Part A

Problem 1. Write a three point summary containing the following:

1) Why is this concept related to optimization?

2) Where do you think this concept will be used or applied while writing your optimization code?

3) One example for each concept with justification.

(Vector Space). Vector spaces are related to optimization because it provides a foundation for components of optimization problems. For an algorithm dealing with vectors, a way to implement error checking is by validating that inputs and outputs follow the axioms of a vector space, for each step of formulating, solving, and find a solution to the problem. An example is finding the feasible regions of an optimization problem, the feasible region is a subset of the vector space, therefore a vector space ensures certain properties when searching the feasible region.

(Linear dependence). Identifying linear dependence in an optimization problem may lead to a reduction of the problem or it may reduce the amount of work done. Linearly dependent entries in matrices cause problems when solving system of equations, such as finding inverse of a matrix during a factorization algorithm. Take the vectors $v = (1, 0)$ and $w = (2, 0)$, the second vector $w = 2v$, the vector w, may not produce any valuable information in a larger problem.

(Basis). A basis of a vector space, provides the minimum number of vectors needed to formalize and work with an optimization problem. The simplex method for linear programming uses the basis of a set of variables to find a solution. In a 2 dimensional problem the unit vectors $(1, 0)$ and $(0, 1)$ can be used to represent any point with addition and scalar multiplication.

(Norm). The norm is used to measure size or distance of a vector, in an optimization problem we can use a norm as an objective function. We can also use norm in an algorithm to compute stopping criteria. Let the vectors $v = (1, 0)$ and $w = (2, 0)$, then the Euclidean norm of $\|v\|_2 = 1$ and $\|w\|_2 = 2$, in a minimization problem we would choose v over w, since $\|v\|_2 < \|w\|_2$.

(Inner product). The inner product of two vectors in an optimization problem tells us how the vectors are related to each other. In gradient descent algorithm we use the inner product to verify the direction of a gradient. We can determine if two vectors are perpendicular, for example if $v \cdot w = 0$ the orthogonal properties under some conditions could signal a local minimum.

(Eigenvalue). Eigenvalues of some Hessian matrix can help describe a function in an optimization problem. In an algorithm, if we need to compute the Hessian of some matrix, we can evaluate critical points. If all eigenvalues of a Hessian are strictly positive, we can can prove there exists a global minimum for a problem.

Part B

Problem 1. Consider the following nonlinear programming problem:

$$\min_{x \in X} \quad (x_1 - 3)^2 + (x_2 - 3)^2$$
$$\text{s.t.} \quad 4x_1^2 + 9x_2^2 \le 36$$
$$x_1^2 + 3x_2 = 3$$

where $x = (x_1, x_2)' \in X = \{x : x_1 \ge -1\}$.

Sketch the feasible region for this optimization problem and identify the optimum solution graphically.
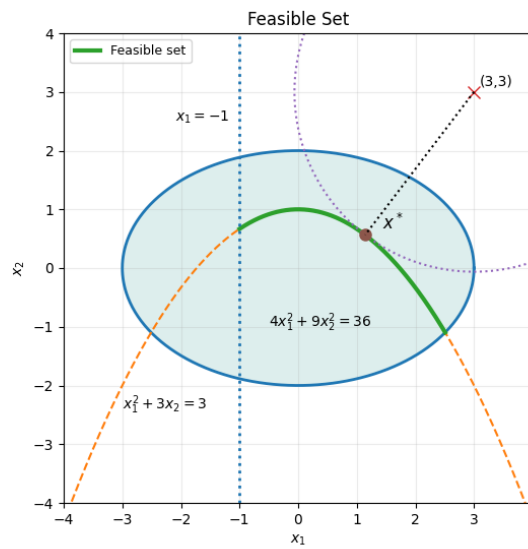

Figure 1: Graphical representation of the feasible region for **Problem 1**, where x^* is the optimal solution, in this case the computed solution is $x^* \approx (1.14, 0.56)$

Problem 2. Sketch the following sets and check whether the set is convex or not.

(a) $S_1 = \{x \in \mathbb{R}^3 | x_1 = x_2, x_2 = x_3\}$, where $x = (x_1, x_2, x_3)'$.

