

Lane-Emden and its IVP

ARTHUR ADAMS, ADAM FRIES, AND ALLAN GAMBOA

*San Francisco State University
Department of Physics and Astronomy*

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Abstract

Solving the Lane-Emden equation can result in an analytic solution for only a few cases of the polytropic index, n . In this paper, we first focus on finding the power series solution for $n = 1$ using the Frobenius method as well as finding the first three terms for an arbitrary index. Finally, we find a numerical solution using the 4th-order Runge-Kutta method for $n = 2$.

Introduction

The description of stellar structure requires four coupled differential equations of pressure, luminosity, density, and mass as functions of radius. Solving these equations requires four sets of initial conditions and at least some knowledge of the properties of the stellar atmosphere [?]. Instead, we consider an approximate model by combining hydrostatic equilibrium with conservation of mass. Here, the outward radiation pressure generated by the star is equally balanced by the gravitational force from the star's mass (??), and the change in mass of the star is dependent on the star's density as a function of radius (??).

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \quad (1)$$

$$\frac{dm}{dr} = 4\pi r^2 \rho(r) \quad (2)$$

After multiplying both sides of (??) by r^2/ρ and differentiating with respect to r , we obtain

$$\frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -G \frac{dm}{dr} = -4\pi r^2 G \rho(r). \quad (3)$$

We now have a second-order differential equation with two unknowns, pressure and density. We can assume a relationship between ρ and P such that

$$P = K\rho^{1+n^{-1}}, \quad \text{where } K = \text{constant}, \quad (4)$$

which is the polytropic equation of state. We then carry out the differentiation after substituting (??) into (??) and obtain a differential equation of a single variable ρ ,

$$\frac{K(n+1)}{4\pi r^2 G n} \frac{d}{dr} \left[r^2 \rho(r)^{(1-n)/n} \frac{d\rho(r)}{dr} \right] = -\rho(r) \quad (5)$$

Here, $\rho(r)$ in (??) is called a polytrope and requires two boundary conditions: $\rho(r) = 0$ at the surface of the star where $r = R$, and $d\rho(r)/dr = 0$ at the boundary $r = 0$ and assumes a constant density at the core of the star [?]. To simplify further, we recast (??) into a dimensionless form by first defining the dimensionless variable ϕ with range $0 \leq \phi \leq 1$ [?]. In doing so we get

$$\rho = \lambda \phi^n, \quad \text{where } \lambda = \text{central density} \quad (6)$$

and

$$\left[\frac{(n+1)K}{4\pi G \lambda^{1-n-1}} \right] \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = -\phi^n. \quad (7)$$

The item in the square brackets of (??) has dimension of length squared. In order to replace r in (??), we relate the constant in square brackets to r by the dimensionless distance variable x such that

$$r = \left[\frac{(n+1)K}{4\pi G \lambda^{1-n-1}} \right]^{1/2} x, \quad (8)$$

and arrive at the Lane-Emden equation

$$\frac{1}{x^2} \frac{d}{dx} \left(x^2 \frac{d\phi}{dx} \right) + \phi^n = 0 \quad (9)$$

with $x = 0$ at the center of the star [?].

Applying the Chain Rule for derivatives, we can express Equation (??) in a quasi-standard form:

$$\frac{d^2\phi}{dx^2} + \frac{2}{x} \frac{d\phi}{dx} + \phi^n = 0. \quad (10)$$

Analytic Solution for $n = 1$

The Lane-Emden equation (??) is of non-linear form and thus cannot be solved in closed form. It is also singular at the origin, a quality that must be considered in the search for a solution. When presented with such a differential equation, a viable option is to represent the solution as a Laurent series, especially if interested in a solution about the singular point. To further generalize the possibility that the singular point may not be isolated one can allow for non-integer powers in the series, in doing so, undertaking the *Frobenius method* (see [?]) and assuming a solution of the form,

$$\phi = (x - x_0)^p \sum_{k=0}^{\infty} a_k (x - x_0)^k \quad (11)$$

where p may be any number.

The problem reduces to one of finding the appropriate coefficients a_k .

In our case, the singular point we are expanding about is $x_0 = 0$. Differentiating our solution,

$$\frac{d\phi}{dx} = \sum_{k=0}^{\infty} a_n(k+p)x^{k+p-1} \quad (12)$$

$$\frac{d^2\phi}{d^2x} = \sum_{k=0}^{\infty} a_k(k+p)(k+p-1)x^{k+p-2} \quad (13)$$

If we are to proceed any further towards an analytical expression, something must be done about the fact that ϕ is raised to the n^{th} power. For our purposes here we will undertake the special case of $n = 1$. With this consideration in place we plug equations ((??)-(??)) into our differential equation ((??)),

$$\begin{aligned} 2 \sum_{k=0}^{\infty} a_k(k+p)x^{k+p-2} + \sum_{k=0}^{\infty} a_k(k+p)(k+p-1)x^{k+p-2} \\ + \sum_{k=0}^{\infty} a_k x^{k+p} = 0 \end{aligned} \quad (14)$$

We realize that the lowest power that appears in this equation, setting $k = 0$ is x^{p-2} . The coefficients corresponding to these power provide us with the following relation.

$$\begin{aligned} 2pa_k + a_k(p-1) &= 0 \\ \text{Ignoring the trivial solution } a_k &= 0 \\ p(p+1) &= 0. \end{aligned} \quad (15)$$

This leads us to the conclusion that the appropriate p values for this differential equation are $p = 0$ and $p = -1$. Yet not both values are physical. Re-examining the transformation ((?)), we see that ϕ directly related to the density of the star. If we are to include the $p = -1$ solution the first term in our series would be,

$$\phi(x)|_{k=0} = a_0 \frac{1}{x}$$

This term diverges at the origin, implying that the density diverges at the origin, leading to an unphysical scenario¹. We thus neglect the $p = -1$ solution.

