A Discrete-Event Fourier Transform

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Abstract

Discrete-Event signals are discontinuous and non-uniformly sampled, making them difficult to analyze using the standard Discrete or Fast Fourier Transforms. However, instead of trying to work around the problematic properties of Discrete-Event signals we can use them to our advantage to develop a transform method that is both more accurate and less computationally expensive than a Fast Fourier Transform of the signal.

Index Terms

Discrete-Event systems, Discrete-Event signals, Fourier transform

I. Introduction

A Discrete-Event System (DES) is a system that has a discrete state-space, and consequently changes state only at discrete points in time [1], [2]. The transitions from one state to another are referred to as *events*. In an "untimed" DES only the order in which events occur is recorded. In a "timed" DES the time at which the events occurred is also recorded. Thus, an event in a timed DES can be thought of as a pair, $e = (t, v)$, consisting of the time, t, when the transition to the state $v \in \{S_0 \ldots S_N\}$ occurred. Since a DES has a discrete state-space, the state trajectories, or signals, associated with a DES are piecewise-constant functions of time (Fig. 1). However, it is generally more convenient to represent each signal as its corresponding sequence of events (Fig. 1). These Discrete-Event (DE) signals can be considered a sampled version of the underlying piecewise-constant signal, in much the same way that a Discrete-Time (DT) signal is a sampled version of an underlying Continuous-Time (CT) signal. The difference is that the DE signal records samples at each state transition rather than at some fixed sampling interval.

Timed DESs are widely used in simulations of networks, queueing systems, manufacturing systems, and digital logic [1], [2]. Timed DESs can also be used to approximate continuous-time systems, and as a consequence have recently begun to be used as the basis for novel numerical integration algorithms and event-based control schemes [3], [4]. There is a substantial existing body of work on Timed DES theory [1], [2], [5], [6]. However, that work focuses on simulation and time-domain analysis of DESs. Although frequency-domain analysis has proved useful for studying CT and DT signals, it does seem not to have been applied to DE signals.

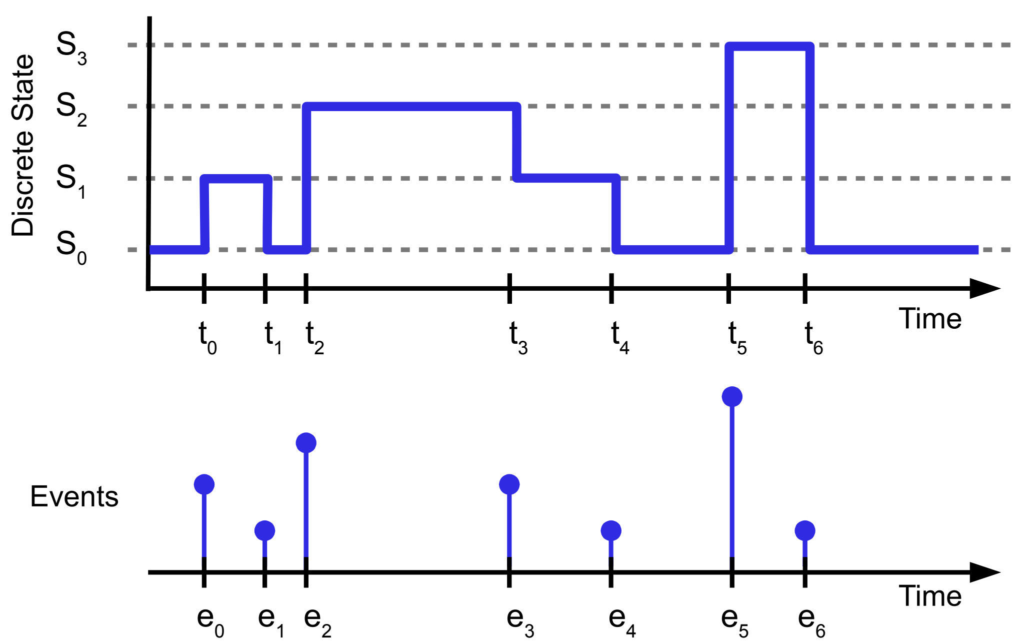

Fig. 1. A discrete-state signal and corresponding discrete-event signal.

