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FEEDBACK STABILIZATION OF TWO DIMENSIONAL MAGNETOHYDRODYNAMIC EQUATIONS*

BY

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Abstract. We prove the local exponential stabilizability for the MHD system in space dimension 2, with internally distributed feedback controllers. These controllers take values in a finite dimensional space which is the unstable manifold of the elliptic part of the linearized operator. They are represented through two scalar functions supported in a subdomain.

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1. Introduction. This paper is concerned with the study of the local exponential stabilization for the magnetohydrodynamic (MHD) equations in space dimension 2, with feedback controllers, localized in a subdomain and taking values in a finite dimensional space. These controllers are scalar functions supported in the given subdomain. The fact that one is able to control the system using scalar functions is due to the possibility of representing the divergence free vector fields using the corresponding stream functions.

The method we use is to linearize the system around the stationary state and then construct a feedback controller stabilizing the linear system. Then we show that the same controller stabilizes, locally in a specified space, the nonlinear system.

The stabilization of the linearized system is obtained via a spectral decomposition of the elliptic part. Thus, we project the system on the stable

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and unstable subspaces corresponding to this decomposition. The unstable subspace is finite dimensional and the corresponding projected system is exactly controllable, as a consequence of the approximate controllability of the original linearized system; one may thus construct a feedback stabilizing this finite dimensional linear system. The projected system on the stable subspace is asymptotically stable and the feedback for the finite dimensional system is stabilizing the initial linearized equations. The approximate controllability of the linearized system is a consequence of the unique continuation result established in [10], for the adjoint system. The feedback controller constructed in the linear case is also stabilizing the nonlinear system.

The stabilization result for the 3-D magnetohydrodynamic equations was treated by Lefter in [7]. The special feature in the two dimensional case is that one may use scalar functions as controllers.

The method of spectral decomposition was used in the study of Navier-Stokes equations with internal controllers by Barbu in [1], Barbu and Triggiani [5]. Barbu, Lasiecka and Triggiani treated in [3] and [4] the stabilization of Navier-Stokes equations with boundary tangential controllers. The stabilization of two dimensional Navier-Stokes equations with Navier slip boundary conditions was studied by Lefter in [8]. We mention also the references [6], [11], [13] for the linearization method in the study of hydrodynamical stability and the existence of invariant manifolds.

The procedure we follow here for the stabilization of the 2-D MHD equations is essentially that used in [7], [8].

2. Preliminaries and main results.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected open set with C^2 boundary $\partial\Omega$ and $\omega \subset\subset \Omega$ an open subset of Ω. Let $Q = \Omega \times (0,\infty)$, $\Sigma = \partial\Omega \times (0,\infty)$, $\mathbf{n}$ is the unit exterior normal to $\partial\Omega$. We consider in the paper the following MHD controlled system:

$$(1)\qquad \begin{cases} \dfrac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y - (B\cdot\nabla)B & \\ \qquad\qquad +\nabla(\dfrac{1}{2}B^2 + p) = f + \chi_\omega \mathcal{B}_1 u & \text{in } Q, \\ \dfrac{\partial B}{\partial t} - \eta\Delta B + (y\cdot\nabla)B - (B\cdot\nabla)y = \mathcal{B}_2 v & \text{in } Q, \\ \nabla\cdot y = 0,\ \nabla\cdot B = 0 & \text{in } Q, \end{cases}$$

with boundary and initial conditions

$$\begin{cases} y = 0,\ B \cdot \mathbf{n} = 0, \text{rot } B = 0 & \text{on } \Sigma, \\ y(\cdot, 0) = y_0,\ B(\cdot, 0) = B_0 & \text{in } \Omega. \end{cases}$$

The functions that appear in the system have the following physical meaning: $y = (y_1, y_2)^T : \Omega \times (0,T) \to \mathbb{R}^2$ is the velocity field, $p : \Omega \times (0,T) \to \mathbb{R}$ is the pressure, $B = (B_1, B_2)^T : \Omega \times (0,T) \to \mathbb{R}^2$ is the magnetic field and $f = (f_1, f_2)^T : \Omega \to \mathbb{R}^2$ represents the density of the exterior forces $((\cdots)^T$ means the matrix transpose). The coefficients ν, η are the positive kinematic viscosity and the magnetic resistivity coefficients. We denote by

$$\text{rot } B = \frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2}$$

the scalar version of the curl operator. We also note the formula curl curl $B = -\Delta B + \nabla \text{div } B$ valid for vector fields in 3-dimensional domains. So, knowing the fact that the solution B will remain divergence free, we wrote $-\Delta B$ instead of curl curl B as is formulated the second equation in the 3 dimensional case (see [7]), keeping in mind that the system models in fact phenomena in a 3 dimensional cylindrical body and the data depend only on x_1, x_2 variables.

