Lyapunov Function and Local Feedback Boundary Stabilization of the Navier-Stokes Equations

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Abstract. We study the local exponential stabilization, near a given steady-state flow, of solutions of the Navier-Stokes equations in a bounded domain. The control is effectuated through a Dirichlet boundary control. We apply a linear feedback controller, provided by an algebraic Riccati equation. We give a class of initial conditions for which a Lyapunov function is obtained. For all $s \in [0, \frac{1}{2}[$, the stabilization of the 2D Navier-Stokes equations is proved for initial condition in $\mathbf{H}^s(\Omega) \cap V_n^0(\Omega)$, where $V_n^0(\Omega)$ is space in which the Stokes operator is defined.

Key words. Navier-Stokes equation, Feedback stabilization, Dirichlet boundary control, Lyapunov function, Riccati equation.

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1 Introduction

Let Ω be a bounded and connected domain in $\mathbb{R}^d$ for $d = 2$ or $d = 3$, with a boundary $\Gamma = \partial\Omega$ of class C^4, and composed of N connected components $\Gamma^{(1)}, \dots, \Gamma^{(N)}$. Let us consider a stationary motion of an incompressible fluid in Ω described by the couple (z_s, p_s), the velocity and the pressure, which is a regular solution to the stationary Navier-Stokes equations:

$$-\nu\Delta z_s + (z_s \cdot \nabla)z_s + \nabla r_s = f, \quad \nabla \cdot z_s = 0 \text{ in } \Omega \quad \text{ and } \quad z_s = v_b \text{ on } \Gamma. \tag{1.1}$$

In this setting, $\nu > 0$ is the viscosity, $f \in \mathbf{H}^1(\Omega)$ and $v_b \in \mathbf{H}^{\frac{5}{2}}(\Gamma)$ obeys $\int_{\Gamma^{(j)}} v_b \cdot n = 0$, for all $j = 1 \dots N$, where n denotes the unit normal vector to Γ, exterior to Ω. Notice that here and in the following, we write in bold the spaces of vector fields: $\mathbf{H}^1(\Omega) = (H^1(\Omega))^d$, $\mathbf{H}^{\frac{5}{2}}(\Gamma) = (H^{\frac{5}{2}}(\Gamma))^d$, etc. We recall that a solution to (1.1) is known to exist in $\mathbf{H}^3(\Omega) \times H^2(\Omega)/\mathbb{R}$ (see [12, Chap. VIII, Thm. 4.1 and Thm. 5.2]).

If z_s is an unstable equilibrium state, and if we assume that at time $t = 0$ the velocity is equal to $z_0 \neq z_s$, then even if z_0 is close to z_s, the resulting unsteady velocity $z(t)$ when $t > 0$ will not necessary stay close to z_s. Hence, in order that $z(t)$ go back to z_s as $t \to \infty$, we are interested in finding a feedback controller which is localized on the boundary Γ. We want to find a pair $(F_\tau, F_n) : \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Gamma) \times \mathbf{L}^2(\Gamma)$ such that the unsteady velocity and pressure (z, r) satisfy the following instationary self-regulated Navier-Stokes equations:

$$\partial_t z - \nu\Delta z + (z \cdot \nabla)z + \nabla r = f, \quad \nabla \cdot z = 0 \ \text{ in } \ (0, \infty) \times \Omega, \quad z(0) = z_0, \tag{1.2}$$

$$\gamma_\tau z = \gamma_\tau v_b + F_\tau(z - z_s) \ \text{ on } \ (0, \infty) \times \Gamma, \tag{1.3}$$

$$\gamma_n z = \gamma_n v_b + F_n(z - z_s) \ \text{ on } \ (0, \infty) \times \Gamma. \tag{1.4}$$

In this setting, the operators γ_τ and γ_n respectively are the tangential and the normal boundary trace operators.

The question of stabilizing the Navier-Stokes equations with a boundary control has already been addressed by A.V. Fursikov in [9, 10], where stabilizability results for the 2D and 3D Navier-Stokes system are proved. With an adequate extension procedure for the initial condition which needs the knowledge of the eigenfunctions of the Oseen operator, the author obtains a boundary control of the form $u = F_0 z_0$ where $F_0 \in \mathcal{L}(\mathbf{X}(\Omega), L^2(0, \infty; \mathbf{L}^2(\Gamma))$ and $\mathbf{X}(\Omega) \subset \mathbf{H}^1(\Omega)$. Despite the law F_0 is referred as a "feedback law", it is not a feedback law in a standard sense. The law F_0 is not pointwise in time.

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To obtain a pointwise (in time) feedback law, another strategy consists in studying an auxiliary optimal control problem. The idea is the following. We linearize (1.2) around (z_s, r_s) and we apply an unknown boundary control $u \in L^2((0,\infty)\times\Gamma)$ to the resulting linear system:

$$\partial_t y - \nu\Delta y + (y\cdot\nabla)z_s + (z_s\cdot\nabla)y + \nabla p = 0 \quad \text{in } (0,\infty)\times\Omega, \tag{1.5}$$

$$\nabla\cdot y = 0 \text{ in } (0,\infty)\times\Omega, \quad y = u \text{ on } (0,\infty)\times\Gamma, \quad y(0) = y_0. \tag{1.6}$$

For an adequate observation operator $\mathcal{C} : \mathcal{D}(\mathcal{C}) \subset \mathbf{L}^2(\Omega) \to \mathcal{Z}$ (a detectablility condition has to be satisfied by the pair composed of the free dynamic operator of system (1.5)-(1.6) and of $\mathcal{C}$, see [14, Chap. 2, (H-5) and Thm. 2.2.2])), the minimizing problem

$$\inf\left\{ \frac{1}{2}\int_0^\infty \|\mathcal{C}y(t)\|_{\mathcal{Z}}^2 dt + \frac{1}{2}\int_0^\infty\int_\Gamma |u(t)|^2 dt \;\middle|\; (y,u) \text{ satisfies } (1.5)-(1.6)\right\} \tag{1.7}$$

provides a linear operator Π which is the unique solution to an algebraic Riccati equation. Hence, one obtains a feedback law F, depending on Π, such that the control $u(t) = -Fy(t)$, for all $t>0$, stabilizes the linear system (1.5)-(1.6). Such a method is followed by J.-P Raymond in [18] where a feedback stabilization result for the two dimensional Navier-Stokes system is obtained. The author uses the feedback law F which is deduced from (1.7) where the observation operator $\mathcal{C}$ is equal to the identity in $\mathbf{L}^2(\Omega)$. The stabilization result is obtained by a fixed point method which is based on the use of regularity results for the optimal coupled system related to (1.7). However, even if the study of the auxiliary control problem remains valid in the three dimensional case, the nonlinear analysis in [18] failed when $d=3$. Indeed, if $d=3$, the nonlinearity of the Navier-Stokes equations imposes to define smoother solutions. In particular, a trace compatibility condition has to be verified by the initial condition. So if we use the feedback law given in [18], then the initial condition must verify the equality $y_0|_\Gamma = Fy_0$. In [17], the author overcomes this difficulty by introducing a feedback law F, which is time dependent in a transitory time interval $[0,t_0]$, and which is such that $F(0)=0$. Then it allows to obtain a stabilization result when $d=3$. Moreover, the feedback law can be calculated with a differential Riccati equation on $[0,t_0]$, and with an algebraic Riccati equation on $]t_0,+\infty[$. Notice that in [18, 17], the use of a smooth function supported in an open subset Γ_c of Γ permits to localize the control in an open part of the boundary.

We propose an alternative method in [1, 2]. We look for a control u which is also solution to an evolution equation on Γ. By using an auxiliary optimal control problem for the whole system satisfied by the new state variable (y,u), we obtain a feedback law for the extended system. Then the initial trace compatibility condition is guaranteed and we obtain a dynamic controller u which stabilizes the Navier-Stokes system in the two and in the three dimensional case.

V. Barbu, I. Lasiecka et R. Triggiani in [5] also obtain a boundary feedback stabilization result for the Navier-Stokes system. When $d=2$, the authors use a cost functional in (1.7) for which the observation term is the norm of $L^2(0,\infty;\mathbf{H}^{\frac{3}{2}-\epsilon}(\Omega))$ for $\epsilon\in]0,\frac{1}{2}[$. When $d=3$ they use a cost functional which involves the norm of the state in $L^2(0,\infty;\mathbf{H}^{\frac{3}{2}+\epsilon}(\Omega))$ for $\epsilon>0$. By this way, they prove that the value function of their optimal control problem is a Lyapunov function for the controlled Navier-Stokes system. Moreover, if $d=2$, the feedback control can be chosen in a finite dimensional space and can be localized in a part of the boundary. However, their approach has two drawbacks: *(i)* the boundary control is supposed to be tangential, *(ii)* there is no constructive method to calculate the feedback law when $d=3$. Indeed, as it is explained in [4, 18], the Riccati equation which is obtained is ill-posed. It is defined in $\mathcal{D}(A_R^2)$, where A_R is the infinitesimal generator of the closed loop system and depends on the unknown R (see [5, Prop. 4.5.1, (4.5.1)]). This difficulty is closely linked to the trace compatibility condition which is necessary to obtain the exponential decrease of the solution of the Navier-Stokes system in the three dimensional case. Indeed, to impose this condition, the authors chose an observation operator with a too high degree of unboundedness ($\mathcal{D}(\mathcal{C}) = \mathbf{H}^{\frac{3}{2}+\epsilon}(\Omega)$) and it does not allow to define the Riccati equation in a classical sense.

In this paper, we discuss the use of the optimal control theory to stabilize the Navier-Stokes system in the two or in the three dimensional case. First, we suggest that the choice of the observation operator $\mathcal{C}$ is not decisive to obtain a well-posed algebraic Riccati equation, and to obtain a feedback law which stabilizes the Navier-Stokes system. Indeed, this has already been underlined in [3] for a distributed control, and if we recall [5], the well posedness of the Riccati equation can be lost if we choose an observation operator with a too high degree of unboundedness. Thus, let us choose the following minimizing problem as a model problem:

$$\inf\left\{ \frac{1}{2}\int_0^\infty \|Py(t)\|^2_{\mathbf{L}^2(\Omega)}dt + \frac{1}{2}\int_0^\infty \int_\Gamma |u(t)|^2 dt \;\middle|\; (y,u) \text{ satisfies } (1.5)-(1.6)\right\}. \tag{1.8}$$

Here, P is the orthogonal projector from $\mathbf{L}^2(\Omega)$ onto $V^0_n(\Omega) = \{y \in \mathbf{L}^2(\Omega) \mid \nabla\cdot y = 0 \text{ in } \Omega,\ \int_\Gamma y\cdot n = 0\}$ (the space in which the Stokes operator is defined). As in [18], the identity in $\mathbf{L}^2(\Omega)$ may also be chosen as an observation operator. But since the solution y to (1.5)-(1.6) is entirely determined by Py and u, the choice of P is preferable (see [17]). Hence, problem (1.8) provides a calculable Riccati operator Π and a feedback law F, which allows to define a closed-loop Navier-Stokes system. In some sense which will be precised later on, it can be proved that $z = z_s + y$ is solution to system (1.2)-(1.3)-(1.4), if and only if, y is solution to the following nonlinear system:

$$Py' + A_\Pi Py + N_\Pi(Py) = 0, \quad y(0) = y_0 \quad \text{and} \quad (I-P)y = (I-P)DFy. \tag{1.9}$$

Here, A_Π is a linear operator which describes the closed-loop dynamic, $N_\Pi(\cdot)$ is a nonlinear operator and D is an adequate lifting operator. The main interest of our approach is that: i) we characterize the domain of A_Π:

$$\mathcal{D}(A_\Pi) = \left\{ y \in \mathbf{H}^2(\Omega)\cap V^0_n(\Omega) \;\mid\; (I - PDF)y \in \mathbf{H}^2(\Omega)\cap \mathbf{H}^1_0(\Omega)\right\},$$

ii) we define a solution to (1.9) for an initial condition $Py_0 \in \mathcal{D}(A_\Pi^{\frac{s}{2}})$ when $s \in [\frac{d-2}{2}, 1]$, iii) we exhibit a Lyapunov function for the closed-loop Navier-Stokes system (1.9). More precisely, we will prove that for all $Py_0 = P(z_s - z_0)$ small enough in $\mathcal{D}(A_\Pi^{\frac{s}{2}})$, the function

$$V_s : \xi \longmapsto \langle P\xi \mid \Pi^{(s)}P\xi\rangle \quad \text{where} \quad \Pi^{(s)} = A_\Pi^{*\frac{s}{2}+\frac{1}{2}}\Pi A_\Pi^{\frac{1}{2}+\frac{s}{2}},$$

is a Lyapunov function for the system (1.9) satisfied by $y = z - z_s$. As a consequence, since we have $\mathcal{D}(A_\Pi^{\frac{s}{2}}) = \mathbf{H}^s(\Omega)\cap V^0_n(\Omega)$ when $s \in [0, \frac{1}{2}[$, for Py_0 small enough in $\mathbf{H}^s(\Omega)\cap V^0_n(\Omega)$ we obtain a 2D stabilisation result generalizing the one of [18] when $s \in [0, \frac{1}{4}]$. However, because trace conditions appear in the definition of $\mathcal{D}(A_\Pi^{\frac{s}{2}})$ when $s \geq \frac{1}{2}$, we only obtain a 3D stabilisation result for a very specific set of initial conditions. Indeed, when $d = 3$ we need to assume that Py_0 is small enough in $\mathcal{D}(A_\Pi^{\frac{s}{2}})$ for $s \in [\frac{1}{2}, 1]$, for which we only have a strict inclusion $\mathcal{D}(A_\Pi^{\frac{s}{2}}) \subsetneq \mathbf{H}^s(\Omega)\cap V^0_n(\Omega)$. More precisely, when $s > \frac{1}{2}$, it can be proved that if $z_0 - z_s$ is small in the space

$$V^s(\Pi,\Omega) = \left\{ y \in \mathbf{H}^s(\Omega) \;\mid\; \nabla\cdot y = 0 \text{ in } \Omega,\ \ \mathcal{T}_\Pi(y) = 0 \text{ on } \Gamma \right\},$$

where

$$\mathcal{T}_\Pi(y) = y|_\Gamma - \partial_n \Pi y + rn \quad \text{and} \quad \left\{\begin{array}{l} \Delta r = \nabla\cdot(z_s\cdot\nabla - (\nabla z_s)^T)\Pi y \ \text{ in } \Omega,\ \displaystyle\int_\Gamma r = 0 \\ \partial_n r = (\nu\Delta - (\nabla z_s)^T + z_s\cdot\nabla)\Pi y\cdot n \quad \text{on} \quad \Gamma, \end{array}\right. \tag{1.10}$$

then we have $P(z_0 - z_s) \in \mathcal{D}(A_\Pi^{\frac{s}{2}})$ and the solution to (1.9) is exponentially stable.

The paper is organized as follows. In Section **2**, we give an abstract formulation for the Navier-Stokes system and we state our main local stabilization result. The Section **3** is dedicated to the optimal control problem which provides a feedback law depending on the solution to an algebraic Riccati equation. The space of initial conditions $\mathcal{D}(A_\Pi^{\frac{s}{2}})$ is characterized in Section **4**, and a Lyapunov function for the Oseen closed-loop system is given in Section **5**. In Section **6**, we apply the feedback law to the nonlinear system, and, by exhibiting a Lyapunov function, we prove a local stabilization result for an initial condition in $\mathcal{D}(A_\Pi^{\frac{s}{2}})$ when $s \geq \frac{d-2}{2}$. The Section **7** deals with a boundary control which is localized in a part of Γ. Some regularity results needed throughout the paper are recalled in an appendix.

2 Preliminaries and main results

2.1 Notations

Let X and Y be two Banach spaces. If A is a closed linear mapping in X, we denote its domain by $\mathcal{D}(A)$ and we denote its spectrum by $\sigma(A)$. Moreover, we denote by $\mathcal{L}(X,Y)$ the space of all bounded operators from X to Y, and we use the shorter expression $\mathcal{L}(X) = \mathcal{L}(X,X)$. For $0 < T \leq \infty$, the space $L^2(0,T;X)$ is the well known Lebesgue space and we also define:

$$W(0,T;X,Y) = \left\{ y \in L^2(0,T;X) \ \mid \ \frac{dy}{dt} \in L^2(0,T;Y) \right\}.$$

When $T \in (0,+\infty)$, it is well known that if X is continuously and densely embedded in Y, then the space $W(0,T;X,Y)$ is continuously embedded in $C([0,T];[X,Y]_{\frac{1}{2}})$ (see [6, Rem. 4.1, p. 95 and Prop. 4.3, p. 99]).