To check convexity, we let $x, y \in S_1$ and some scalar $\alpha \in \mathbb{R}$ then,

$$\alpha x + (1 - \alpha)y = (\alpha x_1 + (1 - \alpha)y_1, \alpha x_2 + (1 - \alpha)y_2, \alpha x_3 + (1 - \alpha)y_3),$$

since $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$, then

$$[\alpha x_1 + (1 - \alpha)y_1] = [\alpha x_2 + (1 - \alpha)y_2] = [\alpha x_3 + (1 - \alpha)y_3].$$

Thus, S_1 is convex.

(b) $S_2 = \{x \in \mathbb{R}^3 | x_1 + x_2 + x_3 = 1\}$, where $x = (x_1, x_2, x_3)'$.

To check convexity, we let $x, y \in S_2$ and some scalar $\alpha \in \mathbb{R}$, since $x_1 + x_2 + x_3 = 1$ and $y_1 + y_2 + y_3 = 1$, we begin by adding each component of αx and $(1 - \alpha)y$,

$$[\alpha x_1 + (1 - \alpha)y_1] + [\alpha x_2 + (1 - \alpha)y_2] + [\alpha x_3 + (1 - \alpha)y_3]$$

$$= \alpha x_1 + \alpha x_2 + \alpha x_3 + (1 - \alpha)y_1 + (1 - \alpha)y_2 + (1 - \alpha)y_3$$

$$= \alpha(x_1 + x_2 + x_3) + (1 - \alpha)(y_1 + y_2 + y_3)$$

$$= \alpha x + (1 - \alpha)y$$

thus, S_2 is convex.

(c) $S_3 = S_1 \cap S_2$.

To check convexity, we know $S_1 \cap S_2$ is the intersection of three equal points that add up to 1, thus $S_3 = \{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$ this is a single point thus by definition it is convex.

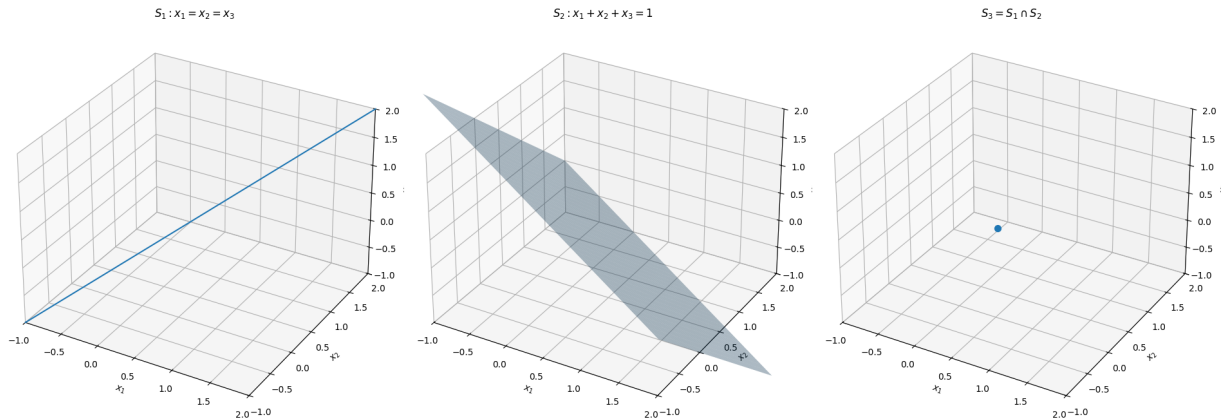**(d)** $S_4 = S_1 \oplus S_2$. Not answered.

Figure 2: Graphical representation each set for **Problem 2**. S_4 is not graphed.

Problem 3. Sketch the following functions and check whether they are convex or not.

(a) $f(x) = x_1^2 + x_2^2$.

(b) $f(x) = -x_1 \ln(x_1) - x_2 \ln(x_2)$.