The functions $u, v : \omega \times (0,T) \to \mathbb{R}$,$u, v \in \mathcal{U} := L^2(0,T; (L^2(\omega)))$ are the controllers and $\chi_\omega : L^2(\omega) \to L^2(\Omega)$ is the operator extending the functions in $L^2(\omega)$ with 0 to the whole Ω. The linear continuous operators $\mathcal{B}_1 : L^2(\omega) \to (L^2(\omega))^2$, $\mathcal{B}_2 : L^2(\omega) \to (L^2(\Omega))^2$ are the control operators and will be described below.

Let φ be be the solution of the following boundary value problem

$$\begin{cases} \Delta\varphi = u - \dfrac{1}{|\omega|} \displaystyle\int_\omega u dx & \text{in } \omega \\ \dfrac{\partial \varphi}{\partial \mathbf{n}} = 0 & \text{on } \partial\omega \end{cases} \tag{2}$$

We define $\mathcal{B}_1 : L^2(\omega) \to (L^2(\omega))^2$ by the formula

$$\mathcal{B}_1 u = \begin{pmatrix} -\dfrac{\partial \varphi}{\partial x_2} \\ \dfrac{\partial \varphi}{\partial x_1} \end{pmatrix}. \tag{3}$$

Let ψ be be the solution of the following boundary value problem

$$\begin{cases} \Delta\psi = \chi_\omega v & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

We define $\mathcal{B}_2 : L^2(\omega) \to (L^2(\Omega))^2$ by the formula

$$\mathcal{B}_2 v = \begin{pmatrix} -\dfrac{\partial\psi}{\partial x_2} \\ \dfrac{\partial\psi}{\partial x_1} \end{pmatrix}. \tag{5}$$

Denote by

$$H = \{z \in (L^2(\Omega))^2 : \nabla \cdot z = 0, z \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

endowed with the L^2 norm and

$$\begin{array}{l} V_1 = H \cap (H_0^1)^2 \\ V_2 = H \cap (H^1)^2 \end{array}$$

endowed with the H^1 norm. We denote by $|\cdot|$ and $(\cdot,\cdot)$ the L^2 norm respectively the L^2 scalar product.

We also recall here the standard estimate on the trilinear term appearing in the Navier-Stokes equations and, consequently in the MHD system (see [12]). Let for $m \geq 0, V^m := H \cap (H^m(\Omega))^2$ with norm $\|\cdot\|_m$. Then the trilinear form

$$b(u,v,w) := \int_\Omega [(u \cdot \nabla)v] \cdot w dx = \int_\Omega \sum_{i,j=1}^{3} u_i \frac{\partial v_j}{\partial x_i} w_j dx$$

is well defined on $(V^1)^3$ and extends to $V^{m_1} \times V^{m_2+1} \times V^{m_3}$ when $m_1+m_2+m_3 \geq 1$ and $m_i \neq 1$ or when at least one of $m_i = 1$ and $m_1 + m_2 + m_3 > 1$. In these situations we have:

$$|b(u,v,w)| \leq C\|u\|_{m_1}\|v\|_{m_2+1}\|w\|_{m_3}. \tag{6}$$

Also, it is antisymmetric in the last two variables: $b(u,v,w) = -b(u,w,v)$.

Consider for a given $f \in (H^{-1}(\Omega))^2$ a steady state variational solution $(\bar{y}, \bar{B}, \bar{p}) \in V_1 \times V_2 \times L^2(\Omega)$ of (1):

$$
\begin{cases}
-\nu\Delta\bar{y} + (\bar{y}\cdot\nabla)\bar{y} + \nabla(\frac{1}{2}\bar{B}^2) - (\bar{B}\cdot\nabla)\bar{B} + \nabla\bar{p} = f & \text{in } \Omega,\\
-\eta\Delta\bar{B} + (\bar{y}\cdot\nabla)\bar{B} - (\bar{B}\cdot\nabla)\bar{y} = 0 & \text{in } \Omega,\\
\nabla\cdot\bar{y} = 0,\ \nabla\cdot\bar{B} = 0 & \text{in } \Omega,\\
\bar{y} = 0,\ \bar{B}\cdot\mathbf{n} = 0, \mathrm{rot}\ \bar{B} = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{7}
$$

