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Backstepping Control Design Lab

For 4th Year Control Engineering Branch

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Backstepping Control Design Lab

1. Introduction:

In control theory, Backstepping is a technique developed circa 1990 by ***Petar V. Kokotovic*** and others for designing stabilizing controls for a special class of nonlinear dynamical systems. These systems are built from subsystems that radiate out from an irreducible subsystem that can be stabilized using some other method. Because of this recursive structure, the designer can start the design process at the known-stable system and "back out" new controllers that progressively stabilize each outer subsystem. The process terminates when the final external control is reached. Hence, this process is known as ***Backstepping***.

2. Non-Standard Backstepping Control:

2.1 Motivated Examples:

In this section two elementary examples are presented to illustrate the basic philosophy and steps required to implement the Backstepping method**.**

A-Linear System Example:

Consider the following linear system:

$$\dot{x}_1 = x_1 + x_2 \quad \textbf{(1)}$$

$$\dot{x}_2 = u \quad \textbf{(2)}$$

Let us consider Eq. (1) as a first sub system and Eq. (2) as a second **sub system**. The control objective is to stabilize the system when starting from any initial condition and regulate the state to the origin. Considering the dynamical system as a set of separate sub systems is the main first step in applying the Backstepping method. Now we will treat each subsystem separately including the stabilization of its own variable.

Accordingly consider Eq. (1) with x_1 as the state variable which it is required to stabilize it via x_2 . x_2 is considered here as a virtual controller, namely we rewrite Eq. (1) as follows:

$$\dot{x}_1 = x_1 + v \tag{3}$$

Eq. (3) is a first order system with v as a control input. Let v be chosen as

$$v = -x_1 - \lambda x_1 = -(1+\lambda)x_1 \,, \qquad \lambda > 0 \tag{4}$$

Hence Eq. (3) becomes:

$$\dot{x}_1 = -\lambda x_1 \tag{5}$$

The state x_1 is an asymptotically stable with the required decay rate according to the value of λ.

Now in order to x_1 to be asymptotically stable with the desired roots as in Eq. (5), the second variable x_2 must be equal to the **virtual control** v. Since x_2 and v are started from different values, then the control effort must be directed to **force** x_2 to **follow** v. Accordingly, the control u could be designed to regulate the following output:

$$y = x_2 - v = x_2 + (1+\lambda)x_1 \tag{6}$$

Differentiate y with time, yields

$$\dot{y} = \dot{x}_2 - \dot{v} = u - \dot{v} \tag{7}$$

Where $\dot{v} = -(1+\lambda)\dot{x}_1 = -(1+\lambda)(x_1+x_2)$. Let $u = \dot{v} - \alpha y$, then Eq. (7) becomes:

$$\dot{y} = -\alpha y, \alpha > 0 \quad \textbf{(8)}$$

The output y goes exponentially asymptotically to zero as $t \rightarrow \infty$. As y approach zero value, $x_2 \rightarrow v$. Finally the control law is

$$u = \dot{v} - \alpha y$$

$$= -(1+\lambda)(x_1+x_2) - \alpha\{x_2 + (1+\lambda)x_1\}$$

$$\boxed{\rightarrow u = -k_1 x_1 - k_2 x_2} \quad \textbf{(9)}$$

Where: $k_1 = (1+\lambda)(1+\alpha)$,

$$k_2 = (1+\lambda+\alpha)$$

The simulation Results are obtained through using Matlab/simulink Ver. (14.9) or (2009b). By selecting the controller parameters as following:

$$\lambda = 5, \alpha = 10$$

The gain values of the controller are found to be:

$$k_1 = 66,\ k_2 = 16$$

The simulation results are shown in figures (1) through (5).

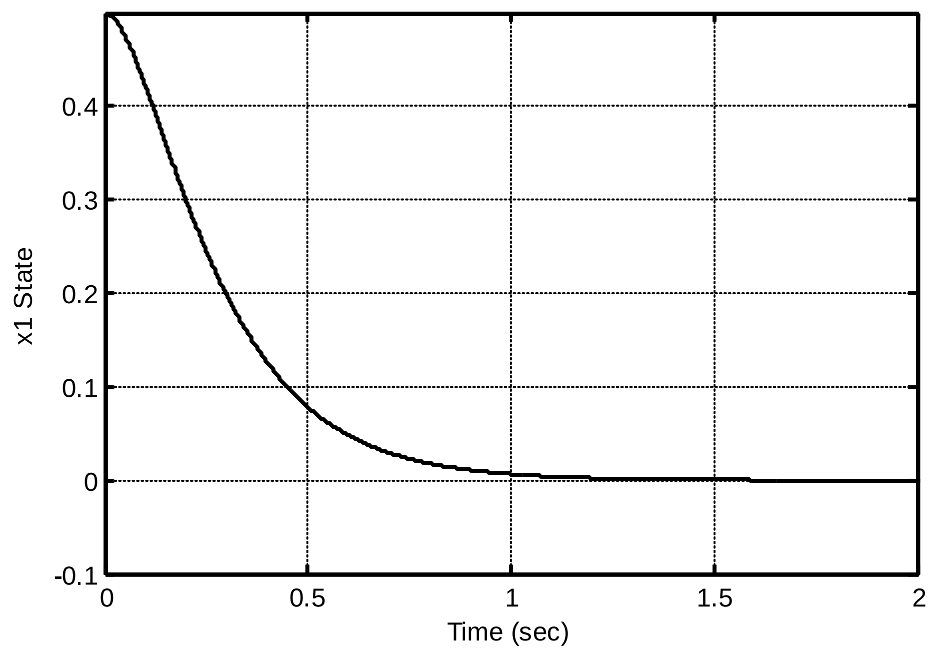

Figure (1): State x_1 time history for example I.

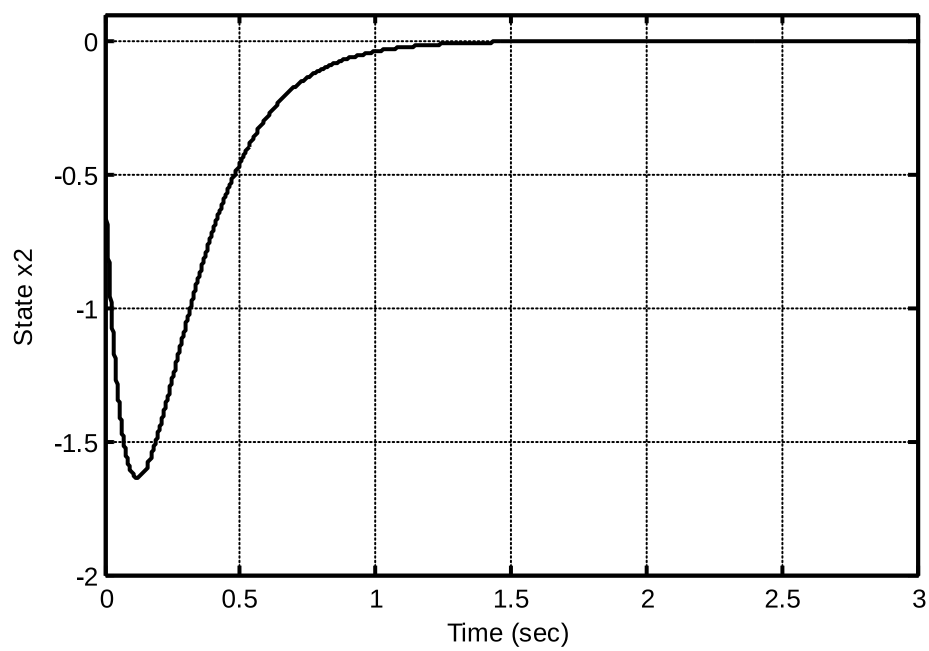

Figure (2): State x_2 time history for example I.

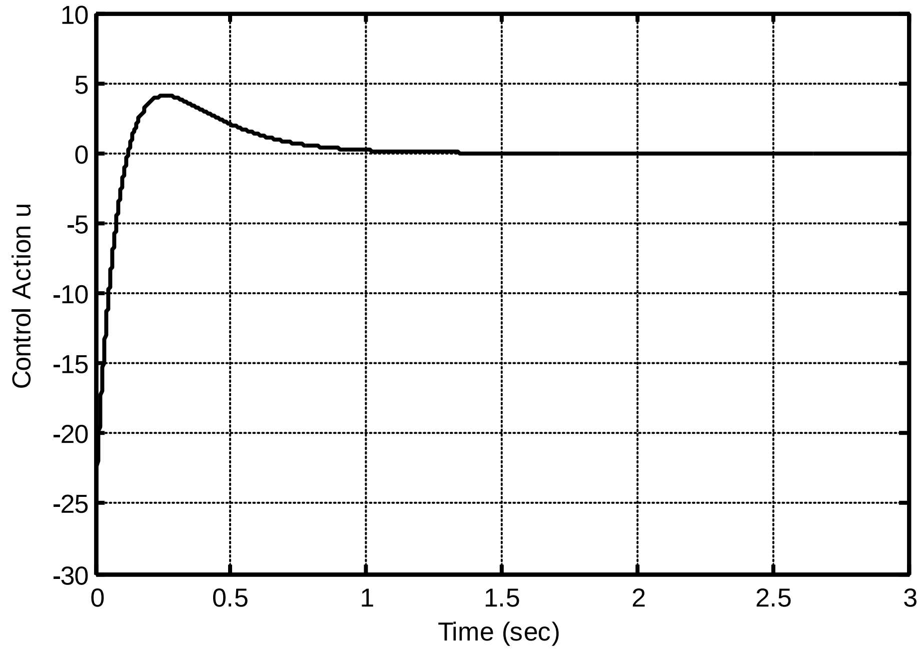

Figure (3): Control Action u time history for example I.

