Chapter 2

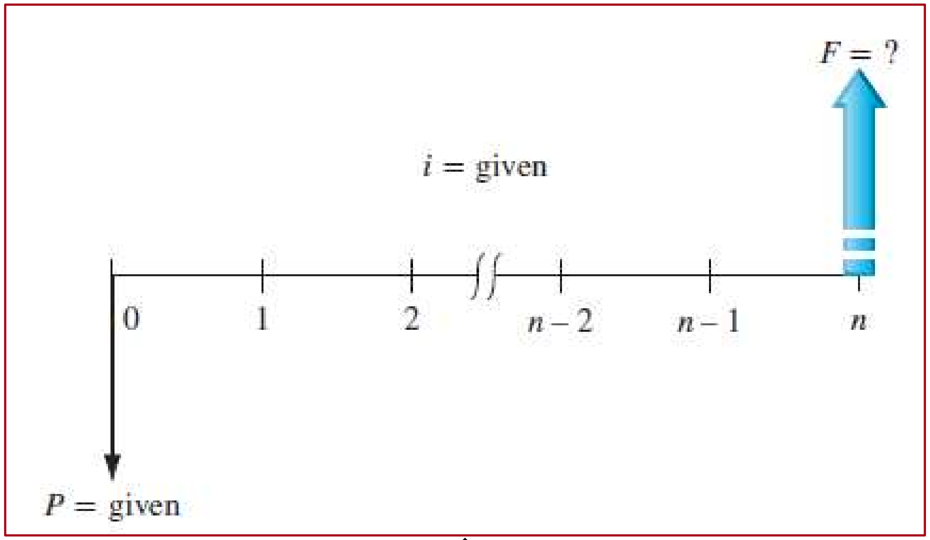
Factors: How Time and Interest Affect Money

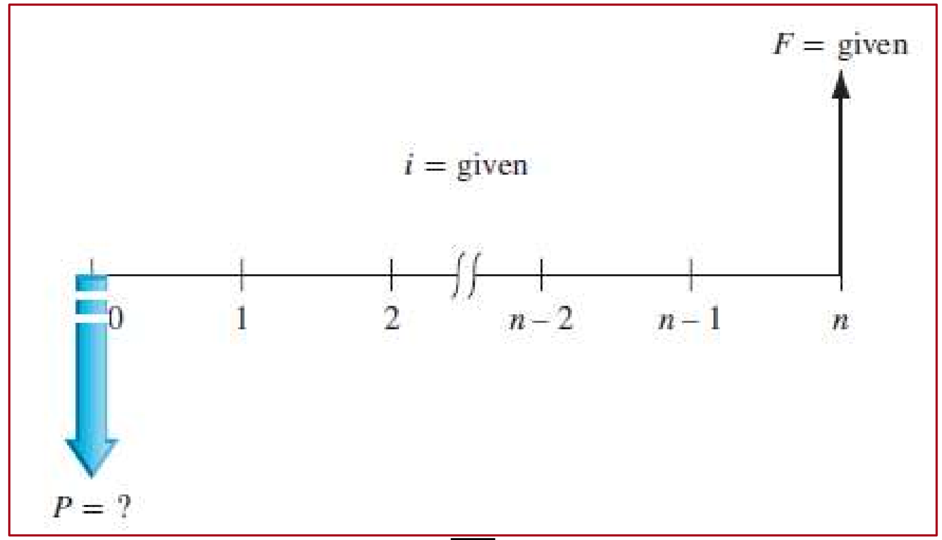
LEARNING OUTCOMES

1. F/P and P/F Factors
2. P/A and A/P Factors
3. F/A and A/F Factors
4. Factor Values
5. Arithmetic Gradient
6. Geometric Gradient
7. Find i or n

Single Payment Factors (F/P and P/F)

Single payment factors involve only P and F. **Cash flow diagrams are as follows:**





Formulas are as follows:

$$F = P(1 + i)^n$$

$$P = F[1/(1 + i)^n]$$

Terms in parentheses or brackets are called *factors.* Values are in tables for i and n values

Factors are represented in *standard factor notation such as (F/P,i,n)*,

where letter to left of slash is what is sought; letter to right represents what is given

Derive (F/P and P/F)

$$F_1 = P + Pi$$
$$= P(1 + i)$$

$$F_2 = F_1 + F_1 i$$
$$= P(1 + i) + P(1 + i)i$$

$$F_2 = P(1 + i + i + i^2)$$
$$= P(1 + 2i + i^2)$$
$$= P(1 + i)^2$$

$$F_3 = F_2 + F_2 i$$

$$F_3 = P(1 + i)^3$$

$$F = P(1 + i)^n$$

$$P = F\left[\frac{1}{(1 + i)^n}\right] = F(1 + i)^{-n}$$

Factor					
Notation	Name	Find/Given	Standard Notation Equation	Equation with Factor Formula	Excel Function
$(F/P,i,n)$	Single-payment compound amount	F/P	$F = P(F/P,i,n)$	$F = P(1 + i)^n$	$= FV(i\%,n,,P)$
$(P/F,i,n)$	Single-payment present worth	P/F	$P = F(P/F,i,n)$	$P = F(1 + i)^{-n}$	$= PV(i\%,n,,F)$

F/P and P/F for Spreadsheets

Future value F is calculated using FV function:

= FV(i%,n,,P)

Present value P is calculated using PV function:

= PV(i%,n,,F)

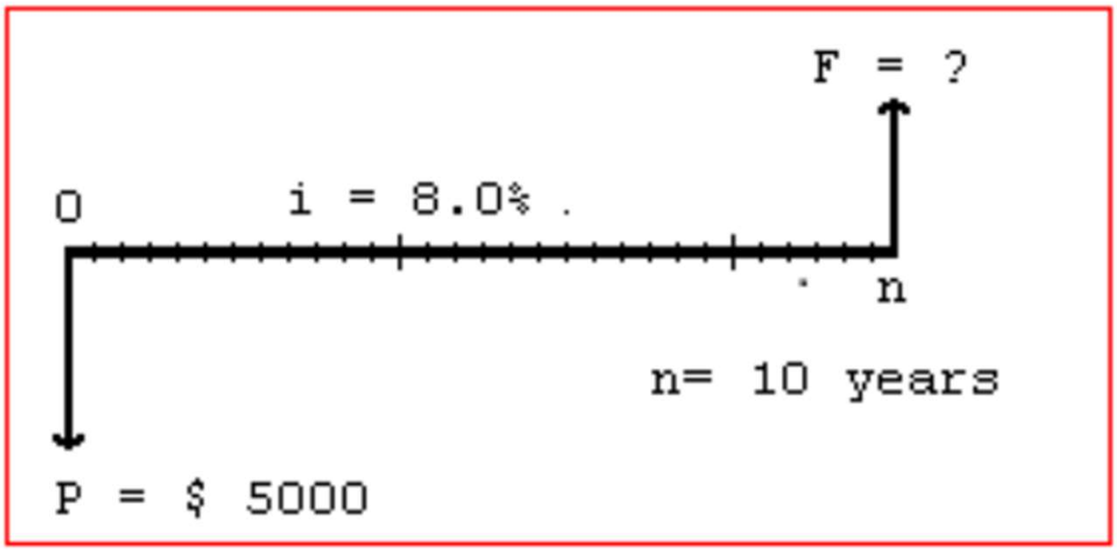
Note the use of double commas in each function

Example: Finding Future Value

A person deposits $5000 into an account which pays interest at a rate of 8% per year. The amount in the account after 10 years is closest to:

(A) $2,792 (B) $9,000 (C) $10,795 (D) $12,165

The cash flow diagram is:



