This support is written to help students follow up during the class sessions and for their individual studies.
It is limited to a general introduction and simple examples, therefore, it will not replace textbook, discussed material and solved problems during class sessions.

Design of machinery
Chapter 7 acceleration analysis

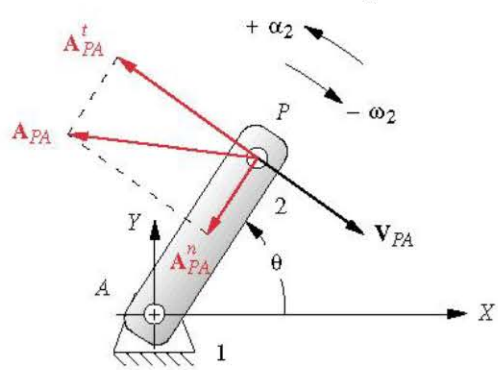
Dr. Jaafar Hallal

Chapter 7 Acceleration analysis

- 7.1 Definition of acceleration

The acceleration is the time rate change of the velocity.

$$F = \sum ma$$



$$\alpha = \frac{d\omega}{dt};$$

$$\mathbf{A} = \frac{d\mathbf{V}}{dt}$$

FIGURE 7-1
Acceleration of a link in pure rotation with a positive (CCW) α_2 and a negative (CW) ω_2

Chapter 7 Acceleration analysis

- 7.1 Definition of acceleration

$$\mathbf{R}_{PA} = pe^{j\theta}$$

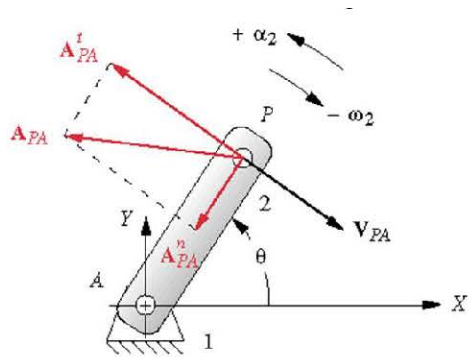
$$\mathbf{V}_{PA} = \frac{d\mathbf{R}_{PA}}{dt} = p\,je^{j\theta}\frac{d\theta}{dt} = p\omega\, je^{j\theta}$$

$$\mathbf{A}_{PA} = \frac{d\mathbf{V}_{PA}}{dt} = \frac{d\left(p\omega\, je^{j\theta}\right)}{dt}$$

$$\mathbf{A}_{PA} = jp\left(e^{j\theta}\frac{d\omega}{dt} + \omega\, je^{j\theta}\frac{d\theta}{dt}\right)$$

$$\mathbf{A}_{PA} = p\alpha\, je^{j\theta} - p\omega^2 e^{j\theta}$$

$$\mathbf{A}_{PA} = \mathbf{A}^t_{PA} + \mathbf{A}^n_{PA}$$



Chapter 7 Acceleration analysis

- 7.1 Definition of acceleration

Acceleration difference (2 points in the same body)

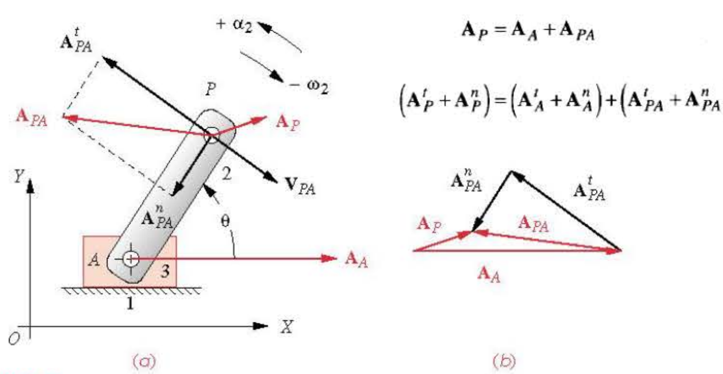


FIGURE 7-2

Acceleration difference in a system with a positive (CCW) α_2 and a negative (CW) ω_2

Chapter 7 Acceleration analysis

- 7.1 Definition of acceleration

Relative acceleration (2 points in different bodies)

$$\mathbf{A}_P = \mathbf{A}_A + \mathbf{A}_{PA}$$

$$\left(\mathbf{A}_P^t + \mathbf{A}_P^n\right) = \left(\mathbf{A}_A^t + \mathbf{A}_A^n\right) + \left(\mathbf{A}_{PA}^t + \mathbf{A}_{PA}^n\right)$$

