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Numerical Stationary Solutions for a Viscous Burgers' Equation*

J. Burns† A. Balogh D.S. Gilliam‡ V.I. Shubov‡

Abstract

This paper is concerned with an interesting numerical anomaly associated with steady state solutions for the viscous Burgers' equation. In particular, we consider Burgers' equation on the interval $(0,1)$ with Neumann boundary conditions. In this work we show that even for moderate values of the viscosity and for certain initial conditions, numerical solutions approach nonconstant shock type stationary solutions. This is rather curious since we also show that the only possible actual stationary solutions are constants. In order to provide a reasonable explanation for this numerical anomaly, we show that the solutions obtained correspond to solutions of a related problem considered recently by L.G. Reyna and M.J. Ward [15].

1 Introduction

In this paper we are concerned with the long time behavior of solutions to Burgers' equation on the interval $(0,1)$ with Neumann boundary conditions. We note that the constants are equilibria for this problem. For the related linearization about zero of the Burgers' equation – one dimensional heat equation with Neumann boundary conditions – it is well known that the steady state temperature is a constant, namely, the mean value of the initial temperature distribution. For the Burgers' equation and small initial data

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this same type of result holds as a consequence of the Center Manifold Theorem. Namely, Burgers' equation with Neumann boundary conditions possesses a one dimensional center manifold (constants) and it can be shown that for a small initial condition the solution converges exponentially to a constant value. In contrast to the heat equation, the steady state constant is not simply the integral of the initial condition, but also depends in some complicated way on both the viscosity parameter and the shape of the initial condition, cf., [6].

Since the Center Manifold Theorem is only a local result, a natural question is whether, for arbitrary initial data, the corresponding solution of Burgers' equation tends to a constant steady state. The answer to this question is still unresolved. In this paper we do answer the intermediate question: If, for a given initial condition, the solution approaches a time independent steady state, even in the $L^2(0,1)$ sense, then is this steady state a constant? The answer, given in Theorem 2.2, is affirmative.

In spite of this result, after considerable numerical testing, one is faced with the reality that, for moderately small viscosity and larger "antisymmetric" (odd about $1/2$ in $(0,1)$) initial conditions, the solutions approach a nonconstant, time independent steady state, cf., [2]. We can only conclude that what we see in practice is simply a numerical anomaly. Nevertheless, due to the relevance of hydrodynamic problems in applications, it is worthwhile endeavor to attempt to understand what these numerical stationary solutions might be.

In particular, numerical calculations have been used to "suggest" that Euler equations do not have unique solution, cf., [12]. The justification for this claim is that a "very fine mesh" is used in the calculation. In this note we suggest that "numerical based" proofs of non-existence must be done with extreme care.

The reason for the anomaly, of course, is that numerical solutions, for a fixed mesh size or degree of approximation, are approaching solutions of the equation that satisfy the boundary conditions to within values that are approximately machine precision zero (or smaller). A similar and related situation can be found in the recent work of L.G. Reyna and M.J. Ward [15] which is, in turn, related to the work of G. Kreiss and H.O. Kreiss [13] on the convergence to steady state of solutions of Burgers' equation with Dirichlet boundary conditions.

The paper is organized into 4 sections. In Section 2 we describe the Burgers' problem, present certain motivating remarks and prove the main result (based on a recent theorem from [4]) concerning the convergence to time independent stationary solutions. In Section 3 we discuss the stationary Burgers' problem and present some related information from [15]. Also in Section 3 we give some numerical illustrations of solutions to the Burgers' problem. Finally in Section 4 we describe a possible mathematical explanation for the observed numerical behavior.

2 Burgers' Equation and Motivating Remarks

Burgers' equation on the interval $(0,1)$ subject to Neumann Boundary Condition is given by the dynamical system

$$\begin{aligned} &w_t - \epsilon w_{xx} + w w_x = 0, \\ &x \in (0,1), \quad t > 0 \\ &w_x(0,t) = w_x(1,t) = 0 \\ &w(x,0) = \phi(x). \end{aligned} \tag{2.1}$$

Clearly, as mentioned in the introduction, $w(x,t) = c$ for $(x,t) \in (0,1) \times [0,\infty)$ for any $c \in \mathbb{R}$ is a stationary solution.

