

FEEDBACK STABILIZATION OF THE 3-D NAVIER-STOKES EQUATIONS BASED ON AN EXTENDED SYSTEM

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Abstract

We study the local exponential stabilization of the 3D Navier-Stokes equations in a bounded domain, around a given steady-state flow, by means of a boundary control. We look for a control so that the solution to the Navier-Stokes equation be a strong solution. In the 3D case, such solutions may exist if the Dirichlet control satisfies a compatibility condition with the initial condition. In order to determine a feedback law satisfying such a compatibility condition, we consider an extended system coupling the Navier-Stokes equations with an equation satisfied by the control on the boundary of the domain. We determine a linear feedback law by solving a linear quadratic control problem for the linearized extended system. We show that this feedback law also stabilizes the nonlinear extended system.

Keywords: Navier-Stokes equation, Feedback stabilization, Riccati equation.

1 Introduction

Let \mathcal{O} and \mathcal{B} two regular bounded domains of class C^∞ in \mathbf{R}^3 such that $\overline{\mathcal{B}} \subset \mathcal{O}$, $\Omega = \mathcal{O} \setminus \overline{\mathcal{B}}$, $\Gamma_e = \partial\mathcal{O}$ and $\Gamma_i = \partial\mathcal{B}$. We have $\Gamma_i \cap \Gamma_e = \emptyset$ and $\partial\Omega = \Gamma_i \cup \Gamma_e$. We consider the motion of an incompressible fluid around the bounded body \mathcal{B} in Ω which is described by the couple (z_e, p_e) , the velocity

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and the pressure, solution to the stationary Navier-Stokes equations

$$\begin{aligned} -\Delta z_e + (z_e \cdot \nabla) z_e + \nabla p_e &= 0 \text{ and } \nabla \cdot z_e = 0 \text{ in } \Omega, \\ z_e &= 0 \text{ on } \Gamma_i, \quad z_e = v_\infty \text{ on } \Gamma_e. \end{aligned}$$

According to [4], if $v_\infty \in H^{\frac{3}{2}}(\Gamma_i; \mathbf{R}^3)$ obeys $\int_{\Gamma_e} v_\infty \cdot n = 0$, such a stationary solution exists in $H^2(\Omega, \mathbf{R}^3) \times H^1(\Omega)/\mathbf{R}$. For an initial condition of the form $z_e + z_0$ and a Dirichlet boundary control u on Γ_i such that $\int_{\Gamma_i} u(t) \cdot n = 0$, the pair $(z + z_e, p + p_e)$ satisfies the instationary Navier-Stokes equations, and (z, p) obeys:

$$\partial_t z - \Delta z + (z \cdot \nabla) z_e + (z_e \cdot \nabla) z + (z \cdot \nabla) z + \nabla p = 0 \text{ in } Q, \quad (1)$$

$$\nabla \cdot z = 0 \text{ in } Q, \quad z = u \text{ on } \Sigma_i, \quad z = 0 \text{ on } \Sigma_e, \quad z(0) = z_0. \quad (2)$$

In this setting $Q = \Omega \times (0, \infty)$, $\Sigma_i = \Gamma_i \times (0, \infty)$, $\Sigma_e = \Gamma_e \times (0, \infty)$ and n denotes the unit normal vector to Γ_i , exterior to Ω . We assume that z_e is an unstable solution of (1)-(2) corresponding to $z_0 = z_e$. Our goal is to find a Dirichlet boundary control u on Γ_i which stabilizes the instationary Navier-Stokes system (1)-(2) for initial data z_0 small enough in an appropriate functional space. To achieve this goal, the three dimensional case is highly demanding in terms of velocity regularity: we need that $z \in L^2(0, \infty; \mathbf{H}^{\frac{3}{2}}(\Omega))$ to obtain a stabilization result. Therefore, we look for a control u regular enough to fit the expected smoothness of z and in particular, the initial compatibility condition $u(0) = z_0|_{\Gamma_i}$ should be satisfied. A way to obtain such compatibility condition is to characterize the trace u as the first component of $(u, \sigma) \in L^2_{loc}([0, \infty); L^2(\Gamma_i; \mathbf{R}^3)) \times L^2_{loc}([0, \infty))$, where (u, σ) is the solution to the time dependent equation:

$$\begin{aligned} \partial_t u &= \Delta_b u + \sigma n + g \quad \text{on } \Sigma_i, \\ u(0) &= z_0|_{\Gamma_i}, \quad \text{and } \int_{\Gamma_i} u(t) \cdot n = 0. \end{aligned}$$

Here Δ_b is a Laplace Beltrami operator and $g \in L^2(\Sigma_i; \mathbf{R}^3)$ is such that $g(t)$ obeys $\int_{\Gamma_i} g(t) \cdot n = 0$. Thus the state (z, u) now satisfy an extended system of two coupled equations with a distributed control g on Γ_i :

$$\partial_t z - \Delta z + (z \cdot \nabla) z_e + (z_e \cdot \nabla) z + (z \cdot \nabla) z + \nabla p = 0 \text{ in } Q, \quad (3)$$

$$\nabla \cdot z = 0 \text{ in } Q, \quad z = u \text{ on } \Sigma_i, \quad z = 0 \text{ on } \Sigma_e, \quad z(0) = z_0, \quad (4)$$

$$\partial_t u - \Delta_b u - \sigma n = g \text{ in } \Sigma_i, \quad \int_{\Gamma_i} u(t) \cdot n = 0, \quad u(0) = z_0|_{\Gamma_i}. \quad (5)$$

In a first step, we consider the linear problem derived from this last coupled system by dropping the nonlinear term $(z \cdot \nabla) z$. We introduce the velocity

space $V_n^0(\Omega) = \{y \in L^2(\Omega; \mathbf{R}^3) \mid \nabla \cdot y = 0 \text{ in } \Omega, y \cdot n = 0 \text{ on } \partial\Omega\}$, and the orthogonal projector P from $L^2(\Omega; \mathbf{R}^3)$ into $V_n^0(\Omega)$. Next we rewrite the extended system as an evolution equation (see section **2.2**) involving a linear unbounded operator \mathcal{A} which is studied in section **4**. Then we state a linear quadratic optimal control problem (see section **2.3**) which provides a distributed feedback controller for the extended system (see section **5**). Finally, we apply the feedback controller to the initial nonlinear system (see section **6**) and we show a local stabilization result.