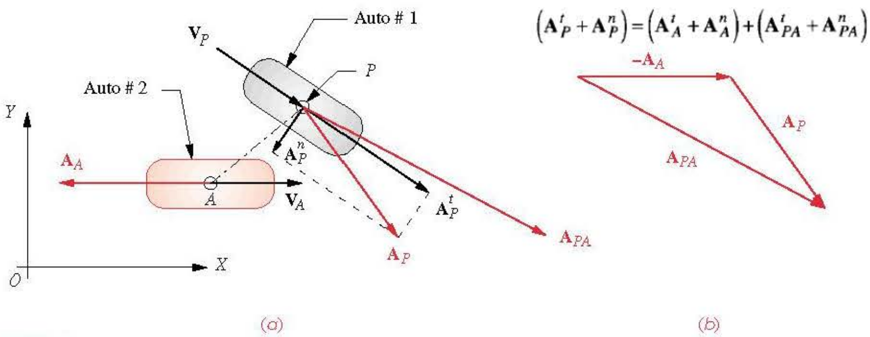


FIGURE 7-3

Relative acceleration

Chapter 7 Acceleration analysis

- 7.3 Analytical solutions for acceleration analysis

The Fourbar pin jointed linkage

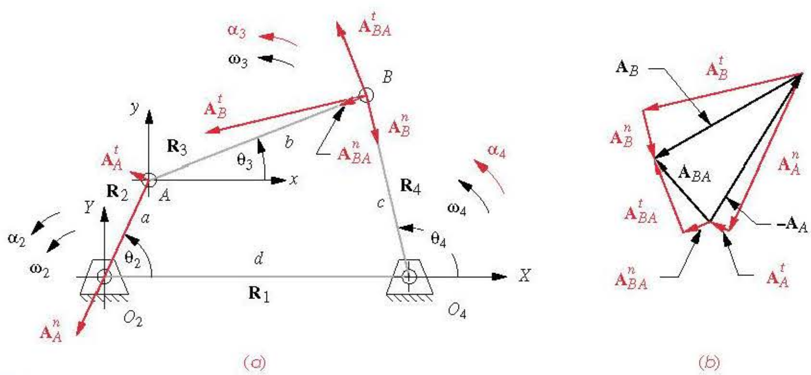


FIGURE 7-5

Position vector loop for a fourbar linkage showing acceleration vectors

Chapter 7 Acceleration analysis

- 7.3 Analytical solutions for acceleration analysis

The Fourbar pin jointed linkage

$$\mathbf{R}_2+\mathbf{R}_3-\mathbf{R}_4-\mathbf{R}_1=0$$

$$ae^{j\theta_2}+be^{j\theta_3}-ce^{j\theta_4}-de^{j\theta_1}=0$$

$$ja\omega_2e^{j\theta_2}+jb\omega_3e^{j\theta_3}-jc\omega_4e^{j\theta_4}=0$$

$$\left(j^2a\omega_2^2e^{j\theta_2}+ja\alpha_2e^{j\theta_2}\right)+\left(j^2b\omega_3^2e^{j\theta_3}+jb\alpha_3e^{j\theta_3}\right)-\left(j^2c\omega_4^2e^{j\theta_4}+jc\alpha_4e^{j\theta_4}\right)=0$$

$$\left(a\alpha_2je^{j\theta_2}-a\omega_2^2e^{j\theta_2}\right)+\left(b\alpha_3je^{j\theta_3}-b\omega_3^2e^{j\theta_3}\right)-\left(c\alpha_4je^{j\theta_4}-c\omega_4^2e^{j\theta_4}\right)=0$$

$$\mathbf{A}_A+\mathbf{A}_{BA}-\mathbf{A}_B=0$$

$$\boxed{\begin{aligned}\mathbf{A}_A&=\left(\mathbf{A}_A^t+\mathbf{A}_A^n\right)=\left(a\alpha_2je^{j\theta_2}-a\omega_2^2e^{j\theta_2}\right)\\\mathbf{A}_{BA}&=\left(\mathbf{A}_{BA}^t+\mathbf{A}_{BA}^n\right)=\left(b\alpha_3je^{j\theta_3}-b\omega_3^2e^{j\theta_3}\right)\\\mathbf{A}_B&=\left(\mathbf{A}_B^t+\mathbf{A}_B^n\right)=\left(c\alpha_4je^{j\theta_4}-c\omega_4^2e^{j\theta_4}\right)\end{aligned}}$$