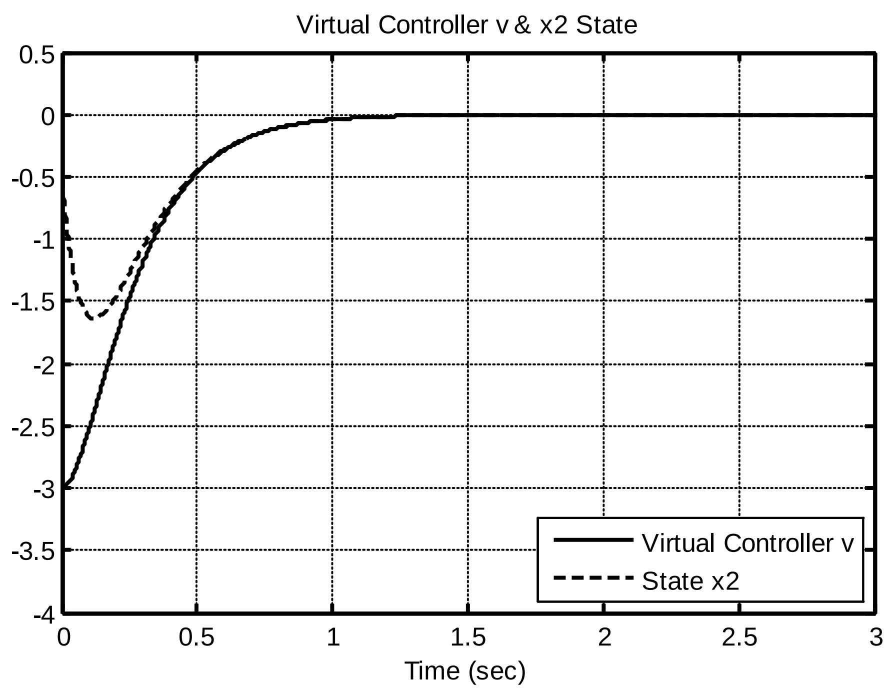

Figure (4): Virtual Controller v and State x_2 time history for example I.

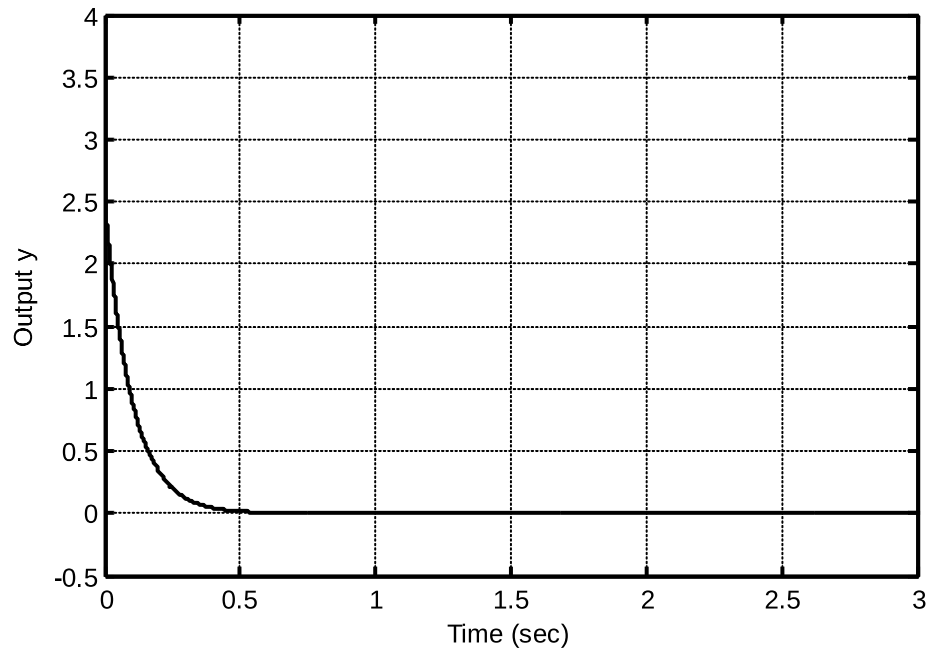

Figure (5): Output y time history for example I.

B-<u>Nonlinear System Example I</u>:

Consider the following linear system:

$$\dot{x}_1 = x_1^2 + x_2 \tag{10}$$

$$\dot{x}_2 = u \tag{11}$$

Eq. (10) can be written in terms of virtual controller as follows:

$$\dot{x}_1 = x_1^2 + v \tag{12}$$

The virtual control may be chosen as

$$v = -x_1^2 - \lambda x_1 \, , \lambda > 0 \tag{13}$$

Accordingly Eq. (12) becomes:

$$\dot{x}_1 = -\lambda x_1 \tag{14}$$

Eq. (14) is asymptotically stable system for positive λ. The other steps required to get the control law is as in the previous example.

H.w1: Complete the design steps for the above example to obtain the control law u.

C-<u>Nonlinear System Example II</u>:

Consider the following linear system:

$$\dot{x}_1 = -x_1^3 + x_2 \quad \textbf{(15)}$$

$$\dot{x}_2 = u \quad \textbf{(16)}$$

Rewrite Eq.(15) in terms of virtual controller as:

$$\dot{x}_1 = -x_1^3 + v \quad \textbf{(17)}$$

The virtual control can be chosen as:

$$v = -x_1 \quad \textbf{(18)}$$

Therefore Eq.(17) becomes:

$$\dot{x}_1 = -x_1^3 - x_1 \quad \textbf{(19)}$$

The state x_1 in Eq.(19) is asymptotically stable (<u>not exponentially</u>) since the right hand side is **negative odd function**. The output y can be written here as:

$$y = x_2 - v = x_2 + x_1 \quad \textbf{(20)}$$

To determine the control law we differentiate y to get:

$$\dot{y} = \dot{x}_2 + \dot{x}_1 = u - x_1^3 + x_2 \quad \textbf{(21)}$$

Let the control law be taken as

$$u = x_1^3 - x_2 - \alpha y = x_1^3 - x_2 - \alpha(x_2 + x_1)$$

$$\boxed{\rightarrow u = x_1^3 - \alpha x_1 - (1+\alpha)x_2} \qquad \textbf{(22)}$$

With the control law as in Eq.(22) the output y is asymptotically stable for positive α.

H.w2: for the following system dynamics:

$$\dot{x}_1 = x_1^2 + x_2 \qquad \textbf{(23)}$$

$$\dot{x}_2 = x_3 \qquad \textbf{(24)}$$

$$\dot{x}_3 = u \qquad \textbf{(25)}$$

Design a control law u to stabilize the above system (Eq, (23)-(25)) based on backstepping control method.

3. Introduction to Lyapunov Function:

Before we present the **Standard Backstepping Method** we need to define the Lyapunov function and how to use it in the stability analysis and controller design for simple 1^{st} order systems.

3.1 Lyapunov Function:

In the theory of ordinary differential equations (ODEs), **Lyapunov functions** are scalar functions that may be used to prove the stability of an equilibrium of an ODE. Named after the Russian mathematician **Aleksandr Mikhailovich Lyapunov**, Lyapunov functions are important to **stability theory** and **control theory**.

For many classes of ODEs, the existence of Lyapunov functions is a necessary and sufficient condition for stability. Whereas there is no general technique for **constructing Lyapunov functions** for ODEs, in many specific cases, the construction of Lyapunov functions is known.

Informally, a Lyapunov function is a function that takes positive values everywhere except at the equilibrium in question, and decreases (or is non-increasing) along **every trajectory** of the ODE. **The principal merit of Lyapunov function-based stability analysis of ODEs is that the actual solution (whether analytical or numerical) of the ODE is not required.**

In the following two elementary examples are presented to illustrate the basic philosophy and step required to implement the Backstepping method.

A-Motivated Example (Stability of a 1^{st} Order Linear System):

Consider the following 1^{st} order differential equation

$$\dot{x} = -x \tag{26}$$

Let the candidate Lyapunov function is

$$V = \frac{1}{2}x^2 \tag{27}$$

where V is a positive definite function, i.e.,

$$V > 0, \forall x \neq 0 \text{ and } V(0) = 0$$

By differentiating V with time, we get

$$\dot{V} = \frac{dV}{dx}\frac{dx}{dt} = x\dot{x} = x(-x) = -x^2$$

Since $\dot{V}$ is negative definite the system in Eq. (26) is asymptotically stable.

