Chapter 3

Control Charts for Variables

3-1. Introduction

- Variable - a single quality characteristic that can be measured on a numerical scale.
- When working with variables, we should monitor both the mean value of the characteristic and the variability associated with the characteristic.

3-2. Control Charts for $\overline{X}$ and R

Notation for variables control charts

- n - size of the sample (sometimes called a subgroup) chosen at a point in time
- m - number of samples selected
- $\overline{x}_i$ = average of the observations in the ith sample (where i = 1, 2, ..., m)
- $\overline{\overline{x}}$ = grand average or “average of the averages (this value is used as the center line of the control chart)

3-2. Control Charts for $\bar{X}$ and R

Notation and values

- R_i = range of the values in the ith sample

$$R_i = x_{max} - x_{min}$$

- $\bar{R}$ = average range for all m samples
- μ is the true process mean
- σ is the true process standard deviation

3-2. Control Charts for $\bar{x}$ and R

Statistical Basis of the Charts

- Assume the quality characteristic of interest is normally distributed with mean μ, and standard deviation, σ.
- If $x_1, x_2, \ldots, x_n$ is a sample of size n, then he average of this sample is

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

- $\bar{x}$ is normally distributed with mean, μ, and standard deviation, $\sigma_{\bar{x}} = \sigma / \sqrt{n}$

3-2. Control Charts for $\overline{X}$ and R

Statistical Basis of the Charts

- The probability is 1 - α that any sample mean will fall between

$$\mu + Z_{\alpha/2}\sigma_{\bar{x}} = \mu + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

and

$$\mu - Z_{\alpha/2}\sigma_{\bar{x}} = \mu - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

- The above can be used as upper and lower control limits on a control chart for sample means, if the process parameters are known.

3-2. Control Charts for $\bar{X}$ and R

Control Limits for the $\bar{X}$ chart

$$\begin{aligned} UCL &= \bar{\bar{x}} + A_2\bar{R} \\ \text{Center Line} &= \bar{\bar{x}} \qquad (3\text{-}4) \\ LCL &= \bar{\bar{x}} - A_2\bar{R} \end{aligned}$$

- A_2 is found in Appendix VI for various values of n.

3-2. Control Charts for $\overline{X}$ and R

<u>Control Limits for the R chart</u>

$$\mathrm{UCL} = D_4\overline{R}$$
$$\text{Center Line} = \overline{R} \qquad (3\text{-}5)$$
$$\mathrm{LCL} = D_3\overline{R}$$

- D_3 and D_4 are found in Appendix VI for various values of *n*.

3-2. Control Charts for $\overline{X}$ and R

Estimating the Process Standard Deviation

- The process standard deviation can be estimated using a function of the sample average range.

$$\hat{\sigma} = \frac{\overline{R}}{d_2}$$

Example 3-1

TABLE 3.1 Flow Width Measurements (microns) for the Hard-Bake Process

Sample Number	Wafers 1	2	3	4	5	$\bar{x}_i$	R_i
1	1.3235	1.4128	1.6744	1.4573	1.6914	1.5119	0.3679
2	1.4314	1.3592	1.6075	1.4666	1.6109	1.4951	0.2517
3	1.4284	1.4871	1.4932	1.4324	1.5674	1.4817	0.1390
4	1.5028	1.6352	1.3841	1.2831	1.5507	1.4712	0.3521
5	1.5604	1.2735	1.5265	1.4363	1.6441	1.4882	0.3706
6	1.5955	1.5451	1.3574	1.3281	1.4198	1.4492	0.2674
7	1.6274	1.5064	1.8366	1.4177	1.5144	1.5805	0.4189
8	1.4190	1.4303	1.6637	1.6067	1.5519	1.5343	0.2447
9	1.3884	1.7277	1.5355	1.5176	1.3688	1.5076	0.3589
10	1.4039	1.6697	1.5089	1.4627	1.5220	1.5134	0.2658
11	1.4158	1.7667	1.4278	1.5928	1.4181	1.5242	0.3509
12	1.5821	1.3355	1.5777	1.3908	1.7559	1.5284	0.4204
13	1.2856	1.4106	1.4447	1.6398	1.1928	1.3947	0.4470
14	1.4951	1.4036	1.5893	1.6458	1.4969	1.5261	0.2422
15	1.3589	1.2863	1.5996	1.2497	1.5471	1.4083	0.3499
16	1.5747	1.5301	1.5171	1.1839	1.8662	1.5344	0.6823
17	1.3680	1.7269	1.3957	1.5014	1.4449	1.4874	0.3589
18	1.4163	1.3864	1.3057	1.6210	1.5573	1.4573	0.3153
19	1.5796	1.4185	1.6541	1.5116	1.7247	1.5777	0.3062
20	1.7106	1.4412	1.2361	1.3820	1.7601	1.5060	0.5240
21	1.4371	1.5051	1.3485	1.5670	1.4880	1.4691	0.2185
22	1.4738	1.5936	1.6583	1.4973	1.4720	1.5390	0.1863
23	1.5917	1.4333	1.5551	1.5295	1.6866	1.5592	0.2533
24	1.6399	1.5243	1.5705	1.5563	1.5530	1.5688	0.1156
25	1.5797	1.3663	1.6240	1.3732	1.6887	1.5264	0.3224
						$\Sigma\bar{x}_i = 37.6400$	$\Sigma R_i = 8.1302$
						$\bar{\bar{x}} = 1.5056$	$\bar{R} = 0.32521$

$$\overline{R} = \frac{\sum_{i=1}^{25} R_i}{25} = \frac{8.1302}{25} = 0.32521$$

$$\text{LCL} = \overline{R} D_3 = 0.32521(0) = 0$$
$$\text{UCL} = \overline{R} D_4 = 0.32521(2.114) = 0.68749$$

$$\overline{\overline{x}} = \frac{\sum_{i=1}^{25} \overline{x}_i}{25} = \frac{37.6400}{25} = 1.5056$$
$$\text{UCL} = \overline{\overline{x}} + A_2 \overline{R} = 1.5056 + (0.577)(0.32521) = 1.69325$$

$$\text{LCL} = \overline{\overline{x}} - A_2 \overline{R} = 1.5056 - (0.577)(0.32521) = 1.31795$$

