Stabilizing a linear systems with saturation through optimal control

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Abstract. We construct a continuous feedback for a saturated system $\dot{x}(t) = Ax(t) + B\sigma(u(t))$. The feedback renders the system asymptotically stable on the whole set of states that can be driven to 0 with an open-loop control. The trajectories of the resulting closed-loop system are optimal for an auxiliary optimal control problem with a convex cost and linear dynamics. The value function for the auxiliary problem, which we show to be differentiable, serves as a Lyapunov function for the saturated system. Relating the saturated system, which is nonlinear, to an optimal control problem with linear dynamics is possible thanks to the monotone structure of saturation.

1 Introduction

Global asymptotic stabilization of a linear system with saturating actuators

$$\dot{x}(t) = Ax(t) + B\sigma(u(t)) \tag{1}$$

can not, in general, be achieved with a linear feedback. Moreover, if an eigenvalue of A has a positive real part and σ is bounded, the set X_0 consisting of all states that can be driven to 0 with an open loop control will not equal to the whole state space. If such eigenvalues are excluded, continuous feedbacks globally stabilizing (1) exist under mild assumptions on σ, as shown by Sontag and Sussmann [19] and Sontag, Sussmann, and Yang [20]. Also then, semiglobal stabilization can be achieved with linear feedback possessing additional properties like robustness and disturbance rejection, see Saberi, Lin, and Teel [18]. For the general case, much work has been devoted to estimating X_0 and to semiglobal stabilization on X_0 (that is, to constructing feedbacks which stabilize (1) on any a priori given compact subset of X_0), see the book by Hu and Lin [11] and the numerous references therein.

Existence of a continuous feedback that renders the saturated system (1) asymptotically stable on the whole set X_0 has not been established. We prove it here, by exhibiting a

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feedback which guarantees that the resulting trajectories of (1) are optimal for the following linear-convex regulator problem:

$$\mathcal{LCR}(x_0): \qquad \text{minimize} \int_0^\infty \frac{1}{2} x(t) \cdot Qx(t) + r(w(t))\, dt \ \text{ s.t. } \left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + Bw(t), \\ x(0) = x_0. \end{array} \right. \tag{2}$$

This problem has no saturation but information about σ is represented through the convex penalty function r. The stabilizing feedback for the saturated system (1) will turn out to be very closely related to the optimal feedback for the $\mathcal{LCR}$. In (2) the control variable is denoted $w(\cdot)$ to distinguish it from the control $u(\cdot)$ in (1) – these are not the same, and Q is any symmetric and positive definite matrix.

Before describing the relationship between the saturation σ and the convex function r appearing in (2), we state the assumptions, which are posed throughout the paper:

(A1) The pair (A, B), where $A \in I\!R^{n\times n}$, $B \in I\!R^{n\times k}$, is controllable.

(A2) The saturation function $\sigma : I\!R^k \mapsto I\!R^k$ has the form $\sigma(u) = (\sigma_1(u_1), \sigma_2(u_2), ..., \sigma_k(u_k))$, where $\sigma_i(0) = 0$, σ_i is nondecreasing on $I\!R$ and strictly increasing on a neighborhood of 0, $i = 1, 2, ..., k$.

Under $(A2)$, there exists a convex function $s : I\!R^k \mapsto I\!R$ with $s(0) = 0$ and with the gradient $\nabla s = \sigma$. Then r is taken to be the convex function conjugate to s in the sense of convex analysis, see Rockafellar [13]. We explain this in detail in Section 2.

Introducing a $\mathcal{LCR}$ as an auxiliary optimal control problem is a natural idea. Feedbacks stabilizing a linear system $\dot{x}(t) = Ax(t) + Bu(t)$ can be found with the help of a $\mathcal{LQR}$ problem. When σ in (1) is the standard saturation, that is $\sigma_i(u_i)$ equals u_i if $-1 \le u_i \le 1$, -1 if $u_i < -1$, and 1 if $u_i > 1$, one can consider a linear-quadratic regulator with a control constraint $|u_i| \le 1$. The constrained $\mathcal{LQR}$ can be equivalently written in the $\mathcal{LCR}$ form (2), with r being quadratic if u satisfies the constraint, and equal to $+\infty$ otherwise (this is a well-known technique in optimization). The use of value functions of auxiliary problems as Lyapunov functions is possible for general nonlinear systems, but need not result in a smooth function, and the resulting stabilizing feedbacks need not be continuous, see Clarke, Ledyaev, Sontag, Subbotin [6] and Clarke, Ledyaev, Rifford, Stern [5]. The expected lack of continuity of optimal feedbacks for problems with nonlinear dynamics was a part of the motivation for an alternate approach to stabilization of a saturated system in [20].