Setting $p = 0$ and finding an expression for the same power of x , we arrive to,

$$2a_{k+2}(k+2) + a_{k+2}(k+2)(k+1) + a_k = 0 \quad (16)$$

Solving for a_{m+1} and searching for a trend

¹All other terms except, $n = 1$ which is a constant, are zero at $x = 0$. None can possibly compensate.

$$\begin{aligned}
a_{k+2} &= \frac{-1}{(k+2)(k+3)} \cdot a_k \\
&= \frac{-1}{(k+2)(k+3)} \frac{-1}{(k)(k+1)} \cdot a_{k-2} \\
&= \frac{(-1)(-1)}{(k+3)(k+2)(k+1)(k)} \cdot a_{k-2}
\end{aligned} \tag{17}$$

The denominator is taking the form of a factorial, the numerator is switching sign with every iteration, and the coefficients are related for every second term.

This leads us to the conclusion that the iterative relation between coefficients is given by,

$$a_k = \frac{(-1)^{k/2}}{(k+1)!} \cdot a_0 \tag{18}$$

for all even k , and the solution for ϕ is

$$\phi(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k} \tag{19}$$

where k now ranges over all non-negative integers.

One aspect of this solution that immediately jumps out is that it is purely even. This though is a consequence of starting with an even differential equation. This property can be shown by considering the differential operator corresponding to our differential equation,

$$D(x) = \frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} + 1 \tag{20}$$

which remains unchanged in the operation $D(-x)$ (i.e. $D(-x) = D(x)$), a property of an even operator.

With further inspection one can also realize that that $\phi(x)/a_0$ is the $\text{sinc}(x)$ function. Thus,

$$\phi(x) = a_0 \text{sinc}(x) = a_0 \frac{\sin(x)}{x} \tag{21}$$

Our result for this special case ($n = 1$) is displayed in figure (??).

Because our solution takes the analytical form shown in (??) we see that $\phi(x)$ is zero whenever $x = n\pi$ where n is any integer ≥ 1 . Thus the first time that ϕ crosses the x -axis is at $x = \pi$. Since $\rho = \lambda\phi^n$ (refer to (??)) this x -intercept is interpreted as the surface of the star due to the fact that at the surface the density is necessarily zero. Consequently, the range of physical interest for $\phi(x)$ is $0 \leq x \leq \pi$.

The properties of this solution can be further uncovered by considering the gradient of the density.² This is plotted on figure ??.

The plot implies that the magnitude of the gradient is greatest at $x \approx 2.09$. However, one would expect that the gradient of the density be greatest near the center since gravity is a

²Essentially $\phi(x)$ in this case.

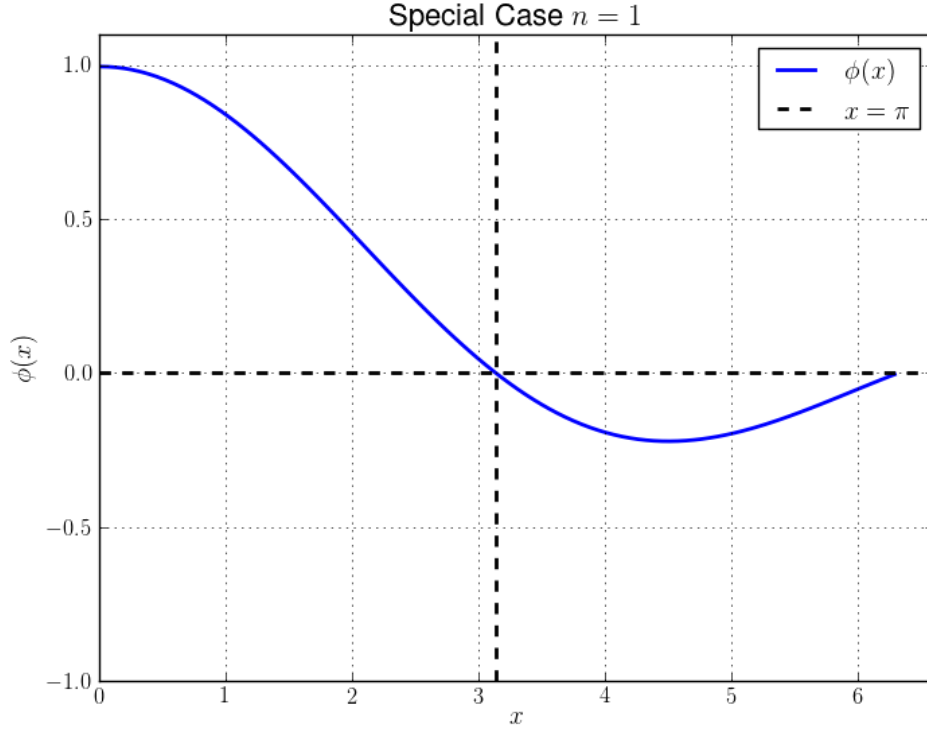


Figure 1: The solution to the Lane-Emden equation (??), for the special case of $n = 1$. The surface of the star interpreted to be when $\phi(x)$ crosses the x -axis is at $x = \pi$ as shown in the figure.

central force and goes as $1/x^2$. The closer one is to the center the greater the force to compact massive objects, this is compensated to some extent by the fact that the mass enclosed is less as one approaches the center. This effect could be explained by the requirement that the star be in hydrostatic-equilibrium as expressed in (??). Pressure has the tendency to disperse objects while gravity is constantly trying to bring them together.

Also unexpected is how abrupt the transition to zero density (i.e the surface) is. One would expect the transition to be asymptotic in nature. From figure ?? the rate of change in density at the surface is nowhere near zero. In fact it is $\approx 75\%$ of the maximum rate of change.

Strictly speaking, our solution takes the following peice-wise form,

$$\phi(x) = \begin{cases} a_0 \frac{\sin(x)}{x} & \text{for } 0 \leq x \leq \pi \\ 0 & \text{for } x > \pi \end{cases} \quad (22)$$

which has a discontinuity at the boundary $x = \pi$. This discontinuity represents the abrupt edge of the star, discussed previously.

