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Asymptotic Stability of Solutions of the Generalized Burgers Equation

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This paper is concerned with the existence and stability analysis of the generalized Burgers equation. For the cylindrical case, the bounded solutions are found to be asymptotically stable to the similarity solution. For the super cylindrical case, the asymptotic limit of the generalized Burgers equation is an error function solution of the linearized Burgers equation. It is shown that this result generalizes the result of Scott [5]. © 1990 Academic Press, Inc.

1. INTRODUCTION

This paper deals with the long time behavior of the bounded solutions of the generalized Burgers equation of the form

$$u_t - g(t)u_{xx} = f(u)u_x, \tag{1.1}$$

where f is a differentiable function in $(-\infty, \infty)$ with $f(0)=0$ and $g(t)>0$ for $t>0$. A special case of (1.1) with $f(u)=u$ has been studied by Scott [5]. In Scott's analysis with $f(u)=u$ the Burgers equation has a Galilean invariance, $u \to u+\lambda$, $x \to x+\lambda t$, which plays a crucial role in the proof of the stability of similarity solutions. This invariance does not occur in the generalized Burgers equation (1.1). For a wide variety of functions f, in Section 4 the bounded monotonic similarity solutions are shown to exist. The stability of similarity solutions is also discussed in Section 4. A sufficient condition for stability is given to replace the absence of the Galilian invariance. Section 3 deals with the super cylindrical case ($g(t)/t \to \infty$ as $t \to \infty$). It is shown that the bounded solutions are asymptotically stable to the linearized solution of the generalized Burgers equation.

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2. The Cauchy Problem for the Generalized Burgers Equation

Consider the Cauchy problem for the generalized Burgers equation

$$u_t - g(t)u_{xx} = f(u)u_x \tag{2.1}$$

$$u(x, 0) = \psi_0(x), \tag{2.2}$$

where f is differentiable in $(-\infty, \infty)$, $g(t) > 0$ for $t > 0$, and g is a Hölder continuous function of exponent α in any finite subinterval S, $0 < \alpha < 1$; that is, for some $A > 0$ we have

$$|g(t_1) - g(t_2)| \leqslant A\,|t_1 - t_2|^\alpha \tag{2.3}$$

for any $t_1, t_2 \in S$, and

$$\begin{gathered}\psi_0(x) \in H^{2+\alpha}(\Omega) \text{ for any bounded interval } \Omega,\\ \text{and } \max_R |\psi_0(x)| < \infty.\end{gathered} \tag{2.4}$$

Here $H^{2+\alpha}(\Omega)$ denotes the Hölder space of order $2+\alpha$ (see [2, 3]).

Note that the above Cauchy problem is a special case of the quasi-linear parabolic equations. Under conditions (2.3)–(2.4), the solution of (2.1)–(2.2) exists and is unique in $H^{2+\alpha, 1+\alpha/2}(R_T)$ for any $T > 0$, where $R_T = R \times [0, T]$ (see Theorem 8.1, Chap. 5, in [3]). A careful examination of the proof of Theorem 8.1 reveals that if $\psi_0(x) \to C \pm D$ as $|x| \to \infty$, then the solution $u(x, t)$ of (2.1)–(2.2) is uniformly bounded and $u(x, t) \to C \pm D$ as $x \to \pm\infty$ for any t. We assume this result in the proof of the stability in Sections 3 and 4.

3. The Super Cylindrical Case ($g(t)/t \to \infty$ as $t \to \infty$)

Let $E(x, t)$ be the solution of the linearized equation of (2.1); that is,

$$u_t - g(t)u_{xx} = 0 \tag{3.1}$$

so that

$$E(x, t) = C + D \operatorname{erf}\left(\frac{x}{2\sqrt{z}}\right), \tag{3.2}$$

where

$$z = \int_0^t g(s)\, ds \qquad \text{and} \qquad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\eta^2}\, d\eta. \tag{3.3ab}$$

THEOREM 3.1. *Let $u(x, t)$ be the solution of the Cauchy problem* (2.1)–(2.2) *with ψ_0 satisfying* (2.4). *Then*

$$|u(x, t) - E(x, t)| \to 0 \qquad \text{uniformly in } x \qquad \text{as} \quad t \to \infty.$$