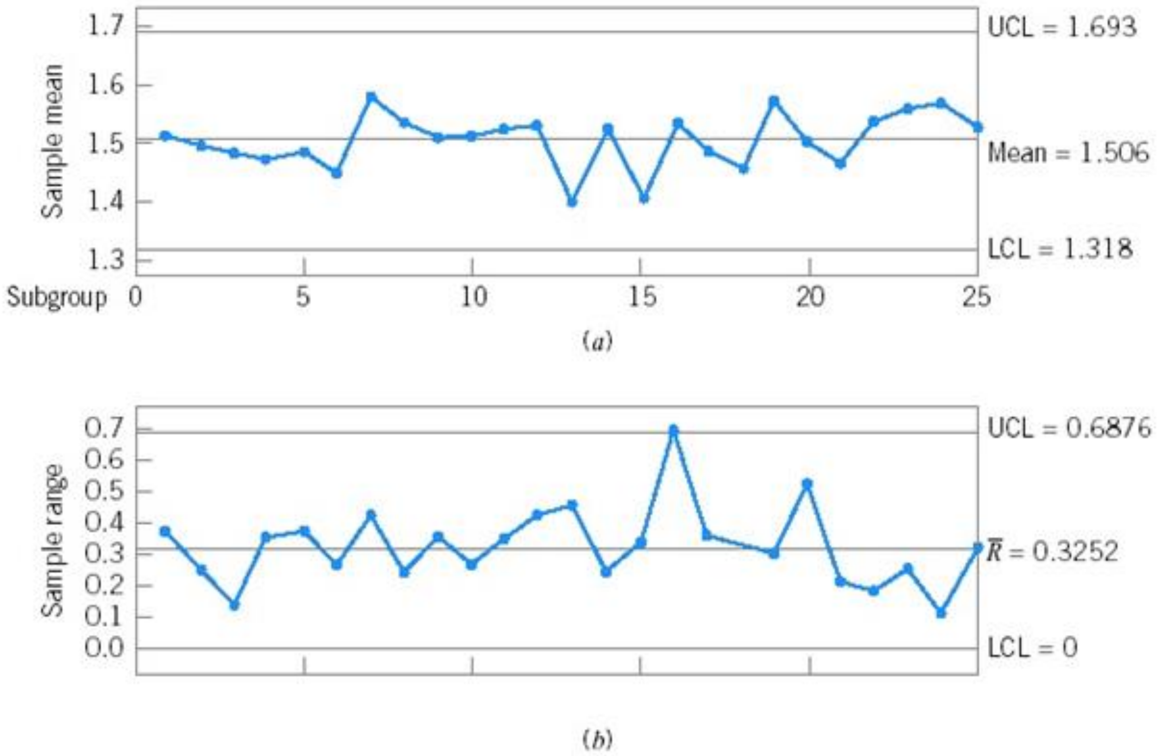


FIGURE 3.1 $\bar{x}$ and R charts (from Minitab) for flow width in the hard-bake process.

$$\hat{\sigma} = \frac{\overline{R}}{d_2} = \frac{0.32521}{2.326} = 0.1398 \text{ microns}$$

Assume spec tolerance is 1.5 +/- 0.5 micron.
Nonconformance probability:

$$\begin{aligned} p &= P\{x < 1.00\} + P\{x > 2.00\} \\ &= \Phi\left(\frac{1.00 - 1.5056}{0.1398}\right) + 1 - \Phi\left(\frac{2.00 - 1.5056}{0.1398}\right) \\ &= \Phi(-3.61660) + 1 - \Phi(3.53648) \\ &\simeq 0.00015 + 1 - 0.99980 \\ &\simeq 0.00035 \end{aligned}$$

3-2. Control Charts for $\overline{X}$ and R

Trial Control Limits

- The control limits obtained from equations (3-4) and (3-5) should be treated as trial control limits.
- If this process is in control for the m samples collected, then the system was in control in the past.
- If all points plot inside the control limits and no systematic behavior is identified, then the process was in control in the past, and the trial control limits are suitable for controlling current or future production.

3-2. Control Charts for $\overline{X}$ and R

Trial control limits and the out-of-control process

- If points plot out of control, then the control limits must be revised.
- Before revising, identify out of control points and look for assignable causes.
  - If assignable causes can be found, then discard the point(s) and recalculate the control limits.
  - If no assignable causes can be found then 1) either discard the point(s) as if an assignable cause had been found or 2) retain the point(s) considering the trial control limits as appropriate for current control.

3-2. Control Charts for $\overline{X}$ and R

Estimating Process Capability

- The x-bar and R charts give information about the capability of the process relative to its *specification limits*.
- Assumes a stable process.
- We can estimate the fraction of nonconforming items for any process where specification limits are involved.
- Assume the process is normally distributed, and x is normally distributed, the fraction nonconforming can be found by solving:

$$P(x < LSL) + P(x > USL)$$

3-2. Control Charts for $\bar{X}$ and R

Process-Capability Ratios (C_p)

- Used to express process capability.
- For processes with both upper and lower control limits,

  Use an estimate of σ if it is unknown.

$$C_p = \frac{USL - LSL}{6\sigma}$$

- If $C_p > 1$, then a low # of nonconforming items will be produced.
- If $C_p = 1$, (assume norm. dist) then we are producing about 0.27% nonconforming.
- If $C_p < 1$, then a large number of nonconforming items are being produced.

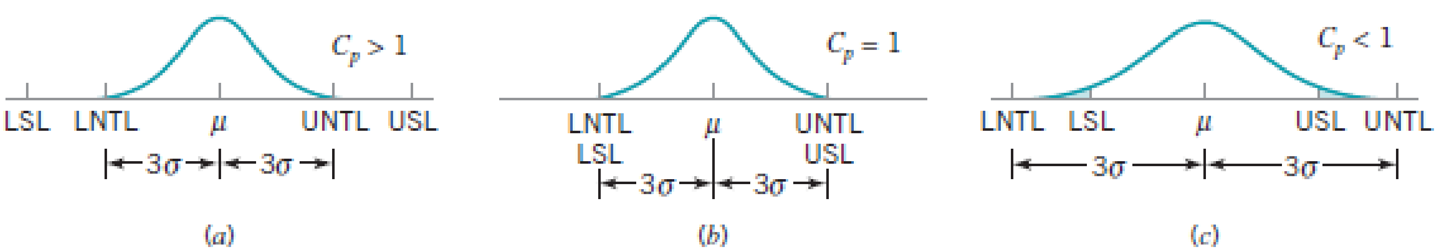
3-2. Control Charts for $\overline{X}$ and R

<u>Process-Capability Ratios (C_p)</u>

- The percentage of the specification band that the process uses up is denoted by

$$\hat{P} = \left(\frac{1}{C_p}\right)100\%$$

**The C_p statistic assumes that the process mean is centered at the midpoint of the specification band – it measures potential capability.