Solution:

$$F = P(F/P,i,n)$$
$$= 5000(F/P,8\%,10)$$
$$= 5000(2.1589)$$
$$= \$10,794.50$$

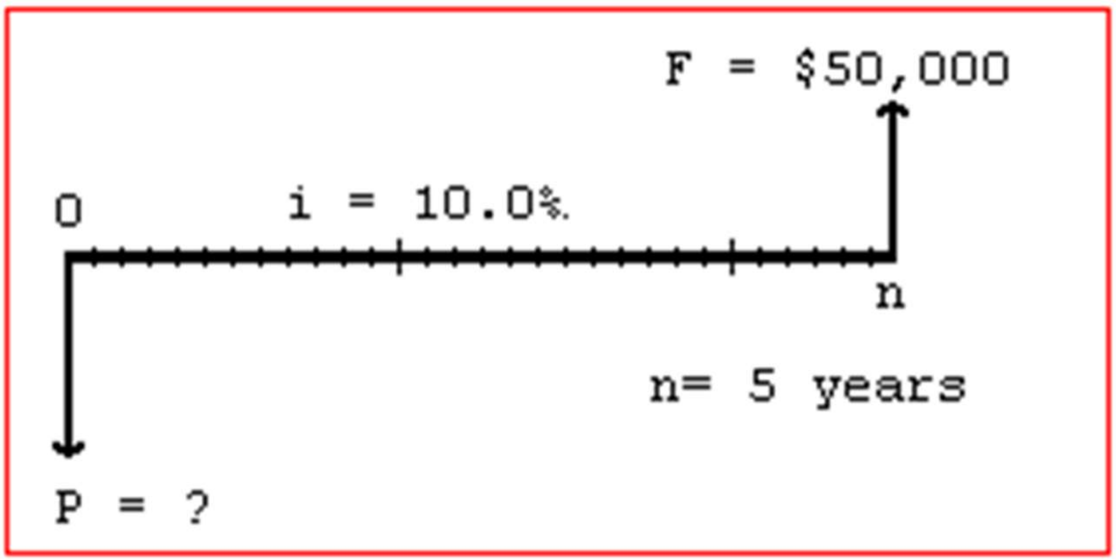
Answer is (C)

Example: Finding Present Value

A small company wants to make a single deposit now so it will have enough money to purchase a backhoe costing \$50,000 five years from now. If the account will earn interest of 10% per year, the amount that must be deposited now is nearest to:

(A) \$10,000 (B) \$ 31,050 (C) \$ 33,250 (D) \$319,160

The cash flow diagram is:



Solution:

$$P = F(P/F,i,n)$$
$$= 50{,}000(P/F,10\%,5)$$
$$= 50{,}000(0.6209)$$
$$= \$31{,}045$$

Answer is (B)

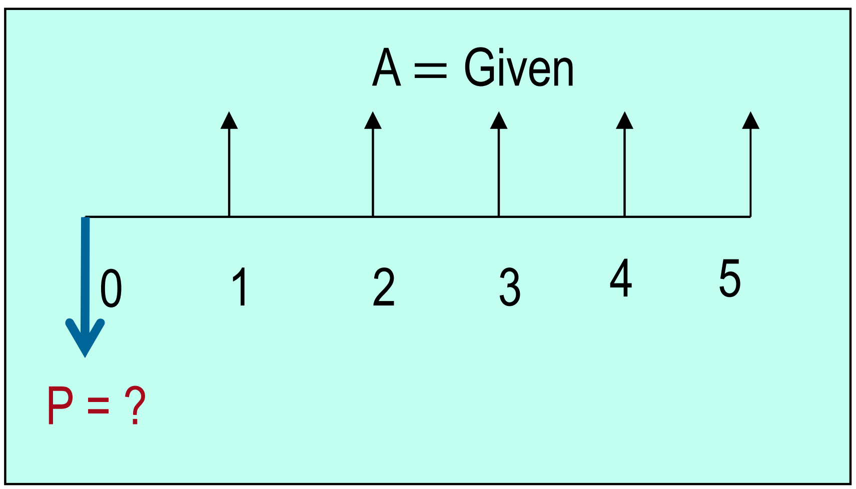
Uniform Series Involving P/A and A/P

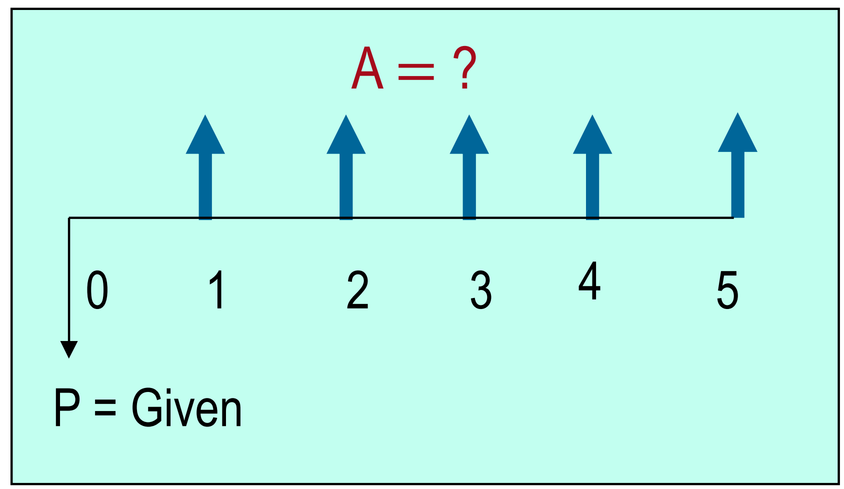
The uniform series factors that involve **P and A** are derived as follows:

(1) Cash flow occurs in ***consecutive*** interest periods

(2) Cash flow amount is ***same*** in each interest period

The cash flow diagrams are:





$P = A(P/A,i,n)$ ← Standard Factor Notation → $A = P(A/P,i,n)$

Note: P is one period *Ahead* of first A value

Derive P/A and A/P

$$P = A\left[\frac{1}{(1+i)^1}\right] + A\left[\frac{1}{(1+i)^2}\right] + A\left[\frac{1}{(1+i)^3}\right] + \cdots + A\left[\frac{1}{(1+i)^{n-1}}\right] + A\left[\frac{1}{(1+i)^n}\right]$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] \qquad i \neq 0$$

$$P = A\left[\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \cdots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\right]$$

$$\frac{P}{1+i} = A\left[\frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \frac{1}{(1+i)^4} + \cdots + \frac{1}{(1+i)^n} + \frac{1}{(1+i)^{n+1}}\right]$$

$$\frac{1}{1+i}P = A\left[\cancel{\frac{1}{(1+i)^2}} + \cancel{\frac{1}{(1+i)^3}} + \cdots + \cancel{\frac{1}{(1+i)^n}} + \frac{1}{(1+i)^{n+1}}\right]$$

$$-\qquad P = A\left[\frac{1}{(1+i)^1} + \cancel{\frac{1}{(1+i)^2}} + \cdots + \cancel{\frac{1}{(1+i)^{n-1}}} + \cancel{\frac{1}{(1+i)^n}}\right]$$

$$\frac{-i}{1+i}P = A\left[\frac{1}{(1+i)^{n+1}} - \frac{1}{(1+i)^1}\right]$$

$$P = \frac{A}{-i}\left[\frac{1}{(1+i)^n} - 1\right]$$

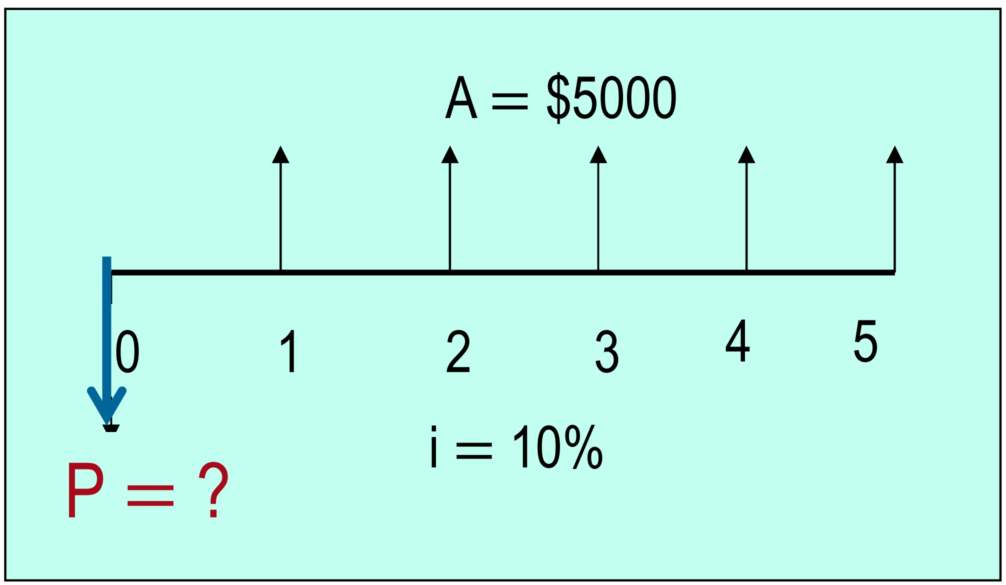
Notation	Factor Name	Find/Given	Factor Formula	Standard Notation Equation	Excel Function
$(P/A,i,n)$	Uniform series present worth	P/A	$\frac{(1+i)^n - 1}{i(1+i)^n}$	$P = A(P/A,i,n)$	= PV(i%,n,A)
$(A/P,i,n)$	Capital recovery	A/P	$\frac{i(1+i)^n}{(1+i)^n - 1}$	$A = P(A/P,i,n)$	= PMT(i%,n,P)

Example: Uniform Series Involving P/A

A chemical engineer believes that by modifying the structure of a certain water treatment polymer, his company would earn an extra $5000 per year. At an interest rate of 10% per year, how much could the company afford to spend now to just break even over a 5 year project period?

