale factor "today." It might be suitable to reference time to the local minimum of $a(t)$ but this would mean that if that local minimum if in the future we would need to speak in terms of negative time, or even a count down towards the end of the universe.

C The Time Evolution of Ω_Λ

From equation 13 we see that Ω_Λ depends on the scale factor in the following way,

$$\Omega_\Lambda = \Omega_{\Lambda,0}\frac{a^2(t)H_0^2}{\dot{a}^2} \tag{45}$$

There is an analytic expression that can be derived here.

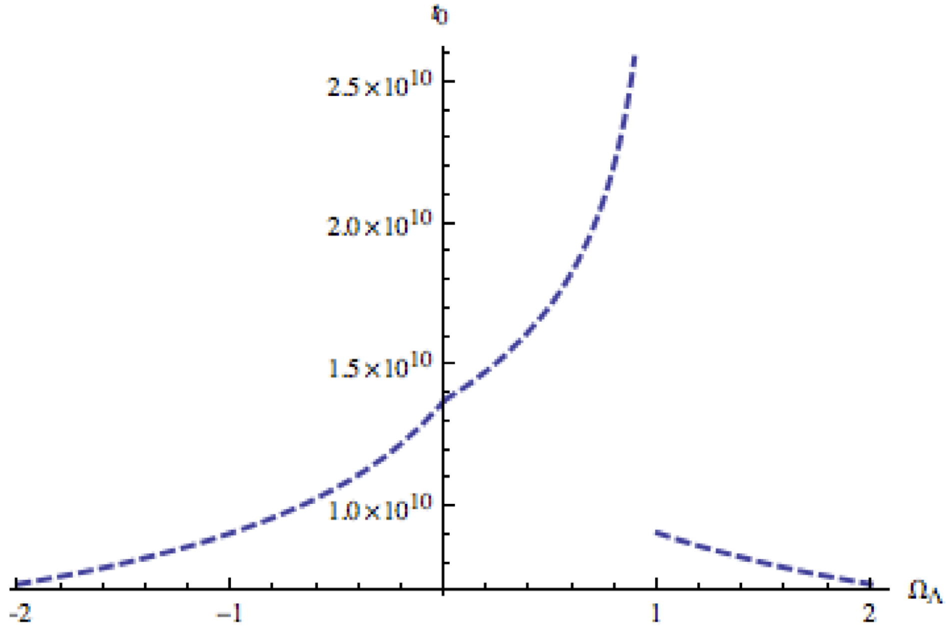

Figure 4: *t_0 vs $\Omega_{\Lambda,0}$ for both positive and negative values of $\Omega_{\Lambda,0}$. Note that the value of $H_0 = 72[km/s \cdot Mpc]$ was used.*

Given that we have, at hand, solutions for $a(t)$ we may directly compute Ω_Λ. On figure 5 (a), (b), and (c) I took into consideration solutions presented in equations 31 and **??** for the 3 regions of interest: $0 < \Omega_{\Lambda,0} < 1$, $\Omega_{\Lambda,0} < 0$, and $1 < \Omega_{\Lambda,0}$ respectively.

From figure (a) it appears that fore the regime $0 < \Omega_{\Lambda,0} < 1$, Ω_Λ is not very dependent on $\Omega_{\Lambda,0}$. It monotonically grows with t. For the other two regimes, displayed in (b) and (c), Ω_Λ displays oscillatory behavior, and it appears that those regimes are inverses of each other. The behavior in time for these two regimes with respect to time appears to be of $\sec(t)$ form. The dependence on $\Omega_{\Lambda,0}$ again appears to be weak

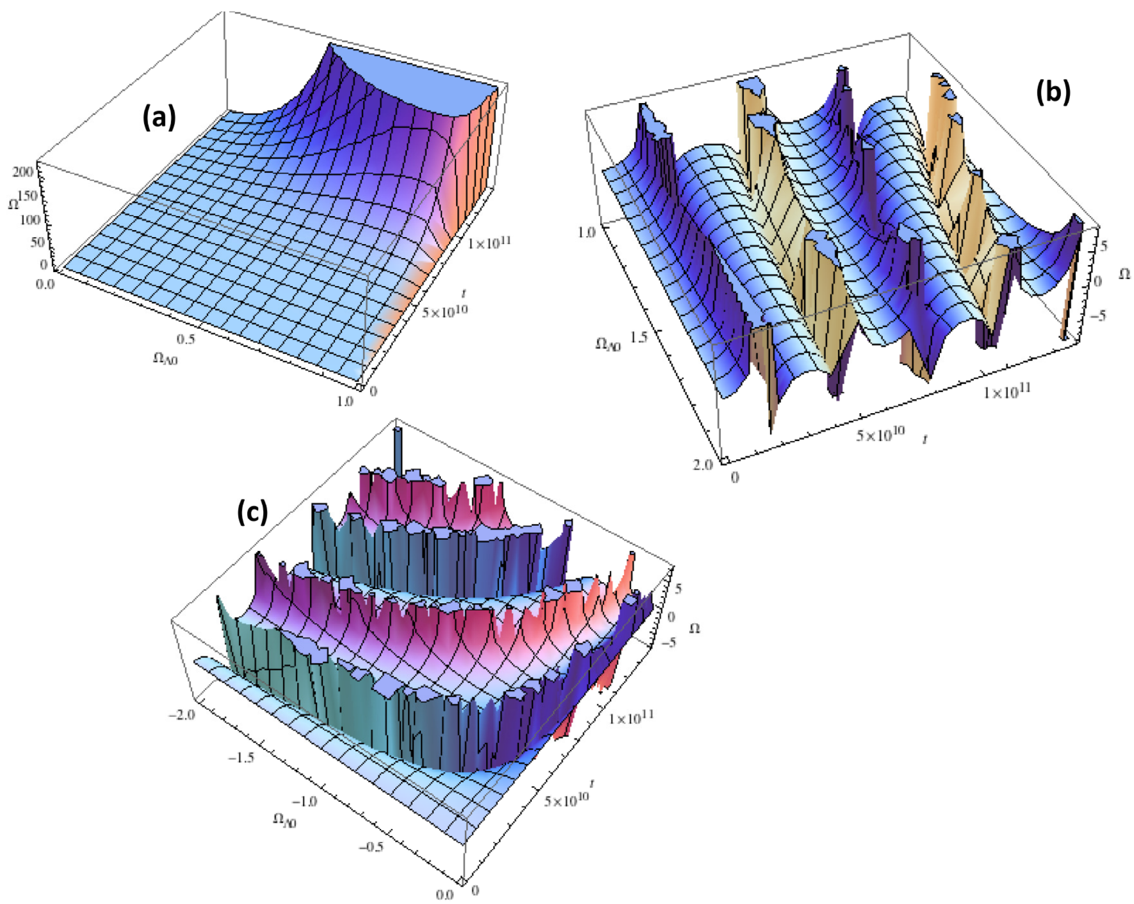

Figure 5: *For (a), (b), and (c) I took into consideration solutions presented in equations 31 and* **??** *for the 3 regions of interest:* $0 < \Omega_{\Lambda,0} < 1$, $\Omega_{\Lambda,0} < 0$, *and* $1 < \Omega_{\Lambda,0}$ *respectively.*

Nicely done and correct for the most part.
Address the comments for a higher grade. 93/100