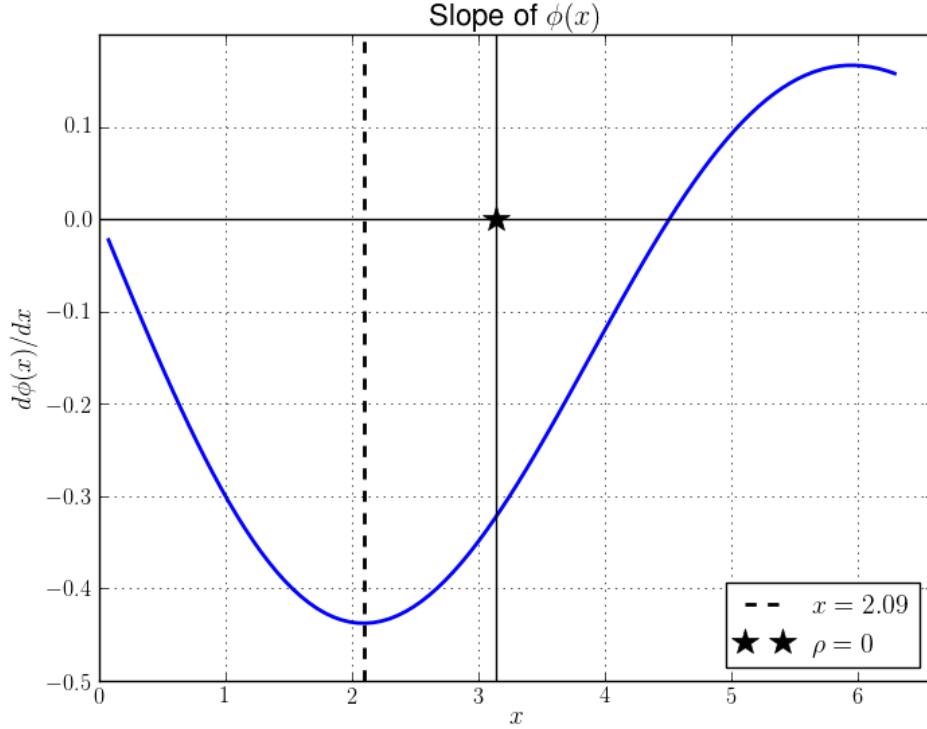


Figure 2: The Gradient to the solution of the Lane-Emden (??), for the special case of $n = 1$. The dotted line notes local minimum of this function at $x \approx 2.09$ which denotes the position where the density of the star is changing the fastest.

Solution for General n

We attempt to find a series solution for $\phi(x)$ in the Lane-Emden equation for general n , where n is a positive integer. Given the possibility of a singularity in the solution at $x = 0$, the Frobenius method allows us to write a series solution for $\phi(x)$ in identical form to Equation (??), accounting for potential singularities (which, by physical constraints, should not exist). Then, by the Lane-Emden equation,

$$\sum_{k=0, \text{even}}^{\infty} a_k (k+p)(k+p-1) x^{k+p-2} + \sum_{k=0, \text{even}}^{\infty} a_k (k+p) x^{k+p-2} + \left[\sum_{k=0, \text{even}}^{\infty} a_k x^{k+p} \right]^n = 0. \quad (23)$$

Physical constraints necessitate that no negative powers of x exist (as this would cause infinite densities at $x = 0$); thus only non-negative values for p are possible. Since $p \geq 0$, the lowest power of the ϕ^n term in Equation (??) will always be greater than those of the first two terms ($pn > p-2$). Thus our solution for p in general will match that found in the $n = 1$ case, namely $p = 0$.

Since $\phi(x)$ is an infinite-degree polynomial with a lowest power of zero, ϕ^n will likewise be

an infinite-degree polynomial with lowest power zero. Therefore, a recursion relation can be found similar to that for the $n = 1$ case, between a_{k+2} and a_k . Furthermore, our general solution will contain only even or odd powers of x .

We can show that the purely even-powered solution is the appropriate choice for $\phi(x)$ in two ways. First and most straightforward is to point out that the first term of our series solutions from the Frobenius method was a constant (zeroth power) term, implying that each successive term must be even. Furthermore, we see also that this is physically appropriate because a constant term is needed for the density of the star at its center to not go to zero (which is an inevitability for a positively odd-powered solution).

Once we have set the condition that the solution to the general Lane-Emden equation in $\phi(x)$ must contain only even powers of x , we can write the first three terms of the series solution to $\phi(x)$ as

$$\phi(x) = a_0 + a_2x^2 + a_4x^4 + \dots \quad (24)$$

Thus we have

$$\phi(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \dots \quad (25a)$$

$$\frac{d\phi}{dx} = 2a_2x + 4a_4x^3 + 6a_6x^5 + \dots \quad (25b)$$

$$\frac{d^2\phi}{dx^2} = 2a_2 + 12a_4x^2 + 30a_6x^4 + \dots \quad (25c)$$

and so the Lane-Emden equation (??) can be written as

$$\begin{aligned} & (2a_2 + 12a_4x^2 + 30a_6x^4 + \dots) + \\ & (4a_2 + 8a_4x^2 + 12a_6x^4 + \dots) + \\ & (a_0 + a_2x^2 + a_4x^4 + \dots)^n = 0. \end{aligned} \quad (26)$$

In order to write a_2 and a_4 in terms of a_0 , we must first consider the final term in Equation (??), which is

$$(a_0 + a_2x^2 + a_4x^4 + \dots) (a_0 + a_2x^2 + a_4x^4 + \dots) \dots n \text{ times}. \quad (27)$$

We note that the only term in the resultant polynomial that is of zeroth order is that which is obtained by multiplying the $n a_0$ terms together. Therefore

$$2a_2 + 4a_2 + a_0^n = 0, \text{ or}$$

$$a_2 = -\frac{a_0^n}{6}. \quad (28)$$

For the x^2 terms we find that the only possible terms are those where one a_2x^2 term from one of the polynomials is multiplied by $n - 1 a_0$ terms in the remaining polynomials. There are $n a_2x^2$ terms, therefore

$$12a_4 + 8a_4 + n a_0^{n-1} a_2 = 0, \text{ or}$$

$$a_4 = \frac{na_0^{2n-1}}{120}. \quad (29)$$

The first three terms can then be written for general n :

$$\phi(x) = a_0 - \frac{a_0^n}{6}x^2 + \frac{na_0^{2n-1}}{120}x^4 + \dots \quad (30)$$

If we set $n = 1$ and compare our result to the first three terms of the analytic solution for $n = 1$ which are

$$\phi(x) = a_0 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \dots \right) \quad (31)$$

we find that the terms agree.

