Review vectorial analysis:

1- scalar field and vector field.

scalar field = function $f(x,y,z)$

$$f(x,y,z) = 3x^2y + xz^3$$

vector field is a vector that has components on x, y, z axis.

$$\vec{v} = v_x\,\vec{i} + v_y\,\vec{j} + v_z\,\vec{k}$$

the components v_x, v_y and v_z are scalar field.

2- Gradient:

The gradient operator on $(\vec{\nabla})$ acts on a function scalar field $\Rightarrow$

$$\overrightarrow{\text{grad}}\, f = \vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

3. Divergence :

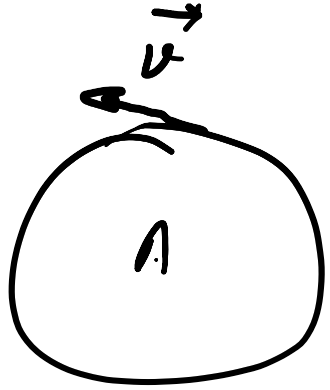
Divergence is defined as:

$$\operatorname{div} \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\vec{V} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}.$$

4- curl or Rotational.

$$\overrightarrow{rot}\ \vec{v} = \frac{\text{circulation of vector } \vec{v} \text{ on a closed frame}}{\text{area inside frame.}}$$



$$= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\vec{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\vec{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\vec{k}$$

5 - Scalar Laplacian:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

$$\vec{\nabla} \cdot \left(\vec{\nabla} f\right)$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

6 - vector Laplacian.

$$\overrightarrow{Lap}\,\vec{v} = \vec{\Delta}\vec{v} = \overrightarrow{grad}\left(div\,\vec{v}\right) - \overrightarrow{rot}\left(\overrightarrow{rot}\,\vec{v}\right)$$

$$= \Delta v_x\,\vec{i} + \Delta v_y\,\vec{j} + \Delta v_z\,k$$

scalar Laplacian.

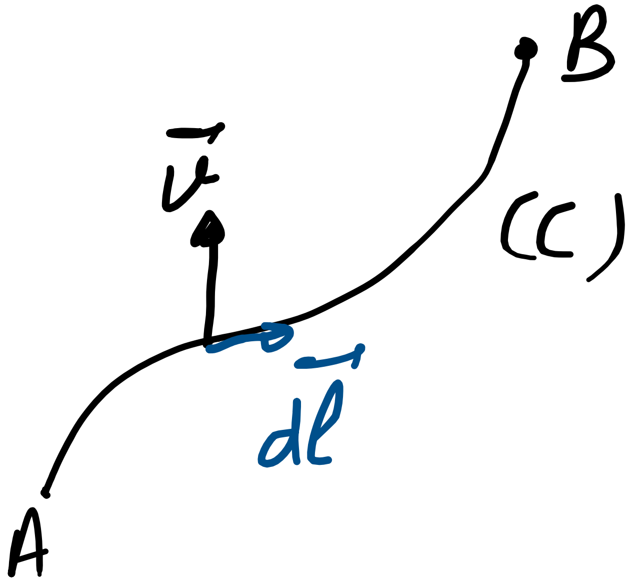
$$\Delta v_x = \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}$$

$$\Delta v_y = \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \;;\; \Delta v_z = \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}.$$

$$\overrightarrow{rot}\,\vec{v} = \vec{\nabla} \wedge \vec{v}$$

8- Stokes theorem and Gauss theorem.

8.1- Circulation of vector



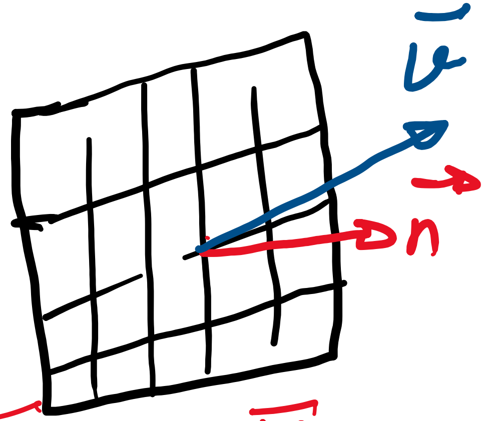
Circulation of $\vec{v}$ through the curve (C) is:

$$C_{AB}(\vec{v}) = \int_{A-B} \vec{v} \cdot \vec{dl}$$

on closed frame.