C_p: Process Capability Ratio (PCR)

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma} \qquad (5\text{-}11)$$

Note: 6σ spread is the basic definition of process capability. 3σ above mean and 3σ below.

If σ is unknown, we can use $\hat{\sigma} = \frac{\bar{\text{R}}}{\text{d}_2}$. $\hat{\sigma}$ in the example is 0.1398.

$$\hat{C}_p = \frac{2.00 - 1.00}{6(0.1398)} = \frac{1.00}{0.8388} = 1.192$$

$$P = \left(\frac{1}{C_p}\right)100\%$$

P : % of specification band the process uses up. *P* can be estimated as:

$$\hat{P} = \left(\frac{1}{\hat{C}_p}\right)100\% = \left(\frac{1}{1.192}\right)100\% = 83.89$$

Revision of Control Limits and Center Lines

Revision of Control Limits and Center Lines. The effective use of any control chart will require **periodic revision** of the control limits and center lines. Some practitioners establish regular periods for review and revision of control chart limits, such as every week, every month, or every 25, 50, or 100 samples. When revising control limits, remember that it is highly desirable to use at least 25 samples or subgroups (some authorities recommend 200–300 individual observations) in computing control limits.

Sometimes the user will replace the center line of the $\bar{x}$ chart with a **target value,** say $\bar{\bar{x}}_0$. If the R chart exhibits control, this can be helpful in shifting the process average to the desired value, particularly in processes where the mean may be changed by a fairly simple adjustment of a manipulatable variable in the process. If the mean is not easily influenced by a simple process adjustment, then it is likely to be a complex and unknown function of several process variables and a target value $\bar{\bar{x}}_0$ may not be helpful, as use of that value could result in many points outside the control limits. In such cases, we would not necessarily know whether the point was really associated with an assignable cause or whether it plotted outside the limits because of a poor choice for the center line.

When the R chart is out of control, we often eliminate the out-of-control points and recompute a revised value of $\bar{R}$. This value is then used to determine new limits and center line on the R chart and new limits on the $\bar{x}$ chart. This will tighten the limits on both charts, making them consistent with a process standard deviation σ consistent with use of the *revised* $\bar{R}$ in the relationship $\bar{R}/d_2$. This estimate of σ could be used as the basis of a preliminary analysis of process capability.

Phase II Operation of Charts

- Use of control chart for monitoring future production, after a set of reliable limits are established, is called *phase II* of control chart usage.
- A run chart showing individuals observations in each sample, called a tolerance chart or tier diagram, may reveal patterns or unusual observations in the data.

3-2. Control Charts for $\overline{X}$ and R

Control Limits, Specification Limits, and Natural Tolerance Limits

- Control limits are functions of the natural variability of the process
- Natural tolerance limits represent the natural variability of the process (usually set at 3-sigma from the mean)
- Specification limits are determined by developers/designers.

3-2. Control Charts for $\overline{X}$ and R

Control Limits, Specification Limits, and Natural Tolerance Limits

- There is no mathematical relationship between control limits and specification limits.
- Do not plot specification limits on the charts
  - Causes confusion between control and capability
  - If individual observations are plotted, then specification limits may be plotted on the chart.

3-2. Control Charts for $\overline{X}$ and R

Guidelines for the Design of the Control Chart

- Specify sample size, control limit width, and frequency of sampling
- if the main purpose of the x-bar chart is to detect moderate to large process shifts, then small sample sizes are sufficient (n = 4, 5, or 6)
- if the main purpose of the x-bar chart is to detect **small** process shifts, larger sample sizes are needed (as much as 15 to 25)…which is often impractical…alternative types of control charts are available for this situation

3-2. Control Charts for $\bar{X}$ and R

Guidelines for the Design of the Control Chart

- If increasing the sample size is not an option, then *sensitizing* procedures (such as warning limits) can be used to detect small shifts…but this can result in increased false alarms.
- R chart is insensitive to shifts in process standard deviation.(the range method becomes less effective as the sample size increases) may want to use S or S^2 chart.

3-2. Control Charts for $\bar{X}$ and R

Guidelines for the Design of the Control Chart

Allocating Sampling Effort

- Choose a larger sample size and sample less frequently? or, Choose a smaller sample size and sample more frequently?
- The method to use will depend on the situation. In general, small frequent samples are more desirable.

3-2. Control Charts for $\overline{x}$ and R

Changing Sample Size on the $\overline{X}$ and R Charts

Changing Sample Size on the $\overline{x}$ and R Charts. We have presented the development of $\overline{x}$ and R charts assuming that the sample size n is constant from sample to sample. However, there are situations in which the sample size n is not constant. One situation is that of **variable sample size on control charts**; that is, each sample may consist of a different number of observations. The $\overline{x}$ and R charts are generally not used in this case because they lead to a changing center line on the R chart, which is difficult to interpret for many users. The $\overline{x}$ and s charts would be preferable in this case.