$$\boxed{\begin{aligned}\mathbf{A}_A&=a\alpha_2\left(-\sin\theta_2+j\cos\theta_2\right)-a\omega_2^2\left(\cos\theta_2+j\sin\theta_2\right)\\\mathbf{A}_{BA}&=b\alpha_3\left(-\sin\theta_3+j\cos\theta_3\right)-b\omega_3^2\left(\cos\theta_3+j\sin\theta_3\right)\\\mathbf{A}_B&=c\alpha_4\left(-\sin\theta_4+j\cos\theta_4\right)-c\omega_4^2\left(\cos\theta_4+j\sin\theta_4\right)\end{aligned}}$$

Chapter 7 Acceleration analysis

- 7.3 Analytical solutions for acceleration analysis

The Fourbar pin jointed linkage

$$\left(a\alpha_2je^{j\theta_2}-a\omega_2^2e^{j\theta_2}\right)+\left(b\alpha_3je^{j\theta_3}\quad b\omega_3^2e^{j\theta_3}\right)-\left(c\alpha_4je^{j\theta_4}-c\omega_4^2e^{j\theta_4}\right)=0$$

real part (x component):

$$-a\alpha_2\sin\theta_2-a\omega_2^2\cos\theta_2-b\alpha_3\sin\theta_3-b\omega_3^2\cos\theta_3+c\alpha_4\sin\theta_4+c\omega_4^2\cos\theta_4=0$$

imaginary part (y component):

$$a\alpha_2\cos\theta_2-a\omega_2^2\sin\theta_2+b\alpha_3\cos\theta_3-b\omega_3^2\sin\theta_3-c\alpha_4\cos\theta_4+c\omega_4^2\sin\theta_4=0$$

$$\boxed{\begin{aligned}\alpha_3&=\frac{CD-AF}{AE-BD}\\\alpha_4&=\frac{CE-BF}{AE-BD}\end{aligned}}$$

Where

$$\left\{\begin{aligned}A&=c\sin\theta_4\\B&=b\sin\theta_3\\C&=a\alpha_2\sin\theta_2+a\omega_2^2\cos\theta_2+b\omega_3^2\cos\theta_3-c\omega_4^2\cos\theta_4\\D&=c\cos\theta_4\\E&=b\cos\theta_3\\F&=a\alpha_2\cos\theta_2-a\omega_2^2\sin\theta_2-b\omega_3^2\sin\theta_3+c\omega_4^2\sin\theta_4\end{aligned}\right.$$

Chapter 7 Acceleration analysis

- 7.3 Analytical solutions for acceleration analysis

The Fourbar slider crank

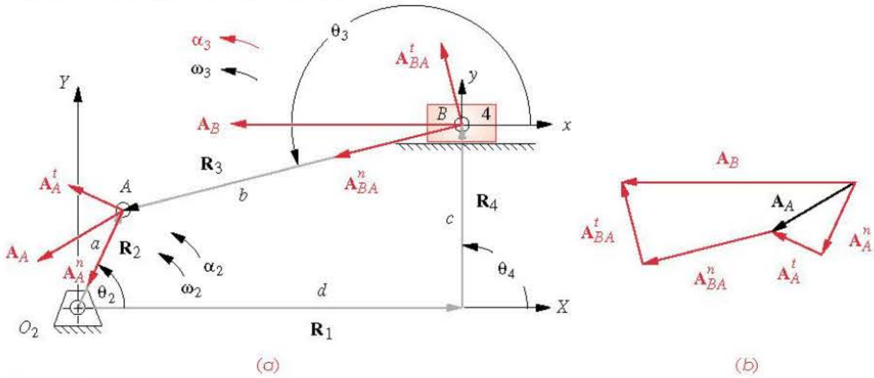


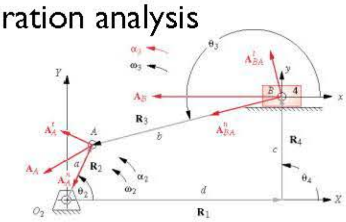
FIGURE 7-6

Position vector loop for a fourbar slider-crank linkage showing acceleration vectors