Next, let us recall that Ω is a bounded and connected domain in $\mathbb{R}^d$, for $d = 2$ or $d = 3$, with a boundary $\Gamma = \partial\Omega$ of class C^4, and composed of N connected components $\Gamma^{(1)}, \ldots, \Gamma^{(N)}$. We will use the usual function spaces $L^2(\Omega)$, $H^s(\Omega)$, $H^s_0(\Omega)$ and $H^{-s}(\Omega) = (H^s_0(\Omega))'$, and we write in bold the spaces of vector fields $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}^s(\Omega) = (H^s(\Omega))^d$, $\mathbf{H}^s_0(\Omega) = (H^s_0(\Omega))^d$ and $\mathbf{H}^{-s}(\Omega) = (H^{-s}(\Omega))^d$. The norms are denoted by $\| \cdot \|_{X(\Omega)}$, where the subscript $X(\Omega)$ refers to the space which is considered, and we denote the scalar product in $\mathbf{L}^2(\Omega)$ by $(\cdot|\cdot)$. Moreover, if $y \in \mathbf{L}^2(\Omega)$ is such that $\nabla \cdot y \in L^2(\Omega)$, then we denote the normal trace of y in $H^{-\frac{1}{2}}(\Gamma)$ by $y \cdot n$ (see [11, III. 3]).

Thus, we introduce the spaces of free divergence functions:

$$\begin{aligned} V^s(\Omega) &= \left\{ y \in \mathbf{H}^s(\Omega) \mid \nabla \cdot y = 0 \text{ in } \Omega, \int_\Gamma y \cdot n = 0 \right\}, \quad s \in [0,2], \\ V^s_n(\Omega) &= \left\{ y \in \mathbf{H}^s(\Omega) \mid \nabla \cdot y = 0 \text{ in } \Omega,\ y \cdot n = 0 \right\}, \quad s \in [0,2]. \end{aligned}$$

We define the interpolation space:

$$V^s_0(\Omega) = \Big[V^2_n(\Omega) \cap \mathbf{H}^1_0(\Omega)\,,\, V^0_n(\Omega) \Big]_{1-s/2}, \quad s \in [0,2]$$

and

$$\begin{aligned} V^s_0(\Omega) &= V^2_0(\Omega) \cap \mathbf{H}^s(\Omega), \quad s > 2 \\ V^{-s}_0(\Omega) &= (V^s_0(\Omega))', \qquad s \geq 0. \end{aligned}$$

In this setting, $(V^s_0(\Omega))'$ is the dual space of $V^s_0(\Omega)$ with respect to the pivot space $V^0_n(\Omega)$. It is equipped with the duality pairing $\langle \cdot | \cdot \rangle_{V^{-s}_0(\Omega), V^s_0(\Omega)}$. The following equalities are well known:

$$\begin{aligned} V^s_0(\Omega) &= V^s_n(\Omega), \quad s \in [0, 1/2[, \\ V^{1/2}_0(\Omega) &= \left\{ y \in V^{1/2}_n(\Omega) \ \mid \ \int_\Omega \rho(x)^{-1} |y|^2 < +\infty \right\}, \\ V^s_0(\Omega) &= \left\{ y \in V^s_n(\Omega) \ \mid \ y|_\Gamma = 0 \right\} \quad \text{if} \quad s > 1/2, \end{aligned}$$

where $\rho(x)$ is the distance from x to Γ. Notice that, according to the above definition, the subscript 0 only means that we have vanishing Dirichlet boundary condition.

Next, we define the spaces of pressures with free mean

$$L^2_0(\Omega) = \left\{ p \in L^2(\Omega) \mid \int_\Omega p = 0 \right\} \quad \text{and} \quad \mathcal{H}^s(\Omega) = H^s(\Omega) \cap L^2_0(\Omega), \qquad s \geq 0,$$

and we recall that the following *Helmholtz Decomposition* holds:

$$\mathbf{L}^2(\Omega) = V^0_n(\Omega) \oplus \nabla \mathcal{H}^1(\Omega).$$

Thus, we introduce the well known Leray projector $P \in \mathcal{L}(\mathbf{L}^2(\Omega), V_n^0(\Omega))$ which is the orthogonal projector from $\mathbf{L}^2(\Omega)$ onto $V_n^0(\Omega)$ (see [11, Chap. III, Thm. 1.1]). It is well known that P can be extended to a bounded linear operator from $\mathbf{H}^{-1}(\Omega)$ onto $V_0^{-1}(\Omega)$) by

$$Py : w \in V_0^1(\Omega) \longmapsto \langle y|w\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)}.$$

About the extension of the Leray projector see [5, App. A].

Next, we define the trace space

$$V^s(\Gamma) \quad = \quad \left\{ y \in \mathbf{H}^s(\Gamma) \mid \int_\Gamma y \cdot n = 0 \right\}, \quad s \in [0,2],$$

and we define the tangential trace $\gamma_\tau \in \mathcal{L}(V^0(\Gamma), \mathbf{L}^2(\Gamma))$ and the normal trace $\gamma_n \in \mathcal{L}(V^0(\Gamma), \mathbf{L}^2(\Gamma))$ by

$$\gamma_\tau v = v - (v \cdot n)n \quad \text{and} \quad \gamma_n v = (v \cdot n)n.$$

Moreover, the boundary normal derivative on Γ of a vector field $v \in \mathbf{H}^2(\Omega)$ is defined by $\partial_n v = (\nabla v)n$.

Finally, we introduce time dependent function spaces. We set $Q_T = (0,T) \times \Omega$, $Q = Q_\infty$, $\Sigma_T = (0,T) \times \Gamma$ and $\Sigma = \Sigma_\infty$, and for $s \geq 0$ and $\sigma \geq 0$ we also define

$$\begin{aligned} V^{s,\sigma}(Q_T) &= L^2(0,T; V^s(\Omega)) \cap H^\sigma(0,T; V^0(\Omega)), \\ V^{s,\sigma}(\Sigma_T) &= L^2(0,T; V^s(\Gamma)) \cap H^\sigma(0,T; V^0(\Gamma)). \end{aligned}$$

2.2 Navier-Stokes and Oseen system with nonhomogeneous boundary conditions

This subsection is devoted to the abstract reformulation of the Navier-Stokes and Oseen systems with nonhomogeneous boundary conditions. First, we define the Stokes operator in $V_n^0(\Omega)$ by

$$\mathcal{D}(A) = V_0^2(\Omega), \quad Ay = -\nu P \Delta y.$$

It is well known that $(\mathcal{D}(A), A)$ is nonnegative, self-adjoint and definite, and that its fractional powers A^θ are well defined and satisfy $\mathcal{D}(A^\theta) = V_0^{2\theta}(\Omega)$ for all $\theta \in [0,1]$. Moreover, it is the infinitesimal generator of an analytic semigroup $(e^{-At})_{t>0}$ on $V_n^0(\Omega)$. Next, we introduce the following trilinear form:

$$b(v_1, v_2, v_3) = \int_\Omega (v_1 \cdot \nabla) v_2 \cdot v_3 \quad \forall (v_1, v_2, v_3) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega).$$

This one is known to satisfy the estimate

$$b(v_1, v_2, v_3) \leq C \|v_1\|_{\mathbf{H}^{s_1}(\Omega)} \|v_2\|_{\mathbf{H}^{1+s_2}(\Omega)} \|v_3\|_{\mathbf{H}^{s_3}(\Omega)} \quad \forall (v_1, v_2, v_3) \in \mathbf{H}^{s_1}(\Omega) \times \mathbf{H}^{1+s_2}(\Omega) \times \mathbf{H}^{s_3}(\Omega), \tag{2.1}$$

where s_1, s_2 and s_3 are real positive numbers such that $s_1+s_2+s_3 \geq \frac{d}{2}$ if $s_i \neq \frac{d}{2}$, $i = 1,2,3$ or $s_1+s_2+s_3 > \frac{d}{2}$ if $s_i = \frac{d}{2}$, for at least one i (see [7, Chap. 6, Prop. 6.1, (6.10)]). Hence, we can define the nonlinear mapping

$$N : V^1(\Omega) \to V_0^{-1}(\Omega), \quad \langle N(y)|w\rangle_{V_0^{-1}(\Omega), V_0^1(\Omega)} = b(y,y,w) \quad \forall (y,w) \in V^1(\Omega) \times V_0^1(\Omega), \tag{2.2}$$

and the linear operator

$$\mathcal{D}(A_0) = V_0^1(\Omega), \quad (A_0 y|w) = b(y, z_s, w) + b(z_s, y, w) \quad \forall (y,w) \in V_0^1(\Omega) \times V_n^0(\Omega).$$

The definition of A_0 is consistent because with (2.1) we have

$$|(A_0 y, w)| \leq \|y\|_{V_0^1(\Omega)} \|z_s\|_{\mathbf{H}^2(\Omega)} \|w\|_{V_n^0(\Omega)}.$$

Hence, we can define the so-called Oseen operator $A + A_0$ with domain $\mathcal{D}(A + A_0) = \mathcal{D}(A) = V_0^2(\Omega)$. Since it can be viewed as a perturbation of A with a perturbation term A_0 with domain $\mathcal{D}(A_0) = \mathcal{D}(A^{\frac{1}{2}})$, we obtain the analyticity of $(e^{-(A+A_0)t})_{t>0}$ on $V^0_n(\Omega)$ from the analyticity of $(e^{-At})_{t>0}$ on $V^0_n(\Omega)$ (see [16, Chap.3, Cor.2.4]). Moreover, there exists λ_0 such that

$$\langle(\lambda_0 + A + A_0)y|y\rangle_{V_0^{-1}(\Omega),V_0^1(\Omega)} \geq \frac{\nu}{2}\|y\|^2_{V_0^1(\Omega)} \quad \forall y \in V_0^1(\Omega),$$

and we have

$$\mathcal{D}((\lambda_0 + A + A_0)^\theta) = \mathcal{D}((\lambda_0 + A + A_0^*)^\theta) = V_0^{2\theta}(\Omega) \quad \forall \theta \in [0, 1],$$

(see [19, (4.2), Lem. 4.1]). In this setting, A_0^* is defined by

$$\mathcal{D}(A_0^*) = V_0^1(\Omega), \quad (A_0^*y|w) = b(w, z_s, y) - b(z_s, y, w) \quad \forall (y, w) \in V_0^1(\Omega) \times V_n^0(\Omega),$$

and it can be proved that $A + A_0^*$ is the adjoint of $A + A_0$ with respect to the pivot space $V^0_n(\Omega)$. Thus, for all $\theta \in [0, 1]$, optimal regularity results ensure that the following mapping

$$\begin{array}{lcl} W(0,T; V_0^{2\theta}(\Omega), V_0^{2(\theta-1)}(\Omega)) & \longrightarrow & L^2(0,T; V_0^{2(\theta-1)}(\Omega)) \times V_0^{2\theta-1}(\Omega)), \\ y & \longmapsto & (\ y' + Ay + A_0 y\ ,\ y(0)\ ), \end{array} \tag{2.3}$$

is an isomorphism (see [6, Chap. 3, Par. 2]).

Next, let us introduce the Dirichlet operator $D : V^0(\Gamma) \longrightarrow V^0(\Omega)$ associated with $\lambda_0 + A + A_0$. For $u \in V^0(\Gamma)$ the function $Du = w$ is defined by

$$Du = w \quad \text{and} \quad \lambda_0 w - \nu\Delta w + (w \cdot \nabla)z_s + (z_s \cdot \nabla)w + \nabla q = 0, \quad \nabla \cdot w = 0, \quad w|_\Gamma = u.$$

The next proposition gives precise statements about D.

Proposition 1 *We define $D : V^0(\Gamma) \longrightarrow V^0(\Omega)$ by $Du = w \in V^0(\Omega)$ for $u \in V^0(\Gamma)$, where $w \in V^0(\Gamma)$ is the unique solution to the equation*

$$\int_\Omega w \cdot f = \int_\Gamma u \cdot (rn - \partial_n\varphi), \quad \forall f \in \mathbf{L}^2(\Omega), \tag{2.4}$$

where $(\varphi, r) \in V_0^2(\Omega) \times H^1(\Omega)$ is the unique pair satisfying

$$\lambda_0\varphi - \nu\Delta\varphi + (\nabla z_s)^T\varphi - (z_s \cdot \nabla)\varphi + \nabla r = f \ \text{ in } \Omega, \quad \nabla \cdot \varphi = 0, \quad \varphi|_\Gamma = 0, \quad \int_\Gamma r = 0. \tag{2.5}$$

If $u \in V^{\frac{1}{2}}(\Gamma)$, then $Du = w \in V^1(\Omega)$ and there exists $q \in L^2_0(\Omega)$ such that (w, p) satisfies:

$$\lambda_0 w - \nu\Delta w + (w \cdot \nabla)z_s + (z_s \cdot \nabla)w + \nabla q = 0, \quad \nabla \cdot w = 0, \quad w|_\Gamma = u. \tag{2.6}$$

In this setting, we have used the notation $(\nabla a)^T b = (\sum_{i=1}^d b_i\partial_{x_j}a_i)_{1\leq j\leq d}$. Moreover, we define the mapping $D^* : V^0(\Omega) \longrightarrow V^0(\Gamma)$ by

$$D^*f = r\,n - \partial_n\varphi, \tag{2.7}$$

where $(\varphi, r) \in V_0^2(\Omega) \times H^1(\Omega)$ is the unique pair satisfying (2.5). Finally, $D : V^0(\Gamma) \longrightarrow V^0(\Omega)$ and $D^* : V^0(\Omega) \longrightarrow V^0(\Gamma)$ obey

$$D \in \mathcal{L}(V^{s-\frac{1}{2}}(\Gamma)), V^s(\Omega)) \quad \forall s \in [0, 2], \tag{2.8}$$

$$D^* \in \mathcal{L}(V_0^s(\Omega), V^{s+\frac{1}{2}}(\Gamma)) \quad \forall s \in [0, 2], \tag{2.9}$$

and $D^* \in \mathcal{L}(V^0(\Omega), V^0(\Gamma))$ is the adjoint of $D \in \mathcal{L}(V^0(\Gamma), V^0(\Omega))$.

Proof. See [19, Appendix 2]. □

Remark 1 *The left equality in (2.6) is understood as an equality in $\mathbf{H}^{-1}(\Omega)$ and the divergence and the trace condition in (2.6) are understood as equalities in $L^2(\Omega)$ and in $\mathbf{L}^2(\Gamma)$ respectively.*

Remark 2 *Because φ in (2.7) satisfies $\nabla\cdot\varphi = 0$, we deduce that its normal derivative $\partial_n\varphi$ is tengential (see [5, Lem. 3.3.1]). Then we have $\gamma_n D^* f = r\, n$ and $\gamma_\tau D^* f = -\partial_n\varphi$ where $(\varphi, r) \in V_0^2(\Omega)\times H^1(\Omega)$ is the unique pair satisfying (2.5).*

Definition 1 *Let us define $B \in \mathcal{L}(V^0(\Gamma); V_0^{-2}(\Omega))$ as follows:*

$$\langle Bu|\varphi\rangle_{V_0^{-2}(\Omega),V_0^2(\Omega)} = (PDu|(\lambda_0 + A + A_0)\varphi) \quad \forall (u,\varphi) \in V^0(\Gamma)\times V_0^2(\Omega).$$

Remark 3 *Let $\widetilde{A+A_0}$ be the extention of $A+A_0$ with domain $V_n^0(\Omega)$ in $(\mathcal{D}((A+A_0)^*))'$ obtained with the extrapolation method (see [6] and [14, Par. 0.3]). Here, $\mathcal{D}((A+A_0)^*)'$ is the dual space of $\mathcal{D}((A+A_0)^*)$ with respect to the pivot space $V_n^0(\Omega)$. Thus, since $(\mathcal{D}(A+A_0^*))' = V_0^{-2}(\Omega)$ we deduce that $B = (\lambda_0 + \widetilde{A+A_0})PD \in \mathcal{L}(V^0(\Gamma), (\mathcal{D}(A+A_0^*))')$. As usual, we will keep the notation $A+A_0$ for $\widetilde{A+A_0}$ and we will write $B = (\lambda_0 + A + A_0)PD \in \mathcal{L}(V^0(\Gamma), V_0^{-2}(\Omega))$.*

Proposition 2 *The adjoint of B is the operator $B^* \in \mathcal{L}(V_0^2(\Omega), V^0(\Gamma))$ defined by*

$$\gamma_\tau B^*\varphi = -\partial_n\varphi \quad \text{and} \quad \gamma_n B^*\varphi = rn \quad \forall\varphi \in V_0^2(\Omega),$$

where $r \in H^1(\Omega)$ satisfies the Neumann problem:

$$\Delta r = \nabla\cdot(z_s\cdot\nabla - (\nabla z_s)^T)\varphi \text{ in } \Omega, \quad \int_\Gamma r = 0 \quad \text{and} \quad \partial_n r = (\nu\Delta - (\nabla z_s)^T + z_s\cdot\nabla)\varphi\cdot n \text{ on } \Gamma.$$