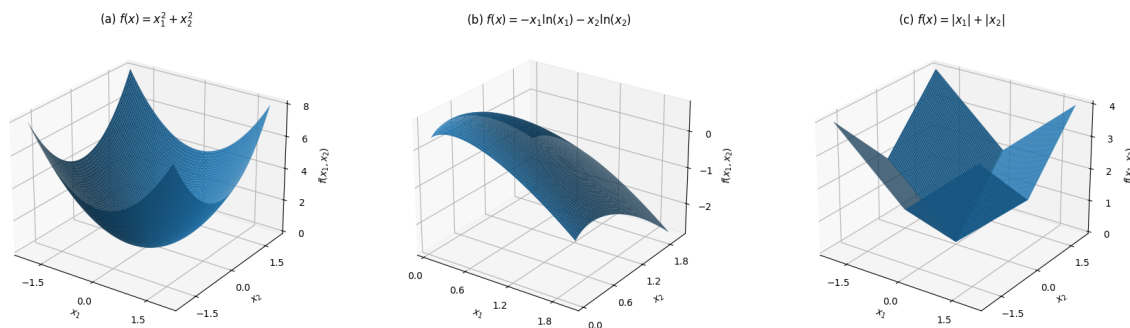**(c)** $f(x) = |x_1| + |x_2|$.

Figure 3: Plots for the functions of **Problem 3**.

By plotting the following functions we observe functions **(a)** and **(c)** are convex functions. While the function for **(b)** is not convex, but a concave function.

Part C

Problem 1. Find the eigenvalues and eigenvectors for the following matrices: How many linearly independent eigenvectors do these matrices have?

(a) $A = \begin{bmatrix} 2 & 7 \\ -1 & -6 \end{bmatrix}$. We first want to find the eigenvalues of A by solving $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2 - \lambda & 7 \\ -1 & -6 - \lambda \end{bmatrix}\right) = (2 - \lambda)(-6 - \lambda) + 7 = \lambda^2 + 4\lambda - 5.$$

Now we solve for λ, in $\lambda^2 + 4\lambda - 5 = 0$.

$$\lambda^2 + 4\lambda - 5 = 0$$
$$(\lambda + 5)(\lambda - 1) = 0.$$
$$\lambda_1 = 1, \qquad \lambda_2 = -5.$$

We now find the corresponding eigenvectors, by solving $(A - \lambda I)\mathbf{v} = 0$.
For $\lambda_1 = 1$,

$$A - I = \begin{bmatrix} 1 & 7 \\ -1 & -7 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} 1 & 7 \\ -1 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad x_1 = -7, x_2 = 1.$$

An eigenvector is,

$$\mathbf{v}_1 = \begin{bmatrix} -7 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = -5$,

$$A + 5I = \begin{bmatrix} 7 & 7 \\ -1 & -1 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 7 & 7 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad x_1 = 1, x_2 = -1.$$

An eigenvector is

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Since A has two distinct eigenvalues, it has 2 linearly independent eigenvectors.

(b) $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. Since B is upper triangular, its eigenvalues are the diagonal entries,

$$\lambda_1 = 3, \qquad \lambda_2 = 1, \qquad \lambda_3 = 1.$$

For $\lambda_1 = 3$, we solve $(B - 3I)\mathbf{v} = 0$:

$$B - 3I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies x_1 = \frac{1}{2}, x_2 = 1, x_3 = 1.$$

An eigenvector is

$$\mathbf{v_1} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}.$$

For $\lambda_{2,3} = 1$, solve $(B - I)\mathbf{v} = 0$:

$$B - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies x_1 = 1, x_2 = 0, x_3 = 0.$$

An eigenvector is

$$\mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Since B has two distinct eigenvalues, it has 2 linearly independent eigenvectors.

Problem 2. Find the global and local minimizers or maximizers (if any) for the following:

(a) $f(x_1, x_2) = x_1^3 - 12x_1x_2 + 8x_2^3, \quad (x_1, x_2) \in \mathbb{R}^2$

We begin by finding the gradient of $f(x_1, x_2)$ and solving $\nabla f = 0$.

$$\frac{\partial f}{\partial x_1} = 3x_1^2 - 12x_2,$$

$$\frac{\partial f}{\partial x_2} = -12x_1 + 24x_2^2,$$

$$\nabla f(x_1, x_2) = \begin{bmatrix} 3x_1^2 - 12x_2 \\ -12x_1 + 24x_2^2 \end{bmatrix}.$$

We now solve $\nabla f = 0$,

$$\begin{bmatrix} 3x_1^2 - 12x_2 \\ -12x_1 + 24x_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From $3x_1^2 - 12x_2 = 0$ we get $x_2 = \frac{x_1^2}{4}$. Substituting into the second equation,

$$-12x_1 + 24\left(\frac{x_1^2}{4}\right)^2 = 0 \implies -12x_1 + \frac{3}{2}x_1^4 = 0 \implies \frac{3}{2}x_1(x_1^3 - 8) = 0,$$

so $x_1 = 0$ or $x_1 = 2$.

