# Anisotropy and scale invariance in dynamical systems

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We present a methodology for estimating both anisotropy and scale invariance in the context of a classical Lagrangian framework. We derive a generalised Lagrangian that can be used to model a spatially discretized field – allowing for the analysis of arbitrary timeseries from any scale free dynamical system that obeys the principle of stationary action.

**Classical Lagrangian field theory:** We begin by defining a classical field $\varphi = \varphi(r)$, where $r \equiv (t, x, y, z)$. Note that this 'four-vector' notation is for convenience only and should not be confused with relativistic definitions, as we will be working purely in terms of a standard Cartesian metric throughout.

We then define a Lagrangian $\mathcal{L}$ to be a function of the space-time position $r$, the volume of the field $\varphi$, and the values of the derivatives of $\varphi$ at $r$: $\varphi_\mu = \partial_\mu \varphi$, such that:

$$\mathcal{L} = \mathcal{L}(r, \varphi, \varphi_\mu), \quad [1]$$

where we have not yet defined any relationship between $r$, $\varphi$ and $\varphi_\mu$.

Given a particular choice of $\varphi(r)$, the action is a functional of $\varphi(r)$, defined via [1] as:

$$S[\varphi(r)] = \int_\Omega d^4 r \, \mathcal{L}(r, \varphi, \varphi_\mu), \quad [2]$$

where $\Omega$ is the integration region that describes the 'trajectory' $\varphi(r)$, defined for all $x, y, z$ between an initial time $t_i$ and final time $t_f$. Note that we now assume that the field $\varphi$ and its derivatives $\varphi_\mu$ are a function of $r$.

The evolution of $\varphi(r)$ between the two field configurations $\varphi(t_i, x, y, z)$ and $\varphi(t_f, x, y, z)$ renders the action stationary with respect to all variations $\delta\varphi(r) = \delta\varphi(t, x, y, z)$ that vanish when $t = t_i$ and $t = t_f$. Using [2], we evaluate this variation as follows:

$$\delta S = \int_\Omega d^4 r \left( \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial \varphi_\mu} \delta\varphi_\mu \right) = \int_\Omega d^4 r \left( \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial \varphi_\mu} \partial_\mu (\delta\varphi) \right)$$

$$= \int_\Omega d^4 r \left( \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \varphi_\mu} \delta\varphi \right) - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \varphi_\mu} \right) \delta\varphi \right), \quad [3]$$

where we convert the middle term on the second line to a surface integral by using the 4-D version of the divergence theorem, such that:

$$\delta S = \int_{\Omega} d^4 r \left( \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \varphi_\mu} \right) \right) \delta\varphi + \int_{\partial\Omega} \frac{\partial \mathcal{L}}{\partial \varphi_\mu} \delta\varphi \, dS_\mu, \qquad [4]$$

where $\partial\Omega$ is the surface of the 4-D volume $\Omega$; $dS_\mu$ is the element of the 3-D surface of $\Omega$; and we assume that $\delta\varphi = 0$ when $t = t_i$ and $t = t_f$, and that all fields decay sufficiently quickly as $(x, y, z) \to \infty$ such that the surface integral in [4] is zero. We therefore obtain:

$$\delta S = \int_{\Omega} d^4 r \left( \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \varphi_\mu} \right) \right) \delta\varphi = 0. \qquad [5]$$

We then see that, since $\delta\varphi$ is arbitrary (except for the constraint that it vanishes at the surface $\partial\Omega$) Hamilton's principle $\delta S = 0$ implies that the fields evolve according to the Euler-Lagrange equation, given by:

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \varphi_\mu} \right) = 0, \qquad [6]$$

or more explicitly:

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\left( \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \varphi}{\partial t} \right)} \right) - \frac{\partial}{\partial x}\left( \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \varphi}{\partial x} \right)} \right) - \frac{\partial}{\partial y}\left( \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \varphi}{\partial y} \right)} \right) - \frac{\partial}{\partial z}\left( \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \varphi}{\partial z} \right)} \right) = 0 \qquad [7]$$