B-<u>Motivated Example (Stability of a 1st Order Nonlinear System)</u>:

Consider the following 1st order nonlinear differential equation:

$$\dot{x} = -f(x) \tag{28}$$

where $f(x) > 0\ \forall x \neq 0$ and $f(0) = 0$. The candidate Lyapunov function is as in Eq. (27), and accordingly $\dot{V}$ becomes:

$$\dot{V} = x\dot{x} = -xf(x) < 0\ \forall x \neq 0$$

This proves that the nonlinear system as given in Eq. (28) is asymptotically stable around the origin.

C-<u>Motivated Example (Controller Design Based On Lyapunov Function)</u>:

Consider the following system

$$\dot{x} = x + u \tag{29}$$

Let the candidate Lyapunov function is as in Eq. (27), then $\dot{V}$ is

$$\dot{V} = x\dot{x} = x\left(\underbrace{x + u}_{-x}\right)$$

By choosing

$$u = -2x \tag{30}$$

we get,

$$\dot{V} = x(-x) = -x^2 < 0 \ \ \forall x \neq 0$$

This proves the asymptotic stability of Eq. (29) with the control u as selected above.

The idea behind selecting u as in Eq. (30) is that we need to make $\dot{V}$ negative definite. This can be accomplished if the bracket $(x + u)$ is negative and odd function as in the following:

$$(x + u\,) = -x \Rightarrow \ u = -2x$$

4. <u>Standard Backstepping Method</u>:

In this section Backstepping method is presented based on a **step-by-step** construction of Lyapunov function. Hence the design of Backstepping control is Lyapunov based or as it named, the standard Backstepping method.

A-<u>Motivated Example</u>:

Consider the following system

$$\dot{x}_1 = x_1 + x_2 \tag{31}$$

$$\dot{x}_2 = u \tag{32}$$

As in the previous design of the Backstepping control x_2 is considered as a virtual controller to x_1 in Eq. (31). Namely we rewrite Eq. (31) as follows:

$$\dot{x}_1 = x_1 + v \tag{33}$$

Eq. (33) is a first order system with v as a control input. Let the candidate Lyapunov function is

$$V_1 = \frac{1}{2}x_1^2 \tag{34}$$

Accordingly $\dot{V}_1$ is

$$\dot{V}_1 = x_1\dot{x}_1 = x_1\left(\underbrace{x_1 + v}_{-x_1}\right)$$

Let $v = -2x_1$, then $\dot{V}$ becomes,

$$\dot{V}_1 = -x_1^2 < 0\ \forall x_1 \neq 0$$

Which means that x_1 decay exponentially asymptotically to the origin when $(x_2 = v = -2x_1)$. This is a first step in the design procedure; the second is by defining the following output,

$$z = x_2 - v = x_2 + 2x_1 \tag{35}$$

Now Eq. (31) is rewritten with replacing x_2 by the new state z according to the transformation (Eq. (35)). Namely,

$$x_2 = z - 2x_1 \tag{36}$$

And therefore Eq. (31) becomes:

$$\dot{x}_1 = -x_1 + z \tag{37}$$

Differentiate z in Eq. (35) with the transformation (Eq. (36)) to get,

$$\dot{z} = \dot{x}_2 + 2\dot{x}_1 = u + 2(x_1 + x_2)$$

$$= u + 2(x_1 + z - 2x_1) = u + 2z - 2x_1$$

The next step is to write the total Lyapunov function as follows:

$$V = V_1 + \frac{1}{2}z^2 = \frac{1}{2}x_1^2 + \frac{1}{2}z^2 \tag{38}$$

Accordingly $\dot{V}$ is,

$$\dot{V} = x_1\dot{x}_1 + z\dot{z} = x_1(-x_1 + z) + z(u + 2z - 2x_1)$$

$$= -x_1^2 + x_1 z + z(u + 2z - 2x_1)$$

$$= -x_1^2 + z(u + 2z - x_1)$$

If

$$u = -3z + x_1 \tag{39}$$

then $\dot{V}$ becomes,

$$\dot{V} = -x_1^2 - z^2$$

$\dot{V}$ is negative definite and thus z and x_1 regulated to the origin $z = x_1 = 0$ asymptotically. As z and x_1 go to zero, x_2 goes to the origin asymptotically too as can be verified from the transformation (Eq. (35)).

Finally the control law which will regulate x_1 and x_2 to the origin asymptotically via the Backstepping design method is

$$\boxed{u = -3(x_2 + 2x_1) + x_1 = -5x_1 - 3x_2} \tag{40}$$

The simulation results for the system with the proposed controller is done in figures (6-10)

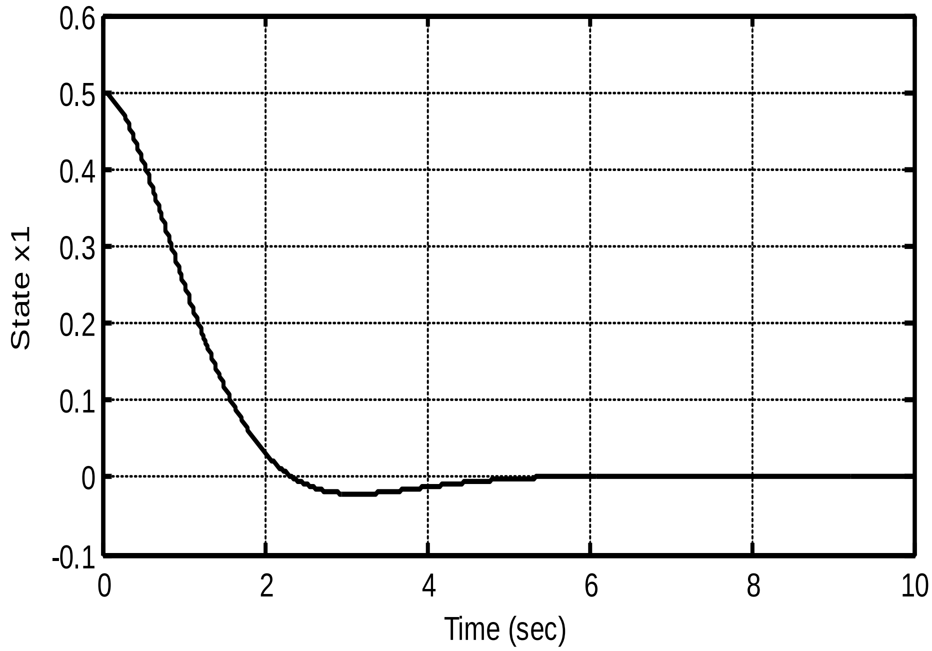

Figure (6): State x_1 time history for Motivated Example A.

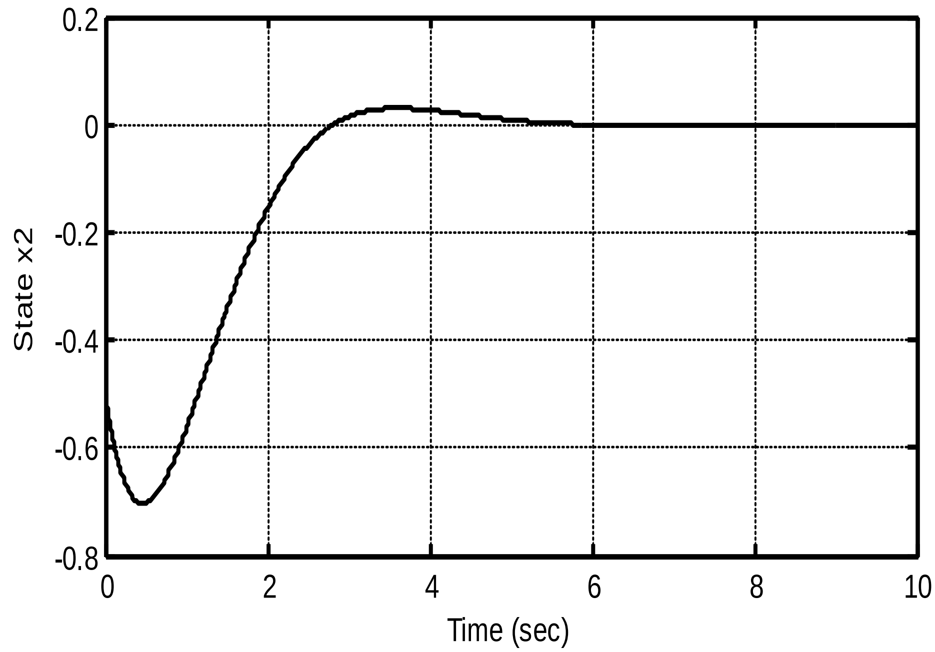

Figure (7): State x_2 time history for Motivated Example A.