The most common approach to frequency-domain analysis of sampled signals is the Discrete Fourier Transform (DFT), typically using some version of the Fast Fourier Transform (FFT). However, the standard DFT assumes uniform sampling of the time-domain signal. But a DE signal consists of samples taken at transition points in a piecewise-constant CT signal, and thus is a non-uniformly sampled representation of an inherently discontinuous signal. A number of different techniques for estimating the spectrum of non-uniformly sampled signals have been developed, such as non-uniform discretization of the Continuous Fourier Transform integral, least-squares estimates, and interpolation and resampling to produce a uniformly sampled signal [7]–[9]. However, this work often assumes that there is no correlation between the signal values and the sample times, whereas DE signals have sample times that are highly correlated to the underlying signal values since samples are taken at signal transition points. Furthermore, most work on Fourier analysis of non-uniformly sampled signals assumes an underlying continuous codomain, which does not hold for DE signals. Fan and Liu [10] developed a technique based on a double-interpolation procedure that is capable of finding the Fourier Transform of general discontinuous, non-uniformly sampled signals. While their technique could, in principle, be applied to DE signals, it is complicated and can be computationally expensive if an accurate transform (implying a high sample rate) is required. Furthermore, the DT nature of the FFT algorithm will introduce erroneous sampling artifacts that are not inherent in the original DE signal.

The central idea of this note is that instead of trying to work around the problematic properties of DE signals, we can use them to our advantage to develop a transform that is both more accurate and less computationally expensive than a Fast Fourier Transform of the signal. The specific contributions that I describe in this note are:

- A semianalytical technique for computing an accurate Fourier Transform of a Discrete-Event signal (Sec. II).
- A variation of the technique suitable for use with periodic Discrete-Event signals (Sec. III).
- An assessment of the computational complexity of our approach, and tradeoffs with the standard FFT (Sec. IV).

II. A Discrete-Event Fourier Transform

A Discrete-Event Fourier Transform (DEFT) should convert a time-domain Discrete-Event signal into a frequency-domain function that accurately reflects the nature of the DE signal. How can we perform this operation?

A naïve approach, such as the generalized DFT known as the "Point Rule" Non-Uniform Time Discrete Fourier Transform (NUT-DFT) [7],

$$X(\omega) = \sum_{i=0}^{N-1} v_i e^{-j\omega t_i}, \tag{1}$$

where t_i is the time of the ith sample and v_i is the corresponding value, works reasonably well for non-uniformly sampled signals with stochastic sampling. But such an approach clearly fails in the case of even a simple DE signal such as a rectangular pulse (Fig. 2).