The special structure of $\mathcal{LCR}$ has important consequences for the value function $V(x_0)$ defined as the optimal value in (2). We now state some of them, details are provided in Section 3. Most importantly, V is a convex function. It is positive definite, has finite values on the open and convex set X_0 while $V(x_0) = +\infty$ if $x_0 \notin X_0$, and its sublevel sets $\{x \,|\, V(x) \le \alpha\}$ are compact for each $\alpha \ge 0$. Finally, we prove it is differentiable on X_0, and then continuity of ∇V on X_0 (which will be the key to continuity of the stabilizing feedback for (1)) follows from a general property of convex functions.

With the differentiability of V established, standard dynamic programming arguments show that the optimal feedback for the $\mathcal{LCR}$ is

$$w = F_{\mathcal{LCR}}(x) = \nabla s(-B^* \nabla V(x)),$$

which is equivalent to $F_{\mathcal{LCR}}(x) = \arg\max_w \{-\nabla V(x(t)) \cdot Bw - r(w)\}$. Optimal trajectories $x(\cdot)$ resulting from applying this optimal feedback to the linear system satisfy

$$\frac{d}{dt} V(x(t)) \leq -\frac{1}{2} x(t) \cdot Qx(t), \tag{3}$$

and hence $x(t) \to 0$ as $t \to \infty$.

Now, the relationship between the saturated system and $\mathcal{LCR}$ should become clear. Since $\nabla s = \sigma$, the *nonsaturated* linear system with the feedback $w = \nabla s(-B^* \nabla V(x))$ is exactly the same as the *saturated* system (1) with the feedback

$$u = F(x) = -B^* \nabla V(x).$$

This means that F is a stabilizing feedback for the saturated system. Moreover, (3) shows that the value function for $\mathcal{LCR}$ serves as a classical Lyapunov function for the saturated system. We state this more precisely in Section 4.

2 Saturation functions as gradients

The key to our approach is expressing the saturation function σ of the saturated linear system (1) as a gradient of a convex function.

Example 2.1 Let $\sigma : \mathbb{R} \mapsto \mathbb{R}$ be a continuous and nondecreasing function, with $\sigma(0) = 0$. Then

$$s(u) = \int_0^u \sigma(t)\, dt$$

defines a differentiable convex function $s : \mathbb{R} \mapsto \mathbb{R}$, with $s(0) = 0$, $s \geq 0$, and, of course, $s' = \sigma$. Other often assumed properties of σ reflect in those of s as follows:

- If $\sigma(u) = 0$ only for $u = 0$, s is positive definite.
- If $\liminf_{u \to 0} \frac{\sigma(u)}{u} > 0$ – equivalently, if for some $\epsilon > 0$, $\delta > 0$, we have $u\sigma(u) \geq \delta u^2$ for $|u| < \epsilon$ – then $s(u)$ is bounded below by $\frac{1}{2}\delta u^2$ if $|u| < \epsilon$, by $-\delta\epsilon u - \frac{1}{2}\delta\epsilon^2$ if $u \leq -\epsilon$, and by $\delta\epsilon u - \frac{1}{2}\delta\epsilon^2$ if $\epsilon < u$.

- If σ is globally Lipschitz with constant l, then $s(u) \leq \frac{l}{2}u^2$.

Here, the important relationship is between strict convexity of s on a neighborhood of 0 and σ being strictly increasing on such a neighborhood.

Statements just made can be easily verified for the standard saturation function $\overline{\sigma} : I\!\!R \mapsto [-1, 1]$, which is the derivative of the following convex function:

$$\overline{s}(u) = \begin{cases} -u - \frac{1}{2} & \text{for} \quad u < -1, \\ \frac{1}{2}u^2 & \text{for} \quad -1 \leq u \leq 1, \\ u - \frac{1}{2} & \text{for} \quad 1 < u. \end{cases} \tag{4}$$

Example 2.2 Suppose $\sigma(u) = (\sigma_1(u_1), \sigma_2(u_2), ...\sigma_k(u_k))$ where each σ_i is nondecreasing on $I\!\!R$, and $\sigma(0) = 0$. With each σ_i we can associate a convex function s_i as outlined in Example 2.1. Then $\sigma = \nabla s$ for $s(u) = \sum_{i=1}^k s_i(u_i)$, this is of course a convex function. Growth properties of s can be easily analyzed in terms of that of σ_i's. In particular, s is strictly convex on a neighborhood of $0 \in I\!\!R^k$ if if and only if each σ_i is strictly increasing on some neighborhood of $0 \in I\!\!R$.