If $x_1 = 0$, then $x_2 = 0$. If $x_1 = 2$, then $x_2 = 1$, thus the critical points are $(0, 0)$ and $(2, 1)$.

Next we perform the second derivative test using the Hessian.

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 6x_1 & -12 \\ -12 & 48x_2 \end{bmatrix}.$$

At $(0, 0)$,

$$\nabla^2 f(0, 0) = \begin{bmatrix} 0 & -12 \\ -12 & 0 \end{bmatrix}, \qquad \det\left(\begin{bmatrix} 0 & -12 \\ -12 & 0 \end{bmatrix}\right) = -144 < 0,$$

so the Hessian is indefinite and $(0, 0)$ is a saddle point.

At $(2, 1)$,

$$\nabla^2 f(2, 1) = \begin{bmatrix} 12 & -12 \\ -12 & 48 \end{bmatrix}.$$

We check the leading principal minors,

$$\Delta_1 = 12 > 0, \qquad \Delta_2 = \det\left(\begin{bmatrix} 12 & -12 \\ -12 & 48 \end{bmatrix}\right) = 576 - 144 = 432 > 0.$$

Thus $\nabla^2 f(2, 1)$ is positive definite, so $(2, 1)$ is a strict local minimizer.

There is no local maximizer, since $\nabla^2 f$ is not negative definite at a critical point.

We now check for global minimizers and maximizers,

$$f(x_1, 0) = x_1^3 \to \infty \quad \text{as } x_1 \to \infty, \qquad f(-x_1, 0) = -x_1^3 \to -\infty \quad \text{as } x_1 \to \infty.$$

Thus, f is unbounded above and below, so it has no global maximizer or global minimizer.

(b) $f(x_1, x_2, x_3) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2, \quad (x_1, x_2, x_3) \in \mathbb{R}^3$

We begin by finding the gradient of $f(x_1, x_2, x_3)$ and solving $\nabla f = 0$.

$$f(x_1, x_2, x_3) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2,$$

$$\frac{\partial f}{\partial x_1} = 2(2x_1 - x_2) \cdot 2 = 8x_1 - 4x_2,$$

$$\frac{\partial f}{\partial x_2} = 2(2x_1 - x_2)(-1) + 2(x_2 - x_3) = -4x_1 + 4x_2 - 2x_3,$$

$$\frac{\partial f}{\partial x_3} = 2(x_2 - x_3)(-1) + 2(x_3 - 1) = -2x_2 + 4x_3 - 2.$$

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 8x_1 - 4x_2 \\ -4x_1 + 4x_2 - 2x_3 \\ -2x_2 + 4x_3 - 2 \end{bmatrix}.$$

We now solve $\nabla f = 0$,

$$\begin{bmatrix} 8x_1 - 4x_2 \\ -4x_1 + 4x_2 - 2x_3 \\ -2x_2 + 4x_3 - 2 \end{bmatrix} x = [0, 0, 0]^T$$

Therefore $(x_1, x_2, x_3) = \left(\frac{1}{2}, 1, 1\right)$ is a critical point.

Next we perform the second derivative test using the Hessian.

$$\nabla^2 f(x_1, x_2, x_3) = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix}.$$

We check the leading principal minors,

$$\Delta_1 = 8 > 0,$$

$$\Delta_2 = \det \begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix} = 32 - 16 = 16 > 0, \qquad \Delta_3 = \det \begin{bmatrix} 8 & -4 & 0 \\ -4 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix} = 96 - 64 = 32 > 0.$$

Thus $\nabla^2 f$ is positive definite, so f is strictly convex and the critical point $\left(\frac{1}{2}, 1, 1\right)$ is a strict local minimizer. Hence $\left(\frac{1}{2}, 1, 1\right)$ is also the unique global minimizer.