We will assume the following hypothesis on the regularity of the stationary solution:

$$
(\mathbf{H}) \qquad \bar{y}, \bar{B} \in W^{1,\infty}(\Omega) \cap H^2(\Omega). \tag{8}
$$

In order to write (1) in an abstract form we define the following two operators (P is the Leray projection):

$$A_1 y = -P\Delta y \text{ for } y \in D(A_1)$$

$$A_2 B = -\Delta B \text{ for } B \in D(A_2)$$

where

$$D(A_1) := (H^2(\Omega))^2 \cap V_1,$$

$$D(A_2) := \{B \in (H^2(\Omega))^2 \cap V_2 \mid \mathrm{rot}\ B = 0 \text{ on } \partial\Omega\}.$$

With no loss of generality we will suppose that $\nu = \eta = 1$ and system (1) may thus be written as

$$
\begin{cases}
y' + A_1 y + P(y\cdot\nabla)y - P(B\cdot\nabla)B = Pf + P(\chi_\omega \mathcal{B}_1 u),\\
B' + A_2 B + (y\cdot\nabla)B - (B\cdot\nabla)y = \mathcal{B}_2 v,\\
y(0) = y_0,\ B(0) = B_0.
\end{cases}
\tag{9}
$$

The main question we address in this paper is to find a feedback control $(u, v) = K(y, B)$ such that, if (y_0, B_0) is in a neighborhood of $(\bar{y}, \bar{B})$ then system (9) admits a global weak solution that satisfies an estimate of the form:

$$\|(y, B) - (\bar{y}, \bar{B})\| \leq Ce^{-\gamma t}\|(y_0 - \bar{y}, B_0 - \bar{B})\|$$

with C, γ positive constants and the norm is in a space that will be specified. Moreover, the feedback control we construct takes values in a finite dimensional space.

Thus, the main result of this paper, that will be proved in §4, concerns the null stabilization of the nonlinear system (1) around the stationary solution satisfying (7):

Theorem 1. *There exist $\delta > 0$, $C > 0$, a neighborhood $\mathcal{V}_\rho$ of 0 in $V_1 \times V_2$, a finite dimensional subspace $U \subset L^2(\omega) \times L^2(\omega)$ and a continuous linear feedback operator $K : H \times H \to U$ such that system* (1) *with $(y_0, B_0) \in \mathcal{V}_\rho$ admits a global weak solution that satisfies:*

$$\|(y(t) - \bar{y}, B(t) - \bar{B})\|_{V_1 \times V_2} \leq C \|(y_0 - \bar{y}, B_0 - \bar{B})\|_{V_1 \times V_2} e^{-\delta t}, \ t > 0. \tag{10}$$

In order to prove this result, we need to study the difference between the solution of (1) and the stationary solution satisfying (7). After renaming by y, B, p, y_0, B_0 the quantities $y - \bar{y}$, $B - \bar{B}$, $p - \bar{p}$, $y_0 - \bar{y}$ and respectively $B_0 - \bar{B}$, we obtain the following system that we have now to stabilize in 0:

$$\begin{cases} y' + A_1 y + P((y \cdot \nabla)\bar{y} + (\bar{y} \cdot \nabla)y - (B \cdot \nabla)\bar{B} - (\bar{B} \cdot \nabla)B) \\ \qquad + P((y \cdot \nabla)y - (B \cdot \nabla)B) = P(\chi_\omega \mathcal{B}_1 u), \\ B' + A_2 B + (y \cdot \nabla)\bar{B} + (\bar{y} \cdot \nabla)B - (B \cdot \nabla)\bar{y} - (\bar{B} \cdot \nabla)y \\ \qquad + (y \cdot \nabla)B - (B \cdot \nabla)y = \mathcal{B}_2 v, \\ y(0) = y_0, \ B(0) = B_0. \end{cases} \tag{11}$$