Chapter 7 Acceleration analysis

- 7.3 Analytical solutions for acceleration analysis

The Fourbar slider crank



$$\mathbf{R}_2 - \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0$$

$$a e^{j\theta_2} - b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0$$

$$ja\omega_2 e^{j\theta_2} - jb\omega_3 e^{j\theta_3} - \dot{d} = 0$$

$$\left(ja\alpha_2 e^{j\theta_2} + j^2 a\omega_2^2 e^{j\theta_2}\right) - \left(jb\alpha_3 e^{j\theta_3} + j^2 b\omega_3^2 e^{j\theta_3}\right) - \ddot{d} = 0$$

$$\left(a\alpha_2\, j e^{j\theta_2} - a\omega_2^2 e^{j\theta_2}\right) - \left(b\alpha_3\, j e^{j\theta_3} - b\omega_3^2 e^{j\theta_3}\right) - \ddot{d} = 0$$

$$\mathbf{A}_B = \mathbf{A}_A + \mathbf{A}_{BA}$$

$$\boxed{\begin{aligned} \mathbf{A}_A &= \left(\mathbf{A}_A^t + \mathbf{A}_A^n\right) = \left(a\alpha_2\, j e^{j\theta_2} - a\omega_2^2 e^{j\theta_2}\right) \\ \mathbf{A}_{BA} &= \left(\mathbf{A}_{BA}^t + \mathbf{A}_{BA}^n\right) = \left(b\alpha_3\, j e^{j\theta_3} - b\omega_3^2 e^{j\theta_3}\right) \\ \mathbf{A}_B &= \mathbf{A}_B^t = \ddot{d} \end{aligned}}$$

Chapter 7 Acceleration analysis

- 7.3 Analytical solutions for acceleration analysis

The Fourbar slider crank

$$\left(a\alpha_2 je^{j\theta_2} - a\omega_2^2 e^{j\theta_2}\right) - \left(b\alpha_3 je^{j\theta_3} - b\omega_3^2 e^{j\theta_3}\right) - \ddot{d} = 0$$

real part (x component):

$$-a\alpha_2 \sin\theta_2 - a\omega_2^2 \cos\theta_2 + b\alpha_3 \sin\theta_3 + b\omega_3^2 \cos\theta_3 - \ddot{d} = 0$$

imaginary part (y component):

$$a\alpha_2 \cos\theta_2 - a\omega_2^2 \sin\theta_2 - b\alpha_3 \cos\theta_3 + b\omega_3^2 \sin\theta_3 = 0$$

$$\boxed{\begin{aligned} \alpha_3 &= \frac{a\alpha_2 \cos\theta_2 - a\omega_2^2 \sin\theta_2 + b\omega_3^2 \sin\theta_3}{b\cos\theta_3} \\ \ddot{d} &= -a\alpha_2 \sin\theta_2 - a\omega_2^2 \cos\theta_2 + b\alpha_3 \sin\theta_3 + b\omega_3^2 \cos\theta_3 \end{aligned}}$$

Chapter 7 Acceleration analysis

- 7.3 Analytical solutions for acceleration analysis

Coriolis acceleration

Slider in rotation

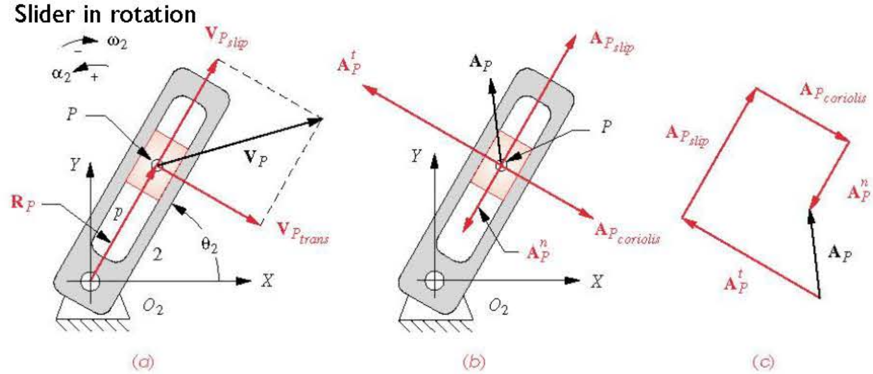


FIGURE 7-7

The Coriolis component of acceleration shown in a system with a positive (CCW) α_2 and a negative (CW) ω_2

