Chapter 4

Nominal and Effective Interest Rates

LEARNING OUTCOMES

1. Understand interest rate statements
2. Use formula for effective interest rates
3. Determine interest rate for any time period
4. Determine payment period (PP) and compounding period (CP) for equivalence calculations
5. Make calculations for single cash flows
6. Make calculations for series and gradient cash flows with PP $\geq$ CP
7. Perform equivalence calculations when PP $<$ CP
8. Use interest rate formula for continuous compounding
9. Make calculations for varying interest rates

Interest Rate Statements

The terms 'nominal' and 'effective' enter into consideration when the interest period is *less than one year*.

New time-based definitions to understand and remember

Interest period (t) – period of time over which interest is expressed. For example,
1% *per month.*

Compounding period (CP) – Shortest time unit over which interest is charged or earned.
For example,10% per year *compounded monthly.*

Compounding frequency (m) – Number of times compounding occurs within the interest period t. For example, at i = 10% per year, compounded monthly, interest would be *compounded 12 times* during the one year interest period.

Understanding Interest Rate Terminology

A ***nominal interest rate (r)*** **is obtained by multiplying an interest rate that is expressed over a short time period by the number of compounding periods in a longer time period: That is:**

r = interest rate per period x number of compounding periods

Example: If i = 1% per month, nominal rate per year is

r = (1)(12) = 12% per year)

Effective interest rates (i) **take compounding into account (effective rates can be obtained from nominal rates via a formula to be discussed later).**

IMPORTANT: Nominal interest rates are essentially simple interest rates. Therefore, they can *never* be used in interest formulas.

Effective rates must *always* be used hereafter in all interest formulas.

More About Interest Rate Terminology

There are 3 general ways to express interest rates as shown below

Sample Interest Rate Statements	Comment
(1) i = 2% per month i = 12% per year	When no compounding period is given, rate is *effective*
(2) i = 10% per year, comp'd semiannually i = 3% per quarter, comp'd monthly	When compounding period is given and it is *not the same* as interest period, it is *nominal*
(3) i = effective 9.4%/year, comp'd semiannually i = effective 4% per quarter, comp'd monthly	When compounding period is given and rate is *specified as effective*, rate *is effective* over stated period

Effective Annual Interest Rates (1)

Nominal rates are converted into effective annual rates via the equation:

$$\mathbf{i_a = (1 + i)^m - 1}$$

where i_a = effective annual interest rate

i = effective rate for one compounding period

m = number times interest is compounded per year

Example: For a nominal interest rate of 12% per year, determine the nominal and effective rates per year for (a) quarterly, and (b) monthly compounding

Solution: (a) Nominal r / year = 12% per year

Nominal r / quarter = 12/4 = 3.0% per quarter

Effective i / year = $(1 + 0.03)^4 - 1$ = 12.55% per year

(b) Nominal r /month = 12/12 = 1.0% per year

Effective i / year = $(1 + 0.01)^{12} - 1$ = 12.68% per year

Effective Annual Interest Rates (2)

Nominal rates can be converted into effective rates for any time period via the following equation:

$$\mathbf{i = (1 + r/m)^m - 1}$$

where i = effective interest rate for any time period

r = nominal rate for same time period as i

m = no. times interest is comp'd in period specified for i

Spreadsheet function is = EFFECT(r%,m) where r = nominal rate per period specified for i

Example: For an interest rate of 1.2% per month, determine the nominal and effective rates (a) per quarter, and (b) per year