The first thing we are doing is to find a feedback that stabilizes the linearized system:

$$\begin{cases} y' + A_1 y + P((y \cdot \nabla)\bar{y} + (\bar{y} \cdot \nabla)y - (B \cdot \nabla)\bar{B} \\ \quad - (\bar{B} \cdot \nabla)B) = P(\chi_\omega \mathcal{B}_1 u), \\ B' + A_2 B + P((y \cdot \nabla)\bar{B} + (\bar{y} \cdot \nabla)B - (B \cdot \nabla)\bar{y} \\ \quad - (\bar{B} \cdot \nabla)y) = \mathcal{B}_2 v, \\ y(0) = y_0, \ B(0) = B_0. \end{cases} \tag{12}$$

We observe (see [2]) that in the second equation one has to introduce a supplementary Leray projection since otherwise we could not obtain a solution (y, B) of (12) with a divergence free B. Denote by $\mathcal{A}$ the following operator:

$$\mathcal{A} \begin{pmatrix} y \\ B \end{pmatrix} = \begin{pmatrix} A_1 y + P((y \cdot \nabla)\bar{y} + (\bar{y} \cdot \nabla)y - (B \cdot \nabla)\bar{B} - (\bar{B} \cdot \nabla)B) \\ A_2 B + P((y \cdot \nabla)\bar{B} + (\bar{y} \cdot \nabla)B - (B \cdot \nabla)\bar{y} - (\bar{B} \cdot \nabla)y) \end{pmatrix}, \tag{13}$$

with $D(\mathcal{A}) = D(A_1) \times D(A_2) \subset H \times H$ and by $\mathcal{B} : (L^2(\omega)) \times (L^2(\omega)) \rightarrow H \times H$

$$\mathcal{B}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} P(\chi_\omega \mathcal{B}_1 u) \\ \mathcal{B}_2 v \end{pmatrix}.$$

Then, the linear controlled system (12) is written in the abstract form

$$\begin{cases} z' + \mathcal{A}z = \mathcal{B}w \\ z(0) = z_0 \end{cases}, \tag{14}$$

where we denoted by $z = (y, B)^T$, $w = (u, v)^T$ and the solution corresponding to the control w will be denoted as z^w.

The stabilization result concerning the linearized MHD system, that will be proved in §3, is the following:

Proposition 1. *i) The operator $-\mathcal{A}$ generates an analytic semigroup in $H \times H$, with compact resolvent.*
ii) The linear system (12) *is approximately controllable in any time T.*
iii) There exist a finite dimensional subspace $U \subset L^2(\omega) \times L^2(\omega)$ and a linear continuous operator $K : H \times H \rightarrow U$ such that the operator $-\tilde{\mathcal{A}} = -\mathcal{A} + \mathcal{B}K$ generates an analytic semigroup of negative type i.e. a semigroup satisfying an estimate of the form:

$$\|e^{-t(\mathcal{A}-\mathcal{B}K)}\| \leq Ce^{-\delta t}, \ t > 0. \tag{15}$$

where C, δ are positive constants. Moreover, for any positive δ there exists such a feedback K with a corresponding change of the constant $C = C(\delta)$ and of the finite dimensional space U.

3. Feedback stabilization of the linearized MHD system. Proof of Proposition 1. The stabilization of the linearized system follows mainly the lines in [7] and [8], particularly the proof of the steps i), iii) below are similar to the corresponding assertions in the cited papers.

i) The operator $\mathcal{A}$ admits the representation $\mathcal{A} = A + A_0$ where

$$A\begin{pmatrix} y \\ B \end{pmatrix} = \begin{pmatrix} A_1 y \\ A_2 B \end{pmatrix},$$

$$A_0\begin{pmatrix} y \\ B \end{pmatrix} = \begin{pmatrix} P((y\cdot\nabla)\bar{y} + (\bar{y}\cdot\nabla)y - (B\cdot\nabla)\bar{B} - (\bar{B}\cdot\nabla)B) \\ P((y\cdot\nabla)\bar{B} + (\bar{y}\cdot\nabla)B - (B\cdot\nabla)\bar{y} - (\bar{B}\cdot\nabla)y) \end{pmatrix}$$

with $D(A) = D(\mathcal{A}) \subset H \times H$ and $D(A_0) = V_1 \times V_2 \subset H \times H$. Remark that, since $\bar{y}, \bar{B} \in W^{1,\infty}(\Omega)$, for $y \in V_1, B \in V_2$ the products of the type $(y \cdot \nabla)\bar{B}, (\bar{B} \cdot \nabla)y$ appearing in the definition of A_0 are in L^2 and it is easy to see that A_0 is closed, A is semi-positive self-adjoint operator and $D(A) \subset D(A_0)$. Moreover, an estimate of the type

$$|A_0 y| \leq \varepsilon |Ay| + C(\varepsilon)|y|$$

is standard to prove (see e.g. [9]) and it implies that $-\mathcal{A}$ is the generator of an analytic semigroup. Compactness of the resolvent is, finally, a consequence of the Rellich theorem on the compact embedding for Sobolev spaces on bounded domains (i.e. $D(\mathcal{A})$ is compactly embedded in $H \times H$).