Now, we explain how the convex function r, representing the control cost in the linear-convex regulator (2), is related to σ. Given a convex function s with $\nabla s = \sigma$ and $s(0) = 0$, we set r to be the convex function conjugate to s in the sense of convex analysis:

$$r(w) = \sup_{u \in \boldsymbol{R}^k} \{w \cdot u - s(u)\}. \tag{5}$$

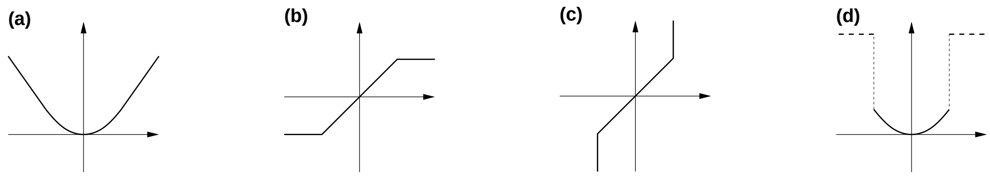
This function is always convex and lower semicontinuous. It need not be finite everywhere – for some w, we may have $r(w) = +\infty$. Also, r need not be differentiable – it's subdifferential ∂r is the set-valued inverse of ∇s (which equals σ, and need not be invertible in the classical sense). We state the definition of the subdifferential in Section 6, for now it can be thought of as a generalization of the gradient, in fact it equals the gradient whenever it is single-valued. Standard reference for these and other concepts of convex analysis is the book by Rockafellar [13].

In many cases of practical interest, r can be found directly. First, observe that the very definition (5) implies that when $s(u) = \sum_{i=1}^k s_i(u_i)$, as in Example 2.2, we have

$$r(w) = \sup_{u \in \boldsymbol{R}^k} \{w \cdot u - s(z)\} = \sum_{i=1}^k \sup_{u_i \in \boldsymbol{R}} \{w_i \cdot u_i - s_i(u_i)\} = \sum_{i=1}^k r_i(w_i),$$

where r_i is the convex conjugate of s_i. That is, r can be found componentwise. We now give some one-dimensional examples.

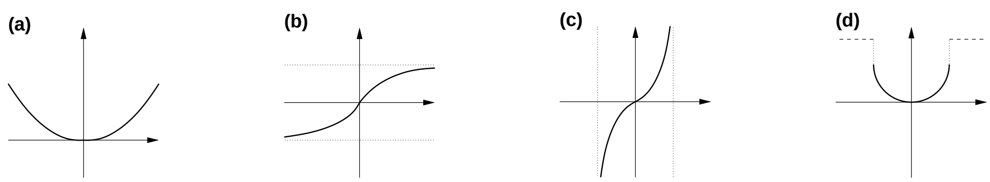
Example 2.3 (standard saturation and the conjugate function)**.** Consider the standard saturation $\overline{\sigma}$, shown below on graph (b). The function $\overline{s}$ given by (4), and shown below on graph (a), can be used to calculate $\overline{r}$ directly from the definition (5). Alternate approach is to look at the set-valued inverse of $\overline{\sigma}$, equal to $\partial\overline{r}$, which is shown on graph (c). Then, it remains to “integrate” $\partial\overline{r}$ to obtain $\overline{r}$, shown on graph (d).



Explicit formulas for $\partial\overline{r}$ and $\overline{r}$ are as follows:

$$\partial\overline{r}(w) = \begin{cases} \emptyset & \text{for} \quad w < -1, \\ (-\infty, 1] & \text{for} \quad w = -1, \\ w & \text{for} \quad -1 < w < 1, \\ [1, +\infty) & \text{for} \quad w = 1, \\ \emptyset & \text{for} \quad w > 1. \end{cases} \qquad \overline{r}(w) = \begin{cases} \frac{1}{2}w^2 & \text{for} \quad w \in [-1,1], \\ +\infty & \text{for} \quad w \notin [-1,1]. \end{cases}$$

Example 2.4 Consider a saturation function $\sigma(u) = \frac{u}{\sqrt{u^2+1}}$, which is a derivative of $s(u) = \sqrt{u^2+1} - 1$. The conjugate r can be found through (5). Alternatively, $\sigma^{-1}(w) = r'(w) = \frac{w}{\sqrt{1-w^2}}$ for $w \in (-1,1)$, while for $w \notin (-1,1)$, $\sigma^{-1}(w) = r'(w) = \emptyset$. Then, $r(w)$ can be found, for any $w \in [-1,1]$, by integrating r'. This leads to $r(w) = 1 - \sqrt{1-w^2}$ on $[-1,1]$, while $r(w) = +\infty$ for $w \notin [-1,1]$. Graph (a) below shows s, (b) shows σ, (c) displays $\sigma^{-1} = r'$, and r is on (d).



Note a slight discrepancy between the set of points where r is finite and the set of points where ∂r, which reduces to r', is nonempty.