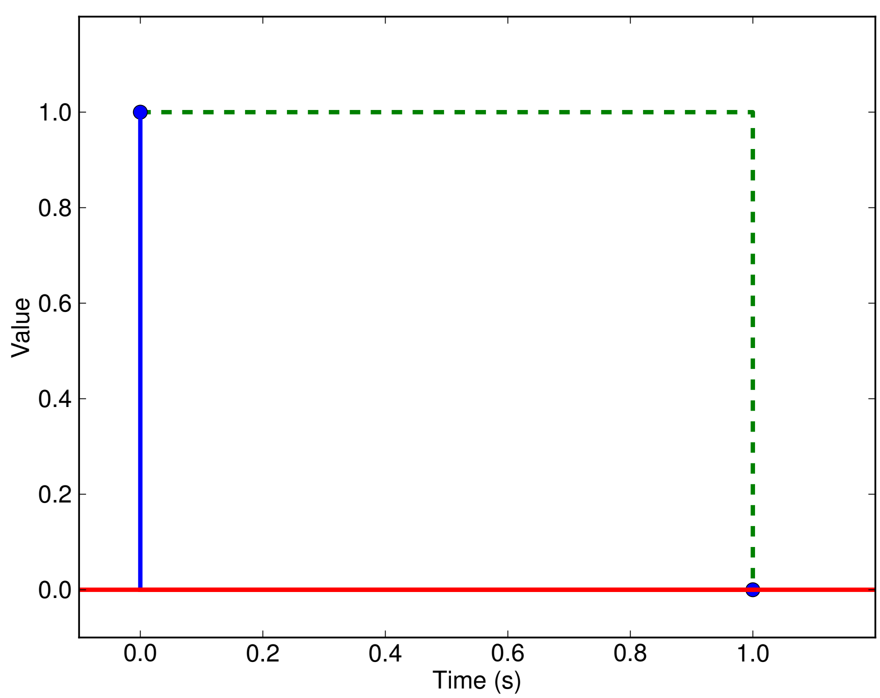

Fig. 2. DE representation of a rectangular pulse

Classical Fourier Transform theory leads us to expect that the transform of a rectangular pulse should be something with a shape resembling a $\mathrm{sinc}(\omega)$ shape. However, equation 1, when applied to a DE signal representing a rectangular pulse,

$$x = \{(0,1),(1,0)\},$$

instead yields

$$X(\omega) = 1 \times e^{-j\omega_k 0} + 0 \times e^{-j\omega_k 1} = 1,$$

which resembles the Fourier Transform of an *impulse* rather than a *rectangle*.

How can we obtain the $\mathrm{sinc}(\omega)$ shape that we expect from the transform of the signal x?

Recalling that a DE signal is simply a compact representation of a piecewise-constant CT signal, where the events mark transitions from one constant value to another, the Fourier Transform of the DE signal can be found by treating the events as defining a CT rectangular pulse that has a magnitude v_0 and is centered at the point halfway between the two events. That is, we let

$$x(t) = v_0 \mathrm{rect}\left(\frac{t-(t_0+t_1)/2}{|t_1-t_0|}\right),$$

where

$$\mathrm{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 0, & |t| \geq \tau/2 \\ 1, & |t| < \tau/2 \end{cases}$$

We can then apply classical Fourier Transform theory to find the Fourier Transform of $x(t)$:

$$X(\omega) = v_0 e^{-j\omega\frac{1}{2}(t_0+t_1)}\ \tau \mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right),$$

where $\tau = |t_1 - t_0|$. The resulting function can be considered a Fourier Transform of the DE signal, x.

Treating a pair of events as defining a rectangular pulse is useful in the specific case of a DE signal that represents a rectangular pulse. However, it also forms the core of a generally applicable technique for finding the Fourier Transform of arbitrary DE signals. Any finite-duration DE signal can be decomposed into a sum of time-shifted rectangular pulses, each with a magnitude corresponding to the value of the event that marks the start of the pulse,

$$x(t) = \sum_{i=0}^{N-1} v_i \mathrm{rect}\left(\frac{t-(t_i+t_{i+1})/2}{|t_{i+1}-t_i|}\right).$$

Then, by the superposition property of the Fourier Transform, the transform of the decomposed DE signal is the sum of the transforms of the component rectangular pulses:

$$X(\omega, t_f) = \sum_{i=0}^{N-1} \tau_i v_i e^{-j\omega(t_i+\tau_i/2)} \mathrm{sinc}\left(\frac{\omega\tau_i}{2\pi}\right), \tag{2}$$

where $\tau_i = |t_{i+1} - t_i|$ and $(t_N, v_N) = (t_f, 0)$. Note that we explicitly include a final time, t_f, at which the signal is assumed to return to its zero state. This ensures that the DE signal to be transformed has a finite duration and can be decomposed into a sum of rectangular pulses, thus eliminating the effects of non-zero final states that produce Dirac delta functions in the Fourier Transform. In practice, the transform of DE signals with non-zero final states can be approximated by making t_f very large.

The DEFT, as defined by equation (2), is a function of a continuous ω. A practical computational implementation of the DEFT will necessarily involve evaluating equation (2) over some set of discrete points, ω_k. Note that this set of points is not in any way related to the number or spacing of the time-domain samples. Thus, unlike the DFT, the frequency resolution of the DEFT can be selected independent of the number of samples, and can even be a nonuniformly-spaced set of frequencies.