$$C(\vec{v}) = \oint \vec{v} \cdot \vec{dl}$$

8.2 - Flux of vector:



$\vec{dA} = \vec{n}\, dA$

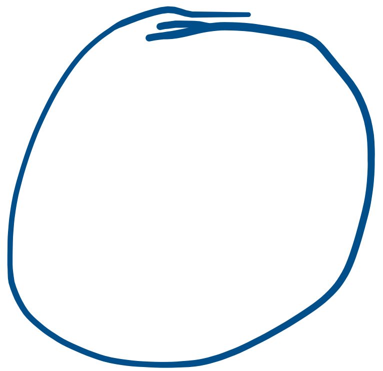
the flux of vector $\vec{v}$ is:

$$\phi_{(s)} = \iint_{(s)} \vec{v} \cdot \vec{dA} = \iint_{(s)} \vec{v} \cdot \vec{n}\, dA.$$

(S) surface.

8.3. Stokes theorem.

$$C(\vec{v}) = \oint \vec{v} \cdot \vec{dl} = \iint \vec{rot}\, \vec{v} \cdot \vec{dA}$$



$$\vec{rot}\,\vec{v} = \frac{\vec{v}\,\vec{dl}}{dA}$$

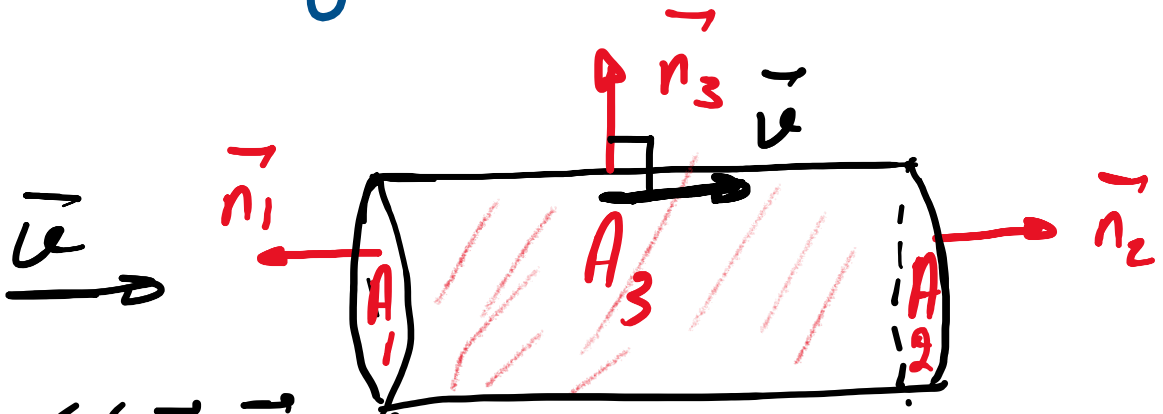
$$\int \vec{v}\,\vec{dl} = \iint \vec{rot}\ \vec{v} \cdot \vec{dA}$$

8.4- Gauss_Ostrogradski theorem.

$$\oiint \vec{v}\, d\vec{A} = \iiint \mathrm{div}\, \vec{v}\, dV.$$

$\oiint \vec{v}\cdot\vec{n}\, dA$ volume.

limited by the dA.



$$\underset{A_1}{\iint} \vec{v}\cdot\vec{n}_1\, dA_1 + \underset{A_2}{\iint} \vec{v}\cdot\vec{n}_2\, dA_2 + \underset{A_3}{\iint} \underbrace{\vec{v}\cdot\vec{n}_3}_{0}\, dA_3 \qquad \vec{n}_3 \perp \vec{v}.$$

in out.