Chapter 7 Acceleration analysis

- 7.3 Analytical solutions for acceleration analysis

Coriolis acceleration

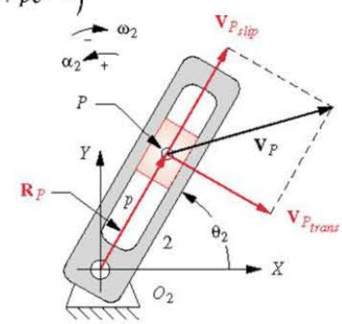
$$\mathbf{R}_P = pe^{j\theta_2}$$

$$\mathbf{V}_P = p\omega_2 je^{j\theta_2} + \dot{p}e^{j\theta_2} \qquad \mathbf{V}_P = \mathbf{V}_{P_{trans}} + \mathbf{V}_{P_{slip}}$$

$$\mathbf{A}_P = \left(p\alpha_2 je^{j\theta_2} + p\omega_2^2 j^2 e^{j\theta_2} + \dot{p}\omega_2 je^{j\theta_2}\right) + \left(\dot{p}\omega_2 je^{j\theta_2} + \ddot{p}e^{j\theta_2}\right)$$

$$\mathbf{A}_P = p\alpha_2 je^{j\theta_2} - p\omega_2^2 e^{j\theta_2} + 2\dot{p}\omega_2 je^{j\theta_2} + \ddot{p}e^{j\theta_2}$$

$$\mathbf{A}_P = \mathbf{A}_{P_{tangential}} + \mathbf{A}_{P_{normal}} + \mathbf{A}_{P_{coriolis}} + \mathbf{A}_{P_{slip}}$$



Chapter 7 Acceleration analysis

- 7.3 Analytical solutions for acceleration analysis

The Fourbar inverted slider crank

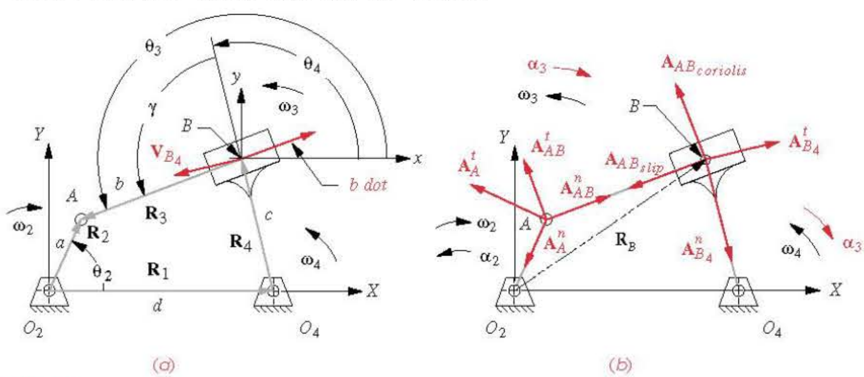


FIGURE 7-8

Acceleration analysis of inversion #3 of the fourbar slider-crank driven with a positive (CCW) α_2 and a negative (CW) ω_2

Chapter 7 Acceleration analysis

- 7.3 Analytical solutions for acceleration analysis

The Fourbar inverted slider crank

$$\mathbf{A}_A = \mathbf{A}_{A_{tangential}} + \mathbf{A}_{A_{normal}}$$

$$\mathbf{A}_{AB} = \mathbf{A}_{AB_{tangential}} + \mathbf{A}_{AB_{normal}} + \mathbf{A}_{AB_{coriolis}} + \mathbf{A}_{AB_{slip}}$$

$$\mathbf{A}_B = \mathbf{A}_{B_{tangential}} + \mathbf{A}_{B_{normal}}$$

$$\theta_3 = \theta_4 \pm \gamma$$

$$\omega_3 = \omega_4;$$

$$\alpha_3 = \alpha_4$$

$$\mathbf{A}_{A_{tangential}} = a\alpha_2\, je^{j\theta_2} \qquad \mathbf{A}_{A_{normal}} = -a\omega_2^2 e^{j\theta_2}$$

$$\mathbf{A}_{B_{tangential}} = c\alpha_4\, je^{j\theta_4} \qquad \mathbf{A}_{B_{normal}} = -c\omega_4^2 e^{j\theta_4}$$

$$\mathbf{A}_{AB_{tangential}} = b\alpha_3\, je^{j\theta_3} \qquad \mathbf{A}_{AB_{normal}} = -b\omega_3^2 e^{j\theta_3}$$

$$\mathbf{A}_{AB_{coriolis}} = 2\dot{b}\omega_3\, je^{j\theta_3} \qquad \mathbf{A}_{AB_{slip}} = \ddot{b}e^{j\theta_3}$$

