Boundary feedback stabilization of the two dimensional Navier-Stokes equations with finite dimensional controllers

Jean-Pierre Raymond *† Laetitia Thevenet *†

Abstract

We study the boundary stabilization of the two-dimensional Navier-Stokes equations about an unstable stationary solution by controls of finite dimension in feedback form. The main novelty is that the linear feedback control law is determined by solving an optimal control problem of finite dimension. More precisely, we show that, to stabilize locally the Navier-Stokes equations, it is sufficient to look for a boundary feedback control of finite dimension, able to stabilize the projection of the linearized equation onto the unstable subspace of the linearized Navier-Stokes operator.

Key words. Dirichlet control, feedback control, stabilization, Navier-Stokes equations, Oseen equations, Riccati equation

AMS subject classifications. 93B52, 93C20, 93D15, 35Q30, 76D55, 76D05, 76D07

1 Introduction.

Control of fluid flows by feedback is a challenging problem both from the theoretical and numerical points of view, see [10, 23] and the references therein. In this paper, we are interested in determining boundary feedback control laws of finite dimension able to stabilize the two dimensional Navier-Stokes equations in a neighbourhood of an unstable stationary solution. The feedback control laws determined in [5, 7], in the case of an internal control, and in [18], in the case of a boundary control, are obtained by solving an algebraic Riccati equation stated in a space of infinite dimension. For numerical calculations an approximation scheme has to be used. In the case when the Reynolds number of the Navier-Stokes equations is large, the dimension N of the discretized equation must be large to have a sufficiently good accuracy. In that case the corresponding algebraic Riccati equation, which is stated in $\mathbb{R}^{N\times N}$, cannot be solved numerically with the existing algorithms because N is too large. In this paper we determine a feedback control law, able to stabilize the Navier-Stokes equations, by solving a finite dimensional Riccati equation (the equation is stated in $\mathbb{R}^{K\times K}$ where K is the dimension of the unstable space of the linearized Navier-Stokes operator). Let us describe more precisely our problem.

Let Ω of class C^4 be a bounded and connected domain in $\mathbb{R}^2$ with boundary Γ, $\nu > 0$, and consider a couple $(\mathbf{w}, \chi)$ – a velocity field and a pressure – solution to the stationary Navier-Stokes equations in Ω:

$$-\nu\Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla\chi = \mathbf{f} \quad \text{and} \quad \operatorname{div}\mathbf{w} = 0 \text{ in } \Omega, \qquad \mathbf{w} = \mathbf{u}_s^\infty \quad \text{on } \Gamma. \tag{1.1}$$

*Université de Toulouse, UPS, Institut de Mathématiques, 31062 Toulouse Cedex 9, France,

†CNRS, Institut de Mathématiques, UMR 5219, 31062 Toulouse Cedex 9, France.

email: raymond@math.univ-toulouse.fr, and Laetitia.Thevenet@math.univ-toulouse.fr.

We assume that $\mathbf{w} \in \mathbf{V}^3(\Omega)$ is an unstable solution of the Navier-Stokes equations. We consider the control system

$$\begin{aligned} &\frac{\partial \mathbf{z}}{\partial t} - \nu\Delta\mathbf{z} + (\mathbf{z}\cdot\nabla)\mathbf{z} + \nabla q = 0, \quad \text{div}\,\mathbf{z} = 0 \quad \text{in } Q_\infty = \Omega\times(0,\infty), \\ &\mathbf{z} = \mathbf{u}_s^\infty + M\hat{\mathbf{u}} \ \text{ on } \Sigma_\infty = \Gamma\times(0,\infty), \quad \mathbf{z}(0) = \mathbf{w} + \mathbf{y}_0 \text{ in } \Omega, \end{aligned} \tag{1.2}$$

and the corresponding system satisfied by $\hat{\mathbf{y}} = \mathbf{z} - \mathbf{w}$

$$\begin{aligned} &\frac{\partial \hat{\mathbf{y}}}{\partial t} - \nu\Delta\hat{\mathbf{y}} + (\hat{\mathbf{y}}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\hat{\mathbf{y}} + (\hat{\mathbf{y}}\cdot\nabla)\hat{\mathbf{y}} + \nabla\hat{p} = 0, \quad \text{div}\,\hat{\mathbf{y}} = 0 \quad \text{in } Q_\infty, \\ &\hat{\mathbf{y}} = M\hat{\mathbf{u}} \ \text{ on } \Sigma_\infty, \quad \hat{\mathbf{y}}(0) = \mathbf{y}_0 \text{ in } \Omega. \end{aligned} \tag{1.3}$$

The operator $M \in \mathcal{L}(L^2(\Gamma;\mathbb{R}^2))$ is used to localize the control in a part of the boundary (see section 2.1). In order to stabilize $\hat{\mathbf{y}}$ with a prescribed exponential decay rate $e^{-\alpha t}, \quad \alpha > 0$, we set:

$$\mathbf{y} = e^{\alpha t}\hat{\mathbf{y}}, \quad p = e^{\alpha t}\hat{p}, \quad \mathbf{u} = e^{\alpha t}\hat{\mathbf{u}}.$$

Then, $\mathbf{y}$ is solution to the system

$$\begin{aligned} &\frac{\partial \mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} - \alpha\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{y} + e^{-\alpha t}(\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = 0, \\ &\text{div}\,\mathbf{y} = 0 \text{ in } Q_\infty, \\ &\mathbf{y} = M\mathbf{u} \ \text{ on } \Sigma_\infty, \quad \mathbf{y}(0) = \mathbf{y}_0 \text{ in } \Omega. \end{aligned} \tag{1.4}$$

In [18], the first author has determined a linear feedback law able to stabilize the nonlinear system (1.4). This law is obtained by solving the Riccati equation of an infinite time horizon control problem. In that approach, the Riccati equation is stated in a space of infinite dimension. In the present paper, we want to find a control $\mathbf{u}$ of finite dimension, that is of the form

$$\mathbf{u}(t,x) = \sum_{i=1}^{n_c} v_i(t)\zeta_i(x) \in L^2(0,\infty;\mathbf{L}^2(\Gamma)), \tag{1.5}$$

able to stabilize equation (1.4) and for which $v_i \in L^2(0,\infty)$ is written in feedback form. (The functions $\{\zeta_i\}_{i=1}^{n_c}$ are not a priori known and have to be determined). Let us explain how we proceed. Following [17, 18], we first write the linearized equation associated with (1.4) in the form

$$\begin{aligned} &P\mathbf{y}' = AP\mathbf{y} + BM\mathbf{u} = AP\mathbf{y} + (\lambda_0 I - A)PD_A M\mathbf{u} \quad \text{in } (0,\infty), \quad P\mathbf{y}(0) = \mathbf{y}_0, \\ &(I-P)\mathbf{y} = (I-P)D_A M\mathbf{u} \quad \text{in } (0,\infty), \end{aligned} \tag{1.6}$$

where P is the so-called Helmholtz or Leray projection operator, A is the linearized Navier-Stokes operator, D_A is a Dirichlet operator and B is a control operator (see section 2). Next, we decompose $\mathbf{V}_n^0(\Omega) = \Big\{\mathbf{y} \in \mathbf{L}^2(\Omega) \mid \text{ div}\,\mathbf{y} = 0 \text{ in } \Omega, \ \mathbf{y}\cdot\mathbf{n} = 0 \text{ on } \Gamma\Big\}$ in the form

$$\mathbf{V}_n^0(\Omega) = \mathbf{Y}_{\alpha^-} \oplus \mathbf{Y}_\alpha,$$

where $\mathbf{Y}_\alpha \subset D(A)$ is the finite dimensional unstable susbspace of A and $\mathbf{Y}_{\alpha^-}$ is the stable subspace. Similarly, we have

$$\mathbf{V}_n^0(\Omega) = \mathbf{Y}^*_{\alpha^-} \oplus \mathbf{Y}^*_\alpha,$$

where $\mathbf{Y}_\alpha^* \subset D(A^*)$ is the finite dimensional unstable susbspace of A^* and $\mathbf{Y}_{\alpha^-}^*$ is the stable subspace. In section 3, we prove that there exist a basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_K\}$ of $\mathbf{Y}_\alpha$ and a basis $\{\varepsilon_1, \cdots, \varepsilon_K\}$ of $\mathbf{Y}_\alpha^*$ such that

$$\Big(\mathbf{e}_i, \varepsilon_j\Big) = \int_\Omega \mathbf{e}_i(x)\varepsilon_j(x)dx = \delta_i^j,$$

where δ_i^j is the Kronecker symbol. This type of result is already established for parabolic equations in [12] and in [11, 13] for linearized Navier-Stokes equations. Let Q denote the projection onto $\mathbf{Y}_\alpha$ along $\mathbf{Y}_{\alpha^-}$. With such a choice for the basis of $\mathbf{Y}_\alpha$ and $\mathbf{Y}_\alpha^*$, we obtain a very simple expression of Q and of Q^*, the adjoint of Q. We consider the system

$$QP\mathbf{y}' = AQP\mathbf{y} + QBM\mathbf{u}, \quad QP\mathbf{y}(0) = Q\mathbf{y}_0. \tag{1.7}$$

In section 4, we prove that equation (1.7) is stabilizable by a control $\mathbf{u}$ of the form (1.5) where $\{\zeta_1, \cdots, \zeta_{n_c}\}$ is a basis of $\mathcal{U} = \mathrm{vect}\Big\{MB^*\varepsilon_j \mid j = 1, \cdots, K\Big\}$. In section 5, we introduce a linear quadratic control problem of which the Riccati equation is

$$\begin{aligned} &\Pi = \Pi^* \in \mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}_\alpha^*), \quad \Pi \geq 0, \\ &\Pi A_\alpha + A_\alpha^*\Pi - \Pi B\widetilde{M}\widetilde{M}^* B^*\Pi + Q^*Q = 0, \end{aligned} \tag{1.8}$$

where $\widetilde{M}$ is defined in (4.9). Let us notice that it is a finite dimensional algebraic Riccati equation. Finally, in section 6 we show that the feedback law

$$v_i(t) = -\Big(\zeta_i, MB^*Q^*\Pi QP\mathbf{y}(t)\Big) = -\int_\Gamma \zeta_i(x)MB^*Q^*\Pi QP\mathbf{y}(t,x)dx,$$

with $\mathbf{u}$ given by (1.5) stabilizes the Navier-Stokes equation locally about $\mathbf{w}$.