Moreover, $B^* \in \mathcal{L}(V_0^2(\Omega), V^0(\Gamma))$ obeys

$$B^* \in \mathcal{L}(V_0^{s+\frac{3}{2}}(\Omega), V^s(\Gamma)) \quad \forall s \in \left]0, \frac{1}{2}\right]. \tag{2.10}$$

Proof. Let $\varphi \in V_0^2(\Omega)$. We start from the expression $B^*\varphi = D^*(\lambda_0 + A + A_0^*)\varphi = -\partial_n\widetilde{\varphi} + rn$, where $(\widetilde{\varphi}, r) \in V_0^2(\Omega)\times H^1(\Omega)$ satisfies

$$\lambda_0\widetilde{\varphi} - \nu\Delta\widetilde{\varphi} + (\nabla z_s)^T\widetilde{\varphi} - (z_s\cdot\nabla)\widetilde{\varphi} + \nabla r = (\lambda_0 + A + A_0^*)\varphi \text{ in } \Omega \quad \text{and} \quad \int_\Gamma r = 0.$$

Hence, by composing this equality by P, we deduce that $\widetilde{\varphi} = \varphi$ and that $\nabla r = (I-P)(\nu\Delta\varphi - (\nabla z_s)^T\varphi + (z_s\cdot\nabla)\varphi)$, which is equivalent to

$$\Delta r = \nabla\cdot(\nu\Delta\varphi - (\nabla z_s)^T\varphi + (z_s\cdot\nabla)\varphi) \text{ in } \Omega \quad \text{and} \quad \partial_n r = (\nu\Delta\varphi - (\nabla z_s)^T\varphi + (z_s\cdot\nabla)\varphi)\cdot n \text{ on } \Gamma.$$

Then we conclude by remarking that $\nabla\cdot\Delta\varphi = 0$. Finally, (2.10) is an easy consequence of $\partial_n \in \mathcal{L}(V_0^{s+\frac{3}{2}}(\Omega), V^s(\Gamma))$ for $s > 0$ and of regularity results for the Laplace problem with a nonhomogeneous Neumann condition. ☐

Proposition 3 *For all $\varepsilon \in]0, \frac{1}{4}[$, we have $(\lambda_0 + A + A_0)^{-\frac{3}{4}-\varepsilon}B \in \mathcal{L}(V^0(\Gamma), V^0(\Omega))$.*

Proof. It is a direct consequence of equality $(\lambda_0 + A + A_0)^{-\frac{3}{4}-\varepsilon}B = (\lambda_0 + A + A_0)^{\frac{1}{4}-\varepsilon}PD$ with $\mathcal{D}((\lambda_0 + A + A_0)^{\frac{1}{4}-\varepsilon}) = V_n^{1/2-2\varepsilon}(\Omega)$, $P \in \mathcal{L}(\mathbf{H}^{1/2-2\varepsilon}(\Omega), V_n^{1/2-2\varepsilon}(\Omega))$ and $D \in \mathcal{L}(V^0(\Gamma), V_n^{1/2-2\varepsilon}(\Omega))$. ☐

We are now in position to define the evolution Oseen system with a nonhomogeneous Dirichlet boundary condition. For a time horizon $0 < T \le \infty$, for an initial condition $Py_0 \in V_n^0(\Omega)$, and for a boundary value $u \in V^{0,0}(\Sigma_T)$, the weak formulation of (1.5)-(1.6) which is given in [19] is

$$\begin{aligned} Py' + APy + A_0Py &= Bu, \quad Py(0) = Py_0, \\ (I-P)y &= (I-P)Du. \end{aligned}$$

Hence, since the solution y is entirely determined by Py and u, the study of the Oseen system can be reduced to the study of the following linear system:

$$y' + Ay + A_0y = Bu, \quad y(0) = y_0 \in V_n^0(\Omega). \tag{2.11}$$

Notice that since (2.3) with $\theta = 0$ is an isomorphism, the solution to (3.2) exists and is unique in $W(0,T;V_n^0(\Omega),V_0^{-2}(\Omega))$. Some regularity results for systems related to (2.11) are collected in an appendix.

Finally, the following proposition gives an abstract formulation for the evolution Navier-Stokes system with a nonhomogeneous Dirichlet boundary condition.

Proposition 4 *Let $(z_s,r_s) \in V^3(\Omega)\times\mathcal{H}^2(\Omega)$ be the solution to (1.1). For $0 < T \le \infty$ and $s \in [0,1]$, if $y_0 \in V^s(\Omega)$ and $u \in L^2(0,T;V^{\frac{1}{2}+s}(\Gamma))$ then the following results hold.*

(i) If there exists $y \in V^{s+1,\frac{s}{2}+\frac{1}{2}}(Q_T)$ solution to the system

$$
\begin{aligned}
&(2.12) \qquad & Py' + APy + A_0Py + N(y) &= Bu, \quad y(0) = y_0 \in V^s(\Omega),\\
&(2.13) \qquad & (I-P)y &= (I-P)Du \in L^2(0,T;\mathbf{L}^2(\Omega)),
\end{aligned}
$$

then there is a unique $p \in H^{-\frac{1}{2}+\frac{s}{2}}(0,T;\mathcal{H}^s(\Omega))$ such that $(z,r) = (y+z_s, p+r_s)$ satisfies

$$
\begin{aligned}
&(2.14) \qquad & &\partial_t z - \nu\Delta z + (z\cdot\nabla)z + \nabla r = f, \quad \nabla\cdot y = 0 \text{ in } (0,T)\times\Omega,\\
&(2.15) \qquad & &z = u + v_b \text{ on } (0,T)\times\Gamma, \;\; z(0) = z_s + y_0 \in V^s(\Omega).
\end{aligned}
$$

(i) Conversely, if $(z,p) \in V^{s+1,\frac{s}{2}+\frac{1}{2}}(Q_T)\times H^{-\frac{1}{2}+\frac{s}{2}}(0,T;\mathcal{H}^s(\Omega))$ is a solution to (2.14)-(2.15), then $y = z - z_s$ satisfies (2.12)-(2.13).

Proof. Let us give a brief sketch of the proof.
(i) Let define $\mathcal{Y}(t) = \int_0^t y(\tau)d\tau$, $\mathcal{N}(\mathcal{Y})(t) = \int_0^t N(y)(\tau)d\tau$ and $\mathcal{U}(t) = \int_0^t u(\tau)d\tau$. We start from the left equality in (2.12) which is written in $L^2(0,T;V_0^{-2}(\Omega))$ and we recall the expression of B in Definition 1. Then we first make an integration by part in space to recover the trace condition $y|_{(0,T)\times\Gamma} = u$, and by integrating over $(0,t)$ the resulting equation we obtain:

$$0 = y - y_0 + (A+A_0)(P\mathcal{Y} - PD\mathcal{U}) - \lambda_0 PD\mathcal{U} + \mathcal{N}(\mathcal{Y}) \in H^{\frac{s}{2}+\frac{1}{2}}(0,T;V_0^{s-1}(\Omega)).$$

Next, from (2.13) we deduce that $P\mathcal{Y} - PD\mathcal{U} = \mathcal{Y} - D\mathcal{U} \in H^1(0,T;V_0^s(\Omega))$. Thus, since (2.6) yields

$$0 = P(\lambda_0 + A + A_0)D\mathcal{U} \in H^1(0,T;V_0^{s-1}(\Omega)),$$

we finally obtain

$$0 = y - y_0 + (A+A_0)\mathcal{Y} + \mathcal{N}(\mathcal{Y}) \in H^{\frac{s}{2}+\frac{1}{2}}(0,T;V_0^{s-1}(\Omega)).$$

Moreover, due to [20, Rem. 1.4(i), Chap. 1, p. 15], for almost each time $t > 0$ there is a unique $\mathcal{P}(t) \in L_0^2(\Omega)$ which satisfies

$$\nabla\mathcal{P}(t) = y(t) - y_0 - \nu\Delta\mathcal{Y}(t) + (\mathcal{Y}(t)\cdot\nabla)z_s + (z_s\cdot\nabla)\mathcal{Y}(t) + \int_0^t (y(\tau)\cdot\nabla)y(\tau)\,d\tau \in \mathbf{H}^{s-1}(\Omega).$$

Hence, by checking each term in the right hand side of this equality we deduce that $\nabla\mathcal{P} \in H^{\frac{1}{2}+\frac{s}{2}}(0,T;\mathbf{H}^{s-1}(\Omega))$. As a consequence, $p = -\frac{d}{dt}\mathcal{P}$ belongs to $H^{-\frac{1}{2}+\frac{s}{2}}(0,T;L_0^2(\Omega))$ and we easily verify that (y,p) satisfies

$$
\begin{aligned}
&(2.16) \qquad & &\partial_t y - \nu\Delta y + (y\cdot\nabla)z_s + (z_s\cdot\nabla)y + (y\cdot\nabla)y + \nabla p = 0,\\
&(2.17) \qquad & &\nabla\cdot y = 0 \text{ in } (0,T)\times\Omega, \;\; y = u \text{ on } (0,T)\times\Gamma, \;\; y(0) = y_0 \in V^s(\Omega).
\end{aligned}
$$

Then the conclusion is straightforward.
(ii) We apply the projector $P \in \mathcal{L}(\mathbf{H}^{-1}(\Omega), V_0^{-1}(\Omega))$ on (2.14) and we easily verify that the steps of *(i)* can be done back. ☐

Remark 4 *In Proposition 4, (2.12) is an evolution system formulated in a weak sense (see [6, Chap. 1, Def. 3.1(v)]). Moreover, the first equality in (2.14) is understood as an equality in the distribution space $\mathcal{D}'(0,T;\mathbf{H}^{-1}(\Omega))$, and in (2.14) and (2.15) the divergence and the trace conditions are understood as equalities in $L^2((0,T)\times\Omega)$ and in $L^2(0,T;\mathbf{L}^2(\Gamma))$ respectively.*

Remark 5 *Let us denote by $C_w([0,T],V^0(\Omega)$ the subspace in $L^\infty(0,T;V^0(\Omega))$ of functions which are continuous from $[0,T]$ into $V^0(\Omega)$ equipped with its weak topology. Then for $Py_0 \in V^0_n(\Omega)$ and $u \in V^{\frac{3}{4},\frac{3}{4}}(\Sigma_T)$ the existence of a solution $y \in L^2(0,T;V^1(\Omega)) \cap C_w([0,T],V^0(\Omega))$ to (2.12)-(2.13) can be obtain from [19, Thm. 5.1]. However, nothing is said there about the regularity of the pressure which appears in the Navier-Stokes equations (see [19, eq. (5.1) and Thm. 5.1]). The technique used in the proof of Proposition 4 to obtain a pressure term in $H^{-\frac{1}{2}+\frac{s}{2}}(0,T;\mathcal{H}^s(\Omega))$ is inspired from [20, Chap. III, Prop. 1.1, p. 266 and p. 307]. To the best of our knowledge, obtaining r in such time negative Sobolev space seems to be new. The reason why we only obtain the pressure in $H^{-\frac{1}{2}+\frac{s}{2}}(0,T;\mathcal{H}^s(\Omega))$ is that we do not have $\partial_t Py \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ but only $\partial_t Py \in L^2(0,T;V_0^{-1}(\Omega))$. This is deeply due to the fact that Dirichlet boundary conditions have to be considered since Ω is bounded (see [15, Chap. 3, Rem. 3.1, 4)]).*

2.3 Main Results

In this paper, we prove a local stabilization result for the closed-loop Navier-Stokes system (1.2)-(1.3)-(1.4) with a feedback law of the form $F = B^*\Pi$, where Π is solution to an algebraic Riccati equation. First, we will prove the following Theorem.

Theorem 1 (Section 3, Theorem 5) *There is a unique nonnegative and self-adjoint operator $\Pi \in \mathcal{L}(V^0_n(\Omega))$, which belongs to $\mathcal{L}(V^0_n(\Omega), V^2_0(\Omega))$, and which is solution to the following Riccati equation:*

$$((A+A_0^*)\Pi\xi|\zeta) + (\xi|(A+A_0^*)\Pi\zeta) + (B^*\Pi\xi|B^*\Pi\zeta)_{V^0(\Gamma)} = (\xi|\zeta) \quad \forall(\xi,\zeta) \in V^0_n(\Omega)\times V^0_n(\Omega). \tag{2.18}$$

Next, since we have a feedback law $B^*\Pi$ which belongs to $\mathcal{L}(V^0_n(\Omega), V^0(\Gamma))$, we can introduce its associated closed-loop linear operator $(\mathcal{D}(A_\Pi), A_\Pi)$ defined in $V^0_n(\Omega)$:

$$\mathcal{D}(A_\Pi) = \big\{y \in V^0_n(\Omega) \ | \ Ay + A_0 y + B(B^*\Pi)y \in V^0_n(\Omega)\big\}, \quad A_\Pi y = Ay + A_0 y + B(B^*\Pi)y.$$

Moreover, in order to characterize the spaces $\mathcal{D}(A_\Pi^{\frac{s}{2}})$ for $s \in [0,2]$, we introduce the function space:

$$V^s_n(\Pi,\Omega) = \left\{ y \in V^s_n(\Omega) \ | \ (I + PDB^*\Pi)y \in V^s_0(\Omega) \right\}, \quad s \in \Big[0\,,\,2\Big]. \tag{2.19}$$

Hence, we will prove the following Theorem.

Theorem 2 (Section 3, Theorem 6 and Section 4, Theorem 7) *The unbounded operator $(\mathcal{D}(A_\Pi), A_\Pi)$ is the infinitesimal generator of an analytic and exponentially stable semigroup on $V^0_n(\Omega)$, and it obeys:*

$$\mathcal{D}(A_\Pi) = V^2_n(\Pi,\Omega).$$

More generally, we have $\mathcal{D}(A_\Pi^{\frac{s}{2}}) = V^s_n(\Pi,\Omega)$ for $s \in [0,2]$.

Next, in order to give an abstract formulation of the closed-loop Navier-Stokes system (1.2)-(1.3)-(1.4) for $F = B^*\Pi$ (see (2.23) below), let us define the spaces

$$V^s(\Pi,\Omega) = \left\{ y \in V^s(\Omega) \ | \ (I + DB^*\Pi)y \in V^s_0(\Omega) \right\}, \quad s \in \Big[0\,,\,2\Big], \tag{2.20}$$

and

$$V^{1+s,\frac{1}{2}+\frac{s}{2}}(\Pi,Q) = L^2(0,\infty;V^{s+1}(\Pi,\Omega)) \cap H^{\frac{1}{2}+\frac{s}{2}}(0,\infty;V^0(\Pi,\Omega)), \quad s \in \Big[0\,,\,1\Big], \tag{2.21}$$

as well as the nonlinear mapping:

$$N_\Pi : V^1_n(\Pi,\Omega) \longrightarrow V_0^{-1}(\Omega), \quad N_\Pi(\xi) = N \circ \big(I - (I-P)D(B^*\Pi)\big)(\xi).$$

Moreover, in order to exhibit a Lyapunov functional, we need to introduce the operator:

$$\Pi^{(s)} : V^s_n(\Pi,\Omega) \longrightarrow (V^s_n(\Pi,\Omega))' \quad \text{and} \quad \Pi^{(s)} = A_\Pi^{*\frac{s}{2}+\frac{1}{2}}\,\Pi\,A_\Pi^{\frac{s}{2}+\frac{1}{2}}.$$

The function V_s, which will be shown to be a Lyapunov functional, is defined on $V^s(\Pi, \Omega)$ as follows:

$$V_s : V^s(\Pi, \Omega) \longrightarrow \mathbb{R}^+ \quad \text{and} \quad V_s(\xi) = \langle \Pi^{(s)} P\xi | P\xi \rangle_{(V_n^s(\Pi,\Omega))', V_n^s(\Pi,\Omega)}. \tag{2.22}$$

The main result of this paper is the following local stabilization result.