The linearization about zero of (2.1) in $L^2(0,1)$ is the one dimensional heat equation with Neumann boundary conditions

$$\begin{aligned} &w_t = \epsilon w_{xx}, \\ &x \in (0,1), \quad t > 0 \\ &w_x(0,t) = w_x(1,t) = 0 \\ &w(x,0) = \phi(x). \end{aligned} \tag{2.2}$$

A well-known consequence of the Fourier representation of the solution to (2.2)

$$w(x,t) = \phi_0 + 2\sum_{n=1}^{\infty} \exp(-n^2\pi^2 t)\phi_n \cos(n\pi x)$$

$$\phi_0 = \int_0^1 \phi(x)\, dx, \quad \phi_n = \int_0^1 \phi(x)\cos(n\pi x)\, dx, \; n = 1, 2, \cdots$$

is that for any $\epsilon > 0$ and every initial condition $\phi \in L^2(0,1)$

$$\lim_{t\to\infty} w(x,t) = \phi_0.$$

A deeper result for (2.1), based on an infinite dimensional version of the Center Manifold Theorem (cf, [7], [9]), is that for small initial data in $H^1(0,1)$, the solution $w(x,t)$ of (2.1) tends to a constant as $t \to \infty$.

More specifically, the result states the following: For each fixed $\epsilon > 0$ and for small enough initial conditions (the size of this ball shrinks with decreasing ϵ) the solution w to (2.1) can be decomposed as $w(x,t) = c(t) +$

$v(x,t)$, where $c(t) \in I\!R$, $v(x,t) \in H^1_m(0,1)$, the subspace of functions of mean zero,

$$H^1_m(0,1) = \{\phi \in H^1(0,1) \ : \ \int_0^1 \phi(x)\,dx = 0\}.$$

Furthermore, there exists a constant $c_{\phi,\epsilon}$ such that

$$c(t)@>t\to\infty>>c_{\phi,\epsilon},$$

and there exist constants γ and $M > 0$, such that $\|v(\cdot,t)\|_{H^1} \ \le \ Me^{-\gamma t}$. Thus the solution to Burgers' equation satisfies

$$w(x,t) = c(t) + v(x,t)@>t\to\infty>>c_{\phi,\epsilon}.$$

The way in which $c_{\phi,\epsilon}$ depends upon ϵ and ϕ is not simple as it is in the case of the heat equation (cf., [6]) and this explicit relationship, to our knowledge, remains an unsolved problem.

Since the Center Manifold Theorem is a local result we still cannot say anything about the long time behavior of solutions (2.1) for larger initial data. In fact, it is not clear without further information that solutions should even exist for all time. The answer to this question is contained in the recent work [4] which for the special case of (2.1) gives the following result.

Theorem 2.1 *[4] For (2.1) with arbitrary initial data $\varphi \in L^2(0,1)$ and $0 < T < \infty$,*

a) There is a unique weak solution

$$w \in L^\infty([0,T], L^2(0,1)) \cap L^2([0,T], H^1(0,1)),$$

b) $w \in H^{2,1}([0,1]\times[t_0,T])$ for any $0 < t_0 < T < \infty$. Here $H^{2,1}([0,1]\times[t_0,T])$ consists of functions possessing square integrable derivatives up to order 2 with respect to x and 1 with respect to t.

c) There is a globally defined dynamical system on the state space $L^2(0,1)$.

d) The dynamics define a nonlinear semigroup $\{T_t, t \ge 0\}$.

* *T_t is continuous in t and $\varphi \in L^2(0,1)$.*
* *T_t is compact for $t > 0$.*
* *There exists a positive continuous monotone increasing function $a(\xi)$, $\xi \ge 0$ such that $a(0) = 0$ and*

$$\|T_t\varphi\| \le a(\|\varphi\|), \ t \in [0,\infty), \ \varphi \in L^2(\Omega),$$

which means that the system is globally Lyapunov stable.

* *There is a global, locally compact attractor.*

We should comment that since the attractor contains all stationary solutions and, as we have already mentioned above, every scalar is a stationary solution, the attractor is unbounded. Due to Theorem 2.1 it is locally compact. The exact composition of the attractor is still not known but we expect that it consists of the one dimensional subspace consisting of constants.

We are now in a position to answer the question posed in the introduction. Even though we cannot, at this time, say that solutions of (2.1) approach a constant steady state, we can prove the following intermediate result.