Solution: (a) Nominal r / quarter = (1.2)(3) = 3.6% per quarter

Effective i / quarter = $(1 + 0.036/3)^3 - 1$ = 3.64% per quarter

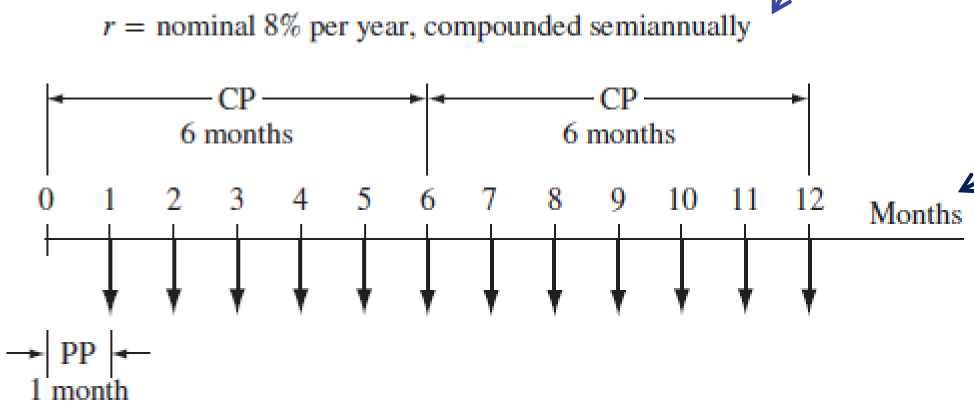
(b) Nominal i /year = (1.2)(12) = 14.4% per year

Effective i / year = $(1 + 0.144 / 12)^{12} - 1$ = 15.39% per year

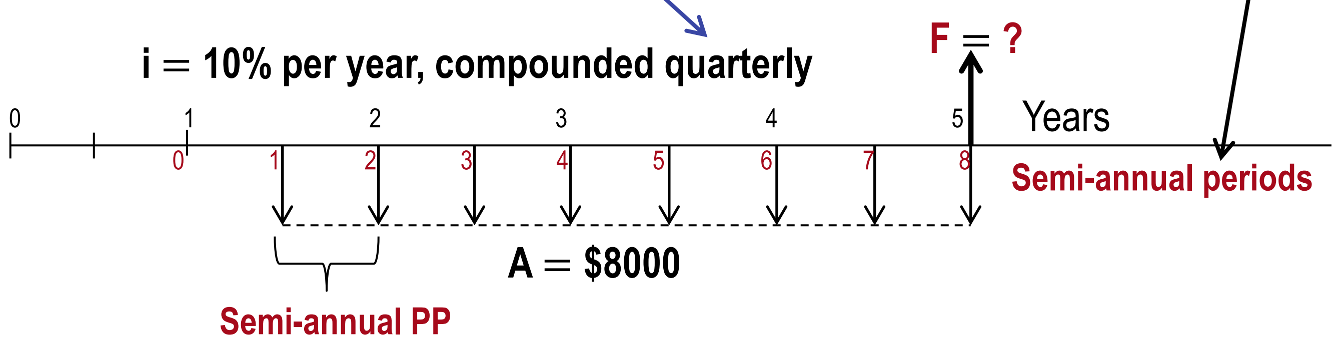
Equivalence Relations: PP and CP

New definition: **Payment Period (PP)** – Length of time between cash flows

In the diagram below, the compounding period (CP) is semiannual and the payment period (PP) is monthly



Similarly, for the diagram below, the CP is quarterly and the payment period (PP) is semiannual



Single Amounts with PP > CP

For problems involving single amounts, the payment period (PP) is usually longer than the compounding period (CP). For these problems, there are an infinite number of i and n combinations that can be used, with only two restrictions:

(1) The i must be an *effective* interest rate, and

(2) The time units on n must be *the same* as those of i
(i.e., if i is a rate per quarter, then n is the number of quarters between P and F)

There are two equally correct ways to determine i and n

Method 1: Determine effective interest rate over the compounding period CP, and set n equal to the number of compounding periods between P and F

Method 2: Determine the effective interest rate for any time period t, and set n equal to the total number of those **same time periods.**

Example: Single Amounts with PP ≥ CP

How much money will be in an account in 5 years if $10,000 is deposited now at an interest rate of 1% per month? Use three different interest rates: (a) monthly, (b) quarterly , and (c) yearly.