Let us situate our result with respect to the literature. The existence of finite dimensional controllers in feedback form able to stabilize a linear parabolic equation has been first established by Triggiani in [21]. For the linearized Navier-Stokes equations, such results have been obtained by Barbu and Triggiani in the case of an internal control. They have also proved a local stabilization result for the Navier-Stokes equations, still with an internal control, by a finite dimensional control in feedback form (see [7]). For that, they follow the approach introduced by Barbu in [5] consisting in looking for a feedback determined by a high gain functional. In the context of the linearized Navier-Stokes equations, we consider functionals J of the form $J(\mathbf{y}, \mathbf{u}) = \int_0^\infty |C\mathbf{y}(t)|^2_{\mathbf{V}_n^0(\Omega)}dt + \int_0^\infty |\mathbf{u}(t)|^2_{\mathbf{V}^0(\Gamma)}dt$. Such a functional will be called a high gain functional if the mapping $\mathbf{y} \mapsto |C\mathbf{y}|_{\mathbf{V}_n^0(\Omega)}$ is a norm strictly stronger than the usual L^2-norm, otherwise it will be called a low gain functional. Even if the feedback law determined in [7], in the case of a distributed control, is of finite dimension, since it is obtained by solving a control problem where the cost is a high gain functional, the underlying Riccati equation is stated over a space of infinite dimension. Moreover, for boundary controls, high gain functionals do not lead to well posed Riccati equations (equations are not well posed in the sense that they are not stated in the domain of the operator A but in the domain of the generator of the closed loop system, and this domain is not known, see [6]). Here, we follow the ideas introduced in [18, 19, 20] consisting in looking for a feedback determined with a low gain. We even carry on this strategy further, since we set $C\mathbf{y} = Q\mathbf{y}$ and $\mathbf{y} \mapsto |Q\mathbf{y}|_{\mathbf{V}_n^0(\Omega)}$ is only a seminorm on $\mathbf{V}_n^0(\Omega)$. Therefore, our goal is to show that a boundary feedback law determined by solving a finite dimensional control problem is able to stabilize locally the Navier-Stokes equations. To the authors knowledge, this type of result is completely new. Even if this result seems interesting, we would like to explain what is its practical interest for numerical computations. Such an approach consisting in decoupling the linearized Navier-Stokes equations into a stable and an unstable part has been used by Ahuja and Rowley in [1] to design reduced order models. Here, our goal is to use this decomposition to define a finite dimensional Riccati equation. Even if the domain of stability of the feedback law determined here is small, the result is still interesting. Indeed, there is an efficient algorithm to solve large scale Riccati equations, the so-called Newton Kleinman method (see [9], and the references therein). The drawback of the Newton-Kleinman method is that it requires an initial guess for which the corresponding closed loop system is stable. The feedback that we determine may provide such an initial guess. By this way, we can hope to enlarge the domain of stability of the feedback law. Let us finally mention that, following [2, 3, 4], it should be possible to define a Lyapunov function of the closed loop system obtained by coupling the Navier-Stokes equations with the finite dimensional feedback control that we have determined.

2 Functional framework

2.1 Notation and assumptions

Let us introduce the following spaces: $H^s(\Omega;\mathbb{R}^N) = \mathbf{H}^s(\Omega)$, $L^2(\Omega;\mathbb{R}^N) = \mathbf{L}^2(\Omega)$, the same notation conventions will be used for trace spaces and for the spaces $H_0^s(\Omega;\mathbb{R}^N)$. We also introduce different spaces of free divergence functions and some corresponding trace spaces:

$$\begin{aligned}
&\mathbf{V}^s(\Omega) = \Big\{\mathbf{y} \in \mathbf{H}^s(\Omega) \mid \ \mathrm{div}\,\mathbf{y} = 0 \ \text{ in } \Omega,\ \langle \mathbf{y}\cdot\mathbf{n}, 1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0\Big\} \quad \text{for } s \geq 0,\\
&\mathbf{V}_n^s(\Omega) = \Big\{\mathbf{y} \in \mathbf{H}^s(\Omega) \mid \ \mathrm{div}\,\mathbf{y} = 0 \ \text{ in } \Omega,\ \mathbf{y}\cdot\mathbf{n} = 0 \text{ on } \Gamma\Big\} \quad \text{for } s \geq 0,\\
&\mathbf{V}_0^s(\Omega) = \Big\{\mathbf{y} \in \mathbf{H}^s(\Omega) \mid \ \mathrm{div}\,\mathbf{y} = 0 \ \text{ in } \Omega,\ \mathbf{y} = 0 \text{ on } \Gamma\Big\} \quad \text{for } s > 1/2,\\
&\mathbf{V}^s(\Gamma) = \Big\{\mathbf{y} \in \mathbf{H}^s(\Gamma) \mid \langle \mathbf{y}\cdot\mathbf{n}, 1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0\Big\} \quad \text{for } s \geq -1/2\,.
\end{aligned}$$

In the above setting $\mathbf{n}$ denotes the unit normal to Γ outward Ω. We shall use the following notation $Q_\infty = \Omega \times (0,\infty)$, and $\Sigma_\infty = \Gamma \times (0,\infty)$. We also set

$$\mathbf{V}^{s,\sigma}(Q_\infty) = H^\sigma(0,\infty;\mathbf{V}^0(\Omega)) \cap L^2(0,\infty;\mathbf{V}^s(\Omega)) \quad \text{for } s,\, \sigma \geq 0,$$

and

$$\mathbf{V}^{s,\sigma}(\Sigma_\infty) = H^\sigma(0,\infty;\mathbf{V}^0(\Gamma)) \cap L^2(0,\infty;\mathbf{V}^s(\Gamma)) \quad \text{for } s,\, \sigma \geq 0.$$

For an open subset Γ_c of Γ, we introduce a weight function $m \in C^2(\Gamma)$ with values in $[0,1]$, with support in Γ_c, equal to 1 in Γ_0, where Γ_0 is an open subset in Γ_c. Associated with this function m we introduce the operator $M \in \mathcal{L}(\mathbf{V}^0(\Gamma))$ defined by

$$M\mathbf{u}(x) = m(x)\mathbf{u}(x) - \frac{m}{\int_\Gamma m}\left(\int_\Gamma m\mathbf{u}\cdot\mathbf{n}\right)\mathbf{n}(x),$$

where $|\Gamma|$ is the $(N-1)$-dimensional Lebesgue measure of Γ. By this way, we can replace the condition $\mathrm{supp}(\mathbf{u}) \subset \Gamma_c$ by considering a boundary condition of the form

$$\mathbf{z} - \mathbf{w} = M\hat{\mathbf{u}} \quad \text{on} \quad \Sigma_\infty.$$

For all $\psi \in H^{1/2+\varepsilon'}(\Omega)$, with $\varepsilon' > 0$, we denote by $c(\psi)$ and $c(m\psi)$ the constants defined by

$$c(\psi) = \frac{1}{|\Gamma|}\int_\Gamma \psi \quad \text{and} \qquad c(m\psi) = \frac{1}{|\Gamma|}\int_\Gamma m\psi. \tag{2.1}$$

Let us recall that P, the so-called Leray or Helmholtz projector, is the orthogonal projection in $\mathbf{L}^2(\Omega)$ onto $\mathbf{V}^0_n(\Omega)$.

2.2 Properties of some operators

In this subsection we briefly recall the definitions and properties of some operators already used in [18]. The proof of these results can be found in [18]. We denote by $(A, D(A))$ and $(A^*, D(A^*))$ the unbounded operators in $\mathbf{V}^0_n(\Omega)$ defined by

$$\begin{aligned} &D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{V}^1_0(\Omega), \quad A\mathbf{y} = \nu P\Delta\mathbf{y} + \alpha\mathbf{y} - P((\mathbf{w}\cdot\nabla)\mathbf{y}) - P((\mathbf{y}\cdot\nabla)\mathbf{w}),\\ &D(A^*) = \mathbf{H}^2(\Omega) \cap \mathbf{V}^1_0(\Omega), \quad A^*\mathbf{y} = \nu P\Delta\mathbf{y} + \alpha\mathbf{y} + P((\mathbf{w}\cdot\nabla)\mathbf{y}) - P((\nabla\mathbf{w})^T\mathbf{y}). \end{aligned}$$

Since $\mathbf{w} \in \mathbf{V}^3(\Omega)$ and $\operatorname{div}\ \mathbf{w} = 0$, we can verify that there exists $\lambda_0 > 0$ in the resolvent set of A satisfying

$$\begin{aligned} &\big((\lambda_0 I - A)\mathbf{y}, \mathbf{y}\big)_{\mathbf{V}^0_n(\Omega)} \ge \frac{\nu}{2}|\mathbf{y}|^2_{\mathbf{V}^1_0(\Omega)} \qquad \text{for all } \mathbf{y} \in D(A),\\ \text{and} \quad &\\ &\big((\lambda_0 I - A^*)\mathbf{y}, \mathbf{y}\big)_{\mathbf{V}^0_n(\Omega)} \ge \frac{\nu}{2}|\mathbf{y}|^2_{\mathbf{V}^1_0(\Omega)} \qquad \text{for all } \mathbf{y} \in D(A^*). \end{aligned} \tag{2.2}$$

Theorem 2.1. *The unbounded operator* $(A - \lambda_0 I)$ (*respectively* $(A^* - \lambda_0 I)$) *with domain* $D(A - \lambda_0 I) = D(A)$ (*respectively* $D(A^* - \lambda_0 I)$) *is the infinitesimal generator of a bounded analytic semigroup on* $\mathbf{V}^0_n(\Omega)$. *Moreover, we have*

$$D((\lambda_0 I - A)^\theta) = D((\lambda_0 I - A^*)^\theta) = [\mathbf{V}^0_n(\Omega), D(A)]_\theta$$

for all $0 \le \theta \le 1$.

Observe that the semigroups $(e^{t(A-\lambda_0 I)})_{t\ge 0}$ and $(e^{t(A^*-\lambda_0 I)})_{t\ge 0}$ are exponentially stable on $\mathbf{V}^0_n(\Omega)$ and that

$$\|e^{t(A-\lambda_0 I)}\|_{\mathcal{L}(\mathbf{V}^0_n(\Omega))} \le Ce^{-\omega t} \quad \text{and} \quad \|e^{t(A^*-\lambda_0 I)}\|_{\mathcal{L}(\mathbf{V}^0_n(\Omega))} \le Ce^{-\omega t},$$

for all $\omega < \nu/2$ (see [8, Chapter 1, Theorem 2.12]).

Let us introduce D_A and D_p, two Dirichlet operators associated with A, defined as follows (see [18, p. 796]). For $\mathbf{u} \in \mathbf{V}^0(\Gamma)$, we set $D_A\mathbf{u} = \mathbf{y}$ and $D_p\mathbf{u} = q$ where $(\mathbf{y}, q)$ is the unique solution in $\mathbf{V}^{1/2}(\Omega) \times (H^{1/2}(\Omega)/\mathbb{R})'$ to the equation

$$\begin{aligned} &\lambda_0\mathbf{y} - \nu\Delta\mathbf{y} - \alpha\mathbf{y} + (\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{w} + \nabla q = 0 \quad \text{in } \Omega,\\ &\operatorname{div}\mathbf{y} = 0 \quad \text{in } \Omega, \quad \mathbf{y} = \mathbf{u} \quad \text{on } \Gamma. \end{aligned}$$

Lemma 2.1. (i) *The operator* D_A *is a bounded operator from* $\mathbf{V}^0(\Gamma)$ *into* $\mathbf{V}^0(\Omega)$, *moreover it satisfies*

$$|D_A\mathbf{u}|_{\mathbf{V}^{s+1/2}(\Omega)} \le C(s)|\mathbf{u}|_{\mathbf{V}^s(\Gamma)} \qquad \text{for all } 0 \le s \le 2\,.$$

(ii) *The operator* $D_A^* \in \mathcal{L}(\mathbf{V}^0(\Omega), \mathbf{V}^0(\Gamma))$, *the adjoint operator of* $D_A \in \mathcal{L}(\mathbf{V}^0(\Gamma), \mathbf{V}^0(\Omega))$, *is defined by*

$$D_A^*\mathbf{g} = -\nu\frac{\partial\mathbf{z}}{\partial\mathbf{n}} + \pi\mathbf{n} - c(\pi)\mathbf{n}, \tag{2.3}$$

where $(\mathbf{z},\pi)$ *is the solution of*

$$\lambda_0\mathbf{z}-\nu\Delta\mathbf{z}-\alpha\mathbf{z}-(\mathbf{w}\cdot\nabla)\mathbf{z}+(\nabla\mathbf{w})^T\mathbf{z}+\nabla\pi=\mathbf{g}\quad \operatorname{div}\mathbf{z}=0\quad \text{in }\Omega,\qquad \mathbf{z}=0\quad \text{on }\Gamma,\tag{2.4}$$

and $c(\pi)$ *is defined by* (2.1).

Lemma 2.2. *The operator* M *obeys the following property:*

$$M=M^*.$$

We introduce the operator $B=(\lambda_0 I-A)PD_A\in\mathcal{L}(\mathbf{V}^0(\Gamma),(D(A^*))')$.