As an example of applying the DEFT, let us consider finding the Fourier Transform of a DE $\mathrm{sinc}(t)$ signal. The DE $\mathrm{sinc}(t)$ signal consists of events that correspond to quantization-level transitions in a CT $\mathrm{sinc}(t)$ signal. The DE signal, as shown in Fig. 3, consists of non-uniformly spaced samples. Evaluating equation (2) over the events in the DE $\mathrm{sinc}(t)$ signal yields the frequency-domain representation shown in Fig. 4.

The result shown in Fig. 4 is not the perfect rectangle we expect from the Fourier Transform of a CT $\mathrm{sinc}(t)$ signal. However, the deviations from a rectangular shape that we see in Fig. 4 are artifacts of the DE nature of the signal being transformed, in the same way that the periodicity of a DT spectrum is an artifact of the DT sampling process. The DEFT produces an accurate frequency-domain representation of the DE signal, which will inevitably be different than the frequency-domain representation of a related CT signal. Reducing the size of the quantization step in the DE signal, which effectively makes the DE signal a better approximation to a CT signal, results in a Fourier Transform that is closer to the expected rectangle, as shown in Fig. 5.

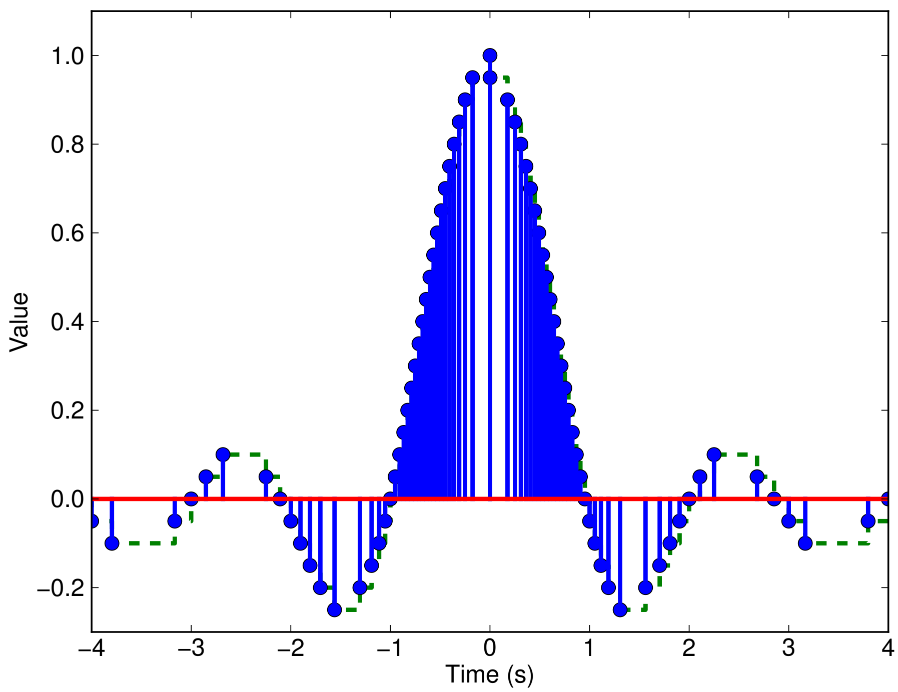

Fig. 3. A DE $\mathrm{sinc}(t)$ signal constructed using a quantization step $\Delta = 0.05$