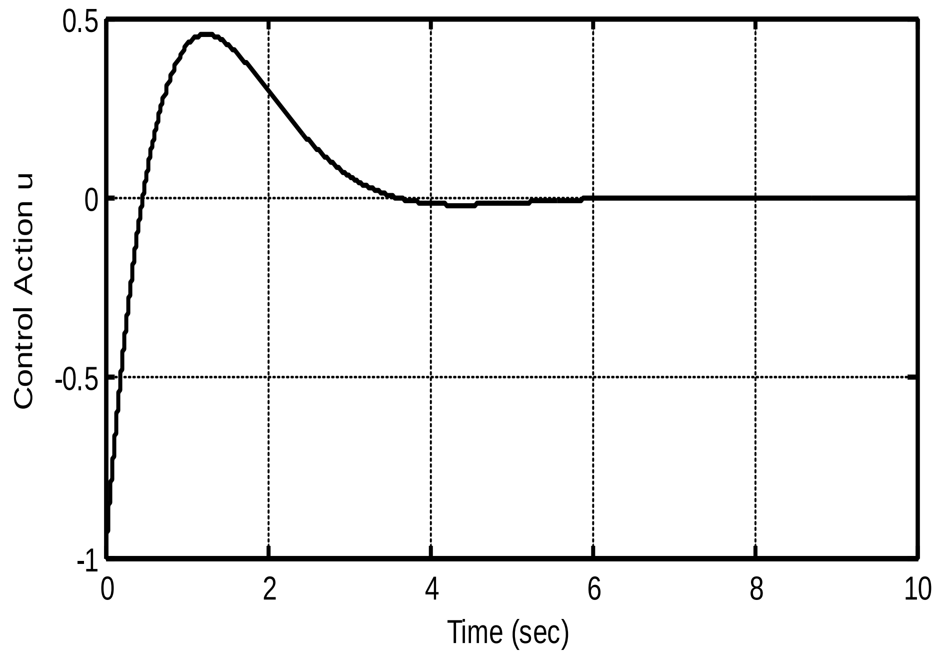

Figure (8): Control Action u time history for Motivated Example A.

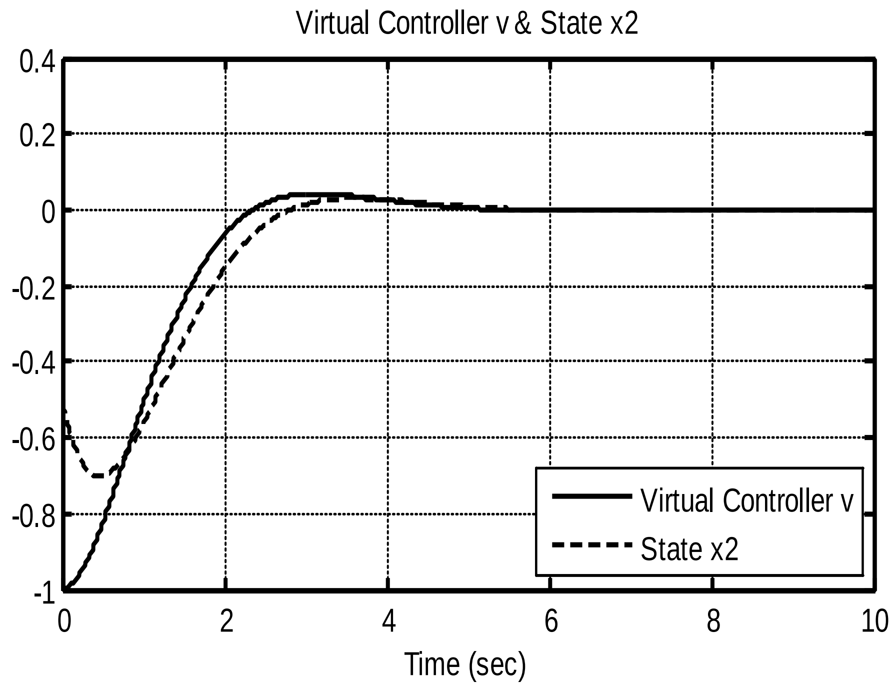

Figure (9): Virtual Controller v and State x_2 time history for Motivated Example A.

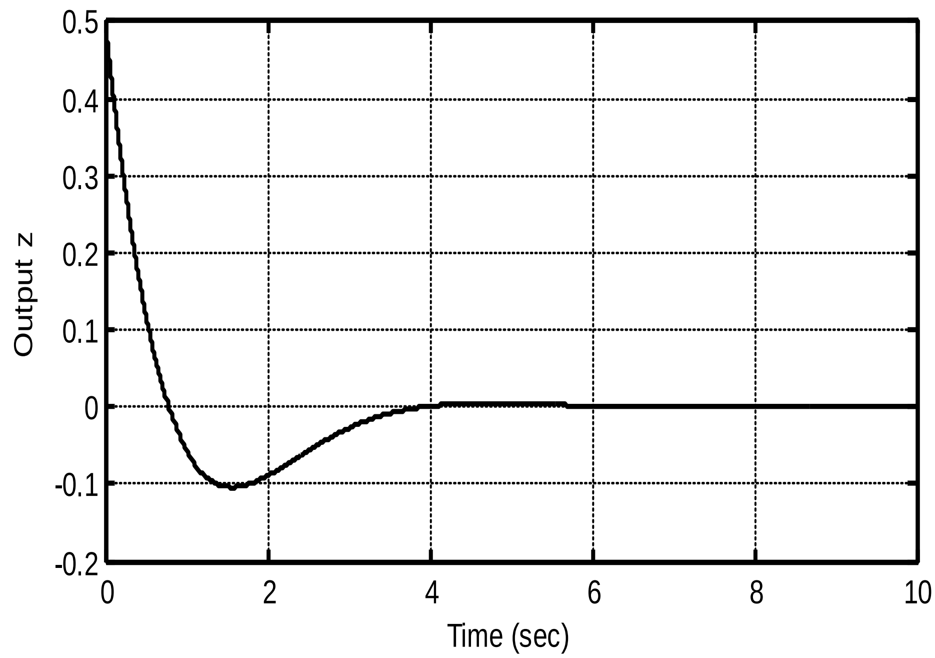

Figure (10): Output z time history for Motivated Example A.

B-Solution of H.w. 2:

For the following system dynamics:

$$\dot{x}_1 = x_1^2 + x_2 \quad \textbf{(23)}$$

$$\dot{x}_2 = x_3 \quad \textbf{(24)}$$

$$\dot{x}_3 = u \quad \textbf{(25)}$$

Design a control law u to stabilize the above system (Eq, (23)-(25)) based on backstepping control method.

B-1: Solution Using Non-Standard Backstepping:

Before we get inside the design process, we will divide our system into two subsystems. Eq. (23) and (24) will represent subsystem 1 and Eq. (24) and (25) will represent subsystems 2.

Step1:

We start our design with subsystem 1 and by taking the scalar system Eq. (23), with x_2 viewed as the control input, namely Eq. (23) will be rewritten in the following way;

$$\dot{x}_1 = x_1^2 + v_1 \quad \textbf{(41)}$$

Where v_1; is a virtual control input to the system [Eq. (41)]. Hence, we proceed to design the virtual control input v_1 to stabilize the system [Eq. (41)] to the origin ($x_1 = 0$). So we formulate the virtual control law v_1 to be;

$$v_1 = -x_1^2 - \lambda x_1, \lambda > 0 \quad \textbf{(42)}$$

It can be seen in Eq. (42) that we cancel the nonlinear unstable term x_1^2, and introduce a stable component $(-\lambda x_1)$ with λ viewed as a design parameter to provide the necessary damping for stabilizing the system. Thus after substituting Eq. (42) into Eq. (41), we obtain:

$$\dot{x}_1 = -\lambda x_1 \quad \textbf{(43)}$$

Now, in order the system dynamics (Eq. (43)) to be exist, state x_2 should mimic υ_1 behavior, therefore we have to backstep by introducing a new output variable (z_2) which is actually a change of variables, i.e.

$$z_2 = x_2 - \upsilon_1$$

$$z_2 = x_2 + x_1^2 + \lambda x_1 \quad \textbf{(44)}$$

Now, we differentiate Eq. (44), we got;

$$\dot{z}_2 = x_3 + (x_1^2 + x_2)(2x_1 + \lambda) \quad \textbf{(45)}$$

From Eq. (44), we can find x_2 with respect to output z_2 in the following way:

$$x_2 = z_2 - x_1^2 - \lambda x_1 \quad \textbf{(46)}$$

Using transformation Eq. (46), we transform subsystem 1 into the form;

$$\dot{x}_1 = -\lambda x_1 + z_2 \quad \textbf{(47)}$$

$$\dot{z}_2 = x_3 + (z_2 - \lambda x_1)(2x_1 + \lambda) \quad \textbf{(48)}$$

Now, it is clear that in order to stabilize the system (Eq. (47)), z_2 should equal to zero (i.e. $x_2 = \upsilon_1$). To achieve this Eq. (48) will be rewritten by seeing x_3 as a control input in the following way:

$$\dot{z}_2 = \upsilon_2 + (z_2 - \lambda x_1)(2x_1 + \lambda) \quad \textbf{(49)}$$

Where υ_2: is another virtual controller to stabilize the system [Eq. (47) and (48)]. Thus we rewrite Eq. (49) as:

$$\dot{z}_2 = \upsilon_2 + \phi_1(x_1, x_2) \qquad \textbf{(50)}$$

Where $\phi_1(x_1, x_2) = (z_2 - \lambda x_1)(2x_1 + \lambda) = 2x_1x_2 + 2x_1^3 + \lambda x_2 + \lambda x_1^2$

We suggest υ_2 with the following form;

$$\upsilon_2 = -\phi_1(x_1, x_2) - \alpha_1 z_2 \qquad \textbf{(51)}$$

Where α_1 is another design parameter, By substituting Eq. (51) into Eq. (50) , yields

$$\dot{z}_2 = -\alpha_1 z_2 \qquad \textbf{(52)}$$

Where υ_2 will cause the z_2 to stabilize to zero and the major consequence is that x_2 will equal υ_1. By this we end step1 and turn to step 2.