Theorem 3 (Section 6, Theorem 10) *Let $s \in [\frac{d-2}{2}, 1]$. There exist $c_0 > 0$ and $\mu_1 > 0$ such that, if $\delta \in (0, \mu_1)$ and*

$$Py_0 \in \mathcal{V}^s_{\Pi,\delta} = \Big\{ y \in V_n^s(\Pi, \Omega) \text{ (defined in (2.19))} \ \mid \ \|y\|_{V_n^s(\Omega)} \le c\delta \Big\}$$

then system

$$Py' + APy + A_0 Py + B(B^*\Pi)Py + N_\Pi(Py) = 0, \qquad Py(0) = Py_0 \in V_n^s(\Pi, \Omega), \tag{2.23}$$

admits a unique solution $y(\cdot, y_0)$ in the set

$$\mathcal{S}^s_{\Pi,\delta} = \Big\{ y \in V^{1+s, \frac{1}{2}+\frac{s}{2}}(\Pi, Q) \text{ (defined in (2.21))} \ \mid \ \|y\|_{V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)} \le \delta \Big\}.$$

Moreover, the function V_s, which is defined by (2.22), is a Lyapunov function of system (2.23): for all $\xi \in V^s(\Pi, \Omega)$ we have $V_s(\xi) \ge C\|\xi\|^2_{V^s(\Pi,\Omega)}$ and there exists $\sigma > 0$ such that, for all $Py_0 \in \mathcal{V}^s_{\Pi,\delta}$ and $y(\cdot, y_0)$ solution to (2.23), the mapping $t \longmapsto V_s(y(t, y_0))$ decreases to 0 and obeys:

$$V_s(y(t, y_0)) \le C\|Py_0\|^2_{V_n^s(\Omega)} e^{-2\sigma t} \quad \forall t \ge 0.$$

From Proposition 2 and Proposition 4, we can deduce another version of the previous stabilization result, in terms of partial differential equations.

Theorem 4 *Let Π be the solution to (2.18), let $f \in \mathbf{H}^1(\Omega)$ and $v_b \in \mathbf{H}^{\frac{5}{2}}(\Gamma)$ be such that $\int_{\Gamma^{(j)}} v_b \cdot n = 0$, for all $j = 1 \dots N$, let $(z_s, r_s) \in V^3(\Omega) \times \mathcal{H}^2(\Omega)$ be a solution to (1.1) and let us consider the system:*

$$\partial_t z - \nu\Delta z + (z \cdot \nabla)z + \nabla r = f \ \text{ and } \ \nabla \cdot z = 0 \ \text{ in } \ (0,\infty) \times \Omega, \quad z(0) = z_0, \tag{2.24}$$

$$\gamma_\tau z = \gamma_\tau v_b + \partial_n \Pi P(z - z_s) \ \text{ on } \ (0,\infty) \times \Gamma, \tag{2.25}$$

$$\gamma_n z = \gamma_n v_b + \psi\, n \ \text{ on } \ (0,\infty) \times \Gamma, \tag{2.26}$$

$$\Delta\psi = \nabla \cdot ((\nabla z_s)^T - z_s \cdot \nabla)\Pi P(z - z_s) \ \text{ in } \ (0,\infty) \times \Omega, \ \int_\Gamma \psi = 0, \tag{2.27}$$

$$\partial_n \psi = (-\nu\Delta + (\nabla z_s)^T - z_s \cdot \nabla)\Pi P(z - z_s) \cdot n \ \text{ on } \ (0,\infty) \times \Gamma. \tag{2.28}$$

There exist $c > 0$ and $\mu_1 > 1$ such that, if $\delta \in (0, \mu_1)$ and

$$P(z_0 - z_s) \in \mathcal{W}^s_\delta = \Big\{ y \in V_n^s(\Pi, \Omega) \text{ (defined in (2.19))} \ \mid \ \|y\|_{V_n^s(\Omega)} \le c\delta \Big\}, \quad s \in \Big[\frac{d-2}{2}, 1\Big], \tag{2.29}$$

then (2.24)-(2.28) admits a unique solution in the set $\{(z_s, r_s)\} + \mathcal{D}^s_\delta$ where

$$\begin{aligned} \mathcal{D}^s_\delta \ = \ \Big\{ & (y, p) \in V^{s+1,\frac{s}{2}+\frac{1}{2}}(Q) \times H^{-\frac{1}{2}+\frac{s}{2}}(0,\infty; \mathcal{H}^s(\Omega)) \\ & \|y\|_{V^{s+1,\frac{s}{2}+\frac{1}{2}}(Q)} \le \delta, \quad \|p\|_{H^{-\frac{1}{2}+\frac{s}{2}}(0,+\infty;H^s(\Omega))} \le \delta(1+\delta) \Big\}. \end{aligned} \tag{2.30}$$

Moreover, we have $P(z - z_s) \in C_b([0, \infty[; V_n^s(\Omega))$ and $(I - P)(z - z_s) \in C_b([0, \infty[; V^{1+s}(\Omega))$, and there exist $C > 0$ and $\sigma > 0$ such that z obeys:

$$\|(I - P)(z(t) - z_s)\|_{V^{1+s}(\Omega)} + \|P(z(t) - z_s)\|_{V_n^s(\Omega)} \le C\|P(z_0 - z_s)\|_{V_n^s(\Omega)}\, e^{-\sigma t} \quad \forall t \ge 0. \tag{2.31}$$

In this setting, $C_b([0, \infty[; X(\Omega))$ is the space of continuous and bounded functions of $t \in [0, \infty[$ with value in $X(\Omega)$.

Finally, we verify that $V^s_n(\Pi,\Omega)=V^s_n(\Omega)$ for $s\in[0,\frac{1}{2}[$ (see remark 9), and from (2.20) and Proposition 2, we verify that:

$$V^s(\Pi,\Omega)=\Big\{y\in V^s(\Omega)\ \mid\ y=\partial_n\Pi y-r(\Pi y)n\ \text{ on }\ \Gamma\Big\},\quad s\in\Big]\frac{1}{2},1\Big], \tag{2.32}$$

where $r(\Pi y)$ is the solution to the following Neumann problem:

$$\Delta r=\nabla\cdot(z_s\cdot\nabla-(\nabla z_s)^T)\Pi y\ \text{ in }\ \Omega,\ \int_\Gamma r=0\quad \partial_n r=(\nu\Delta-(\nabla z_s)^T+z_s\cdot\nabla)\Pi y\cdot n\quad\text{on}\quad\Gamma. \tag{2.33}$$

Hence, we deduce the following Corollary.

Corollary 1 *(i) if $d=2$ and $s\in[0,\frac{1}{2}[$, then there exist $c>0$ and $\mu_1>0$ such that, if $\delta\in(0,\mu_1)$ and $P(z_0-z_s)\in V^s_n(\Omega)$ obeys $\|P(z_0-z_s)\|_{V^s_n(\Omega)}\le c$, then (2.24)-(2.28) admits a unique solution (z,r) in the set $\{(z_s,r_s)\}+\mathcal{D}^s_\delta$, and z obeys (2.31).*
(ii) If $d=2$ or $d=3$ and if $s\in]\frac{1}{2},1]$, then there exist $c>0$ and $\mu_1>0$ such that, if $\delta\in(0,\mu_1)$ and $z_0-z_s\in V^s(\Pi,\Omega)$ obeys $\|z_0-z_s\|_{V^s(\Omega)}\le c\delta$, then (2.24)-(2.28) admits a unique solution (z,r) in the set $\{(z_s,r_s)\}+\mathcal{D}^s_\delta$, and z obeys (2.31).
Here, $V^s(\Pi,\Omega)$ is defined by (2.32)-(2.33) and $\mathcal{D}^s_\delta$ is defined by (2.30).

Remark 6 *The first equality in (2.24) is understood as an equality in the distribution space $\mathcal{D}'(0,\infty;\mathbf{H}^{-1}(\Omega))$, the divergence condition in (2.24) is understood as an equality in $L^2((0,\infty)\times\Omega)$ and (2.25)-(2.26) are understood as equalities in $L^2(0,\infty;\mathbf{L}^2(\Gamma))$. Finally, (2.27) and (2.28) are understood as equalities in $L^2(0,\infty;(H^1(\Omega)/\mathbb{R})')$ and in $L^2(0,\infty;H^{-\frac{1}{2}}(\Omega))$ respectively.*

Remark 7 *Theorem 4 when $s\in[0,\frac{1}{4}]$ extends the stabilization results of [18, Thm. 6.1 and Thm. 6.7]. Notice that, in the case of tengential feedback control, an analogous result is also proved in [4, Thm. 3.1.3] with a fixed point argument.*

Remark 8 *Let us underline that in Theorem 4, it is necessary to assume Ω of class C^4, and to have a stationary state z_s in $V^3(\Omega)$. In particular, such an assumption is needed in the proof of Lemma 1 in section* **4***, see Remark 10.*

3 Optimal Control Problem Stated Over an Infinite Time Horizon

By following the path of [18], we obtain a feedback law from an auxiliary optimal control problem stated over an infinite time horizon. Let $y_0\in V^0_n(\Omega)$ and let us consider the following minimizing problem:

$$\inf\Big\{\ \mathcal{J}(y,u)\ \mid\ (y,u)\in W(0,\infty;V^0_n(\Omega),V^{-2}_0(\Omega))\times L^2(0,\infty;V^0(\Gamma))\ \text{ satisfies }\ (3.2)\Big\}, \tag{3.1}$$

where

$$y'+Ay+A_0y=Bu,\quad y(0)=y_0\in V^0_n(\Omega), \tag{3.2}$$

and where the cost functional $\mathcal{J}$ is defined by

$$\mathcal{J}(y,u)=\frac{1}{2}\int_0^\infty\int_\Omega|y|^2+\frac{1}{2}\int_0^\infty\int_\Gamma|u|^2. \tag{3.3}$$

Theorem 5 *For all $y_0\in V^0_n(\Omega)$, the problem (3.1) admits a unique solution (y_{y_0},u_{y_0}). The optimal control obeys $u_{y_0}=-B^*\Phi_{y_0}$, where $(y_{y_0},\Phi_{y_0})\in W(0,\infty;V^0_n(\Omega),V^{-2}_0(\Omega))\times W(0,\infty;V^2_0(\Omega),V^0_n(\Omega))$ is the unique solution to the following system:*

$$(\mathcal{S}_{y_0})\left\{\begin{array}{lll} y'+Ay+A_0y & =-BB^*\Phi, & y(0)=y_0\in V^0_n(\Omega),\\ -\Phi'+A\Phi+A_0^*\Phi & =y, & \Phi(\infty)=0,\\ \Phi(t) & =\Pi\,y(t) & \forall t\ge 0.\end{array}\right.$$

In this setting, Π *is the unique nonnegative and self-adjoint operator of* $\mathcal{L}(V^0_n(\Omega))$*, which also belongs to* $\mathcal{L}(V^0_n(\Omega), V^2_0(\Omega))$*, solution to the following Riccati equation:*

$$((A + A_0^*)\Pi\xi|\zeta) + (\xi|(A + A_0^*)\Pi\zeta) + (B^*\Pi\xi|B^*\Pi\zeta)_{V^0(\Gamma)} = (\xi|\zeta) \quad \forall(\xi,\zeta) \in V^0_n(\Omega) \times V^0_n(\Omega). \tag{3.4}$$

Moreover, Π *obeys*

$$\langle \Pi y_0 | y_0 \rangle_{V^1_0(\Omega), V_0^{-1}(\Omega)} = 2\mathcal{J}(y_{y_0}, u_{y_0}) = 2\inf\Big\{ \mathcal{J}(y,u) \ \mid \ (y,u) \ \text{ satisfies } \ (3.2)\Big\}. \tag{3.5}$$

Proof. This theorem can be deduced from an obvious adaptation of Theorem 4.1, of Lemma 4.2 and of Theorem 4.5 in [18] (there R_A and M shall be replace by the identity in $\mathbf{V}^0(\Gamma)$). □

The fact that Π belongs to $\mathcal{L}(V^0_n(\Omega), V^2_0(\Omega))$ (and then that $B^*\Pi$ is bounded from $V^0_n(\Omega)$ onto $V^0(\Gamma)$) ensures that $B(B^*\Pi)$ is well defined in $\mathcal{L}(V^0_n(\Omega), V_0^{-2}(\Omega))$. As a consequence, the optimal state $y_{y_0} \in W(0,\infty; V^0_n(\Omega), V_0^{-2}(\Omega))$ is solution to (S_{y_0}), if and only if, it is solution to the following evolution equation:

$$y' + Ay + A_0 y + B(B^*\Pi)y = 0, \qquad y(0) = y_0 \in V^0_n(\Omega). \tag{3.6}$$

Hence, let us introduce the closed-loop linear operator $(\mathcal{D}(A_\Pi), A_\Pi)$ associated with the evolution system (3.6).

Definition 2 *Let us define the linear operator* $(\mathcal{D}(A_\Pi), A_\Pi)$ *in* $V^0_n(\Omega)$ *as follows:*

$$\mathcal{D}(A_\Pi) = \big\{ y \in V^0_n(\Omega) \mid Ay + A_0 y + B(B^*\Pi)y \in V^0_n(\Omega) \big\}, \tag{3.7}$$

$$A_\Pi y = Ay + A_0 y + B(B^*\Pi)y. \tag{3.8}$$

Theorem 6 *The linear operator* $(\mathcal{D}(A_\Pi), A_\Pi))$ *is the infinitesimal generator of an analytic and exponentially stable semigroup on* $V^0_n(\Omega)$*. For* $y_0 \in V^0_n(\Omega)$ *the optimal trajectory* y_{y_0} *is the unique solution to the evolution equation (3.6) and there exist* $C > 0$ *and* $\sigma > 0$ *such that*

$$\|y_{y_0}(t)\|_{V^0_n(\Omega)} \le Ce^{-\sigma t}\|y_0\|_{V^0_n(\Omega)} \quad \forall t \ge 0. \tag{3.9}$$

Proof. See [18, Theorem 4.5]. □

4 A class of initial conditions

Here, for $f \in L^2(0,\infty; V_0^{-2}(\Omega))$ and for $y_0 \in V^0_n(\Omega)$, we consider the solution to the following evolution equation:

$$y' + Ay + A_0 y + B(B^*\Pi)y = f, \qquad y(0) = y_0. \tag{4.10}$$

The goal of this section is to prove that for an initial condition y_0 in $V^s_n(\Pi,\Omega)$, and for a nonhomogeneous term f in $L^2(0,\infty; V_0^{-1+s}(\Omega))$, the solution to (4.10) belongs to $W(0,\infty; V^{s+1}_n(\Omega), V^{s-1}_0(\Omega))$ for $s \ge \frac{d-2}{2}$ (see Corollary 2). The obtention of such a regularity for the solution to the linear system (4.10) is motivated by the nonlinear analysis of Section **6**. From Theorem 7 we already know that $(e^{-A_\Pi t})_{t>0}$ is analytic on $V^0_n(\Omega)$ where $A_\Pi = A + A_0 + B(B^*\Pi)$ is the linear operator introduced in Definition 2. Then, in order to apply maximal regularity results for linear evolution equation defined with the infinitesimal generator of an analytic semigroup, we need to characterize the domains $\mathcal{D}(A_\Pi)$ and $\mathcal{D}(A^*_\Pi)$.

First, let us introduce the following function spaces depending on the operator Π.

Definition 3 *For all* $s \in [0,2]$ *we define the spaces*

$$V^s_n(\Pi,\Omega) = \Big\{ y \in V^s_n(\Omega) \mid (I + PDB^*\Pi)y \in V^s_0(\Omega) \Big\}, \tag{4.11}$$

$$V^{-s}_n(\Pi,\Omega) = V^{-s}_0(\Omega). \tag{4.12}$$

Remark 9 *Since we have $V_0^s(\Omega) = V_n^s(\Omega)$ when $s \in [0, \frac{1}{2}[$, for $y \in V_n^s(\Omega)$ the compatibility condition $(I + PDB^*\Pi)y \in V_0^s(\Omega)$ in (4.11) is always satisfied. As a consequence, when $s \in [0, \frac{1}{2}[$ we have the equality $V_n^s(\Pi, \Omega) = V_n^s(\Omega)$.*

Thus, we state two preliminary lemmas.