Another situation is that of making a **permanent** (or **semipermanent**) **change** in the sample size because of cost or because the process has exhibited good stability and fewer resources are being allocated for process monitoring. In this case, it is easy to recompute, the new control limits directly from the old ones without collecting additional samples based on the new sample size. Let

$$\overline{R}_{\text{old}} = \text{average range for the old sample size}$$
$$\overline{R}_{\text{new}} = \text{average range for the new sample size}$$
$$n_{\text{old}} = \text{old sample size}$$
$$n_{\text{new}} = \text{new sample size}$$
$$d_2(\text{old}) = \text{factor } d_2 \text{ for the old sample size}$$
$$d_2(\text{new}) = \text{factor } d_2 \text{ for the new sample size}$$

3-2. Control Charts for $\overline{X}$ and R

Changing Sample Size on the $\overline{X}$ and R Charts

Control Limits

$\overline{x}-\text{chart}$

$$\text{UCL} = \overline{\overline{x}} + A_2\left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\overline{R}_{\text{old}}$$

$$\text{LCL} = \overline{\overline{x}} - A_2\left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\overline{R}_{\text{old}}$$

$R-chart$

$$UCL = D_4\left[\frac{d_2(new)}{d_2(old)}\right]\overline{R}_{old}$$

$$CL = \overline{R}_{new} = \left[\frac{d_2(new)}{d_2(old)}\right]\overline{R}_{old}$$

$$LCL = \max\left\{0, D_3\left[\frac{d_2(new)}{d_2(old)}\right]\overline{R}_{old}\right\}$$

From Example 1, we know that

$$n_{\text{old}} = 5 \qquad \overline{R}_{\text{old}} = 0.32521$$

and from Appendix Table VI we have

$$d_2(\text{old}) = 2.326 \qquad d_2(\text{new}) = 1.693$$

Therefore, the new control limits on the $\overline{x}$ chart are found from equation (6.12) as

$$\begin{aligned} \text{UCL} &= \overline{\overline{x}} + A_2\left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\overline{R}_{\text{old}} \\ &= 1.5056 + (1.023)\left[\frac{1.693}{2.326}\right](0.32521) \\ &= 1.5056 + 0.2422 = 1.7478 \end{aligned}$$

and

$$\begin{aligned} \text{LCL} &= \overline{\overline{x}} - A_2\left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\overline{R}_{\text{old}} \\ &= 1.5056 - (1.023)\left[\frac{1.693}{2.326}\right](0.32521) \\ &= 1.5056 - 0.2422 = 1.2634 \end{aligned}$$

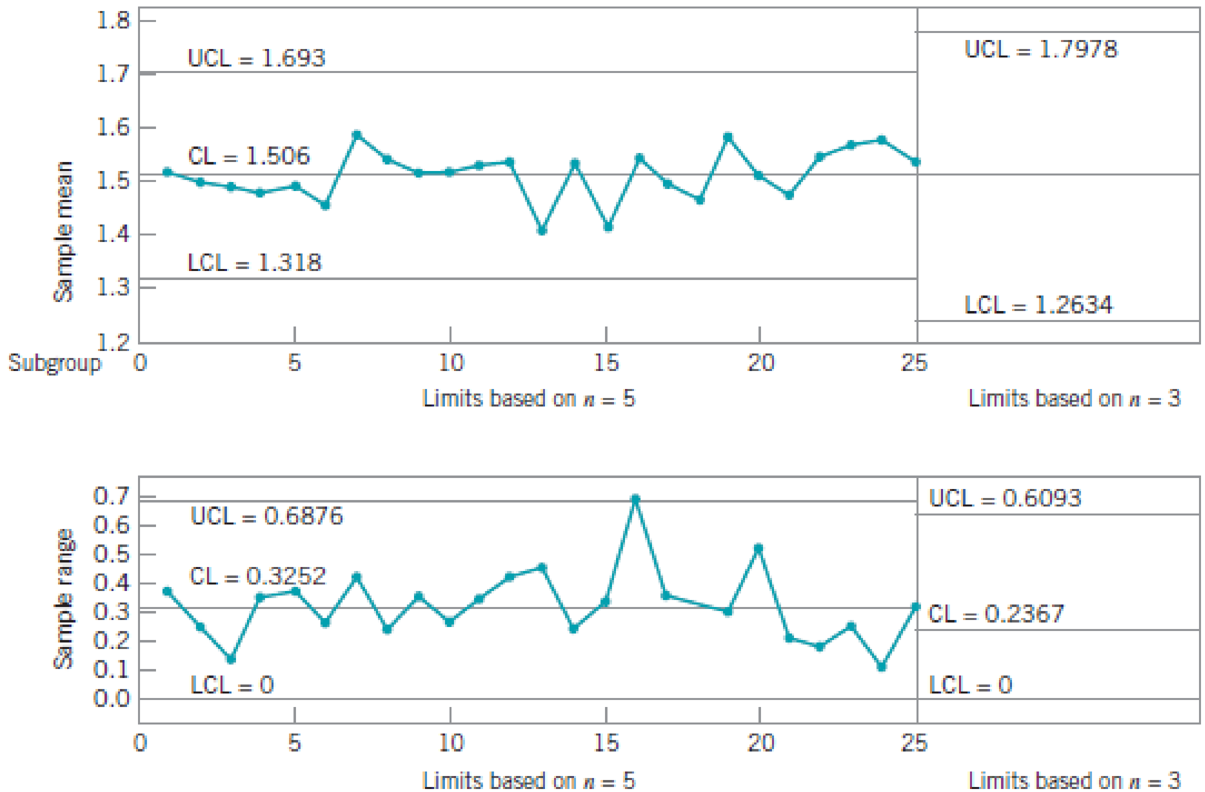
For the R chart, the new parameters are given by equation

$$\begin{aligned} \text{UCL} &= D_4\left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\overline{R}_{\text{old}} \\ &= (2.574)\left[\frac{1.693}{2.326}\right](0.32521) \\ &= 0.6093 \end{aligned}$$

$$\begin{aligned} \text{CL} = \overline{R}_{\text{new}} &= \left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\overline{R}_{\text{old}} \\ &= \left[\frac{1.693}{2.326}\right](0.32521) \\ &= 0.2367 \end{aligned}$$

$$\begin{aligned} \text{LCL} &= \max\left\{0, D_3\left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\overline{R}_{\text{old}}\right\} \\ &= 0 \end{aligned}$$

Figure shows the new control limits. Note that the effect of reducing the sample size is to *increase* the width of the limits on the $\bar{x}$ chart (because $\sigma/\sqrt{n}$ is smaller when $n = 5$ than when $n = 3$) and to *lower* the center line and the upper control limit on the R chart (because the expected range from a sample of $n = 3$ is smaller than the expected range from a sample of $n = 5$).