Numerical Solutions for $n = 2$

The Runge-Kutta method (RK4) is an effective numerical tool in the search for a solution to a given differential equation. It works by making four successive approximations to the desired solution from a starting point (x_0) , typically defined by the initial conditions pertaining to the problem. The motivation behind the RK4 method consists of finding the appropriate constants such that

$$y_{n+1} = y_n + a\eta_1 + b\eta_2 + c\eta_3 + d\eta_4 \quad (32)$$

agrees with a Taylor series expansion up to the fifth term, or out to 4th order [?]. To find y_{n+1} from y_n , we evaluate four successive slopes of $f(x, y)$ given by

$$y' = f(x, y) \quad (33)$$

and take a weighted average. In order, we follow the following prescription [?] for evaluating the four slopes given a step-size h :

1. At position (x_0, y_0) , the slope is

$$\eta_1 = f(x_0, y_0) \quad (34)$$

2. The x -coordinate is incremented by $h/2$ so that the next y -coordinate is

$$y_1 = y_0 + \eta_1 \frac{h}{2} \quad (35)$$

and the second slope is then

$$\eta_2 = f(x_0 + h/2, y_1) \quad (36)$$

3. Using the same x -position, we find another y -coordinate using the slope we previously calculated, η_2

$$y_3 = y_0 + \eta_2 \frac{h}{2} \quad (37)$$

to get the next successive slope

$$\eta_3 = f(x_0 + h/2, y_3) \quad (38)$$

4. The last slope is obtained at the $x = x_0 + h$, where again we use our previously calculated slope η_3 to determine the corresponding y -coordinate,

$$y_4 = y_0 + \eta_3 h \quad (39)$$

and the slope at that position is

$$\eta_4 = f(x_0 + h, y_4) \quad (40)$$

The slopes at the midpoint ($x + h/2$) are weighted twice as much as the other two slopes in the averaging and the successive solution of y is then

$$y_{n+1} = y_n + (\eta_1 + 2\eta_2 + 2\eta_3 + \eta_4) \frac{h}{6}. \quad (41)$$

Because the RK4 method works best for 1st-order differential equations, and since we wish to find a numerical solution to a 2nd-order differential equation, we need to add an addition adjustment to our algorithm. The Lane-Emden equation of $n = 2$ can be written as

$$\phi'' + \frac{2}{x}\phi' + \phi^2 = 0 \quad (42)$$

which can be reduced to two 1st-order differential equations. First, we assign

$$v = \phi' = f(x, y, v) \quad (43)$$

and then (??) becomes

$$v' = -\frac{2}{x}v - \phi^2 = g(x, y, v) \quad (44)$$

To solve (??), we must now evaluate the increments of ϕ and v with the following updated recipe [?] for finding the slopes:

$$\begin{aligned} \eta_1 &= hf(x_0, y_0, v_0) \\ m_1 &= hg(x_0, y_0, v_0) \\ \eta_2 &= hf(x_0 + h/2, y_0 + \eta_1/2, v_0 + m_1/2) \\ m_2 &= hg(x_0 + h/2, y_0 + \eta_1/2, v_0 + m_1/2) \\ \eta_3 &= hf(x_0 + h/2, y_0 + \eta_2/2, v_0 + m_2/2) \\ m_3 &= hg(x_0 + h/2, y_0 + \eta_2/2, v_0 + m_2/2) \\ \eta_4 &= hf(x_0 + h, y_0 + \eta_3, v_0 + m_3) \\ m_4 &= hg(x_0 + h, y_0 + \eta_3, v_0 + m_3) \end{aligned} \quad (45)$$

Just as y is incremented by the averaged η , v is incremented by the similarly averaged m . As a sanity check, we plotted the numerical solution of the Lane-Emden equation for the case of $n = 1$ and checked the result against our series solution found earlier. We find that for a step-size of $h = 0.1$, the numerical solution agrees with our series solution and we may proceed further to the case of $n = 2$.

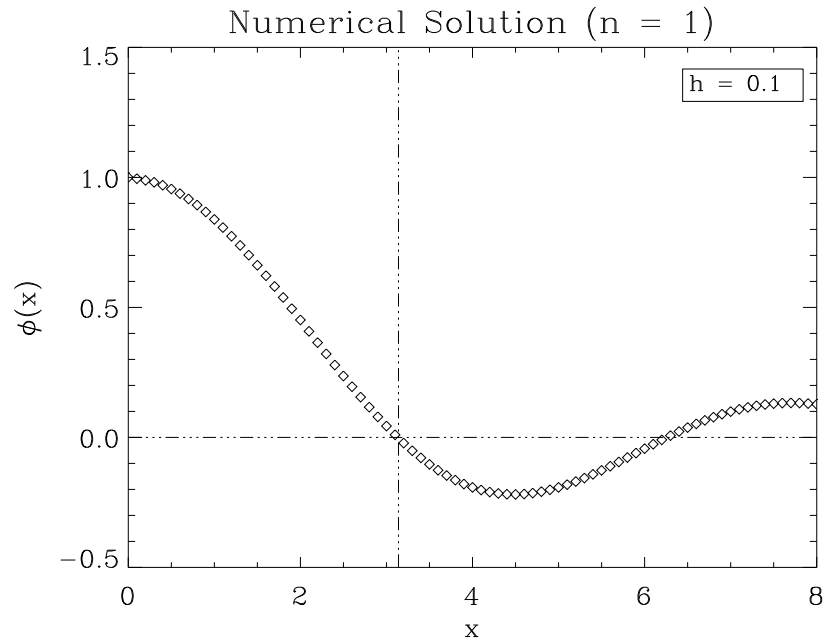


Figure 3: The numerical solution of the Lane-Emden equation of polytropic index $n = 1$.