(A) $11,170 (B) 13,640 (C) $15,300 (D) $18,950

The cash flow diagram is as follows:



Solution:

$$P = 5000(P/A,10\%,5)$$
$$= 5000(3.7908)$$
$$= \$18,954$$

Answer is (D)

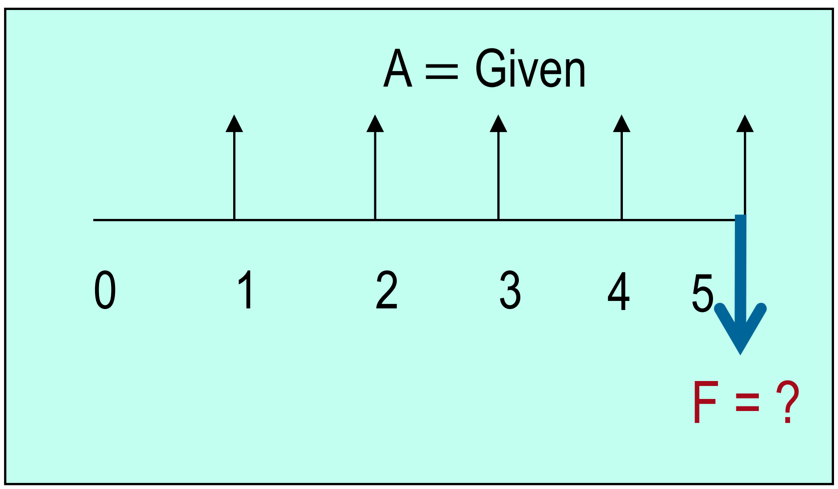
Uniform Series Involving F/A and A/F

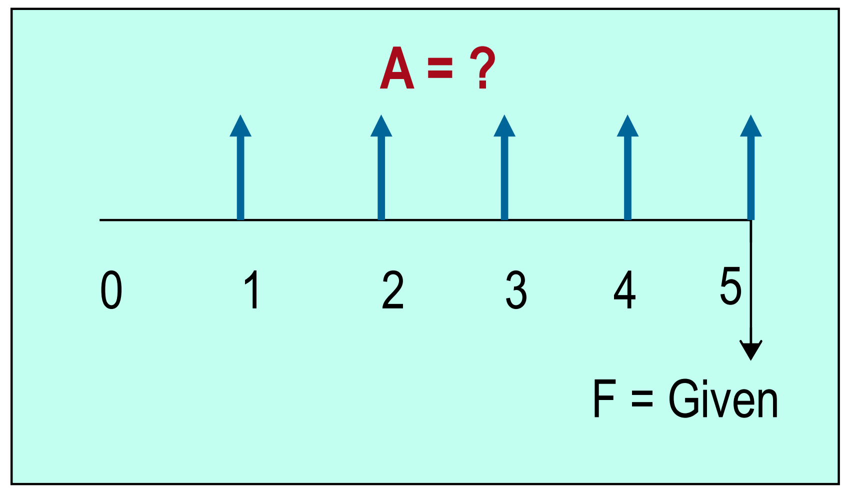
The uniform series factors that involve **F and A** are derived as follows:

(1) Cash flow occurs in ***consecutive*** interest periods

(2) Last cash flow occurs in ***same*** period as F

Cash flow diagrams are:





$F = A(F/A,i,n)$ ← Standard Factor Notation → $A = F(A/F,i,n)$

Note: F takes place in the *same* period as last A

Derive F/A and A/F

$$A = F\left[\frac{1}{(1+i)^n}\right]\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right]$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right]$$

$$(F/A,i,n) = [(1+i)^n]\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = \frac{(1+i)^n - 1}{i}$$

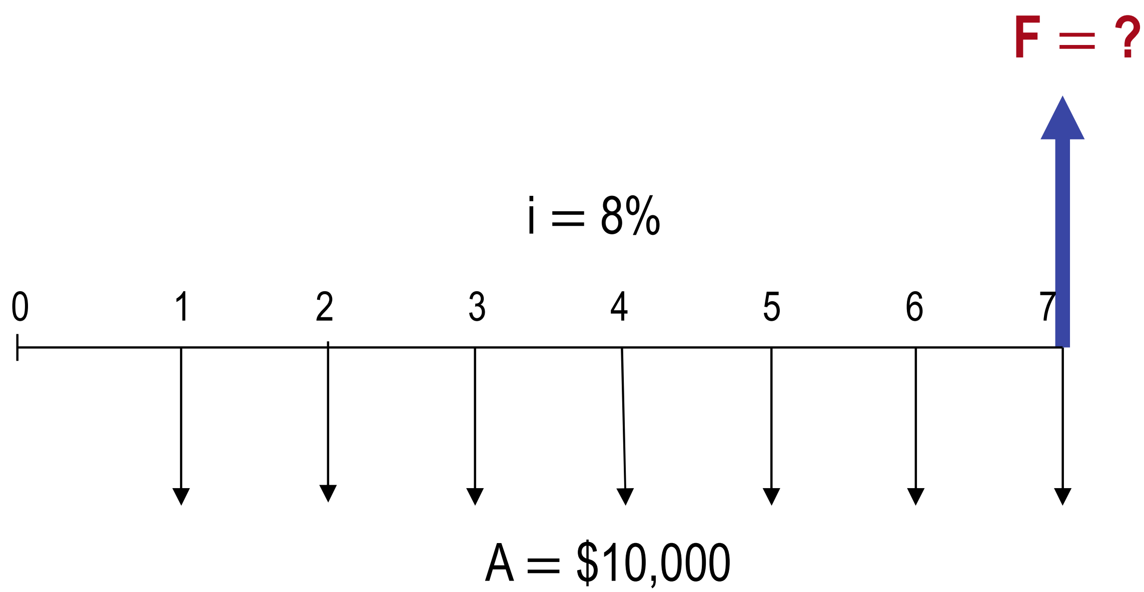
Notation	Factor Name	Find/Given	Factor Formula	Standard Notation Equation	Excel Functions
$(F/A,i,n)$	Uniform series compound amount	F/A	$\frac{(1+i)^n - 1}{i}$	$F = A(F/A,i,n)$	$= \mathrm{FV}(i\%,n,A)$
$(A/F,i,n)$	Sinking fund	A/F	$\frac{i}{(1+i)^n - 1}$	$A = F(A/F,i,n)$	$= \mathrm{PMT}(i\%,n,F)$

Example: Uniform Series Involving F/A

An industrial engineer made a modification to a chip manufacturing process that will save her company \$10,000 per year. At an interest rate of 8% per year, how much will the savings amount to in 7 years?