Theorem 2.2 *Fix $\epsilon > 0$ and $\phi \in L^2(0,1)$. Let $w(\cdot,t)$ be a weak solution of (2.1). If there is a function $h \in L^2(0,1)$ such that*

$$\lim_{t\to\infty} \|w(\cdot,t) - h(\cdot)\|_{L^2(0,1)} @ > t \to \infty >> 0$$

Then $h(\cdot) = c_{\phi,\epsilon}$ for some constant $c_{\phi,\epsilon}$.

Note that because it is only assumed that the solution $w(x,t)$ converge in the L^2-norm as t tends to infinity, we cannot immediately conclude that h is a stationary solution of Burgers' equation. However, based on the result stated above from [4], we can conclude that such a limit must be a stationary solution.

Proof: Fix $\epsilon > 0$, and let $\phi \in L^2(0,1)$. From Theorem 2.1 and our hypothesis that the solution converges to a time independent function in the L^2 sense, we conclude that

$$\lim_{t_1\to\infty} T_{t_1}(\phi) = h.$$

Also, since for any $t > 0$, T_t is continuous and T_t is a semigroup, we have

$$T_t(h) = \lim_{t_1\to\infty} T_t(T_{t_1}(\phi)) = \lim_{t_1\to\infty} T_{t_1+t}(\phi) = h.$$

Therefore for every $t > 0$

$$T_t(h) = h.$$

That is, h is a weak stationary solution of the system (2.1).

On the other hand, a weak stationary solution must satisfy

$$\left(-\epsilon v_x + \frac{v^2}{2}\right)_x = 0, \tag{2.3}$$

where all the derivatives are understood as weak derivatives and the equality holds a.e.. One possibility is that v is a constant, in which case we have,

$$c_0 = \frac{v^2}{2}. \tag{2.4}$$

Clearly a constant provides a stationary solution since, in addition, it satisfies the boundary conditions.

Now, the only distributional solution of the equation

$$\psi' = 0$$

is a constant, we see that any other stationary solution to (2.1) must satisfy

$$-\epsilon v_x + \frac{v^2}{2} = c_0, \;\; c_0 \in \mathbb{R}. \tag{2.5}$$

This equation can be solved explicitly and we obtain:

$$v(x) = \sqrt{2c_0}\tanh\left(\frac{\sqrt{2c_0}}{2\epsilon}(c_1 - x)\right), \tag{2.6}$$

where c_0 and c_1 are arbitrary constants.

A straightforward calculation gives

$$v_x(x) = -\frac{c_0}{\epsilon}\operatorname{sech}^2\left(\frac{\sqrt{2c_0}}{2\epsilon}(c_1 - x)\right), \tag{2.7}$$

which cannot vanish at $x = 0$ or $x = 1$ (unless $c_0 = 0$). Thus the only stationary solutions are constants. Q.E.D.

3 Numerical Stationary Solutions

For fixed ϵ and for small initial data numerical approximation of the solutions to (2.1) supports the conclusion of the Center Manifold Theorem, namely, solutions tend to a constant as t tends to infinity. But for small ϵ and "certain" initial data (not to small), the numerical solution converges to a nonconstant function, cf. [2]. These same nonconstant steady state limits are readily obtained using many different numerical algorithms and on various different computer platforms. We are lead to conjecture the existence of some type of *Numerical Stationary Solutions* for the problem (2.1).

One class of initial data for which we obtain this anomaly are functions in the class $\mathcal{S}$ consisting of "antisymmetric" functions, that is, functions that are odd about $x = 1/2$ in the interval $(0, 1)$,

$$\mathcal{S} = \left\{\phi \in L^2(0,1) \,:\, \phi(x) = -\phi(1-x)\right\}. \tag{3.1}$$

For initial data $\phi \in \mathcal{S}$, a straightforward consequence of Theorem 2.1, is that $w(\cdot, t) \in \mathcal{S}$ for all t. This can easily be seen from the uniqueness and the fact that if $\phi \in \mathcal{S}$ and $w(x,t)$ is the solution of (2.1), then the function $z(x,t) = -w(1-x,t)$ also satisfies (2.1) and hence

$$w(x,t) = -w(1-x,t)$$

i.e., $w(\cdot, t) \in \mathcal{S}$. Note that a continuous function ϕ in $\mathcal{S}$ must satisfy $\phi(1/2) = 0$ and so, for $t > 0$ a solution with initial data $\phi \in \mathcal{S}$ will satisfy $w(1/2, t) = 0$ for all $t > 0$. Thus if

$$\lim_{t \to 0} w(x,t) = c_{\phi,\epsilon}$$

exists then the constant $c_{\phi,\epsilon}$ must be zero.

