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Stability of Equilibrium Solution to a Nonlinear Parabolic Equation under a 3-Point Boundary Condition

Bruce D. Calvert

Department of Mathematics, University of Auckland
Private Bag 92019, Auckland, New Zealand
calvert@math.auckland.ac.nz

Abstract

We consider a nonlinear parabolic equation in one space variable, satisfying a three point boundary condition, such as the heat equation when the temperature at the end is controlled by a sensor at the the point η. We show that the integral solution, in the space of continuous functions satisfying the boundary values, converges to the equilibrium solution. This answers a question which had been posed for nonlinear Laplacians such as p-Laplacians, but answered only for the linear case.

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1 Introduction

In [2] the author considers the Cauchy problem on $[0, \infty) \times [0, 1]$,

$$u_t(t,x) = (g(u_x))_x(t,x) - f(x) \tag{1}$$
$$u(t,0) = 0 \tag{2}$$
$$u(t,\eta) = \beta u(t,1) \tag{3}$$
$$u(0,x) = u_0(x) \tag{4}$$

where $\eta \in (0, 1)$ and $\beta > 1$ are given, along with f and g. It is supposed that $g : (a, b) \to \mathbb{R}$ is an increasing homeomorphism, and $-\infty \leq a < 0 < b \leq \infty$.

There, it is shown that we have an integral solution to the Cauchy problem $du/dt = Lu - f$ with initial value u_0, in the space of continuous functions satisfying (2) and (3), where Lu is the nonlinear Laplacian $(g(u_x))_x$. The question asked is: does the solution converge to the equilibrium solution, $L^{-1}f$? In [3] it is shown that this holds if L is linear, i.e. for some $k \in \mathbb{R}$, $g(x) = kx$, by considering the spectrum of L. This paper shows that we have convergence to $L^{-1}f$ without linearity, by considering initial values which are super-solutions or sub-solutions of $Lu = f$.

In response to the three point boundary condition, which does not sit well with the use of weak solutions and Sobolev spaces, we use the concept of an integral solution in the sup norm. The techniques of the proof of the main theorem, in particular super and sub solutions, go back to [6]. That paper used T-accretivity for generators of nonlinear semigroups in spaces with uniformly convex dual. We find there are a number of papers, [8], [9], [10], [11], [19], [22], [12], [1] and [20], which deal with attractors for differential equations somewhat like (1), but with neither the three point boundary condition, nor the use of integral solutions. The papers [7] and [16] give examples of instability. There are papers which do consider integral solutions, and their asymptotic behavior, and T-accretive generators of nonlinear semigroups, such as [23], [15], [13] , [14] and [17]. Each of the first four use the idea of sub and super-solutions, much as this paper does, but the details are surprisingly different. The approach of [21] should be noted, as it gives stability for some integral solutions of nonlinear evolution equations.

2 Preliminaries

Definition 1 *Suppose $\beta \in (1, \infty)$ and $\eta \in (0, 1)$ are given. Suppose $g : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, and is C^1. Let X denote the Banach space of continuous functions $u : [0, 1] \to \mathbb{R}$, satisfying $u(0) = 0$ and $u(\eta) = \beta u(1)$, under the sup norm.*

We define a nonlinear operator in X.

Definition 2 *Let $D(L)$ consist of $u \in X$ which have first and second continuous derivatives on [0,1], i.e. one sided derivatives at the endpoints. For $u \in D(L)$ let $Lu = (g(u_x))_x = g'(u_x)u_{xx}$.*

We recall the definition of integral solution (of type 0) of a Cauchy problem from [18] Definition 5.1. Let E be a Banach space over $\mathbb{R}$, let $x \in E$, let A be a nonlinear operator in E, not necessarily single valued, with domain $D(A)$ and range $R(A)$.

Definition 3 *We consider the Cauchy problem:*

$$(CP; x)_\infty \qquad \begin{cases} (d/dt)u(t) \in Au(t) \\ u(0) = x \end{cases} \qquad (0 \le t < \infty). \tag{5}$$

Definition 4 *Let E be a Banach space over $\mathbb{R}$, let $x \in X$, let A be a nonlinear operator in E, with domain $D(A)$ and range $R(A)$. Then $u : [0,\infty) \to E$ is an integral solution (of type 0) of the Cauchy Problem $(CP;x)_\infty$ if u satisfies (i) through (iii):*

(i) $u(0) = x$.

(ii) u is continuous.