Chapter 7 Acceleration analysis

- 7.3 Analytical solutions for acceleration analysis

The Fourbar inverted slider crank

$$\alpha_4 = \frac{a\left[\alpha_2\cos(\theta_3-\theta_2)+\omega_2^2\sin(\theta_3-\theta_2)\right]+c\omega_4^2\sin(\theta_4-\theta_3)-2\dot{b}\omega_3}{b+c\cos(\theta_3-\theta_4)}$$

$$\ddot{b} = -\frac{\left\{\begin{array}{c} a\omega_2^2\left[b\cos(\theta_3-\theta_2)+c\cos(\theta_4-\theta_2)\right]+a\alpha_2\left[b\sin(\theta_2-\theta_3)-c\sin(\theta_4+\theta_2)\right] \\ +2\dot{b}c\omega_4\sin(\theta_4-\theta_3)-\omega_4^2\left[b^2+c^2+2bc\cos(\theta_4-\theta_3)\right] \end{array}\right\}}{b+c\cos(\theta_3-\theta_4)}$$

Chapter 7 Acceleration analysis

- 7.4 Acceleration analysis of the geared Fivebar linkage

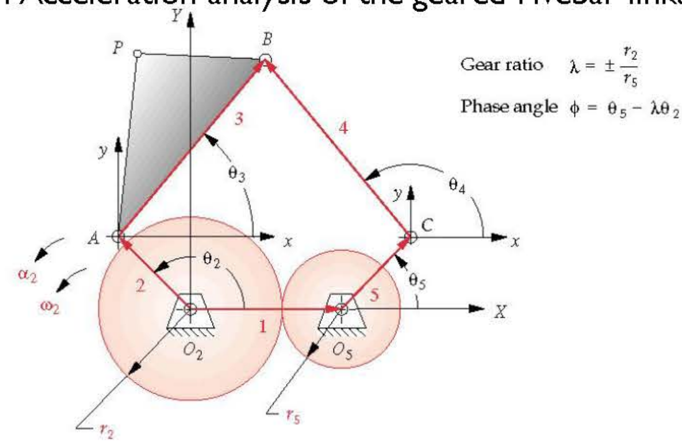


FIGURE P7-4
Configuration and terminology for Problems 7-9 and 7-60

Chapter 7 Acceleration analysis

- 7.4 Acceleration analysis of the geared Fivebar linkage

$$a\omega_2 je^{j\theta_2} + b\omega_3 je^{j\theta_3} - c\omega_4 je^{j\theta_4} - d\omega_5 je^{j\theta_5} = 0$$

$$\left(a\alpha_2 je^{j\theta_2} - a\omega_2^2 e^{j\theta_2}\right) + \left(b\alpha_3 je^{j\theta_3} - b\omega_3^2 e^{j\theta_3}\right)$$
$$-\left(c\alpha_4 je^{j\theta_4} - c\omega_4^2 e^{j\theta_4}\right) - \left(d\alpha_5 je^{j\theta_5} - d\omega_5^2 e^{j\theta_5}\right) = 0$$

$\theta_5 = \lambda\theta_2 + \phi;$

$\omega_5 = \lambda\omega_{2;}$

$\alpha_5 = \lambda\alpha_2$

real:

$$-a\alpha_2 \sin\theta_2 - a\omega_2^2 \cos\theta_2 - b\alpha_3 \sin\theta_3 - b\omega_3^2 \cos\theta_3$$
$$+ c\alpha_4 \sin\theta_4 + c\omega_4^2 \cos\theta_4 + d\alpha_5 \sin\theta_5 + d\omega_5^2 \cos\theta_5 = 0$$

imaginary:

$$a\alpha_2 \cos\theta_2 - a\omega_2^2 \sin\theta_2 + b\alpha_3 \cos\theta_3 - b\omega_3^2 \sin\theta_3$$
$$- c\alpha_4 \cos\theta_4 + c\omega_4^2 \sin\theta_4 - d\alpha_5 \cos\theta_5 + d\omega_5^2 \sin\theta_5 = 0$$

