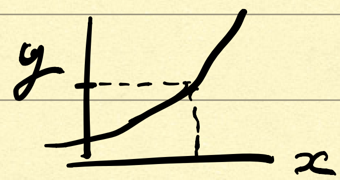
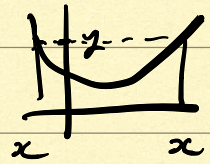
$X \sim p_X(x)$ AND $Y = g(X)$ THEN

$$p_Y(y) = \sum_k p_X(g_k^{-1}(y)) \cdot \left(\left| \frac{dg}{dx_k} \right| \right)^{-1}$$

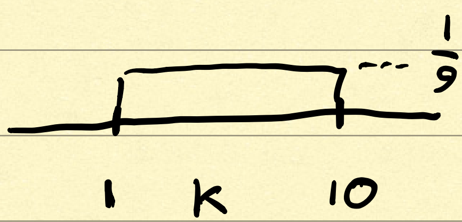
WHERE $\{x_k\}$ IS THE PRE-IMAGE OF y

IF $g(x)$ IS MONOTONIC 

(NOT )

THEN $p_Y(y) = p_X(g^{-1}(y)) \left| \frac{dg}{dx} \right|^{-1}$ ← $x = g^{-1}(y)$

EX



$p_K(k)$

$K \sim$ UNIFORM $[1, 10]$

K - RATE

T - MEAN TIME

$T = \frac{1}{K}$

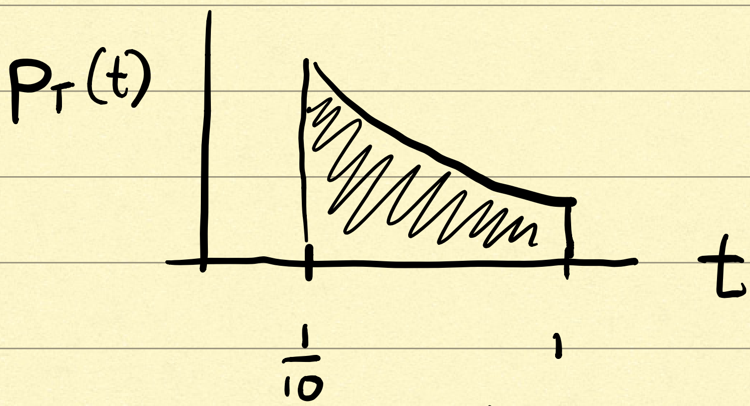
WHAT IS $p_T(t)$?

$$g(k) = \frac{1}{k} \qquad g^{-1}(t) = \frac{1}{t}$$

$$\frac{dg}{dk} = \frac{-1}{k^2}$$

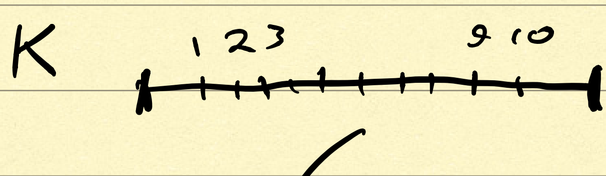
$$\left.\frac{dg}{dk}\right|_{k=g^{-1}(t)} = -t^2$$

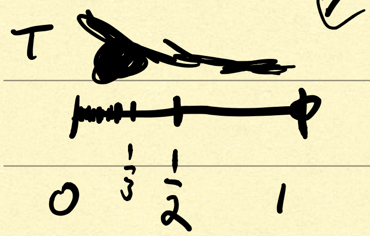
$$p_T(t) = \begin{cases} \frac{1}{9} \cdot t^{-2} & \frac{1}{10} < t < 1 \\ 0 & \text{ELSE} \end{cases}$$



CHECK : $$\int_{\frac{1}{10}}^{1} p_T(t)\,dt = \int_{\frac{1}{10}}^{1} \frac{1}{9} t^{-2}\,dt = -\frac{1}{9}\frac{1}{t}\Big|_{t=\frac{1}{10}}^{t=1}$$