(a) For monthly rate, 1% is effective [$n = (5 \text{ years}) \times (12 \text{ CP per year} = 60]$

$$F = 10{,}000(F/P,1\%,60) = \$18{,}167$$

months

effective i per month

i and n must *always* have same time units

(b) For a quarterly rate, effective i/quarter $= (1 + 0.03/3)^3 - 1 = 3.03\%$

$$F = 10{,}000(F/P,3.03\%,20) = \$18{,}167$$

quarters

effective i per quarter

i and n must *always* have same time units

(c) For an annual rate, effective i/year $= (1 + 0.12/12)^{12} - 1 = 12.683\%$

$$F = 10{,}000(F/P,12.683\%,5) = \$18{,}167$$

years

effective i per year

i and n must *always* have same time units

Series with PP ≥ CP

For series cash flows, ***first step*** is to determine ***relationship*** between PP and CP

When PP $\geq$ CP, the ***only*** procedure (2 steps) that can be used is as follows:

(1) First, find effective i per PP
Example: if PP is in quarters, ***must*** find effective ***i/quarter***

(2) Second, determine n, the number of A values involved
Example: quarterly payments for 6 years yields n $= 4 \times 6 = 24$

Note: Procedure when PP $<$ CP is discussed later

Example: Series with PP $\geq$ CP

How much money will be accumulated in 10 years from a deposit of \$500 every 6 months if the interest rate is 1% per month?

Solution: First, find relationship between PP and CP

PP = ***six months***, CP = ***one month***; Therefore, **PP > CP**

Since PP > CP, find effective i per PP of six months

Step 1. $i/6 \text{ months} = (1 + 0.06/6)^6 - 1 = 6.15\%$

Next, determine n (number of 6-month periods)

Step 2: $n = 10(2) = 20$ **six month periods**

Finally, set up equation and solve for F

$F = 500(F/A,6.15\%,20) = \$18,692$ (by factor or spreadsheet)

Series with PP < CP

Two policies: (1) interperiod cash flows earn *no interest* (most common)
(2) interperiod cash flows earn *compound interest*

For policy (1), ***positive cash flows*** are moved ***to beginning of the interest period*** in which they occur and ***negative cash flows*** ***are moved to the*** ***end of the interest period***

Note: The condition of PP < CP with no interperiod interest is the *only situation in which* the actual cash flow diagram is changed

For policy (2), cash flows are ***not moved*** and equivalent P, F, and A values are determined using the ***effective interest rate per payment period***

Example: Series with PP < CP

A person deposits $100 per month into a savings account for 2 years. If $75 is withdrawn in months 5, 7 and 8 (in addition to the deposits), construct the cash flow diagram to determine how much will be in the account after 2 years at i = 6% per year, compounded quarterly. Assume there is no interperiod interest.

Solution: Since PP < CP with no interperiod interest, the cash flow diagram must be ***changed using quarters as the time periods***

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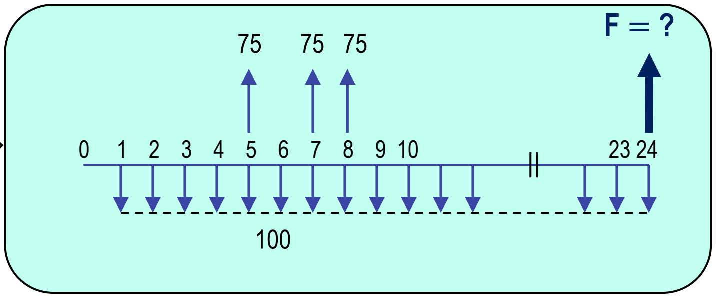

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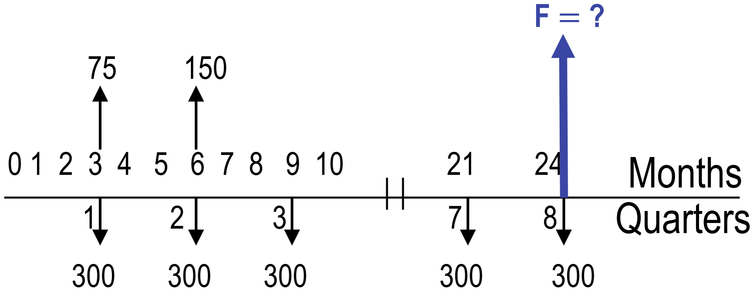


Continuous Compounding

When the interest period is infinitely small, interest is *compounded continuously.* Therefore, PP > CP and m increases.