(A) \$45,300 (B) \$68,500 (C) \$89,228 (D) \$151,500

The cash flow diagram is:



Solution:

$$F = 10{,}000(F/A,8\%,7)$$
$$= 10{,}000(8.9228)$$
$$= \$89{,}228$$

Answer is (C)

Factor Values for Untabulated i or n

3 ways to find factor values for untabulated i or n values

1. **Use formula**
2. **Use spreadsheet function with corresponding P, F, or A value set to 1**
3. **Linearly interpolate in interest tables**

Formula or spreadsheet function is fast and a ccurate
Interpolation is only approximate

Example: Untabulated i

Determine the value for (F/P, 8.3%,10)

Formula: $F = (1 + 0.083)^{10} = 2.2197$ ← **OK**

Spreadsheet: $= FV(8.3\%, 10, , 1) = 2.2197$ ← **OK**

Interpolation:

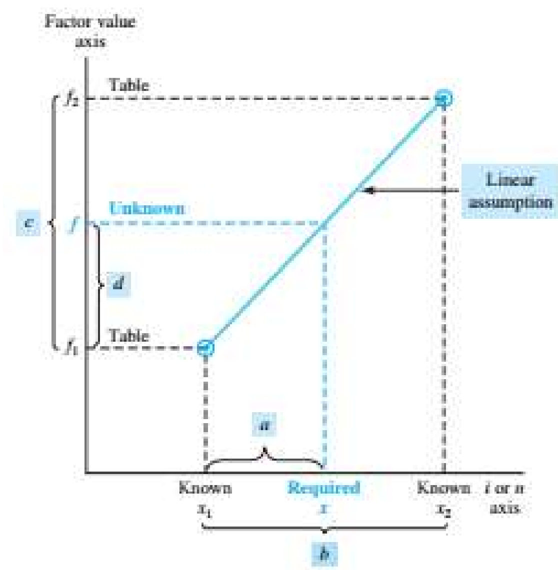
8% ------ 2.1589
8.3% ------ x
9% ------ 2.3674

$$x = 2.1589 + [(8.3 - 8.0)/(9.0 - 8.0)][2.3674 - 2.1589]$$
$$= 2.2215$$

← **(Too high)**

Absolute Error $= \mathbf{2.2215 - 2.2197 = 0.0018}$

	A	B	C	D
1	i =	n =		
2	3.25%	25.00	Enter requested i and n	
3			Value obtained	
4	Factor	Value	with this function	
5	P/F	0.44952	`= -PV(A2,B2,,1)	
6	P/A	16.93786	`= -PV(A2,B2,1)	
7				
8	F/P	2.22460	`= -FV(A2,B2,,1)	
9	F/A	37.67993	`= -FV(A2,B2,1)	
10				
11	A/F	0.02654	`= -PMT(A2,B2,,1)	
12	A/P	0.05904	`= -PMT(A2,B2,1)	
13				



etermine the P/A factor value for $i = 7.75\%$ and $n = 10$ years, using the three methods de
ribed previously.

olution

ctor formula: Apply the P/A factor relation in Equation [2.8] or from the summary page a
e front of the text. Showing 5-decimal accuracy,

$$(P/A,7.75\%,10) = \frac{(1+i)^n - 1}{i(1+i)^n} = \frac{(1.0775)^{10} - 1}{0.0775(1.0775)^{10}} = \frac{1.10947}{0.16348}$$
$$= 6.78641$$

readsheet: Utilize the spreadsheet function in Figure 2–9, that is, = −PV(7.75%,10,1), t
splay 6.78641.

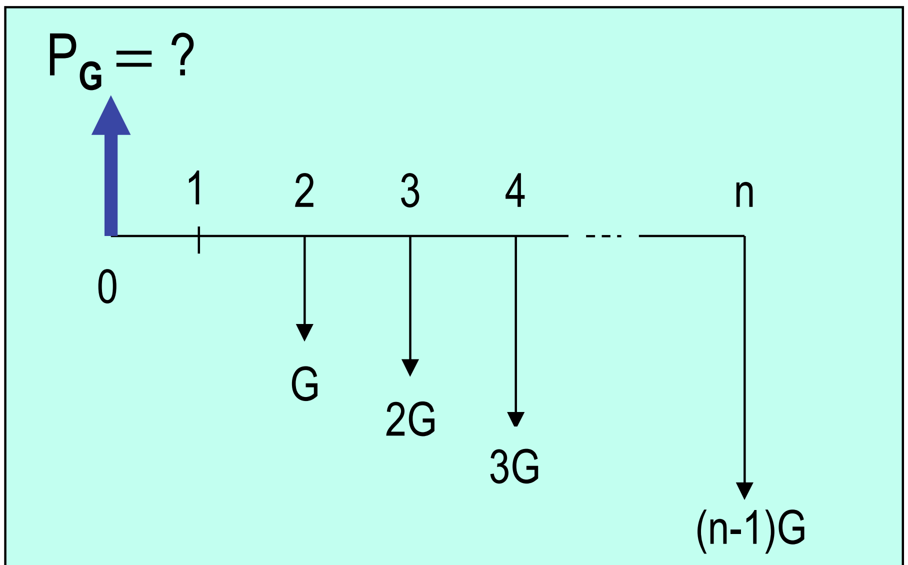
near interpolation: Use Figure 2–10 as a reference for this solution. Apply the Equa
on [2.16] and [2.17] sequence, where x is the interest rate i, the bounding interest rates ar
$= 7\%$ and $i_2 = 8\%$, and the corresponding P/A factor values are $f_1 = (P/A,7\%,10) = 7.023$
d $f_2 = (P/A,8\%,10) = 6.7101$. With 4-place accuracy,

$$f = f_1 + \frac{(i - i_1)}{(i_2 - i_1)}(f_2 - f_1) = 7.0236 + \frac{(7.75 - 7)}{(8 - 7)}(6.7101 - 7.0236)$$
$$= 7.0236 + (0.75)(-0.3135) = 7.0236 - 0.2351$$
$$= 6.7885$$

Arithmetic Gradients

Arithmetic gradients *change* by the *same amount* each period

The cash flow diagram for the P_G of an arithmetic gradient is:



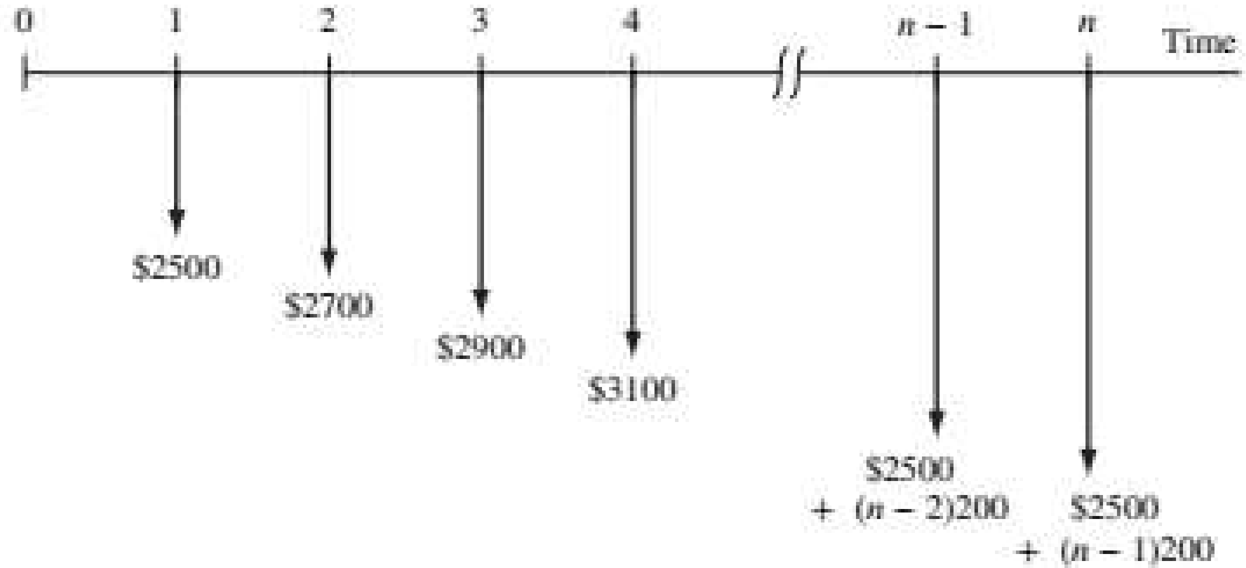
Standard factor notation is

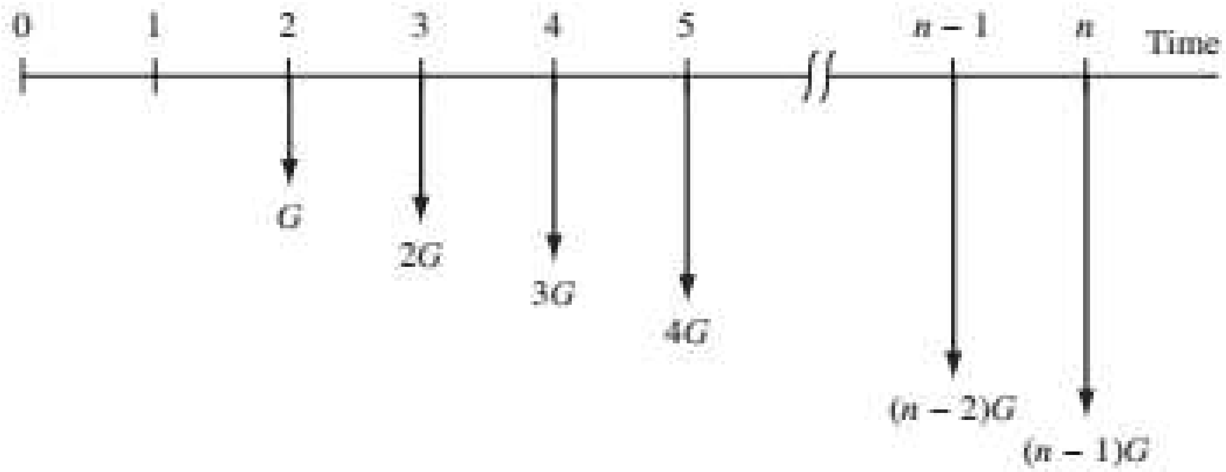
$$P_G = G(P/G, i, n)$$

G starts between periods 1 and 2
(not between 0 and 1)