Proposition 2.1. *The operator adjoint* $B^*\in\mathcal{L}(D(A^*),\mathbf{V}^0(\Gamma))$ *satisfies* $B^*\mathbf{\Phi}=D_A^*(\lambda_0 I-A^*)\mathbf{\Phi}$ *and*

$$B^*\mathbf{\Phi}=-\nu\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}}+\psi\mathbf{n}-c(\psi)\mathbf{n},$$

for all $\mathbf{\Phi}\in D(A^*)$, *with*

$$\nabla\psi=(I-P)\Big[\nu\Delta\mathbf{\Phi}+\alpha\mathbf{\Phi}+(\mathbf{w}\cdot\nabla)\mathbf{\Phi}-(\nabla\mathbf{w})^T\mathbf{\Phi}\Big],$$

and $c(\psi)$ *defined by* (2.1). *Moreover, the following estimate holds*

$$|B^*\mathbf{\Phi}|_{\mathbf{V}^{s-3/2}(\Gamma)}\le C|\mathbf{\Phi}|_{\mathbf{V}^s(\Omega)\cap\mathbf{V}_0^1(\Omega)},$$

for all $\mathbf{\Phi}\in\mathbf{V}^s(\Omega)\cap\mathbf{V}_0^1(\Omega)$ *with* $s\ge 2$.

3 Projected systems

In order to introduce the generalized eigenfunctions of the operator A, we consider the complexified space

$$V_n^0(\Omega)=\mathbf{V}_n^0(\Omega)\oplus i\mathbf{V}_n^0(\Omega).$$

The first equation in (1.6) may be extended to spaces of functions with complex values as follows

$$Py'=APy+BMu,\quad Py(0)=y_0,\tag{3.1}$$

where y_0, y and u are now functions with complex values.

3.1 The resolvent of the operator A

We first study the resolvent of the operator A.

Lemma 3.1. *The resolvent of* A *is compact and the spectrum of* A *is discrete.*

Proof. See [11, Lemma 3.1]. □

Now, we give a decomposition of the resolvent of A by using Laurent series. Let λ_1 belong to the spectrum of A. For λ in the neighbourhood of λ_1, the resolvent of A can be expressed in a Laurent series:

$$R(\lambda,A)=\sum_{k=-\infty}^{+\infty}(\lambda-\lambda_1)^kR_k\quad\text{with}\quad R_k=\frac{1}{2\pi i}\int_{|\lambda-\lambda_1|=\varepsilon}\frac{R(\lambda,A)}{(\lambda-\lambda_1)^{k+1}}d\lambda,\tag{3.2}$$

and $\varepsilon>0$ small enough.

Lemma 3.2. *The expansion (3.2) of the resolvent in a Laurent series in a neighbourhood of* λ_1 *contains finitely many terms with negative power of* $\lambda - \lambda_1$*, that is:*

$$R(\lambda, A) = \sum_{k=-m(\lambda_1)}^{+\infty} (\lambda - \lambda_1)^k R_k. \tag{3.3}$$

Proof. The proof is done in [12, Lemma 3.3]. □

Since the spectrum of A is a pointwise spectrum, we may always choose $\alpha > 0$ such that

$$\cdots \leq \Re\lambda_{N_\alpha+1} < 0 < \Re\lambda_{N_\alpha} \leq \cdots \leq \Re\lambda_1$$

for some $N_\alpha \in \mathbb{N}^*$. We consider the continuous contour γ_0 in the complex half-plane $\{\lambda \in \mathbb{C} \mid \Re\lambda \leq 0\}$ made up of a segment of the line $\{\Re\lambda = 0\}$ and the two branches of γ on the rays $\{\ \text{Arg}\lambda = \pm\theta\}$. Thanks to this new contour, we obtain another expression of the semigroup given in the following lemma.

Lemma 3.3. *The semigroup* e^{tA} *may be written in the form*

$$e^{tA} = \frac{1}{2\pi i}\int_{\gamma_0} (\lambda I - A)^{-1} e^{\lambda t} d\lambda + \sum_{j=1}^{N_\alpha} e^{\lambda_j t} \sum_{n=1}^{m(\lambda_j)} \frac{t^{n-1}}{(n-1)!} R_{-n}(\lambda_j).$$

Proof. See [12, p. 603]. □

3.2 Canonical Systems

For $1 \leq j \leq N_\alpha$, we set

$$E(\lambda_j) = \text{Ker } (A - \lambda_j I) \quad \text{ and } \quad \ell(j) = \text{ dim } E(\lambda_j).$$

$E(\lambda_j)$ is the eigenspace associated to the eigenvalue λ_j and $\ell(j)$ is the geometric multiplicity of λ_j. We also introduce the generalized eigenspace

$$G(\lambda_j) = \text{Ker } ((A - \lambda_j I)^{m(\lambda_j)}) \quad \text{ and } \quad N(\lambda_j) = \text{ dim } G(\lambda_j),$$

where $m(\lambda_j)$ is the multiplicity of the pole λ_j in the resolvent (see Lemma 3.2) and $N(\lambda_j)$ is the algebraic multiplicity of λ_j. Let us define the multiplicity of an eigenvector.

Definition 3.1. *(See [11, 12]). We say that* $(e_1^k, e_2^k, \cdots, e_r^k)$ *forms a chain of generalized eigenvectors, when the following relations hold*

$$(\lambda_j I - A)e_1^k = 0, \quad (\lambda_j I - A)e_2^k + e_1^k = 0, \quad \cdots, \quad (\lambda_j I - A)e_r^k + e_{r-1}^k = 0.$$

If the maximal order of the chain of generalized eigenvectors corresponding to e_1^k *is* m *then the number* m *is called the multiplicity of the eigenvector* e_1^k*.*

We consider a special case of generalized eigenvectors.

Definition 3.2. *(See [11, 12]). A basis of* $G(\lambda_j)$ *of the form*

$$\left\{e_i^k \mid k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k\right\}$$

is called a canonical system if

- $\left\{e_1^k \mid k=1,\cdots,\ell(j)\right\}$ *is a basis of* $E(\lambda_j)$,
- e_1^1 *is an eigenvector with maximum possible multiplicity* m_1,
- e_1^k *is an eigenvector with maximum possible multiplicity* m_k *such that* e_1^k *is not linearly expressible in terms of* $e_1^1,\cdots,e_1^{k-1}$,
- *for* $k=1,\cdots,\ell(j)$ *and* $i=2,\cdots,m_k$, $(A-\lambda_j I)e_i^k = e_{i-1}^k$.

Obviously, we have $m(\lambda_j)=\max(m_1,\cdots,m_{\ell(j)})$. We remark that if λ_j is an eigenvalue of the operator A with multiplicity $m(\lambda_j)$, then $\overline{\lambda_j}$ is an eigenvalue of the operator A^* with the same multiplicity. That is why, we can define another canonical system associated to $\overline{\lambda_j}$ for A^*:

$$\left\{\varepsilon_i^k \mid k=1,\cdots,\ell(j),\ i=1,\cdots,m_k\right\},$$

where

- $\left\{\varepsilon_1^k \mid k=1,\cdots,\ell(j)\right\}$ is a basis of Ker $(A^*-\overline{\lambda_j}I)$,
- ε_1^1 is an eigenvector with maximum possible multiplicity m_1,
- ε_1^k is an eigenvector with maximum possible multiplicity m_k such that ε_1^k is not linearly expressible in terms of $\varepsilon_1^1,\cdots,\varepsilon_1^{k-1}$,
- for $k=1,\cdots,\ell(j)$ and $i=2,\cdots,m_k$, $(A^*-\overline{\lambda_j}I)\varepsilon_i^k=\varepsilon_{i-1}^k$.

In what follows, we denote by $\Big(\cdot,\cdot\Big)$ the complex inner product in $V_n^0(\Omega)$, that is

$$\Big(f,g\Big)=\int_\Omega f\,\overline{g}\,dx.$$

Definition 3.3. *For a couple* $(e_i,\varepsilon_j)\in(V_n^0(\Omega))^2$, *we denote by* $e_i\varepsilon_j$ *the operator defined by*

$$(e_i\varepsilon_j)f=\Big(f,\varepsilon_j\Big)\,e_i$$

for all $f\in V_n^0(\Omega)$.

Theorem 3.1. *For any canonical system* $\left\{\varepsilon_i^k \mid k=1,\cdots,\ell(j),\ i=1,\cdots,m_k\right\}$ *of* A^* *corresponding to the eigenvalue* $\overline{\lambda_j}$, *there is a uniquely determined canonical system* $\left\{e_i^k \mid k=1,\cdots,\ell(j),\ i=1,\cdots,m_k\right\}$ *of* A *for* λ_j *such that the principal part of the resolvent can be expressed in the following way*

$$\sum_{p=-m(\lambda_j)}^{-1}(\lambda-\lambda_j)^p R_p(\lambda_j)$$
$$=\sum_{k=1}^{\ell(j)}\left(\frac{e_1^k\varepsilon_1^k}{(\lambda-\lambda_j)^{m_k}}+\frac{e_1^k\varepsilon_2^k+e_2^k\varepsilon_1^k}{(\lambda-\lambda_j)^{m_k-1}}+\cdots+\frac{e_1^k\varepsilon_{m_k}^k+e_2^k\varepsilon_{m_k-1}^k+\cdots+e_{m_k}^k\varepsilon_1^k}{(\lambda-\lambda_j)}\right).$$

Proof. See [12, Theorem 3.1] or [11, 15]. □

3.3 The complex projected system

We consider the space $Z_\alpha = \oplus_{j=1}^{N_\alpha} G(\lambda_j)$. We denote by $N = \sum_{j=1}^{N_\alpha} N(\lambda_j)$ its dimension. With [14, p.178-182], we first notice that the space $V_n^0(\Omega)$ can be decomposed as follows:

$$V_n^0(\Omega) = Z_{\alpha^-} \oplus Z_\alpha$$

where Z_{α^-} is the stable space of A, that is to say $Z_{\alpha^-} \cap D(A)$ is invariant under A. The space Z_α will be equipped with the norm

$$|y|_{Z_\alpha} = |y|_{V_n^0(\Omega)}.$$

Let γ_α be a simple closed curve enclosing $(\lambda_1, \cdots, \lambda_{N_\alpha})$ but no other point of the spectrum of A, and oriented counterclockwise. The operator

$$P_\alpha = \frac{1}{2\pi i} \int_{\gamma_\alpha} (\lambda I - A)^{-1} d\lambda$$

is the projection onto Z_α parallel to Z_{α^-} (see [14, p. 178-182]).

Lemma 3.4. *For all $j \in \{1, \cdots, N_\alpha\}$, we consider $\left\{\varepsilon_i^k(\overline{\lambda_j}) \mid k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k\right\}$, canonical system of A^* corresponding to the eigenvalue $\overline{\lambda_j}$, and the canonical system of A for λ_j determined in Theorem 3.1 $\left\{e_i^k(\lambda_j) \mid k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k\right\}$. Then, for all $(p, q) \in \{1, \cdots, N_\alpha\}^2$ we have*

$$\left(e_{\ell_p}^{k_p}(\lambda_p), \varepsilon_{\ell_q}^{k_q}(\overline{\lambda_q})\right) = \delta_{m_{k_p}+1-\ell_p, k_p, \lambda_p}^{\ell_q,\, k_q,\, \lambda_q},$$

where the Kronecker symbol is equal to 1 if $\lambda_q = \lambda_p$, $k_q = k_p$ and $\ell_q = m_{k_p} + 1 - \ell_p$, and is equal to 0 otherwise.