2 Extended system and optimal control problem

2.1 Functional framework

Let us define the spaces of free divergence functions

$$\begin{aligned} V^s(\Omega) &= \{y \in H^s(\Omega; \mathbf{R}^3) \mid \nabla \cdot y = 0 \text{ in } \Omega, \int_{\partial\Omega} y \cdot n = 0\}, \quad s \geq 0, \\ V_n^s(\Omega) &= \{y \in H^s(\Omega; \mathbf{R}^3) \mid \nabla \cdot y = 0 \text{ in } \Omega, y \cdot n = 0 \text{ on } \partial\Omega\}, \quad s \geq 0, \end{aligned}$$

and the corresponding trace spaces with a free mean normal component

$$V^s(\Gamma_i) = \{y \in H^s(\Gamma_i; \mathbf{R}^3) \mid \int_{\Gamma_i} y \cdot n = 0\}, \quad V^{-s}(\Gamma_i) = V^s(\Gamma_i)', \quad s \geq 0.$$

We denote by $V_0^s(\Omega)$ the interpolation space $[V^2(\Omega) \cap H_0^1(\Omega; \mathbf{R}^3), V_n^0(\Omega)]_{1-s/2}$ for $0 \leq s \leq 2$ and $V^{-s}(\Omega) = V_0^s(\Omega)'$ its dual counterpart with respect to the pivot space $V_n^0(\Omega)$. It is well known that

$$\begin{aligned} V_0^s(\Omega) &= V_n^s(\Omega), \quad 0 \leq s < \frac{1}{2}, \\ V_0^{\frac{1}{2}}(\Omega) &= \{y \in V_n^{\frac{1}{2}}(\Omega) \mid \int_{\Omega} \rho(x)^{-1} |y|^2 < +\infty\}, \\ V_0^s(\Omega) &= \{y \in V_n^s(\Omega) \mid y = 0 \text{ on } \partial\Omega\}, \quad \frac{1}{2} < s \leq 2, \end{aligned}$$

where $\rho(x)$ is the distance from x to $\partial\Omega$. Notice that, according to the above definition, we have $V_0^s(\Omega) = V^s(\Omega) \cap H_0^1(\Omega; \mathbf{R}^3)$ for $1 \leq s \leq 2$. Finally, for $0 < T \leq \infty$, and X_1 and X_2 two Banach spaces, we introduce the function space

$$W(0, T; X_1, X_2) = L^2(0, T; X_1) \cap H^1(0, T; X_2).$$

2.2 Abstract formulation of the extended system

In this section, we state an abstract weak formulation for the system

$$\partial_t z - \Delta z + (z \cdot \nabla) z_e + (z_e \cdot \nabla) z + \kappa(z \cdot \nabla) z + \nabla p = 0 \text{ in } Q, \quad (6)$$

$$\nabla \cdot z = 0 \text{ in } Q, \quad z = u \text{ on } \Sigma_i, \quad z = 0 \text{ on } \Sigma_e, \quad z(0) = z_0 \in V^0(\Omega), \quad (7)$$

$$\partial_t u - \Delta_b u - \sigma n = g \text{ in } \Sigma_i, \quad u(0) = u_0 \in V^{-\frac{1}{2}}(\Gamma_i). \quad (8)$$

Equation (6) corresponds to the Oseen equation if $\kappa = 0$, and to the Navier-Stokes equation if $\kappa = 1$. Equation (6), the left hand side of (8) and (7) are satisfied in the sense of distributions. We observe that σ plays the role of the Lagrange multiplier associated with the constraint $\int_{\Gamma_i} u \cdot n$.

By using transformations developed in [6] we are going to give an equivalent formulation of (6)-(7)-(8). First we define the unbounded operator $(\mathcal{D}(A), A) = (V_0^2(\Omega), A(x, \partial))$ where $A(x, \partial) = P\Delta - P(\nabla z_e) - P(z_e \cdot \nabla)$. We choose $\lambda_0 > 0$ such that $\langle (\lambda_0 - A)y | y \rangle \geq \frac{1}{2} \|y\|_{V_0^1(\Omega)}^2$ for all $y \in V_0^1(\Omega)$, and we introduce the Dirichlet operator $D \in \mathcal{L}(V^0(\Gamma_i), V^0(\Omega))$ associated with $\lambda_0 - A$: $Du = w$ where $u \in V^{\frac{1}{2}}(\Omega)$ satisfies

$$\lambda_0 w - A(x, \partial)w = 0, \quad \nabla \cdot w = 0 \quad \text{in } \Omega, \quad w = u \text{ on } \Gamma_i \quad \text{and} \quad w = 0 \text{ on } \Gamma_e.$$

Thus, for $z \in L^{12/5}(\Omega; \mathbf{R}^3)$ we define $b(z, z) \in L^2(0, T; \mathcal{D}(A^*)')$ by

$$\langle b(z, z) | v \rangle = \int_{\Omega} (\nabla v) z \cdot z, \quad \forall v \in \mathcal{D}(A^*).$$

Finally, we define the unbounded operator $(\mathcal{D}(A_b), A_b) = (V^2(\Gamma_i), P_b \Delta_b)$ in $V^0(\Gamma_i)$, where $P_b \in \mathcal{L}(L^2(\Gamma_i, \mathbf{R}^3), V^0(\Gamma_i))$ denotes the orthogonal projector from $L^2(\Gamma_i; \mathbf{R}^3)$ into $V^0(\Gamma_i)$.