(iii) For every $r \in (0,\infty)$ and $t \in (r,\infty)$, and every $x_0 \in D(A)$, $y_0 \in Ax_0$,

$$\|u(t) - x_0\|^2 - \|u(r) - x_0\|^2 \leq 2\int_r^t \langle y_0, u(\xi) - x_0\rangle_s \, d\xi \tag{6}$$

Here we use the semi-inner product; for x, y in a real Banach space E, $\langle y, x\rangle_s = \sup\{(y,f) : f \in F(x)\}$, where F is the duality map, $F(x) = \{x^* \in E^* : (x,x^*) = \|x\|^2 = \|x^*\|^2\}$.

We obtain integral solutions when A is dissipative, and other conditions hold. We use [18], Definition 2.4.

Definition 5 *Let E be a Banach space over $\mathbb{R}$, let $x \in X$, let A be a nonlinear operator in E, not necessarily single valued, with domain $D(A)$ and range $R(A)$. We say A is dissipative to mean that for $x, y \in D(A)$, and $x' \in A(x)$ and $y' \in A(y)$, and $\lambda > 0$,*

$$\|(x - \lambda x') - (y - \lambda y'\| \geq \|x - y\|. \tag{7}$$

Proposition 1 *For all $f \in X$, the operator $u \mapsto Lu - f$, denoted $L - f$, is dissipative in X, and $R(-(L-f) + \lambda I) = X$ for all $\lambda > 0$, and $D(L-f)$ is dense in X.*

Proof. The first two assertions come from [2], Th 8.2. The third can be seen by by approximating, in $C[0,1]$, a given element x of X by a polynomial, and adding a small polynomial of degree 1 to get an approximation to x in X. ∎

Theorem 1 *For all $x \in X$, and for all $f \in X$, the Cauchy problem $(CP;x)_\infty$ for $L - f$ has a unique integral solution $u(t;x) : [0,\infty) \to X$ of type 0, and the following hold:*

$$\begin{aligned} \|u(t;x) - u(t;y)\| &\leq \|u(s;x) - u(s;y)\| \qquad (0 \leq s < t < \infty, x, y \in X) && (8)\\ \|u(t;x) - u(s;x)\| &\leq \|Lx - f\| \, |t - s| \qquad (0 \leq s < t < \infty, x \in D(L)). && (9) \end{aligned}$$

Proof. This is immediate from [18] Th 5.10, given Proposition 1. ∎

Proposition 2 *$R(L) = X$ and L is injective. The inverse $(-L)^{-1}$ is order preserving, and continuous.*

Proof. Theorem 3.1 of [2] shows $L : D(L) \to X$ is bijective. Theorem 4.1 of [2] shows $(-L)^{-1}$ is order preserving. Given $f \in X$, Theorem 2.1 of [2] defines k_2 by $R(k_2) = 0$, where

$$R(k) := (\beta - 1)\int_{s=0}^{\eta} g^{-1}(k + \int_s^1 f)\, ds + \beta \int_{s=\eta}^{1} g^{-1}(k + \int_s^1 f)\, ds. \qquad (10)$$

Since $f \mapsto R(k)$ is continuous, and the solution k_2 exists by the intermediate value theorem, or a one dimensional topological degree, we have k_2 a continuous function of f, and $u = (-L)^{-1}f$, given by equations (2.5), (2.6) and (2.7) of [2], is a continuous function of k_2. ■

3 Main Result

Theorem 2 *Suppose $g : \mathbb{R} \to \mathbb{R}$ is a C^1 diffeomorphism. For all $x \in X$, and for all $f \in X$, the integral solution $u(t; x) : [0, \infty) \to X$ to the Cauchy problem $(CP; x)_\infty$ for $L - f$ converges to $L^{-1}f$ as $t \to \infty$.*

Proof. Equation (8) gives nonexpansiveness, for all $t \geq 0$, of the operator $x \mapsto u(t, x)$, denoted $U(t)$. Hence we need only show $U(t)x = u(t; x) \to L^{-1}f$ for $x \in D(L)$, this being dense. Let $x \in D(L)$ be given. Note X is a Banach lattice under the usual partial order. Let v and w be defined by:

$$-(Lv - f) = [-(Lx - f)]^+$$

and

$$-(Lw - f) = -[-(Lx - f)]^-.$$

By Proposition 2, $w \leq x \leq v$, and by construction, $Lv - f \leq 0$ and $Lw - f \geq 0$. We have versions of equations (8) and (9) in which we replace the norm by the norm of the positive part;

$$\|[u(t; x) - u(t; y)]^+\| \leq \|[u(s; x) - u(s; y)]^+\| \qquad (0 \leq s < t < \infty, x, y \in X). \qquad (11)$$
$$\|[u(t; x) - u(s; x)]^+\| \leq \|[Lx - f]^+\|\, |t - s| \qquad (0 \leq s < t < \infty, x \in D(L)). \qquad (12)$$

