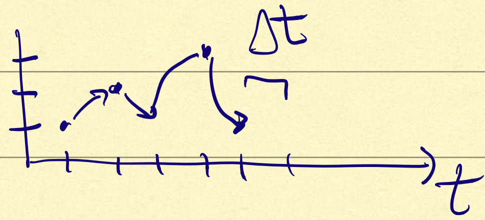
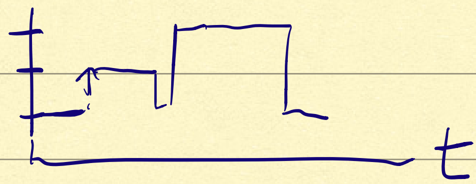
REVIEW

RANDOM VARIABLES $\quad X \sim p_x(x)$

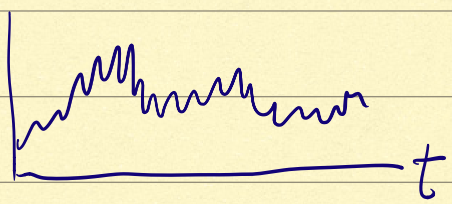
STOCHASTIC PROCESS



MARKOV CHAINS



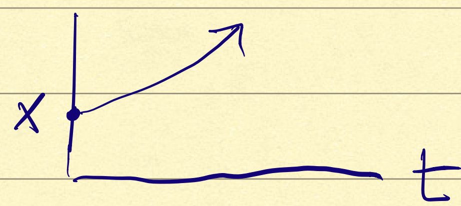
POISSON, CONTINUOUS TIME MARKOV CHAIN



STOCHASTIC DIFFERENTIAL EQUATION

PARAMETRIC NOISE ~ HETEROGENEITY

EX $\quad \dfrac{dy}{dt} = Ay$



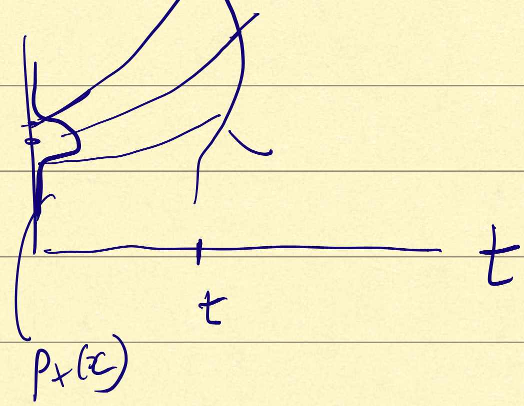
$y(0) = X$

$y(t) = X e^{+At}$

SUPPOSE $\quad X \sim p_x(x)$

X

THEN



WHAT IS $y \sim p_y(y)$?

$\downarrow$

$y(t) \sim p_y(y; t)$

SUPPOSE $Y = g(X)$ AND $X \sim p_x(x)$.

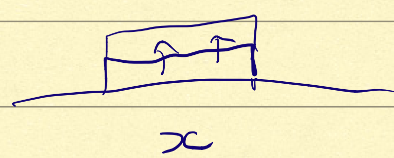
$\uparrow$ SOME FUNCTION

THEN WHAT IS $p_Y(y)$?

SUPPOSE $Y = cX$

IT IS **NOT** JUST

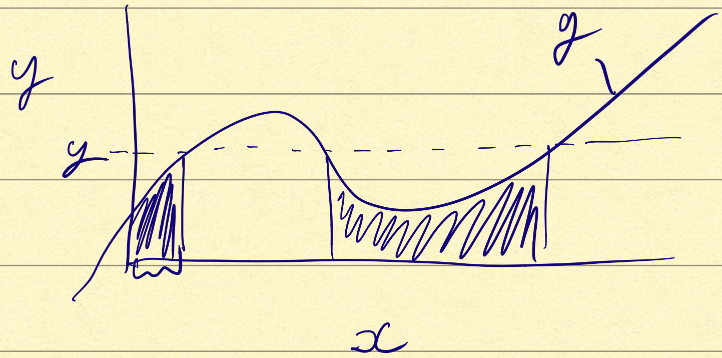
$p_Y(y) = c\, p_x(x)$ ...