Proof. We give a proof which is similar to that by Scott [5] for the special case $f(u) = u$. The solution of (2.1)–(2.2) can be written in terms of $E(x, t)$ as

$$\begin{aligned} u(x, z) = E(x, z) &- \int_{-\infty}^{\infty} \frac{1}{2(\pi z)^{1/2}} [\psi_0(\tau) - E(\tau, 0)] \exp\left[-\frac{(x-\tau)^2}{4z}\right] d\tau \\ &- \int_0^z \int_{-\infty}^{\infty} \frac{F(u(\tau, \xi))(x-\tau) \exp[-(x-\tau)^2/4(z-\xi)]}{4G(\xi)\sqrt{\pi}\,(z-\xi)^{3/2}} \, d\tau \, d\xi, \end{aligned} \tag{3.4}$$

where

$$G(z) = G\left(\int_0^t g(s)\, ds\right) = g(t) \qquad \text{and} \qquad F'(u) = f(u). \tag{3.5ab}$$

Since $u(x, t)$ is uniformly bounded in $R \times [0, \infty)$, it follows from (3.4) that

$$\begin{aligned} |u(x, z) - E(x, z)| &\leqslant \int_{-\infty}^{\infty} \frac{|\psi_0(\tau) - E(\tau, 0)|}{2\sqrt{\pi z}} \exp\left[-\frac{(x-\tau)^2}{4z}\right] d\tau \\ &\quad + \int_0^z \int_{-\infty}^{\infty} \frac{K_0\, |x-\tau| \exp[-(x-\tau)^2/4(z-\xi)]}{G(\xi)(z-\xi)^{3/2}} \, d\tau \, d\xi \\ &\leqslant \int_{-\infty}^{\infty} \frac{|\psi_0(\tau) - E(\tau, 0)|}{2\sqrt{\pi z}} \exp\left[-\frac{(x-\tau)^2}{4z}\right] d\tau \\ &\quad + K \int_0^t \left(\int_\xi^t g(s)\, ds\right)^{-1/2} d\xi. \end{aligned} \tag{3.6}$$

For any $\varepsilon > 0$, we can find $T > 0$ such that

$$g(t) \geqslant \frac{t}{\varepsilon} \qquad \text{for any} \quad t \geqslant T \tag{3.7}$$

and $X > 0$ such that

$$|\psi_0(\tau) - E(\tau, 0)| \leqslant \varepsilon \qquad \text{for} \quad |\tau| \geqslant X. \tag{3.8}$$

Then it follows from (3.6)–(3.8) that

$$\begin{aligned}
&|u(x, z) - E(x, z)| \\
&\quad \leqslant \left(\int_{|\tau| \geqslant X} + \int_{|\tau| \leqslant X} \right) \frac{|\psi_0(\tau) - E(\tau, 0)|}{2\sqrt{\pi z}} \exp\left[-\frac{(x-\tau)^2}{4z} \right] dz \\
&\quad + K \left[\int_0^T \frac{(2\varepsilon)^{1/2}}{(t^2 - T^2)^{1/2}} d\xi + \int_T^t \frac{(2\varepsilon)^{1/2}}{(t^2 - \xi^2)^{1/2}} d\xi \right] \\
&\quad \leqslant 2\varepsilon + (2\varepsilon)^{1/2} K \left\{ \frac{T}{(t^2 - T^2)^{1/2}} + \frac{\pi}{2} - \arcsin \frac{T}{t} \right\} \qquad \text{for large } t.
\end{aligned}$$

This completes the proof.

This shows that the asymptotic limit of the solution of (2.1)–(2.5) is an error function solution of the linearized equation (2.1) provided $g(t)/t \to \infty$ as $t \to \infty$.

4. The Cylindrical Case ($g(t)/t \to \beta \neq 0$ as $t \to \infty$)

Consider the similarity solution of (2.1) for $g(t) = \beta t$ of the form

$$u(x, t) \equiv \Omega\left(\frac{x}{t}\right) \equiv \Omega(\xi). \tag{4.1}$$

Substituting (4.1) into (2.1) gives

$$\beta \Omega'' = -\Omega'(f(\Omega)) + \xi). \tag{4.2}$$

For the special case when $f(\Omega) = \Omega$, Eq. (4.2) has implicit solutions of the form

$$\begin{aligned}
\Omega(\xi) &= -\xi + 2\sqrt{\beta}\, H(\eta), \qquad \eta \geqslant \eta_0 \\
\xi &= \gamma + \sqrt{\beta} \int_{\eta_0}^{\eta} \frac{ds}{H(s)},
\end{aligned} \tag{4.3}$$