Lemma 1 *The operator $\Pi \in \mathcal{L}(V_n^0(\Omega))$ obeys*

$$\Pi \in \mathcal{L}(V_n^s(\Pi, \Omega), V_0^{s+2}(\Omega)) \quad \forall s \in [-2, 1]. \tag{4.13}$$

Proof. First, in order to prove (4.13) when $s = 1$, we assume that $y_0 \in V_n^1(\Pi, \Omega)$ and we define $(\widetilde{y}, \widetilde{\Phi}) = (e^{-\lambda_0(\cdot)}y_{y_0}, e^{-\lambda_0(\cdot)}\Phi_{y_0})$, where (y_{y_0}, Φ_{y_0}) is the unique solution to $(\mathcal{S}_{y_0})$ (see Theorem 5). Hence, we easily verify that

$$\begin{aligned}
\widetilde{y}' + A\widetilde{y} + A_0^*\widetilde{y} + \lambda_0\widetilde{y} &= -Bu, \quad \widetilde{y}(0) = y_0 \in V_n^1(\Omega), \\
u &= -B^*\widetilde{\Phi}. \\
-\widetilde{\Phi}' + A\widetilde{\Phi} + A_0^*\widetilde{\Phi} + \lambda_0\widetilde{\Phi} &= (I + 2\lambda_0\Pi)\widetilde{y}, \quad \widetilde{\Phi}(\infty) = 0.
\end{aligned}$$

Moreover, since we have $\widetilde{\Phi}(0) = \Phi(0) = \Pi y_0$ and by recalling (4.11), we obtain the compatibility condition which will allow to use the Lemma 6 of the Appendix:

$$y_0 - PDu(0) = y_0 + PDB^*\widetilde{\Phi}(0) = y_0 + PDB^*\Pi y_0 \in V_0^1(\Omega).$$

Thus, let us prove that the mapping $y_0 \longmapsto \widetilde{\Phi}(0) = \Pi y_0$ is continuous from $V_n^1(\Pi, \Omega)$ onto $V_0^3(\Omega)$. In a first step, with the same bootstrap argument used to prove Corollary 4.3 in [18], we can prove that

$$\|\widetilde{\Phi}\|_{V^{\frac{7}{2}-\varepsilon, \frac{7}{4}-\frac{\varepsilon}{2}}(Q)} \le C_0\|y_0\|_{V_n^{\frac{1}{2}-\varepsilon}(\Omega)} \quad \forall \varepsilon \in \left]0, \frac{1}{2}\right[. \tag{4.14}$$

In a second step, we obtain $\Pi \in \mathcal{L}(V_n^1(\Pi, \Omega), V_0^3(\Omega))$ from (4.14) with the following calculation:

$$\begin{aligned}
\|\widetilde{\Phi}(0)\|_{V_0^3(\Omega)} &\le C_1\|\widetilde{\Phi}\|_{W(0,\infty;V_0^4(\Omega),V_0^2(\Omega))} \le C_2\|\widetilde{\Phi}\|_{V^{4,2}(Q)} \le C_3\|\widetilde{y}\|_{V^{2,1}(Q)} \\
&\le C_4(\|B^*\widetilde{\Phi}\|_{V^{\frac{3}{2},\frac{3}{4}}(\Sigma)} + \|y_0\|_{V_n^1(\Omega)}) \le C_5(\|\widetilde{\Phi}\|_{V^{\frac{7}{2}-\varepsilon,\frac{7}{4}-\frac{\varepsilon}{2}}(Q)} + \|y_0\|_{V_n^1(\Omega)}),
\end{aligned}$$

where we have used the continuous embeddings $W(0, T; V_0^4(\Omega), V_n^2(\Omega)) \hookrightarrow C([0, T]; V_0^3(\Omega))$ and $V^{4,2}(Q) \hookrightarrow W(0, T; V_0^4(\Omega), V_n^2(\Omega))$ for $T > 0$, the estimate (8.13) with $s = 2$, the estimate (8.12) with $s = \frac{3}{2}$ and the estimate (8.11) with $s = \frac{3}{2} - \varepsilon$. Finally, with an obvious duality argument we deduce that $\Pi \in \mathcal{L}(V_0^{-2}(\Omega), V_n^0(\Omega))$ from $\Pi \in \mathcal{L}(V_n^0(\Omega), V_0^2(\Omega))$, and (4.13) follows by interpolation. $\square$

Remark 10 *In the proof of Lemma 1, the assumtions Ω of class C^4, and $z_s \in V^3(\Omega)$, are needed to obtain the estimate $\|\widetilde{\Phi}\|_{V^{4,2}(Q)} \le C\|\widetilde{y}\|_{V^{2,1}(Q)}$ from Lemma 7 in the Appendix. Indeed, Lemma 7 is deduced from [18, Lem. 8.5] which, for the solution to the stationary system*

$$\lambda_0\Phi + A\Phi + A_0^*\Phi = f \in V_n^2(\Omega),$$

uses the estimates $\|\Phi\|_{V^4(\Omega)} \le C\|f\|_{V_n^2(\Omega)}$ (see [18, Lem. 8.4]), and for which the assumtions Ω of class C^4 and $z_s \in V^3(\Omega)$ are needed.

Lemma 2 *For all $\theta \in [0, 1]$, the mapping $I + PDB^*\Pi : V_n^{2\theta}(\Pi, \Omega) \longrightarrow V_0^{2\theta}(\Omega)$ is an isomorphism.*

Proof. *(i) Isomorphism $I + PDB^*\Pi : V_n^0(\Omega) \longrightarrow V_n^0(\Omega)$.*
First, from (4.13) with $s = 0$, from (2.10) with $s = \frac{1}{2}$, and from (2.8) with $s = 1$, we deduce that

$$PDB^*\Pi \in \mathcal{L}(V_n^0(\Omega), V_n^1(\Omega)). \tag{4.15}$$

Then with the compact embedding $V^1_n(\Omega) \subset\subset V^0_n(\Omega)$, we obtain that $I + PDB^*\Pi$ is a compact perturbation of the identity. As a consequence, $I + PDB^*\Pi : V^0_n(\Omega) \longrightarrow V^0_n(\Omega)$ is an isomorphism if and only if it is injective. Hence, we first assume that $y \in V^0_n(\Omega)$ is such that $y + PDB^*\Pi y = 0$ and by taking the scalar product with $(\lambda_0 + A + A_0^*)\Pi y$ we obtain

$$((A + A_0^*)\Pi y|y)_{V^0_n(\Omega)} + \lambda_0(\Pi y|y)_{V^0_n(\Omega)} + \|B^*\Pi y\|^2_{V^0(\Gamma)} = 0. \tag{4.16}$$

Moreover, (3.4) with $\zeta = \xi$ yields the equality

$$((A + A_0^*)\Pi y|y) + \frac{1}{2}\|B^*\Pi y\|^2_{V^0(\Gamma)} = \frac{1}{2}\|y\|^2_{V^0_n(\Omega)},$$

and from (4.16) we deduce that $\frac{1}{2}\|y\|^2_{V^0_n(\Omega)} + \lambda_0(\Pi y|y)_{V^0_n(\Omega)} + \frac{1}{2}\|B^*\Pi y\|^2_{V^0(\Gamma)} = 0$. Then we necessary have $y = 0$, and it proves that $I + PDB^*\Pi : V^0_n(\Omega) \longrightarrow V^0_n(\Omega)$ is an isomorphism.
(ii) Isomorphism $I + PDB^*\Pi : V^2_n(\Pi,\Omega) \longrightarrow V^2_0(\Omega)$.
First, $I + PDB^*\Pi \in \mathcal{L}(V^2_n(\Pi,\Omega), V^2_0(\Omega))$ is a straigtforward consequence of (4.11) with $s = 2$. Next, let us prove that $(I + PDB^*\Pi)^{-1} \in \mathcal{L}(V^2_0(\Omega), V^2_n(\Pi,\Omega))$. Let $y \in V^0_n(\Omega)$ and $f \in V^2_0(\Omega)$ be such that

$$y + PDB^*\Pi y = f \in V^2_0(\Omega). \tag{4.17}$$

Hence, since in *(i)* we have shown that $I + PDB^*\Pi : V^0_n(\Omega) \longrightarrow V^0_n(\Omega)$ is an isomorphism, we have

$$\|y\|_{V^0_n(\Omega)} \le C\|f\|_{V^0_n(\Omega)}. \tag{4.18}$$

Moreover, with (4.15), (4.17) and (4.18) we obtain that y belongs to $V^1_n(\Pi,\Omega)$ from the following calculation:

$$\|y\|_{V^1_n(\Omega)} \le \|f\|_{V^1_0(\Omega)} + \|PDB^*\Pi y\|_{V^1_n(\Omega)} \le \|f\|_{V^1_0(\Omega)} + C_1\|y\|_{V^0_n(\Omega)} \le C_2\|f\|_{V^1_0(\Omega)}. \tag{4.19}$$

Next, from (4.13) with $s = 1$, from (2.10) with $s = \frac{3}{2}$ and from (2.8) with $s = 2$, we deduce that

$$PDB^*\Pi \in \mathcal{L}(V^1_n(\Omega), V^2_n(\Omega)). \tag{4.20}$$

Thus, (4.20) with (4.17) and (4.19) yields $y \in V^2_n(\Pi,\Omega)$ and

$$\|y\|_{V^2_n(\Omega)} \le \|f\|_{V^2_0(\Omega)} + \|PDB^*\Pi y\|_{V^2_n(\Omega)} \le \|f\|_{V^2_0(\Omega)} + C_3\|y\|_{V^1_n(\Omega)} \le C_4\|f\|_{V^2_0(\Omega)}. \tag{4.21}$$

Then we have shown that $(I + PDB^*\Pi)^{-1} \in \mathcal{L}(V^2_0(\Omega), V^2_n(\Pi,\Omega))$ and we conclude that $I + PDB^*\Pi : V^2_n(\Pi,\Omega) \longrightarrow V^2_0(\Omega)$ is an isomorphism.
(iii) Isomorphism $I + PDB^*\Pi : V^{2\theta}_n(\Pi,\Omega) \longrightarrow V^{2\theta}_0(\Omega)$.
Since $I + PDB^*\Pi : V^2_n(\Pi,\Omega) \longrightarrow V^2_0(\Omega)$ and $I + PDB^*\Pi : V^0_n(\Omega) \longrightarrow V^0_n(\Omega)$ are isomorphisms, we obtain that $I + PDB^*\Pi : V^{2\theta}_n(\Pi,\Omega) \longrightarrow V^{2\theta}_0(\Omega)$ is an isomorphism for $\theta \in]0,1[$ by interpolation. ☐

Next, in the Theorem 7 thereafter, we are going prove that for all $s \in [0,2]$ we have the equality $\mathcal{D}(A_\Pi^{\frac{s}{2}}) = V^s_n(\Pi,\Omega)$. To the best of our knowledge, a result of such a kind is new. Indeed, the analysis of [18] does not provide a precise characterization of $\mathcal{D}(A_\Pi)$, and the approach in [14, Chap. 2] is too general and only allows to characterize $\mathcal{D}(A_\Pi)$ in terms of the domain of the free dynamic operator $A + A_0$. The only main information on $\mathcal{D}(A_\Pi)$ obtained there is the inclusion:

$$\mathcal{D}(A_\Pi) \subset \mathcal{D}((\lambda_0 + A + A_0)^{\frac{1}{4}-\varepsilon}) = \mathcal{D}(A^{\frac{1}{4}-\varepsilon}) \quad \text{for} \quad \varepsilon \in \left]0, \frac{1}{4}\right[,$$

which is a consequence of [14, Chap. 2, Thm. 2.2.1, (a_6)], where the following equality is proved:

$$\mathcal{D}(A_\Pi) = \left\{ y \in \mathcal{D}(A^{\frac{1}{4}-\varepsilon}) \mid (\lambda_0 + A + A_0)^{\frac{1}{4}-\varepsilon} y - (\lambda_0 + A + A_0)^{-\frac{3}{4}-\varepsilon} BB^*\Pi y \in \mathcal{D}(A^{\frac{3}{4}+\varepsilon}) \right\} \quad \forall \varepsilon \in \left]0, \frac{1}{4}\right[.$$

Moreover, even when they consider the heat equation with Dirichlet boundary control in [14, Chap. 3, Par. 3.1 and 3.2], the authors do not provide a complete characterization of the domain of the closed loop operator.

Theorem 7 *The unbounded operator $(\mathcal{D}(A_\Pi), A_\Pi)$, which is defined in Definition 4, obeys*

$$\sup\left\{Re(\lambda) \mid \ \lambda \in \sigma(A_\Pi)\right\} < 0, \tag{4.22}$$

and

$$\mathcal{D}({A_\Pi}^\theta) = [\mathcal{D}(A_\Pi), V_n^0(\Omega)]_{1-\theta} = V_n^{2\theta}(\Pi, \Omega) \quad \forall \theta \in [0, 1]. \tag{4.23}$$

Proof. *(i) Inequality (4.22).*
The results follows from Theorem 7 and [6, Chap. 1, Thm. 2.2 and Prop. 2.9].
(i) Characterization of $\mathcal{D}(A_\Pi)$.
From $A_\Pi = A + A_0 + BB^*\Pi$ and $B = (\lambda_0 + A + A_0)PD$ we deduce that

$$A_\Pi = (\lambda_0 + A + A_0)(I + PDB^*\Pi) - \lambda_0. \tag{4.24}$$

Hence, the continuous embedding $V_n^2(\Pi, \Omega) \hookrightarrow \mathcal{D}(A_\Pi)$ is a direct consequence of (4.25) with $I + PDB^*\Pi \in \mathcal{L}(V_n^2(\Pi, \Omega), V_0^2(\Omega))$ and $V_0^2(\Omega) = \mathcal{D}(\lambda_0 + A + A_0)$. Next, from (4.25) we have

$$A_\Pi y = (\lambda_0 + A + A_0)(y + PDB^*\Pi y) - \lambda_0 y \in V_0^2(\Omega) \quad \forall y \in \mathcal{D}(A_\Pi), \tag{4.25}$$

and with $\mathcal{D}(\lambda_0 + A + A_0) = V_0^2(\Omega)$ it yields

$$(I + PDB^*\Pi)y \in V_0^2(\Omega) \quad \forall y \in \mathcal{D}(A_\Pi). \tag{4.26}$$

Moreover, according to Lemma 2 the mapping $I + PDB^*\Pi : V_n^2(\Pi, \Omega) \to V_0^2(\Omega)$ is an isomorphism, so with $V_0^2(\Omega) = \mathcal{D}(\lambda_0 + A + A_0)$ and (4.25) we can make the following calculation:

$$\begin{aligned}
\|y\|_{V_0^2(\Omega)} &\leq C_1\|(I + PDB^*\Pi)y\|_{V_0^2(\Omega)} \\
&\leq C_2\|(\lambda_0 + A + A_0)(I + PDB^*\Pi)y\|_{V_n^0(\Omega)} \\
&\leq C_2(\|A_\Pi y\|_{V_n^0(\Omega)} + \lambda_0\|y\|_{V_n^0(\Omega)}.
\end{aligned}$$

Then by recalling (4.26), the continuous embedding $\mathcal{D}(A_\Pi) \hookrightarrow V_n^2(\Pi, \Omega)$ is proved.
(iii) Characterization of $\mathcal{D}(A_\Pi^\theta)$ when $\theta \in]0, 1[$.
Since we have the equality $\mathcal{D}(A_\Pi) = V_n^2(\Pi, \Omega)$, to prove (4.23) it is sufficient to prove

$$\mathcal{D}(A_\Pi^\theta) = [\mathcal{D}(A_\Pi), V_n^0(\Omega)]_{1-\theta} \quad \text{and} \quad \mathcal{D}(A_\Pi^{*\theta}) = [\mathcal{D}(A_\Pi^*), V_n^0(\Omega)]_{1-\theta} \quad \forall \theta \in [0, 1]. \tag{4.27}$$

Moreover, according to [21], to prove (4.27) is equivalent to prove that the holomorphic-function

$$z \in \{z \in \mathbb{C} \mid Re(z) > 0\} \longmapsto {A_\Pi}^{-z} \in \mathcal{L}(V_n^0(\Omega)),$$

can be extended to a strong continuous function from $\{z \in \mathbb{C} \mid Re(z) \geq 0\}$ in $\mathcal{L}(V_n^0(\Omega))$. Then from the perturbation equality

$$(t + A_\Pi)^{-1} = (t + A)^{-1} - (t + A)^{-1}(A_0 + B(B^*\Pi))(t + A_\Pi)^{-1} \quad \forall t \geq 0,$$

and from the equalities

$$A^{-z} = \frac{\sin \pi z}{\pi} \int_0^{+\infty} t^{-z}(t + A)^{-1} dt \quad \text{and} \quad A_\Pi^{-z} = \frac{\sin \pi z}{\pi} \int_0^{+\infty} t^{-z}(t + A_\Pi)^{-1} dt,$$

we deduce that

$$A_\Pi^{-z} = A^{-z} - I(z) \quad \text{and} \quad I(z) = \frac{\sin \pi z}{\pi} \int_0^{+\infty} t^{-z}(t + A)^{-1}(A_0 + BB^*\Pi)(t + A_\Pi)^{-1} dt.$$

Moreover, since A is maximal accretive, the interpolation equality $\mathcal{D}(A^\theta) = [\mathcal{D}(A), V_n^0(\Omega)]_{1-\theta}$ is true (see [6, Chap. 1, Prop. 6.1]) and from [21] we deduce that A^{-z} can be extended to a strong continuous function from $\{z \in \mathbb{C} \mid Re(z) \geq 0\}$ in $\mathcal{L}(V_n^0(\Omega))$. Then A^{-z} is bounded independently on $z \in \{z \in \mathbb{C} \mid Re(z) > 0\}$ in a neigborhood of 0, and by virtue of [13, Ch. 17, Thm. 17.9.1], it remains to show that $z \longmapsto I(z)$ is bounded independently on $z \in \{z \in \mathbb{C} \mid Re(z) > 0\}$ in a neigborhood of 0. Let ρ and σ be respectively the real and the imaginary part of $z \in \{z \in \mathbb{C} \mid Re(z) \geq 0\}$. Since A and A_Π are infinitesimal generators of analytic semigroups on $V_n^0(\Omega)$, for $\varepsilon \in]0, \frac{1}{4}[$ we have the following resolvent inequalities:

$$\|(t + A_\Pi)^{-1}\|_{V_n^0(\Omega)} \leq \frac{C}{1+t} \quad \text{and} \quad \|(t + A)^{-1}A^{\frac{3}{4}+\varepsilon}\|_{V_n^0(\Omega)} \leq \frac{C}{(1+t)^{\frac{1}{4}-\varepsilon}} \quad \forall t \geq 0.$$

Then with $A^{-\frac{3}{4}-\varepsilon}(A_0 + B(B^*\Pi)) \in \mathcal{L}(V_n^0(\Omega))$, the following calculation

$$\begin{aligned} \|I(z)\|_{\mathcal{L}(V_n^0(\Omega))} &= \left\| \frac{\sin \pi z}{\pi} \int_0^{+\infty} t^{-z} \big((t + A)^{-1}A^{\frac{3}{4}+\varepsilon}\big)\big(A^{-\frac{3}{4}-\varepsilon}(A_0 + BB^*\Pi)\big)(t + A_\Pi)^{-1} dt \right\|_{\mathcal{L}(V_n^0(\Omega))} \\ &\leq Ce^{\pi\sigma} \int_0^{\infty} \frac{dt}{t^{-\rho}(1+t)^{\frac{5}{4}-\varepsilon}}, \end{aligned}$$

allows to conclude. □

We are now in position to characterize the adjoint of A_Π.