This is because cash flow in year 1 is usually not equal to G and is handled separately as a *base amount* (shown on next slide)

Note that P_G is located Two Periods Ahead of the first change that is equal to G





$$CF_n = \text{base amount} + (n-1)G$$

Example

A local university has initiated a logo-licensing program with the clothier Holister, Inc. Estimated fees (revenues) are \$80,000 for the first year with uniform increases to a total of \$200,000 by the end of year 9. Determine the gradient and construct a cash flow diagram that identifies the base amount and the gradient series.

Solution

The year 1 base amount is $CF_1 = \$80{,}000$, and the total increase over 9 years is

$$CF_9 - CF_1 = 200{,}000 - 80{,}000 = \$120{,}000$$

Equation [2.18], solved for G, determines the arithmetic gradient.

$$G = \frac{(CF_9 - CF_1)}{n - 1} = \frac{120{,}000}{9 - 1}$$
$$= \$15{,}000 \text{ per year}$$

$$
\begin{aligned}
P &= G(P/F,i,2) + 2G(P/F,i,3) + 3G(P/F,i,4) + \cdots \\
&\quad + [(n-2)G](P/F,i,n-1) + [(n-1)G](P/F,i,n)
\end{aligned}
$$

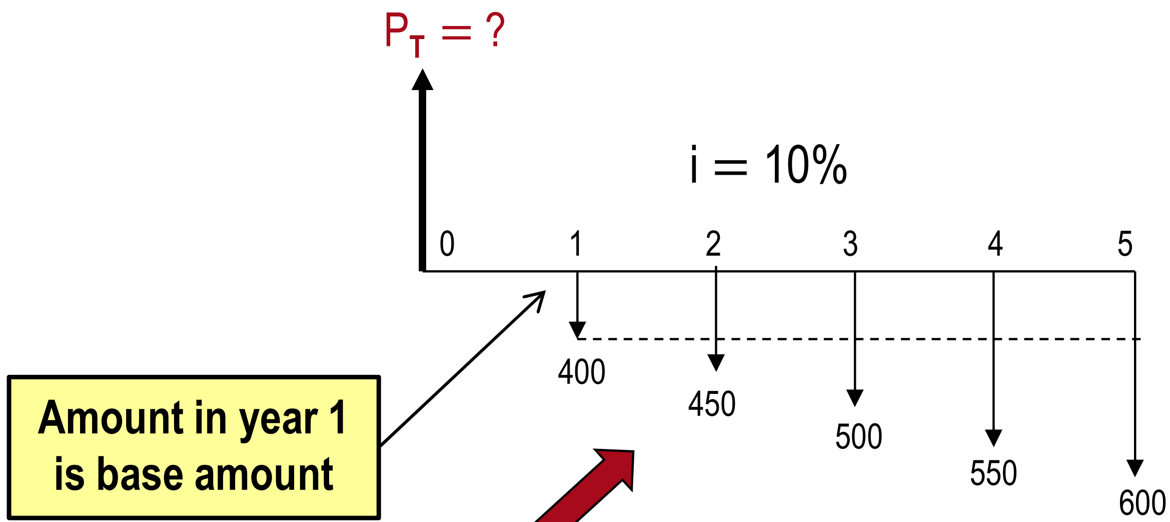
$$P = G\left[\frac{1}{(1+i)^2} + \frac{2}{(1+i)^3} + \frac{3}{(1+i)^4} + \cdots + \frac{n-2}{(1+i)^{n-1}} + \frac{n-1}{(1+i)^n}\right]$$

$$P\,(1+i)^1 = G\left[\frac{1}{(1+i)^1} + \frac{2}{(1+i)^2} + \frac{3}{(1+i)^3} + \cdots + \frac{n-2}{(1+i)^{n-2}} + \frac{n-1}{(1+i)^{n-1}}\right]$$

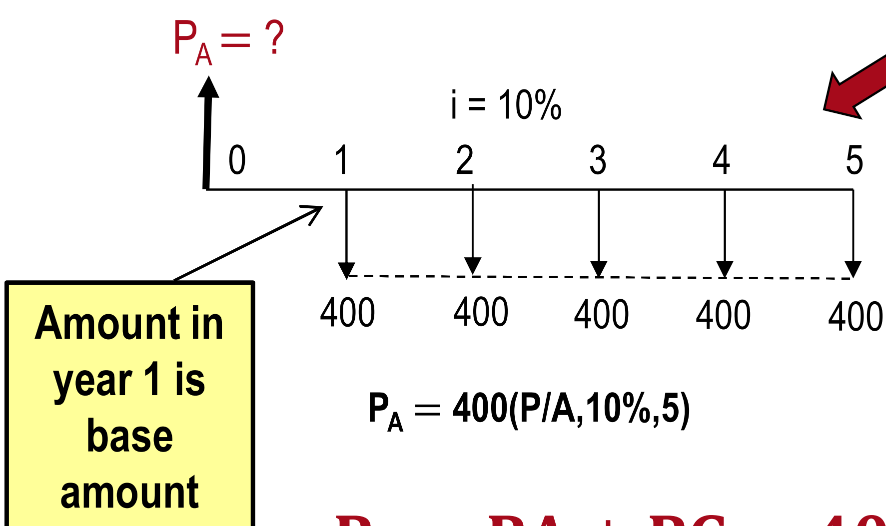
$$iP = G\left[\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\right] - G\left[\frac{n}{(1+i)^n}\right]$$

$$P_G = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]$$

Typical Arithmetic Gradient Cash Flow

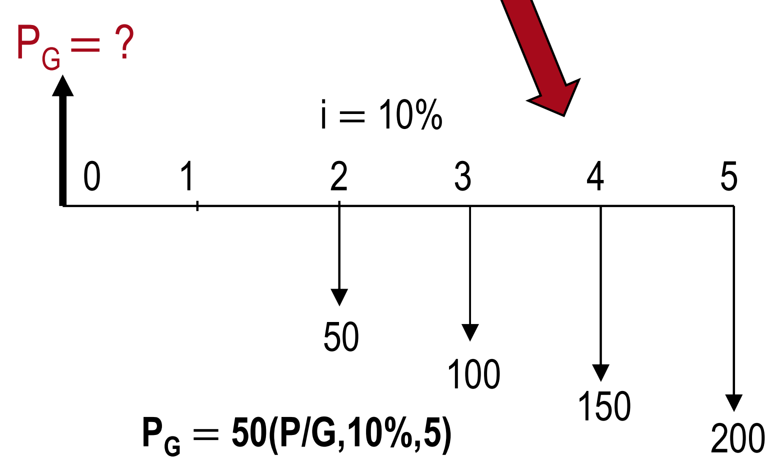


This diagram = this *base amount* plus this gradient



$P_A = 400(P/A,10\%,5)$

+



$P_G = 50(P/G,10\%,5)$

$$P_T = PA + PG = 400(P/A, 10\%, 5) + 50(P/G, 10\%, 5)$$

The equivalent uniform annual series *AG* for an arithmetic gradient *G* is found by multiplying the present worth in Equation by the (*A/P,i,n*) formula

$$A_G = G(P/G,i,n)(A/P,i,n)$$
$$= G(A/G,i,n)$$

$$A_G = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

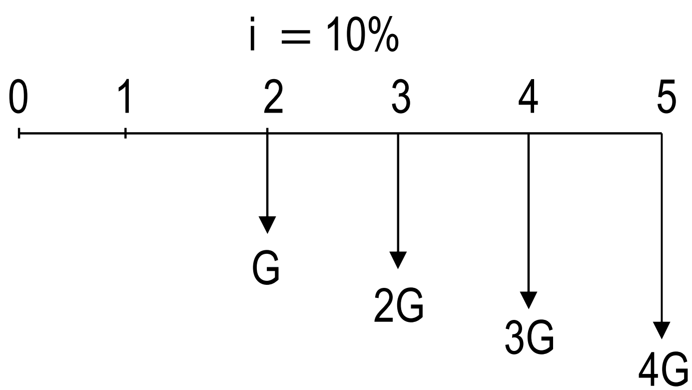
$$A_G = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right]$$