8.5- Reynolds transport theorem:

let $f(\vec{r},t)$ be scalar quantity (depends on position and time).

the quantity F is defined as $F(t)=\iiint\limits_{V_s} f(\vec{r},t)\,\underbrace{dV}_{\text{element of volume}}$.

$M=\int dm = \iiint \rho\, dV.$

V_s : volume of the system

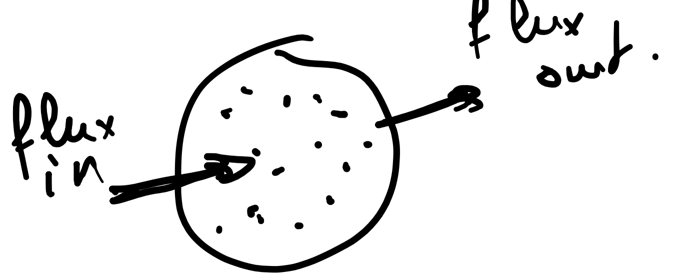
$$\frac{dF}{dt}=\frac{d}{dt}\iiint\limits_{V_s} f(\vec{r},t)\,dV$$

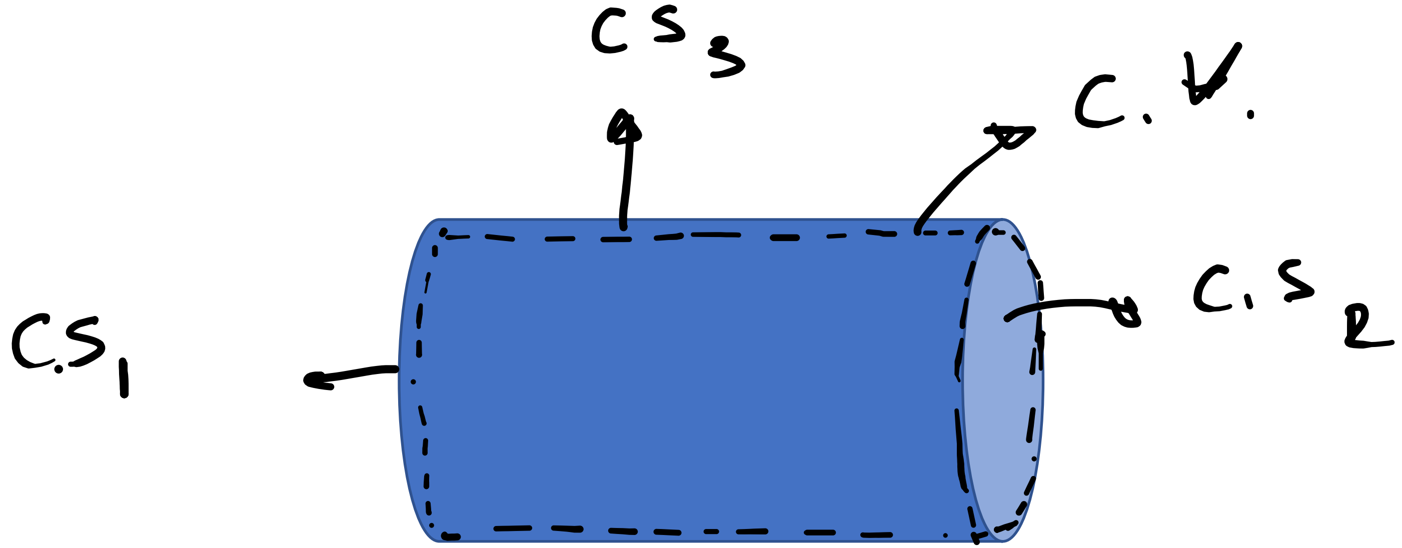
the Reynold theorem is defined as:

$$\frac{dF}{dt} = \frac{d}{dt} \iiint_{V_s} f(\vec{r},t)\, dV.$$

$$= \iiint_{C.V} \frac{\partial f}{\partial t}\, dv + \oiint_{C.S} f \cdot \vec{v} \cdot \vec{n}\, dA$$

C.V control volum





$$\frac{dF}{dr} = \iiint_{C.V} \frac{\partial f}{\partial t}\, dV + \oiint_{C.S} f\, \vec{v} \cdot \vec{n}\, dA$$