Take limit as $m \rightarrow \infty$ to find the effective interest rate equation

$$\mathbf{i = e^{r} - 1}$$

Example: **If a person deposits \$500 into an account every 3 months at an interest rate of 6% per year, compounded continuously, how much will be in the account at the end of 5 years?**

Solution: Payment Period: PP = 3 months

Nominal rate per *three months*: r = 6%/4 = 1.50%

Effective rate per 3 months: $i = e^{0.015} - 1 = 1.51\%$

$$\mathbf{F = 500(F/A, 1.51\%, 20) = \$11,573}$$

Varying Rates

When interest rates vary over time, use the interest rates associated with their respective time periods to find P

Example: Find the present worth of $2500 deposits in years 1 through 8 if the interest rate is 7% per year for the first five years and 10% per year thereafter.

Solution: $P = 2{,}500(P/A,7\%,5) + 2{,}500(P/A,10\%,3)(P/F,7\%,5)$

$= \$14{,}683$

An equivalent annual worth value can be obtained by replacing each cash flow amount with 'A' and setting the equation equal to the calculated P value

$14{,}683 = A(P/A,7\%,5) + A(P/A,10\%,3)(P/F,7\%,5)$

$A =$ **\$2500 per year**

Summary of Important Points (1)

Must understand: interest period, compounding period, compounding frequency, and payment period

Always use *effective rates* in interest formulas

$$\mathbf{i = (1 + r/m)^m - 1}$$

Interest rates are stated different ways; must know how to get effective rates

For single amounts, make sure units on i and n are the same

Summary of Important Points (2)

For uniform series with PP $\geq$ CP, find effective i over PP

For uniform series with PP $<$ CP and no interperiod interest, move cash flows to match compounding period

For continuous compounding, use $i = e^r - 1$ to get effective rate

For varying rates, use stated i values for respective time periods

Example 1

Janice is an engineer with Southwest Airlines. She purchased Southwest stock for $6.90 per share and sold it exactly 1 year later for $13.14 per share. She was very pleased with her investment earnings. Help Janice understand exactly what she earned in terms of

a) effective annual rate and

b) effective rate for quarterly compounding, and for monthly compounding. Neglect any commission fees for purchase and selling of stock and any quarterly dividends paid to stockholders

Solution 1

(a) The effective annual rate of return i_a has a compounding period of 1 year, since the stock purchase and sales dates are exactly 1 year apart. Based on the purchase price of \$6.90 per share and using the definition of interest rate in Equation [1.2],

$$i_a = \frac{\text{amount of increase per 1 year}}{\text{original price}} \times 100\% = \frac{6.24}{6.90} \times 100\% = 90.43\% \text{ per year}$$

(b) For the effective annual rates of 90.43% per year, compounded quarterly, and 90.43%, compounded monthly, apply Equation [4.4] to find corresponding effective rates on the basis of each compounding period.

Quarter: $m = 4$ times per year $\quad i = (1.9043)^{1/4} - 1 = 1.17472 - 1 = 0.17472$

This is 17.472% per quarter, compounded quarterly.

Month: $m = 12$ times per year $\quad i = (1.9043)^{1/12} - 1 = 1.05514 - 1 = 0.05514$

This is 5.514% per month, compounded monthly.

Example 2

Tesla Motors manufactures high-performance battery electric vehicles. An engineer is on a Tesla committee to evaluate bids for new-generation coordinate-measuring machinery to be directly linked to the automated manufacturing of high-precision vehicle components. Three bids include the interest rates that vendors will charge on unpaid balances. To get a clear understanding of finance costs, Tesla management asked the engineer to determine the effective semiannual and annual interest rates for each bid. The bids are as follows:

(*a*) Determine the effective rate for each bid on the basis of semiannual periods.
(*b*) What are the effective annual rates? These are to be a part of the final bid selection.
(*c*) Which bid has the lowest effective annual rate?

Solution 2

(a) Convert the nominal rates to a semiannual basis, determine m, then use Equation [4.7] to calculate the effective semiannual interest rate i. For bid 1,

$$r = 9\% \text{ per year} = 4.5\% \text{ per 6 months}$$

$$m = 2 \text{ quarters per 6 months}$$

$$\text{Effective } i\% \text{ per 6 months} = \left(1 + \frac{0.045}{2}\right)^2 - 1 = 1.0455 - 1 = 4.55\%$$

Table 4–4 (left section) summarizes the effective semiannual rates for all three bids.