The set $\operatorname{dom} r = \{w \in I\!R^n \mid r(w) < +\infty\}$ need not equal $I\!R^n$. In fact $r(w) = +\infty$ whenever $w \notin \overline{\operatorname{rge}\sigma}$ (the closure of the range of σ). Infinite values of r introduce a control constraint to the linear-convex regulator – feasible controls must satisfy $w(t) \in \operatorname{dom} r$. For the standard saturation, as expected, this means $w(t) \in \operatorname{dom}\overline{r} = \operatorname{rge}\overline{\sigma} = [-1,1]$. In general $\operatorname{rge}\sigma \subset \operatorname{dom} r$ (the equality $\operatorname{rge}\sigma = \operatorname{dom} r$ fails in Example 2.4). For details, see the beginning of Section 24 in Rockafellar [13].

To summarize this section, we state the following.

Fact 2.5 (saturation and convex functions)**.** Given a saturation function σ as in Assumption $(A2)$, there exist convex functions $s : I\!\!R^k \mapsto [0, +\infty)$ and $r : I\!\!R^k \mapsto [0, +\infty]$ related to each other by (5) and such that:

(i) s is differentiable, $\nabla s = \sigma$, $s(0) = 0$, and s is strictly convex on some neighborhood of 0;

(ii) r is positive definite and on some neighborhood of 0, it has finite values.

3 The value function for $\mathcal{LCR}$

The value function of the linear-convex regulator,

$$V(x_0) = \inf \left\{ \int_0^\infty \frac{1}{2} x(t) \cdot Qx(t) + r(u(t))\, dt \mid \dot{x}(t) = Ax(t) + Bu(t),\ x(t) = x_0 \right\} \tag{6}$$

with the minimization carried out over all locally integrable controls $u : [0, +\infty)$, is obviously positive definite. It may occur that for some $x_0 \in I\!\!R^n$, $V(x_0) = +\infty$; this is the case when no control makes the integral in (6) finite.

A key property of V is that it is a convex function on $I\!\!R^n$. This is a consequence of a general principle that value functions for convex optimization problems are convex, see Rockafellar [15]. Here, since a composition of an affine map with a convex function is convex, and the trajectory $x(\cdot)$ depends affinely on x_0 and $u(\cdot)$, the integral in (6) is a convex function of x_0 and $u(\cdot)$. Minimizing it with respect to $u(\cdot)$ yields a convex function of x_0. Convexity can also be verified directly through the definition of convexity, the infinite values just require some extra care.

A consequence of the value function being convex is that the level sets of V, being $\{x \in I\!\!R^n \mid V(x) \leq r\}$, are convex and bounded, for each $r \in I\!\!R$. Boundedness follows from the existence of a single nonempty and bounded level set (8.7.1 in [13]), in this case $\{x \in I\!\!R^n \mid V(x) \leq 0\}$ which reduces to a single point 0. In turn, boundedness implies that any process $(\bar{x}(\cdot), \bar{u}(\cdot))$ for which the integral in (6) is finite satisfies $\bar{x}(t) \to 0$ as $t \to \infty$. Indeed, from the definition of the value function it follows that

$$V(\bar{x}(t)) \leq \int_t^\infty \frac{1}{2} \bar{x}(t) \cdot Q\bar{x}(t) + r(\bar{u}(t))\, dt,$$

for every $t \geq 0$, and the integral above tends to 0 as $t \to +\infty$.

We now argue that the set $\operatorname{dom} V = \{x \in I\!\!R^n \mid V(x) < +\infty\}$ is open. By continuity of r at 0 and controllability of (A, B), $\operatorname{dom} V$ contains some neighbourhood N of 0. Pick any $x_0 \in \operatorname{dom} V$, and let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a process for which the integral in (6) is finite. For some $T > 0$, $\bar{x}(T) \in N$. Thus any x_0' from some neighbourhood of x_0 can be driven into

N by the control $\bar{u}(\cdot)$ truncated to $[0, T]$. This shows that $V(x_0')$ is finite, and thus $\operatorname{dom} V$ is open.

The main result of this section claims the smoothness of V, which will turn out to be the key to the continuity of the stabilizing feedback for the saturated system. The proof is in Section 6.