where

$$H(\eta) = \pm [\eta - \eta_0 \exp\{-2(\eta - \eta_0)\}]^{1/2} \tag{4.4}$$

and γ and η_0 are constants with $\eta_0 > -\frac{1}{2}$. Also Ω is monotonic in $(-\infty, \infty)$ and

$$\Omega \to \gamma \pm \sqrt{\beta}\, \Delta(\eta_0) \qquad \text{as} \quad \xi \to \pm\infty. \tag{4.5}$$

Details about this solution can be found in [4, 5]. However, the solution of (4.2) for any f is not yet known. In the following we discuss solutions of (4.2).

LEMMA 4.1. *Any non-trivial solution of* (4.2) *is monotone in its maximal interval.*

Proof. Let Ω be a solution of (4.2). If $\Omega'(\xi_0)=0$ at some point ξ_0, then $\Omega''(\xi_0)=0$ and therefore Ω is a constant solution. Hence if Ω is a non-trivial solution of (4.2) then Ω is either increasing or decreasing.

COROLLARY 4.1. *Any bounded solution Ω of* (4.2) *which is defined in* $(-\infty, \infty)$ *has an inflexion point ξ_0 which satisfies the equation*

$$f(\Omega(\xi_0))+\xi_0=0. \tag{4.6}$$

Consider next the initial value problem (4.2) with the initial conditions

$$f(\Omega(\xi_0))=-\xi_0, \qquad \Omega'(\xi_0)=a, \tag{4.7ab}$$

where ξ_0 and $a(\neq 0)$ are arbitrary constants. We integrate (4.2) for Ω satisfying (4.7ab) over $[\xi_0, \xi]$ and $[\xi, \xi_0]$ to obtain

$$\beta\Omega'(\xi)=\beta a-F(\Omega(\xi))+F(\Omega(\xi_0))-\int_{\xi_0}^{\xi}\Omega'(s)s\,ds \tag{4.8a}$$

or

$$\beta\Omega'(\xi)=\beta a-F(\Omega(\xi))+F(\Omega(\xi_0))-\xi\Omega(\xi)+\xi_0\Omega(\xi_0)$$
$$+\int_{\xi_0}^{\xi}\Omega(s)\,ds, \qquad \text{for} \quad \xi\geqslant\xi_0, \tag{4.8b}$$

and

$$\beta\Omega'(\xi)=\beta a-F(\Omega(\xi))+F(\Omega(\xi_0))+\int_{\xi}^{\xi_0}\Omega'(s)s\,ds, \tag{4.9a}$$

or

$$\beta\Omega'(\xi)=\beta a-F(\Omega(\xi))+F(\Omega(\xi_0))-\xi\Omega(\xi)+\xi_0\Omega(\xi_0)$$
$$-\int_{\xi}^{\xi_0}\Omega(s)\,ds, \qquad \text{for} \quad \xi\leqslant\xi_0, \tag{4.9b}$$

respectively, where F satisfies (3.6).

We next prove the following Lemma:

LEMMA 4.2. *If* $a>0$ *and* $f(u)\to\pm\infty$ *as* $u\to\pm\infty$, *then the solution of* (4.2)–(4.7) *is bounded in* $(-\infty,\infty)$ *for any* ξ_0.

Proof. Since $a>0$, it follows from Lemma 4.1 that $\Omega'(\xi)>0$ for any ξ in the interval of existence of Ω. It follows from (4.8b) and (4.9b) that if Ω is bounded in any finite subinterval of its interval of existense, Ω' is also bounded. It also follows from the sign of $\Omega''(\xi)$ in (4.2) and the conditions imposed on f that Ω is bounded in any finite subinterval of its interval of existence. Therefore Ω is defined in $(-\infty,\infty)$. Suppose $\Omega(\xi)\to\infty$ as $\xi\to\infty$. Then the right-hand side of (4.8a) approaches $-\infty$ as $\xi\to\infty$, which contradicts the fact that $\Omega'(\xi)>0$ for any ξ. Similarly, we can conclude from (4.9a) that Ω is bounded as $\xi\to-\infty$.