$$= \frac{1}{9}(10-1) = 1 \checkmark$$





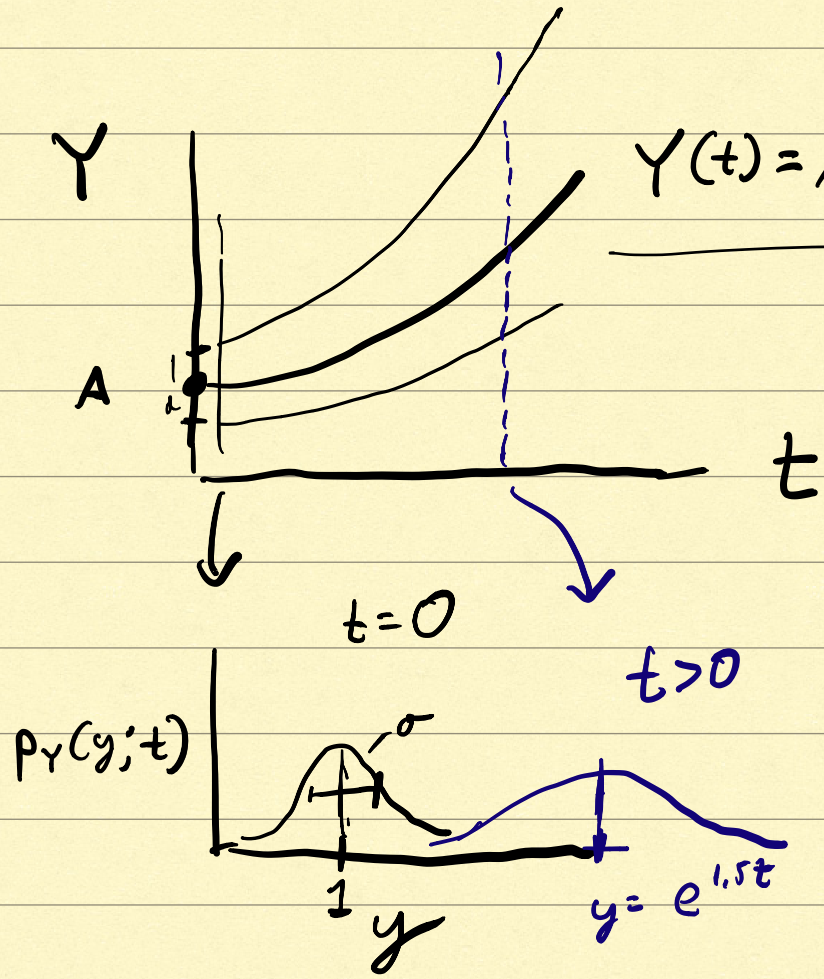
EX HETEROGENEITY / PARAMETRIC NOISE

$$\frac{dY}{dt} = (R-1)Y \qquad R = 2.5$$

$$Y(0) = A \qquad A \sim p_A(a)$$

$$p_A(a) = \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{(a-1)^2}{2\sigma_0^2}}$$

$$Y(t) = A e^{+1.5t}$$



$$Y = g(A) = g(A;t)$$

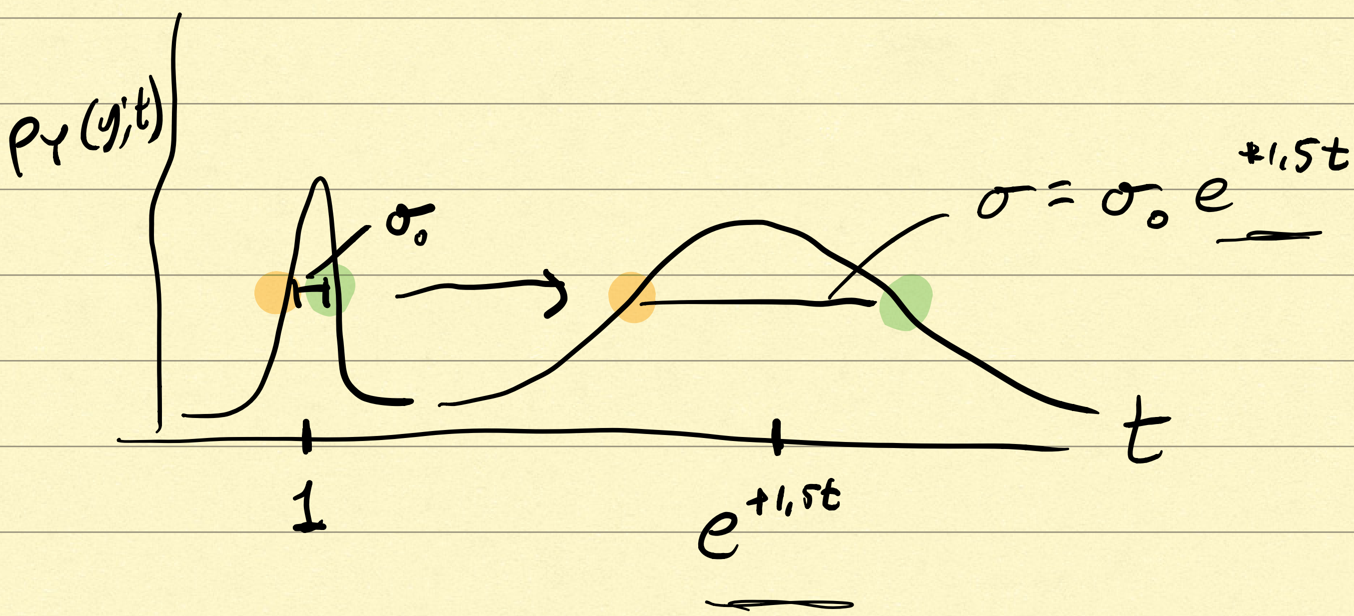
$$g^{-1}(y) = \frac{y}{e^{+1.5t}} = y e^{-1.5t}$$