**Scale transformations:** We define a scale transformation as a mapping from original points $(\varphi, r)$ to scaled points $(\varphi_s, x_s) = (\lambda\varphi, \lambda^\alpha x)$ in 5-D space with axes labelled according to: $(\varphi, r) = (\varphi, t, x, y, z)$, where $\lambda$ and $\alpha$ are constants, such that:

$$\varphi_s(\lambda^\alpha r) = \lambda\varphi(r) \quad \Longrightarrow \quad \varphi_s(r) = \lambda\varphi(\lambda^{-\alpha} r), \qquad [8]$$

where $\lambda^\alpha r$ is shorthand for $(\lambda^{\alpha_t} t, \lambda^{\alpha_x} x, \lambda^{\alpha_y} y, \lambda^{\alpha_z} z)$.

Differentiating [8], we obtain:

$$\varphi_{s\mu}(r) = \partial_\mu\big(\varphi_s(r)\big) = \lambda^{1-\alpha_\mu}\varphi_\mu(\lambda^{-\alpha}r), \tag{9}$$

where $\lambda^{1-\alpha_\mu}$ depends only on the $\mu^{th}$ component of the vector of exponents given by: $\alpha = \big(\alpha_t, \alpha_x, \alpha_y, \alpha_z\big)$.

Using [2], [8], and [9], we see that the scaled action is given by:

$$S[\varphi_s(r)] = \int_{\lambda^{\alpha_t}t_i}^{\lambda^{\alpha_t}t_f} dt \iiint_{all\ x,y,z} dxdydz\, \mathcal{L}\left(r, \lambda\varphi(\lambda^{-\alpha}r), \lambda^{1-\alpha_\mu}\varphi_\mu(r)\right), \tag{10}$$

where we assume that the bounding surface recedes to infinity and hence the rescaling of the spatial coordinates has no effect upon the limits of the spatial integration.

We then change variables in [10] according to: $r' = \lambda^{-\alpha}r$, such that:

$$S[\varphi_s(r)] = \lambda^{\sum_i \alpha_i} \int_{t_i}^{t_f} dt' \iiint_{all\ x',y',z'} dx'dy'dz' \mathcal{L}\left(\lambda^{\alpha}r, \lambda\varphi(r'), \lambda^{1-\alpha_\mu}\varphi_\mu(r')\right), \tag{11}$$

where $\lambda^{\sum_i \alpha_i}$ is the Jacobian that accounts for the change in integration variables; $\sum_i \alpha_i = \alpha_t + \alpha_x + \alpha_y + \alpha_z$; and the integrals are now over the same space-time region $\Omega$ as in the original un-scaled action in [2], which means that we can re-write [11] using the same simple notation:

$$S[\varphi_s(r)] = \lambda^{\sum_i \alpha_i} \int_\Omega d^4r\, \mathcal{L}\left(\lambda^{\alpha}r, \lambda\varphi(r), \lambda^{1-\alpha_\mu}\varphi_\mu(r)\right) \tag{12}$$

**Scale freeness:** The action $S[\varphi(r)]$ is said to be scale free if, for any choice of $\varphi(r)$, i.e. not just choices satisfying the Euler-Lagrange equation [6], the following relationship holds:

$$S[\varphi_s(r)] = \lambda^n S[\varphi(r)], \tag{13}$$

where $n$ is a constant.

Or more explicitly, using [2], [12], and [13], the condition for scale freeness is that:

$$\lambda^{\sum_i \alpha_i - n} \int_\Omega d^4r\, \mathcal{L}\big(\lambda^{\alpha}r, \lambda\varphi, \lambda^{1-\alpha_\mu}\varphi_\mu\big) = \int_\Omega d^4r\, \mathcal{L}\big(r, \varphi, \varphi_\mu\big). \tag{14}$$

**Generalised scale free Lagrangians:** We can write a generalised form of [1] as a power series expansion, where we henceforth assume for the sake of simplicity that we are dealing with a 2-D system with purely implicit space-time dependence:

$$\mathcal{L} = \sum_{a,b,c,d} C_{a,b,c,d}\, \varphi^{a} {\varphi_t}^{b} {\varphi_x}^{c} {\varphi_y}^{d}, \qquad [15]$$

where $a, b, c,$ and $d$ are constants; and $C_{a,b,c,d}$ is an arbitrary expansion coefficient.