Definition 1 *We shall say that $(z, u) \in L^2(0, T; V^0(\Omega)) \times L^2(0, T; V^{-\frac{1}{2}}(\Gamma_i))$ if $\kappa = 0$, or $(z, u) \in L^2(0, T; V^0(\Omega) \cap L^{12/5}(\Omega, \mathbf{R}^3)) \times L^2(0, T; V^{-\frac{1}{2}}(\Gamma_i))$ if $\kappa = 1$, is a weak solution to (6)-(7)-(8) if and only if it obeys the system:*

$$(Pz)' = APz + (\lambda_0 - A)PDu + \kappa b(z, z) \in L^2(0, T; \mathcal{D}(A^*)'), \quad (9)$$

$$u' = A_b u + g \in L^2(0, T; \mathcal{D}(A_b)'), \quad (10)$$

$$Pz(0) = Pz_0 \in V_n^0(\Omega), \quad u(0) = u_0 \in V^{-\frac{1}{2}}(\Gamma_i), \quad (11)$$

$$(I - P)z = (I - P)Du \in L^2(0, T; V^0(\Omega)), \quad u_0 \cdot n = z_0 \cdot n. \quad (12)$$

Theorem 2 *Let $(z, p, u, \sigma) \in W(0, T; V^1(\Omega), V^{-1}(\Omega)) \times L^2(0, T; L^2(\Omega)/\mathbf{R}) \times W(0, T; V^{\frac{1}{2}}(\Gamma_i), V^{-\frac{3}{2}}(\Gamma_i)) \times L^2(0, T)$. Then (z, p, u, σ) satisfies (6)-(7)-(8) if and only if (z, u) satisfy (9)-(10)-(11)-(12).*

According to [6], the right hand side of (12) is equivalent to $(I - P)z_0 = (I - P)Du_0$. Then (12) ensures that the couple (z, z_0) is entirely determined by its projected part (Pz, Pz_0) and the boundary values (u, u_0) . In the following we only consider the new 'extended' state $Y = (Pz, u)$ and the initial condition $Y_0 = (Pz_0, u_0)$. We define $\mathcal{H}^0 = V_n^0(\Omega) \times V^{-\frac{1}{2}}(\Gamma_i)$ and an

adequate unbounded operator $(\mathcal{D}(\mathcal{A}), \mathcal{A})$ in $\mathcal{H}^0 - \mathcal{A}$ is defined by (19), (20) and studied in section 4. We introduce the bilinear operator B

$$B(Y_1, Y_2) = \begin{pmatrix} b(y_1 + (I - P)Du_1, y_2 + (I - P)Du_2) \\ 0 \end{pmatrix}. \quad (13)$$

Theorem 3 *Let $(z, p, u, \sigma) \in W(0, T; V^1(\Omega), V^{-1}(\Omega)) \times L^2(0, T; L^2(\Omega)/\mathbf{R}) \times W(0, T; V^{\frac{1}{2}}(\Gamma_i), V^{-\frac{3}{2}}(\Gamma_i)) \times L^2(0, T)$. Then (z, p, u, σ) satisfies (6)-(7)-(8) if and only if (12) holds true and the state $Y = (Pz, u)$ satisfies*

$$Y' = \mathcal{A}Y + \kappa B(Y, Y) + \begin{pmatrix} 0 \\ g \end{pmatrix} \in L^2(0, T; \mathcal{D}(\mathcal{A}^*)'), \quad Y(0) = Y_0 \in \mathcal{H}^0. \quad (14)$$

2.3 The extended system and the linear quadratic control problem

The feedback control law is obtained by studying the control problem

$$(\mathcal{Q})_{z_0, u_0}^\infty \quad \inf\{ \mathcal{I}(g) \mid g \in L^2(0, \infty; V^0(\Gamma_i)) \},$$

where

$$\mathcal{I}(g) = \frac{1}{2} \int_0^\infty \int_\Omega |\nabla z_g|^2 + \frac{1}{2} \int_0^\infty \int_{\Gamma_i} |g|^2,$$

and $(z_g, u_g) \in W(0, \infty; V^1(\Omega) \times V^{\frac{1}{2}}(\Gamma_i), V^{-1}(\Omega) \times V^{-\frac{3}{2}}(\Gamma_i))$ satisfies (6)-(7)-(8). If we introduce $\mathcal{C}Y = \nabla(y + (I - P)Du) \in \mathcal{Z}$ with $\mathcal{Z} = L^2(\Omega; \mathbf{R}^9)$ - \mathcal{C} is studied in section 4 lemma 9 - we can rewrite $(\mathcal{Q})_{z_0, u_0}^\infty$ in the form:

$$(\mathcal{P}_{Y_0}^\infty) \quad \inf\{ \mathcal{J}(g) \mid g \in L^2(0, \infty; V^0(\Gamma_i)) \},$$

where

$$\mathcal{J}(g) = \frac{1}{2} \int_0^\infty \|\mathcal{C}Y_g\|_{\mathcal{Z}}^2 + \frac{1}{2} \int_0^\infty \int_{\Gamma_i} |g|^2,$$

and Y_g satisfies (14) for $\kappa = 0$.