TABLE 4–4 Effective Semiannual and Annual Interest Rates for Three Bid Rates, Example 4.5

	Semiannual Rates			Annual Rates		
Bid	Nominal r per 6 Months, %	CP per 6 Months, m	Equation [4.7], Effective i, %	Nominal r per Year, %	CP per Year, m	Equation [4.7], Effective i, %
1	4.5	2	4.55	9	4	9.31
2	6.0	2	6.09	12	4	12.55
3	4.4	6	4.48	8.8	12	9.16

(b) For the effective annual rate, the time basis in Equation [4.7] is 1 year. For bid 1,

$$r = 9\% \text{ per year} \qquad m = 4 \text{ quarters per year}$$

$$\text{Effective } i\% \text{ per year} = \left(1 + \frac{0.09}{4}\right)^4 - 1 = 1.0931 - 1 = 9.31\%$$

The right section of Table 4–4 includes a summary of the effective annual rates.

(c) Bid 3 includes the lowest effective annual rate of 9.16%, which is equivalent to an effective semiannual rate of 4.48% when interest is compounded monthly.

Example 3

A dot-com company plans to place money in a new venture capital fund that currently returns 18% per year, compounded daily. What effective rate is this (a) yearly and (b) semiannually?

(*a*) Use Equation [4.7], with $r = 0.18$ and $m = 365$.

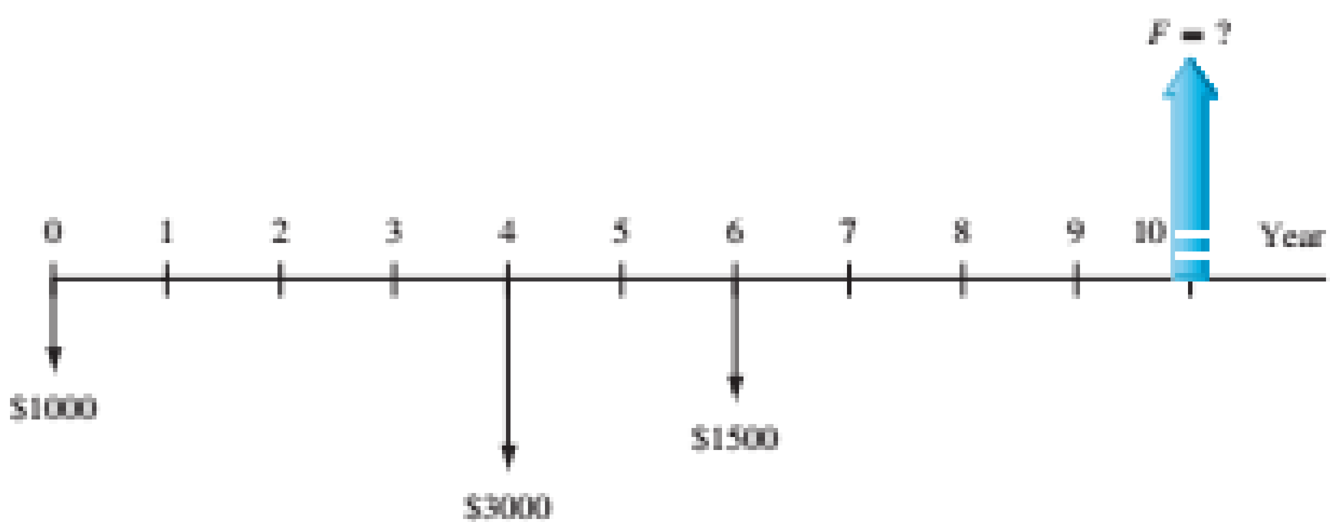
$$\text{Effective } i\% \text{ per year} = \left(1 + \frac{0.18}{365}\right)^{365} - 1 = 19.716\%$$

(*b*) Here $r = 0.09$ per 6 months and $m = 182$ days.

$$\text{Effective } i\% \text{ per 6 months} = \left(1 + \frac{0.09}{182}\right)^{182} - 1 = 9.415\%$$

Example 4

Over the past 10 years, Gentrack has placed varying sums of money into a special capital accumulation fund. The company sells compost produced by garbage-to-compost plants in the United States and Vietnam. Figure 4–5 is the cash flow diagram in $1000 units. Find the amount in the account now (after 10 years) at an interest rate of 12% per year, compounded semiannually



Solution 4

Method 1: Use the semiannual CP to express the effective semiannual rate of 6% per 6-month period. There are $n = (2)$(number of years) semiannual periods for each cash flow. Using tabulated factor values, the future worth by Equation [4.9] is

$$
\begin{aligned}
F &= 1000(F/P,6\%,20) + 3000(F/P,6\%,12) + 1500(F/P,6\%,8) \\
&= 1000(3.2071) + 3000(2.0122) + 1500(1.5938) \\
&= \$11,634 \quad (\$11.634 \text{ million})
\end{aligned}
$$

Method 2: Express the effective annual rate, based on semiannual compounding.