Proof. Let p be in $\{1, \cdots, N_\alpha\}$. With the definition of P_α and (3.2) for $k = -1$, we have

$$P_\alpha = \sum_{j=1}^{N_\alpha} \frac{1}{2\pi i} \int_{|\lambda - \lambda_j| = \varepsilon} (\lambda I - A)^{-1} d\lambda = \sum_{j=1}^{N_\alpha} R_{-1}(\lambda_j).$$

Let us set $z = e_{\ell_p}^{k_p}(\lambda_p)$ with $k_p \in \{1, \cdots, \ell(p)\}$ and $\ell_p \in \left\{1, \cdots, m_{k_p}\right\}$. We have $P_\alpha(z) = z$ since z belongs to Z_α. Due to Theorem 3.1, we obtain

$$R_{-1}(\lambda_j)(z) = \sum_{k=1}^{\ell(j)} \left(z, \varepsilon_{m_k}^k(\overline{\lambda_j})\right) e_1^k(\lambda_j) + \left(z, \varepsilon_{m_k-1}^k(\overline{\lambda_j})\right) e_2^k(\lambda_j) + \cdots + \left(z, \varepsilon_1^k(\overline{\lambda_j})\right) e_{m_k}^k(\lambda_j).$$

Since $\left\{e_i^k(\lambda_j) \mid j = 1, \cdots, N_\alpha,\ k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k\right\}$ is a basis of Z_α, we clearly obtain the result and the proof is complete. $\square$

Remark 3.1. *With Lemma 3.4, we obtain*

$$P_\alpha z = \sum_{j=1}^{N_\alpha} \sum_{k=1}^{\ell(j)} \sum_{i=1}^{m_k} \left(z, \varepsilon_{m_k+1-i}^k(\overline{\lambda_j})\right) e_i^k(\lambda_j).$$

Since $\dim(Z_\alpha) < \infty$, we can extend in a continuous way the operator P_α to $(D(A^*))'$ as follows:

$$P_\alpha z = \sum_{j=1}^{N_\alpha} \sum_{k=1}^{\ell(j)} \sum_{i=1}^{m_k} \left\langle z, \varepsilon^k_{m_k+1-i}(\overline{\lambda_j}) \right\rangle_{(D(A^*))', D(A^*)} e^k_i(\lambda_j) \quad \text{for all} \quad z \in (D(A^*))'.$$

We notice that the operator P_α belongs to $\mathcal{L}((D(A^*))', Z_\alpha)$.

Remark 3.2. *Due to Lemma 3.3, we notice that*

$$P_\alpha e^{At} = \sum_{j=1}^{N_\alpha} e^{\lambda_j t} \sum_{n=1}^{m(\lambda_j)} \frac{t^{n-1}}{(n-1)!} R_{-n}(\lambda_j).$$

3.4 The real projected system

Since the operator A has real coefficients, λ_j and $\overline{\lambda_j}$ either both are or both are not eigenvalues of A. Moreover $e^k_i(\lambda_j)$ is a corresponding eigenfunction (or a generalized eigenfunction) if and only if $\overline{e^k_i(\lambda_j)}$ is an eigenfunction (or a generalized eigenfunction) associated with $\overline{\lambda_j}$. A similar assertion applies to eigenvalues and generalized functions of the operator A^*. Due to that, we can choose a canonical system $(\overline{e^k_i(\lambda_j)})_{i,k}$ associated to $\overline{\lambda_j}$ such that

$$\overline{e^k_i(\lambda_j)} = e^k_i(\overline{\lambda_j}).$$

As a consequence, if λ_j is real, the chosen canonical system associated to λ_j is real too. Let us consider the sets:

$$\mathcal{F}_1 = \{j \in \{1, \cdots, N_\alpha\} \mid \Im\lambda_j > 0\}, \qquad \text{and} \qquad \mathcal{F}_2 = \{j \in \{1, \cdots, N_\alpha\} \mid \Im\lambda_j = 0\}.$$

Then, we set

$$\mathcal{B}_{1,1} = \left\{ \Re e^k_i(\lambda_j), \quad \Im(e^k_i(\lambda_j)) \mid j \in \mathcal{F}_1,\ k = 1, \cdots, \ell(j),\ i = 1 \cdots m_k \right\},$$

and

$$\mathcal{B}_{1,2} = \left\{ e^k_i(\lambda_j) \mid j \in \mathcal{F}_2,\ k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k \right\}.$$

The family $\mathcal{B}_1 = \mathcal{B}_{1,1} \cup \mathcal{B}_{1,2}$ is linearly independent. We rewrite this family in the form

$$\mathbf{e}_1, \cdots, \mathbf{e}_K. \tag{3.4}$$

From Lemma 3.4, we consider the unique canonical system $\left\{\varepsilon^k_i \mid k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k\right\}$ corresponding to $\left\{e^k_i \mid k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k\right\}$. For the eigenvalues and generalized eigenfunctions of the operator A^*, we also have:

$$\overline{\varepsilon^k_i(\lambda_j)} = \varepsilon^k_i(\overline{\lambda_j}).$$

Therefore, we consider the sets

$$\mathcal{B}_{2,1} = \left\{ \Re\varepsilon^k_{m_k+1-i}(\lambda_j),\ \Im\varepsilon^k_{m_k+1-i}(\lambda_j) \mid j \in \mathcal{F}_1,\ k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k \right\},$$

and

$$\mathcal{B}_{2,2} = \left\{ \varepsilon^k_{m_k+1-i}(\lambda_j) \mid j \in \mathcal{F}_2,\ k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k \right\}.$$

The family $\mathcal{B}_2 = \mathcal{B}_{2,1} \cup \mathcal{B}_{2,2}$ is linearly independent. Moreover, from Lemma 3.4, and following the proof of [12, Lemma 6.2], we can rewrite the family $\mathcal{B}_2$ in the form

$$\varepsilon_1, \cdots, \varepsilon_K, \tag{3.5}$$

so that

$$\Big(\mathbf{e}_i, \varepsilon_j\Big) = \delta_i^j, \tag{3.6}$$

where δ_i^j is a Kronecker symbol and $\Big(\cdot, \cdot\Big)$ is the inner product in the real space $\mathbf{V}_n^0(\Omega)$. Let us consider the operator Q defined by

$$Q\mathbf{f} = \Re(P_\alpha \mathbf{f}), \quad \text{for all } \mathbf{f} \in \mathbf{V}_n^0(\Omega).$$

From the definition of P_α, we can define $Q \in \mathcal{L}(\mathbf{V}_n^0(\Omega))$ by

$$Q\mathbf{f} = \sum_{j=1}^{K} \Big(\mathbf{f}, \varepsilon_j\Big)\, \mathbf{e}_j \quad \text{for all } \mathbf{f} \in \mathbf{V}_n^0(\Omega).$$

Proposition 3.1. *The adjoint operator $Q^* \in \mathcal{L}(\mathbf{V}_n^0(\Omega))$ is defined by*

$$Q^*\mathbf{f} = \sum_{j=1}^{K} \Big(\mathbf{f}, \mathbf{e}_j\Big)\, \varepsilon_j \quad \textit{for all} \quad \mathbf{f} \in \mathbf{V}_n^0(\Omega).$$

Proof. This proposition is a direct consequence of the definition of the operator Q. □

We can decompose the real space $\mathbf{V}_n^0(\Omega)$ as follows:

$$\mathbf{V}_n^0(\Omega) = \mathbf{Y}_{\alpha^-} \oplus \mathbf{Y}_\alpha$$

where $\mathbf{Y}_\alpha$ is defined by

$$\mathbf{Y}_\alpha = \text{vect}\Big\{\mathbf{e}_j \mid j = 1, \cdots, K\Big\},$$

and $\mathbf{Y}_{\alpha^-} \cap D(A)$ is invariant under A. The operator Q is the projection onto $\mathbf{Y}_\alpha$ parallel to $\mathbf{Y}_{\alpha^-}$. Similarly, we can decompose the real space $\mathbf{V}_n^0(\Omega)$ as follows:

$$\mathbf{V}_n^0(\Omega) = \mathbf{Y}^*_{\alpha^-} \oplus \mathbf{Y}^*_\alpha$$

where $\mathbf{Y}^*_\alpha$ is defined by

$$\mathbf{Y}^*_\alpha = \text{vect}\Big\{\varepsilon_j \mid j = 1, \cdots, K\Big\},$$

and $\mathbf{Y}^*_{\alpha^-} \cap D(A^*)$ is invariant under A^*. The operator Q^* is the projection onto $\mathbf{Y}^*_\alpha$ parallel to $\mathbf{Y}^*_{\alpha^-}$.

Proposition 3.2. *We can characterize the space $\mathbf{Y}_{\alpha^-}$ as follows:*

$$\mathbf{Y}_{\alpha^-} = \Big\{\mathbf{f} \in \mathbf{V}_n^0(\Omega) \mid \Big(\mathbf{f}, \mathbf{g}\Big) = 0 \quad \textit{for all} \quad \mathbf{g} \in \mathbf{Y}^*_\alpha\Big\}.$$

Proof. Let $\mathbf{f}$ belong to $\mathbf{Y}_{\alpha^-}$. We have $Q\mathbf{f} = 0$. From the definition of Q, since $\{\mathbf{e}_1, \cdots, \mathbf{e}_K\}$ is a basis of $\mathbf{Y}_\alpha$, we obtain $\Big(\mathbf{f}, \varepsilon_j\Big) = 0$, for all $j = 1, \cdots, K$, and therefore

$$\mathbf{Y}_{\alpha^-} \subset \Big\{\mathbf{f} \in \mathbf{V}^0_n(\Omega) \mid \Big(\mathbf{f}, \mathbf{g}\Big) = 0 \quad \text{for all} \quad \mathbf{g} \in \mathbf{Y}^*_\alpha\Big\}.$$

Let $\mathbf{f} \in \mathbf{V}^0_n(\Omega)$ be such that $\Big(\mathbf{f}, \mathbf{g}\Big) = 0$ for all $\mathbf{g} \in \mathbf{Y}^*_\alpha$. Obviously, we have $\Big(\mathbf{f}, \varepsilon_j\Big) = 0$, for all $j = 1, \cdots, K$. From the definition of the operator Q, we obtain $Q\mathbf{f} = 0$. Thus, we have $\mathbf{f} = (I - Q)\mathbf{f}$ and the proof is complete. □

Remark 3.3. *Similarly, we can characterize the space* $\mathbf{Y}^*_{\alpha^-}$ *as follows:*

$$\mathbf{Y}^*_{\alpha^-} = \Big\{\mathbf{\Phi} \in \mathbf{V}^0_n(\Omega) \mid \Big(\mathbf{\Phi}, \mathbf{h}\Big) = 0 \quad \textit{for all} \quad \mathbf{h} \in \mathbf{Y}_\alpha\Big\}.$$

Corollary 3.1. *We can identify the dual of* $\mathbf{Y}_\alpha$ *with* $\mathbf{Y}^*_\alpha$*, and the dual of* $\mathbf{Y}_{\alpha^-}$ *with* $\mathbf{Y}^*_{\alpha^-}$*.*

Proof. The space $\mathbf{V}^0_n(\Omega)$ is identified with its dual. Moreover, let $\mathbf{f}$ belong to $\mathbf{Y}_\alpha \subset \mathbf{V}^0_n(\Omega)$. For all $\mathbf{g}$ in $\mathbf{V}^0_n(\Omega)$, with Remark 3.3, we have

$$\Big(\mathbf{f}, \mathbf{g}\Big) = \Big(\mathbf{f}, Q^*\mathbf{g}\Big),$$

and we deduce that the dual of $\mathbf{Y}_\alpha$ can be identified with $\mathbf{Y}^*_\alpha$ since $Q^*(\mathbf{V}^0_n(\Omega)) = \mathbf{Y}^*_\alpha$. The proof is similar to show that the dual of $\mathbf{Y}_{\alpha^-}$ can be identified with $\mathbf{Y}^*_{\alpha^-}$. □

From their definitions, Q and Q^* are linear and continuous from $\mathbf{V}^0_n(\Omega)$ to respectively $\mathbf{Y}_\alpha$ and $\mathbf{Y}^*_\alpha$. Since $\mathbf{Y}_\alpha$ and $\mathbf{Y}^*_\alpha$ are spaces of finite dimension, we can extend the operator Q to $(D(A^*))'$ and Q^* to $(D(A))'$ in the following way.