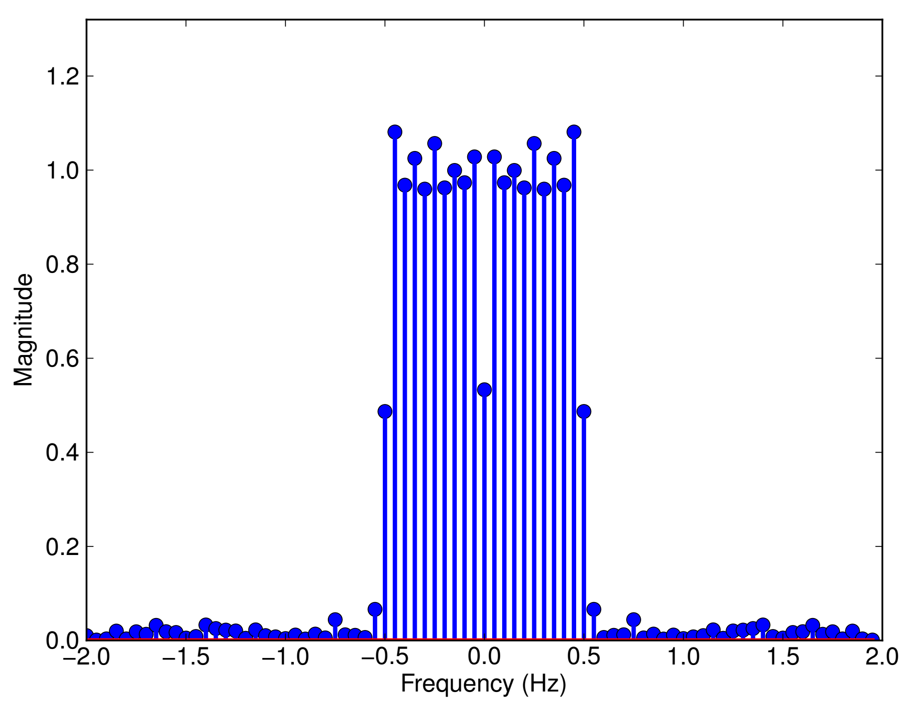

Fig. 4. DEFT of the DE $\mathrm{sinc}(t)$ signal, computed with $\omega_k = 2\pi 0.05k$

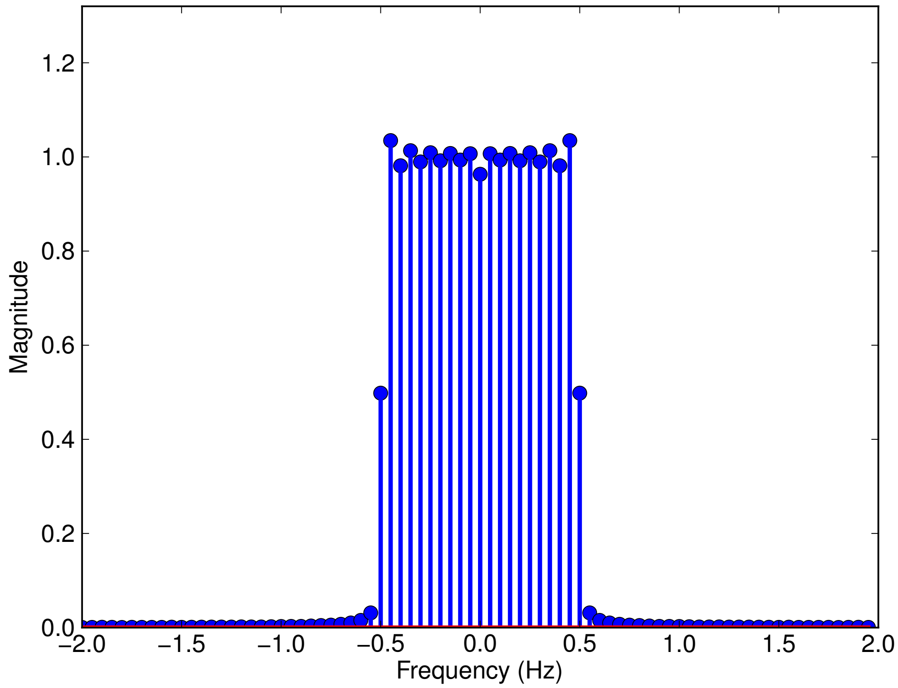

Fig. 5. DEFT of a DE $\mathrm{sinc}(t)$ signal with quantization step $\Delta = 0.001$

III. A Discrete-Event Fourier Transform for Periodic Signals

The DEFT developed in section II provides a way to find the Fourier Transform of arbitrary finite-duration DE signals through superposition of the Fourier Transforms of rectangular pulses. We can apply a similar, but not identical, approach to find the Fourier Transform of infinite-duration periodic DE signals. There are two key differences between the DEFT described in section II and a DEFT for periodic signals:

1) Since we are dealing with periodic signals, instead of assuming a terminal state that is a return to the zero state, we assume that the terminating state is a return to the initial state.
2) The periodic nature of the DE signal means that its frequency-domain representation will be a discrete spectrum containing non-zero values only at integer multiples of the fundamental frequency.