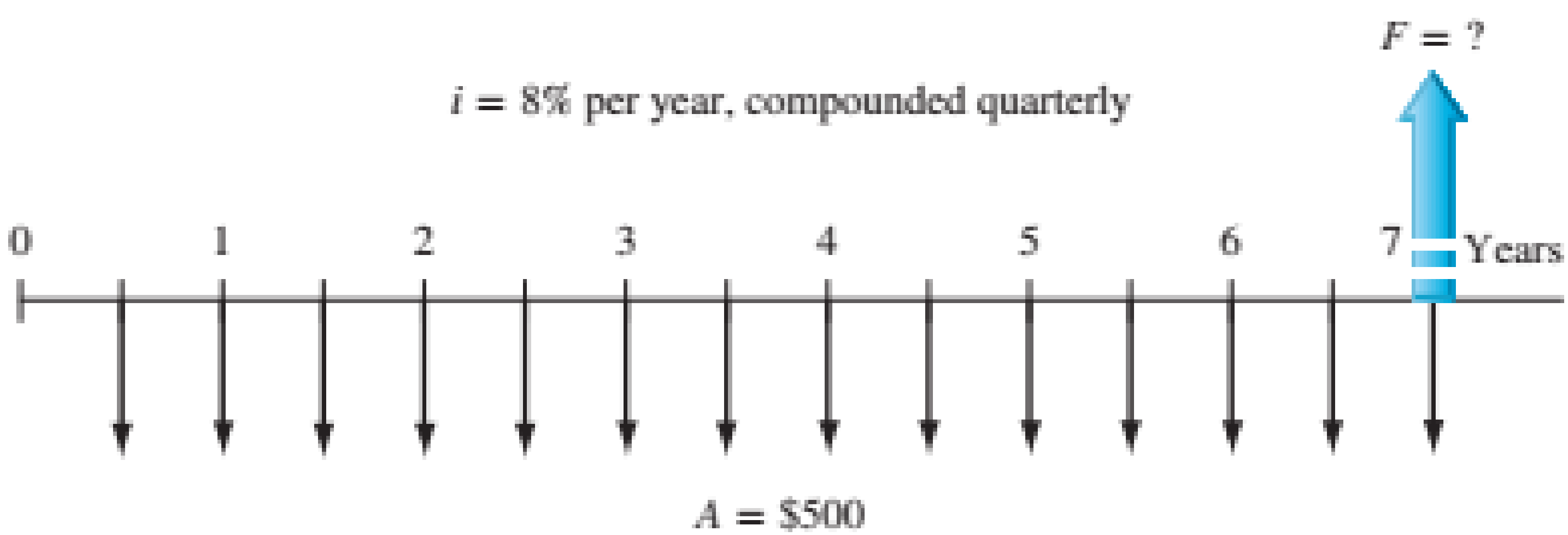
$$
\text{Effective } i\% \text{ per year} = \left(1 + \frac{0.12}{2}\right)^2 - 1 = 12.36\%
$$

The n value is the actual number of years. Use the factor formula $(F/P,i,n) = (1.1236)^n$ and Equation [4.9] to obtain the same answer as above.

$$
\begin{aligned}
F &= 1000(F/P,12.36\%,10) + 3000(F/P,12.36\%,6) + 1500(F/P,12.36\%,4) \\
&= 1000(3.2071) + 3000(2.0122) + 1500(1.5938) \\
&= \$11,634 \quad (\$11.634 \text{ million})
\end{aligned}
$$

Example 5

For the past 7 years, Excelon Energy has paid $500 every 6 months for a software maintenance contract. What is the equivalent total amount after the last payment, if these funds are taken from a pool that has been returning 8% per year, compounded quarterly?