$$\frac{dg}{da} = e^{1.5t}$$

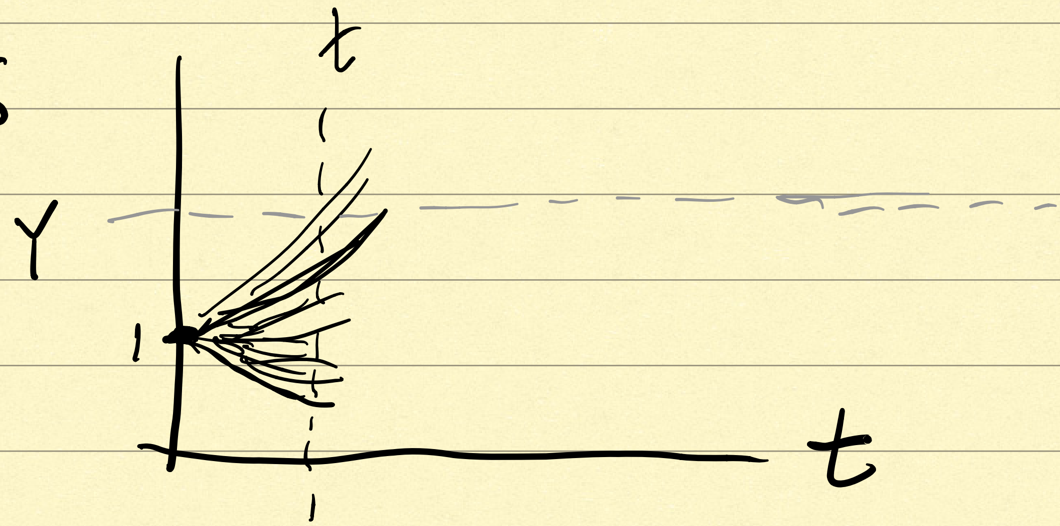
$$\left.\frac{dg}{da}\right|_{a=g^{-1}(y)} = e^{1.5t} \qquad p_X(g^{-1}(y)) \cdot \left|\frac{dg}{da}\right|^{-1}$$

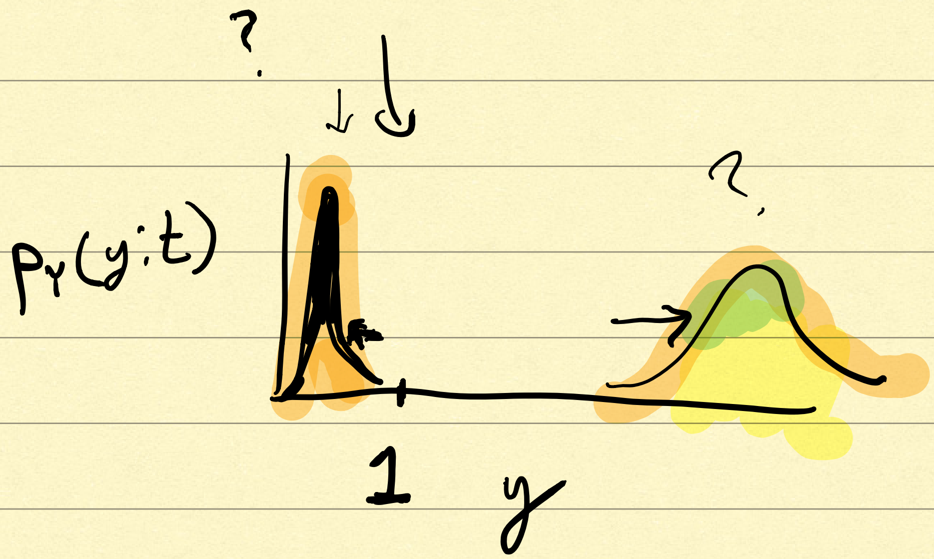
$$p_Y(y;t) = \frac{1}{\sqrt{2\pi}\,\sigma_0} \exp\left(-\frac{(y e^{-1.5t} - 1)^2}{2\sigma_0^2}\right) e^{-1.5t}$$