The fact that $-\mathcal{A}$ has compact resolvent and generates an analytic semigroup implies that its spectrum $\sigma(\mathcal{A})$ is discrete, with no finite accumulation points and is contained in an angular domain $V_{\alpha,\theta} := \{z \in \mathbb{C} : \arg(z - \alpha) \in (-\theta, \theta)\}$ with some $\alpha \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$.

ii) Approximate controllability in time T for problem (14) is equivalent to the unique continuation property for the dual equation, i.e.

$$-\xi'(t) + \mathcal{A}^*\xi(t) = 0 \text{ and } \mathcal{B}^*\xi(t) = 0\,, t \in (0,T) \Rightarrow \xi(t) = 0,\ t \in (0,T) \tag{16}$$

Let $\xi = \begin{pmatrix} \zeta \\ C \end{pmatrix}$ with ζ, C written as colon vectors. Then, the dual equation (16) may be rewritten as :

$$\begin{cases} -\zeta_t - \Delta\zeta + (\nabla C)\bar{B} - (\nabla\zeta)\bar{y} + (\nabla\bar{B})^T C & \\ \quad +(\nabla\bar{y})^T\zeta + \nabla\pi = 0 & \text{in } Q, \\ -C_t - \Delta C + (\nabla\zeta)\bar{B} - (\nabla C)\bar{y} - (\nabla\bar{B})^T\zeta & \\ \quad -(\nabla\bar{y})^T C + \nabla\rho = 0 & \text{in } Q, \\ \nabla\cdot\zeta = 0,\ \nabla\cdot C = 0 & \text{in } Q, \\ \zeta = 0,\ C\cdot\mathbf{n} = 0,\ \text{rot } C = 0 & \text{on } \Sigma. \end{cases} \tag{17}$$

Here, terms like ∇C or $(\nabla\bar{B})^T$, which is the transpose of $\nabla\bar{B}$, are Jacobian matrices and belong to $\mathcal{M}_{2\times2}(\mathbb{R})$, while terms of the type $(\nabla C)\bar{B}$ are simply the matrix products and belong to $\mathcal{M}_{2\times1}(\mathbb{R})$. Observe also that $(\nabla C)\bar{B} = (\bar{B}\cdot\nabla)C$.

In order to make explicit the unique continuation property we need to compute $\mathcal{B}^* = \begin{pmatrix} \mathcal{B}_1^* \\ \mathcal{B}_2^* \end{pmatrix}$. It is easy to see that

$$\mathcal{B}_1^*\zeta = \left(\varphi - \frac{1}{\mid \omega \mid}\int_\omega \varphi dx\right)\Big|_\omega,$$

where $\zeta = (-\frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_1})^T$ and

$$\mathcal{B}_2^* C = -\psi \mid_\omega,$$

where ψ is the stream function of C, $C = (-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1})^T$ with $\psi = 0$ on the boundary $\partial\Omega$.

It is now easy to see that the unique continuation property that has to be proved reads:

$$\varphi \equiv cst(t),\ \psi \equiv 0 \text{ in } \omega \times (0,T) \Longrightarrow \zeta \equiv 0, C \equiv 0 \text{ in } Q \tag{18}$$

reduces to the unique continuation property:

$$\zeta \equiv 0,\ C \equiv 0 \text{ in } \omega \times (0,T) \Longrightarrow \zeta \equiv 0, C \equiv 0 \text{ in } Q. \tag{19}$$

In order to prove the unique continuation property (19), we apply the divergence operator to (17) and obtain a system of parabolic-elliptic equations in ζ, π, C, ρ and the conclusion is a consequence of the following result proved in [10], Theorem 2.1:

Consider systems of the type

$$\begin{cases} \dfrac{\partial y_i}{\partial t} + L_i y_i + \displaystyle\sum_{k=1}^m \sum_{l=1}^n \gamma_{ikl}(x,t)\frac{\partial y_k}{\partial x_l} + \sum_{k=1}^m \varsigma_{ik}(x,t) y_k = 0,\ 1 \le i \le p \\ L_i y_i + \displaystyle\sum_{k=1}^m \sum_{l=1}^n \gamma_{ikl}(x,t)\frac{\partial y_k}{\partial x_l} + \sum_{k=1}^m \varsigma_{ik}(x,t) y_k = 0, \quad p \le i \le m \end{cases} \tag{20}$$

where L_i are linear second order homogeneous uniformly elliptic operators with C^1 coefficients, $\gamma_{ikl} \in L^\infty_{loc}(Q)$, $\varsigma_{ik} \in L^\infty(0,T; L^n_{loc}(\Omega))$.

The unique continuation result of [10] says that if $y_i = 0$ on $\omega \times (0,T), i = \overline{1,m}$, then $y_i \equiv 0$ on $\Omega \times (0,T), i = \overline{1,m}$.

iii) The spectral decomposition we are doing now consists in separating the spectrum of $\mathcal{A}$ in a stable part and an unstable one. Let $\delta > 0$ be such that $\sigma(\mathcal{A}) \cap \{\lambda | \text{ Re } \lambda = \delta\} = \phi$. Let $\sigma_1 = \sigma(\mathcal{A}) \cap \{\lambda | \text{ Re } \lambda < \delta\}$, $\sigma_2 = \sigma(\mathcal{A}) \cap \{\lambda | \text{ Re } \lambda > \delta\}$. It is clear that σ_1 is a finite set and $\sigma_2 \subset V_{\delta,\theta'}$ or some $\theta' \in (0, \frac{\pi}{2})$. Correspondingly, the complexified space $(H \times H)^c$ is decomposed as a direct sum of two closed subspaces $H_1 \bigoplus H_2$, subspaces which are invariant for $\mathcal{A}$ (we denoted also by $\mathcal{A}$ the complexified operator) and $\sigma(\mathcal{A}|H_i) = \sigma_i, i = 1, 2$. Of course H_1 is finite dimensional and let $N = \dim H_1$. Denote by P_N the projection onto H_1 given by the direct sum $H_1 \bigoplus H_2$ and by $Q_N = I - P_N$. Then, with $z_1 = P_N z$ and $z_2 = Q_N z$, equation (14) projects into two equations:

$$z_1' + \mathcal{A} z_1 = P_N \mathcal{B} w \tag{21}$$

$$z_2' + \mathcal{A} z_2 = Q_N \mathcal{B} w. \tag{22}$$

The operator $-\mathcal{A}_2 = -Q_N \mathcal{A}$ generates on H_2 a stable analytic semigroup that satisfies:

$$|e^{-t \mathcal{A}_2} z_2^0| \leq C e^{-\delta t} |z_2^0|. \tag{23}$$

Equation (21) is a finite dimensional linear equation in the space H_1. Moreover, equation (21) is exactly controllable in any time T. Indeed, we proved that equation (14) is approximately controllable in any time T, so the set $\{z^w(T) : w \in L^2(0, T; (L^2(\omega)))^2\}$ is dense in $H \times H$. So the projection of this set, through P_N, on H_1, which is finite dimensional, is the whole space that is $\{z_1^w(T) : w \in L^2(0, T; (L^2(\omega))^2)\} = H_1$. Moreover, if we choose as $U \subset (L^2(\omega))^2$ an N dimensional subspace such that Im $P_N \mathcal{B} = P_N \mathcal{B}(U)$ the pair $(\mathcal{A}_1, P_N \mathcal{B})$ remains exactly controllable in any time T and thus is completely stabilizable (see [14]), i.e. for any $\delta_1 > 0$ there exits a linear operator $K_1 : H_1 \to U$ and a constant $C = C(\delta_1)$ such that

$$\|e^{-t(\mathcal{A}_1 - P_N \mathcal{B} K_1)}\| \leq C e^{-\delta_1 t}. \tag{24}$$