Step2:

Now we consider subsystem 2 (Eq. (24) and (25), in order to achieve the above control objectives x_3 should act like υ_2. Thus again we have to backstep by introducing a new output variable z_3, which represent a change of variables as follows;

$$z_3 = x_3 - \upsilon_2$$

$$z_3 = x_3 + \phi_1(x_1, x_2) + \alpha_1 z_2$$

$$z_3 = x_3 + \phi_2(x_1, x_2) \qquad \textbf{(53)}$$

Where $\phi_2(x_1, x_2) = \alpha_1\lambda x_1 + (\lambda + \alpha_1)x_2 + 2x_1x_2 + (\lambda + \alpha_1)x_1^2 + 2x_1^3$

Differentiating Eq. (53) with respect to time we got;

$$\dot{z}_3 = \dot{x}_3 + \frac{d\phi_2(x_1,x_2)}{dt} \tag{54}$$

where the term $\frac{d\phi_2(x_1,x_2)}{dt}$ can be found by using the chain rule in the following way:

$$\frac{d\phi_2(x_1,x_2)}{dt} = \frac{\partial\phi_2}{\partial x_1}.\frac{\partial x_1}{\partial t} + \frac{\partial\phi_2}{\partial x_2}.\frac{\partial x_2}{\partial t}$$

$$\frac{d\phi_2(x_1,x_2)}{dt} = [\alpha_1\lambda + 2x_2 + [2(\lambda+\alpha_1)x_1 + 6x_1^2](x_1^2 + x_2)](x_1^2 + x_2) + [(\lambda+\alpha_1) + 2x_1]x_3 \tag{55}$$

Using Eq. (53) to find x_3 with respect to z_3, in the following way;

$$x_3 = z_3 + \phi_2(x_1,x_2) \tag{56}$$

Substituting Eq. (56) back into Eq. (55), so that Eq. (54) will be;

$$\dot{z}_3 = u + \phi_3(x_1,x_2) \tag{57}$$

Where

$$\phi_3(x_1,x_2) = [\alpha_1\lambda + 2x_2 + [2(\lambda+\alpha_1)x_1 + 6x_1^2](x_1^2 + x_2)](x_1^2 + x_2) + [(\lambda+\alpha_1) + 2x_1]\big(z_3 + \phi_2(x_1,x_2)\big) \tag{58}$$

The subsystem 2 will be transformed using Eq. (56), yields

$$\dot{x}_2 = z_3 + \phi_2(x_1,x_2) \tag{59}$$

$$\dot{z}_3 = u + \phi_3(x_1,x_2) \tag{60}$$

In order to guarantee that $(x_3 = v_2)$, z_3 should be stabilize to origin and this will turn to $x_2 = -\phi_2(x_1,x_2) = v_1$. Actually this certifies our design steps, and by choosing the control action u as;

$$\boxed{u = -\phi_3(x_1, x_2) - \alpha_2 z_3} \tag{61}$$

Where α_2 is a design parameter. Eventually by substituting the Eq. (61) back into Eq. (60) we found;

$$\dot{z}_3 = -\alpha_2 z_3 \tag{62}$$

Where the term $(-\alpha_2 z_3)$ is added to provide the necessary damping to stabilize the system.

B-2: Solution Using Standard Backstepping:

In this section we repeat the design of controller using standard backstepping control, where the design is done based on constructing a Lyapunov function in each design step in the following way;

Step1:

Consider subsystem 1 and by taking the scalar system Eq. (23), with x_2 viewed as the control input, so Eq. (23) will be rewritten in the following way;

$$\dot{x}_1 = x_1^2 + \upsilon_1 \tag{63}$$

Where υ_1; is a virtual control input to the system [Eq. (63)]. Now in order to design the virtual control input υ_1 to stabilize the system [Eq. (63)] to the origin ($x_1 = 0$), we candidate the following Lyapunov function;

$$V_1 = \frac{1}{2} x_1^2 \tag{64}$$

Accordingly $\dot{V}_1$ can be found to be;

$$\dot{V}_1 = x_1 \dot{x}_1 = x_1 \left(\underbrace{x_1^2 + \upsilon_1}_{-\lambda x_1} \right) \tag{65}$$

To insure the negative definiteness of $\dot{V}_1$, we suggest the following Equality;

$$x_1^2 + \upsilon_1 = -\lambda x_1, \lambda > 0 \tag{66}$$

Where the term $(-\lambda x_1)$ is chosen to assure providing the necessary damping to the system with λ as a design parameter. The virtual control law υ_1 can be found as;

$$\upsilon_1 = -x_1^2 - \lambda x_1 \tag{67}$$

Then $\dot{V}_1$ will be

$$\dot{V}_1 = x_1\dot{x}_1 = -\lambda x_1^2 < 0 \; \forall x_1 \neq 0 \tag{68}$$

Which means that x_1 will decay exponentially asymptotically to the origin when $(x_2 = v_1 = -x_1^2 - \lambda x_1)$. To do this state x_2 should act like v_1 behavior, therefore we have to backstep by introducing a new output variable (z_2) which is actually a change of variables, i.e.

$$z_2 = x_2 - v_1$$

$$z_2 = x_2 + x_1^2 + \lambda x_1 \tag{69}$$

Now, we differentiate Eq. (69), we got;

$$\dot{z}_2 = x_3 + (x_1^2 + x_2)(2x_1 + \lambda) \tag{70}$$

From Eq. (69), we can find x_2 with respect to output z_2 in the following way:

$$x_2 = z_2 - x_1^2 - \lambda x_1 \tag{71}$$

Using transformation Eq. (71), we transform subsystem 1 into the form;

$$\dot{x}_1 = -\lambda x_1 + z_2 \tag{72}$$

$$\dot{z}_2 = x_3 + (z_2 - \lambda x_1)(2x_1 + \lambda) \tag{73}$$

Step2:

Now, it is clear that in order to stabilize the system (Eq. (72)), z_2 should equal to zero (i.e. $x_2 = v_1$). To achieve this Eq. (73) will be rewritten by seeing x_3 as a control input in the following way:

$$\dot{z}_2 = v_2 + (z_2 - \lambda x_1)(2x_1 + \lambda) \tag{74}$$

Where υ_2: is another virtual controller to stabilize the system [Eq. (72) and (73)]. Thus we rewrite Eq. (74) as:

$$\dot{z}_2 = \upsilon_2 + \phi_1(x_1, x_2) \qquad \textbf{(75)}$$

Where $\phi_1(x_1, x_2) = (z_2 - \lambda x_1)(2x_1 + \lambda) = 2x_1x_2 + 2x_1^3 + \lambda x_2 + \lambda x_1^2$

The next step is to write the total Lyapunov function for step as follows:

$$V_2 = V_1 + \frac{1}{2}z_2^2 = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 \qquad \textbf{(76)}$$

Accordingly $\dot{V}_2$ is

$$\dot{V}_2 = x_1\dot{x}_1 + \ z_2\dot{z}_2$$

$$\dot{V}_2 = x_1(z_2 - \lambda x_1) + z_2\big(\upsilon_2 + \phi_1(x_1, x_2)\big)$$

$$\dot{V}_2 = -\lambda x_1^2 + z_2\left(\underbrace{\upsilon_2 + x_1 + \phi_1(x_1, x_2)}_{-\alpha_1 z_2}\right) \qquad \textbf{(77)}$$

To insure the negative definiteness of $\dot{V}_2$, we suggest the following Equality;

$$\upsilon_2 + x_1 + \phi_1(x_1, x_2) = -\alpha_1 z_2, \alpha_1 > 0 \qquad \textbf{(78)}$$

Where the term $(-\alpha_1 z_2)$ is chosen to assure providing the necessary damping to the system with α_1 as a design parameter. The second virtual control law υ_2 can be found as;

$$\upsilon_2 = -x_1 - \phi_1(x_1, x_2) - \alpha_1 z_2$$

$$\upsilon_2 = \phi_2(x_1, x_2) \qquad \textbf{(79)}$$

Where $\phi_2(x_1, x_2) = -(1 + \alpha_1\lambda)x_1 - (\lambda + \alpha_1)x_1^2 - 2x_1^3 - (\lambda + \alpha_1)x_2 - 2x_1x_2$

By substituting Eq. (79) into Eq. (75), we get;

$$\dot{z}_2 = -x_1 - \alpha_1 z_2 \qquad \textbf{(80)}$$

The above system will decay exponentially asymptotically to the origin when $(x_2 = \upsilon_1 = -x_1^2 - \lambda x_1)$.