Recalculated control limits for the hard-bake process in Example 6.1 to reflect changing the sample size from $n = 5$ to $n = 3$.

3-2.1 Charts Based on Standard Values

When it is possible to specify standard values for the process mean and standard deviation, we may use these standards to establish the control charts for $\bar{x}$ and R without analysis of past data. Suppose that the standards given are μ and σ. Then the parameters of the $\bar{x}$ chart are

$$\begin{aligned} \text{UCL} &= \mu + 3\frac{\sigma}{\sqrt{n}} \\ \text{Center line} &= \mu \\ \text{LCL} &= \mu - 3\frac{\sigma}{\sqrt{n}} \end{aligned}$$

The quantity $3/\sqrt{n} = A$, say, is a constant that depends on n, which has been tabulated in Appendix Table VI. Consequently, we could write the parameters of the $\bar{x}$ chart as

$$\begin{aligned} \text{UCL} &= \mu + A\sigma \\ \text{Center line} &= \mu \\ \text{LCL} &= \mu - A\sigma \end{aligned}$$

To construct the R chart with a standard value of σ, recall that $\sigma = R/d_2$, where d_2 is the mean of the distribution of the relative range. Furthermore, the standard deviation of R is $\sigma_R = d_3\sigma$, where d_3 is the standard deviation of the distribution of the relative range. Therefore, the parameters of the control chart are

$$\boxed{\begin{aligned} \text{UCL} &= d_2\sigma + 3d_3\sigma \\ \text{Center line} &= d_2\sigma \\ \text{LCL} &= d_2\sigma - 3d_3\sigma \end{aligned}}$$

It is customary to define the constants

$$\begin{aligned} D_1 &= d_2 - 3d_3 \\ D_2 &= d_2 + 3d_3 \end{aligned}$$

These constants are tabulated in Appendix Table VI. Thus, the parameters of the R chart with standard σ given are

$$\boxed{\begin{aligned} \text{UCL} &= D_2\sigma \\ \text{Center line} &= d_2\sigma \\ \text{LCL} &= D_1\sigma \end{aligned}}$$

3-2.2 The Effects of Non-normality $\overline{X}$ and R

- In general, the $\overline{X}$ chart is insensitive (robust) to small departures from normality.
- The R chart is more sensitive to nonnormality than the $\overline{x}$ chart
- For 3-sigma limits, the probability of committing a type I error is 0.00461on the R-chart. (Recall that for $\overline{x}$, the probability is only 0.0027).

3-3.1 Construction and Operation of $\bar{x}$ and S Charts

- First, S^2 is an "unbiased" estimator of σ^2
- Second, S is NOT an unbiased estimator of σ
- S is an unbiased estimator of $c_4\,\sigma$

  where c_4 is a constant
- The standard deviation of S is $\sigma\sqrt{1-c_4^2}$

- In statistics, the bias (or bias function) of an estimator is the difference between this estimator's expected value and the true value of the parameter being estimated. An estimator or decision rule with zero bias is called unbiased. Otherwise the estimator is said to be biased.

3-3.1 Construction and Operation of $\overline{x}$ and S Charts

- If a standard σ is given the control limits for the S chart are:

$$\mathrm{UCL} = B_6\boldsymbol{\sigma}$$

$$\mathrm{CL} = c_4\boldsymbol{\sigma}$$

$$\mathrm{LCL} = B_5\boldsymbol{\sigma}$$

- B_5, B_6, and c_4 are found in the Appendix for various values of *n*.

3-3.1 Construction and Operation of $\bar{x}$ and S Charts

No Standard Given

- If σ is unknown, we can use an average sample standard deviation, $\bar{S} = \frac{1}{m}\sum_{i=1}^{m} S_i$

$$UCL = B_4 \bar{S}$$

$$CL = \bar{S}$$

$$LCL = B_3 \bar{S}$$

3-3.1 Construction and Operation of $\overline{x}$ and S Charts

$\overline{x}$ Chart when Using S

The upper and lower control limits for the $\overline{x}$ chart are given as

$$UCL = \overline{\overline{x}} + A_3 \overline{S}$$

$$CL = \overline{\overline{x}}$$

$$LCL = \overline{\overline{x}} - A_3 \overline{S}$$

where A_3 is found in the Appendix

3-3.1 Construction and Operation of $\overline{x}$ and S Charts

Estimating Process Standard Deviation

- The process standard deviation, σ can be estimated by

$$\hat{\sigma} = \frac{\overline{S}}{c_4}$$

Example 3-2

TABLE 3.2 Inside Diameter Measurements (mm) for Automobile Engine Piston Rings

Sample Number	Observations					$\bar{x}_i$	S_i
1	74.030	74.002	74.019	73.992	74.008	74.010	0.0148
2	73.995	73.992	74.001	74.011	74.004	74.001	0.0075
3	73.988	74.024	74.021	74.005	74.002	74.008	0.0147
4	74.002	73.996	73.993	74.015	74.009	74.003	0.0091
5	73.992	74.007	74.015	73.989	74.014	74.003	0.0122
6	74.009	73.994	73.997	73.985	73.993	73.996	0.0087
7	73.995	74.006	73.994	74.000	74.005	74.000	0.0055
8	73.985	74.003	73.993	74.015	73.988	73.997	0.0123
9	74.008	73.995	74.009	74.005	74.004	74.004	0.0055
10	73.998	74.000	73.990	74.007	73.995	73.998	0.0063
11	73.994	73.998	73.994	73.995	73.990	73.994	0.0029
12	74.004	74.000	74.007	74.000	73.996	74.001	0.0042
13	73.983	74.002	73.998	73.997	74.012	73.998	0.0105
14	74.006	73.967	73.994	74.000	73.984	73.990	0.0153
15	74.012	74.014	73.998	73.999	74.007	74.006	0.0073
16	74.000	73.984	74.005	73.998	73.996	73.997	0.0078
17	73.994	74.012	73.986	74.005	74.007	74.001	0.0106
18	74.006	74.010	74.018	74.003	74.000	74.007	0.0070
19	73.984	74.002	74.003	74.005	73.997	73.998	0.0085
20	74.000	74.010	74.013	74.020	74.003	74.009	0.0080
21	73.982	74.001	74.015	74.005	73.996	74.000	0.0122
22	74.004	73.999	73.990	74.006	74.009	74.002	0.0074
23	74.010	73.989	73.990	74.009	74.014	74.002	0.0119
24	74.015	74.008	73.993	74.000	74.010	74.005	0.0087
25	73.982	73.984	73.995	74.017	74.013	73.998	0.0162
						$\Sigma =$ 1850.028	0.2351
						$\bar{\bar{x}} =$ 74.001	$\bar{s} = 0.0094$