Theorem 3.1 (differentiability of V)**.** *The value function V is differentiable at every point of $\operatorname{dom} V$ and $\|\nabla V(x_i)\| \to +\infty$ for any sequence of points $x_i \in \operatorname{dom} V$ converging to a point not in $\operatorname{dom} V$. The gradient ∇V is continuous on $\operatorname{dom} V$. The function V is strictly convex.*

Some results guaranteeing differentiability of value functions in similar settings exist, but do not directly apply here. Benveniste and Scheinkman [2], and Gota and Montrucchio [10] require r to be differentiable and the optimal controls to be interior in some sense, neither assumption is met here.[1] Rockafellar [14] showed that if the (maximized) Hamiltonian is strictly concave in x, strictly convex in p, the value function is differentiable. The Hamiltonian for $\mathcal{LCR}$ is

$$H(x, p) = p \cdot Ax - \frac{1}{2} x \cdot Qx + s(B^* p). \tag{7}$$

It is not strictly convex in p unless B is invertible and s is strictly convex everywhere. Barbu [1] incorporated a controllable linear system to the framework of [14], under additional growth properties of H. Ideas from [14] and [1] can be combined to show that V is differentiable on a neighborhood of 0. Then, writing $\mathcal{LCR}$ as a finite time problem with a terminal penalty V and applying the technique of Theorem 3.1 in Goebel [9] could be used to translate this to a global statement. Instead, we give a more direct proof in Section 6. We rely in part on a duality result of Goebel [7] and clearly illustrate how local strict convexity can yield a global differentiability statement.

Corollary 3.2 (optimal feedback for $\mathcal{LCR}$)**.** *The mapping $F_{\mathcal{LCR}} : \operatorname{dom} V \to \mathbb{R}^k$ defined by $F_{\mathcal{LCR}}(x) = \nabla s(-B^* \nabla V(x))$ is the optimal feedback for $\mathcal{LCR}$. That is, for any $x_0 \in \operatorname{dom} V$, the process $(\bar{x}(\cdot), \bar{w}(\cdot))$ with $\bar{x}(\cdot)$ being a solution to $\bar{x}(0) = x_0$, $\dot{x}(t) = A\bar{x}(t) + B\bar{w}(t)$ and $\bar{w}(t) = F_{\mathcal{LCR}}(\bar{x}(t))$, is optimal for $\mathcal{LCR}(x_0)$.*

We outline the standard argument. The value function V satisfies the Hamilton-Jacobi equation

$$H(x, -\nabla V(x)) = 0 \quad \text{for all } x \in \operatorname{dom} V, \tag{8}$$

[1]These two works represent a wide body of theoretical economics research devoted to optimal control on infinite time intervals, a good source of references is the book by Carlson, Haurie, and Leizarowitz [4].

where H is given by (7). In the current setting, this can easily be deduced from the proof of Theorem 3.1 in Section 6, as H is constant along Hamiltonian trajectories (solutions to (17)) and $H(0,0)=0$. From the definition of r in terms of s in (5), one can see that

$$r(\nabla s(u)) = \nabla s(u) \cdot u - s(u).$$

The Hamilton-Jacobi equation and the equation above show that

$$\frac{d}{dt}V(\bar{x}(t)) = -\frac{1}{2}\bar{x}(t) \cdot Q\bar{x}(t) - r(\bar{w}(t)), \tag{9}$$

which implies both that $\bar{x}(t) \to 0$ as $t \to 0$ and that $x(\cdot)$ is optimal for $\mathcal{LCR}(x_0)$. The latter follows from integrating (9) on $[0,+\infty)$ and comparing the result with the definition of $V(x_0)$. Additionally, this shows that the optimal control $\bar{w}(\cdot)$ is continuous and $\bar{w}(t) \to 0$ as $t \to \infty$.

4 Stabilizing feedback for saturated systems

We are now ready to state our main result.

Theorem 4.1 (stabilizing feedback for saturated systems)**.** *Consider the system*

$$\dot{x}(t) = Ax(t) + B\sigma(u(t)) \tag{10}$$

under assumptions (A1) and (A2). Let X_0 be the set of all $x_0 \in \mathbb{R}^n$ for which there exists a piecewise continuous control $u : [0,+\infty) \mapsto \mathbb{R}^k$ such that the solution of (10) with $x(0) = x_0$ converges to 0. Let $Q \in \mathbb{R}^{n\times n}$ be any symmetric and positive definite matrix.

Then, there exists a continuous mapping $F : X_0 \mapsto \mathbb{R}^k$ and a convex, positive definite, and differentiable function $V : X_0 \mapsto \mathbb{R}$ such that, for any $x_0 \in X_0$, the solution $x(\cdot)$ to

$$\dot{x}(t) = Ax(t) + B\sigma(F(x(t))) \tag{11}$$

with $x(0) = x_0$ satisfies

$$\frac{d}{dt}V(x(t)) \leq -\frac{1}{2}x(t) \cdot Qx(t) \tag{12}$$

so that $x(t) \to 0$ as $t \to +\infty$.

As may be now expected, justification of Theorem 4.1 hinges upon translating the optimal feedback for $\mathcal{LCR}$ to a stabilizing feedback for the saturated system. First, we need to relate the set where the value function V is finite to X_0.