3 Main result

Theorem 4 *Let Π_2 and Π_3 be the operator defined in (35). Consider the following coupled system,*

$$\partial_t z - \Delta z + (z \cdot \nabla)z_e + (z_e \cdot \nabla)z + (z \cdot \nabla)z + \nabla p = 0 \text{ in } Q, \quad (15)$$

$$\nabla \cdot z = 0 \text{ in } Q, \quad z = u \text{ in } \Sigma, \quad z = 0 \text{ in } \Sigma_e, \quad (16)$$

$$\partial_t u - \Delta_b u + \Pi_3 u - \sigma n = -\Pi_2 Pz \text{ in } \Sigma, \quad (17)$$

$$z(0) = z_0 \in V^{\frac{1}{2}}(\Omega), \quad u(0) = u_0 \in V^0(\Gamma_i). \quad (18)$$

There exists $c_0 > 0$ and $\mu_0 > 0$ such that, if $\delta \in (0, \mu_0)$ and

$$(z_0, u_0) \in \mathcal{W}_\delta = \{(z_0, u_0) \in V^{\frac{1}{2}}(\Omega) \times V^0(\Gamma_i) \mid z_0 - Du_0 \in V_0^{\frac{1}{2}}(\Omega), \\ \|u_0\|_{V^0(\Gamma_i)} + \|Pz_0\|_{V^{\frac{1}{2}}(\Omega)} \leq c_0\delta\},$$

then, (15)-(16)-(17)-(18) admit a unique solution in the set

$$\mathcal{D}_\delta = \left\{ (z, p, u, \sigma) \in W(0, +\infty; V^{\frac{3}{2}}(\Omega), V^{-\frac{1}{2}}(\Omega)) \times L^2(0, \infty; H^{\frac{1}{2}}(\Omega)/\mathbf{R}) \right. \\ \times W(0, +\infty; V^1(\Gamma_i), V^{-1}(\Gamma_i)) \times L^2(0, \infty) \mid \\ \|z\|_{L^2(0, +\infty; V^{\frac{3}{2}}(\Omega))} + \|u\|_{L^2(0, +\infty; V^1(\Gamma_i))} + \|\sigma\|_{L^2(0, \infty)} \leq \delta, \\ \left. \|p\|_{L^2(0, +\infty; H^{\frac{1}{2}}(\Omega))} \leq \delta(1 + \delta) \right\}.$$

Moreover, (z, u) obeys

$$\|z(t)\|_{V^{\frac{1}{2}}(\Omega)} + \|u(t)\|_{V^0(\Gamma_i)} \leq C(\|u_0\|_{V^0(\Gamma_i)} + \|Pz_0\|_{V^{\frac{1}{2}}(\Omega)}) e^{-\eta t}, \quad t \geq 0.$$

4 The operator \mathcal{A}

The goals of this section are:

- to give a definition of the unbounded operator $(\mathcal{D}(\mathcal{A}), \mathcal{A})$ in \mathcal{H}^0 .
- to characterize the function spaces for which the mapping $Y \mapsto (Y' - \mathcal{A}Y, Y(0))$ is an isomorphism, in order to have optimal regularity results for the extended system (14) when $\kappa = 0$.
- to characterize the functional spaces for which the mapping $Q \mapsto (-Q' - \mathcal{A}^*Q, Q(T))$ is an isomorphism, in order to study the backward adjoint equation which appears in the characterization of the solution to $(\mathcal{P}_{Y_0}^\infty)$ - see part 4.

Theorem 5

We define the unbounded operator $(\mathcal{D}(\mathcal{A}), \mathcal{A})$ in $\mathcal{H}^0 = V_n^0(\Omega) \times V^{-\frac{1}{2}}(\Gamma_i)$ by

$$\mathcal{D}(\mathcal{A}) = \left\{ (y, u) \in V_n^2(\Omega) \times V^{\frac{3}{2}}(\Gamma_i) \mid (y - PDu) \in V_0^2(\Omega) \right\}, \quad (19)$$

$$\mathcal{A} = \begin{pmatrix} A(x, \partial) & (\lambda_0 - A(x, \partial))PD \\ 0 & P_b \Delta_b \end{pmatrix}. \quad (20)$$

The domain $\mathcal{D}(\mathcal{A})$ is dense in \mathcal{H}^0 , and \mathcal{A} generates an analytic semigroup in \mathcal{H}^0 . Moreover, for $0 \leq \theta \leq 1$, the identifications below hold

$$\begin{aligned}\mathcal{D}((\lambda_0 - \mathcal{A})^\theta) &= [\mathcal{D}(\mathcal{A}), \mathcal{H}^0]_{1-\theta}, \\ &= \{(y, u) \in V_n^{2\theta}(\Omega) \times V^{2\theta-\frac{1}{2}}(\Gamma_i) \mid (y - P Du) \in V_0^{2\theta}(\Omega)\}.\end{aligned}\tag{21}$$

The unbounded operator $(\mathcal{D}(\mathcal{A}^*), \mathcal{A}^*)$ in $\mathcal{H}_*^0 = V_n^0(\Omega) \times V^{\frac{1}{2}}(\Gamma_i)$ is defined by

$$\begin{aligned}\mathcal{D}(\mathcal{A}^*) &= V_0^2(\Omega) \times V^{\frac{5}{2}}(\Gamma_i), \\ \mathcal{A}^* &= \begin{pmatrix} A^*(x, \partial) & 0 \\ D^*(\lambda - A^*(x, \partial)) & P_b \Delta_b \end{pmatrix}.\end{aligned}$$