The feedback K, that we use to stabilize the linear system (14), is defined as

$$K = \text{ Re } \tilde{K}, \ \tilde{K} = K_1 \circ P_N.$$

We denote by $z^{\tilde{K}}, z_1^{\tilde{K}}, z_2^{\tilde{K}}$ the corresponding solutions of (14),(21) respectively (22). The only estimate to put in evidence is on the corresponding solution of (22) because we have by the complete stabilization of (21) that

$$|z_1^{\tilde{K}}(t)| \leq C e^{-\delta_1 t} |z_1^0|. \tag{25}$$

Variations of constants formula gives

$$z_2^{\tilde{K}}(t) = e^{-t\ \mathcal{A}_2} z_2^0 + \int_0^t e^{-(t-s)\mathcal{A}_2} Q_N \mathcal{B} K_1 z_1(s) ds.$$

Passing to the norm and using the estimates (23) and (25) we obtain

$$|z_2^{\tilde{K}}(t)| \leq Ce^{-\delta t}|z_2^0| + \int_0^t Ce^{-\delta(t-s)} e^{-\delta_1 s}|z_1^0| ds,$$

from where, for a $\delta_1 > \delta$ and a constant $C = C(\delta, \delta_1)$,

$$|z_2^{\tilde{K}}(t)| \leq Ce^{-\delta t}|z_0|.$$

This, together with (25), give (15) and we conclude the proof of the theorem. □

4. Local stabilization of the MHD system. Proof of Theorem 1. For the stabilization of the nonlinear system we need the following fundamental lemma and its corollary. This is probably a well known result but since we could not find a classical reference for it, we mention here the author's paper [8] (see also [7]) where a proof of this result is given.

Lemma 1. *Let $\tilde{H}$ be a Hilbert space with norm $|\cdot|$ and scalar product $(\cdot,\cdot)$ and let $-\tilde{\mathcal{A}}$ be the generator of an analytic semigroup of negative type satisfying an estimate of the type* (15) *and such that $D(\tilde{\mathcal{A}}) = D(\tilde{\mathcal{A}}^*)$. The following facts hold:*

1. *The quadratic functional*

$$h(z) = \int_0^\infty |\tilde{\mathcal{A}}^{\frac{1}{2}} e^{-t\tilde{\mathcal{A}}} z|^2 dt$$

is finite for all $z \in \tilde{H}$ and defines an equivalent norm in $\tilde{H}$.

2. *Given $\alpha > 0$ the quadratic form*

$$h(z) = \int_0^\infty |\tilde{\mathcal{A}}^{\alpha+\frac{1}{2}} e^{-t\tilde{\mathcal{A}}} z|^2 dt$$

defines an equivalent norm in $D(\mathcal{A}^\alpha)$ that is

$$h(z) \sim |\tilde{\mathcal{A}}^\alpha z|^2.$$

Moreover,

$$h(z) = (Rz, z)$$

where R is an unbounded self-adjoint operator in H with continuous embedding $D(\mathcal{A}^{2\alpha}) \subset D(R)$. For $z \in D(\mathcal{A})$ we have

$$(Rz, \tilde{\mathcal{A}}z) = \frac{1}{2}|\tilde{\mathcal{A}}^{\alpha+\frac{1}{2}}z|^2. \tag{26}$$

We are now in a position to stabilize the nonlinear system (11). We choose $\alpha = \frac{1}{2}$ and construct the self-adjoint operator R using the Lemma 1. We introduce the feedback K and system (11) may be rewritten as

$$\begin{cases} z' + \tilde{\mathcal{A}}z = g(z) \\ z(0) = z_0 \end{cases} \tag{27}$$

where the nonlinearity g is

$$g\begin{pmatrix} y \\ B \end{pmatrix} = \begin{pmatrix} P((B\cdot\nabla)B - (y\cdot\nabla)y) \\ P((B\cdot\nabla)y - (y\cdot\nabla)B) \end{pmatrix}.$$

The idea, the same as in [1] or [5], is to multiply equation (27), scalarly in $\tilde{H}$, with Ry and integrate.