The nonconstant solutions to the stationary Burgers' equation (not the boundary conditions) given in (2.6) form a two parameter family depending on the parameters c_0 and c_1. In order that such a function be in $\mathcal{S}$ it follows that $c_1 = 1/2$. Define,

$$h(x) = \sqrt{2c_0} \tanh\left(\frac{\sqrt{2c_0}}{2\epsilon}(1/2 - x)\right), \ h \in \mathcal{S}. \tag{3.2}$$

We now demonstrate that for suitable initial data and c_0 the functions (3.2) are actually *numerical stationary solutions* to (2.1), i.e., they satisfy the Burgers' equation and they approximately satisfy the boundary conditions (to within exponentially small terms).

Namely, the functions in (3.2) satisfy (2.5) and

$$h'(x) = -\frac{c_0}{\epsilon} \operatorname{sech}^2\left(\frac{\sqrt{2c_0}}{2\epsilon}(1/2 - x)\right), \tag{3.3}$$

which for small ϵ and/or large c_0 gives

$$h'(0) = h'(1) = -\frac{c_0}{\epsilon} \operatorname{sech}^2\left(\frac{\sqrt{2c_0}}{4\epsilon}\right) = -\gamma, \tag{3.4}$$

where γ is an exponentially small positive number.

There is no reason to believe that numerical solutions to Burgers' equation should approach a function of the type (3.2), especially in light of Theorem 2.2 which suggests they should approach a constant. Nevertheless, this does happen for larger initial data and/or smaller ϵ.

One possible numerical explanation for this behavior can be found in the work of L.G. Reyna and M.J. Ward [15]. The authors are primarily interested in the following problem

$$\begin{aligned} &w_t - \epsilon w_{xx} + w w_x = 0, \\ &x \in (0,1), \quad t > 0 \\ &w(0,t) = \alpha > 0, \\ &w(1,t) = -\alpha \\ &w(x,0) = \phi(x). \end{aligned} \tag{3.5}$$

It is shown in [15] that, just as above, there is a one parameter family of solutions of the associated stationary equation

$$-\epsilon u_{xx} + u u_x = 0 \tag{3.6}$$

given by

$$u(x) = -\alpha \tanh(\alpha\epsilon^{-1}(x - x_0)/2), \quad x_0 \in (0,1),$$

but these functions only satisfy the boundary conditions to within exponentially small terms for all x_0.

In order to obtain a problem for which the boundary conditions are satisfied exactly, the authors in [15] replace the problem (3.5) by the problem

$$\begin{aligned} &w_t - \epsilon w_{xx} + w w_x = 0, \\ &x \in (0,1), \quad t > 0 \\ &-\epsilon w_x(0,t) + \kappa[w(0,t) - \alpha] = 0 \\ &\epsilon w_x(1,t) + \kappa[w(1,t) + \alpha] = 0 \\ &w(x,0) = \phi(x), \text{ with } \alpha,\ \kappa > 0. \end{aligned} \tag{3.7}$$

For (3.7), the associated stationary problem

$$\begin{aligned} &-\epsilon u_{xx} + u u_x = 0, \\ &-\epsilon u_x(0) + \kappa[u(0) - \alpha] = 0 \\ &\epsilon u_x(1) + \kappa[u(1) + \alpha] = 0 \end{aligned} \tag{3.8}$$

has a nontrivial stationary solution given by

$$u(x) = -\beta \tanh(\beta\epsilon^{-1}(x - 1/2)/2) \tag{3.9}$$

provided that β is chosen to satisfy the transcendental equation

$$-\frac{\beta^2}{2}\operatorname{sech}^2\left(\frac{\beta\epsilon^{-1}}{4}\right) + \kappa\left[\alpha - \beta\tanh\left(\frac{\beta\epsilon^{-1}}{4}\right)\right] = 0. \tag{3.10}$$

For $\epsilon \approx 0$ the authors give the following asymptotic formula

$$\beta \sim \alpha + 2\alpha\left(1 - \frac{\alpha}{\kappa}\right)e^{-\alpha\epsilon^{-1}/2} + \cdots .$$

Furthermore, they show that the largest eigenvalue of this linearization about the equilibrium solution satisfies

$$\lambda_0 \sim -2\alpha^2 \left(1 - \frac{\alpha}{\kappa}\right) e^{-\alpha\epsilon^{-1}/2} + \cdots$$

which is negative for $\kappa > \alpha$ and positive for $\kappa < \alpha$.