It is the adjoint of $(\mathcal{D}(\mathcal{A}), \mathcal{A})$ with respect to the pivot space $V_n^0(\Omega) \times V^0(\Gamma_i)$. The domain $\mathcal{D}(\mathcal{A}^*)$ is dense in \mathcal{H}_*^0 and \mathcal{A}^* generates an analytic semigroup in \mathcal{H}_*^0 . Finally, for $0 \leq \theta \leq 1$, the identifications below hold

$$\mathcal{D}((\lambda_0 - \mathcal{A}^*)^\theta) = [\mathcal{D}(\mathcal{A}^*), \mathcal{H}_*^0]_{1-\theta} = V_0^{2\theta}(\Omega) \times V^{\frac{1}{2}+2\theta}(\Gamma_i).\tag{22}$$

Moreover, for $0 \leq \theta \leq 1$, we define the function spaces

$$\mathcal{H}^{2\theta} = [\mathcal{D}(\mathcal{A}), \mathcal{H}^0]_{1-\theta}, \quad \mathcal{H}_*^{2\theta} = [\mathcal{D}(\mathcal{A}^*), \mathcal{H}_*^0]_{1-\theta}, \quad \mathcal{H}^{-2\theta} = (\mathcal{H}_*^{2\theta})', \quad \mathcal{H}_*^{-2\theta} = (\mathcal{H}^{2\theta})'.$$

Then, as a consequence of the analyticity of the semigroups $(e^{At})_{t \geq 0}$ and $(e^{A^*t})_{t \geq 0}$ respectively in \mathcal{H}^0 and in \mathcal{H}_*^0 , we can state a general isomorphism Theorem (see [1], Chap.3, Thm 2.2, p.166):

Theorem 6 For every $0 \leq \theta \leq 1$, the mappings below are isomorphisms:

$$\begin{aligned}W(0, T; \mathcal{H}^{2\theta}, \mathcal{H}^{2(\theta-1)}) &\rightarrow L^2(0, T; \mathcal{H}^{2(\theta-1)}) \times [\mathcal{H}^{2\theta}, \mathcal{H}^{2(\theta-1)}]_{\frac{1}{2}}, \\ Y &\mapsto (Y' - \mathcal{A}Y, Y(0)), \\ W(0, T; \mathcal{H}_*^{2\theta}, \mathcal{H}_*^{2(\theta-1)}) &\rightarrow L^2(0, T; \mathcal{H}_*^{2(\theta-1)}) \times [\mathcal{H}_*^{2\theta}, \mathcal{H}_*^{2(\theta-1)}]_{\frac{1}{2}}, \\ Q &\mapsto (-Q' - \mathcal{A}^*Q, Q(T)).\end{aligned}$$

Next we determine the spaces $[\mathcal{H}^{2\theta}, \mathcal{H}^{2(\theta-1)}]_{\frac{1}{2}}$ of initial conditions.

Lemma 7 For all $0 \leq \theta \leq 1$ the following characterization holds:

$$[\mathcal{H}^{2\theta}, \mathcal{H}^{2(\theta-1)}]_{\frac{1}{2}} = \mathcal{H}^{2\theta-1}.\tag{23}$$

Finally, a direct application of Theorem 6 with (23) ensures the existence of a unique solution Y to the extended linear system (14) when $\kappa = 0$.

Theorem 8 *Let $g \in L^2(0, T; V^0(\Gamma_i))$ and $Y_0 \in \mathcal{H}^0$. There exists a unique solution $Y \in L^2(0, T; \mathcal{H}^0)$ to the extended system*

$$Y' = \mathcal{A}Y + \begin{pmatrix} 0 \\ g \end{pmatrix} \in L^2(0, T; \mathcal{D}(\mathcal{A}^*)'), \quad Y(0) = Y_0 \in \mathcal{H}^0. \quad (24)$$

Moreover, Y belongs to $W(0, T; \mathcal{H}^1, \mathcal{H}^{-1})$. More generally, if we assume that $Y_0 \in \mathcal{H}^{2\theta}$, $0 \leq \theta \leq \frac{1}{2}$, then Y belongs to $W(0, T; \mathcal{H}^{2\theta+1}, \mathcal{H}^{2\theta-1})$.

We now treat the backward adjoint equation which appears in the characterization of the solution to $(\mathcal{P}_{Y_0}^\infty)$.

Lemma 9 *Let us define $\mathcal{C} \in \mathcal{L}(\mathcal{H}^1, \mathcal{Z})$ with $\mathcal{Z} = L^2(\Omega, \mathbf{R}^9)$ by $\mathcal{C} : Y \rightarrow \nabla(y + (I - P)Du)$. Then*

$$\|\mathcal{C} \cdot\|_{\mathcal{Z}} \sim \|\cdot\|_{\mathcal{H}^1} \quad \text{and} \quad \mathcal{C}^* \mathcal{C} \in \mathcal{L}(\mathcal{H}^{\theta+1}, \mathcal{H}_*^{\theta-1}), \quad 0 \leq \theta \leq 1. \quad (25)$$

Then Theorem 6 with (25) leads to the following theorem.