Remarks. (i) The condition in Lemma 4.2 is satisfied if f is an increasing odd function with $f(u)\to\infty$ as $u\to\infty$. In this case, the solution Ω of (4.2)–(4.4) is odd when $\xi_0=0$.

(ii) If f is an even function which is increasing in $(0,\infty)$, $f(0)=0$, and $f(u)\to\infty$ as $u\to\infty$, the solution Ω for any $a\neq 0$ is bounded in (ξ_0,∞). This follows from the proof of Lemma 4.2.

(iii) If f is as in (i) and $a<(d/d\xi)f^{-1}(-\xi_0)$, then solution Ω is unbounded. This follows from the concavity of Ω included in (4.2).

(iv) If f is as in (ii) and $|a|>|(d/d\xi)f^{-1}(-\xi_0)|$, then the solution Ω is unbounded where $(d/d\xi)[f^{-1}(\xi_0)]$ denotes the slope of the tangent line to the graph of $f(\Omega)+\xi=0$ at ξ_0.

The following lemma deals with the existence of bounded solutions of (4.2) which are bounded by upper and lower solutions.

LEMMA 4.3a. *Equation* (4.2) *has infinitely many bounded decreasing solutions* Ω *satisfying* $p(\xi)\leqslant\Omega(\xi)\leqslant q(\xi)$, $-\infty<\xi<\infty$, *where* $p(\xi)=c+d$, $c<0$,

$$q(\xi)=\begin{cases} e, & \xi\leqslant\xi_1 \\ b(\xi-\xi_1)^2+e, & \xi_1<\xi\leqslant\xi_0, \qquad b<0 \\ c\tanh\xi+d, & \xi>\xi_0\end{cases}$$

$$b=\frac{c}{2}\frac{\operatorname{sech}^2\xi_0}{(\xi_0-\xi_1)},\qquad e=c\tanh\xi_0+d-\frac{c}{2}(\xi_0-\xi_1)\operatorname{sech}^2\xi_0$$

and ξ_0 *and* ξ_1 *are chosen such that*

$$2\beta\tanh\xi\leqslant f(c\tanh\xi+d)+\xi\qquad \textit{for}\quad \xi\geqslant\xi_0$$

and

$$-(\xi-\xi_1)[f(b(\xi-\xi_1)^2+e)+\xi]\leqslant\beta \qquad \textit{for} \quad \xi_1\leqslant\xi\leqslant\xi_0.$$

Proof. It can easily be shown that q satisfies the inequality

$$\limsup_{h\to 0}\frac{q'(\xi+h)-q'(\xi-h)}{2h}\leqslant -\frac{1}{\beta}q'(\xi)[f(q(\xi))+\xi]$$
$$\text{for} \quad -\infty<\xi<\infty.$$

Therefore q is an upper solution of (4.2), where the upper and lower solutions are in the sense of Bernfeld and Lakshmikantam [1]. The function

$$F(\xi,\Omega,\Omega')=-\frac{1}{\beta}\Omega'[f(\Omega)+\xi]$$

satisfies Nagumo's conditions in any finite interval with respect to p and q. Then, by Theorem 1.7.2 on page 45 in [1], Eq. (4.2) has infinitely many bounded decreasing solutions Ω with $p(\xi)\leqslant\Omega(\xi)\leqslant q(\xi)$.

LEMMA 4.3b. *Equation* (4.2) *has infinitely many bounded increasing solutions* Ω *satisfying* $p(\xi)\leqslant\Omega(\xi)\leqslant q(\xi)$, $-\infty<\xi<\infty$, *where* $q(\xi)=c+d$, $c>0$,

$$p(\xi)=\begin{cases} e & \xi\leqslant\xi_1\\ b(\xi-\xi_1)^2+e, & \xi_1<\xi\leqslant\xi_0, \quad b>0\\ c\tanh\xi+d, & \xi>\xi_0\end{cases}$$

$$b=\frac{c}{2}\frac{\operatorname{sech}^2\xi_0}{(\xi_0-\xi_1)}, \qquad e=c\tanh\xi_0+d-\frac{c}{2}(\xi_0-\xi_1)\operatorname{sech}^2\xi_0$$

and ξ_0, ξ_1 *are chosen such that*

$$2\beta\tanh\xi\leqslant f(c\tanh\xi+d)+\xi \qquad \textit{for} \quad \xi\geqslant\xi_0$$

and

$$-(\xi-\xi_1)[f\{b(\xi-\xi_1)^2+e\}+\xi]\leqslant\beta \qquad \textit{for} \quad \xi_1\leqslant\xi\leqslant\xi_0.$$

Proof. The proof is similar to that of Lemma 4.3a.