Solution 5

The cash flow diagram is shown in Figure 4–6. The payment period (6 months) is longer than the compounding period (quarter); that is, PP > CP. Applying the guideline, we need to determine an **effective semiannual interest rate.** Use Equation [4.7] with $r = 4\%$ per 6-month period and $m = 2$ quarters per semiannual period.

$$\text{Effective } i\% \text{ per 6 months} = \left(1 + \frac{0.04}{2}\right)^2 - 1 = 4.04\%$$

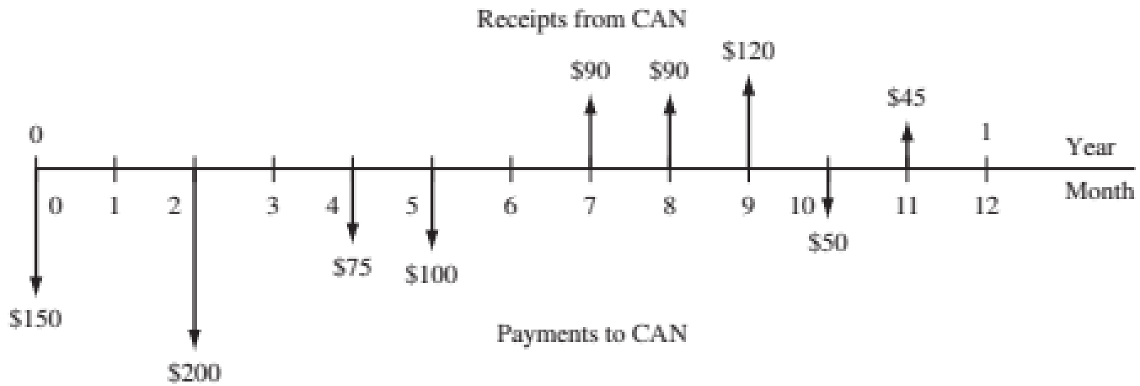
The effective semiannual interest rate can also be obtained from Table 4–3 by using the r value of 4% and $m = 2$ to get $i = 4.04\%$.

The value $i = 4.04\%$ seems reasonable, since we expect the effective rate to be slightly higher than the nominal rate of 4% per 6-month period. The total number of semiannual payment periods is $n = 2(7) = 14$. The relation for F is

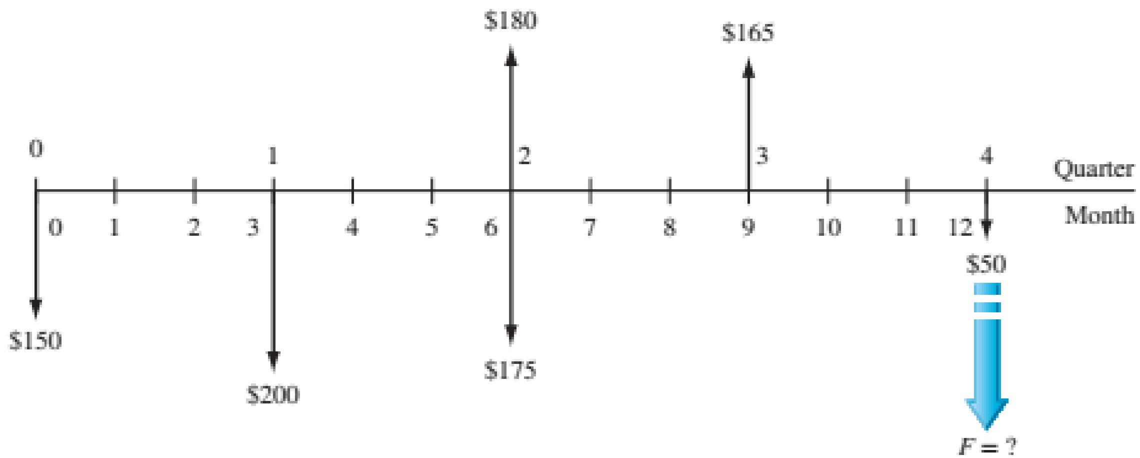
$$\begin{aligned} F &= A(F/A,4.04\%,14) \\ &= 500(18.3422) \\ &= \$9171.09 \end{aligned}$$