We obtain

$$\frac{d}{dt}(Rz, z) + (\tilde{\mathcal{A}}z, Rz) = (g(z), Rz). \tag{28}$$

Standard estimates using the inequality (6) for the trilinear term b (with $m_1 = m_2 = 1, m_3 = 0$) show that, for $z, \zeta \in D(\tilde{\mathcal{A}}^{\frac{1}{2}}) = V_1 \times V_2$, one has

$$|(g(z), \zeta)| \leq C\|z\|_1\|z\|_2|\zeta| \leq C|\tilde{\mathcal{A}}^{\frac{1}{2}}||\mathcal{A}z||\zeta|. \tag{29}$$

Since $D(\tilde{\mathcal{A}}^{2\alpha}) \subset D(R)$ continuously, there exists a constant C such that for $\zeta \in D(\mathcal{A}^{2\alpha})$

$$|R\zeta| \leq C|\tilde{\mathcal{A}}^{2\alpha}\zeta|. \tag{30}$$

Using (29) with $\zeta = Rz$, we obtain from (28), (30) and (26) that

$$\frac{d}{dt}(Rz, z) + |\tilde{\mathcal{A}}z|^2 \leq C(Rz, z)^{\frac{1}{2}}|\tilde{\mathcal{A}}z|^2 \tag{31}$$

With $\rho = \dfrac{1}{4C^2}$ it is easy to see that the set $\mathcal{V}_\rho = \{z_0 : (Rz_0, z_0) < \rho\}$ is invariant under the flow generated by (27). Moreover, for $z_0 \in \mathcal{V}_\rho$ one has

$$\frac{d}{dt}(Rz, z) + \frac{1}{2}|\tilde{\mathcal{A}}z|^2 \leq 0$$

and thus, since $c(Rz, z) \leq \|z\|^2 \leq C|\tilde{\mathcal{A}}z|^2$, with some constant $\delta > 0$,

$$\frac{d}{dt}(Rz, z) + \delta(Rz, z) \leq 0.$$

Integrating the last inequality one finds that

$$|(Rz, z)| \leq |(Rz_0, z_0)|e^{-\delta t}.$$

The proof is complete since the norm in $D(\tilde{\mathcal{A}}^{\frac{1}{2}})$ is equivalent with the norm induced from $V_1 \times V_2$. $\square$

REFERENCES

1. Barbu, V. – *Feedback stabilization of Navier-Stokes equations*, ESAIM Control Optim. Calc. Var., 9(2003), 197–206 (electronic).
2. Barbu, V.; Havârneanu, T.; Popa, C.; Sritharan, S.S. – *Local exact controllability for the magnetohydrodynamic equations revisited*, Adv. Differential Equations, 10(2005), 481–504.
3. Barbu, V.; Lasiecka, I.; Triggiani, R. – *Abstract settings for tangential boundary stabilization of NavierStokes equations by high- and low-gain feedback controllers,* Nonlinear Analysis, 64(2006), 2704–2746.
4. Barbu, V.; Lasiecka, I.; Triggiani, R. – *Tangential Boundary Stabilization of Navier-Stokes Equations*, Mem. Amer. Math. Soc. 181, 2006.
5. Barbu, V.; Triggiani, R. – *Internal stabilization of Navier-Stokes equations with finite-dimensional controllers,* Indiana Univ. Math. J., 53(2004), 1443–1494.
6. Ladyženskaja, O.A.; Solonnikov, V.A. – *The linearization principle and invariant manifolds for problems of magnetohydrodynamics* (Russian), Boundary value problems of mathematical physics and related questions in the theory of functions, 7. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI), 38(1973), 46–93.
7. Lefter, C. – *Feedback stabilization of magnetohydrodynamic equations,* to appear.
8. Lefter, C. – *Feedback stabilization of Navier-Stokes equations with Navier-slip boundary conditions,* Nonlinear Analysis TMA, 79(2009), 553-562.

9. Pazy, A. – *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, 44, Springer-Verlag, New York, 1983.
10. Saut, J.-C.; Scheurer, B. – *Unique continuation for some evolution equations,* J. Differential Equations, 66(1987), 118–139.
11. Sritharan, S.S. – *Invariant Manifold Theory for Hydrodynamic Transition*, Pitman Research Notes in Mathematics Series, 241, Longman Scientific & Technical, Harlow, 1990.
12. Temam, R. – *Navier-Stokes Equations*, Theory and numerical analysis, Reprint of the 1984 edition. AMS Chelsea Publishing, Providence, RI, 2001,
13. Yudovich, V.I. – *The Linearization Method in Hydrodynamical Stability Theory*, Translations of Mathematical Monographs, 74, American Mathematical Society, Providence, RI, 1989.
14. Zabczyk, J. – *Mathematical Control Theory: An Introduction*, Systems & Control: Foundations & Applications, Birkhäuser Boston Inc., Boston, MA, 1992.

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