Proposition 5 *The adjoint of A_Π with respect to the pivot space $V_n^0(\Omega)$ is defined by*

$$\mathcal{D}(A_\Pi^*) = V_0^2(\Omega) \quad \textit{and} \quad A_\Pi^* = A + A_0^* + (B^*\Pi)^*B^*. \tag{4.28}$$

Moreover, A_Π^* obeys

$$\mathcal{D}((A_\Pi^*)^\theta) = [\mathcal{D}(A_\Pi^*), V_n^0(\Omega)]_{1-\theta} = V_0^{2\theta}(\Omega) \quad \forall \theta \in [0, 1]. \tag{4.29}$$

Proof. First, for $w \in \mathcal{D}(A_\Pi^*)$ and from the definition of the adjoint domain we have

$$|(A_\Pi y|w)_{V_n^0(\Omega)}| = |(Ay + A_0 y + B(B^*\Pi)y|w)_{V_n^0(\Omega)}| \leq C_w \|y\|_{V_n^0(\Omega)} \quad \forall y \in \mathcal{D}(A_\Pi).$$

Then with $B = (\lambda_0 + A + A_0)PD$ and $\mathcal{D}(A_\Pi) = V_n^2(\Pi, \Omega)$ we deduce that

$$|((\lambda_0 + A + A_0)(I + PDB^*\Pi)y|w)_{V_n^0(\Omega)}| \leq C_w \|y\|_{V_n^0(\Omega)} \quad \forall y \in V_n^2(\Pi, \Omega). \tag{4.30}$$

Moreover, since we know from Lemma 2 that the mapping $I + PDB^*\Pi : V_n^2(\Pi, \Omega) \to V_0^2(\Omega)$ is an isomorphism, we deduce that (4.30) is equivalent to

$$|((\lambda_0 + A + A_0)y|w)_{V_n^0(\Omega)}| \leq C_w \|y\|_{V_n^0(\Omega)} \quad \forall y \in V_0^2(\Omega). \tag{4.31}$$

Thus, with $V_0^2(\Omega) = \mathcal{D}(\lambda_0 + A + A_0)$ inequality (4.31) means that $w \in \mathcal{D}(\lambda_0 + A + A_0^*)$. Then from $\mathcal{D}(\lambda_0 + A + A_0^*) = V_0^2(\Omega)$ we deduce that $\mathcal{D}(A_\Pi{}^*) \subset V_0^2(\Omega)$. Conversely, if $w \in V_0^2(\Omega)$ we easily verify that for all $y \in \mathcal{D}(A_\Pi)$ we have

$$\begin{aligned} (A_\Pi y|w) &= (Ay + A_0 y + B(B^*\Pi)y|w) \\ &= \langle Ay + A_0 y + BB^*\Pi y|w\rangle_{V_0^{-2}(\Omega), V_0^2(\Omega)} \\ &= (y|(A + A_0^*)w) + (B^*\Pi y|B^*w)_{V^0(\Gamma)} \\ &= (y|(A + A_0^* + (B^*\Pi)^*B^*)w), \end{aligned}$$

and the inclusion $V_0^2(\Omega) \subset \mathcal{D}(A_\Pi{}^*)$ follows. Finally, the equality $\mathcal{D}(A_\Pi^*) = V_0^2(\Omega)$ is proved, and (4.29) follows from the right equality in (4.27). □

From (4.22) in Theorem 7, we know that $A_\Pi^{-1} : V_n^0(\Omega) \longrightarrow \mathcal{D}(A_\Pi)$ is an isomorphism. Then as usual, the extrapolation method permits to extend the definition of A_Π^{-1} to a bounded linear operator from $\mathcal{D}(A_\Pi^*)'$ onto $V_n^0(\Omega)$ (see [14, Chap. 0, 0.3]). Moreover, $\|A_\Pi^{-1} \cdot\|_{V_n^0(\Omega)}$ defines a norm which is equivalent to the one of

$\mathcal{D}(A_\Pi^*)'$. Hence, from $\mathcal{D}(A_\Pi^*)' = V_0^{-2}(\Omega)$ we deduce that $A_\Pi^{-1} \in \mathcal{L}(V_0^{-2}(\Omega), V_n^0(\Omega))$. Then by interpolation, we deduce that $A_\Pi^{-\theta}$ defines a bounded linear operator from $V_0^{-2\theta}(\Omega)$ onto $V_n^0(\Omega)$ for all $\theta \in [0,1]$, and that the following norm equivalence holds:

$$\|A_\Pi^{-\theta} \cdot \|_{V_n^0(\Omega)} \sim \| \cdot \|_{V_0^{-2\theta}(\Omega)} \quad \forall \theta \in [0,1]. \tag{4.32}$$

Finally, we deduce the main result of this section.

Corollary 2 *Let $f \in L^2(0,\infty; V_0^{-1+s}(\Omega))$ and $y_0 \in V_n^s(\Pi,\Omega)$ for $s \in [-1,1]$. Then the solution to the following evolution equation*

$$y' + Ay + A_0 y + B(B^*\Pi)y = f, \qquad y(0) = y_0, \tag{4.33}$$

belongs to $W(0,\infty; V_n^{1+s}(\Pi,\Omega), V_0^{-1+s}(\Omega))$ and obeys the following estimate:

$$\|y\|_{W(0,\infty;V_n^{1+s}(\Pi,\Omega),V_0^{-1+s}(\Omega))} \leq C(\|f\|_{L^2(0,\infty;V_0^{-1+s}(\Omega))} + \|y_0\|_{V_n^s(\Pi,\Omega)}). \tag{4.34}$$

Proof. Maximal regularity results for analytic semigroup which may be found in [6, Chap. 3, Thm. 2.2] ensures that the following mapping is an isomorphism:

$$\begin{array}{ccc} W(0,\infty; [\mathcal{D}(A_\Pi), V_n^0(\Omega)]_{\frac{1}{2}-\frac{s}{2}}, [\mathcal{D}(A_\Pi^*), V_n^0(\Omega)]'_{\frac{s}{2}+\frac{1}{2}}) & \rightarrow & L^2(0,\infty; [\mathcal{D}(A_\Pi^*), V_n^0(\Omega)]_{\frac{s}{2}+\frac{1}{2}}) \times C_{s,\frac{1}{2}}(\Pi) \\ y & \mapsto & (y' + A_\Pi y\,,\, y(0)), \end{array}$$

where $C_{s,\frac{1}{2}}(\Pi) = \big[[\mathcal{D}(A_\Pi), V_n^0(\Omega)]_{\frac{1}{2}-\frac{s}{2}}, [\mathcal{D}(A_\Pi^*), V_n^0(\Omega)]_{\frac{s}{2}+\frac{1}{2}}\big]_{\frac{1}{2}}$. Notice that we can set $T=\infty$ in [6, Chap. 3, Thm. 2.2] because $-A_\Pi$ is of negative type (see also [6, Chap. 1, Thm. 3.1, (i)]). Then we conclude with (4.23) and (4.29). $\square$

5 A Lyapunov Function for the Closed-Loop Oseen System.

Here, we assume that $s \in [-1,1]$ and we consider the closed-loop Oseen system:

$$y' + A_\Pi y = 0, \quad y(0) = y_0 \in V_n^s(\Pi,\Omega), \tag{5.1}$$

for which, according to Theorem 7, the solution is known to be exponentially stable. Hence, a natural question arising is: can we exhibit a Lyapunov function for system (5.1)? More precisely, is there exists a function $\widetilde{V}_s$ such that

$$\left\{ \begin{array}{l} \widetilde{V}_s : V_n^s(\Pi,\Omega) \longrightarrow \mathbb{R}^+ \quad \text{and} \quad \widetilde{V}_s(\xi) \geq C\|\xi\|^2_{V_n^s(\Pi,\Omega)} \quad \text{for all } \xi \in V_n^s(\Pi,\Omega), \\ \text{for all } y_0 \in V_n^s(\Pi,\Omega) \text{ and } y(\cdot,y_0) \text{ solution to (5.1)}, \ t \longmapsto \widetilde{V}_s(y(t,y_0)) \text{ decreases to } 0. \end{array} \right. \tag{5.2}$$

In order to construct such a function $\widetilde{V}_s$, we first need to define the following operator.

Definition 4 *For $s \in [-1,1]$, we define the following linear operator:*

$$\Pi^{(s)} : V_n^s(\Pi,\Omega) \longrightarrow (V_n^s(\Pi,\Omega))' \quad \text{and} \quad \Pi^{(s)} = A_\Pi^{*\frac{s}{2}+\frac{1}{2}}\, \Pi\, A_\Pi^{\frac{s}{2}+\frac{1}{2}}.$$

Thus, from (3.5), it can be proved that the bilinear form $(\cdot|\cdot)_{\Pi,s}$ defined by

$$(\xi|\zeta)_{\Pi,s} = \langle \Pi^{(s)}\xi|\zeta\rangle_{(V_n^s(\Pi,\Omega))',V_n^s(\Pi,\Omega)} \quad \forall (\xi,\zeta) \in V_n^s(\Pi,\Omega) \times V_n^s(\Pi,\Omega), \tag{5.3}$$

is a scalar product on $V_n^s(\Pi,\Omega)$ for which A_Π is accretive. In fact, from the algebraic Riccati equation (3.4), it can be shown that

$$(A_\Pi \xi|\xi)_{\Pi,s} \geq \sigma(\xi|\xi)_{\Pi,s} \qquad \text{where} \quad \sigma > 0.$$

As a consequence, by using the new scalar product (5.3) with equation (5.1), we deduce that the mapping

$$\xi \longmapsto \widetilde{V}_s(\xi) = (\xi|\xi)_{\Pi,s}$$

is convenient, and that $t \mapsto \widetilde{V}_s(y(t,y_0))$ has an exponential rate of decrease equal to $2\sigma > 0$.

Lemma 3 *For all $s \in [-1,1]$, the bilinear form $(\cdot|\cdot)_{\Pi,s}$ defined by (5.3) is a scalar product on $V_0^s(\Omega)$. If we define $\|\xi\|_{\Pi,s} = \big((\xi|\xi)_{\Pi,s}\big)^{1/2}$, then the following norm equivalence holds,*

$$\|\cdot\|_{\Pi,s} \sim \|\cdot\|_{V_n^s(\Pi,\Omega)}, \tag{5.4}$$

and we also have:

$$(A_\Pi \cdot |\cdot)_{\Pi,s} \sim \|\cdot\|^2_{V_n^{1+s}(\Pi,\Omega)}. \tag{5.5}$$

Proof. *(i) Norm equivalence (5.4).*
From (4.23) and from the equality $\|\xi\|_{\Pi,s} = \|A_\Pi^{\frac{s}{2}}\xi\|_{\Pi,0}$ for all $\xi \in V_n^s(\Pi,\Omega)$, we deduce that proving (5.4) for $s \in [-1,1]$ can be reduced to proving (5.4) for $s = 0$. First, the existence of $C_1 > 0$ such that $\|\cdot\|_{\Pi,0} \leq C_1\|\cdot\|_{V_n^0(\Omega)}$ is a straightforward consequence of $\Pi^{(0)} \in \mathcal{L}(V_n^0(\Omega))$. Next, the existence of $C_2 > 0$ such that $\|\cdot\|_{\Pi,0} \geq C_2\|\cdot\|_{V_n^0(\Omega)}$ follows from the following calculation where $\zeta = A_\Pi^{\frac{1}{2}}\xi$:

$$\begin{array}{rcl}
\|\xi\|^2_{V_n^0(\Omega)} & = & \|A_\Pi^{-\frac{1}{2}}\zeta\|^2_{V_n^0(\Omega)} \\
\text{(by (4.32) with } \theta = 1/2) & \leq & C_1'\|\zeta\|^2_{V_0^{-1}(\Omega)} \\
 & \leq & C_2'\|y_\zeta\|^2_{W(0,+\infty;V_n^0(\Omega),V_0^{-2}(\Omega))} \\
\text{(with } y_\zeta' = -Ay_\zeta - A_0y_\zeta - BB^*\Pi y_\zeta) & \leq & C_3'(\|y_\zeta\|^2_{L^2(0,+\infty;V_n^0(\Omega))} + \|B(B^*\Pi)y_\zeta\|^2_{L^2(0,+\infty;V_0^{-2}(\Omega))} \\
\text{(by (3.5))} & \leq & 2C_2\mathcal{J}(y_\zeta,u_\zeta) = C_2\langle \Pi\zeta|\zeta\rangle_{V_0^1(\Omega),V_0^{-1}(\Omega)} \\
\text{(with } \zeta = A_\Pi^{\frac{1}{2}}\xi \in V_0^{-1}(\Omega)) & = & C_2\langle \Pi A_\Pi^{\frac{1}{2}}\xi|A_\Pi^{\frac{1}{2}}\xi\rangle_{V_0^1(\Omega),V_0^{-1}(\Omega)} \\
 & = & C_2\|\xi\|^2_{\Pi,0}.
\end{array}$$

(ii) Norm equivalence (5.5).
If we replace ξ and ζ by $A_\Pi^{\frac{s}{2}+\frac{1}{2}}\xi \in V_n^0(\Omega)$ in (4.13), then we obtain the following equality:

$$(A_\Pi\xi|\xi)_{\Pi,s} = \langle A_\Pi\xi|\Pi^{(s)}\xi\rangle_{V_0^{s-1}(\Omega),V_0^{1-s}(\Omega)} = \frac{1}{2}\|A_\Pi^{\frac{s}{2}+\frac{1}{2}}\xi\|^2_{V_n^0(\Omega)} + \frac{1}{2}\|B^*\Pi A_\Pi^{\frac{s}{2}+\frac{1}{2}}\xi\|^2_{V^0(\Gamma)} \quad \forall\xi \in V_n^{1+s}(\Omega).$$

Then since $B^*\Pi \in \mathcal{L}(V_n^0(\Omega), V^0(\Gamma))$, (5.5) follows from the previous equality. □

Theorem 8 *For all $s \in [-1,1]$, the mapping $\widetilde{V}_s : V_n^s(\Pi,\Omega) \longrightarrow \mathbb{R}^+$ defined by $\widetilde{V}_s(\xi) = \|\xi\|^2_{\Pi,s}$ satisfies (5.2).*

Proof. From (5.4) and (5.5) with the continuous embedding $V_n^{1+s}(\Pi,\Omega) \hookrightarrow V_n^s(\Pi,\Omega)$, we obtain the existence of $\sigma > 0$ such that $(A_\Pi\xi|\xi)_{\Pi,s} \geq \sigma\|\xi\|^2_{\Pi,s}$. Hence, by evaluating $(y' + A_\Pi y|y)_{\Pi,s}$, with (5.1) it finally yields:

$$\frac{d}{dt}\|y(t)\|^2_{\Pi,s} + 2\sigma\|y(t)\|^2_{\Pi,s} \leq 0.$$

□

6 Stabilization of the nonlinear equation.

In this section, we prove that for initial conditions belonging to an adequate neighborhood of the origin, the feedback law $B^*\Pi$ stabilizes the Navier-Stokes system. We recall that $d = 2$ or $d = 3$ is the dimension of the geometrical domain Ω, and we fix $s \in [\frac{d-2}{2}, 1]$. Then we consider the following nonlinear system:

$$\begin{array}{rcl}
Py' + APy + A_0Py + N(y) + B(B^*\Pi)Py & = & 0, \qquad Py(0) = Py_0 \in V_n^s(\Pi,\Omega), \qquad (6.1) \\
(I-P)y & = & -(I-P)DB^*\Pi Py, \qquad (6.2)
\end{array}$$

where the nonlinear mapping $N : V^1(\Omega) \to V_0^{-1}(\Omega)$ is defined by (2.2). First, a consequence of Corollary 2 is that for sufficiently small Py_0 in $V_n^s(\Pi,\Omega)$, there exists a unique solution of (6.1)-(6.2) which belongs to $V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)$.