Conservation of mass: $\frac{dM}{dt} = 0$

M: mass of system. $\quad dM = \rho \, dV$.

$$M = \int dM$$

$$M = \iiint_{V_s} \rho \, dV \Rightarrow \frac{dM}{dt} = \frac{d}{dt} \iiint_{V_s} \rho \, dV$$

$$\frac{dM}{dt} = \iiint_{C.V} \frac{\partial \rho}{\partial t} \, dV + \oiint_{S.S} \rho \, \vec{v} \cdot \vec{n} \, dA$$

$$\frac{dM}{dt} = \iiint_{C.V} \frac{\partial \rho}{\partial t}\, dV + \oiint_{S.S} \underbrace{\rho \vec{v}} \cdot \vec{n}\, dA$$

$$= \iiint_{C.V} \frac{\partial \rho}{\partial t}\, dV + \iiint_{C.V} \operatorname{div}(\rho \vec{v})\, dV .$$

cons of mass 0

$$0 = \iiint_{C.V} \underbrace{\left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v})\right]}_{0} dV \Rightarrow \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0.$$

for incompressible fluid $\Rightarrow \rho = \text{constant}.$

$$\frac{\partial \rho}{\partial t} = 0.$$

$$\operatorname{div}(\rho \vec{v}) = \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}$$

$$= \rho\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right] = \rho \operatorname{div} \vec{v}.$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{v}) = 0.$$

$$0 + \rho\, \mathrm{div}(\vec{v}) = 0 \Rightarrow \mathrm{div}(\vec{v}) = 0.$$

Incompressible fluid

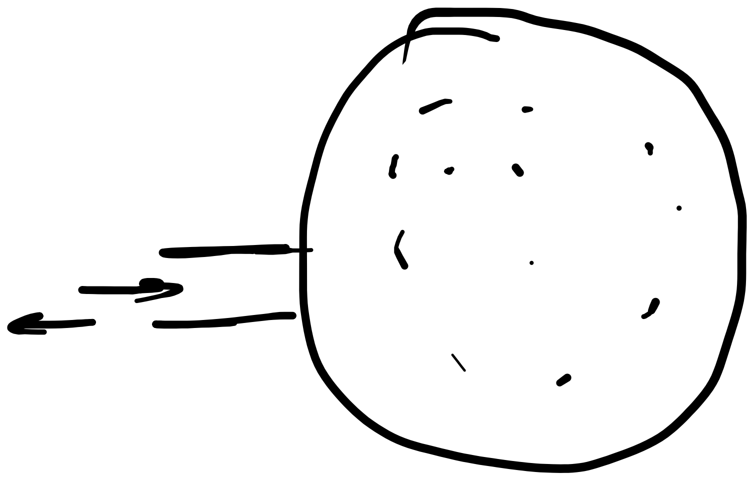
$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0.$$

$$\mathrm{div}(\overrightarrow{\mathrm{rot}}\, \vec{u}) = 0.$$

for incompressible fluid.

$$\vec{v} = \overline{\mathrm{rot}}\ \vec{u}.$$

$$\frac{dM}{dt} = \iiint_{C.V} \frac{\partial \rho}{\partial t}\, dV. \quad + \oiint_{C.S} \rho \vec{v} . \vec{n}\, dA$$



$$N = \iiint \rho\, dV.$$

$$\frac{dN}{dt} = \iiint_{C.V} \frac{\partial \rho}{\partial t}\, dV + \oiint_{C.S} \rho \vec{v} . \vec{n}\, dA$$

volume. flux

Example: the expression of flow velocity in pipe is:

$$\vec{v} = \overbrace{3xy^2}^{v_x}\vec{i} \overbrace{-y^3}^{v_y}\vec{j} + \overbrace{xy}^{v_z}\vec{k}.$$

is the fluid compressible??

$$\mathrm{div}(\vec{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 3y^2 + (-3y^2) + 0$$
$$= 0.$$

$\Rightarrow$ incompressible fluid.