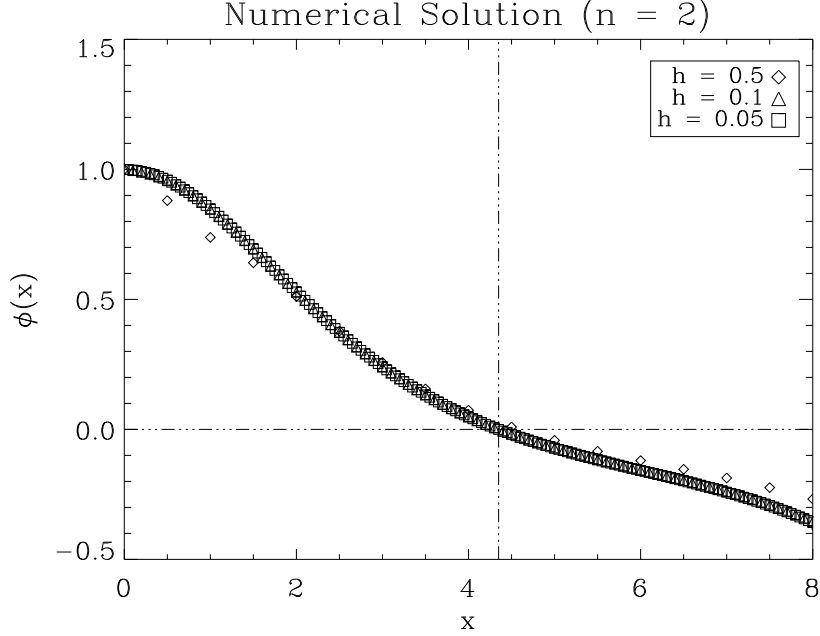


Figure 4: The numerical solutions for $n = 2$ using the RK4 method for three different step sizes are shown. The solutions intersect the x -axis at ~ 4.35 . The value of the dimensionless distance where $\phi(x) = 0$ corresponds to the radius of the star.

Based on the algorithm described by (??), we wrote a procedure in IDL (Appendix B) in order to generate an array of solutions for the Lane-Emden equation for $n = 2$ and plotted our results in figure (??) for three different step-sizes. We notice that for $h = 0.5$, our solutions for $\phi(x)$ diverge slightly from both the other smaller step-sizes, which appear to produce almost identical results.

To get a sense of how our numerical solutions vary with step-size, we evaluated $\phi(x = 1)$ for the case of $n = 1$ for a range of h and plotted the results in figure (??). We notice that the output converges rapidly with smaller and smaller step-sizes. However, once we choose $h = 0.1$, we can expect approximately the same precision as $h = 0.001$. According to table (??), once h is smaller than 0.1, the percent difference between subsequent diminishing step-sizes begins to oscillate toward zero at the cost of computing time with minor improvement.

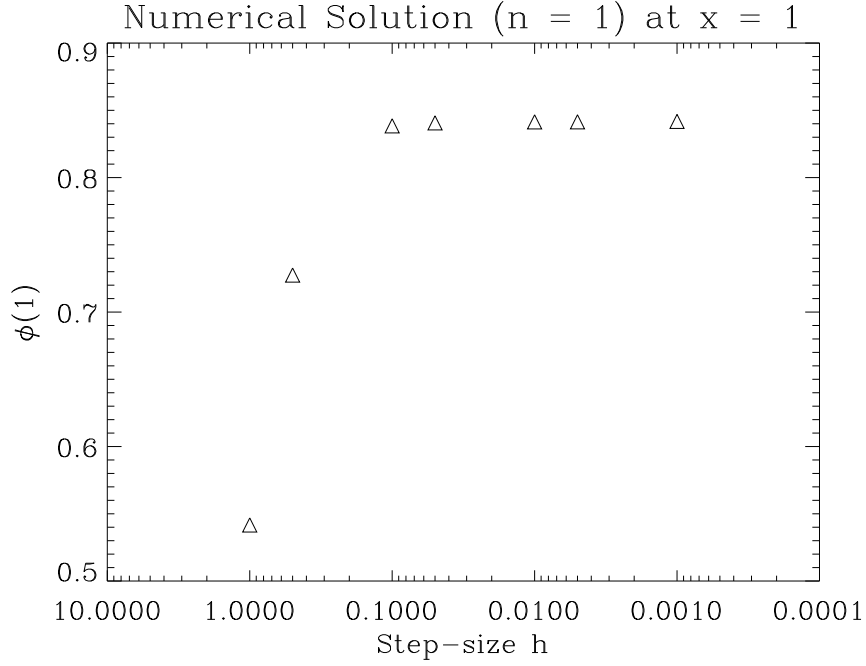


Figure 5: The figure shows $\phi(1)$ evaluated with varying step-size h .

| h | x | $\phi(x)_{n+1}$ | ϕ_n | %-difference |
|------------|---------|-----------------|----------|--------------|
| 1.00000 | 1.00000 | 0.727498 | 0.541667 | 34.3073 |
| 0.500000 | 1.00000 | 0.838454 | 0.727498 | 15.2518 |
| 0.100000 | 1.00000 | 0.840666 | 0.838454 | 0.263796 |
| 0.0500000 | 1.00000 | 0.841419 | 0.840666 | 0.0895205 |
| 0.0100000 | 1.00000 | 0.841452 | 0.841419 | 0.00393861 |
| 0.00500000 | 1.00000 | 0.841769 | 0.841452 | 0.0377341 |

Table 1: $\phi(1)$ is evaluated for varying step-size and the percent difference between results for neighboring h 's are reported.

After comparing our RK4 solution against the $n = 1$ series solution and choosing an appropriate step-size, we are confident in our result for the solutions to the case of $n = 2$. We can now make a comparison between our approximate three term series solution for arbitrary n and our RK4 method. Figure (??) depicts the case for $n = 1$. We can see that up to $x \simeq 2$, the two solutions agree but diverge significantly thereafter. We can image that finding more than three terms in our series solution would agree more closely with our RK4 solution.