$$\bar{\bar{x}} = \frac{1}{25}\sum_{i=1}^{25}\bar{x}_i = \frac{1}{25}(1850.028) = 74.001$$

$$\bar{s} = \frac{1}{25}\sum_{i=1}^{25}s_i = \frac{1}{25}(0.2351) = 0.0094$$

For $\bar{x}$ chart:

$$\text{UCL} = \bar{\bar{x}} + A_3\bar{s} = 74.001 + (1.427)(0.0094) = 74.014$$

$$\text{CL} = \bar{\bar{x}} = 74.001$$

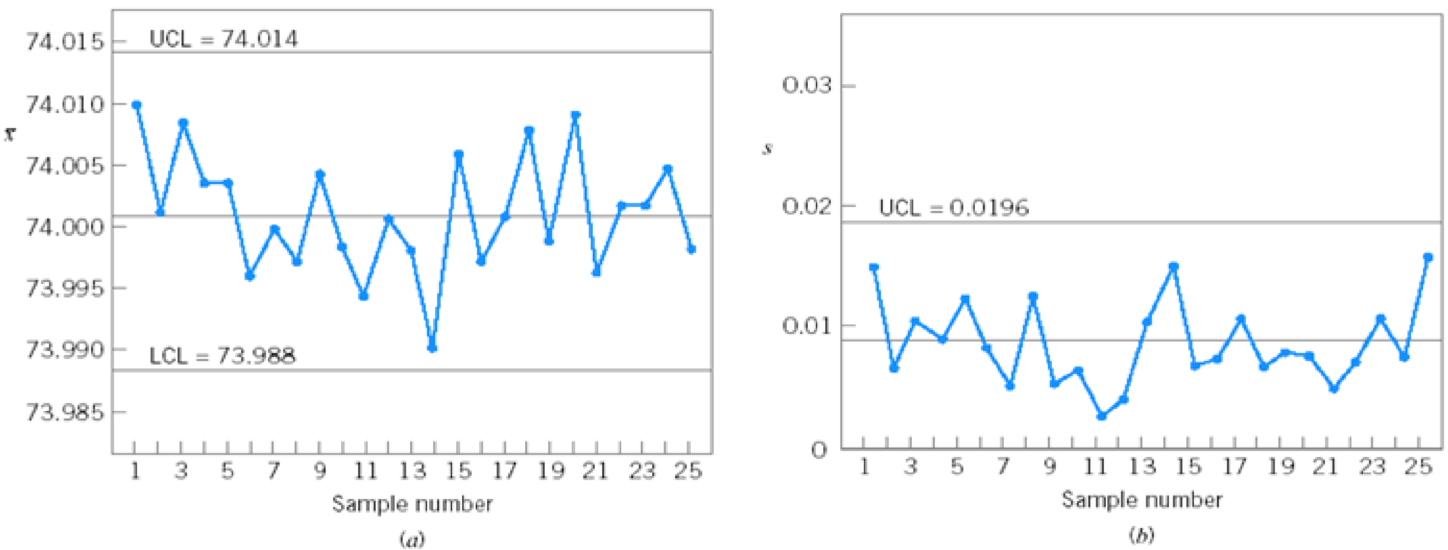
$$\text{LCL} = \bar{\bar{x}} - A_3\bar{s} = 74.001 - (1.427)(0.0094) = 73.988$$

For s chart:

$$\text{UCL} = B_4\bar{s} = (2.089)(0.0094) = 0.0196$$

$$\text{CL} = \bar{s} = 0.0094$$

$$\text{LCL} = B_3\bar{s} = (0)(0.0094) = 0$$



(*a*) The $\bar{x}$ chart with control limits based on $\bar{s}$. (*b*) The s control chart.

3-3.2 The $\overline{x}$ and S Control Charts with Variable Sample Size

- The $\overline{x}$ and S charts can be adjusted to account for samples of various sizes.
- A "weighted" average is used in the calculations of the statistics.

m = the number of samples selected.

n_i = size of the ith sample

3-3.2 The $\overline{x}$ and S Control Charts with Variable Sample Size

- The grand average can be estimated as:

$$\overline{\overline{x}} = \frac{\sum_{i=1}^{m} n_i x_i}{\sum_{i=1}^{m} n_i}$$

- The average sample standard deviation is:

$$\overline{S} = \frac{\sum_{i=1}^{m} (n_i - 1) S_i^2}{\sum_{i=1}^{m} n_i - m}$$

3-3.2 The $\overline{x}$ and S Control Charts with Variable Sample Size

- Control Limits

$$\mathrm{UCL} = \overline{\overline{x}} + A_3\overline{S}$$
$$\mathrm{CL} = \overline{\overline{x}}$$
$$\mathrm{LCL} = \overline{\overline{x}} - A_3\overline{S}$$

$$\mathrm{UCL} = B_4\overline{S}$$
$$\mathrm{CL} = \overline{S}$$
$$\mathrm{LCL} = B_3\overline{S}$$

- If the sample sizes are *not equivalent* for each sample, then
  - there can be control limits for each point (control limits may differ for each point plotted)

$\bar{x}$ and s Control Charts with Variable Sample Size

$$\bar{\bar{x}} = \frac{\sum_{i=1}^{m} n_i \bar{x}_i}{\sum_{i=1}^{m} n_i} \tag{5-30}$$

$$\bar{s} = \left[\frac{\sum_{i=1}^{m} (n_i - 1) s_i^2}{\sum_{i=1}^{m} n_i - m} \right]^{1/2} \tag{5-31}$$