Then, f has no local or global maximizer, since $\nabla^2 f$ is positive definite, f is convex so it has no local maxima, and

$$\lim_{x \to \infty} f(x_1, x_2, x_3) = \infty,$$

so f is unbounded above and has no global maximizer.

Problem 3. Let $f(x) = c_1x_1^2 + c_2x_2^2 + c_3x_3^2 + c_4x_1x_2 + c_5x_1x_3 + c_6x_2x_3$.

(a) Represent the function $f(x)$ defined on $\mathbb{R}^3$, i.e., $f : \mathbb{R}^3 \to \mathbb{R}$ in quadratic form.

We want to represent $f(x)$ in quadratic form $f(x) = x^\top Q x$, where Q is a symmetric matrix.

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Since $x^\top Q x = q_{11}x_1^2 + q_{22}x_2^2 + q_{33}x_3^2 + 2q_{12}x_1x_2 + 2q_{13}x_1x_3 + 2q_{23}x_2x_3$,

We assign the following, $c_1 = q_{11}, c_2 = q_{22}, c_3 = q_{33}$. Notice, to assign c_4, c_5, c_6 and to make Q symmetric we do the following, $c_4 = 2q_{12} \Rightarrow \frac{c_4}{2} = q_{12}$. Thus $\frac{c_5}{2} = q_{13}$ and $\frac{c_6}{2} = q_{23}$. Thus,

$$Q = \begin{bmatrix} c_1 & \frac{c_4}{2} & \frac{c_5}{2} \\ \frac{c_4}{2} & c_2 & \frac{c_6}{2} \\ \frac{c_5}{2} & \frac{c_6}{2} & c_3 \end{bmatrix}.$$

Hence,

$$f(x) = x^\top Q x.$$

(b) Derive its first derivative/gradient

We derive the first derivative and gradient of $f(x)$,

$$\frac{\partial f}{\partial x_1} = 2c_1x_1 + c_4x_2 + c_5x_3,$$

$$\frac{\partial f}{\partial x_2} = 2c_2x_2 + c_4x_1 + c_6x_3,$$

$$\frac{\partial f}{\partial x_3} = 2c_3x_3 + c_5x_1 + c_6x_2.$$

$$\nabla f(x) = \begin{bmatrix} 2c_1x_1 + c_4x_2 + c_5x_3 \\ c_4x_1 + 2c_2x_2 + c_6x_3 \\ c_5x_1 + c_6x_2 + 2c_3x_3 \end{bmatrix}.$$

Equivalently, since Q is symmetric,

$$\nabla f(x) = (Q + Q^\top)x = 2Qx.$$

(c) Derive its Hessian.

$$\nabla^2 f(x) = \begin{bmatrix} 2c_1 & c_4 & c_5 \\ c_4 & 2c_2 & c_6 \\ c_5 & c_6 & 2c_3 \end{bmatrix}.$$

Equivalently,

$$\nabla^2 f(x) = Q + Q^\top = 2Q.$$

Problem 4.

(a) Show that the function $f(x, y, z) = \exp(x^2 + y^2 + z^2) - x^4 - y^6 - z^6$ has a global minimizer on $\mathbb{R}^n$.

Not Answered.

(b) For the function $f(x, y) = x^3 \exp(3y) - 3x \exp(y)$, show that only one critical point exists and that this point is not a global minimizer in $\mathbb{R}^2$.

To find the critical points, we compute the gradient,

$$\frac{\partial f}{\partial x} = 3x^2 - 3e^y,$$

$$\frac{\partial f}{\partial y} = 3e^{3y} - 3xe^y,$$

$$\nabla f(x, y) = \begin{bmatrix} 3x^2 - 3e^y \\ 3e^{3y} - 3xe^y \end{bmatrix}.$$

We now solve $\nabla f = 0$,

$$\begin{bmatrix} 3x^2 - 3e^y \\ 3e^{3y} - 3xe^y \end{bmatrix} x = [0, 0]^\top$$

From $3x^2 - 3e^y = 0 \implies x^2 = e^y$, and from $3e^{3y} - 3xe^y = 0 \implies e^{3y} = xe^y \implies e^{2y} = x$.