The formulation of the DEFT for periodic signals follows from the differences just noted. Much like the standard DFT, we compute the transform over a finite segment of the time-domain signal. For a signal with a period T the DEFT can be expressed as

$$X(\omega_k) = \frac{1}{T} \sum_{i=0}^{N-1} \tau_i v_i e^{-j\omega_k (t_i + \tau_i/2)} \mathrm{sinc}\left(\frac{\omega_k \tau_i}{2\pi}\right) \tag{3}$$

where $(t_N, v_N) = (T, v_0)$ and $\omega_k = 2\pi k/T$ for $k \in \mathbb{Z}$.

As an example of the application of this DEFT, consider finding the Fourier Transform of a DE $\sin(t)$ signal, such as the $1\,\mathrm{Hz}$ signal shown in Fig. 6. Applying equation (3) to this signal yields the discrete spectrum shown in Fig. 7. As expected, Fig. 7 shows large peaks at $f = \pm 1\,\mathrm{Hz}$. A few harmonics are also visible. These harmonics are attributable to the quantized nature of DE signal, and are reduced in magnitude as the quantization step is reduced. As with the standard DFT, the frequency resolution of the periodic DEFT may be improved by increasing the number of samples over which the transform is computed.

IV. Computational Complexity

Evaluation of a single addend in the DEFT sum (equations (2) and (3)) involves 2 additions, 8 multiplications, 1 evaluation of a sinc function, and 1 evaluation of a complex exponential function. Evaluation of the complex exponential will typically be the most computationally expensive operation. For a DE signal containing N events, evaluation of the DEFT sum at a single frequency will require $O(N)$ operations, where "operation" in this case refers to computing a complex exponential. A DEFT evaluated at M discrete frequencies will thus require $O(MN)$ operations. Note that the complexity is dependent on the choice of frequency resolution, which is *independent* of the number of samples.

In contrast to the DEFT, finding the Fourier Transform of non-uniformly sampled signals using interpolation followed by an FFT will have a complexity of $O(M \log M)$. Although the operations counted in determining the complexity of the FFT are complex multiplications rather than complex exponentials, the difference between the two kinds of operations represents a constant factor and can be neglected when considering only the big-O complexity. To generate an accurate approximation of the Fourier Transform using interpolation and the FFT it may be necessary to make M large during interpolation. For signals with a number of events $N < \log M$ the DEFT provides lower computational complexity than FFT-based methods.

V. Conclusion

This note contributes to the theory of Discrete-Event Systems by developing a semi-analytical Fourier Transform for Discrete-Event signals. This Discrete-Event Fourier Transform uses the superposition property of the Fourier Transform to compute the frequency-domain representation of a DE signal as a sum of scaled sinc functions, where each sinc is the transform of a time-domain rectangular pulse that represents a single segment of the piecewise constant DE signal. This approach allows the frequency-domain representation of a DE signal to be computed directly, instead of using interpolation to approximate the DE signal as a DT signal and performing an FFT. The DEFT may be useful in application areas such as event-driven signal processing systems [11] and the analysis of event-based control systems [12].

After the work described here was completed, I became aware of what appears to be a related approach developed by Aeschlimann [13] for analyzing the spectra of zero-order hold reconstructions of non-uniformly sampled signals. However, Aeschlimann is not directly concerned with Discrete-Event systems, and thus treats the approach as an analysis technique rather than a transform method. Furthermore, he assumes that the value for an inter-event interval is defined by the amplitude of the sample at the *end* of the interval, which differs from the typical way Discrete-Event signals are represented, and he works with signals in which values are paired with delays rather than absolute times. Aeschlimann also does not discuss the handling of signal start or end states, transforms for the periodic case, or the computational complexity of the approach.