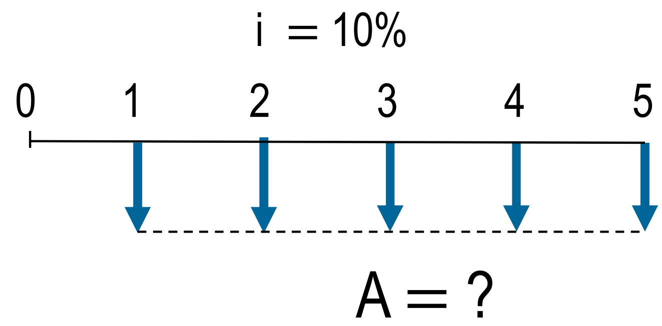
An ***F/G* factor** (*arithmetic gradient future worth factor*) to calculate the future worth *FG* of a gradient series can be derived by multiplying the *P/G* and *F/P* factors

$$F_G = G\left(\frac{1}{i}\right)\left[\left(\frac{(1+i)^n - 1}{i}\right) - n\right]$$

Converting Arithmetic Gradient to A

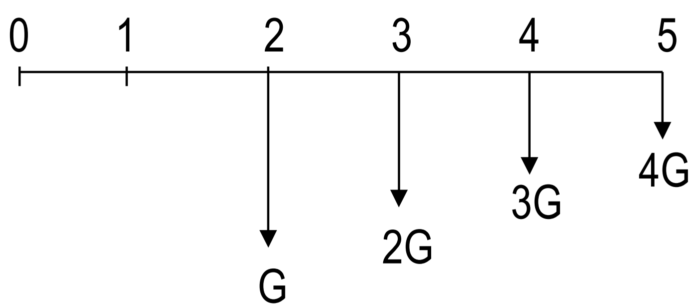
Arithmetic gradient can be converted into equivalent A value using G(A/G,i,n)





General equation when base amount is involved is

$$\mathbf{A = base\ amount + G(A/G, i, n)}$$



For decreasing gradients, change plus sign to minus

$$\mathbf{A = base\ amount - G(A/G, i, n)}$$

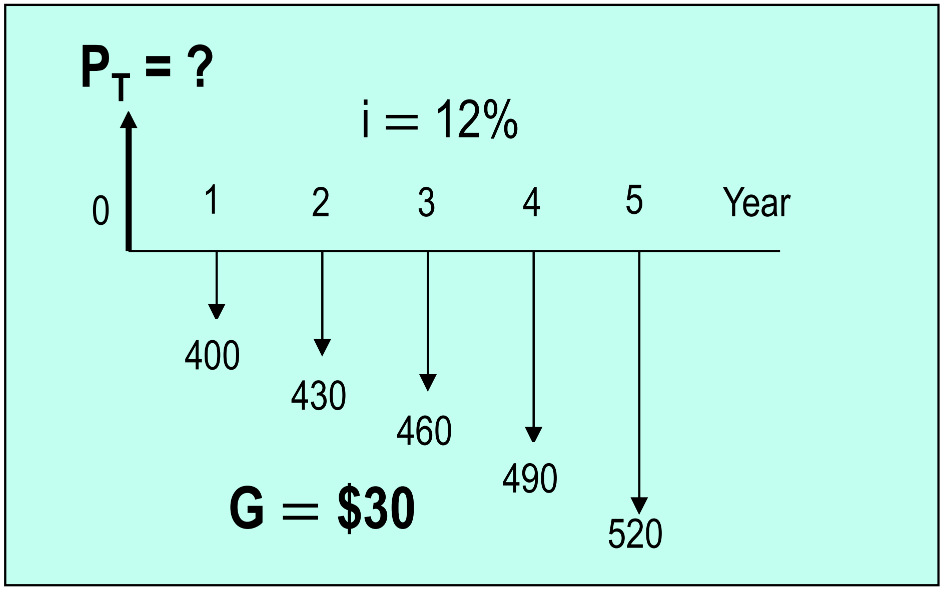
There is no direct, single-cell spreadsheet function to calculate *PG* or *AG* for an arithmetic gradient. Use the NPV function to display *PG* and the PMT function to display *AG* after entering all cash flows (base and gradient amounts) into contiguous cells. General formats for these functions are

$$= \text{NPV}(i\%, \textit{second\_cell:last\_cell}) + \textit{first\_cell}$$
$$= \text{PMT}(i\%, n, \textit{cell\_with\_}P_G)$$

Example: Arithmetic Gradient

The present worth of \$400 in year 1 and amounts increasing by \$30 per year through year 5 at an interest rate of 12% per year is closest to:

(A) \$1532 (B) \$1,634 (C) \$1,744 (D) \$1,829



Solution:

$$P_T = 400(P/A,12\%,5) + 30(P/G,12\%,5)$$
$$= 400(3.6048) + 30(6.3970)$$
$$= \$1{,}633.83$$

Answer is (B)

The cash flow could also be converted into an **A** value as follows:

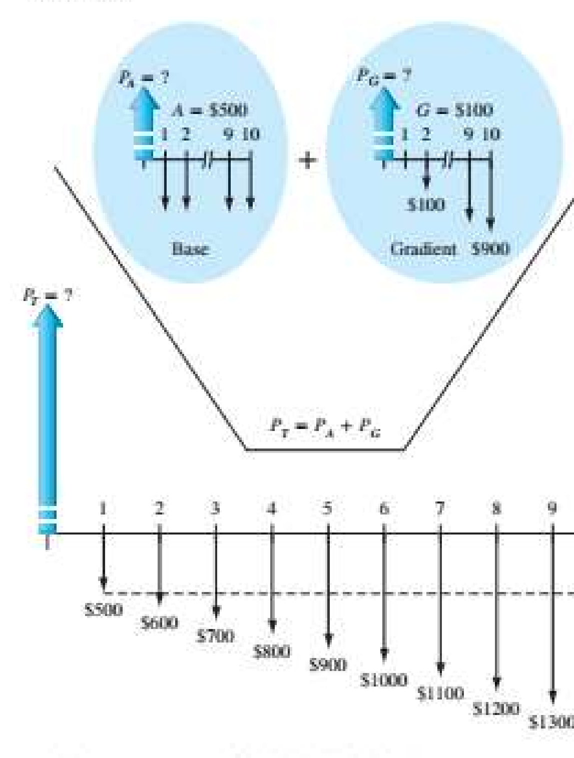
$$A = 400 + 30(A/G,12\%,5)$$
$$= 400 + 30(1.7746)$$
$$= \$453.24$$

Example

Neighboring parishes in Louisiana have agreed to pool road tax resources already designated for bridge refurbishment. At a recent meeting, the engineers estimated that a total of \$500,000 will be deposited at the end of next year into an account for the repair of old and safety-questionable bridges throughout the area. Further, they estimated that the deposits will increase by \$100,000 per year for only 9 years thereafter, then cease. Determine the equivalent (*a*) present worth, and (*b*) annual series amounts, if public funds earn at a rate of 5% per year.