Theorem 9 *Let $s \in [\frac{d-2}{2}, 1]$. There exist $c_0 > 0$ and $\mu_0 > 0$ such that, if $\delta \in (0, \mu_0)$ and*

$$Py_0 \in \mathcal{V}^s_{\Pi,\delta} = \Big\{ y \in V^s_n(\Pi, \Omega) | \quad \|y\|_{V^s_n(\Omega)} < c_0\delta \Big\}, \tag{6.3}$$

system (6.1)-(6.2) admits a unique solution in the set

$$\mathcal{S}^s_\delta = \Big\{ y \in V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q) \,|\, \|y\|_{V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)} \leq \delta \Big\}. \tag{6.4}$$

Proof. The proof is divided in two parts. The case $s > 0$ is treated in the first part and the case $d = 2$ and $s = 0$ is treated in the second part.
(i) Existence and uniqueness when $s > 0$.
First, we consider the mapping

$$\Psi : z \in V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q) \to y_z \quad \text{where} \quad \begin{cases} Py_z' + A_\Pi Py_z &= -N(z), \quad Py_z(0) = Py_0 \in V^s_n(\Pi, \Omega), \\ (I-P)y_z &= (I-P)D(B^*\Pi)Py_z, \end{cases}$$

and we seek $c_0 > 0$ and $\mu_0 > 0$ such that, for every $Py_0 \in \mathcal{V}^s_{\Pi,\delta}$ with $\delta \in (0, \mu_0)$, Ψ is a contraction in $\mathcal{S}^s_\delta$. First, from (6.2) with $B^*\Pi \in \mathcal{L}(V^0_n(\Omega), V^0(\Gamma)) \cap \mathcal{L}(V^{1+s}(\Omega), V^{1+s}(\Gamma))$ and $D \in \mathcal{L}(V^0(\Gamma), V^0(\Omega)) \cap \mathcal{L}(V^{1+s}(\Gamma), V^{1+s}(\Omega))$ we easily deduce that

$$\|(I-P)\Psi(z)\|_{V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)} \leq C\|P\Psi(z)\|_{V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)}. \tag{6.5}$$

Then, inequality (6.5), estimate (4.34), (2.1) with $(s_1, s_2, s_3) = (s, s, 1-s)$ and the continuous embedding $W(0,\infty; V^{1+s}_n(\Pi,\Omega), V^{-1+s}_0(\Omega)) \hookrightarrow V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)$ provide a constant $C_0 > 0$ such that

$$\|\Psi(z)\|_{V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)} \leq C_0(\|z\|_{L^\infty(0,\infty;V^s(\Omega))}\|z\|_{L^2(0,\infty;V^{1+s}(\Omega))} + \|Py_0\|_{V^s_n(\Omega)}). \tag{6.6}$$

Moreover, since $s > 0$ we have the continuous embedding $V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q) \hookrightarrow L^\infty(0,\infty; V^s(\Omega))$ and it yields the existence of $C_1 > 0$ such that

$$\|\Psi(z)\|_{V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)} \leq C_0(C_1\|z\|^2_{V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)} + \|Py_0\|_{V^s_n(\Omega)}).$$

Then, since $z \in \mathcal{S}^s_\delta$ and $y_0 \in \mathcal{V}^s_{\Pi,\delta}$, we deduce that

$$\|\Psi(z)\|_{V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)} \leq C_0(C_1\mu_0 + c_0)\delta. \tag{6.7}$$

Next, as for (6.5), if z_1 and z_2 belong to S^s_δ we easily verify that

$$\|(I-P)\Psi(z_1) - (I-P)\Psi(z_2)\|_{V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)} \leq C\|P\Psi(z_1) - P\Psi(z_2)\|_{V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)}, \tag{6.8}$$

and that $y = P\Psi(z_1) - P\Psi(z_2)$ is solution to

$$y' + A_\Pi y = b(z_1 - z_2, z_1) + b(z_2, z_1 - z_2), \quad y(0) = 0.$$

Then from (6.8), (4.34), and from (2.1) with $(s_1, s_2, s_3) = (s, s, 1-s)$, we deduce that there exists $C_2 > 0$ such that

$$\begin{aligned} \|\Psi(z_1) - \Psi(z_2)\|_{V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)} \quad &\leq \quad C_2(\|z_1 - z_2\|_{L^\infty(0,\infty;V^s(\Omega))}\|z_1\|_{L^2(0,\infty;V^{1+s}(\Omega))} \\ &\qquad + \|z_2\|_{L^\infty(0,\infty;V^s(\Omega))}\|z_1 - z_2\|_{L^2(0,\infty;V^{1+s}(\Omega))}). \end{aligned} \tag{6.9}$$

Finally, by invoking the continuous embedding $V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q) \hookrightarrow L^\infty(0,\infty; V^s(\Omega))$ and the fact that z_1 and z_2 belong to S^s_δ, we obtain the existence of $C_3 > 0$ such that

$$\|\Psi(z_1) - \Psi(z_2)\|_{V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)} \leq C_2C_3\mu_0\|z_1 - z_2\|_{V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)}. \tag{6.10}$$

If we choose $\mu_0 = \min(\frac{1}{2C_0C_1}, \frac{1}{2C_2C_3})$ and $c_0 < \frac{1}{2C_0}$, from (6.7) and (6.10) we deduce that Ψ is a contraction in S^s_δ and that (6.1)-(6.2) admits a unique solution.
(ii) Existence and uniqueness in the two dimensional case when $s = 0$.
For all $z \in V^{1,\frac{1}{2}}(Q)$ and $v \in V^1_0(\Omega)$, a standard integration by part yields $b(z(t), z(t), v) = -b(z(t), v, z(t))$. Then from (2.1) with $(s_1, s_2, s_3) = (\frac{1}{2}, 0, \frac{1}{2})$ and from the Cauchy-Schwarz inequality we deduce that

$$\|N(z)(t)\|_{L^2(0,\infty;V_0^{-1}(\Omega))} \leq C\|z\|^2_{L^4(0,\infty;V^{\frac{1}{2}}(\Omega))}$$

Then from the continuous embeddings $H^{\frac{1}{4}}(0,\infty) \hookrightarrow L^4(0,\infty)$ and $V^{1,\frac{1}{2}}(Q) \hookrightarrow H^{\frac{1}{4}}(0,\infty; V^{\frac{1}{2}}(\Omega))$ we finally deduce that

$$\|\Psi(z)\|_{V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)} \leq C_0(C_1\|z\|^2_{V^{1,\frac{1}{2}}(Q)} + \|Py_0\|_{V_n^0(\Omega)}),$$

and (6.7) is obtained for $s = 0$. Moreover, we obtain (6.10) for $s = 0$ similarly and we conclude as in the case where $s > 0$. □

Notice that formulation (6.1)-(6.2) has been choosed for simplicity, in order to fit the formulation (2.12)-(2.13) given in Proposition 4. In fact, if we consider the space (2.21) as the state space, only evolution system (6.1) is needed. More precisely, $y \in V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)$ is solution to (6.1)-(6.2), if and only if, $y \in V^{1+s,\frac{1}{2}+\frac{s}{2}}(\Pi, Q)$ is solution to

$$(6.11) \qquad Py' + APy + A_0Py + N_\Pi(Py) + B(B^*\Pi)Py = 0, \qquad Py(0) = Py_0 \in V^s_n(\Pi, \Omega),$$

where the nonlinear mapping N_Π is defined by

$$(6.12) \qquad N_\Pi : V^1_n(\Pi, \Omega) \longrightarrow V_0^{-1}(\Omega), \quad N_\Pi(\xi) = N \circ \big(I - (I - P)DB^*\Pi\big)(\xi).$$

Indeed, if $y \in V^{1+s,\frac{1}{2}+\frac{s}{2}}(Q)$ is solution to (6.1)-(6.2), we deduce that $N(y)$ belongs to $V_0^{s-1}(\Omega)$, and optimal regularity results given by corollary 2 ensures that $Py \in W(0, +\infty; V^{s+1}_n(\Pi, \Omega), V^{s-1}_0(\Omega))$. Hence, from $W(0, +\infty; V^{s+1}_n(\Pi, \Omega), V^{s-1}_0(\Omega)) \hookrightarrow L^2(0, \infty; V^{s+1}_n(\Pi, \Omega)) \cap H^{\frac{1}{2}+\frac{s}{2}}(0, \infty; V^0_n(\Pi, \Omega))$ we deduce that $(I + PDB^*\Pi)Py \in L^2(0, \infty; V^{s+1}_0(\Omega)) \cap H^{\frac{1}{2}+\frac{s}{2}}(0, \infty; V^0_n(\Omega))$, and with (6.2) we finally obtain:

$$(6.13) \qquad (I + DB^*\Pi)y \in L^2(0, \infty; V^{s+1}_0(\Omega)) \cap H^{\frac{1}{2}+\frac{s}{2}}(0, \infty; V^0_n(\Omega)).$$

Then we deduce that $y \in V^{1+s,\frac{1}{2}+\frac{s}{2}}(\Pi, Q)$ and we conclude that y is solution to (6.11) by remarking that the equality $N_\Pi(Py) = N(y)$ follows from (6.2). Conversely, if $y \in V^{1+s,\frac{1}{2}+\frac{s}{2}}(\Pi, Q)$ is solution to (6.11), then we obtain (6.2) by remarking that composing (6.13) by $I - P$ gives zero.

As a consequence, we only need to consider equation (6.11) which only depends on Py, and analogously as in the linear case, provided that the initial condition Py_0 is small enough in $V^s_n(\Pi, \Omega)$, one can prove that $V_s = \widetilde{V}_s \circ P(\cdot) = \|P \cdot\|^2_{\Pi,s}$ (see Lemma 3) is a Lyapunov function for the nonlinear system (6.12). The proof relies on an adequate estimate of $(N_\Pi(\xi)|\xi)_{\Pi,s}$, which holds for $s \geq \frac{d-2}{2}$.

Lemma 4 *Let us consider the nonlinear mapping N_Π defined by (6.12). For all $s \in [\frac{d-2}{2}, 1]$, the following estimate holds:*

$$(6.14) \qquad (N_\Pi(\xi)|\xi)_{\Pi,s} \leq C_s\|\xi\|_{\Pi,s}(A_\Pi\xi|\xi)_{\Pi,s} \quad \forall \xi \in V^{1+s}_n(\Pi, \Omega).$$

Proof. First, as a direct consequence of (4.13), (4.23) and of (4.29), we deduce the following preliminary regularizing property:

$$(6.15) \qquad \Pi^{(s)} \in \mathcal{L}(V^{1+s}_n(\Pi, \Omega), V^{1-s}_0(\Omega)) \quad \forall s \in [0, 1].$$

Thus, if $s > 0$, from (2.1) for $(s_1, s_2, s_3) = (s, s, 1 - s)$, with (6.12) and

$$(6.16) \qquad (I - (I - P)DB^*\Pi) \in \mathcal{L}(V^s_n(\Omega), V^s(\Omega)) \cap \mathcal{L}(V^{s+1}_n(\Omega), V^{s+1}(\Omega))$$

and from (6.15), we obtain

$$(N_\Pi(\xi)|\xi)_{\Pi,s} = |\langle N_\Pi(\xi)|\Pi^{(s)}\xi\rangle_{V_0^{s-1}(\Pi,\Omega),V_0^{1-s}(\Pi,\Omega)}| \le C_s\|\xi\|_{V_n^s(\Omega)}\|\xi\|^2_{V_n^{1+s}(\Omega)} \quad \forall \xi \in V_n^{1+s}(\Pi,\Omega). \tag{6.17}$$

Then (6.14) follows from (5.4) and (5.5). Next, to treat the case $s = 0$, it will be preferable to use:

$$\Pi^{(0)} \in \mathcal{L}(V_n^{\frac{1}{2}}(\Pi,\Omega), V_0^{\frac{1}{2}}(\Omega)). \tag{6.18}$$

Hence, we invoke (2.1) for $(s_1, s_2, s_3) = (\frac{1}{2}, 0, \frac{1}{2})$ and (6.18), with (6.12) and (6.16) where $s = 0$, to obtain the estimate

$$(N_\Pi(\xi)|\xi)_{\Pi,0} = |(N_\Pi(\xi)|\Pi^{(0)}\xi)| \le C\|\xi\|^2_{V_n^{1/2}(\Pi,\Omega)}\|\xi\|_{V_n^1(\Pi,\Omega)} \quad \forall \xi \in V_n^1(\Pi,\Omega).$$

Thus, the interpolation inequality $\|\cdot\|_{V_n^{1/2}(\Omega)} \le C\|\cdot\|^{1/2}_{V_n^0(\Omega)}\|\cdot\|^{1/2}_{V_n^1(\Omega)}$ yields (6.17) with $s = 0$, and (6.14) follows from (5.4) and (5.5). □

We are now in position to prove a local stabilization result.

Theorem 10 *Let $s \in [\frac{d-2}{2}, 1]$. There exist $c_0 > 0$ and $\mu_1 > 0$ such that, if $\delta \in (0, \mu_1)$ and $Py_0 \in \mathcal{V}^s_{\Pi,\delta}$, then system (6.11) admits a unique solution $y(\cdot, y_0)$ in the set $\mathcal{S}^s_{\Pi,\delta} = \mathcal{S}^s_\delta \cap V^{1+s,\frac{1}{2}+\frac{s}{2}}(\Pi, Q)$. Moreover, we have $y \in C_b([0,\infty[; V^s(\Pi,\Omega))$, and the function V_s, which is defined by (2.22), is a Lyapunov function of system (6.11): for all $\xi \in V^s(\Pi,\Omega)$ we have $V_s(\xi) \ge C\|\xi\|^2_{V^s(\Pi,\Omega)}$ and there exists $\sigma > 0$ such that, for all $Py_0 \in \mathcal{V}^s_{\Pi,\delta}$ and $y(\cdot, y_0) \in \mathcal{S}^s_{\Pi,\delta}$ solution to (6.11), the mapping $t \longmapsto V_s(y(t, y_0))$ decreases to 0 and obeys:*

$$V_s(y(t,y_0)) \le C\|Py_0\|^2_{V_n^s(\Omega)}e^{-2\sigma t} \quad \forall t \ge 0. \tag{6.19}$$

In this setting, the sets $\mathcal{V}^s_{\Pi,\delta}$ and $\mathcal{S}^s_\delta$ are defined in (6.3) and (6.4), and $C_b([0,\infty[; V^s(\Pi,\Omega))$ is the space of continuous and bounded functions of $t \in [0,\infty[$ with value in $V^s(\Pi,\Omega)$.