It turns out that numerically this problem is related to our problem, at least to within exponentially small error terms in satisfying the boundary conditions. Most importantly, and for reasons which we cannot explain, *for small ϵ and larger initial conditions, numerical solutions to (2.1) can approach a stationary function given by (3.9) for suitable choices of α and β.*

The main difference in these problems is that for (2.1) we cannot, at this time, predict the values of α and hence β in (3.9) and (3.10). This translates into not knowing the appropriate limiting value

$$\lim_{t\to\infty} w(0,t) = v(0). \tag{3.11}$$

Here $v(0)$ is the value at $x = 0$ of the limiting stationary solution given in (2.3) which turns out to be strongly dependent on the initial condition for (2.1). The value of $v(0)$ is related to the indeterminacy of the constant c_0 in (2.5),

$$c_0 = -\epsilon v_x(0) + \frac{v^2(0)}{2}$$

which in turn is related to the values of α in (3.7) and β in (3.9).

The upshot is that if we numerically solve (2.1) to obtain a value $v(0)$ from (3.11) and define $\alpha = v(0)$ then, for almost any value of $\kappa > \alpha$, and β determined from (3.10), we see that, to within exponentially small terms, the stationary solution obtained numerically coincides with (3.9).

We now present several numerical examples of the above discussion. These numerical exercises were carried out using a Gear method. For relatively large values of ϵ and and not to large initial conditions ϕ, numerical solutions behave as predicted by by Theorem 2.2. We consider $\epsilon = .1$ and initial conditions $\phi(x) = C\cos(\pi x)$ and $\phi(x) = C(1/2 - x)^3$ (note that these ϕ are antisymmetric so we expect the solution to converge to zero). For smaller values of C we obtain the results depicted in Figures 1 and 2.

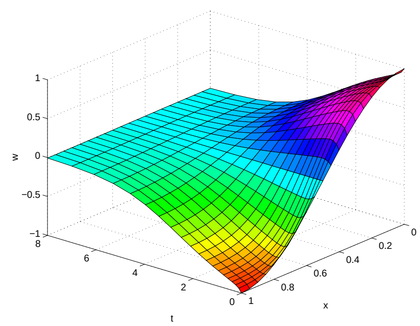


Figure 1: $\epsilon = .1$, $\phi = \cos(\pi x)$, trajectories tend to zero

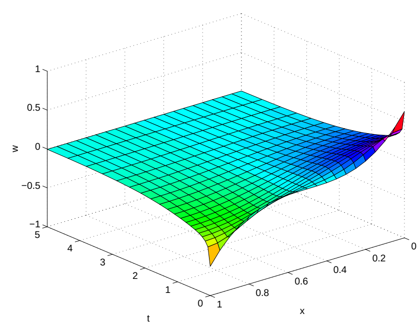


Figure 2: $\epsilon = .1$, $\phi(x) = 5(1/2 - x)^3$, trajectories tend to zero

On the other hand, for the same $\epsilon = .1$ and slightly larger values of C in the initial conditions we have the results depicted in Figures 3 and 4.

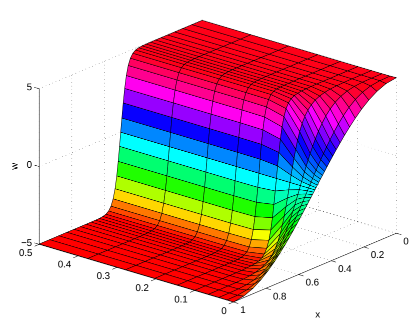


Figure 3: $\phi(x) = 5\cos(\pi x)$ trajectories do not tend to zero

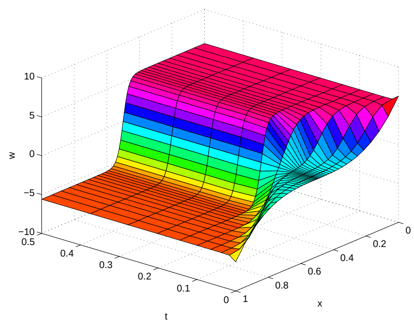
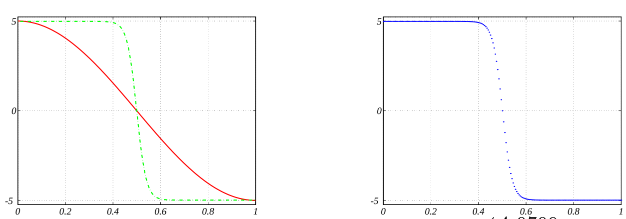