LEMMA 4.4. *Let* $u(x,t)$ *be the solution of* (2.1) *with the initial condition* $u(x,0)=\psi_0(x)=\Omega(x)$, *where* $\Omega(\xi)=\Omega(x/t)$ *is a similarity solution of* (4.2) *with* $\Omega(\xi)\to C\pm D$ *as* $\xi\to\pm\infty$. *Then*

$$|u(x,t)-\Omega(x/t)|\to 0 \textit{ uniformly in } x \qquad \textit{as} \quad t\to\infty.$$

Proof. The proof follows the same lines as in [5].
Let $v=\Omega\{x/(t+1)\}$ and $w=u-v$. Then $w(x,0)=0$ and

$$w_z-\left[w\,\frac{F(u)-F(v)}{(u-v)G}\right]_x=W_{xx}+H_x, \tag{4.10}$$

where

$$H(x,z)=\frac{G-\beta(t+1)}{G}\,v_x, \tag{4.11}$$

and z and G are defined as in (3.3a) and (3.5). The second term on the left-hand side in (4.10) is defined because f is differentiable.
Let $\Gamma(x,t;\xi,\tau)$ be the fundamental solution for

$$\phi_z-\left[\frac{F(u)-F(v)}{(u-v)G}\right]\phi_x=\phi_{xx},\qquad z\geqslant 0. \tag{4.12}$$

Then the solution ψ of

$$\psi_z-\left[\frac{F(u)-F(v)}{(u-v)G}\right]\psi_x=\psi_{xx}+H(x,z), \tag{4.13}$$

with $\psi(x,0)=0$, can be represented in the integral form

$$\psi(x,z)=\int_0^z\int_{-\infty}^{\infty}\Gamma(x,z;\xi,\tau)H(\xi,\tau)\,d\xi\,d\tau. \tag{4.14}$$

Furthermore, $w=\psi_x$ is a solution of (4.10). By the uniqueness of the linear parabolic problem (4.10) with $w(x,0)=0$, we conclude that $w=\psi_x$ is the unique solution of (4.10).
Next we find an estimate of w. We have the inequality

$$\left|\frac{\partial\Gamma}{\partial x}(x,z;\xi,\tau)\right|\leqslant K_1\,\frac{\exp[-K_2(x-\xi)^2/(z-\tau)]}{(z-\tau)}, \tag{4.15}$$

where K_1 and K_2 are constants independent of z, x, ξ, and τ (see the derivation of (6.13), p. 24, in [2]). It follows from the definition of H in (4.11) that

$$H(x,z)=o\left(\frac{1}{t}\right)=o(z^{-1/2})\ \text{uniformly in } x\qquad\text{as}\quad z\to\infty. \tag{4.16}$$

Let

$$I(z)=\int_0^z\max_{-\infty<x<\infty}|H(x,s)|\,ds.$$

Define $\Phi(x, z) = I(z) \pm \psi(x, z)$. We have $\Phi(x, 0) = 0$ and $L\Phi \leqslant 0$, where

$$L \equiv \frac{\partial^2}{\partial x^2} + \frac{F(u) - F(v)}{(u-v)G} \frac{\partial}{\partial x} - \frac{\partial}{\partial z}.$$

Using Theorem 9, p. 43 of Friedman [2], we find

$$I(z) \pm \psi(x, z) \geqslant 0 \qquad \text{for any } x \text{ and } z.$$

This implies that

$$|\psi(x, z)| \leqslant I(z) = o(z^{1/2}) \text{ uniformly in } x \qquad \text{as} \quad z \to \infty. \tag{4.17}$$

Write $\psi(x, z)$ as

$$\begin{aligned} \psi(x, z) = \int_{-\infty}^{\infty} &\Gamma\left(x, z; \xi, \frac{z}{2}\right) \psi\left(\xi, \frac{z}{2}\right) d\xi \\ &+ \int_{z/2}^{Z} \int_{-\infty}^{\infty} \Gamma(x, z; \xi, \tau) H(\xi, \tau)\, d\xi\, d\tau. \end{aligned} \tag{4.18}$$