Step3:

Now we consider subsystem 2 (Eq. (24) and (25), in order to achieve the above control objectives x_3 should act like υ_2. Thus again we have to backstep by introducing a new output variable z_3, which represent a change of variables as follows;

$$z_3 = x_3 - \upsilon_2$$

$$z_3 = x_3 - \phi_2(x_1, x_2) \qquad \textbf{(81)}$$

We need to find x_3 with respect to z_3 using Eq. (81) as follows:

$$x_3 = z_3 + \phi_2(x_1, x_2)$$

Differentiating Eq. (81) with respect to time we got;

$$\dot{z}_3 = \dot{x}_3 - \frac{d\phi_2(x_1,x_2)}{dt} \qquad \textbf{(83)}$$

Where the term $\frac{d\phi_2(x_1,x_2)}{dt}$ can be found by using the chain rule in the following way:

$$\frac{d\phi_2(x_1, x_2)}{dt} = \frac{\partial\phi_2}{\partial x_1} \cdot \frac{\partial x_1}{\partial t} + \frac{\partial\phi_2}{\partial x_2} \cdot \frac{\partial x_2}{\partial t}$$

$$\frac{d\phi_2(x_1,x_2)}{dt} = [-(1+\alpha_1\lambda) - 2x_2 - [2(\lambda+\alpha_1)x_1 + 6x_1^2](x_1^2+x_2)](x_1^2+x_2) - [(\lambda+\alpha_1)+2x_1]x_3 \quad \textbf{(84)}$$

The subsystem 2 will be transformed using Eq. (81), yields

$$\dot{x}_2 = z_3 + \phi_2(x_1,x_2) \quad \textbf{(85)}$$

$$\dot{z}_3 = u - \phi_3(x_1,x_2) \quad \textbf{(86)}$$

Where $\phi_3(x_1,x_2) = [-(1+\alpha_1\lambda) - 2x_2 - [2(\lambda+\alpha_1)x_1 + 6x_1^2](x_1^2+x_2)](x_1^2 + x_2) - [(\lambda+\alpha_1)+2x_1]\big(z_3+\phi_2(x_1,x_2)\big)$

In order to guarantee that $(x_3 = \upsilon_2)$, and this can be done by choosing the following candidate composite lyapunov as;

$$V_3 = V_2 + \frac{1}{2}z_3^2 \quad \textbf{(87)}$$

Differentiating V_3 with respect to time yields;

$$\dot{V}_3 = \frac{dV_2}{dt} + z_3\dot{z}_3$$

$$\dot{V}_3 = -\lambda x_1^2 - \alpha_1 z_2^2 + z_3\left(\underbrace{u-\phi_3(x_1,x_2)}_{-\alpha_2 z_3}\right) \quad \textbf{(88)}$$

By imposing the following Equality;

$$u - \phi_3(x_1,x_2) = -\alpha_2 z_3, \alpha_2 > 0 \quad \textbf{(89)}$$

Where the term $(-\alpha_2 z_3)$ is chosen to assure providing the necessary damping to the system with α_2 as a design parameter. The final control law u can be found as;

$$\boxed{u = -\alpha_2 z_3 + \phi_3(x_1,x_2)} \quad \textbf{(90)}$$

Eventually by substituting the Eq. (90) back into Eq. (86) we found;

$$\dot{z}_3 = -\alpha_2 z_3 \quad \textbf{(91)}$$

By choosing the design parameters to be:

$$\lambda = 5, \alpha_1 = \alpha_2 = 2$$

The simulation results are obtained and shown in Fig. (11)-(18).

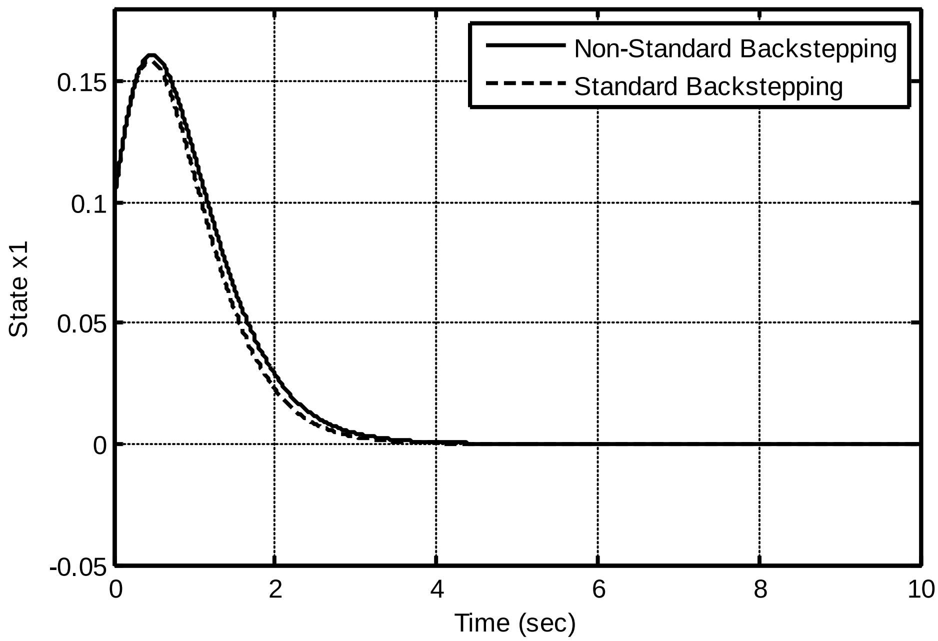

Figure (11): State x_1 Time History For both Standard and Non-Standard Backstepping Design.

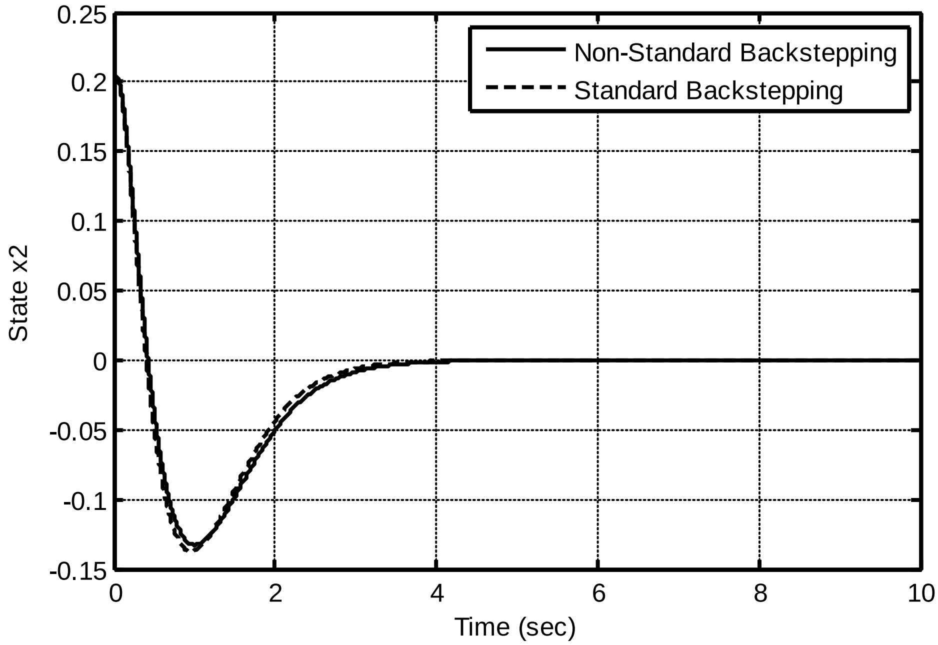

Figure (12): State x_2 Time History For both Standard and Non-Standard Backstepping Design.