We then use [8] and [9] to scale-transform [15] as follows:

$$\mathcal{L} \to \mathcal{L}_s = \lambda^{a+(1-\alpha_t)b+(1-\alpha_x)c+(1-\alpha_y)d}\mathcal{L}\,, \qquad [16]$$

which, using [14], together with the fact that the scale factor $\lambda$ is arbitrary, means that $\mathcal{L}$ is scale free if the following condition holds:

$$a + (1-\alpha_t)b + (1-\alpha_x)c + \left(1-\alpha_y\right)d - n = 0, \qquad [17]$$

which we use, together with [15], to write an expression for a generalised scale free Lagrangian as follows:

$$\mathcal{L} = \varphi^{n} \sum_{b,c,d} C_{b,c,d}\, \varphi^{(\alpha_t-1)b+(\alpha_x-1)c+(\alpha_y-1)d} {\varphi_t}^{b} {\varphi_x}^{c} {\varphi_y}^{d}, \qquad [18]$$

**Spatial discretization:** For the purpose of analysing timeseries we must spatially discretize [18] such that continuous spatial gradients transform to differences between field values at contiguous nodes according to:

$$\begin{aligned} \varphi_x &\to \varphi^{\{x\}} - \varphi \\ \varphi_y &\to \varphi^{\{y\}} - \varphi \end{aligned} \qquad [19]$$

where $\varphi^{\{x\}}$ and $\varphi^{\{y\}}$ indicate the field strengths (at a given time) one ‘step’ in the $x$ and $y$ directions from $\varphi$, respectively.

Using [19], we write the spatially discretized form of [18] as follows:

$$\mathcal{L} = \varphi^{n} \sum_{b,c,d} C_{b,c,d}\, \varphi^{(\alpha_t - 1)b + (\alpha_x - 1)c + (\alpha_y - 1)d} {\varphi_t}^{b} \left(\varphi^{\{x\}} - \varphi\right)^{c} \left(\varphi^{\{y\}} - \varphi\right)^{d}, \qquad [20]$$

which means that our system is now a function of field strengths at spatially contiguous points in the 2-D grid: $\mathcal{L} = \mathcal{L}\left(\varphi, \varphi_t, \varphi^{\{x\}}, \varphi^{\{y\}}\right)$ and hence the Euler-Lagrange equation [7] now reads:

$$\frac{\partial \mathcal{L}}{\partial \varphi} + \frac{\partial \mathcal{L}}{\partial \varphi^{\{x\}}} + \frac{\partial \mathcal{L}}{\partial \varphi^{\{y\}}} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \varphi_t}\right) = 0\,. \qquad [21]$$

which we apply to [20] to obtain the equation of motion.

We re-define the spatial scaling factor in the y-dimension such that it relates to the scaling factor in the x-dimension as follows: $\alpha_y = \alpha_x + \delta y$, where $\delta y$ is a constant. We are therefore able to test for isotropy $\left(\alpha_y = \alpha_x\right)$ using Bayesian model reduction by setting the prior variance of $\delta y$ to zero, where $\delta y$ is given a prior mean of zero.

**Scale invariance:** We obtain the condition for scale invariance by setting $n = 0$ in [14]:

$$\lambda^{\sum_i \alpha_i} \int_{\Omega} d^4 r\, \mathcal{L}\left(\lambda^{\alpha} r, \lambda\varphi, \lambda^{1-\alpha_\mu}\varphi_\mu\right) = \int_{\Omega} d^4 r\, \mathcal{L}\left(r, \varphi, \varphi_\mu\right), \qquad [22]$$

which describes a system in which there is a symmetry under change of scale.

We are therefore also able to test for scale invariance $(n = 0)$ during model reduction by setting the prior variance of $n$ to zero, where $n$ is given a prior mean of zero. If we find that there is evidence for scale invariance (scale symmetry) then we can calculate a corresponding conservation law via Noether's theorem[1].

**Infinitesimal scale transformations:** In order to use Noether's theorem we must first formulate [22] in terms of arbitrarily small variations, such that any change of scale can be described via the successive application of infinitesimal transformations.