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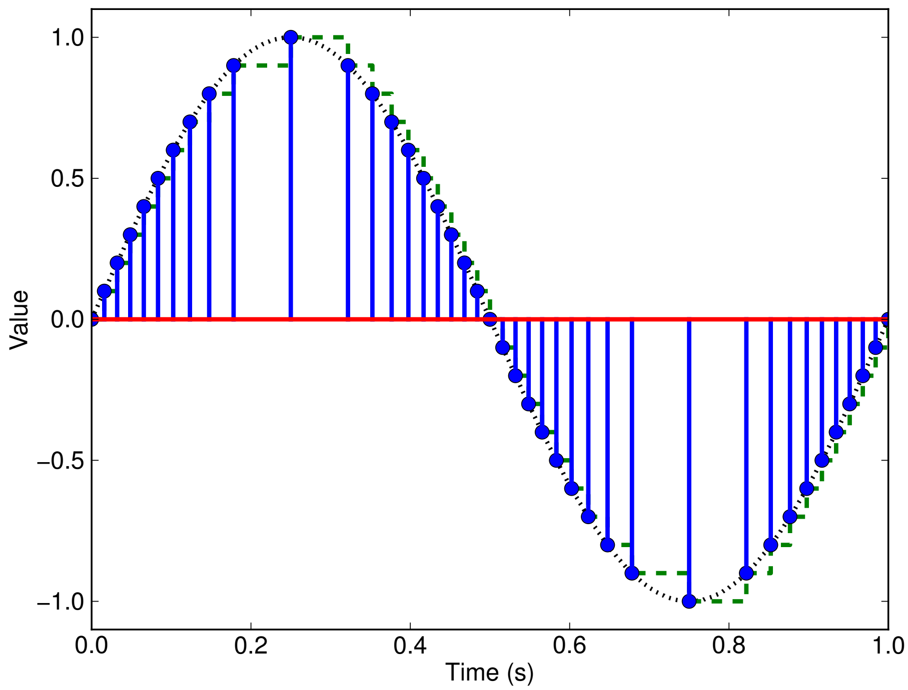

Fig. 6. A 1 Hz DE $\sin(t)$ signal with quantization step $\Delta = 0.1$

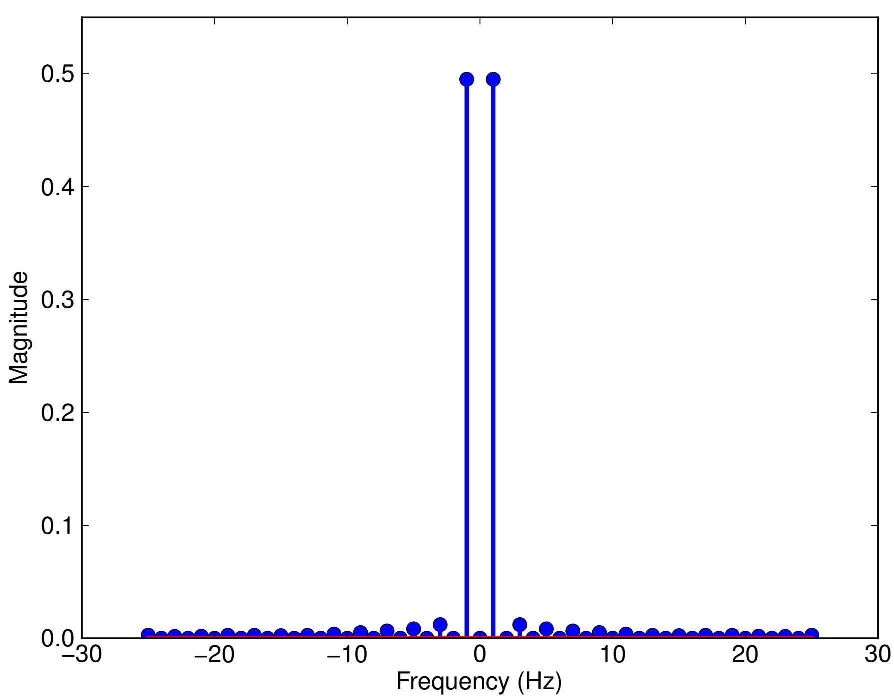

Fig. 7. DEFT of a DE $\sin(t)$ signal with quantization step $\Delta = 0.1$