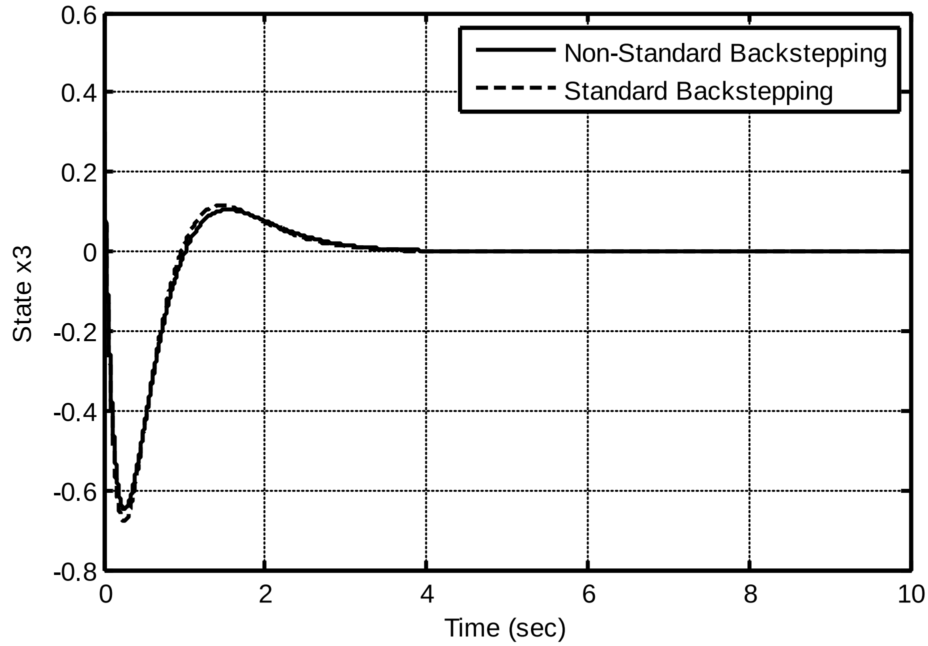

Figure (13): State x_3 Time History For both Standard and Non-Standard Backstepping Design.

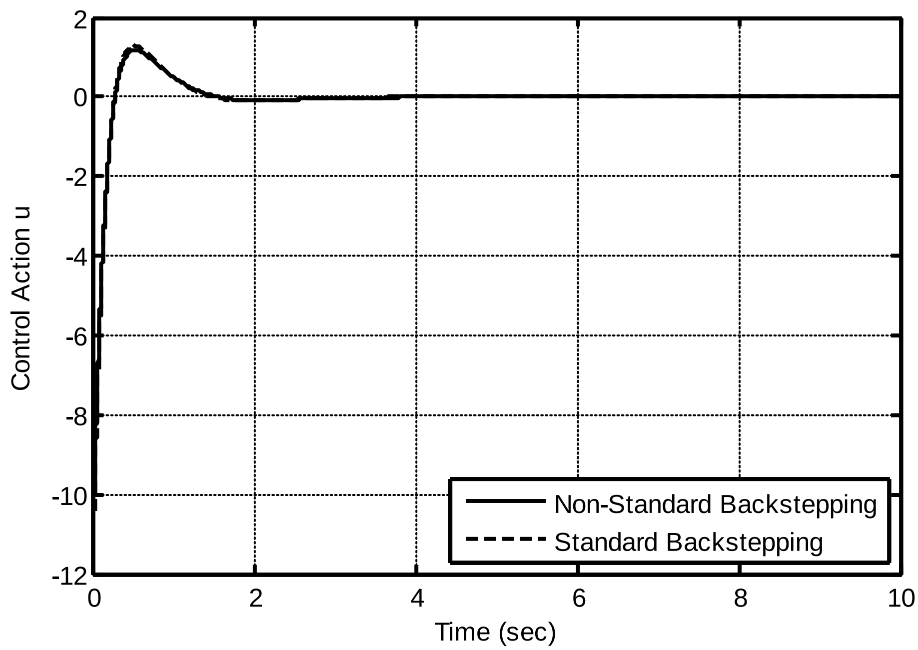

Figure (14): Control Action u Time History For both Standard and Non-Standard Backstepping Design.

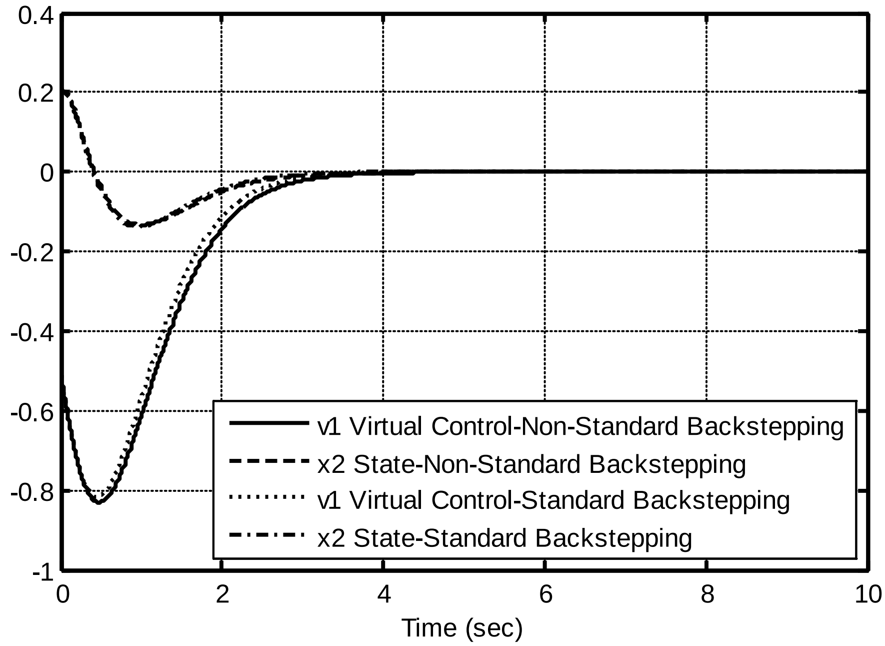

Figure (15): Virtual Control Action v_1 versus x_2 state Time History for both Standard and Non-Standard Backstepping Design.

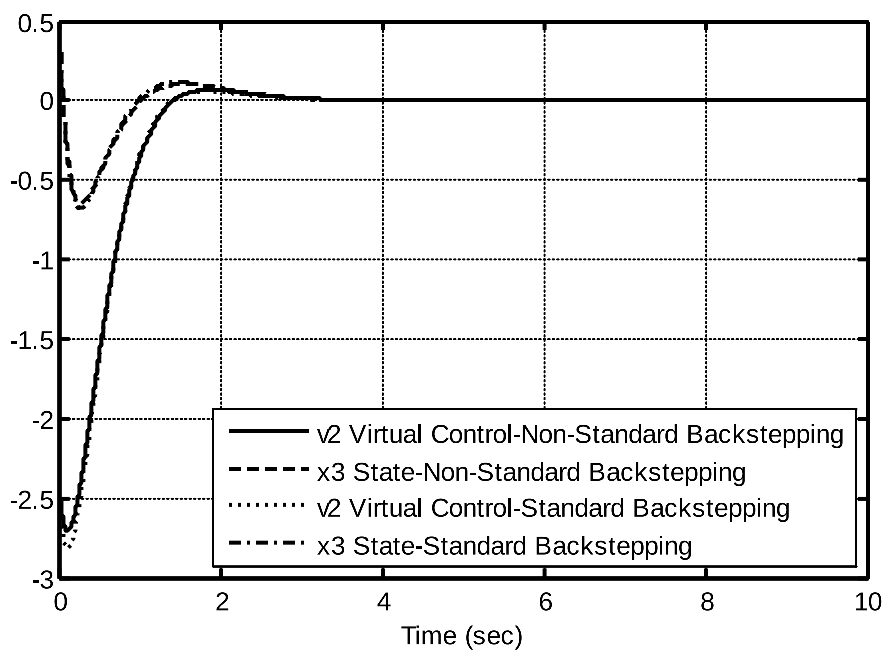

Figure (16): Virtual Control Action v_2 versus x_3 state Time History for both Standard and Non-Standard Backstepping Design.

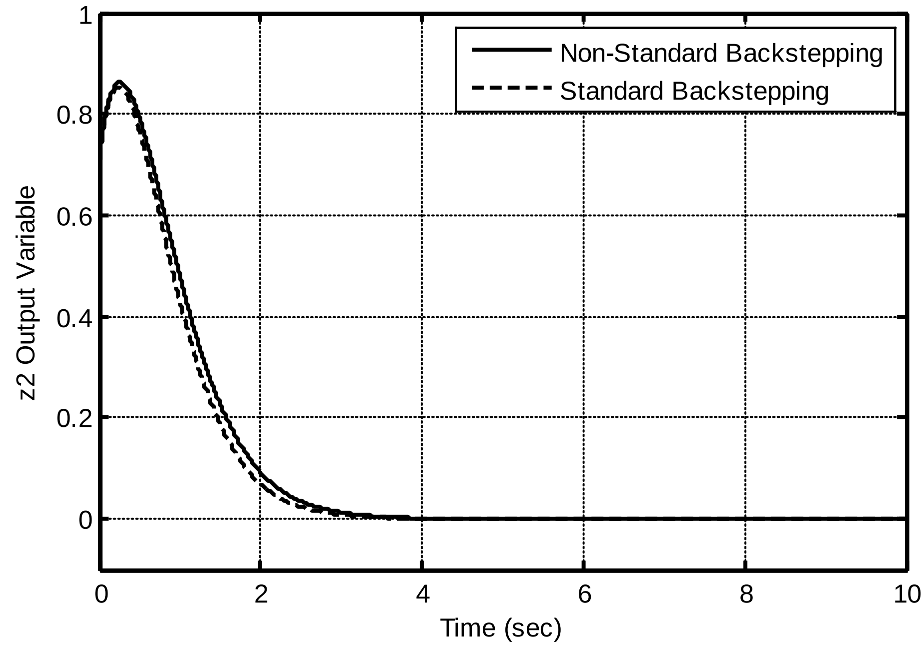

Figure (17): Output Function z_2 Time History For both Standard and Non-Standard Backstepping Design.