Lemma 4.2 *$X_0 = \operatorname{dom} V$.*

Proof. Fix $x_0 \in X_0$. Then there exists a piecewise continuous control such that the resulting solution of the saturated system, originating at x_0, converges to 0. As σ is continuously invertible around 0 and (A, B) is controllable, x_0 can be steered to 0 by a piecewise continuous control $u(\cdot)$ in finite time, say $T > 0$. Then the control $w(t) = \sigma(u(t))$ on $[0, T]$ and $w(t) = 0$ for $r > T$ and the resulting trajectory of $\dot{x}(t) = Ax(t) + Bw(t)$ yields a finite cost in (6). Indeed, as $\operatorname{rge}\sigma \subset \operatorname{dom} r$ (see the discussion at the end of Section 2), $r(w(\cdot))$ is piecewise continuous, $x(\cdot)$ is continuous, and both are 0 outside a compact interval. Thus, $V(x_0) < +\infty$ which means that $X_0 \subset \operatorname{dom} V$.

On the other hand, if $x_0 \in \operatorname{dom} V$, then the solution of

$$\dot{x}(t) = Ax(t) + B\nabla s(-B^*\nabla V(x(t))) \tag{13}$$

converges to 0, see Corollary 3.2. By construction, $\nabla s = \sigma$, so $u(t) = -B^*\nabla V(x(t))$ is the control required by the definition of X_0. Thus $\operatorname{dom} V \subset X_0$. □

Proof of Theorem 4.1. Given the system (10) and a matrix Q as assumed, let V be the value function (6) with the convex function r given by (5) and s such that $s(0) = 0$, $\nabla s = \sigma$. Corollary 3.2 and the discussion following it show that for any point $x_0 \in \operatorname{dom} V$, so by Lemma 4.2, for any point $x_0 \in X_0$, any solution $x(\cdot)$ to (13) with $x(0) = x_0$ satisfies (12). As by construction $\nabla s = \sigma$, the mapping $F : X_0 \mapsto \mathbb{R}^k$ defined by

$$F(x) = -B^*\nabla V(x) \tag{14}$$

satisfies the conclusions of Theorem 4.1. Continuity was established in Theorem 3.1. □

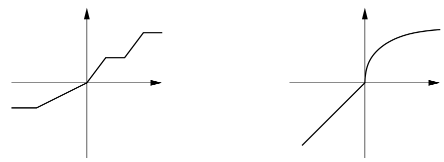
5 Comments and extensions

We now make several comments regarding our main result, Theorem 4.1, and the constructions leading up to it.

(*i*) The stabilizing feedback F for the saturated system is not the same as the optimal feedback for $\mathcal{LCR}$. However, by construction, trajectories of the saturated system with $u(t) = F(x(t))$ agree with optimal trajectories for the linear-convex regulator.

(*ii*) The optimal feedback $F_{\mathcal{LCR}}$ for the linear-convex regulator is related to the stabilizing feedback F by $F_{\mathcal{LCR}}(x) = \sigma(F(x))$, and when σ is invertible, $F(x) = \sigma^{-1}(F_{\mathcal{LCR}}(x))$. When σ is not invertible, the relationship $F(x) = \sigma^{-1}(F_{\mathcal{LCR}}(x))$ is not valid even in the set-valued sense, as then σ^{-1} is not single-valued.

(*iii*) The construction of F does not rely on considering σ^{-1}, not even on a subset of $\operatorname{rge}\sigma$ on which σ is invertible (this was, for example, the approach of [19]). Partly due to this, F is continuous even when the saturation σ is not invertible on "large" subsets of $\operatorname{rge}\sigma$. Furthermore, we do not request that σ be Lipschitz, differentiable at 0, or bounded. Examples of saturations we allow are sketched below.



(*iv*) $\mathcal{LCR}$ is a convex optimization problem. From the numerical computation viewpoint, such problems have many advantages over their nonconvex counterparts, see the book by Boyd and Vandenberghe [3]. A seemingly more obvious choice of an auxiliary control problem, with a convex or even quadratic cost and the dynamics provided directly by the saturated system, does not lead to a convex problem and is unlikely to yield a regular feedback or even a regular value function (which needs not be convex in such a case).

(*v*) An approach different from ours, but with some favorable convex structure, would be to find a Lyapunov function $\widetilde{V}$ for the saturated system as a solution to the Hamilton-Jacobi inequality

$$\inf_u \nabla\widetilde{V}(x)\cdot(Ax+B\sigma(u)) \le -\frac{1}{2}x\cdot Qx,$$

which translates to $\widetilde{H}(x,-\nabla\widetilde{V}(x))\ge 0$ for

$$\widetilde{H}(x,p) = p\cdot Ax - \frac{1}{2}x\cdot Qx + \sup_{w\in\operatorname{rge}\sigma} p\cdot Bw \ge 0.$$

This Hamiltonian is concave in x, convex in p, similarly to (7) corresponding to $\mathcal{LCR}$. However, it does not have finite values everywhere unless σ is bounded. Also, it is not clear if solutions are smooth ($\widetilde{H}$ is not strictly convex in p anywhere). Furthermore, recovering the stabilizing feedback for the saturated system would need to involve σ^{-1} in some way.