Theorem 10 *Let $Y \in L^2(0, T; \mathcal{H}^1)$. There exists a unique solution $Q \in L^2(0, T; \mathcal{H}_*^0)$ to the backward equation*

$$-Q' = \mathcal{A}^* Q + \mathcal{C}^* \mathcal{C} Y, \quad Q(T) = 0. \quad (26)$$

Moreover, Q belongs to $W(0, T; \mathcal{H}_^1, \mathcal{H}_*^{-1})$. More generally, if we assume that $Y \in L^2(0, T; \mathcal{H}^{2\theta})$, $\frac{1}{2} \leq \theta \leq 1$, then Q belongs to $W(0, T; \mathcal{H}_*^{2\theta}, \mathcal{H}_*^{2(\theta-1)})$.*

5 Resolution of the optimal control problem

5.1 The finite time horizon case

Let $0 < T < \infty$ be a finite time horizon. To deal with the optimal control problem $(\mathcal{P}_{Y_0}^T)$ we first study the following problem:

$$(\mathcal{P}_\xi^T) \quad \inf\{\mathcal{J}_T(g) \mid g \in L^2(0, T; V^0(\Gamma_i))\}, \quad (27)$$

where

$$\mathcal{J}_T(g) = \frac{1}{2} \int_0^T \|CY_g\|_{\mathcal{Z}}^2 + \frac{1}{2} \int_0^T \int_{\Gamma_i} |g|^2,$$

and $Y_g \in W(0, T; \mathcal{H}, \mathcal{H}^{-1})$ is the solution to

$$Y' = \mathcal{A}Y + \begin{pmatrix} 0 \\ g \end{pmatrix}, \quad Y(0) = \xi \in \mathcal{H}^0. \quad (28)$$

We introduce the projection operator $\Lambda : (f, g) \in \mathcal{H}_*^0 \mapsto (0, g)$. The problem (\mathcal{P}_ξ^T) admits a unique solution $(0, g_{\xi, T})$ where $(0, g_{\xi, T}) = -\Lambda Q_{\xi, T}$ and $(Y_{\xi, T}, Q_{\xi, T})$ is the unique solution to the system

$$(S^T) \quad \begin{cases} Y' &= \mathcal{A}Y - \Lambda Q, & Y(0) = \xi \in \mathcal{H}^0, \\ -Q' &= \mathcal{A}^*Q + \mathcal{C}^*\mathcal{C}Y, & Q(T) = 0. \end{cases}$$

Finally, we denote by $\Pi(T) \in \mathcal{L}(\mathcal{H}^0, \mathcal{H}_*^0)$, the mapping

$$\Pi(T) : \xi \mapsto Q_{\xi, T}(0).$$

5.2 The infinite time horizon case

Since, for every ξ and $0 < T < \infty$, the solution to (\mathcal{P}_ξ^T) has been characterized, we are in position to study the optimal control problem (\mathcal{P}_ξ^∞) and the regularity of its solution in function of the regularity of ξ . The problem (\mathcal{P}_ξ^∞) is defined by

$$(\mathcal{P}_\xi^\infty) \quad \inf\{\mathcal{J}(g) \mid g \in L^2(0, \infty; V^0(\Gamma_i))\}, \quad (29)$$

where

$$\mathcal{J}(g) = \frac{1}{2} \int_0^\infty \|\mathcal{C}Y_g\|_Z^2 + \frac{1}{2} \int_0^\infty \int_{\Gamma_i} |g|^2, \quad Y_g \in W(0, \infty; \mathcal{H}, \mathcal{H}^{-1}) \text{ satisfies (28).}$$

Using a null controllability result stated in [3] we can show that there exists a control $g \in L^2(0, \infty; V^0(\Gamma_i))$ such that $\mathcal{J}(g) < +\infty$. This gives us the existence of a unique solution g_ξ to (\mathcal{P}_ξ^∞) .

Theorem 11 *The problem (\mathcal{P}_ξ^∞) admits a unique solution g_ξ where $g_\xi = -\psi_\xi$ and $(Y_\xi, Q_\xi) = ((y_\xi, u_\xi), (\Phi_\xi, \psi_\xi)) \in W(0, \infty; \mathcal{H}^1, \mathcal{H}^{-1}) \times W(0, \infty; \mathcal{H}_*^1, \mathcal{H}_*^{-1})$ is the unique solution to the system:*

$$(S) \quad \begin{cases} Y' &= \mathcal{A}Y - \Lambda Q, & Y(0) = \xi \in \mathcal{H}^0, \\ -Q' &= \mathcal{A}^*Q + \mathcal{C}^*\mathcal{C}Y, & Q(\infty) = 0. \end{cases}$$

Moreover, there exists $\Pi \in \mathcal{L}(\mathcal{H}^*, \mathcal{H}_*^0)$, $\Pi^* = \Pi$ such that

$$Q_\xi = \Pi Y_\xi, \quad \mathcal{J}(g_\xi) = \frac{1}{2} \langle \Pi \xi | \xi \rangle_{\mathcal{H}_*^0, \mathcal{H}^0}. \quad (30)$$

The control g_ξ has been calculated as the limit of $g_{\xi, T}$ when $T \rightarrow \infty$, where $g_{\xi, T}$ is the unique solution to (\mathcal{P}_ξ^T) .

We now focus on the properties of Π . First, from the limit $\Pi(T) \rightarrow \Pi$ as $T \rightarrow \infty$ we can show that Π satisfies an algebraic Riccati equation. Next, Theorems 8 and 10 for (S) lead to sharp regularity result for Π .

Theorem 12 Π satisfies a Riccati equation: $\forall(\xi, \zeta) \in \mathcal{H}^1 \times \mathcal{H}^1$,

$$\langle \Pi \xi | \mathcal{A} \zeta \rangle_{\mathcal{H}_*^1, \mathcal{H}^{-1}} + \langle \mathcal{A} \xi | \Pi \zeta \rangle_{\mathcal{H}^{-1}, \mathcal{H}_*^1} + (\mathcal{C} \xi | \mathcal{C} \zeta)_{\mathcal{Z}} - (\Lambda \Pi \xi | \Lambda \Pi \zeta)_{V^0(\Gamma_i)} = 0, \quad (31)$$

and the regularizing property $\Pi \in \mathcal{L}(\mathcal{H}^{2\theta}, \mathcal{H}_*^{2\theta})$, $0 \leq \theta \leq \frac{1}{2}$.