Remark 3.4. *We define the operators* $Q \in \mathcal{L}((D(A^*))', \mathbf{Y}_\alpha)$ *and* $Q^* \in \mathcal{L}((D(A))', \mathbf{Y}^*_\alpha)$ *by*

$$Q\mathbf{f} = \sum_{j=1}^{K} \Big\langle \mathbf{f}, \varepsilon_j \Big\rangle_{(D(A^*))', D(A^*)} \mathbf{e}_j \quad \textit{and} \quad Q^*\mathbf{f} = \sum_{j=1}^{K} \Big\langle \mathbf{f}, \mathbf{e}_j \Big\rangle_{(D(A))', D(A)} \varepsilon_j,$$

for all $\mathbf{f}$ *in* $(D(A^*))' = (D(A))'$*.*

Definition 3.4. *We consider the subspace* $\mathcal{U}$ *of* $L^2(\Gamma)$ *defined by*

$$\mathcal{U} = \text{vect}\Big\{MB^*\varepsilon_j \mid j = 1, \cdots, K\Big\}.$$

We denote by $\{\zeta_1, \cdots \zeta_{n_c}\}$ *a basis of* $\mathcal{U}$*.*

Remark 3.5. *From Remark 3.4 and the definition of the space* $\mathcal{U}$ *we can deduce that* MB^*Q^* *belongs to* $\mathcal{L}((D(A))', \mathcal{U})$*.*

For notational simplicity, we still denote by A_α the restriction of A to $\mathbf{Y}_\alpha$. Let A_{α^-} be the unbounded operator in $\mathbf{Y}_{\alpha^-}$ defined by

$$D(A_{\alpha^-}) = D(A) \cap \mathbf{Y}_{\alpha^-}, \quad A_{\alpha^-}\mathbf{y} = A\mathbf{y} \quad \text{for all} \quad \mathbf{y} \in D(A_{\alpha^-}).$$

It is easy to check that the adjoint of $(A_{\alpha^-}, D(A_{\alpha^-}))$ is the unbounded operator $(A^*_{\alpha^-}, D(A^*_{\alpha^-}))$ in $\mathbf{Y}^*_{\alpha^-}$ defined by

$$D(A^*_{\alpha^-}) = D(A^*) \cap \mathbf{Y}^*_{\alpha^-}, \quad A^*_{\alpha^-}\mathbf{y} = A^*\mathbf{y} \quad \text{for all} \quad \mathbf{y} \in D(A^*_{\alpha^-}).$$

The space $D(A)$ (respectively $D(A^*)$) is equipped with the norm $\mathbf{y} \mapsto |(\lambda_0 I - A)\mathbf{y}|_{\mathbf{V}^0_n(\Omega)}$ (respectively $\mathbf{y} \mapsto |(\lambda_0 I - A^*)\mathbf{y}|_{\mathbf{V}^0_n(\Omega)}$). Let us recall that $D(A) = D(A^*)$, and actually the two norms are equivalent. The space $D(A_{\alpha^-})$ is a closed subspace of $D(A)$ and it is dense in $\mathbf{Y}_{\alpha^-}$. Thus, we shall equip $D(A_{\alpha^-})$ with the norm of $D(A)$. Similarly, $D(A^*_{\alpha^-})$ is closed in $D(A^*)$ and dense in $\mathbf{Y}^*_{\alpha^-}$, and $D(A^*_{\alpha^-})$ will be equipped with the norm of $D(A^*)$.

The operator $A^*_{\alpha^-}$ can be considered either as an unbounded operator in $\mathbf{Y}^*_{\alpha^-}$ (or in $\mathbf{V}^0_n(\Omega)$) with domain $D(A^*_{\alpha^-})$ or as an isomorphism from $D(A^*_{\alpha^-})$ to $\mathbf{Y}^*_{\alpha^-}$. Similarly, the operator A_{α^-} can be considered either as an unbounded operator in $\mathbf{Y}_{\alpha^-}$ (or in $\mathbf{V}^0_n(\Omega)$) with domain $D(A_{\alpha^-})$ or as an isomorphism from $D(A_{\alpha^-})$ to $\mathbf{Y}_{\alpha^-}$. Since $A^*_{\alpha^-} \in \mathrm{Isom}(D(A^*_{\alpha^-}), \mathbf{Y}^*_{\alpha^-})$, we have $(A^*_{\alpha^-})^* \in \mathrm{Isom}(\mathbf{Y}_{\alpha^-}, (D(A^*_{\alpha^-}))')$ (here, $\mathrm{Isom}(E, F)$ denotes the space of isomorphisms from E onto F). The operator $(A^*_{\alpha^-})^*$ can also be viewed as an unbounded operator in $(D(A^*_{\alpha^-}))'$ with domain $\mathbf{Y}_{\alpha^-}$. As in [8, Chapter 3, p. 160], it can be shown that this unbounded operator is the extension of A_{α^-} to $(D(A^*_{\alpha^-}))'$. For simplicity, it will be still denoted by A_{α^-}. Let us observe that we have the following decomposition

$$D(A) = \mathbf{Y}_\alpha \oplus D(A_{\alpha^-}) \quad \text{and} \quad D(A^*) = \mathbf{Y}^*_\alpha \oplus D(A^*_{\alpha^-}). \tag{3.7}$$

In Proposition 3.4, we are going to see that $(I-Q)$ belongs to $\mathcal{L}((D(A^*))', (D(A^*_{\alpha^-}))')$. For that, we need a precise characterization of $(D(A^*_{\alpha^-}))'$, which is given in Proposition 3.3.

Lemma 3.5. *The space $(D(A^*))'$ can be decomposed as follows:*

$$(D(A^*))' = \mathbf{Y}_\alpha \oplus \overline{\mathbf{Y}_{\alpha^-}}^{|\cdot|_{(D(A^*))'}},$$

where $\overline{\mathbf{Y}_{\alpha^-}}^{|\cdot|_{(D(A^*))'}}$ denotes the closure of $\mathbf{Y}_{\alpha^-}$ in the norm $(D(A^*))'$.

Proof. Step 1. We first prove the identity $\overline{\mathbf{Y}_{\alpha^-}}^{|\cdot|_{(D(A^*))'}} = E_\alpha$, where

$$E_\alpha = \left\{ \mathbf{f} \in (D(A^*))' \mid \Big\langle \mathbf{f}, \mathbf{g} \Big\rangle_{(D(A^*))', D(A^*)} = 0 \quad \text{for all } \mathbf{g} \in \mathbf{Y}^*_\alpha \cap D(A^*) \right\}.$$

Let $\mathbf{f}$ belong to $\overline{\mathbf{Y}_{\alpha^-}}^{|\cdot|_{(D(A^*))'}}$. There exists $(\mathbf{f}_n)_{n\in\mathbb{N}}$, such that $\mathbf{f}_n$ belongs to $\mathbf{Y}_{\alpha^-}$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \mathbf{f}_n = \mathbf{f}$ in $(D(A^*))'$. From Proposition 3.2, for all $n \in \mathbb{N}$, we have

$$\Big\langle \mathbf{f}_n, \mathbf{g} \Big\rangle_{(D(A^*))', D(A^*)} = \Big(\mathbf{f}_n, \mathbf{g} \Big) = 0 \quad \text{for all} \quad \mathbf{g} \in \mathbf{Y}^*_\alpha \cap D(A^*).$$

We show that $\overline{\mathbf{Y}_{\alpha^-}}^{|\cdot|_{(D(A^*))'}} \subset E_\alpha$ by passing to the limit in the previous identity when n tends to infinity. Let $\mathbf{f}$ belong to E_α. There exists $(\mathbf{f}_n)_{n\in\mathbb{N}}$, such that $\mathbf{f}_n$ belongs to $\mathbf{V}^0_n(\Omega)$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \mathbf{f}_n = \mathbf{f}$ in $(D(A^*))'$. For all $n \in \mathbb{N}$, we have $(I-Q)\mathbf{f}_n \in \mathbf{Y}_{\alpha^-}$. Moreover, from the definition of E_α and (3.6), we can check that $Q\mathbf{f} = 0$. Thus, we have $(I-Q)\mathbf{f}_n - \mathbf{f} = (I-Q)(\mathbf{f}_n - \mathbf{f})$. It follows that $(I-Q)\mathbf{f}_n$ tends to $\mathbf{f}$ in $(D(A^*))'$ when n tends to infinity and the equality $\overline{\mathbf{Y}_{\alpha^-}}^{|\cdot|_{(D(A^*))'}} = E_\alpha$ is proved.

Step 2. We show that $(D(A^*))' = \mathbf{Y}_\alpha \oplus E_\alpha$. From Remark 3.4, for all $\mathbf{f} \in (D(A^*))'$, we have $Q\mathbf{f} \in \mathbf{Y}_\alpha$. Since $(I-Q)\mathbf{f} \in E_\alpha$ for all $\mathbf{f} \in (D(A^*))'$, the proof is complete. $\square$

Since the space $D(A_{\alpha^-})$ is continuously and densely imbedded in $\mathbf{Y}_{\alpha^-}$ and $D(A^*_{\alpha^-})$ is continuously and densely imbedded in $\mathbf{Y}^*_{\alpha^-}$, by duality we have

$$\mathbf{Y}^*_{\alpha^-} \hookrightarrow (D(A_{\alpha^-}))' \quad \text{and} \quad \mathbf{Y}_{\alpha^-} \hookrightarrow (D(A^*_{\alpha^-}))',$$

with dense and continuous imbeddings if $(D(A_{\alpha^-}))'$ is equipped with the dual norm of $D(A_{\alpha^-})$ and if $(D(A^*_{\alpha^-}))'$ is equipped with the dual norm of $D(A^*_{\alpha^-})$. Since the two norms $|\cdot|_{D(A)}$ and $|\cdot|_{D(A^*)}$ are equivalent, we have the following proposition.

Proposition 3.3. *We have the identity* $(D(A^*_{\alpha^-}))' = \overline{\mathbf{Y}_{\alpha^-}}^{|\cdot|_{(D(A^*))'}}$.

Proposition 3.4. *The operator* $(I-Q)$ *is linear and continuous from* $(D(A^*))'$ *to* $(D(A^*_{\alpha^-}))'$.