(*a*) The cash flow diagram of this conventional arithmetic gradient series from the perspective of the parishes is shown in Figure 2–16. According to Equation [2.19], two computations must be made and added: the first for the present worth of the base amount P_A and the second for the present worth of the gradient P_G. The total present worth P_T occurs in year 0. This is illustrated by the partitioned cash flow diagram in Figure 2–17. In \$1000 units, the total present worth is

Cash flow series with a conventional arithmetic gradient (in \$1000 units)
Example 2.9.



$$
\begin{aligned}
P_T &= 500(P/A,5\%,10) + 100(P/G,5\%,10) \\
&= 500(7.7217) + 100(31.6520) \\
&= \$7026.05 \quad (\$7,026,050)
\end{aligned}
$$

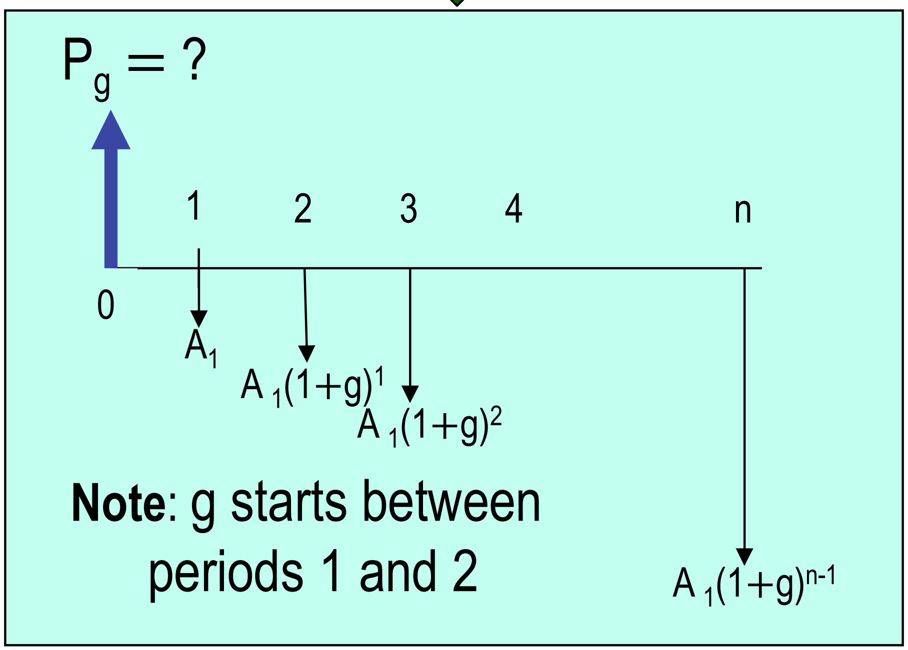
Here, too, it is necessary to consider the gradient and the base amount separately. The total annual series A_T is found by Equation [2.20] and occurs in years 1 through 10.

$$
\begin{aligned}
A_T &= 500 + 100(A/G,5\%,10) = 500 + 100(4.0991) \\
&= \$909.91 \text{ per year} \quad (\$909,910)
\end{aligned}
$$

Geometric Gradients

Geometric gradients **change by the *same percentage* each period**

Cash flow diagram for present worth of geometric gradient



There are *no tables* for geometric factors

Use following equation for $g \neq i$:

$$P_g = A_1\{1 - [(1+g)/(1+i)]^n\}/(i - g)$$

where: A_1 = cash flow in period 1
g = rate of increase

If $g = i$, $P_g = A_1 n/(1+i)$

Note: If g is negative, change signs in front of both g values

A **geometric gradient** series is a cash flow series that either increases or decreases by a **constant percentage** each period. The uniform change is called the **rate of change.**

g = **constant rate of change,** in decimal form, by which cash flow values increase or decrease from one period to the next. The gradient g can be + or −.

A_1 = **initial cash flow in period 1** of the geometric series

P_g = **present worth** of the entire geometric gradient series, including the initial amount A_1

$$P_g = \frac{A_1}{(1+i)^1} + \frac{A_1(1+g)}{(1+i)^2} + \frac{A_1(1+g)^2}{(1+i)^3} + \cdots + \frac{A_1(1+g)^{n-1}}{(1+i)^n}$$

$$= A_1\left[\frac{1}{1+i} + \frac{1+g}{(1+i)^2} + \frac{(1+g)^2}{(1+i)^3} + \cdots + \frac{(1+g)^{n-1}}{(1+i)^n}\right]$$

$$P_g\left(\frac{1+g}{1+i} - 1\right) = A_1\left[\frac{(1+g)^n}{(1+i)^{n+1}} - \frac{1}{1+i}\right]$$

$$P_g = A_1\left[\frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g}\right] \qquad g \neq i$$

The $(P/A,g,i,n)$ factor calculates P_g in period $t = 0$ for a geometric gradient series **starting in period 1** in the amount A_1 and increasing by a constant rate of g each period.

$$P_g = A_1(P/A,g,i,n)$$

$$(P/A,g,i,n) = \begin{cases} \dfrac{1 - \left(\dfrac{1+g}{1+i}\right)^n}{i-g} & g \neq i \\[2ex] \dfrac{n}{1+i} & g = i \end{cases}$$

A coal-fired power plant has upgraded an emission control valve. The modification costs only \$8000 and is expected to last 6 years with a \$200 salvage value. The maintenance cost is expected to be high at \$1700 the first year, increasing by 11% per year thereafter. Determine the equivalent present worth of the modification and maintenance cost by hand and by spreadsheet at 8% per year.

The cash flow diagram (Figure 2–22) shows the salvage value as a positive cash flow and all costs as negative. Use Equation [2.35] for $g \neq i$ to calculate P_g. Total P_T is the sum of three present worth components.

$$
\begin{aligned}
P_T &= -8000 - P_g + 200(P/F,8\%,6) \\
&= -8000 - 1700\left[\frac{1-(1.11/1.08)^6}{0.08-0.11}\right] + 200(P/F,8\%,6) \\
&= -8000 - 1700(5.9559) + 126 = \$-17{,}999
\end{aligned}
$$