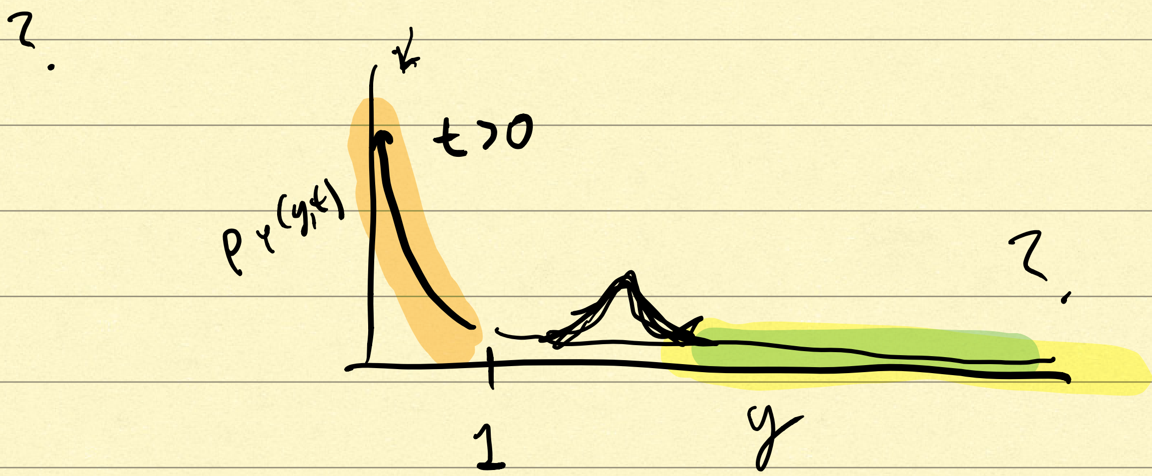
$$= \frac{1}{\sqrt{2\pi}\,\sigma_0 e^{+1.5t}} \exp\left(-\frac{(y - e^{+1.5t})^2}{2(\sigma_0 e^{+1.5t})^2}\right)$$

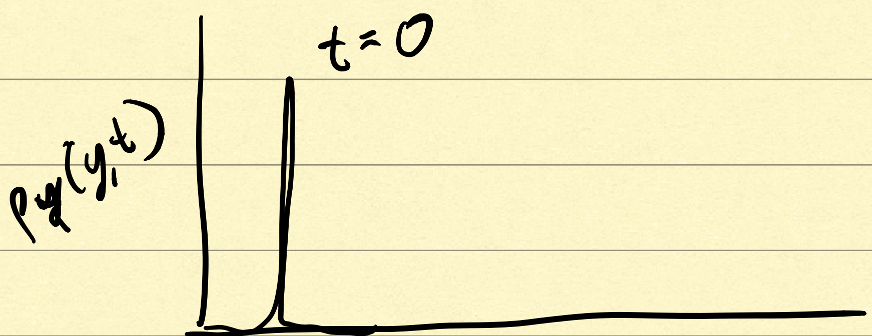


PS 6









$1 \qquad y$

$$\boxed{p_Y(y;t) = p_R\left(\frac{\ln(y)}{t} + 1\right) \cdot \frac{1}{ty}}$$

CHECK $\displaystyle\int_0^\infty p_Y(y;t)\,dy = 1$

$t \gg 1$?

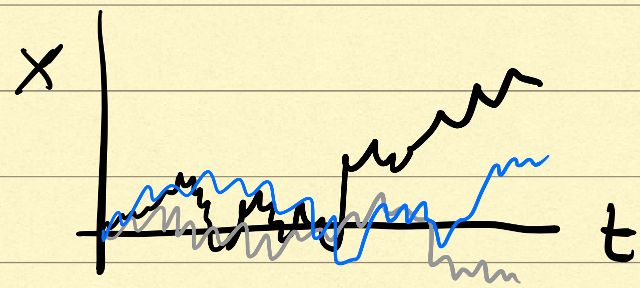
$$\mathbb{P}(Y > 1) = \int_1^\infty p_Y(y;t)\,dy$$

$$\mathbb{P}(Y < 1) = \int_0^1 p_Y(y;t)\,dy$$

STOCHASTIC DIFFERENTIAL EQUATIONS

$X(t)$ – CONTINUOUS-TIME, CONTINUOUS STATE SPACE STOCHASTIC PROCESS

EX.