Proof. Due to Proposition 3.3 and Lemma 3.5, we have $(I-Q)\mathbf{f} \in (D(A^*_{\alpha^-}))'$ for all $\mathbf{f}$ in $(D(A^*))'$. We notice that for all $\mathbf{\Phi} \in D(A^*_{\alpha^-})$, we have $\mathbf{\Phi} = (I-Q^*)\mathbf{\Phi}$. Then, we have

$$\Big\langle (I-Q)\mathbf{f}, \mathbf{\Phi} \Big\rangle_{(D(A^*_{\alpha^-}))', D(A^*_{\alpha^-})} = \Big\langle \mathbf{f}, \mathbf{\Phi} \Big\rangle_{(D(A^*))', D(A^*)}$$

for all $\mathbf{\Phi} \in D(A^*_{\alpha^-})$, and the proof is complete. □

Let us set

$$\mathbf{y}_\alpha = Q\mathbf{y} \quad \text{and} \quad \mathbf{y}_{\alpha^-} = (I-Q)\mathbf{y}.$$

The linearized equation

$$\mathbf{y}' = A\mathbf{y} + BM\mathbf{u} \quad \text{in } (0,\infty), \quad \mathbf{y}(0) = \mathbf{y}_0, \tag{3.8}$$

may be split as follows:

$$\begin{aligned} &{\mathbf{y}_\alpha}' = A_\alpha \mathbf{y}_\alpha + QBM\mathbf{u} \quad \text{in } (0,\infty), \quad \mathbf{y}_\alpha(0) = Q\mathbf{y}_0, \\ &{\mathbf{y}_{\alpha^-}}' = A_{\alpha^-}\mathbf{y}_{\alpha^-} + (I-Q)BM\mathbf{u} \quad \text{in } (0,\infty), \quad \mathbf{y}_{\alpha^-}(0) = (I-Q)\mathbf{y}_0. \end{aligned} \tag{3.9}$$

4 Stabilizability of the real projected system by finite dimensional controls

In this section, we study the controllability of the projected system

$${\mathbf{y}_\alpha}' = A_\alpha \mathbf{y}_\alpha + QBM\mathbf{u} \quad \text{in } (0,\infty), \quad \mathbf{y}_\alpha(0) = Q\mathbf{y}_0. \tag{4.1}$$

A classical way to prove that is to use the well-known Kalman controllability criterion. We look for $(\zeta_1, \cdots, \zeta_{n_c})$ and $(v_1, \cdots, v_{n_c})$, so that the system

$${\mathbf{y}_\alpha}' = A_\alpha \mathbf{y}_\alpha + \sum_{i=1}^{n_c} v_i(t) QBM\zeta_i \quad \text{in } (0,\infty), \quad \mathbf{y}_\alpha(0) = Q\mathbf{y}_0, \tag{4.2}$$

is stabilizable. Since $(\mathbf{e}_j)_{j=1}^K$ is a basis of $\mathbf{Y}_\alpha$, we have

$$\mathbf{y}_\alpha = \sum_{j=1}^K y_j\, \mathbf{e}_j.$$

Moreover, using Remark 3.4, we have

$$QBM\zeta_i = \sum_{j=1}^K \Big\langle BM\zeta_i, \varepsilon_j \Big\rangle_{(D(A^*))', D(A^*)} \mathbf{e}_j.$$

Denoting by $y_r = \mathrm{col}[y_1, \cdots, y_K]$ and $V = \mathrm{col}[v_1, \cdots, v_{n_c}]$, equation (4.2) can be written as follows

$$y_r' = \Lambda y_r + CV \quad \text{in } (0,\infty), \tag{4.3}$$

where

$$
\Lambda = \begin{pmatrix} \Lambda_1 & & & \\ & \Lambda_2 & 0 & \\ 0 & & \ddots & \\ & & & \Lambda_{N_\alpha} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \left(\zeta_1, MB^*\varepsilon_1\right)_{\mathbf{V}^0(\Gamma)} & \cdots & \left(\zeta_{n_c}, MB^*\varepsilon_1\right)_{\mathbf{V}^0(\Gamma)} \\ \vdots & & \vdots \\ \left(\zeta_1, MB^*\varepsilon_K\right)_{\mathbf{V}^0(\Gamma)} & \cdots & \left(\zeta_{n_c}, MB^*\varepsilon_K\right)_{\mathbf{V}^0(\Gamma)} \end{pmatrix}.
$$

From the Kalman controllability criterion, the pair $\{\Lambda, C\}$ is controllable if and only if

$$
\operatorname{rank}[C, \Lambda C, \Lambda^2 C, \cdots, \Lambda^{K-1} C] = K. \tag{4.4}
$$

If we assume that the family

$$
\{MB^*\varepsilon_j \mid j = 1, \cdots, K\} \tag{4.5}
$$

is linearly independent, then (4.4) is satisfied. Indeed, considering an arbitrarily sequence of scalars $\{r_j\}_{j=1}^K$, there exists a unique vector $\zeta \in \mathbf{V}^0(\Gamma)$ such that

$$
\left(\zeta, MB^*\varepsilon_i\right)_{\mathbf{V}^0(\Gamma)} = r_i, \quad i = 1, \cdots, K,
$$

(see for instance [7, p. 1465]). Applying this method for every columns of C, we can prove that

$$
\operatorname{rank}(C) = K. \tag{4.6}
$$

Remark 4.1. *The linearly independence of the family (4.5) is not a necessary condition to the controllability. Let us consider the case where the family (4.5) is linearly dependent. To simplify, let us choose $K = 2$, $n_c = 1$ and*

$$
\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.
$$

The matrix $C \in \mathbb{R}^{2\times 1}$ is given by

$$
C = \begin{pmatrix} \left(\zeta_1, MB^*\varepsilon_1\right)_{\mathbf{V}^0(\Gamma)} \\ \left(\zeta_1, MB^*\varepsilon_2\right)_{\mathbf{V}^0(\Gamma)} \end{pmatrix}.
$$

We can find a vector $\zeta_1 \in \mathbf{V}^0(\Gamma)$ such that $(\zeta_1, MB^*\varepsilon_2)_{\mathbf{V}^0(\Gamma)} \neq 0$. With such a function ζ_1, we can check that the Kalman criterion (4.4) is satisfied.

To prove that the system (4.1) is stabilizable by a control of finite dimension, we are going to use the fact that the system (3.8) is stabilizable by a control of infinite dimension.

Proposition 4.1. *There exists a control $\mathbf{u} \in L^2(0, \infty; \mathcal{U})$ such that the solution $\mathbf{y}_\alpha$ to (4.1) obeys*

$$
|\mathbf{y}_\alpha(t)|_{\mathbf{V}^0_n(\Omega)} \leq C e^{-\rho t} |\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}.
$$

for some $\rho > 0$.

Proof. *Step* 1. It has been proved in [18, Theorem 4.5], that there exists a control $\tilde{\mathbf{u}}$ supported in an open subset Γ_c of Γ, such that the solution $\mathbf{y}_{\tilde{\mathbf{u}}}$ of (3.8) obeys $|\mathbf{y}_{\tilde{\mathbf{u}}}(t)|_{\mathbf{V}^0_n(\Omega)} \leq Ce^{-\rho t}|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}$ with $\rho > 0$. Since Q is a continuous operator from $\mathbf{V}^0_n(\Omega)$ to $\mathbf{Y}_\alpha$, we have

$$|Q\mathbf{y}_{\tilde{\mathbf{u}}}(t)|_{\mathbf{V}^0_n(\Omega)} \leq Ce^{-\rho t}|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}. \tag{4.7}$$

Since Q and A_α commute, we have

$$Q\mathbf{y}_{\tilde{\mathbf{u}}}(t) = e^{A_\alpha t}Q\mathbf{y}_0 + \int_0^t e^{A_\alpha(t-\tau)}QBM\tilde{\mathbf{u}}(\tau)d\tau.$$

Moreover, the solution to the real projected system (4.1) obeys

$$\mathbf{y}_\alpha(t) = e^{A_\alpha t}Q\mathbf{y}_0 + \int_0^t e^{A_\alpha(t-\tau)}QBM\mathbf{u}(\tau)d\tau.$$

We look for $\mathbf{u} \in L^2(0,\infty;\mathcal{U})$ such that $\mathbf{y}_\alpha = Q\mathbf{y}_{\tilde{\mathbf{u}}}$. Both solutions are the same if

$$QBM\mathbf{u}(\tau) = QBM\tilde{\mathbf{u}}(\tau), \quad \text{for all} \quad \tau \in (0,\infty). \tag{4.8}$$

Step 2. From the definition of the operator Q, condition (4.8) is satisfied if

$$\Big(\mathbf{u}(\tau) - \tilde{\mathbf{u}}(\tau), MB^*\varepsilon_j\Big)_{\mathbf{V}^0(\Gamma)} = \Big\langle BM\mathbf{u}(\tau) - BM\tilde{\mathbf{u}}(\tau), \varepsilon_j\Big\rangle_{(D(A^*))', D(A^*)} = 0, \ \forall j = 1, \cdots, K,$$

and all $\tau \in (0,\infty)$. We choose $\mathbf{u}(\tau)$ equal to the orthogonal projection in $\mathbf{V}^0(\Gamma)$ of $\tilde{\mathbf{u}}(\tau)$ onto $\mathcal{U}$. Thus, condition (4.8) is satisfied and both solutions $Q\mathbf{y}_{\tilde{\mathbf{u}}}$ and $\mathbf{y}_\alpha$ to (4.1) are equal. Finally, with (4.7), we obtain the desired estimate. □

We denote by $\widetilde{M} \in \mathcal{L}(\mathbb{R}^{n_c}, \mathbf{V}^0(\Gamma))$ the operator defined by

$$(\widetilde{M}\mathbf{v})(x) = \sum_{i=1}^{n_c} v_i M\zeta_i(x) \quad \text{for all} \quad \mathbf{v} = (v_1, \cdots, v_{nc}) \in \mathbb{R}^{n_c}. \tag{4.9}$$

Proposition 4.2. *The adjoint operator $\widetilde{M}^*$ belongs to $\mathcal{L}(\mathbf{V}^0(\Gamma), \mathbb{R}^{n_c})$ and for all $\mathbf{u} \in \mathbf{V}^0(\Gamma)$, we have*

$$(\widetilde{M}^*\mathbf{u})_i = \Big(\zeta_i, M\mathbf{u}\Big)_{\mathbf{V}^0(\Gamma)}, \quad \textit{for all } i = 1, \cdots, n_c.$$

Proof. This is a direct consequence of the definition of $\widetilde{M}$. □

5 Feedback control of the real projected system

The aim of this section is to study the finite dimensional control problem $(\mathcal{P}^\infty_{\mathbf{y}_0})$ stated in subsection 5.2. More precisely we want to characterize its optimal solution via a feedback law defined thanks to a finite dimensional Riccati equation (see (5.9)). Equation (5.9) is stated over a finite dimensional space since $(\mathcal{P}^\infty_{\mathbf{y}_0})$ is a finite dimensional control problem. To achieve this goal we could obviously use results from the existing literature [24, 16]. In (5.9) we look for a solution Π belonging to $\mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}^*_\alpha)$ because we have not identified the dual of $\mathbf{Y}_\alpha$ with itself. The approach in [24, 16] consists in looking for an operator Π defined in a space which is identified with its dual. Here we follow the lines of [18] where, by studying a family of finite time horizon control problems $(\mathcal{P}^k_{\mathbf{y}_0})$, we clearly understand why Π belongs to $\mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}^*_\alpha)$, and why we recover results which are very similar to that in [24, 16].

In what follows we only state the results since the proof can easily be adapted from [18].