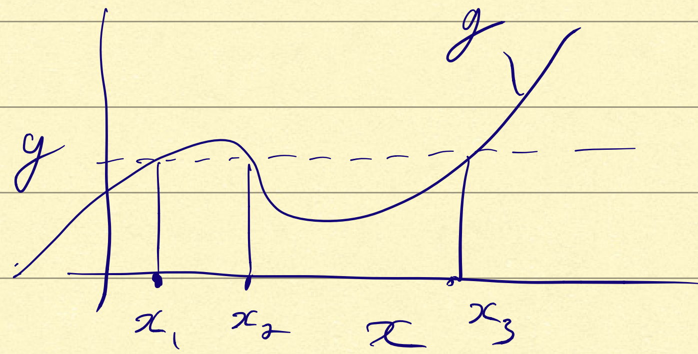


FOR A GENERAL FUNCTION $g(x)$

DEFINE $I_y = \{x : g(x) \le y\}$ INDEX SET OF y

$\{x_k\} = \{x : g(x) = y\}$ PRE-IMAGE OF y





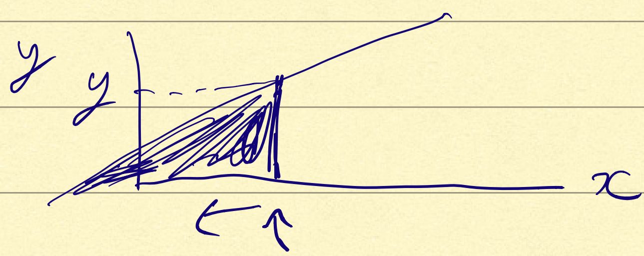
CUMULATIVE $F_Y(y) = \mathbb{P}(Y \le y)$

$= \mathbb{P}(x \in I_y)$

$= \int_{I_y} p_X(x)\, dx$

EX $y = ax + b$

$a > 0$



$I_y = \{x \le \frac{y-b}{a}\}$

$$x \le \frac{y-b}{a}$$

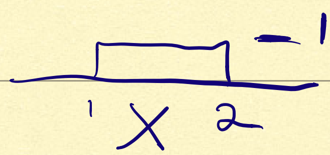
GIVEN $F_X(x)$, THEN

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right)$$

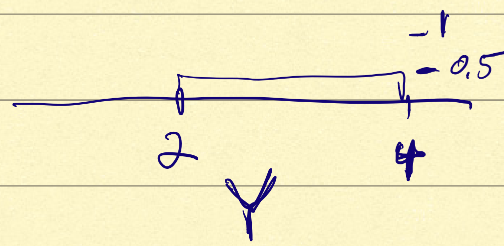
$$p_Y(y) = \frac{d}{dy} F_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{a}\frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = \frac{1}{a}\, p_X\left(\frac{y-b}{a}\right)$$

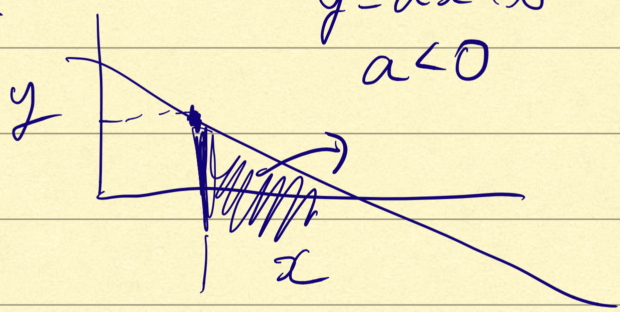
CHECK:



$Y = 2X$



EX



$y = ax + b$

$a < 0$

$$F_Y(y) = \mathbb{P}\left(X \ge \frac{y-b}{a}\right)$$

$$= 1 - F_X\left(\frac{y-b}{a}\right)$$

$$x \geq \frac{y-b}{a}$$

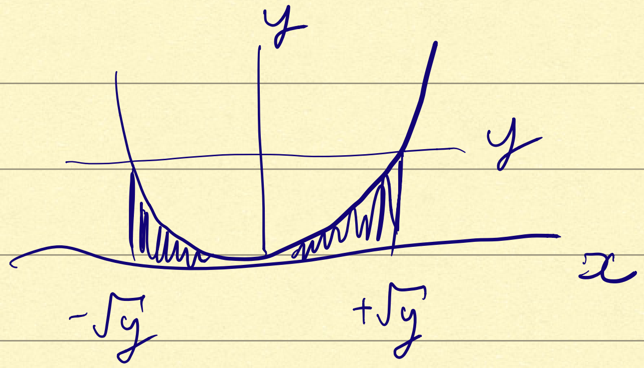
$$p_Y(y) = \frac{d}{dy}\left(1 - F_X\left(\frac{y-b}{a}\right)\right)$$

$$= \left(-\frac{1}{a}\right)\frac{d}{dy} F_X\left(\frac{y-b}{a}\right)$$

$$= -\frac{1}{a} \, p_X\left(\frac{y-b}{a}\right)$$

$$= \left|\frac{1}{a}\right| p_X\left(\frac{y-b}{a}\right)$$

EX $Y = X^2$



$$I_y = \begin{cases} [-\sqrt{y}, \sqrt{y}] & y \geq 0 \\ \text{EMPTY} & y < 0 \end{cases}$$

$$F_Y(y) = \begin{cases} \left(F_X(\sqrt{y}) - F(-\sqrt{y})\right) & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$p_Y(y) = \begin{cases} p_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + p_X(-\sqrt{y}) \frac{1}{2\sqrt{y}} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