We therefore re-define the scale factor $\lambda$ to lie close to unity, such that:

$$\lambda = 1 + \epsilon, \qquad [23]$$

where $\epsilon$ is arbitrarily small and hence:

$$\lambda^{\alpha} = (1 + \epsilon)^{\alpha} \approx 1 + \epsilon\alpha\,. \qquad [24]$$

Using [23] and [24] we write the integrand on the left-hand side of [22] as follows:

$$\left(1 + \epsilon \sum_i \alpha_i\right) \mathcal{L}\big(r + \epsilon\alpha r, \varphi + \epsilon\varphi, \varphi_\mu + \epsilon(1 - \alpha_\mu)\varphi_\mu\big)\,, \qquad [25]$$

which, expanding to first order, reads:

$$\left(1 + \epsilon \sum_i \alpha_i\right) \mathcal{L} + \epsilon \sum_i \alpha_i r_i \frac{\partial \mathcal{L}}{\partial r_i} + \epsilon\varphi \frac{\partial \mathcal{L}}{\partial \varphi} + \epsilon \sum_i (1 - \alpha_i)\varphi_i \frac{\partial \mathcal{L}}{\partial \varphi_i}\,, \qquad [26]$$

where all fields $\varphi$ and their derivatives are evaluated at $r$; and all Lagrangians $\mathcal{L}$ and their derivatives evaluated at $\big(r, \varphi(r), \varphi_\mu(r)\big)$.

**Scale invariance for infinitesimal scale transformations:** Using [22] and [26] we see that the action is invariant under change of scale if and only if the following condition holds:

$$\int_\Omega d^4 r \left\{ \sum_i \alpha_i\, \mathcal{L} + \sum_i \alpha_i r_i \frac{\partial \mathcal{L}}{\partial r_i} + \varphi \frac{\partial \mathcal{L}}{\partial \varphi} + \sum_i (1 - \alpha_i)\varphi_i \frac{\partial \mathcal{L}}{\partial \varphi_i} \right\} = 0. \qquad [27]$$

where the partial derivatives of $\mathcal{L}$ are derivatives of the seven-argument function:

$$\mathcal{L}(r, \varphi, \varphi_\mu) = \mathcal{L}\big(t, x, y, \varphi, \varphi_t, \varphi_x, \varphi_y\big). \qquad [28]$$

In defining these partial derivatives, all seven arguments are treated as independent variables, with any derivative calculated with the other six held constant.

However, once we choose a specific field configuration $\varphi(r) = \varphi(t, x, y)$ we can also define a three-variable function $\mathcal{L}^{\varphi}(r) = \mathcal{L}^{\varphi}(t, x, y)$ via [1], in which the partial derivatives of $\mathcal{L}^{\varphi}$ with respect to the three components of $r$ are defined holding the other two components fixed, such that:

$$\frac{\partial \mathcal{L}^{\varphi}}{\partial r_j} = \frac{\partial \mathcal{L}}{\partial r_j} + \frac{\partial \mathcal{L}}{\partial \varphi}\frac{\partial \varphi}{\partial \nu} + \sum_i \frac{\partial \mathcal{L}}{\partial \varphi_i}\varphi_{ji}, \qquad [29]$$

where $\varphi_{ji} = \frac{\partial^2 \varphi(r)}{\partial r_j \partial r_i}$; and we note that the total derivative with respect to $r_\nu$ is evaluated with $r_\nu$ $(j \neq i)$ held constant, but not with $\varphi$ and $\partial\varphi$ held constant. Therefore, to distinguish between the three and seven-variable partials we will use straight $d$ notation for the three-variable derivatives (keeping in mind that they are nevertheless still partials).