Chapter 7 Acceleration analysis

- 7.4 Acceleration analysis of the geared Fivebar linkage

$$\alpha_3 = \frac{\begin{bmatrix} -a\alpha_2 \sin(\theta_2-\theta_4) - a\omega_2^2\cos(\theta_2-\theta_4) \\ -b\omega_3^2\cos(\theta_3-\theta_4) + d\omega_5^2\cos(\theta_5-\theta_4) \\ +d\alpha_5\sin(\theta_5-\theta_4) + c\omega_4^2 \end{bmatrix}}{b\sin(\theta_3-\theta_4)}$$

$$\alpha_4 = \frac{\begin{bmatrix} a\alpha_2 \sin(\theta_2-\theta_3) + a\omega_2^2\cos(\theta_2-\theta_3) \\ -c\omega_4^2\cos(\theta_3-\theta_4) - d\omega_5^2\cos(\theta_3-\theta_5) \\ +d\alpha_5\sin(\theta_3-\theta_5) + b\omega_3^2 \end{bmatrix}}{c\sin(\theta_4-\theta_3)}$$

$$\begin{aligned} \mathbf{A}_A &= a\alpha_2(-\sin\theta_2 + j\cos\theta_2) - a\omega_2^2(\cos\theta_2 + j\sin\theta_2) \\ \mathbf{A}_{BA} &= b\alpha_3(-\sin\theta_3 + j\cos\theta_3) - b\omega_3^2(\cos\theta_3 + j\sin\theta_3) \\ \mathbf{A}_C &= c\alpha_5(-\sin\theta_5 + j\cos\theta_5) - c\omega_5^2(\cos\theta_5 + j\sin\theta_5) \\ \mathbf{A}_B &= \mathbf{A}_A + \mathbf{A}_{BA} \end{aligned}$$

Chapter 7 Acceleration analysis

- 7.5 Acceleration of any point on a linkage

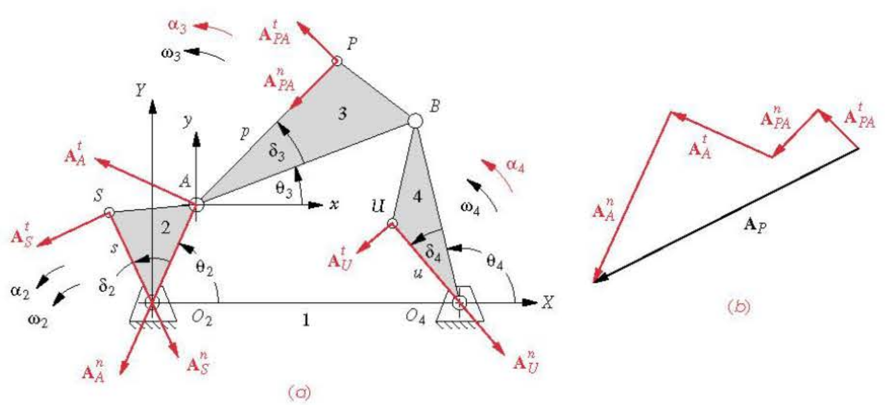


FIGURE 7-9

Finding the acceleration of any point on any link

Chapter 7 Acceleration analysis

- 7.5 Acceleration of any point on a linkage

$$\mathbf{R}_{SO_2} = \mathbf{R}_S = se^{j(\theta_2+\delta_2)} = s[\cos(\theta_2+\delta_2) + j\sin(\theta_2+\delta_2)]$$

$$\mathbf{V}_S = jse^{j(\theta_2+\delta_2)}\omega_2 = s\omega_2[-\sin(\theta_2+\delta_2) + j\cos(\theta_2+\delta_2)]$$

$$\begin{aligned}\mathbf{A}_S &= s\alpha_2\, je^{j(\theta_2+\delta_2)} - s\omega_2^2 e^{j(\theta_2+\delta_2)} \\ &= s\alpha_2\{-\sin(\theta_2+\delta_2) + j\cos(\theta_2+\delta_2)\} \\ &\quad - s\omega_2^2\{\cos(\theta_2+\delta_2) + j\sin(\theta_2+\delta_2)\}\end{aligned}$$

Chapter 7 Acceleration analysis

- 7.5 Acceleration of any point on a linkage

$$\mathbf{R}_{UO_4} = ue^{j(\theta_4+\delta_4)} = u\{\cos(\theta_4+\delta_4) + j\sin(\theta_4+\delta_4)\}$$

$$\mathbf{V}_U = jue^{j(\theta_4+\delta_4)}\omega_4 = u\omega_4[-\sin(\theta_4+\delta_4) + j\cos(\theta_4+\delta_4)]$$

$$\begin{aligned}\mathbf{A}_U &= u\alpha_4\, je^{j(\theta_4+\delta_4)} - u\omega_4^2 e^{j(\theta_4+\delta_4)} \\ &= u\alpha_4[-\sin(\theta_4+\delta_4) + j\cos(\theta_4+\delta_4)] \\ &\quad - u\omega_4^2[\cos(\theta_4+\delta_4) + j\sin(\theta_4+\delta_4)]\end{aligned}$$