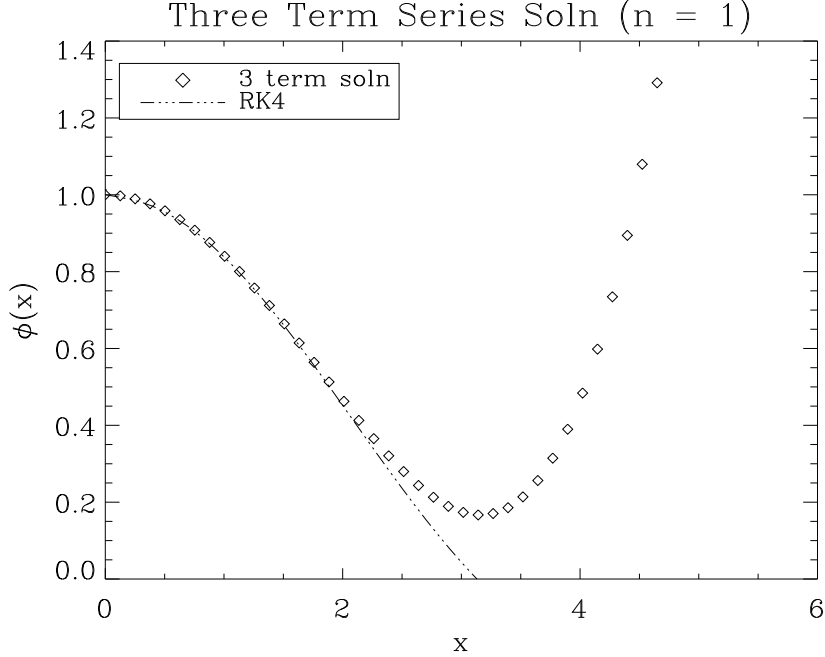


Figure 6: A comparison between the first three terms to our solution for an arbitrary n (setting $n = 1$, here) and the RK4 method for the case of $n = 1$ is depicted in this plot. We notice that the two solutions begin to diverge at around $x = 2$.

Conclusion

We have shown that for $n = 1$, the Lane-Emden equation can produce an analytic solution of $\text{sinc}x$. This solution is zero for when the dimensionless quantity x is $\simeq 3.14$ which puts a constraint on the radius of the star we are modeling. When considering the case of an arbitrary polytropic index, we discovered that finding the first three terms of the series solution did not produce results that matched well to our numerical solution for $n = 1$, and that to improve accuracy, we needed to define the series solutions with more than three terms. Finally, when we considered our numerical solution for $n = 2$, we found that while decreasing the step-size h allowed the solution to converge, that no real gain was made by any further decrease in h below $h = 0.1$.

A Result Tables RK4 Method

Table 2: Results for solving the Lane-Emden equation for $n = 2$ using the RK4 method are shown for step-size $h = 0.1$. For brevity we have omitted every second solution in the table.

| x | ϕ | v | η_1 | η_2 | η_3 | η_4 | m_1 | m_2 | m_3 | m_4 |
|---------|------------|------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|--------------|
| 0.10000 | 0.995008 | -0.0668338 | 0.00000 | -0.00500000 | 0.00500000 | -0.0299501 | -0.100000 | 0.100000 | -0.299501 | 0.497999 |
| 0.30000 | 0.981271 | -0.0942693 | -0.00525634 | -0.00751449 | -0.00711091 | -0.00926610 | -0.0451630 | -0.0370915 | -0.0400976 | -0.0345515 |
| 0.50000 | 0.956027 | -0.156937 | -0.0126877 | -0.0142224 | -0.0141724 | -0.0156646 | -0.0306938 | -0.0296947 | -0.0297688 | -0.0287439 |
| 0.70000 | 0.919152 | -0.210734 | -0.0184912 | -0.0198173 | -0.0197640 | -0.0210409 | -0.0265223 | -0.0254557 | -0.0254965 | -0.0243706 |
| 0.90000 | 0.872423 | -0.255342 | -0.0234256 | -0.0245196 | -0.0244587 | -0.0254965 | -0.0218794 | -0.0206630 | -0.0207092 | -0.0194568 |
| 1.10000 | 0.817775 | -0.289843 | -0.0273890 | -0.0282283 | -0.0281639 | -0.0289441 | -0.0167871 | -0.0154985 | -0.0155514 | -0.0142536 |
| 1.30000 | 0.757267 | -0.313953 | -0.0303187 | -0.0308979 | -0.0308343 | -0.0313555 | -0.0115827 | -0.0103109 | -0.0103679 | -0.00910954 |
| 1.50000 | 0.692948 | -0.328055 | -0.0322208 | -0.0325508 | -0.0324918 | -0.0327686 | -0.00659955 | -0.00542009 | -0.00547803 | -0.00432906 |
| 1.70000 | 0.626730 | -0.333075 | -0.0331626 | -0.0332677 | -0.0332161 | -0.0332752 | -0.00210206 | -0.00106969 | -0.00112550 | -0.000134116 |
| 1.90000 | 0.560306 | -0.330302 | -0.0332572 | -0.0331705 | -0.0331278 | -0.0330036 | 0.00173433 | 0.00258787 | 0.00253673 | 0.00334463 |
| 2.10000 | 0.495086 | -0.321218 | -0.0326453 | -0.0324042 | -0.0323710 | -0.0321011 | 0.00482302 | 0.00548682 | 0.00544211 | 0.00606033 |
| 2.30000 | 0.432181 | -0.307337 | -0.0314784 | -0.0311208 | -0.0310968 | -0.0307189 | 0.00715333 | 0.00763306 | 0.00759573 | 0.00803327 |
| 2.50000 | 0.372406 | -0.290092 | -0.0299051 | -0.0294665 | -0.0294509 | -0.0289996 | 0.00877159 | 0.00908439 | 0.00905465 | 0.00933069 |
| 2.70000 | 0.316303 | -0.270755 | -0.0280613 | -0.0275733 | -0.0275648 | -0.0270705 | 0.00976091 | 0.00993056 | 0.00990805 | 0.0100473 |
| 2.90000 | 0.264181 | -0.250400 | -0.0260646 | -0.0255535 | -0.0255508 | -0.0250386 | 0.0102231 | 0.0102759 | 0.0102599 | 0.0102888 |
| 3.10000 | 0.216156 | -0.229887 | -0.0240117 | -0.0234985 | -0.0235004 | -0.0229901 | 0.0102643 | 0.0102264 | 0.0102160 | 0.0101600 |
| 3.30000 | 0.172190 | -0.209873 | -0.0219784 | -0.0214791 | -0.0214843 | -0.0209909 | 0.00985558 | 0.00988058 | 0.00987469 | 0.00975688 |
| 3.50000 | 0.132134 | -0.190837 | -0.0200209 | -0.0195471 | -0.0195547 | -0.0190887 | 0.00947617 | 0.00932448 | 0.00932216 | 0.00916198 |
| 3.70000 | 0.0957575 | -0.173102 | -0.0181792 | -0.0177386 | -0.0177477 | -0.0173162 | 0.00881127 | 0.00862959 | 0.00862996 | 0.00844321 |
| 3.90000 | 0.0627799 | -0.156870 | -0.0164790 | -0.0160764 | -0.0160864 | -0.0156935 | 0.00805122 | 0.00785263 | 0.00785494 | 0.00765386 |
| 4.10000 | 0.0328886 | -0.142247 | -0.0149354 | -0.0145733 | -0.0145836 | -0.0142314 | 0.00724238 | 0.00703668 | 0.00704030 | 0.00683399 |
| 4.30000 | 0.00575759 | -0.129268 | -0.0135552 | -0.0132343 | -0.0132445 | -0.0129335 | 0.00641876 | 0.00621295 | 0.00621740 | 0.00601225 |
| 4.50000 | -0.0189411 | -0.117920 | -0.0123392 | -0.0120590 | -0.0120691 | -0.0117985 | 0.00560402 | 0.00540277 | 0.00540766 | 0.00520787 |