Example 3-3

TABLE 3.3 Inside Diameter Measurements (mm) on Automobile Engine Piston Rings

Sample Number			Observations			$\bar{x}_i$	s_i
1	74.030	74.002	74.019	73.992	74.008	74.010	0.0148
2	73.995	73.992	74.001			73.996	0.0046
3	73.988	74.024	74.021	74.005	74.002	74.008	0.0147
4	74.002	73.996	73.993	74.015	74.009	74.003	0.0091
5	73.992	74.007	74.015	73.989	74.014	74.003	0.0122
6	74.009	73.994	73.997	73.985		73.996	0.0099
7	73.995	74.006	73.994	74.000		73.999	0.0055
8	73.985	74.003	73.993	74.015	73.988	73.997	0.0123
9	74.008	73.995	74.009	74.005		74.004	0.0064
10	73.998	74.000	73.990	74.007	73.995	73.998	0.0063
11	73.994	73.998	73.994	73.995	73.990	73.994	0.0029
12	74.004	74.000	74.007	74.000	73.996	74.001	0.0042
13	73.983	74.002	73.998			73.994	0.0100
14	74.006	73.967	73.994	74.000	73.984	73.990	0.0153
15	74.012	74.014	73.998			74.008	0.0087
16	74.000	73.984	74.005	73.998	73.996	73.997	0.0078
17	73.994	74.012	73.986	74.005		73.999	0.0115
18	74.006	74.010	74.018	74.003	74.000	74.007	0.0070
19	73.984	74.002	74.003	74.005	73.997	73.998	0.0085
20	74.000	74.010	74.013			74.008	0.0068
21	73.982	74.001	74.015	74.005	73.996	74.000	0.0122
22	74.004	73.999	73.990	74.006	74.009	74.002	0.0074
23	74.010	73.989	73.990	74.009	74.014	74.002	0.0119
24	74.015	74.008	73.993	74.000	74.010	74.005	0.0087
25	73.982	73.984	73.995	74.017	74.013	73.998	0.0162

$$\bar{\bar{x}} = \frac{\sum_{i=1}^{25} n_i \bar{x}_i}{\sum_{i=1}^{25} n_i} = \frac{5(74.010) + 3(73.996) + \cdots + 5(73.998)}{5 + 3 + \cdots + 5}$$

$$= \frac{8362.075}{113} = 74.001$$

$$\bar{s} = \left[\frac{\sum_{i=1}^{25} (n_i - 1) s_i^2}{\sum_{i=1}^{25} n_i - 25} \right]^{1/2} = \left[\frac{4(0.0148)^2 + 2(0.0046)^2 + \cdots + 4(0.0162)^2}{5 + 3 + \cdots + 5 - 25} \right]^{1/2}$$

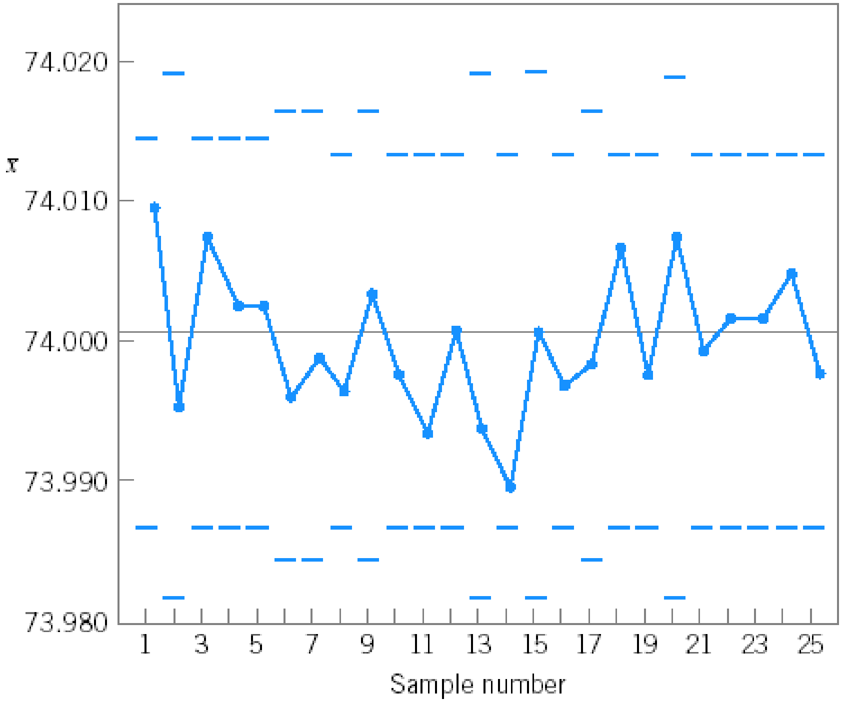
$$= \left[\frac{0.009324}{88} \right]^{1/2} = 0.0103$$

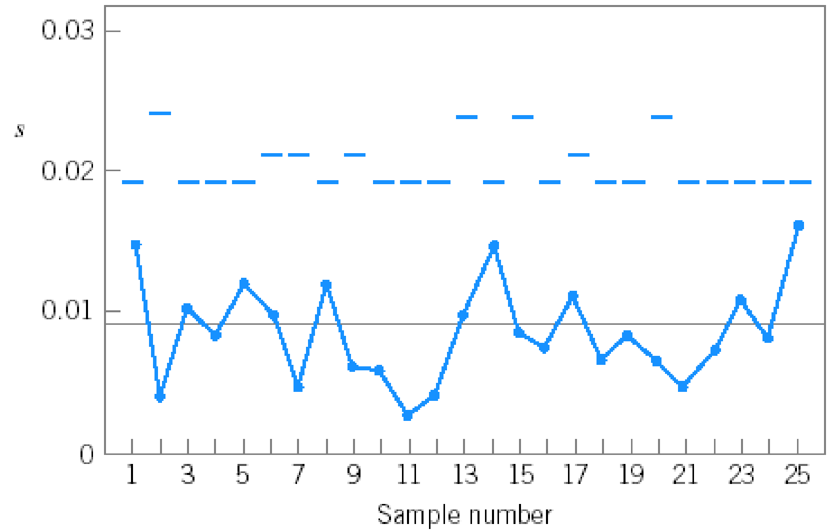
For $\bar{x}$ chart:

$$\text{UCL} = 74.001 + (1.427)(0.0103) = 74.016$$
$$\text{CL} = 74.001$$
$$\text{LCL} = 74.001 - (1.427)(0.0103) = 73.986$$

For s chart:

$$\text{UCL} = (2.089)(0.0103) = 0.022$$
$$\text{CL} = 0.0098$$
$$\text{LCL} = 0(0.0103) = 0$$





The $\bar{x}$ and s control charts for piston-ring data with variable sample size, Example 3.3

3-3.3 The S^2 Control Chart

- There may be situations where the process variance itself is monitored. An S^2 chart is

$$UCL = \frac{\overline{S}^2}{n-1}\chi^2_{\alpha/2,n-1}$$

$$CL = \overline{S}^2$$

$$LCL = \frac{\overline{S}^2}{n-1}\chi^2_{1-(\alpha/2),n-1}$$

where $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-(\alpha/2),n-1}$ are points found from the chi-square distribution.