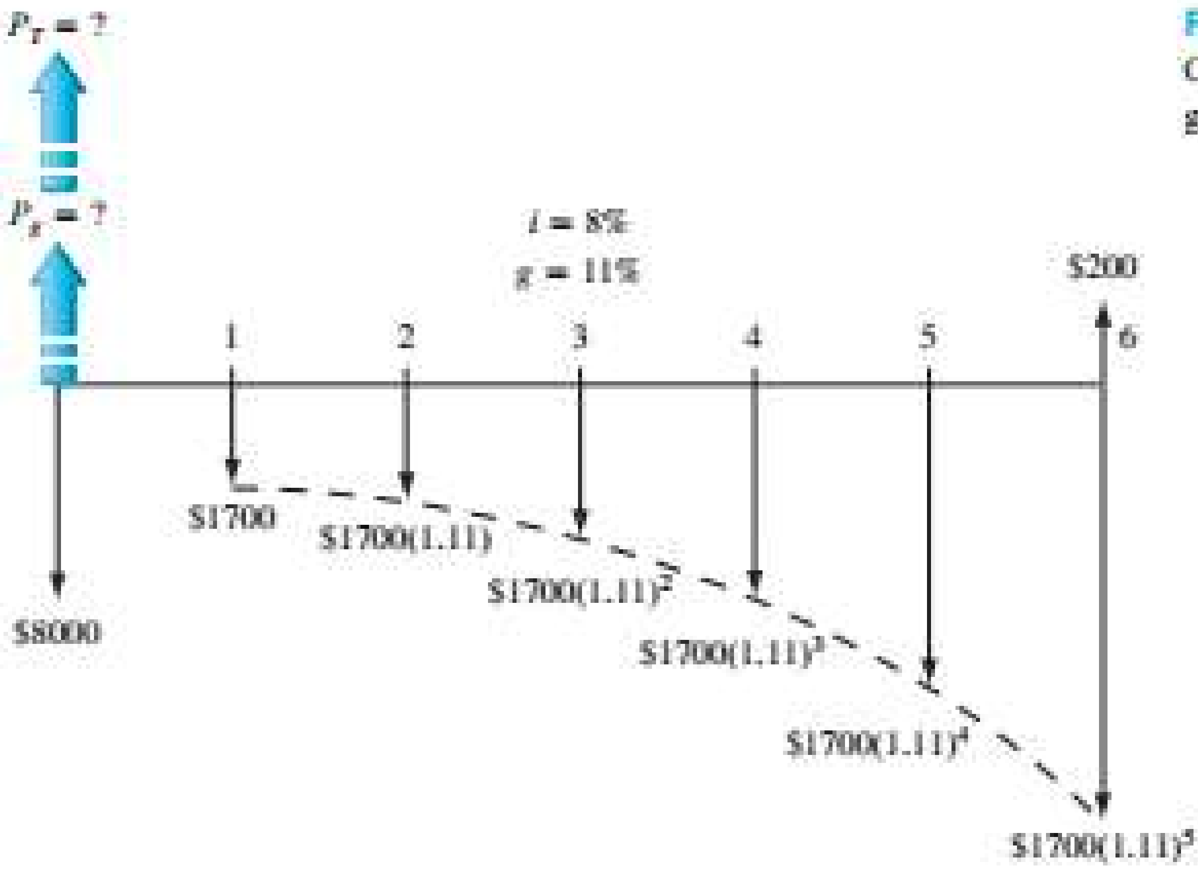


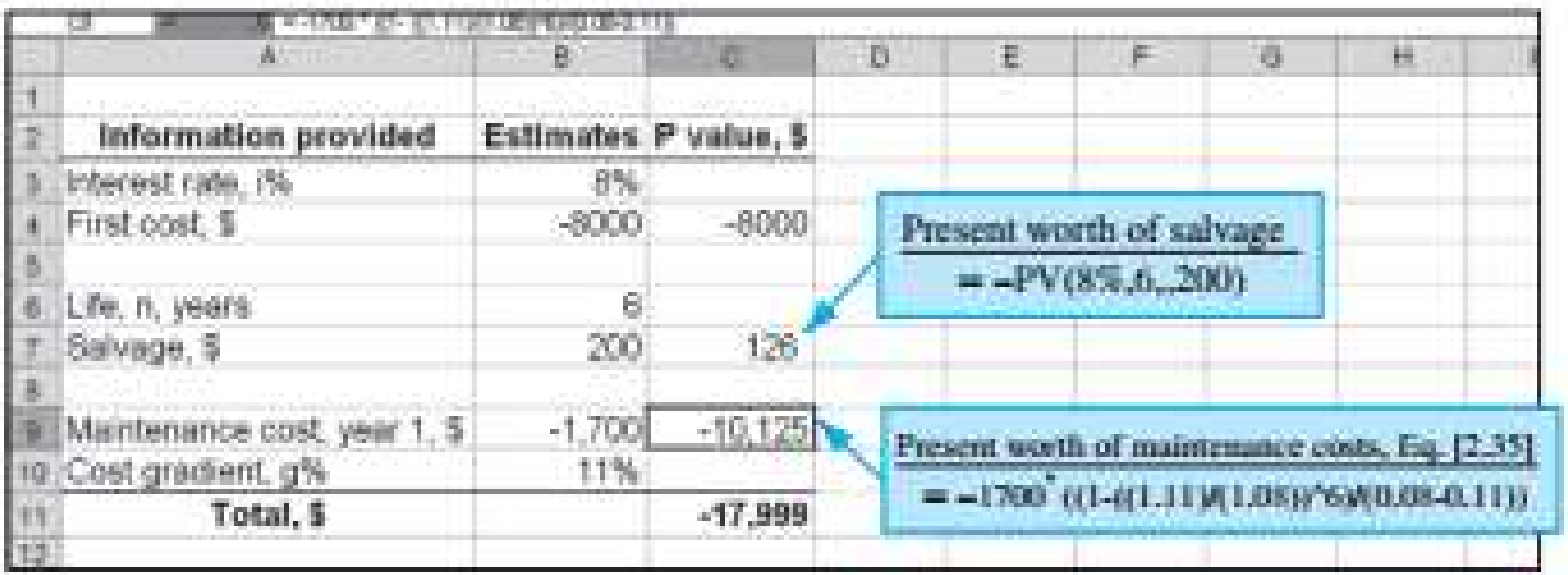
Figure 2–22
Cash flow diagram of a geometric gradient, Example 2.11.

Solution by Spreadsheet

Figure 2–23 details the spreadsheet operations to find the geometric gradient present worth P_g and total present worth P_T. To obtain $P_T = \$-17{,}999$, three components are summed—first cost, present worth of estimated salvage in year 6, and P_g. Cell tags detail the relations for the second and third components; the first cost occurs at time 0.

Comment

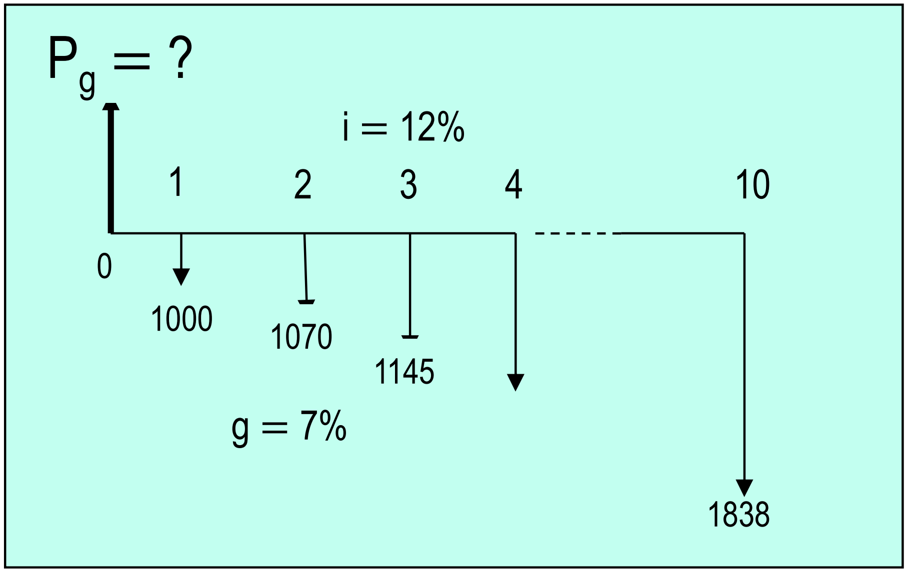
The relation that calculates the $(P/A,g,i\%,n)$ factor is rather complex, as shown in the cell tag and formula bar for C9. If this factor is used repeatedly, it is worthwhile using cell reference formatting so that A_1, i, g, and n values can be changed and the correct value is always obtained. Try to write the relation for cell C9 in this format.



Example: Geometric Gradient

Find the present worth of $1,000 in year 1 and amounts increasing by 7% per year through year 10. Use an interest rate of 12% per year.

(a) $5,670 **(b) $7,333** **(c) $12,670** **(d) $13,550**



Solution:

$$P_g = 1000[1-(1+0.07/1+0.12)^{10}]/(0.12 - 0.07)$$
$$= \$7,333$$

Answer is (b)

To find A, multiply P_g by (A/P,12%,10)

Unknown Interest Rate i

Unknown interest rate problems involve solving for i, given n and 2 other values (P, F, or A)

(Usually requires a trial and error solution or interpolation in interest tables)

Procedure: Set up equation with all symbols involved and solve for i

A contractor purchased equipment for $60,000 which provided income of $16,000 per year for 10 years. The annual rate of return of the investment was closest to:

(a) 15% **(b) 18%** **(c) 20%** **(d) 23%**

Solution: Can use either the P/A or A/P factor. Using A/P:

$$60{,}000(A/P, i\%, 10) = 16{,}000$$

$$(A/P, i\%, 10) = 0.26667$$

From A/P column at n =10 in the interest tables, i is between 22% and 24% Answer is (d)

Unknown Recovery Period n

Unknown recovery period problems involve solving for n, given i and 2 other values (P, F, or A)

(Like interest rate problems, they usually require a trial & error solution or interpolation in interest tables)

Procedure: Set up equation with all symbols involved and solve for n

A contractor purchased equipment for $60,000 that provided income of $8,000 per year. At an interest rate of 10% per year, the length of time required to recover the investment was closest to:

(a) 10 years **(b) 12 years** **(c) 15 years** **(d) 18 years**

Solution: Can use either the P/A or A/P factor. Using A/P:

$$60{,}000(A/P, 10\%, n) = 8{,}000$$
$$(A/P, 10\%, n) = 0.13333$$

From A/P column in i = 10% interest tables, n is between 14 and 15 years **Answer is (c)**

Summary of Important Points

In P/A and A/P factors, P is *one period ahead* of first A

In F/A and A/F factors, F is in *same period as last* A

To find untabulated factor values, best way is to use *formula or spreadsheet*

For arithmetic gradients, gradient G starts between *periods 1 and 2*

Arithmetic gradients have 2 parts, *base amount* (year 1) and *gradient amount*

For geometric gradients, gradient g starts been *periods 1 and 2*

In geometric gradient formula, A_1 is amount in *period 1*

To find unknown i or n, *set up equation involving all terms* and solve for i or n