Figure 4: $\phi(x) = 50(1/2 - x)^3$, trajectories do not tend to zero

On the left in Figures 5 and 7 we have plots of the initial conditions and the corresponding solution for large t. On the right we have plotted the corresponding stationary solution (3.9) for suitable β. In Figures 6 and 8 we have plotted the difference between the numerical stationary solution at $t = .5$ and the corresponding hyperbolic tangent function (3.9).



$\phi(x) = 5\cos(\pi x)$ $\qquad$ $4.9799\tanh\left(\frac{4.9799}{2\epsilon}(1/2 - x)\right)$

Figure 5:

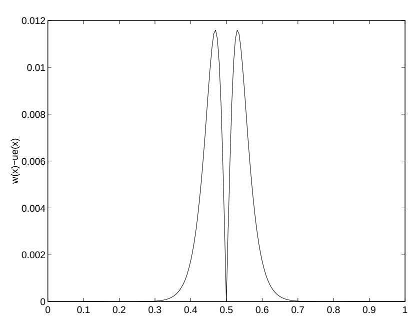
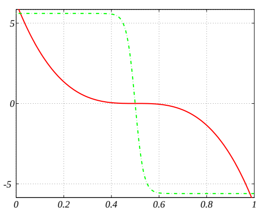
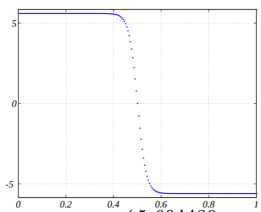


Figure 6:



$\phi(x) = \phi(x) = 50(1/2 - x)^3$



$5.604439 \tanh\left(\frac{5.604439}{2\epsilon}(1/2 - x)\right)$

Figure 7:

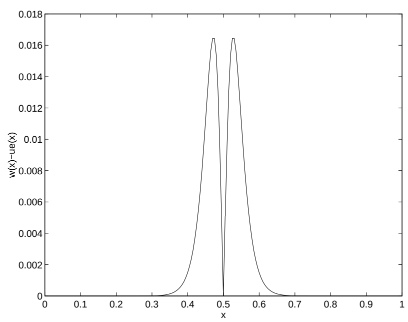


Figure 8:

4 An Alternate Explanation

As noted in section 3 the function (3.2) satisfies stationary Burgers' equation and the derivative at $x = 0, 1$ satisfies (3.4). For a small positive number γ let us consider replacing the stationary problem (2.5) with Neumann boundary conditions by the following problem:

$$-\epsilon h_x + \frac{h^2}{2} = c_0, \;\; c_0 \in I\!R \tag{4.1}$$

where we seek a solution in the class $\mathcal{S}$ subject to the boundary conditions

$$h'(0) = h'(1) = -\gamma. \tag{4.2}$$

It is easy to see the problem (4.1), (4.2) can only have solutions in the form (3.2) where c_0 is chosen to satisfy the boundary condition (4.2).

In the space $\mathcal{S}$ (odd functions about 1/2) there are exactly two solutions of (2.5) for γ small enough: Namely, there exist $c_0^< \approx 0$ and $c_0^> \gg 0$ giving

$$h^<(x) = \sqrt{2c_0^<} \tanh\left(\frac{\sqrt{2c_0^<}}{2\epsilon}(1/2 - x)\right) \tag{4.3}$$

$$h^>(x) = \sqrt{2c_0^>} \tanh\left(\frac{\sqrt{2c_0^>}}{2\epsilon}(1/2 - x)\right) \tag{4.4}$$

and these functions satisfy

$$h_x(0) = -\gamma, \quad h_x(1) = -\gamma$$

with $c_0^<$ and $c_0^>$ chosen to satisfy

$$\frac{c_0}{\epsilon} \operatorname{sech}^2\left(\frac{\sqrt{2c_0}}{4\epsilon}\right) = \gamma. \tag{4.5}$$