Using [29] we can now write the condition for scale invariance in [27] as follows:

$$\int_\Omega d^4 r \left\{ \varphi \frac{\partial \mathcal{L}}{\partial \varphi} + \sum_i (1-\alpha_i)\varphi_i \frac{\partial \mathcal{L}}{\partial \varphi_i} + \sum_i \alpha_i r_i \left( \frac{\partial \mathcal{L}}{\partial r_i} - \frac{\partial \mathcal{L}}{\partial \varphi}\varphi_i - \sum_j \frac{\partial \mathcal{L}}{\partial \varphi_j}\varphi_{ij} \right) + \sum_i \alpha_i \mathcal{L} \right\} = 0, \qquad [30]$$

or equivalently:

$$\int_\Omega d^4 r \left\{ \left( \varphi - \sum_i \alpha_i r_i \varphi_i \right) \frac{\partial \mathcal{L}}{\partial \varphi} + \sum_i (1-\alpha_i)\varphi_i \frac{\partial \mathcal{L}}{\partial \varphi_i} + \sum_i \left( \alpha_i r_i \frac{\partial \mathcal{L}}{\partial r_i} + \alpha_i \mathcal{L} \right) - \sum_{j,i} \alpha_i r_i \frac{\partial \mathcal{L}}{\partial \varphi_j}\varphi_{ij} \right\} = 0. \qquad [31]$$

**Noether's theorem and scale invariance:** In order to satisfy Noether's theorem we next stipulate that $\varphi$ satisfies the Euler-Lagrange equation, which now takes the form:

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \sum_i \frac{d}{dr_i}\left( \frac{\partial \mathcal{L}}{\partial \varphi_i} \right) = 0, \qquad [32]$$

which we use to re-write the integrand of [31] as follows:

$$\left( \varphi - \sum_i \alpha_i r_i \varphi_i \right) \sum_j \frac{d}{dr_j}\left( \frac{\partial \mathcal{L}}{\partial \varphi_j} \right) + \sum_j (1-\alpha_j)\varphi_j \frac{\partial \mathcal{L}}{\partial \varphi_j} - \sum_{j,i} \alpha_i r_i \varphi_{ij} \frac{\partial \mathcal{L}}{\partial \varphi_j} + \sum_j \left( \alpha_j r_j \frac{\partial \mathcal{L}}{\partial \varphi_j} + \alpha_j \mathcal{L} \right), \qquad [33]$$

where the first three terms re-arrange to:

$$\sum_j \left(\varphi - \sum_i \alpha_i r_i\, \varphi_i\right) \frac{d}{dr_j}\left(\frac{\partial \mathcal{L}}{\partial \varphi_j}\right) + \sum_j \varphi_j \frac{\partial \mathcal{L}}{\partial \varphi_j} - \sum_{j,i} \alpha_i r_i\, \varphi_{ij} \frac{\partial \mathcal{L}}{\partial \varphi_j} - \sum_j \alpha_j \varphi_j \frac{\partial \mathcal{L}}{\partial \varphi_j}, \qquad [34]$$

which we then see can be written as the following total ('partial') derivative:

$$\sum_j \frac{d}{dr_j}\left[\left(\varphi - \sum_i \alpha_i r_i\, \varphi_i\right)\frac{\partial \mathcal{L}}{\partial \varphi_j}\right]. \qquad [35]$$

Similarly, the last term in [33] can be written as:

$$\sum_i \frac{d}{dr_i}[\alpha_i r_i \mathcal{L}] = \sum_i \left(\alpha_i r_i \frac{\partial \mathcal{L}}{\partial r_i} + \alpha_i \mathcal{L}\right). \qquad [36]$$

Therefore, using [35] and [36], the scale invariance condition in [33] says that:

$$\int_\Omega d^4 r \sum_j \frac{d}{dr_j}\left[\left(\varphi - \sum_i \alpha_i r_i\, \varphi_i\right)\frac{\partial \mathcal{L}}{\partial \varphi_j} + \alpha_j r_j \mathcal{L}\right] = 0. \qquad [37]$$

which we can re-write using partial notation for the four-variable partial derivatives:

$$\int_\Omega d^4 r \sum_j \frac{\partial}{\partial j} N_j = 0, \qquad [38]$$

which, together with [37], describes the Noether charge density:

$$N_j = \left(\varphi - \sum_i \alpha_i r_i\, \varphi_i\right)\frac{\partial \mathcal{L}}{\partial \varphi_j} + \alpha_j r_j \mathcal{L}, \qquad [39]$$

which is conserved by virtue of scale invariance in a classical Lagrangian field.

**References**


1 Noether, E. Invariante Variationsprobleme. Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physikalische Klasse, 235-257 (1918).