Next we set $\mathcal{A}_\Pi = (\Lambda \Pi - \mathcal{A})$, so that the optimal trajectory Y_ξ satisfy

$$Y' + \mathcal{A}_\Pi Y = 0, \quad Y(0) = \xi \in \mathcal{H}^0. \quad (32)$$

From a classical result due to Datko in [5, Chap 4, Thm 4.1], we show the exponential stability of $e^{-\mathcal{A}_\Pi t} Y_0$. Let $\alpha > 0$, the positivity of \mathcal{A}_Π allows us to define its fractional power \mathcal{A}_Π^α and we respectively identify $\mathcal{D}(\mathcal{A}_\Pi^\alpha)$ and $\mathcal{D}(\mathcal{A}_\Pi^{*\alpha})$ with $\mathcal{D}((\lambda_0 - \mathcal{A})^\alpha)$ and $\mathcal{D}((\lambda_0 - \mathcal{A}^*)^\alpha)$. Thus (21) and (22) ensures a characterization $\mathcal{D}(\mathcal{A}_\Pi^\alpha)$ and $\mathcal{D}(\mathcal{A}_\Pi^{*\alpha})$.

Theorem 13 The unbounded operator $(\mathcal{D}(-\mathcal{A}_\Pi), -\mathcal{A}_\Pi)$ generates an analytic exponentially stable semigroup in \mathcal{H}^0 and the characterizations below hold

$$\mathcal{D}(\mathcal{A}_\Pi^\theta) = \mathcal{H}^{2\theta}, \quad \mathcal{D}(\mathcal{A}_\Pi^{*\theta}) = \mathcal{H}_*^{2\theta}, \quad 0 \leq \theta \leq 1.$$

Next we define new norms in the spaces \mathcal{H}^α and $\mathcal{H}^{1+\alpha}$ which are essential to study the stabilization of the Navier-Stokes equation (see section 6).

Definition 14 We define $\Pi^{(\alpha)} \in \mathcal{L}(\mathcal{H}^\alpha, \mathcal{H}_*^{-\alpha})$ by $\Pi^{(\alpha)} = \mathcal{A}_\Pi^{*\frac{\alpha}{2}} \Pi \mathcal{A}_\Pi^{\frac{\alpha}{2}}$.

Theorem 15 The operator $\Pi^{(\alpha)}$ has the regularizing property:

$$\Pi^{(\alpha)} \in \mathcal{L}(\mathcal{H}^{2\theta+\alpha}, \mathcal{H}_*^{2\theta-\alpha}), \quad 0 \leq \theta \leq \frac{1}{2}. \quad (33)$$

Definition 16 We define the two mappings \mathcal{N}_α and $\mathcal{R}_{1+\alpha}$ by

$$\begin{aligned} \mathcal{N}_\alpha(\xi) &= (\langle \Pi^{(\alpha)} \xi | \xi \rangle_{\mathcal{H}_*^{-\alpha}, \mathcal{H}^\alpha})^{\frac{1}{2}}, \quad \xi \in \mathcal{H}^\alpha, \\ \mathcal{R}_{1+\alpha}(\xi) &= (\langle \mathcal{A}_\Pi \xi | \Pi^{(\alpha)} \xi \rangle_{\mathcal{H}^{-1+\alpha}, \mathcal{H}_*^{1-\alpha}})^{\frac{1}{2}}, \quad \xi \in \mathcal{H}^{1+\alpha}. \end{aligned}$$

Theorem 17 \mathcal{N}_α and $\mathcal{R}_{1+\alpha}$ define norms respectively on \mathcal{H}^α and $\mathcal{H}^{1+\alpha}$,

$$\mathcal{N}_\alpha(\cdot) \sim \|\cdot\|_{\mathcal{H}^\alpha}, \quad \mathcal{R}_{1+\alpha}(\cdot) \sim \|\cdot\|_{\mathcal{H}^{1+\alpha}}. \quad (34)$$

We shall point out that the expression of $\mathcal{R}_{1+\alpha}(\xi)$ is explicitly given by

$$\langle \mathcal{A}_\Pi \xi | \Pi^{(\alpha)} \xi \rangle_{\mathcal{H}^{-1+\alpha}, \mathcal{H}_*^{1-\alpha}} = \frac{1}{2} \|\mathcal{C} \mathcal{A}_\Pi^{\frac{\alpha}{2}} \xi\|_{\mathcal{Z}}^2 + \frac{1}{2} \|\Lambda \Pi \mathcal{A}_\Pi^{\frac{\alpha}{2}} \xi\|_{V^0(\Gamma_i)}^2, \quad \forall \xi \in \mathcal{H}^{1+\alpha},$$

which follows from (31) in which we have replaced ξ and ζ by $\mathcal{A}_\Pi^{\frac{\alpha}{2}} \xi$.

We finish this section by giving the PDE formulation of the closed loop system (32).