3-4. The Shewhart Control Chart for Individual Measurements

- **What if you could not get a sample size greater than 1 (n =1)? Examples include**
  - Automated inspection and measurement technology is used, and every unit manufactured is analyzed.
  - The production rate is very slow, and it is inconvenient to allow samples sizes of N > 1 to accumulate before analysis
  - Repeat measurements on the process differ only because of laboratory or analysis error, as in many chemical processes.
- **The X and MR charts are useful for samples of sizes n = 1.**

3-4. The Shewhart Control Chart for Individual Measurements

Moving Range Chart

- The moving range (MR) is defined as the absolute difference between two successive observations:

$$MR_i = |x_i - x_{i-1}|$$

which will indicate possible shifts or changes in the process from one observation to the next.

3-4. The Shewhart Control Chart for Individual Measurements

X and Moving Range Charts

- The X chart is the plot of the individual observations. The control limits are

$$\mathrm{UCL} = \bar{x} + 3\frac{\overline{MR}}{d_2}$$

$$\mathrm{CL} = \bar{x}$$

$$\mathrm{LCL} = \bar{x} - 3\frac{\overline{MR}}{d_2}$$

where $$\overline{MR} = \frac{\sum_{i=1}^{m} MR_i}{m}$$

3-4. The Shewhart Control Chart for Individual Measurements

X and Moving Range Charts

- The control limits on the moving range chart are:

$$\mathrm{UCL} = D_4 \overline{\mathrm{MR}}$$

$$\mathrm{CL} = \overline{\mathrm{MR}}$$

$$\mathrm{LCL} = 0$$

3-4. The Shewhart Control Chart for Individual Measurements

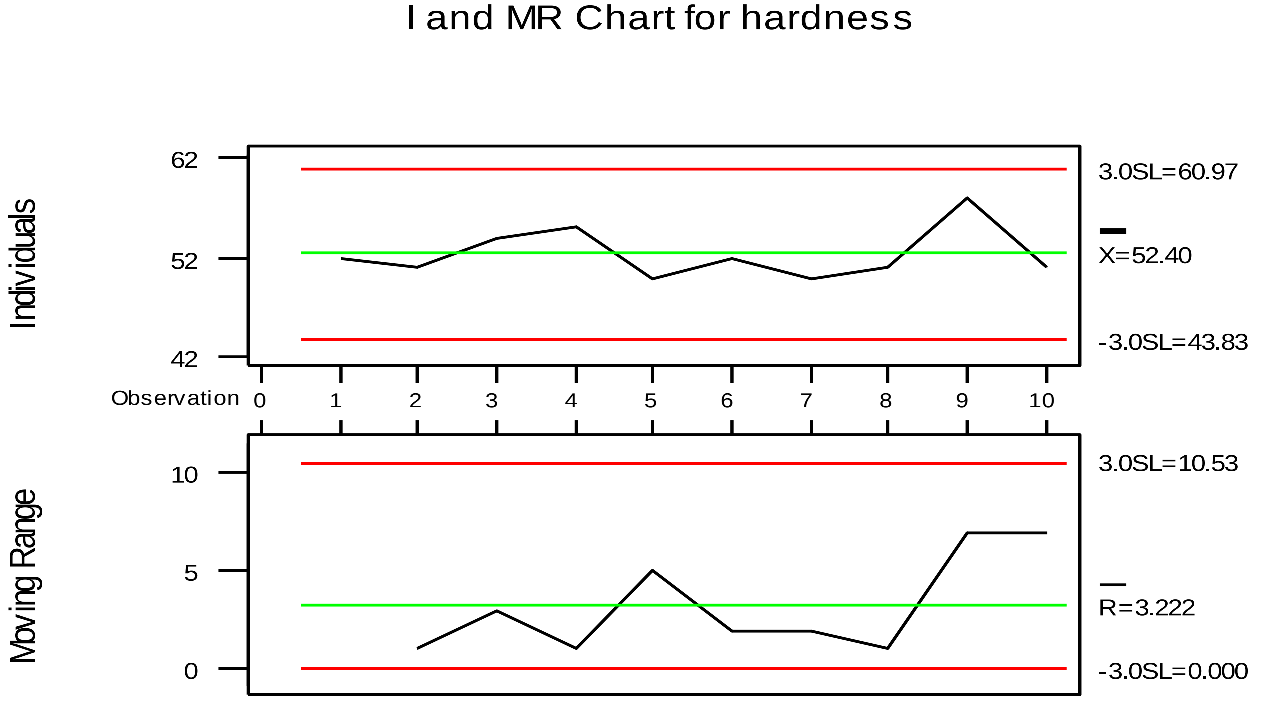
Example

Ten successive heats of a steel alloy are tested for hardness. The resulting data are

Heat	Hardness	Heat	Hardness
1	52	6	52
2	51	7	50
3	54	8	51
4	55	9	58
5	50	10	51

3-4. The Shewhart Control Chart for Individual Measurements

Example



3-4. The Shewhart Control Chart for Individual Measurements

Interpretation of the Charts

- X Charts can be interpreted similar to $\overline{x}$ charts. MR charts cannot be interpreted the same as $\overline{x}$ or R charts.
- Since the MR chart plots data that are "correlated" with one another, then looking for patterns on the chart does not make sense.
- MR chart cannot really supply useful information about process variability.
- More emphasis should be placed on interpretation of the X chart.