Table 3: Results for solving the Lane-Emden equation for $n = 2$ using the RK4 method are shown for step-size $h = 0.05$. For brevity we have included every fourth solution in the table.

| x | ϕ | v | η_1 | η_2 | η_3 | η_4 | m_1 | m_2 | m_3 | m_4 |
|----------|------------|------------|-------------|-------------|-------------|-------------|-------------|--------------|--------------|--------------|
| 0.050000 | 0.998751 | -0.0333542 | 0.00000 | -0.00125000 | 0.00125000 | -0.00749688 | -0.0500000 | 0.0500000 | -0.149938 | 0.249750 |
| 0.250000 | 0.988766 | -0.0817594 | -0.00326438 | -0.00367946 | -0.00367385 | -0.00408479 | -0.0166032 | -0.0163788 | -0.0164081 | -0.0162048 |
| 0.450000 | 0.966110 | -0.143883 | -0.00644551 | -0.00682306 | -0.00681821 | -0.00719114 | -0.0151021 | -0.0149081 | -0.0149126 | -0.0147080 |
| 0.650000 | 0.931674 | -0.199265 | -0.00930798 | -0.00963989 | -0.00963344 | -0.00995916 | -0.0132766 | -0.0130186 | -0.0130237 | -0.0127575 |
| 0.850000 | 0.886996 | -0.246051 | -0.0117622 | -0.0120373 | -0.0120298 | -0.0122977 | -0.0110072 | -0.0107042 | -0.0107103 | -0.0104026 |
| 1.050000 | 0.833935 | -0.282964 | -0.0137351 | -0.0139469 | -0.0139389 | -0.0141430 | -0.00847456 | -0.00815125 | -0.00815811 | -0.00783357 |
| 1.250000 | 0.774526 | -0.309506 | -0.0151921 | -0.0153389 | -0.0153309 | -0.0154701 | -0.00587376 | -0.00555366 | -0.00556098 | -0.00524251 |
| 1.450000 | 0.710831 | -0.325903 | -0.0161359 | -0.0162204 | -0.0162130 | -0.0162904 | -0.00337850 | -0.00308104 | -0.00308844 | -0.00279476 |
| 1.650000 | 0.644807 | -0.332971 | -0.0166009 | -0.0166290 | -0.0166224 | -0.0166443 | -0.00112350 | -0.000862603 | -0.000869711 | -0.000613961 |
| 1.850000 | 0.578201 | -0.331933 | -0.0166438 | -0.0166237 | -0.0166183 | -0.0165932 | 0.000802484 | 0.00101871 | 0.00101220 | 0.00122264 |
| 2.050000 | 0.512492 | -0.324239 | -0.0163352 | -0.0162763 | -0.0162721 | -0.0162093 | 0.00235556 | 0.00252427 | 0.00251857 | 0.00268146 |
| 2.250000 | 0.448859 | -0.311407 | -0.0157508 | -0.0156625 | -0.0156594 | -0.0155684 | 0.00352998 | 0.00365249 | 0.00364772 | 0.00376481 |
| 2.450000 | 0.388182 | -0.294891 | -0.0149645 | -0.0148558 | -0.0148538 | -0.0147433 | 0.00434852 | 0.00442906 | 0.00442525 | 0.00450103 |
| 2.650000 | 0.331065 | -0.276003 | -0.0140441 | -0.0139228 | -0.0139217 | -0.0137994 | 0.00485237 | 0.00489684 | 0.00489394 | 0.00493445 |
| 2.850000 | 0.277867 | -0.255859 | -0.0130480 | -0.0129206 | -0.0129203 | -0.0127927 | 0.00509210 | 0.00510708 | 0.00510500 | 0.00511683 |
| 3.050000 | 0.228745 | -0.235366 | -0.0120240 | -0.0118960 | -0.0118962 | -0.0117684 | 0.00512058 | 0.00511256 | 0.00511119 | 0.00510079 |
| 3.250000 | 0.183696 | -0.215226 | -0.0110098 | -0.0108851 | -0.0108858 | -0.0107617 | 0.00498804 | 0.00496297 | 0.00496218 | 0.00493539 |
| 3.450000 | 0.142595 | -0.195956 | -0.0100335 | -0.00991502 | -0.00991594 | -0.00979841 | 0.00473908 | 0.00470207 | 0.00470174 | 0.00466358 |
| 3.650000 | 0.105229 | -0.177913 | -0.00911470 | -0.00900442 | -0.00900554 | -0.00889637 | 0.00441125 | 0.00436650 | 0.00436651 | 0.00432107 |
| 3.850000 | 0.0713297 | -0.161325 | -0.00826635 | -0.00816549 | -0.00816671 | -0.00806706 | 0.00403473 | 0.00398556 | 0.00398582 | 0.00393629 |
| 4.050000 | 0.0405905 | -0.146320 | -0.00749596 | -0.00740514 | -0.00740641 | -0.00731685 | 0.00363280 | 0.00358168 | 0.00358211 | 0.00353088 |
| 4.250000 | 0.0126891 | -0.132950 | -0.00680693 | -0.00672637 | -0.00672765 | -0.00664834 | 0.00322255 | 0.00317126 | 0.00317180 | 0.00312058 |
| 4.450000 | -0.0127019 | -0.121213 | -0.00619979 | -0.00612939 | -0.00613064 | -0.00606147 | 0.00281593 | 0.00276567 | 0.00276627 | 0.00271619 |