Theorem 18 *The operator Π can be rewritten as follows,*

$$\Pi = \begin{pmatrix} \Pi_1 & \Pi_2^* \\ \Pi_2 & \Pi_3 \end{pmatrix}, \quad \begin{aligned} \Pi_1 &\in \mathcal{L}(V_n^0(\Omega)), \\ \Pi_2 &\in \mathcal{L}(V_n^0(\Omega), V^{\frac{1}{2}}(\Gamma_i)), \\ \Pi_3 &\in \mathcal{L}(V^{-\frac{1}{2}}(\Gamma_i), V^{\frac{1}{2}}(\Gamma_i)), \end{aligned} \quad (35)$$

where Π_1 and Π_3 are positive, definite and self-adjoint operators. Then $Y = (Pz, u) \in W(0, \infty; \mathcal{H}^1, \mathcal{H}^{-1})$ satisfies (32) if and only if $(z, p, u, \sigma) \in W(0, \infty; V^1(\Omega), V^{-1}(\Omega)) \times L^2(0, \infty; L^2(\Omega)/\mathbf{R}) \times W(0, \infty; V^{\frac{1}{2}}(\Gamma_i), V^{-\frac{3}{2}}(\Gamma_i)) \times L^2(0, \infty)$ satisfies

$$\begin{aligned} \partial_t z - \Delta z + (z \cdot \nabla) z_e + (z_e \cdot \nabla) z + \nabla p &= 0, \quad \nabla \cdot z = 0 \text{ in } Q, \\ \partial_t u - \Delta_b u + \Pi_3 u - \sigma n &= -\Pi_2 Pz, \quad z = u \text{ in } \Sigma_i, \quad z = 0 \text{ in } \Sigma_e, \\ z(0) = z_0 &\in V^0(\Omega), \quad u(0) = u_0 \in V^{-\frac{1}{2}}(\Gamma_i), \quad z_0 \cdot n = u_0 \cdot n. \end{aligned}$$

6 Stabilization of the Navier-Stokes equations

We now come back to the stabilization of the Navier-Stokes system. We now consider the nonlinear system (15)-(16)-(17)-(18) which we can rewrite in the abstract formulation

$$Y' + \mathcal{A}_\Pi Y = B(Y, Y), \quad Y(0) = Y_0 \in \mathcal{H}^{\frac{1}{2}}. \quad (36)$$

Finally, Theorem 4 is a direct consequence of the following result.

Theorem 19 *There exists $c_0 > 0$ and $\mu_0 > 0$ such that, if $\delta \in (0, \mu_0)$ and $Y_0 \in \mathcal{V}_\delta = \{Y \in \mathcal{H}^{\frac{1}{2}} \mid \|Y\|_{\mathcal{H}^{\frac{1}{2}}} < c_0 \delta\}$ then, (36) admit a unique solution in the set $\mathcal{S}_\delta = \{Y \in W(0, \infty; \mathcal{H}^{\frac{3}{2}}, \mathcal{H}^{-\frac{1}{2}}) \mid \|Y\|_{W(0, \infty; \mathcal{H}^{\frac{3}{2}}, \mathcal{H}^{-\frac{1}{2}})} \leq \delta\}$. Moreover, there exists $\eta > 0$ such that*

$$\|Y(t)\|_{\mathcal{H}^{\frac{1}{2}}} \leq C \|Y_0\|_{\mathcal{H}^{\frac{1}{2}}} e^{-\eta t}. \quad (37)$$

Proof. Here, we give a brief sketch of the proof of the stability result. We multiply the left hand side of (36) by $\Pi^{(\frac{1}{2})} Y(t)$. According to (34) with $\alpha = \frac{1}{2}$ we obtain

$$\frac{1}{2} \frac{d}{dt} \mathcal{N}_{\frac{1}{2}}^2(Y(t)) + \mathcal{R}_{\frac{3}{2}}^2(Y(t)) = \langle B(Y(t), Y(t)) | \Pi^{(\frac{1}{2})} Y(t) \rangle.$$

Then we invoke a classical estimation - see [2, Chap.6, 6.9 and 6.10] - to obtain $|\langle B(Y(t), Y(t)) | \Pi^{(\frac{1}{2})} Y(t) \rangle| \leq C \|Y(t)\|_{\mathcal{H}^{\frac{1}{2}}} \|Y(t)\|_{\mathcal{H}^{\frac{1}{2}}} \|\Pi^{(\frac{1}{2})} Y(t)\|_{\mathcal{H}^{\frac{1}{2}}}$. According to (33) with $\alpha = \theta = \frac{1}{2}$ it yields

$$|\langle B(Y(t), Y(t)) | \Pi^{(\frac{1}{2})} Y(t) \rangle| \leq C \|Y(t)\|_{\mathcal{H}^{\frac{1}{2}}} \|Y(t)\|_{\mathcal{H}^{\frac{3}{2}}}^2.$$

Thus (34) gives us $C_0 > 0$ such that

$$\frac{d}{dt} \mathcal{N}_{\frac{1}{2}}^2(Y(t)) + 2(1 - C_0 \mathcal{N}_{\frac{1}{2}}(Y(t))) \mathcal{R}_{\frac{3}{2}}^2(Y(t)) \leq 0. \quad (38)$$

It is obvious to see that if $\mathcal{N}_{\frac{1}{2}}(Y_0)$ is small enough, we can choose $\mathcal{N}_{\frac{1}{2}}(Y_0) < \frac{1}{4C_0}$, so that the mapping $t \rightarrow \mathcal{N}_{\frac{1}{2}}(Y(t))$ be a nonincreasing function with values less than $\frac{1}{4C_0}$. This gives the existence of δ , and (37) follows from (38).

References

- [1] A. Bensoussan, G. Da Prato, M.C. Delfour, S.K. Mitter *Representation and Control of Infinite Dimensional Systems, Volume I*. Birkhauser, Boston, 1992.
- [2] P. Constantin, C. Foias. *Navier-Stokes Equations*. University of Chicago Press, Chicago Lectures in Mathematics, 1988.
- [3] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov, J.-P. Puel. Local exact controllability of the Navier-Stokes system. to appear in *J. Math. Pures et Appliquées.*
- [4] G.P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. II., Nonlinear Steady Problems*. Springer, New York, 1994.
- [5] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, New York, 1983.
- [6] J.-P. Raymond. Stokes and Navier-Stokes Equations with Nonhomogeneous Boundary Conditions, preprint, 2005.