Example 6

Last year AllStar Venture Capital agreed to invest funds in Clean Air Now (CAN), a start-up company in Las Vegas that is an outgrowth of research conducted in mechanical engineering at the University of Nevada–Las Vegas. The product is a new filtration system used in the process of carbon capture and sequestration (CCS) for coal-fired power plants. The venture fund manager generated the cash flow diagram in Figure 4–9a in \$1000 units from AllStar's perspective. Included are payments (outflows) to CAN made over the first year and receipts (inflows) from CAN to AllStar. The receipts were unexpected this first year; however, the product has great promise, and advance orders have come from eastern U.S. plants anxious to become zero-emission coal-fueled plants. The interest rate is 12% per year, compounded quarterly, and AllStar uses the no-interperiod-interest policy. How much is AllStar in the "red" at the end of the year?



(*a*)



(*b*)

With no interperiod interest considered, Figure 4–9*b* reflects the moved cash flows. All negative cash flows (payments to CAN) are moved to the end of the respective quarter, and all positive cash flows (receipts) are moved to the beginning of the respective quarter. Calculate the F value at $12\%/4 = 3\%$ per quarter.

$$F = 1000[-150(F/P,3\%,4) - 200(F/P,3\%,3) + (-175 + 180)(F/P,3\%,2)$$
$$+ 165(F/P,3\%,1) - 50]$$
$$= \$-262{,}111$$

Example 7

1. For an interest rate of 18% per year, compounded continuously, calculate the effective monthly and annual interest rates.
2. An investor requires an effective return of at least 15%. What is the minimum annual nominal rate that is acceptable for continuous compounding?

Example 7

(a) The nominal monthly rate is $r = 18\%/12 = 1.5\%$, or 0.015 per month. By Equation [4.11], the effective monthly rate is

$$i\% \text{ per month} = e^r - 1 = e^{0.015} - 1 = 1.511\%$$

Similarly, the effective annual rate using $r = 0.18$ per year is

$$i\% \text{ per year} = e^r - 1 = e^{0.18} - 1 = 19.722\%$$

(b) Solve Equation [4.11] for r by taking the natural logarithm.

$$e^r - 1 = 0.15$$

$$e^r = 1.15$$

$$\ln e^r = \ln 1.15$$

$$r = 0.13976$$

Therefore, a rate of 13.976% per year, compounded continuously, will generate an effective 15% per year return. The general formula to find the nominal rate, given the effective continuous rate i, is $r = \ln(1 + i)$.

Example 8

CE, Inc. leases large earth tunneling equipment. The net profit from the equipment for each of the last 4 years has been decreasing, as shown below. Also shown are the annual rates of return on invested capital. The return has been increasing. Determine the present worth *P* and equivalent uniform series *A* of the net profit series. Take the annual variation of rates of return into account.

Year	1	2	3	4
Net Profit	\$70,000	\$70,000	\$35,000	\$25,000
Annual Rate	7%	7%	9%	10%

Example 8

Figure 4–10 shows the cash flows, rates for each year, and the equivalent P and A. Equation [4.12] is used to calculate P. Since for both years 1 and 2, the net profit is $70,000 and the annual rate is 7%, the P/A factor can be used for these 2 years only.

$$\begin{aligned} P &= [70(P/A,7\%,2) + 35(P/F,7\%,2)(P/F,9\%,1) \\ &\quad + 25(P/F,7\%,2)(P/F,9\%,1)(P/F,10\%,1)](1000) \\ &= [70(1.8080) + 35(0.8013) + 25(0.7284)](1000) \\ &= \$172,816 \end{aligned} \qquad [4.14]$$

To determine an equivalent annual series, substitute the symbol A for all net profit values on the right side of Equation [4.14], set it equal to $P = \$172,816$, and solve for A. This equation accounts for the varying i values each year. See Figure 4–10 for the cash flow diagram transformation.

$$\$172,816 = A[(1.8080) + (0.8013) + (0.7284)] = A[3.3377]$$

$$A = \$51,777 \text{ per year}$$

Comment

If the average of the four annual rates, that is, 8.25%, is used, the result is $A = \$52,467$. This is a $690 per year overestimate of the equivalent annual net profit.