(*vi*) The role of semiconcavity of a Lyapunov function in stabilization of a general nonlinear system was stressed by Rifford [12]. We note that the value function for $\mathcal{LCR}$, and consequently the Lyapunov function for the saturated system we obtain here, need not be semiconcave (some convex functions are). Consider a one-dimensional system $\dot{x}(t)=\overline{\sigma}(u(t))$ with standard saturation and set $Q=1$. Solving $H(x,-\nabla V(x))=0$ yields

$$\nabla V(x) = \begin{cases} -\frac{1}{2}(x^2+1) & \text{if} \quad x<-1, \\ x & \text{if} \quad -1\le x\le 1, \\ \frac{1}{2}(x^2+1) & \text{if} \quad 1<x. \end{cases}$$

Consequently, V has cubic growth and $V(x) - \alpha x^2$ is never concave on $\mathbb{R}$ (it is on compact subsets though). This example also shows that the value function is not piecewise quadratic when one considers a regulator with quadratic cost and piecewise affine dynamics.

(vii) The componentwise structure of σ as in assumption ($A2$) is not necessary for our main result, as long as the conclusions of Fact 2.5 remain valid. Corollary 5.1 makes this precise, and Example 5.2 shows a saturation function without the componentwise structure.

Corollary 5.1 *Conclusions of Theorem 4.1 hold for any σ such that functions s, r as described in Fact 2.5 exist.*

This is true since the statements in Section 3 and the proof of Theorem 3.1 in Section 6 only invoke Fact 2.5. Lemma 4.2 requires that σ^{-1} be continuous around 0. But $\sigma^{-1} = \nabla r$ there (differentiability of r around 0 is implied by strict convexity of s around 0), moreover, ∇r is continuous (gradient of any differentiable convex function is). We now give an example of σ which satisfies the assumption of Corollary 5.1, but does not have the componentwise structure. For such saturation functions, calculating s and r is less simple, and makes use of calculus rules for conjugate convex functions, see [13] or Chapter 11 in Rockafellar and Wets [16].

Example 5.2 (projection onto a convex set)**.** The standard saturation $\overline{\sigma}$ on $\mathbb{R}$ can be thought of as a projection of u onto $[-1, 1]$ – for any u, $\overline{\sigma}(u)$ is the point in $[-1, 1]$ closest to u. In general, if C is a nonempty, closed, and convex set in $\mathbb{R}^k$, the projection onto it, denoted P_C, is a well-defined continuous mapping, with Lipschitz constant 1; see for example [16], 2.35 and 12.20. Then also $P_C = \nabla s$ for a convex function r given by

$$s(u) = \inf_{z \in \mathbb{R}^k} \left\{ \sup_{c \in C} z \cdot c + \frac{1}{2}\|u - z\|^2 \right\}. \tag{15}$$

This formula becomes much clearer for particular choices of C. For example, consider C to be the unit ball in $\mathbb{R}^k$. The map P_C is an identity for points in C, and a radial projection onto the unit sphere for points outside it (that is, $P_C(u) = u/\|u\|$). Then $\sup_{c \in C} u \cdot c = \|u\|$, and $s(u) = \frac{1}{2}\|u\|^2$ for $\|u\| \le 1$, $\|u\| - 1/2$ for $\|u\| > 1$. When $k = 1$, this reduces to the function $\overline{r}$ corresponding to standard saturation. Note also that this s is strictly convex around 0, in fact this property is present whenever 0 is in the interior of C.

For any convex C, the conjugate function r of (15) can be found through 11.4 and 11.23 in [16]:

$$r(w) = \begin{cases} \frac{1}{2}\|w\|^2 & \text{for} \quad w \in C, \\ +\infty & \text{for} \quad w \notin C. \end{cases}$$

Again, the standard saturation leads to a special instance of the formula above.

6 Proof of Theorem 3.1

Differentiability of V, as described in the first sentence of Theorem 3.1, is equivalent to the subdifferential of V, denoted ∂V and defined as

$$\partial V(x) = \left\{ p \in I\!R^n \mid V(x') \geq V(x) + p \cdot (x' - x) \text{ for all } x' \in I\!R^n \right\}, \tag{16}$$

being single valued where nonempty; Rockafellar [13], Theorem 26.1.