We thus have

$$\begin{aligned} |w| = \left|\frac{\partial \psi}{\partial x}\right| \leqslant \int_{-\infty}^{\infty} &\left|\Gamma_x\left(x, z; \xi, \frac{z}{2}\right)\right| \left|\psi\left(\xi, \frac{z}{2}\right)\right| d\xi \\ &+ \int_{z/2}^{z} \int_{-\infty}^{\infty} |\Gamma_x(x, z; \xi, \tau)|\, |H(\xi, \tau)|\, d\xi\, d\tau. \end{aligned}$$

In view of (4.15), (4.16), and (4.17), the last inequality can be reduced in the form $|w| = o(1)$ uniformly in x as $z \to \infty$. Obviously $|\Omega[x/(t+1)] - \Omega(x/t)| \to 0$ as $t \to \infty$. Therefore the proof of Lemma 4.4 is complete.

THEOREM 4.1. *Let $\Omega(x/t)$ be a similarity solution of* (4.2) *where $\Omega(\xi) \to C \pm D$ as $|\xi| \to \infty$. Let u be the solution of* (2.1)–(2.2) *with $\psi_0(x) \geqslant \Omega(x)$ or $\psi_0(x) \leqslant \Omega(x)$ for any x and*

$$\int_{-\infty}^{\infty} |\psi_0(x) - \Omega(x)|\, dx < \infty. \tag{4.19}$$

Then

$$|u(x, t) - \Omega(x/t)| \to 0 \textit{ uniformly in } x \qquad \textit{as} \quad t \to \infty.$$

Proof. In general, the Galilean transformation which is needed in the proof of Scott [5] for the case $f(u) = u$ does not satisfy (2.1)–(2.2). The conditions on ψ_0 as stated above are sufficient for the stability of Ω. Let

$U(x, t)$ be the solution of (2.1) with $U(x, 0) = \Omega(x)$. Let $w = u - U$. Then $w(x, 0)$ is either non-negative or non-positive for any x, and w satisfies the equation

$$w_t - \left[w\left(\frac{F(u) - F(U)}{u - U}\right)\right]_x = g(t) w_{xx}. \tag{4.20}$$

By Theorem 9 Friedman [2, p. 43], we have $w(x, t)$ is either non-negative or non-positive for any x and any $t > 0$. Let $\Gamma(x, z; \xi, \tau)$ be the fundamental solution of (4.20). Then $w(x, z)$ can be written in the integral form

$$w(x, z) = \int_{-\infty}^{\infty} \Gamma(x, z; \xi, z/2) w(\xi, z/2)\, d\xi.$$

Using (4.15) and the boundedness of w, we have

$$|w_x(x, t)| \leqslant K/t \tag{4.21}$$

for any x and any t.

Integrating (4.20) with respect to x over $(-\infty, \infty)$, and using the fact that w_x and $F(u) - F(U) \to 0$ as $|x| \to \infty$ for any t, we obtain

$$\frac{d}{dt} \int_{-\infty}^{\infty} w(x, t)\, dx = 0. \tag{4.22}$$

We then integrate this equation with respect to t over $[0, t]$ to obtain

$$\int_{-\infty}^{\infty} w(x, t)\, dx = \int_{-\infty}^{\infty} w(x, 0)\, dx = A. \tag{4.23}$$

Because of the assumption imposed on ψ_0, A is finite and independent of t. We integrate (4.21) over $[x_0, x]$ to find

$$|w(x, t)| \geqslant |w(x_0, t)| - \frac{K(x - x_0)}{t}.$$

For a fixed t, we choose x_0 as the maximum point of $|w(x, t)|$, integrate the last inequality over $[x_0, x_0 + (t/K)\, |w(x_0, t)|]$, and deduce

$$\int_{x_0}^{x_0 + (t/K)|w(x_0, t)|} |w(x, t)|\, dx \geqslant \frac{t}{2K} |w(x_0)|^2.$$

Combining this result with (4.23), we obtain

$$|w(x_0)| \leqslant \left(\frac{2K\, |A|}{t}\right)^{1/2}. \tag{4.24}$$

Since A and K are independent of t, we conclude from (4.24) that $|w(x, t)| \to 0$ uniformly in x as $t \to \infty$. So the application of Lemma 4.4 gives $|u(x, t) - \Omega(x/t)| \to 0$ uniformly in x as $t \to \infty$.

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