5.1 A finite time horizon control problem

For all $\mathbf{y}_0$ in $\mathbf{Y}_\alpha$, we consider the following optimal control problem

$$(\mathcal{P}^k_{\mathbf{y}_0}) \qquad \inf\Big\{I_k(\mathbf{y},\mathbf{v}) \ \mid\ (\mathbf{y},\mathbf{v}) \text{ satisfies (5.1)}, \mathbf{v}\in L^2(0,k;\mathbb{R}^{n_c})\Big\},$$

where

$$I_k(\mathbf{y},\mathbf{v}) = \frac{1}{2}\int_0^k\int_\Omega |Q\mathbf{y}|^2\,dxdt + \frac{1}{2}\int_0^k |\mathbf{v}(t)|^2\,dt,$$

and

$$\mathbf{y}' = A_\alpha\mathbf{y} + QB\widetilde{M}\mathbf{v} \quad \text{in } (0,k), \qquad \mathbf{y}(0)=\mathbf{y}_0. \tag{5.1}$$

Remark 5.1. *To simplify the notations, we shall denote by $\mathbf{y}$ the solution to equation (5.1), but we have to keep in mind that it represents $\mathbf{y}_\alpha = Q\mathbf{y}$. Then, of course we have $Q\mathbf{y}=\mathbf{y}$ and the cost functional can also be written*

$$I_k(\mathbf{y},\mathbf{v}) = \frac{1}{2}\int_0^k\int_\Omega |\mathbf{y}|^2\,dxdt + \frac{1}{2}\int_0^k |\mathbf{v}(t)|^2\,dt.$$

Proposition 5.1. *Problem $(\mathcal{P}^k_{\mathbf{y}_0})$ admits a unique solution $(\mathbf{y},\mathbf{v})$ where*

$$\mathbf{v}(t) = -\widetilde{M}^*B^*Q^*\mathbf{\Phi}(t), \tag{5.2}$$

and $\mathbf{\Phi}$ is solution to the equation

$$-\mathbf{\Phi}' = A^*_\alpha\mathbf{\Phi} + Q^*Q\mathbf{y} \quad \text{in } (0,k), \quad \mathbf{\Phi}(k)=0. \tag{5.3}$$

Conversely the system

$$\begin{aligned} &\mathbf{y}' = A_\alpha\mathbf{y} - QB\widetilde{M}\widetilde{M}^*B^*Q^*\mathbf{\Phi} \quad \text{in } (0,k), \quad \mathbf{y}(0)=\mathbf{y}_0,\\ &-\mathbf{\Phi}' = A^*_\alpha\mathbf{\Phi} + Q^*Q\mathbf{y} \quad \text{in } (0,k), \quad \mathbf{\Phi}(k)=0 \end{aligned} \tag{5.4}$$

admits a unique solution $(\mathbf{y},\mathbf{\Phi})\in C^1(0,k;\mathbf{Y}_\alpha)\times C^1(0,k;\mathbf{Y}^*_\alpha)$ and $(\mathbf{y},-\widetilde{M}^*B^*Q^*\mathbf{\Phi})$ is the optimal solution to $(\mathcal{P}^k_{\mathbf{y}_0})$.

Proof. The proof follows the lines of [18, Theorem 3.1]. □

As in [18, Corollary 3.8], from this proposition, we obtain the following corollary.

Corollary 5.1. *The value of the infimum of $(\mathcal{P}^k_{\mathbf{y}_0})$ is given by*

$$\inf(\mathcal{P}^k_{\mathbf{y}_0}) = \frac{1}{2}\Big\langle \mathbf{y}_0, \mathbf{\Phi}(0)\Big\rangle_{\mathbf{Y}_\alpha,\mathbf{Y}^*_\alpha},$$

where $(\mathbf{y},\mathbf{\Phi})$ is solution to system (5.4).

We define the operator $\Pi(k)\in\mathcal{L}(\mathbf{Y}_\alpha,\mathbf{Y}^*_\alpha)$ by

$$\Pi(k)\mathbf{y}_0 = \mathbf{\Phi}(0),$$

where $(\mathbf{y},\mathbf{\Phi})$ is solution to system (5.4).

Theorem 5.1. *The solution $(\mathbf{y},\mathbf{v})$ to problem $(\mathcal{P}^k_{\mathbf{y}_0})$ belongs to $C^1([0,k];\mathbf{Y}_\alpha)\times C([0,k];\mathbb{R}^{n_c})$ and it obeys the feedback formula*

$$\mathbf{v}(t) = -\widetilde{M}^*B^*Q^*\Pi(t)\mathbf{y}(t).$$

Moreover, the optimal cost is given by

$$J(\mathbf{y},\mathbf{v}) = \frac{1}{2}\Big\langle \mathbf{y}_0, \Pi(k)\mathbf{y}_0\Big\rangle_{\mathbf{Y}_\alpha,\mathbf{Y}^*_\alpha}.$$

5.2 An infinite time horizon control problem

For all $\mathbf{y}_0$ in $\mathbf{Y}_\alpha$, we now consider the infinite time horizon control problem

$$(\mathcal{P}^\infty_{\mathbf{y}_0}) \qquad \inf\Big\{ I(\mathbf{y},\mathbf{v}) \ \mid \ (\mathbf{y},\mathbf{v}) \text{ satisfies } (5.5), \mathbf{v}\in L^2(0,\infty;\mathbb{R}^{n_c})\Big\},$$

where

$$I(\mathbf{y},\mathbf{v}) = \frac{1}{2}\int_0^\infty \int_\Omega |Q\mathbf{y}|^2\, dxdt + \frac{1}{2}\int_0^\infty |\mathbf{v}(t)|^2_{\mathbb{R}^{n_c}}\, dt,$$

and

$$\mathbf{y}' = A_\alpha \mathbf{y} + QB\widetilde{M}\mathbf{v} \quad \text{in } (0,\infty), \qquad \mathbf{y}(0)=\mathbf{y}_0. \tag{5.5}$$

Theorem 5.2. *For all $\mathbf{y}_0$ in $\mathbf{Y}_\alpha$, problem $(\mathcal{P}^\infty_{\mathbf{y}_0})$ admits a unique solution $(\mathbf{y}_{\mathbf{y}_0},\mathbf{v}_{\mathbf{y}_0})$. Moreover, denoting by $(\mathbf{y}^k_{\mathbf{y}_0},\mathbf{v}^k_{\mathbf{y}_0})$ the solution to the finite time horizon control problem $(\mathcal{P}^k_{\mathbf{y}_0})$, and by $(\tilde{\mathbf{y}}^k_{\mathbf{y}_0},\tilde{\mathbf{v}}^k_{\mathbf{y}_0})$ their extensions by zero to the interval (k,∞) we have*

$$(\tilde{\mathbf{y}}^k_{\mathbf{y}_0},\tilde{\mathbf{v}}^k_{\mathbf{y}_0}) \to (\mathbf{y}_{\mathbf{y}_0},\mathbf{v}_{\mathbf{y}_0}), \qquad \text{in } \ L^2(0,\infty;\mathbf{Y}_\alpha)\times L^2(0,\infty;\mathbb{R}^{n_c}).$$

Proof. We have proved in subsection 4 that there exists a control $\mathbf{v}$ such that the projected system (5.5) is stabilizable by finite dimensional controllers. This implies that $(\mathbf{y}_\mathbf{v},\mathbf{v})$ obeys

$$I(\mathbf{y}_\mathbf{v},\mathbf{v})<\infty,$$

where $\mathbf{y}_\mathbf{v}$ is the solution of equation (5.5). The existence of a unique solution $(\mathbf{y}_{\mathbf{y}_0},\mathbf{v}_{\mathbf{y}_0})$ to $(\mathcal{P}^\infty_{\mathbf{y}_0})$ follows from classical arguments. The convergence of $(\tilde{\mathbf{y}}^k_{\mathbf{y}_0},\tilde{\mathbf{v}}^k_{\mathbf{y}_0})$ towards $(\mathbf{y}_{\mathbf{y}_0},\mathbf{v}_{\mathbf{y}_0})$ follows the proof of [18, Theorem 4.1]. □

Theorem 5.3. *There exists $\Pi\in\mathcal{L}(\mathbf{Y}_\alpha,\mathbf{Y}^*_\alpha)$ satisfying $\Pi=\Pi^*\geq 0$ and*

$$\inf(\mathcal{P}^\infty_{\mathbf{y}_0}) = \frac{1}{2}\Big\langle \mathbf{y}_0,\Pi\mathbf{y}_0\Big\rangle_{\mathbf{Y}_\alpha,\mathbf{Y}^*_\alpha}.$$

Proof. See [18, Theorem 4.1]. The operator Π is obtained as the limit of $\Pi(k)$ when k tends to infinity. □

Theorem 5.4. *For every $\mathbf{y}_0$ in $\mathbf{Y}_\alpha$, the system*

$$\begin{aligned}
&\mathbf{y}' = A_\alpha\mathbf{y} - QB\widetilde{M}\widetilde{M}^*B^*Q^*\mathbf{\Phi} \quad \text{in } (0,\infty), \quad \mathbf{y}(0)=\mathbf{y}_0\\
&-\mathbf{\Phi}' = A^*_\alpha\mathbf{\Phi} + Q^*Q\mathbf{y} \quad \text{in } (0,\infty), \quad \mathbf{\Phi}(\infty)=0\\
&\mathbf{\Phi}(t) = \Pi\mathbf{y}(t) \quad \text{for all } t\in(0,\infty),
\end{aligned} \tag{5.6}$$

admits a unique solution $(\mathbf{y},\mathbf{\Phi})$ in $H^1(0,\infty;\mathbf{Y}_\alpha)\times H^1(0,\infty;\mathbf{Y}^*_\alpha)$. This solution satisfies

$$\|\mathbf{y}\|_{H^1(0,\infty;\mathbf{Y}_\alpha)} + \|\mathbf{\Phi}\|_{H^1(0,\infty;\mathbf{Y}^*_\alpha)} \leq C|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}.$$

The pair $(\mathbf{y},-\widetilde{M}^*B^*Q^*\mathbf{\Phi})$ is the solution of problem $(\mathcal{P}^\infty_{\mathbf{y}_0})$.

Proof. The existence of a solution follows the lines of [18].

Step 1. We prove the uniqueness of this solution. We denote by $(\mathbf{y},\mathbf{v})$ the solution to problem $(\mathcal{P}^\infty_{\mathbf{y}_0})$. Adapting the proof of [18, Lemma 4.2], we can check that

$$\mathbf{v} = -\widetilde{M}^*B^*Q^*\mathbf{\Phi},$$

where $(\mathbf{y},\mathbf{\Phi})$ is solution to (5.4). Thus, with Theorem 5.3, we have

$$\int_0^\infty |\mathbf{y}(t)|^2_{\mathbf{Y}_\alpha} dt + \int_0^\infty |\widetilde{M}^* B^* Q^* \mathbf{\Phi}(t)|^2_{\mathbb{R}^{n_c}} = \frac{1}{2}\Big\langle \mathbf{y}_0, \Pi \mathbf{y}_0 \Big\rangle_{\mathbf{Y}_\alpha, \mathbf{Y}^*_\alpha}. \tag{5.7}$$

It follows that if $\mathbf{y}_0 = 0$, then $\mathbf{y} = 0$ and since $\mathbf{\Phi}(t) = \Pi \mathbf{y}(t)$, the uniqueness is proved.