B Source Code

```
#A routine for plotting quantities relevant to
#part a of problem 3.28. Written in Python 2.7

from numpy import *
import pylab as pyl
from matplotlib import *
import math

#Defining my starting variables
xfin = 2*pi
x = linspace(0,xfin,100); a0=1;phi=0; N = 50#number of terms

#Building phi with a for loop

for i in range(N):

    phi = phi + a0*cos(i*pi)*x**(2.*i)/math.factorial(2.*i+1.)

#finding the x intercept
count = 0
for i in phi:
    if i<0:
        ind1=count-1
        break
    count = count +1

xint = str(x[ind])[0:5]
#plotting \phi(x)
rc('text', usetex=True)
pyplot.figure()
p1=pyplot.plot(x,phi,linewidth=2)
p2=pyplot.plot([pi]*2,[-1,max(phi)+.1*max(phi)], '--k',linewidth=2)
pyplot.plot([0,xfin+.05*xfin],[0,0], '--k',linewidth=2)
pyplot.xlim([0,xfin+.05*xfin])
pyplot.ylim([-1,max(phi)+.1*max(phi)])
#pyplot.plot([0,0],[min(phi)+.7*min(phi),max(phi)+.2*max(phi)], '--k',linewidth=2)
pyplot.grid(True)
pyplot.title(r'Special Case $n=1$', fontsize=16)
pyplot.xlabel(r'$x$', fontsize=14)
pyplot.ylabel(r'$\phi(x)$', fontsize=14)
pyplot.legend([p1[0],p2[0]], [r'\phi(x)', r'$x=\pi$'])
```



```

pyplot.savefig('LanEmdNeq1.png')
pyplot.show()

#the slope of phi, derived from its analytical form  $a_0 \sin(x)/x$ 

Dphi = a0*(x*cos(x)-sin(x))/x**2

#finding finding the local minimum of the slope

for i in array(range(len(Dphi)))+2:
    if count>0:
        if Dphi[i]-Dphi[i-1]>0:
            ind=i-1
            break

#plotting  $d\phi(x)/dx$ 
pyplot.figure()
pyplot.plot(x,Dphi,linewidth=2)
pyplot.plot([pi]*2,[-.5,.2],'-k',linewidth=1)
p1=pyplot.plot([x[ind]]*2,[-.5,.2], '--k',linewidth=2)
p2 = pyplot.plot([pi],0,'*k', markersize=15)
pyplot.plot([0,xfin+.05*xfin],[0,0],'-k',linewidth=1)
pyplot.xlim([0,xfin+.05*xfin])
pyplot.ylim([- .5,.2])
pyplot.title(r'Slope of  $\phi(x)$ ',fontsize=16)
pyplot.xlabel(r'$x$',fontsize=14)
pyplot.ylabel(r'$d\phi(x)/dx$',fontsize=14)
pyplot.grid(True)
xint = str(x[ind])[0:4]
pyplot.legend([p1[0],p2[0]], [r'$x='+xint+'$',r'$\rho=0$'],loc=4)
pyplot.savefig('SlopePhi.png')
pyplot.show()

```

```

FUNCTION rk4le, h, N
;+
; NAME: rk4le
;
; PURPOSE:
;   Iteratively solves a first order differential equation
;   using the 4th-order Runge-Kutta method.
;
; EXPLANATION:
;   This procedure calls on two functions which are the constituent
;   pieces of a 2nd-order differential equation. For example, if
;    $y'' + y' = 0$ , then  $v = y'$  is our first constituent, and
;    $v' + v = 0$  is our second constituent, and this procedure works
;   to solve both pieces.
;
; CALLING SEQUENCE:
;   xymatrix = rk4le(h, N)
;
; INPUTS:
;   h - step-size per iteration
;   N - numbers of iterations
;
; OUTPUTS:
;   Returns a 2 by N dimensional matrix containing the solution and its
;   independent variable.
;
; EXAMPLE:
;   IDL> xy = rk4le(0.1, 100.)
;   IDL> x = xy[:,0]
;   IDL> y = xy[:,1]
;
; RESTRICTIONS:
;   This procedure was designed to cater to solving the Lane-Emden
;   equation and initial conditions are currently hardcoded in.
;   More generality will come in the future.
; PROCEDURES CALLED:
;   None
;
; REVISION HISTORY:
;   Written by Adam Fries, Allan Gamboa, and Arthur Adams
;   Oct 2012 based off of 'Mathematics for Physicists'
;   by Susan M. Lea pgs. 202-205
;-

;; set up Lane-Emden initial conditions

```



```

PRO learb_n, n
;;
;; This code was created to plot the 3 term series
;; solution against our RK4 solution for n = 1
;;
x = findgen(100)/100*4*!pi
a0 = 1.

phi = a0*(1. - (a0^n)/6.*x^2. + n*a0^(2.*n - 1.)/120.*x^4.)

phir = rk4le(0.1, 100)

title = '!6Three Term Series Soln (n = 1)'
xtitle = '!6x'
ytitle = '!7u!6(x)'
plot, x, phi, charsize = 1.6, psym = 4, yr = [0, 1.25], xr = [0,6] $
    , title = title, xtitle = xtitle, ytitle = ytitle
oplot, phir[:,0], phir[:,1], line = 4
legend, ['!63 term soln', 'RK4'], psym = [4,0], line = [0,4] $
    , charsize = 1.4

end

```