Then substituting x into the first equation $x^2 = e^y$, we see $e^{4y} = e^y \implies 4y = y \implies 3y = 0$. Thus, $y = 0$, substituting into $x = e^{2y} \implies x = e^0 = 1$.

Therefore $(x, y) = (1, 0)$ is a unique critical point. To show it is not a global minimizer we evaluate the critical point,

$$f(1, 0) = 1^3 - 3 = -2$$

but, when $y = 0$ and as $x \to -\infty$, then $f(x, 0) \to -\infty < -2$.

Thus, f is unbounded below, so there is no global minimum, and hence the critical point is not a global minimizer.

Problem 5.

(a) Give an example for an optimization problem in your own words. Try to describe a problem that has real implications.

An optimization problem can be deciding which gas station to fill your tank at. There are many factors to consider, such as distance from each gas station, price of each gas station, your vehicle's MPG. Essentially, is it worth traveling farther for a cheaper price? This could have real implications for any companies that wants to minimize their expenses on gas, think delivery drivers, truck/freight companies, or even a police department's patrol fleet.

(b) Identify all the information relevant to formulate meaningful constraints.

Suppose there is a finite set of gas stations $i \in \{1, \ldots, G\}$. For each station i, let p_i be the price per gallon and let d_i be the one way distance in miles from your current location to station i.

Your vehicle has tank capacity c (gallons) and initial fuel level c_0 with $0 \le c_0 \le c$. Your vehicle gets n miles per gallon, so the fuel required to drive one way to station i is $q_i = \frac{d_i}{n}$ (gallons).

Let $y_i \in \{0,1\}$ indicate whether station i is chosen (1) or not (0).

The problem then is to choose one gas station that minimizes the total cost under the following constraints,

- You must be able to reach the chosen station using the initial fuel, $q_i \leq c_0$.

- Arriving at station i leaves fuel $c_0 - q_i \geq 0$.

- You must fill the tank to capacity at the chosen station, the number of gallons purchased at station i is $x_i = c - c_0 + q_i$. Which satisfies $0 \leq x_i \leq c$ whenever $q_i \leq c_0$.

- After filling to full capacity c, returning home consumes another q_i gallons, so the final fuel level $c_f = c - q_i$, which must satisfy $c_f \geq 0$.

(c) Formulate/Derive the mathematical model of the stated optimization problem.

$$\min \sum_{i=1}^{G} p_i x_i$$

$$\text{s.t.} \sum_{i=1}^{G} y_i = 1$$

$$q_i y_i \leq c_0 y_i$$
$$x_i = (c - c_0 + q_i)\, y_i$$
$$y_i \in \{0,1\}, \ x_i \geq 0 \qquad \forall i, i = 1, \ldots, G.$$

Problem 6. Show that the matrix $\begin{pmatrix} x^4 & x^3 & x^2 \\ x^3 & x^2 & x \\ x^2 & x & 1 \end{pmatrix}$ is positive semi-definite but not positive definite.

We check the principal minors,

$$\Delta_1 = x^4 > 0, \qquad \forall x$$

$$\Delta_2 = \det\left(\begin{pmatrix} x^4 & x^3 \\ x^3 & x^2 \end{pmatrix}\right) = x^4 x^2 - x^3 x^3 = x^6 - x^6 = 0,$$

$$\Delta_3 = \det\left(\begin{pmatrix} x^4 & x^3 & x^2 \\ x^3 & x^2 & x \\ x^2 & x & 1 \end{pmatrix}\right) = x^4(x^2 - x^2) - x^3(x^3 - x^3) + x^2(x^4 - x^4) = 0,$$

Thus the matrix is positive semi-definite but not positive definite, since the 2nd and 3rd principal minors are equal to 0.

Problem 7. Let $g(x)$ be differentiable on $\mathbb{R}^m$ with continuous first partial derivatives. Let A be an $m \times n$ matrix. Define f on $\mathbb{R}^n$ by $f(y) = g(Ay)$. Compute $\nabla f, \nabla g,$ and $\nabla^2 f$

Not Answered.

Problem 8. Show that the collection of all polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

defined on the interval $[a, b]$ with real coefficients $a_i \in \mathbb{R}$, forms a real vector space.