These come from the resolvents $J_\lambda := (I - \lambda[L - f])^{-1}$, for $\lambda > 0$, satisfying

$$[(I - \lambda[L - f])^{-1}x - (I - \lambda[L - f])^{-1}y]^+\| \leq \|[x - y]^+\|$$

for all $x, y \in X$, since for $t > 0$, $(I - (t/n)[L - f])^{-n}x \to U(t)x$ as $n \to \infty$. It follows from (12) that $U(t)v$ is a decreasing function of t, and similarly, $U(t)w$ is an increasing function of t. Moreover, for all t,

$$U(t)w \leq U(t)x \leq U(t)v. \qquad (13)$$

Hence, we need only show $U(t)w \to L^{-1}f$ and $U(t)v \to L^{-1}f$ as $t \to \infty$. Now given $t \geq 0$ and n a positive integer,

$$(L-f)(I-(t/n)[L-f])^{-n}w \geq 0.$$

Let us use s as space variable since x has denoted a point in X. Write $a = a(t,n) := (I-(t/n)[L-f])^{-n}w$. We have

$$\|(L-f)a\| \leq \|(L-f)w\| \tag{14}$$

and so there is a constant $K_0 > 0$ such that $\|(g(a_s))_s\| \leq K_0$ for all t, n. After adding a constant to g we may assume $g(0) = 0$. Now $a \in D(L)$, and a has an extremum at an interior point of [0,1], so there is $d \in (0,1)$ with $a_s(d) = 0$. By integrating $(g(a_s))_s$ this implies $\|g(a_s)\| \leq K_0$. Since $(g^{-1})'$ is continuous, and $g(a_s)$ is bounded, there is $K_1 > 0$ such that $(g^{-1})'g(a_s)(s) \leq K_1$ for all t, n and s. Hence $g'(a_s(s)) \geq 1/K_1$ for all t, n and s. Now on [0,1], by (14),

$$g'(a_s)a_{ss} \geq -\|f^-\| - \|(L-f)w\|.$$

Hence, there is $K_2 > 0$ such that for all t, n,

$$a_{ss} \geq -K_2$$

and therefore $s \mapsto a(s) + (s^2/2)K_2$ has second derivative greater than or equal to zero, so is convex. Then the limit as $n \to \infty$, $s \mapsto U(t)w(s) + (s^2/2)K_2$, is convex. Write $h = s \mapsto (s^2/2)K_2$. We now have $U(t)w + h : [0,1] \to \mathbb{R}$ being increasing in t, bounded above from (13), and convex. There is a pointwise limit of $U(t)w + h : [0,1] \to \mathbb{R}$ which we denote by $w_\infty + h$, which must be convex, hence continuous on $(0,1)$. One checks that $w_\infty + h$ is continuous at 0 and at 1. Thus w_∞ is continuous, and $U(t)w \nearrow w_\infty$ pointwise. Hence $U(t)w \to w_\infty$ in norm.

We will show $w_\infty = L^{-1}(f)$.

Now Theorem 4.7 of [18] says that for every $z \in X$,

$$\lim_{\lambda \to 0+} \lambda^{-1}\|J_\lambda z - z\|) = \liminf_{h \to 0+} h^{-1}\|U(h)z - z\|. \tag{15}$$

Apply this to w_∞, since for $h > 0$, $U(h)w_\infty = w_\infty$, to get

$$\lim_{\lambda \to 0+} (L-f)J_\lambda w_\infty = 0. \tag{16}$$

Since $LJ_\lambda w_\infty \to f$, and L^{-1} is continuous, $J_\lambda w_\infty \to L^{-1}f$. Since $J_\lambda w_\infty \to w_\infty$, we have $w_\infty = L^{-1}(f)$, or $U(t)w \to L^{-1}f$.

Similarly, $U(t)v \to L^{-1}f$. ■

Remark. There are many papers giving the existence of multiple solutions for three point boundary value problems, e.g. [4], [5]. It is sometimes natural to regard these as equilibrium points in a dynamical system, in other words to look at a parabolic BVP associated with an elliptic BVP. In this paper and in [2] and [3], this question has been addressed, but we have one equilibrium point only, and it is globally asymptotically stable. I would hope that some of the three point BVPs with multiple solutions can be looked at as giving equilibrium points of dynamical systems, and the various types of stability and lack of stability of these equilibrium points can be investigated.

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