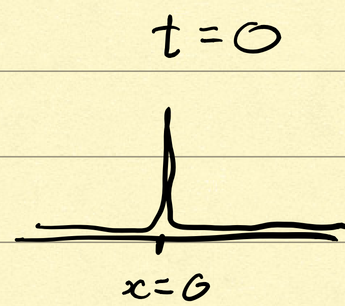


DIFFUSION $X(t)$

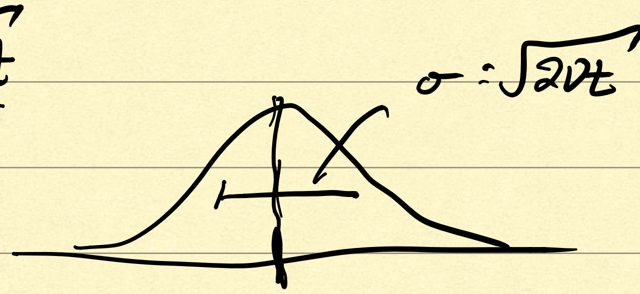
$p_X(x;t)$ SATISFIES

$$\frac{\partial p_X}{\partial t} = D \frac{\partial^2 p_X}{\partial x^2}$$

IF $p_X(x,0) = \delta(x)$



THEN $p_X(x;t) = \frac{1}{\sqrt{2\pi D t}} e^{-\frac{x^2}{2Dt}}$



IF $D = 1$

$B(t)$ – BROWNIAN MOTION

GENERALIZE $X(t) = \sigma B(t)$

$$\frac{dX}{dt} = \underline{\underline{f(x)}} + \sigma \underline{W(t)}$$

$$W(t) = \frac{dB}{dt}$$

$\downarrow$

WHITE NOISE

PROBLEM: $B(t)$ IS NOT DIFFERENTIABLE!

OLD CALCULUS $X(t) \approx a_0 + a_1 t + a_2 t^2 + \dots$

NEW CALCULUS $X(t) \approx a_0 + a_{\frac{1}{2}} t^{\frac{1}{2}} + a_1 t + a_2 t^2$

① $\underbrace{dX} = \underbrace{f(x)\,dt} + \sigma \underbrace{dB}$

$$\frac{dX}{dt} = f(x) + \sigma \underline{\xi(t)}$$

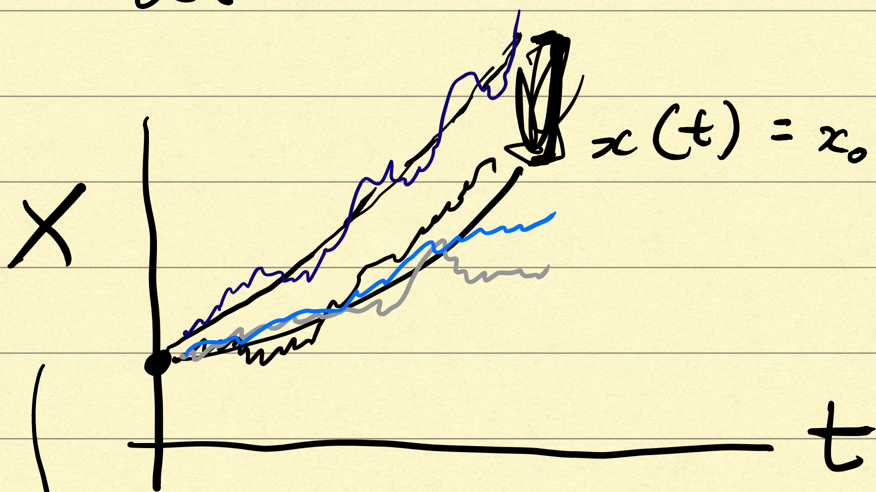
② IN STOCHASTIC CALCULUS, THE NEW CHAIN RULE GETS A BONUS TERM.

$$dx = ax\,dt + \underbrace{\sigma x\,dB}$$

BLACK-SCHOLES MODEL

$$\frac{dx}{dt} = ax$$

$$x(t) = x_0 e^{at}$$



- ITO
- THORP
- BLACK
- SCHOLES
- SIMMONS

MFPT

$$-1 = D\frac{\partial^2}{\partial x^2}T$$