Project 6.1: FitzHugh-Nagumo

The phenomenon of excitability exists in many biological systems, including in the electrophysiology of neurons. The FitzHugh-Nagumo equations

$$\frac{dv}{dt} = v - \frac{1}{3}v^3 - w \tag{1}$$
$$\frac{dw}{dt} = \epsilon\left(v + a - bw\right) \tag{2}$$

describe neuron electrophysiology where, roughly speaking, v is the electrical potential (voltage) across the cell's membrane, and w is the activity of ion pumps. The parameters ϵ, a and b represent properties of the ion pumps. The model has been nondimensionalized. Both v and w can be negative or positive.

1. Confirm that for $\epsilon = 0.08$, $a = 0.5$, $b = 0.2$, the system is oscillatory.
2. Confirm that for $\epsilon = 0.08$, $a = 1.0$, $b = 0.2$. the system is excitable. Specifically, if you choose initial conditions $v(0) = -1.5, w(0) = -0.5$, the system evolves directly towards a stable steady state, but if you choose initial conditions $v(0) = -0.0, w(0) = -0.5$, the system moves away from the steady state, before eventually converging towards the steady state.

Assume the neuron is at rest (at its steady state), and another cell injects a current into it. The current is injected between $t = 40$ and $t = 47$, and has a strength of $I_0 = 1.0$. In the model, this means

$$\frac{dv}{dt} = v - \frac{1}{3}v^3 - w + I(t) \tag{3}$$
$$\frac{dw}{dt} = \epsilon\left(v + a - bw\right) \tag{4}$$

where

$$I(t) = \begin{cases} I_0 & t_{\text{start}} < t < t_{\text{stop}} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

or, in Matlab,

```
I0 = 1.0;
tStart = 40;
tStop = 47;
I =@(t) I0*(t>tStart).*(t<tStop);
```

3. At the excitable parameters from above ($a = 1.0$), simulate the system with initial conditions at the steady state (or very close), and simulate an injection at $t = 40$ as above.

Neurons are connected in a neural network. Suppose there are ten cells, each with membrane potential and ion pump activity obeying the FitzHugh-Nagumo equations for $v_i(t)$ and $w_i(t)$ where $i = 1..10$ indexes the cells. The cells are electrically connected so that

$$\frac{dv_i}{dt} = v_i - \frac{1}{3}v_i^3 - w_i + I_i(t) + D\left(v_{i-1} - 2v_i + v_{i+1}\right) \tag{6}$$
$$\frac{dw_i}{dt} = \epsilon\left(v_i + a - bw_i\right) \tag{7}$$

where $D = 0.9$ is a new parameter that described the electrical connectivity of the neighboring cells. The ion pumps are not connected between cells, so the w equation is unchanged. For simplicity, let's assume the cells are connected in a ring, so that

$$\frac{dv_1}{dt} = v_1 - \frac{1}{3}v_1^3 - w_1 + I_1(t) + D\left(v_{10} - 2v_1 + v_2\right) \tag{8}$$

and similarly for the tenth cell.

4. Write Matlab code to simulate these ten cells. There will be 20 equations v, w for each cell.

   (a) Assume there is no injection current. We expect all ten cells to settle at the same steady state. Make two plots. First, plot a time series of the membrane potential of all ten cells as a function of time. Second, make a movie of the voltage in all ten cells, where the horizontal axis is the cell number.

```
% movie
for nt=1:numel(T)
    figure(5); clf; hold on; box on;
    plot(X(nt,1:10));
    set(gca,'ylim', [-2.5,2.5])
    xlabel('Cell');
    ylabel('Voltage')
    drawnow;
end
```

   (b) Now assume that the fourth cell (and only the fourth cell) receives an injection current $I(t)$ as above, between $t = 40$ and $t = 47$. Make a time series with all ten cells. Make a movie with voltage for all cells. [1]

   Hint: The most elegant and extensible code design choice would be to write the functions for a general number of cells N_{cell}. If you cannot figure out how to do that, in this case, it is possible to make it work with an inelegant solution of writing out all 20 ODEs. Make a judicious choice between elegance and getting things done.

[1] This type of behavior is called an *excitable traveling wave pulse*. These are unlike harmonic pulses familiar from sound waves, vibrations, light. For example, when two excitable pulses collide, they annihilate.