Proof. We let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ and we will show $f(x)$ forms a real vector space on the interval $[a, b]$ with real coefficients $a_i \in \mathbb{R}$. Then let $g(x)$ be of the same form with real coefficients $b_i \in \mathbb{R}$.

Notice, $f(x) + g(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \ldots + (a_1 + b_1)x + (a_0 + b_0)$, since $\mathbb{R}$ is closed under addition each coefficient $(a_i + b_i) \in \mathbb{R}$ and $(b_i + a_i) \in \mathbb{R}$. Then, by commutative laws, swapping the coefficients shows $f(x) + g(x) = g(x) + f(x)$, proving additive commutativity.

Now, let $h(x)$ be of the same form with real coefficients $c_i \in \mathbb{R}$ then, $f(x) + (g(x) + h(x)) = (a_n + (b_n + c_n))x^n + (a_{n-1} + (b_{n-1} + c_{n-1}))x^{n-1} + \ldots + (a_0 + (b_0 + c_0))$, by the associative laws of $\mathbb{R}$, $a_i + (b_i + c_i) \equiv (a_i + b_i) + c_i$. Thus $f(x) + (g(x) + h(x)) = (f(x) + g(x)) + h(x)$ proving additive associativity.

Now, define $0(x) = 0$, for all $x \in [a, b]$, then $(f + 0)(x) = f(x) + 0 = f(x)$, thus there exists a unique zero polynomial that satisfies the additive identity.

Next, we consider $f(x) + (-1)f(x) = a_n x^n + \ldots + a_0 + (-1)(a_n x^n + \ldots + a_0) = a_n x^n + \ldots + a_0 - a_n x^n - \ldots - a_0 = 0$, since all terms cancel, proving the additive inverse.

Multiplying $1 \cdot f(x) = (1) \cdot (a_n x^n + \ldots + a_0) = a_n x^n + \ldots + a_0 = f(x)$, thus the polynomial form does not change when multiplying by 1, proving the multiplicative identity.

Let $\alpha \in \mathbb{R}$ then, $\alpha(f(x) + g(x)) = \alpha(a_n x^n + \ldots + a_1 x + a_0 + b_n x^n + \ldots + b_1 x + b_0)$,

$$= \alpha a_n x^n + \ldots + \alpha a_0 + \alpha b_n x^n + \ldots + \alpha b_0,$$

$$= \alpha(a_n x^n + \ldots + a_0) + \alpha(b_n x^n + \ldots + b_0),$$

$$= \alpha f(x) + \alpha g(x).$$

Thus proving scalar distributivity.

Let $\beta \in \mathbb{R}$ then, $(\alpha + \beta)f(x) = (\alpha + \beta)(a_n x^n + \ldots + a_0)$

$$= \alpha a_n x^n + \ldots + \alpha a_0 + \beta a_n x^n + \ldots + \beta a_0$$

$$= \alpha(a_n x^n + \ldots + a_0) + \beta(a_n x^n + \ldots + a_0)$$

$$= \alpha f(x) + \beta f(x).$$

Thus proving polynomial distributivity.

Finally to prove multiplicative associativity, since $\alpha, \beta \in \mathbb{R}$ and we know $\mathbb{R}$ and polynomials are closed under multiplication, it is easy to see that,

$$(\alpha\beta)f(x) = (\alpha\beta)a_n x^n + \ldots + (\alpha\beta)a_0,$$

$$= \alpha(\beta a_n x^n + \ldots + \beta a_0),$$

$$= \alpha(\beta f(x)).$$

Proving multiplicative associativity.

Therefore, the set of all real-coefficient polynomial functions on $[a, b]$ satisfies the vector space axioms over $\mathbb{R}$, and thus forms a real vector space. ■

Appendix

A2

All files and code can be found on Github.

`https://github.com/allanpaiz/csce-590/tree/main/hw2`

Sources

1. **Use of AI**   During the completion of this assignment, I used ChatGPT 2.5 (OpenAI) as an assistive tool. I reviewed and edited all AI assisted outputs, and take full responsibility for the accuracy of my work.

2. Lecture notes from **CSCE 590** : *Optimization* by Narayanan, Vignesh.

3. Lecture notes from **MATH 344** : *Applied Linear Algebra* by Linz, William.