Proof. Let $c_0 > 0$ and $\mu_0 > 0$ be the ones given in Theorem 9 and let $0 \le \mu_1 \le \mu_0$. For $\delta \in (0, \mu_1)$ and for an initial condition $Py_0 \in \mathcal{V}^s_{\Pi,\delta}$, we consider the solution $y \in \mathcal{S}^s_\delta$ to (6.1)-(6.2), which is also solution in $\mathcal{S}^s_{\Pi,\delta}$ to (6.11). If $s = 0$, for all $T > 0$ we deduce that $Py \in W(0,T; V_n^1(\Pi,\Omega), V_0^{-1}(\Omega)) \hookrightarrow C([0,T]; V_n^0(\Pi,\Omega))$ from (6.1), and we obtain $y \in C([0,T]; V^1(\Pi,\Omega))$ from (6.2). If $s > 0$, for all $T > 0$ we deduce that $y \in C([0,T]; V^s(\Pi,\Omega))$ from the continuous embedding $V^{1+s,\frac{1}{2}+\frac{s}{2}}(\Pi,Q)) \hookrightarrow C([0,T]; V^s(\Pi,\Omega))$. Next, we multiply the first equation in (6.1) by $\Pi^{(s)}Py(t)$ and we obtain

$$\frac{1}{2}\frac{d}{dt}\|Py(t)\|^2_{\Pi,s} + (A_\Pi Py(t)|Py(t))_{\Pi,s} = (N_\Pi(Py(t))|Py(t))_{\Pi,s}. \tag{6.20}$$

Thus, from (6.14), we deduce the existence of $C_s > 0$ such that:

$$\frac{d}{dt}\|Py(t)\|^2_{\Pi,s} + 2(1 - C_s\|Py(t)\|_{\Pi,s})(A_\Pi Py(t)|Py(t))_{\Pi,s} \le 0. \tag{6.21}$$

If we choose y_0 so that $\|Py_0\|_{\Pi,s} < \frac{1}{2C_s}$, then the mapping $t \longmapsto \|Py(t)\|_{\Pi,s}$ is a nonincreasing function with values less than $\frac{1}{2C_s}$. As a consequence, for $C_1 > 0$ and $\sigma > 0$ such that $\|\cdot\|_{V_0^s(\Omega)} \le C_1\|\cdot\|_{\Pi,s}$ and $2\sigma\|\cdot\|^2_{\Pi,s} \le (A_\Pi\cdot|\cdot)_{\Pi,s}$, if we choose $\mu_1 = \min(\mu_0, \frac{1}{2C_1C_s})$, then we have $\|Py_0\|_{\Pi,s} \le \mu_1 \le \frac{1}{2C_s}$ and (6.21) yields:

$$\frac{d}{dt}\|Py(t)\|^2_{\Pi,s} + 2\sigma\|Py(t)\|^2_{\Pi,s} \le 0 \quad \forall t \in (0,\infty). \tag{6.22}$$

Finally, (6.22) with (5.4) yields (6.19). □

Proof of Theorem 4

Let $s \in [\frac{d-2}{2}, 1]$, and let $c_0 > 0$ and $\mu_1 > 0$ be the ones given in Theorem 10. For $\delta \in (0, \mu_1)$ and for $Py_0 \in \mathcal{V}^s_{\Pi,\delta}$, we consider the solution $y \in \mathcal{S}^s_{\Pi,\delta}$ to (6.11) which is also solution in $\mathcal{S}^s_\delta$ to (6.1)-(6.2). Since from Proposition 4 the formulation (6.1)-(6.2) is equivalent to

$$\begin{aligned}&\partial_t z - \nu\Delta z + (z\cdot\nabla)z + \nabla r = f, \quad \nabla\cdot z = 0 \text{ in } (0,T)\times\Omega,\\ &z = v_b - B^*\Pi P y \text{ on } (0,\infty)\times\Gamma, \;\; z(0) = z_s + y_0,\end{aligned}$$

which, by Proposition 2 is equivalent to (2.24)-(2.28), Theorem 10 yields the existence and the uniqueness of

$$(z, r) = (z_s + y, r_s + p) \in \big\{(z_s, r_s)\big\} + \mathcal{S}^s_\delta \times H^{-\frac{1}{2}+\frac{s}{2}}(0, +\infty; \mathcal{H}^s(\Omega)),$$

solution to (2.24)-(2.25)-(2.26)-(2.27)-(2.28). Moreover, since $y = z - z_s$ satisfies

$$\partial_t y - \nu\Delta y + (y\cdot\nabla)z_s + (z_s\cdot\nabla)y + (y\cdot\nabla)y + \nabla(r - r_s) = 0, \tag{6.23}$$

from $y \in V^{1+s,\frac{s}{2}+\frac{1}{2}}(Q)$ we deduce that $\partial_t y \in H^{-\frac{1}{2}+\frac{s}{2}}(0,\infty; V^0_n(\Omega))$. Thus, by checking each term in (6.23) we successively obtain

$$\nabla(r - r_s) \in H^{-\frac{1}{2}+\frac{s}{2}}(0, +\infty; V^0_n(\Omega)) + L^2(0, +\infty; \mathbf{H}^{s-1}(\Omega)),$$

and

$$\begin{aligned}\|r - r_s\|_{H^{-\frac{1}{2}+\frac{s}{2}}(0,+\infty;H^s(\Omega))} &\leq c_1\|\nabla(r - r_s)\|_{H^{-\frac{1}{2}+\frac{s}{2}}(0,+\infty;\mathbf{H}^{s-1}(\Omega))}\\ &\leq c_1(\|\partial_t y\|_{H^{-\frac{1}{2}+\frac{s}{2}}(0,+\infty;V^0_n(\Omega))} + \|(y\cdot\nabla)y\|_{L^2(0,+\infty;\mathbf{H}^{s-1}(\Omega))})\\ &+ \|\nu\Delta y + (y\cdot\nabla)z_s + (z_s\cdot\nabla)y\|_{L^2(0,+\infty;\mathbf{H}^{s-1}(\Omega))}\\ &\leq c_2(\|y\|_{W(0,+\infty;V^{s+1}_0(\Omega)V^{s-1}_0(\Omega))} + \|y\|^2_{W(0,+\infty;V^{s+1}_0(\Omega),V^{s-1}_0(\Omega))}),\end{aligned}$$

where c_1 and c_2 are two positive constants. Then for c_0 given in (6.3) we can choose $c = c_0(\max(c_2, \sqrt{c_2}, 1))^{-1}$ in (7.9). Finally, (2.31) follows from (6.19) with (6.2) and $DB^*\Pi \in \mathcal{L}(V^s_0(\Omega), V^{1+s}(\Omega))$. $\square$

7 Feedback control localized in a part of the boundary.

In the previous sections, we deal with a boundary control acting on the whole boundary Γ. Nevertheless, it is possible to treat the case of a boundary control which is localized in an open subset of Γ. By following the idea of [18] we introduce a weight function $m \in C^2(\Gamma)$ with values in $[0, 1]$, with support in $\Gamma_m \subset \Gamma$ and equal to 1 in Γ_1, where Γ_1 is an open subset of Γ_m. Hence, we define the operator $M \in \mathcal{L}(\mathbf{L}^2(\Gamma); V^0(\Gamma))$ as follows:

$$M : v \longmapsto m\Big(v - \Big(\int_\Gamma m\Big)^{-1}\Big(\int_\Gamma mv\cdot n\Big)n\Big) \quad \forall v \in \mathbf{L}^2(\Gamma). \tag{7.1}$$

Then we now consider the following evolution system with nonhomogeneous Dirichlet boundary condition localized on Γ_m:

$$y' + Ay + A_0 y = BMu, \quad y(0) = y_0 \in V^{-1}_0(\Omega). \tag{7.2}$$

From the null controllability result for distributed control stated in [8], we can obtain a null controllability result for a control localized on Γ_1 by using an extension of the domain Ω. Then for $\mathcal{J}$ defined in (3.3), the minimizing problem

$$\inf\Big\{\ \mathcal{J}(y, u) \ \mid\ (y, u) \in W(0,\infty; V^0_n(\Omega), V^{-2}_0(\Omega)) \times L^2(0,\infty; V^0(\Gamma)) \ \text{ satisfies } \ (7.2)\Big\}$$

admits a unique solution $(\widehat{y}, \widehat{u})$, and $\widehat{y}$ is the solution to the closed loop system

$$y' + Ay + A_0 y = BM^2(B^*\Pi_M)y, \quad y(0) = y_0 \in V^{-1}_0(\Omega).$$

Here Π_M is the unique solution to an algebraic Riccati equation. Then as in sections **3** and **6** we can prove the following theorem.

Theorem 11 *The following results hold.*
(i) There is a unique nonnegative and self-adjoint operator $\Pi_M \in \mathcal{L}(V_n^0(\Omega))$, which belongs to $\mathcal{L}(V_n^0(\Omega), V_0^2(\Omega))$, and which is solution to the following Riccati equation:

$$((A + A_0^*)\Pi_M\xi|\zeta) + (\xi|(A + A_0^*)\Pi_M\zeta) + (MB^*\Pi_M\xi|MB^*\Pi_M\zeta)_{V^0(\Gamma)} = (\xi|\zeta) \ \forall(\xi,\zeta) \in V_n^0(\Omega) \times V_n^0(\Omega). \tag{7.3}$$

(ii) Let Π_M be the solution to (7.3), let $f \in \mathbf{H}^1(\Omega)$ and $v_b \in \mathbf{H}^{\frac{5}{2}}(\Gamma)$ be such that $\int_{\Gamma(j)} v_b \cdot n = 0$, for all $j = 1 \ldots N$, and let $(z_s, r_s) \in V^3(\Omega) \times \mathcal{H}^2(\Omega)$ be a solution to (1.1). If we define the space of initial conditions

$$V_n^s(\Pi_M, \Omega) = \left\{ y \in V_n^s(\Omega) \ | \ (I + PDM^2B^*\Pi_M)y \in V_0^s(\Omega) \right\}, \quad s \in \left[\frac{d-2}{2}, 1\right],$$

and if we consider the system

$$\partial_t z - \nu\Delta z + (z\cdot\nabla)z + \nabla r = f \ \text{ and } \ \nabla\cdot z = 0 \ \text{ in } \ (0,\infty)\times\Omega, \ z(0) = z_0, \tag{7.4}$$
$$\gamma_\tau z = \gamma_\tau v_b + m^2\partial_n\Pi_M P(z - z_s) \ \text{ on } \ (0,\infty)\times\Gamma, \tag{7.5}$$
$$\gamma_n z = \gamma_n v_b + M^2(\psi n) \ \text{ on } \ (0,\infty)\times\Gamma, \tag{7.6}$$
$$\Delta\psi = \nabla\cdot(\nabla z_s^T - z_s\cdot\nabla)\Pi_M P(z - z_s) \ \text{ in } \ (0,\infty)\times\Omega, \tag{7.7}$$
$$\partial_n\psi = (-\nu\Delta + \nabla z_s^T - z_s\cdot\nabla)\Pi_M P(z - z_s)\cdot n \ \text{ on } \ (0,\infty)\times\Gamma, \tag{7.8}$$

then the following result holds. There exist $c > 0$ and $\mu_1 > 0$ such that, if $\delta \in (0, \mu_1)$ and

$$P(z_0 - z_s) \in \mathcal{W}_\delta^s = \left\{ y \in V_n^s(\Pi_M, \Omega) \ | \ \|y\|_{V_n^s(\Omega)} \le c\delta \right\}, \tag{7.9}$$

then (7.4)-(7.8) admits a unique solution in the set $\{(z_s, r_s)\} + \mathcal{D}_\delta^s$ where

$$\begin{aligned} \mathcal{D}_\delta^s \ = \ \Big\{ & (y,p) \in V^{s+1,\frac{s}{2}+\frac{1}{2}}(Q) \times H^{-\frac{1}{2}+\frac{s}{2}}(0,\infty;\mathcal{H}^s(\Omega)) \\ & \|y\|_{V^{s+1,\frac{s}{2}+\frac{1}{2}}(Q)} \le \delta, \quad \|p\|_{H^{-\frac{1}{2}+\frac{s}{2}}(0,+\infty;H^s(\Omega))} \le \delta(1+\delta) \Big\}. \end{aligned}$$

Moreover, there exist $C > 0$ and $\sigma > 0$ such that z obeys

$$\|(I - P)(z(t) - z_s)\|_{V^{1+s}(\Omega)} + \|P(z(t) - z_s)\|_{V_n^s(\Omega)} \le C\|P(z_0 - z_s)\|_{V_n^s(\Omega)}\, e^{-\sigma t} \quad \forall t \ge 0. \tag{7.10}$$

(iii) If $s \in [0, \frac{1}{2}[$ we have $V_n^s(\Pi_M, \Omega) = V_n^s(\Omega)$, and if $s \in]\frac{1}{2}, 1]$ we have $V_n^s(\Pi_M, \Omega) \supset P(V^s(\Pi_M, \Omega))$, where

$$V^s(\Pi, \Omega) = \left\{ y \in V^s(\Omega) \quad | \quad \mathcal{T}_{\Pi_M}(y) = 0 \ \text{ on } \Gamma \right\},$$

and where

$$\mathcal{T}_{\Pi_M}(y) = y|_\Gamma - m^2\partial_n\Pi y + M^2(rn) \quad \text{ and } \quad \left\{ \begin{array}{l} \Delta r = \nabla\cdot(z_s\cdot\nabla - (\nabla z_s)^T)\Pi_M y \ \text{ in } \ \Omega, \\ \partial_n r = (\nu\Delta - (\nabla z_s)^T + z_s\cdot\nabla)\Pi_M y\cdot n \quad \text{on} \quad \Gamma. \end{array} \right.$$

As a consequence, if $d = 2$ and $s \in [0, \frac{1}{2}[$, then there exist $c > 0$ and $\mu_1 > 0$ such that, if $\delta \in (0, \mu_1)$ and

$$P(z_0 - z_s) \in \mathcal{W}_\delta^s = \left\{ y \in V_n^s(\Omega) \ | \ \|y\|_{V_n^s(\Omega)} \le c\delta \right\},$$

then (2.24)-(2.28) admits a unique solution in the set $\{(z_s, r_s)\} + \mathcal{D}_\delta^s$ and z obeys (7.10).

Remark 11 *Notice that, from (7.1), for all $v \in \mathbf{L}^2(\Gamma)$ we easily verify that $M^2(v) = m^2 v - m\mathcal{C}(m, v\cdot n)n$, where for all $\phi \in L^2(\Gamma)$, $\mathcal{C}(m,\phi) = mc(m,\phi) + c(m, m\phi) + c(m,m)c(m,\phi)$ and $c(m,\phi) = \big(\int_\Gamma m\big)^{-1}\int_\Gamma m\phi$. As a consequence, in (7.6), the expression of $M^2(\psi n)$ is given by:*

$$M^2(\psi\, n) = m\Big(m\psi - \mathcal{C}(m,\psi)\Big)n \quad \text{ where } \quad \mathcal{C}(m,\psi) = mc(m,\psi) + c(m, m\psi) + c(m,m)c(m,\psi).$$

8 Appendix

We recall that $z_s \in V^3(\Omega)$ and that Ω is of class C^4. Here, we collect some regularity results for the state and the adjoint state.

Lemma 5 *For $s \geq 0$ and $0 < \varepsilon < \frac{1}{2}$, if $\Phi \in V^{s+2,\frac{s}{2}+1}(Q)$ then we have*

$$\|B^*\Phi\|_{V^{s+\frac{1}{2}-\varepsilon,\frac{s}{2}+\frac{1}{4}-\frac{\varepsilon}{2}}(\Sigma)} \leq C\|\Phi\|_{V^{s+2,\frac{s}{2}+1}(Q)}. \tag{8.11}$$

Proof. It is an easy consequence of $B^* \in \mathcal{L}(V_0^{s+2}(\Omega), V^{s+\frac{1}{2}-\varepsilon}(\Gamma)) \cap \mathcal{L}(V_0^{\frac{3}{2}+\varepsilon}(\Omega), V^0(\Gamma))$ and of the continuous embedding $V^{s+2,\frac{s}{2}+1}(Q) \hookrightarrow L^2(0,\infty; V_0^{s+2}(\Omega)) \cap H^{\frac{s}{2}+\frac{1}{4}-\frac{\varepsilon}{2}}(0,\infty; V_0^{\frac{3}{2}+\varepsilon}(\Omega))$. $\square$

Lemma 6 ([19], Theorem. 4.1 (iii) and (iv)) *For $s \in [1/2, 2]\backslash\{1\}$, let $y_0 \in V_n^{s-\frac{1}{2}}(\Omega)$ and $u \in V^{s,\frac{s}{2}}(\Sigma)$. If $s \geq 1$ we also assume that $y_0 - PDu(0) \in V_0^{s-\frac{1}{2}}(\Omega)$. The solution y to the equation*

$$y' + Ay + A_0 y + \lambda_0 y = Bu, \quad y(0) = y_0,$$

obeys

$$\|y\|_{V^{s+\frac{1}{2},\frac{s}{2}+\frac{1}{4}}(Q)} \leq C(\|u\|_{V^{s,\frac{s}{2}}(\Sigma)} + \|y_0\|_{V_0^{s-\frac{1}{2}}(\Omega)}). \tag{8.12}$$

Lemma 7 ([18], Lemma 8.5) *For $s \in [0, 2]$, let $f \in V^{s,\frac{s}{2}}(Q)$. The solution Φ to the equation*

$$-\Phi' + A\Phi + A_0\Phi + \lambda_0\Phi = f, \quad \Phi(\infty) = 0,$$

obeys

$$\|\Phi\|_{V^{s+2,\frac{s}{2}+1}(Q)} \leq C\|f\|_{V^{s,\frac{s}{2}}(Q)}. \tag{8.13}$$

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