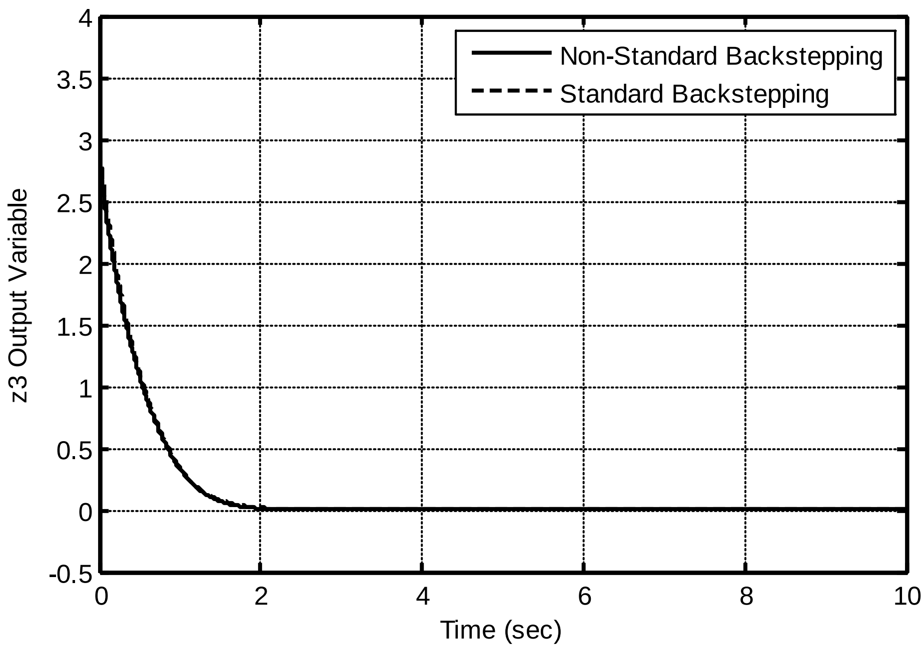

Figure (18): Output Function z_3 Time History For both Standard and Non-Standard Backstepping Design.

5. Case Study: Mathematical Model of The Electro-Hydraulic Actuator System:

The Electro-Hydraulic Actuator system response will be studied in order to realize the dynamic problems, so a mathematical equations should be represented for the Electro-Hydraulic Actuator system basic components which is consists of a 4/3 way servo valve with double rod double acting cylinder and other components as shown in Fig. (19);

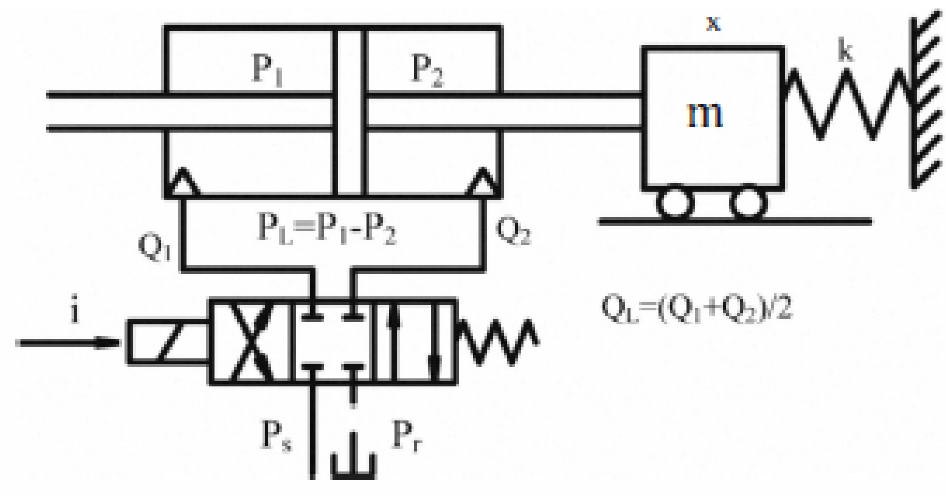

Figure (19). The Electro-Hydraulic Actuator system schematic diagram.

The model dynamics for the cylinder can be described, via Newton's Law, by the following equation

$$m\ddot{x} = P_L\Omega - b\dot{x} - kx \tag{92}$$

Where;

x represent the displacement of the actuator,

m is the mass of the load,

$P_L = P_1 - P_2$ is the load pressure of the cylinder,

P_1 and P_2 are the pressure of the actuator of chamber 1 and chamber 2 respectively,

$\mathbf{\Omega}$ is the ram area of the cylinder,

b represents the viscous damping coefficient,

k is the effective bulk modulus of spring.

The load pressure of the cylinder can be represented with the following equation:

$$\frac{V_t}{4\beta_e}\dot{P}_L = -\Omega\dot{x} - C_{tm}P_L + Q_L \tag{93}$$

Where;

V_t is the total volume of the cylinder and the hoses between the cylinder and the servo valve,

β_e is the effective bulk modulus,

C_{tm} is the coefficient of the total internal leakage of the cylinder due to pressure, and $Q_L = (Q_1 + Q_2)/2$ is the load flow.

Q_L is related to the spool valve displacement of the servo valve, as in equation below:

$$Q_L = C_d w x_v \sqrt{\frac{(P_s - sgn\,(x_v)P_L)}{\rho}} \tag{94}$$

Where;

C_d is the discharge coefficient,

w is the spool valve area gradient,

P_s is the supply pressure of the fluid,

ρ is the fluid Density,

x_v is the spool valve displacement of the servo valve, as in the following equation:

$$\tau_v \dot{x}_v = -x_v + K_v u_o \tag{94}$$

Where the spool valve displacement x_v is related to the current input i, τ_v and K_v are the time constant and gain of the servo-valve respectively.

Here we omit the spool dynamics, as described in Eq. (94), and consider only that the spool follows the command signal u_o; namely

$$x_v = K_v u_o \tag{95}$$

By defining

$$x_1 = x - x_d, \qquad x_2 = \dot{x} \quad \text{and} \quad x_3 = P_L,$$

where x_d is the desired actuator displacement, the mathematical model in Eq. (92) becomes:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= a_1 x_1 + a_2 x_2 + a_3 x_3 \\ \dot{x}_3 &= b_2 x_2 + b_3 x_3 + u \end{aligned} \tag{96}$$

where

$$a_1 = -\frac{k}{m}, a_2 = -\frac{b}{m}, a_3 = \frac{\Omega}{m},$$

$$b_2 = -\frac{4\beta_e \Omega}{V_t}, b_3 = -\frac{4\beta_e C_{tm}}{V_t} \text{ and}$$

$$u = \frac{4\beta_e C_d w K_v}{V_t\sqrt{\rho}}\sqrt{P_s - sgn(u_o)x_3}\,u_o$$

To determine u_o, note that the sign of u_o is equal to the sign of u, therefore:

$$u_o = \frac{V_t\sqrt{\rho}}{4\beta_e C_d w K_v\sqrt{P_s - sgn\,(u_o)x_3}}u \tag{97}$$

The system parameters value is given in the following table:

Table (1): The actuator parameters

The parameter	Description	The value (SI units)
b	Viscous damping coefficient.	19.84*10^3 m/s
Ω	Ram area of the cylinder.	(5550/1000000) m^2
Vt	Total volume of the cylinder and the hoses between the cylinder and the servo valve.	(1.75*10^6)/((10^3)^3)m^3
Ctm	Coefficient of the total internal leakage of the cylinder due to pressure.	(15/(10^3)^5) m^5/Ns
K	Effective bulk modulus of spring.	70*10^3 N/m
β	Effective bulk modulus.	(700*(10^3)^2) N/m^2
C_d w/$\sqrt{\rho}$	C_d is the discharge coefficient, w is the spool valve area gradient and ρ is the fluid density.	3.42*10^4/(10^3)^3)m^3$\sqrt{Ns}$
m	Mass of the load.	20~250 Kg
Ps	Supply pressure of the fluid.	10 MPa
k_v	Gain of the servo-valve.	0.03

Student Task: our task here is to design a controller based on the Backstepping control method for the Electro-Hydraulic Actuator system as modeled above with a desired displacement $x_d = 2\ mm$.

6. References:

We recommend to our students the following references for further reading;

[1] H. K. Khalil, “Nonlinear Systems”, 3rd Edition, Prentise Hall, USA, 2002.

[2] R. Sepulchre, M. Jankovic, and P.V. Kokotovic, “Constructive Nonlinear Control”, 1st Edition, Springer, USA, 1997.