The graph of ∂V is a stable "manifold" for the Hamiltonian dynamical system. That is, if $p_0 \in -\partial V(x_0)$ then there exist locally absolutely continuous arcs $x(\cdot)$, $p(\cdot)$ on $[0, +\infty)$ with $x(0) = x_0$, $p(0) = p_0$, $x(t) \to 0$, $p(t) \to 0$ as $t \to +\infty$, and such that

$$\dot{x}(t) = Ax(t) + B\nabla s(B^* p(t)), \qquad \dot{p}(t) = -A^* p(t) + Qx(t). \tag{17}$$

This follows from the conjugacy relationship $V(x_0) = W^*(-x_0)$ (recall (18)), where W is the value function of the "dual" control problem:

$$W(p_0) = \inf \left\{ \int_0^{+\infty} s(B^* p(t)) + \frac{1}{2} z(t) \cdot Qz(t)\, dt \mid \dot{p}(t) = -A^* p(t) - z(t),\ p(0) = p_0 \right\}. \tag{18}$$

This conjugacy follows from a general result of Goebel [7], with special cases previously studied by Rockafellar [14] and Goebel [8]. (Discussion at the end of Section 3 suggests that in the current setting, the conjugacy could be deduced from [14] by incorporating some techniques of Barbu [1] and a finite-time conjugacy result of Rockafellar and Wolenski [17]. The general case in [7] relied in part on [17] and a limiting argument.) With the conjugacy present, $p_0 \in \partial V(x_0)$ implies $V(x_0) + W(p_0) = -x_0 \cdot p_0$ (for any x, p, $V(x) + W(p) \geq -x \cdot p$), see [13], 23.5. Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an optimal process for $V(x_0)$, $(\bar{p}(\cdot), \bar{w}(\cdot))$ be an optimal process for $W(p_0)$. Then $\bar{x}(\cdot)$ and $\bar{p}(\cdot)$ have the desired properties, this follows from Proposition 2.2 in Rockafellar [14] (while [14] deals with strictly convex Hamiltonians, this assumption is not used in 2.2).

Pick any $p_0' \in -\partial V(x_0')$, $p_0'' \in -\partial V(x_0'')$, and suppose that $(x_0' - x_0'') \cdot (p_0' - p_0'') = 0$. We will argue that this implies that $x_0' = x_0''$ and $p_0' = p_0''$, which in particular means that ∂V is single-valued when nonempty, and also that ∂V is strictly monotone which translates to strict convexity of V. Let arcs $x'(\cdot)$, $p'(\cdot)$ on $[0, +\infty)$ originate at x_0', p_0', satisfy (17), and converge to 0; let $x''(\cdot)$, $p''(\cdot)$ correspond similarly to x_0'', p_0''. Define $f(t) = (x'(t) - x''(t)) \cdot (p'(t) - p''(t))$. Then

$$\begin{aligned} \frac{d}{dt} f(t) &= (x'(t) - x''(t)) \cdot Q(x'(t) - x''(t)) \\ &+ \left[B^* p'(t) - B^* p''(t) \right] \cdot \left[\nabla s(B^* p'(t)) - \nabla s(B^* p''(t)) \right]. \end{aligned}$$

The second term above is nonnegative. This reflects the general fact that ∇s is a monotone operator, see [13], 24.9, but can be also verified directly thanks to the componentwise structure of $\nabla s = \sigma$ (each component is nondecreasing).

As $f(0) = 0$ and $\lim_{t\to\infty} f(t) = 0$, we must have $f'(t) = 0$ for all t. In particular, $x'(t) = x''(t)$ for all t. As $p'(t)$, $p''(t)$ go to 0, there exists $T > 0$ such that for all $t > T$, $B^*p'(t)$ and $B^*p''(t)$ are in a convex neighborhood of 0 on which s is strictly convex. For such t, $[B^*p'(t) - B^*p''(t)]\cdot[\nabla s(B^*p'(t)) - \nabla s(B^*p''(t))] = 0$ implies that $B^*p'(t) = B^*p''(t)$ (this is immediate when ∇s has the componentwise structure, as then each component is strictly increasing). As we have $\dot{p}'(t) - \dot{p}''(t) = -A^*(p'(t) - p''(t))$ and $(-A^*, B^*)$ is detectable, $p'(t) = p''(t)$ for all $t > T$. But then $\dot{p}'(t) - \dot{p}''(t) = -A^*(p'(t) - p''(t))$ implies that $p'(t) = p''(t)$ for all t. In particular, we get $p_0' = p_0''$ as well as $x_0' = x_0''$.

Any convex function differentiable on an open set has a continuous gradient there, 25.5.1 in [13], this shows the last statement of the theorem.

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