Chapter 7 Acceleration analysis

- 7.5 Acceleration of any point on a linkage

$$\mathbf{R}_{PA} = pe^{j(\theta_3+\delta_3)} = p\left[\cos(\theta_3+\delta_3) + j\sin(\theta_3+\delta_3)\right]$$

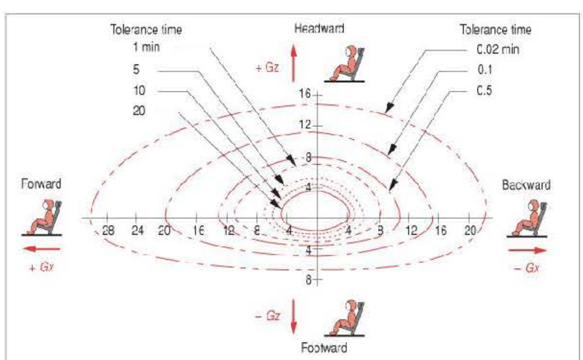
$$\mathbf{V}_{PA} = jpe^{j(\theta_3+\delta_3)}\omega_3 = p\omega_3\left[-\sin(\theta_3+\delta_3) + j\cos(\theta_3+\delta_3)\right]$$

$$\begin{aligned}\mathbf{A}_{PA} &= p\alpha_3\, je^{j(\theta_3+\delta_3)} - p\omega_3^2\, e^{j(\theta_3+\delta_3)} \\ &= p\alpha_3\left[-\sin(\theta_3+\delta_3) + j\cos(\theta_3+\delta_3)\right] \\ &\qquad - p\omega_3^2\left[\cos(\theta_3+\delta_3) + j\sin(\theta_3+\delta_3)\right]\end{aligned}$$

$$\mathbf{A}_P = \mathbf{A}_A + \mathbf{A}_{PA}$$

Chapter 7 Acceleration analysis

- 7.6 Human tolerance of acceleration



Average levels of linear acceleration, in different directions that can be tolerated on a voluntary basis for specfiec periods. Each curve shows the average G load that can be tolerated for the time indicated. The data points obtained were actually those on the axes, the lines as such are extrapolated from the data points to form the concentric figures.

(Adapted from reference [1], Fig. 17-17, p. 505, reprinted with permission)

FIGURE 7-10

Human tolerance of acceleration

Chapter 7 Acceleration analysis

- 7.7 Jerk

The Jerk is the time rate change of the acceleration

$$\varphi = \frac{d\alpha}{dt} \qquad \mathbf{J} = \frac{d\mathbf{A}}{dt}$$

For example a Fourbar linkage have:

$$\varphi_3 = \frac{A - B - C + D - E - F + G + H\varphi_4}{K}$$

$$\varphi_4 = \frac{KN - KL - KM - KP - KQ + AR - BR - CR + DR - ER - FR + GR + KS + KT}{KU - HR}$$

Chapter 7 Acceleration analysis

- 7.7 Jerk

Where:

$$\begin{array}{lll} A = a\omega_2^3 \sin\theta_2 & D = b\omega_3^3 \sin\theta_3 & G = 3c\omega_4\alpha_4 \cos\theta_4 \\ B = 3a\omega_2\alpha_2 \cos\theta_2 & E = 3b\omega_3\alpha_3 \cos\theta_3 & H = c\sin\theta_4 \\ C = a\varphi_2 \sin\theta_2 & F = c\omega_4^3 \sin\theta_4 & K = b\sin\theta_3 \end{array}$$

$$\begin{array}{lll} L = a\omega_2^3 \cos\theta_2 & P = b\omega_3^3 \cos\theta_3 & S = c\omega_4^3 \cos\theta_4 \\ M = 3a\omega_2\alpha_2 \sin\theta_2 & Q = 3b\omega_3\alpha_3 \sin\theta_3 & T = 3c\omega_4\alpha_4 \sin\theta_4 \\ N = a\varphi_2 \cos\theta_2 & R = b\cos\theta_3 & U = c\cos\theta_4 \end{array}$$