GENERAL FORMULA

LET $X \sim p_X(x)$ AND $Y = g(X)$

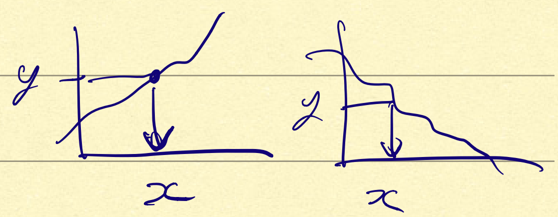
THEN $p_Y(y) = \sum_k p_X(x_k(y)) \cdot \left|\frac{dx_k}{dy}\right|$

WHERE $\{x_k\}$ IS THE PRE-IMAGE OF y

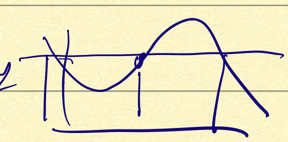
IF $\frac{dg}{dx} \neq 0$ THEN

$$p_Y(y) = \sum_k p_X(g_k^{-1}(y)) \cdot \left(\left|\frac{dg}{dx}\right|_{x_k}\right)^{-1}$$

IF $g(x)$ IS MONOTONIC



NOT



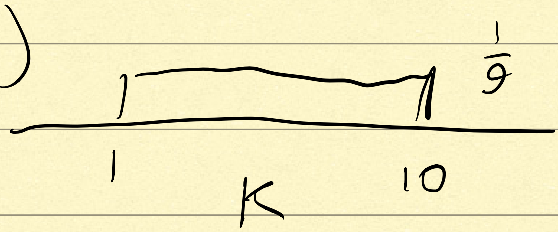
THEN

$$p_Y(y) = p_X(g^{-1}(y)) \cdot \left|\left(\frac{dg}{dx}\right)^{-1}\right|$$

EX $K \sim \text{UNIFORM}(1, 10)$

K - RATE CONSTANT (UNITS s^{-1})

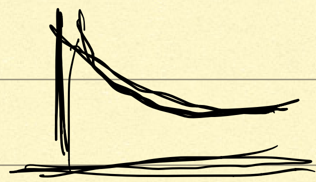
$p_K(k)$



$$p_K(k) = \begin{cases} \frac{1}{9} & \underline{1 \le k \le 10} \\ 0 & \text{ELSE} \end{cases}$$

LET $T = \frac{1}{K}$ - MEAN TIME (UNITS s)

$$g(k) = \frac{1}{k}$$



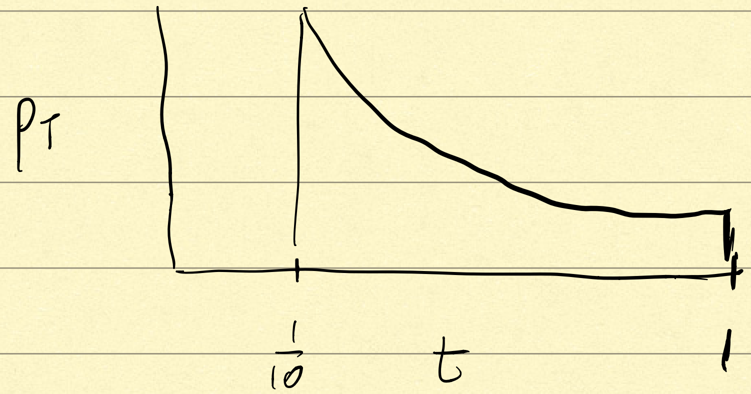
$$g^{-1}(t) = \frac{1}{t}$$

$$\frac{dg}{dk} = -\frac{1}{\underline{k^2}} = \frac{-1}{\left(\frac{1}{t}\right)^2} = -t^2$$

$$\Rightarrow p_T(t) = p_K\left(g^{-1}(t)\right) \cdot \left| \left.\frac{dg}{dk}\right|_t^{-1} \right|$$

$$= \begin{cases} \frac{1}{9} \cdot \frac{1}{t^2} & \frac{1}{10} < t < 1 \end{cases}$$

(O ELSE



WHY?