To see that there are exactly two such values of c_0 for small γ let $s = \sqrt{c_0}/(2\sqrt{2}\epsilon)$ so that equation (3.4) becomes

$$s^2 \operatorname{sech}^2(s) = \frac{\gamma}{8\epsilon}.$$

The function $f(s) = s^2 \operatorname{sech}^2(s)$ has a critical value at $s_0 \approx 1.2$. This allows us to conclude that the maximum value of f is $M_\epsilon = 8\epsilon s_0^2 \operatorname{sech}^2(s_0)$. From the graph of f in Figure 7, it is clear that this maximum imposes a smallness constraint on γ. Namely, in order for the boundary condition in (3.4) to be satisfied, we need

$$\gamma \leq M_\epsilon.$$

For fixed ϵ and γ sufficiently small, we see that there are two solutions given by (4.3) and (4.4) and both functions satisfy the boundary conditions

$$h_x(1) = h_x(0) = -\gamma.$$

The solution $h^<$ is very nearly the zero function, whereas the solution $h^>$ is not usually small.

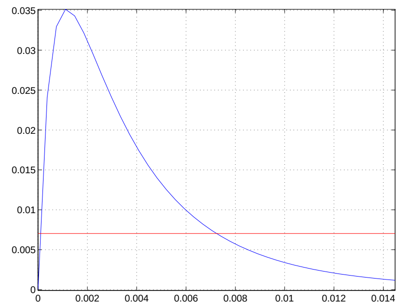


Figure 9: Graph of f, for $\epsilon = .01$ and $\gamma = .007$

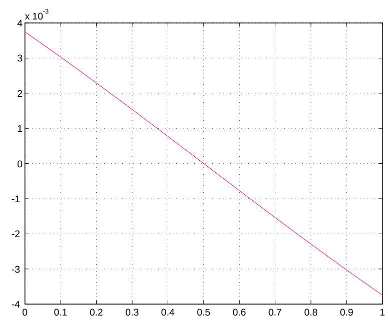


Figure 10: Graph of $h^{<}$, for $\epsilon = .01$ and $c_0 = 7.7287e-05$

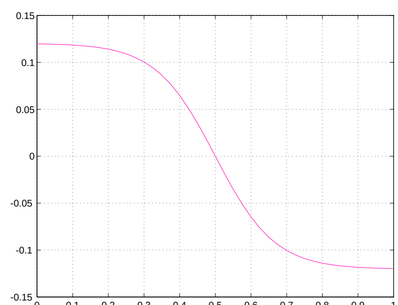


Figure 11: Graph of $h^{>}$, for $\epsilon = .01$ and $c_0 = .0072$

In each of Figures 10 and 11 there is actually two functions plotted, h computed numerically and also from the formula (3.2).

A complete analysis of the mathematical validity of these stationary solutions for Burgers equation would involve a careful analysis of the long time behavior of solutions to the dynamical system:

$$w_t - \epsilon w_{xx} + w w_x = f_\gamma$$
$$w_x(0,t) = w_x(1,t) = 0,$$
$$w(x,0) = \phi(x),$$
$$f_\gamma = \gamma(\delta_0 - \delta_1) \in H^{-1}(0,1)$$

where by δ_a we denote the δ-function concentrated at $x = a$, and by $H^{-1}(0,1)$ we denote the dual of $H^1(0,1)$ which consists of all distributions from $H^{-1}(I\!R)$ whose support belongs to $[0,1]$. For small γ and small initial conditions ϕ result from [3] imply the global in time existence of solutions to the above system and the existence of a compact local attractor. Unfortunately, for larger initial conditions the results of [3] do not apply for f_γ as above.

Furthermore, it is not easy to numerically test whether these are actually the stationary solutions obtained in Section 2 since realistic values of γ, for most problems of interest, a much smaller than machine precision zero.

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Interdisciplinary Center for Applied Mathematics and Department of Mathematics, VPI & SU, Blacksburg, VA, 24061-0131

Department of Mathematics, Texas Tech University, Lubbock, TX 79409

Department of Mathematics, Texas Tech University, Lubbock, TX 79409

Department of Mathematics, Texas Tech University, Lubbock, TX 79409

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