Step 2. We prove the final estimate. Let us denote by $(\mathbf{y},\mathbf{\Phi})$ the solution of system (5.6). From (5.7), we have

$$\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{Y}_\alpha)} \leq C|\mathbf{y}_0|_{\mathbf{Y}_\alpha}.$$

Moreover, since $\mathbf{\Phi} = \Pi\mathbf{y}$ and $\Pi \in \mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}^*_\alpha)$, it follows that

$$\|\mathbf{\Phi}\|_{L^2(0,\infty;\mathbf{Y}^*_\alpha)} \leq C|\mathbf{y}_0|_{\mathbf{Y}_\alpha}.$$

From the equation satisfied by $\mathbf{\Phi}$, we deduce that $\mathbf{\Phi}$ belongs to $H^1(0,\infty;\mathbf{Y}^*_\alpha)$, and

$$\|\mathbf{\Phi}\|_{H^1(0,\infty;\mathbf{Y}^*_\alpha)} \leq C(\|\mathbf{\Phi}\|_{L^2(0,\infty;\mathbf{Y}^*_\alpha)} + \|\mathbf{y}\|_{L^2(0,\infty;\mathbf{Y}_\alpha)}).$$

Since $\mathbf{\Phi} \in H^1(0,\infty;\mathbf{Y}^*_\alpha)$, we can verify that $\mathbf{v} = -\widetilde{M}^* B^* Q^* \mathbf{\Phi}$ belongs to $H^1(0,\infty;\mathbb{R}^{n_c})$ and

$$\|\mathbf{v}\|_{H^1(0,\infty;\mathbb{R}^{n_c})} \leq C\|\mathbf{\Phi}\|_{H^1(0,\infty;\mathbf{Y}^*_\alpha)}. \tag{5.8}$$

Then, obviously we have $B\widetilde{M}\mathbf{v} \in H^1(0,\infty;(D(A^*))')$ and

$$\|B\widetilde{M}\mathbf{v}\|_{H^1(0,\infty;(D(A^*))')} \leq C\|\mathbf{\Phi}\|_{H^1(0,\infty;\mathbf{Y}^*_\alpha)}.$$

Finally, since Q and is linear continuous from $(D(A^*))'$ to $\mathbf{Y}_\alpha$ we obtain

$$\|QB\widetilde{M}\mathbf{v}\|_{H^1(0,\infty;\mathbf{Y}_\alpha)} \leq C\|\mathbf{\Phi}\|_{H^1(0,\infty;\mathbf{Y}^*_\alpha)}.$$

Using the equation satisfied by $\mathbf{y}$, we deduce that $\mathbf{y}$ belongs to $H^1(0,\infty;\mathbf{Y}_\alpha)$, and

$$\|\mathbf{y}\|_{H^1(0,\infty;\mathbf{Y}_\alpha)} \leq C(\|\mathbf{\Phi}\|_{L^2(0,\infty;\mathbf{Y}^*_\alpha)} + \|\mathbf{y}\|_{L^2(0,\infty;\mathbf{Y}_\alpha)}).$$

With all these estimates, we obtain

$$\|\mathbf{y}\|_{H^1(0,\infty;\mathbf{Y}_\alpha)} + \|\mathbf{\Phi}\|_{H^1(0,\infty;\mathbf{Y}^*_\alpha)} \leq C(\|\mathbf{\Phi}\|_{L^2(0,\infty;\mathbf{Y}^*_\alpha)} + \|\mathbf{y}\|_{L^2(0,\infty;\mathbf{Y}_\alpha)}) \leq C|\mathbf{y}_0|_{\mathbf{Y}_\alpha},$$

and the proof is complete. $\square$

Remark 5.2. *From estimate (5.8) and Theorem 5.4, we deduce that*

$$\|\mathbf{v}\|_{H^1(0,\infty;\mathbb{R}^{n_c})} \leq C|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}$$

for every $\mathbf{y}_0$ in $\mathbf{Y}_\alpha$, where $\mathbf{v} = -\widetilde{M}^* B^* Q^* \mathbf{\Phi}$ and $\mathbf{\Phi}$ solution to the system (5.6). By iterating the argument used in the previous proof, we can prove that $(\mathbf{y},\mathbf{\Phi})$ actually belongs to $H^r(0,\infty;\mathbf{Y}_\alpha)\times H^r(0,\infty;\mathbf{Y}^*_\alpha)$ for all $r > 0$.

Let us consider the algebraic Riccati equation

$$\begin{aligned} &\Pi \in \mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}^*_\alpha), \quad \Pi = \Pi^*, \quad \Pi \geq 0, \\ &\Pi A_\alpha + A^*_\alpha \Pi - \Pi B\widetilde{M}\widetilde{M}^* B^* \Pi + Q^* Q = 0. \end{aligned} \tag{5.9}$$

Let us make some comments. We shall say that $\Pi = \Pi^* \geq 0$ when

$$\Big\langle \Pi \mathbf{y}, \mathbf{z} \Big\rangle_{\mathbf{Y}_\alpha^*, \mathbf{Y}_\alpha} = \Big\langle \mathbf{y}, \Pi \mathbf{z} \Big\rangle_{\mathbf{Y}_\alpha, \mathbf{Y}_\alpha^*} \quad \text{and} \quad \Big\langle \Pi \mathbf{y}, \mathbf{y} \Big\rangle_{\mathbf{Y}_\alpha^*, \mathbf{Y}_\alpha} \geq 0$$

for all $\mathbf{y}$, $\mathbf{z} \in \mathbf{Y}_\alpha$. We shall say that an operator $\Pi \in \mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}_\alpha^*)$ obeys the second equation in (5.9) when

$$\Big\langle \Pi A_\alpha \mathbf{y}, \mathbf{z} \Big\rangle_{\mathbf{Y}_\alpha^*, \mathbf{Y}_\alpha} + \Big\langle A_\alpha^* \Pi \mathbf{y}, \mathbf{z} \Big\rangle_{\mathbf{Y}_\alpha^*, \mathbf{Y}_\alpha} - \big(\widetilde{M}^* B^* \Pi \mathbf{y}, \widetilde{M}^* B^* \Pi \mathbf{z}\big)_{\mathbb{R}^{n_c}} + \Big\langle Q^* Q \mathbf{y}, \mathbf{z} \Big\rangle_{\mathbf{Y}_\alpha^*, \mathbf{Y}_\alpha} = 0,$$

for all $\mathbf{y}$, $\mathbf{z} \in \mathbf{Y}_\alpha$.

To prove that the operator $\Pi \in \mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}_\alpha^*)$, determined in Theorem 5.3, is the unique solution to equation (5.9), it is sufficient to adapt the classical proofs to the case where $\mathbf{Y}_\alpha$ is not identified with its dual (see e.g. [24, 16]).

From Theorem 5.4, it follows that, for all $\mathbf{y}_0 \in \mathbf{Y}_\alpha$, the evolution equation

$$\mathbf{y}' = A_\alpha \mathbf{y} - QB\widetilde{M}\widetilde{M}^* B^* Q^* \Pi \mathbf{y} \quad \text{in } (0, \infty), \quad \mathbf{y}(0) = \mathbf{y}_0, \tag{5.10}$$

admits at least one weak solution belonging to $H^1(0, \infty; \mathbf{Y}_\alpha)$. Moreover, we can check that this solution is unique. Due to Theorem 5.4, this solution is equal to $\mathbf{y}_{\mathbf{y}_0}$ where $(\mathbf{y}_{\mathbf{y}_0}, \mathbf{v}_{\mathbf{y}_0})$ is the solution of problem $(\mathcal{P}^\infty_{\mathbf{y}_0})$. Then, with Theorem 5.3 we have

$$\int_0^\infty |\mathbf{y}_{\mathbf{y}_0}(t)|^2_{\mathbf{Y}_\alpha} \leq C|\mathbf{y}_0|^2 < \infty. \tag{5.11}$$

Let us define the operator $A_\pi \in \mathcal{L}(\mathbf{Y}_\alpha)$ by:

$$A_\pi \mathbf{y} = A_\alpha \mathbf{y} - QB\widetilde{M}\widetilde{M}^* B^* Q^* \Pi \mathbf{y} \quad \text{for all} \quad \mathbf{y} \in \mathbf{Y}_\alpha.$$

Remark 5.3. *The semigroup $(e^{tA_\pi})_{t\geq 0}$ satisfies:*

$$|e^{tA_\pi}\mathbf{f}|_{\mathbf{V}^0_n(\Omega)} \leq Ce^{-\beta t}|\mathbf{f}|_{\mathbf{V}^0_n(\Omega)}, \quad \textit{for all} \quad \mathbf{f} \in \mathbf{Y}_\alpha, \tag{5.12}$$

for some $\beta > 0$.

Proof. The operator A_α belongs to $\mathcal{L}(\mathbf{Y}_\alpha)$. From Remark 3.5, $MB^*Q^*\Pi$ is linear continuous from $\mathbf{Y}_\alpha$ to $\mathcal{U}$. It follows that $B\widetilde{M}\widetilde{M}^* B^* Q^* \Pi$ belongs to $\mathcal{L}(\mathbf{Y}_\alpha, (D(A^*))')$. Using Remark 3.4, A_π belongs to $\mathcal{L}(\mathbf{Y}_\alpha)$. The estimate (5.12) is a consequence of (5.11). □

Let us come back to the equation satisfied by $\mathbf{y}$. From Theorem 5.4, we can give the expression of the feedback control

$$\mathbf{v} = -\widetilde{M}^* B^* Q^* \Pi \mathbf{y}. \tag{5.13}$$

Thus, the linearized equation becomes

$$\mathbf{y}' = A\mathbf{y} - B\widetilde{M}\widetilde{M}^* B^* Q^* \Pi Q \mathbf{y},$$

that is to say, using the definition of $\widetilde{M}$

$$\mathbf{y}' = A\mathbf{y} + BM\mathbf{u},$$

with

$$\mathbf{u} = \sum_{i=1}^{n_c} v_i \zeta_i \quad \text{and } \mathbf{v} = (v_1, \cdots, v_{n_c}) = -\widetilde{M}^* B^* Q^* \Pi Q \mathbf{y}. \tag{5.14}$$

Remark 5.4. *From the definition of $\widetilde{M}^*$, we deduce the expression of the feedback law*

$$v_i(t) = -\Big(\zeta_i, MB^*Q^*\Pi QP\mathbf{y}(t)\Big) = -\int_\Gamma \zeta_i(x) MB^*Q^*\Pi QP\mathbf{y}(t,x)dx, \quad \textit{for all } i = 1 \cdots n_c.$$

6 Stabilizabilition of the Navier-Stokes equations by finite dimensional controllers in Feedback form

In this section, using the expression of the feedback control given by (5.14), we shall consider the system

$$\begin{aligned} &P\mathbf{y}' = AP\mathbf{y} - \textstyle\sum_{i=1}^{n_c} \big(\zeta_i, MB^*Q^*\Pi QP\mathbf{y}\big)BM\zeta_i + PF(\mathbf{y}), \qquad P\mathbf{y}(0) = \mathbf{y}_0, \\ &(I-P)\mathbf{y} = -\textstyle\sum_{i=1}^{n_c} \big(\zeta_i, MB^*Q^*\Pi QP\mathbf{y}\big)(I-P)D_A M\zeta_i \quad \text{ in } (0,\infty), \end{aligned} \tag{6.1}$$

where

$$F(\mathbf{y}) = -e^{-\alpha t}(\mathbf{y}\cdot\nabla)\mathbf{y}.$$

Writing $\mathbf{f} = F(\mathbf{y})$ and $\mathbf{y}$ instead of $P\mathbf{y}$, we first have to study the nonhomogeneous equation

$$\mathbf{y}' = A\mathbf{y} - \sum_{i=1}^{n_c} \big(\zeta_i, MB^*Q^*\Pi Q\mathbf{y}\big)BM\zeta_i + \mathbf{f}, \qquad \mathbf{y}(0) = \mathbf{y}_0.$$

We recall that this equation may be written in the form

$$\mathbf{y}' = A\mathbf{y} - B\widetilde{M}\widetilde{M}^*B^*Q^*\Pi Q\mathbf{y} + \mathbf{f}, \qquad \mathbf{y}(0) = \mathbf{y}_0. \tag{6.2}$$

To study such an equation, we will need the following lemma.

Lemma 6.1. *Let X be a Hilbert space, and suppose that $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup of negative type. Then, the mapping*

$$\begin{array}{ccc} L^2(0,\infty;X)\cap H^1(0,\infty;(D(\mathcal{A}^*))') & \mapsto & L^2(0,\infty;(D(\mathcal{A}^*))')\times[(D(\mathcal{A}^*))',X]_{1/2} \\ \mathbf{y} & \mapsto & (\mathbf{y}'-\mathcal{A}\mathbf{y},\mathbf{y}(0)) \end{array}$$

is an isomorphism.

Proof. The proof is a direct consequence of on [8, Chapter 3, p. 165] and [8, Chapter 1-3, p. 108 and p. 80]. ☐

In the following, it is useful to introduce the notation

$$\mathcal{V}^\theta(\Omega) = D\left((\lambda_0 I - A^*)^{\theta/2}\right) \quad \text{and} \quad \mathcal{V}^{-\theta}(\Omega) = (\mathcal{H}^\theta(\Omega))' \qquad \text{for} \quad 0\le\theta\le 2.$$

6.1 Studying of the linearized problem with a nonhomogeneous source term

In this subsection, we study equation (6.2). We assume that

$$\mathbf{f}\in L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega)), \quad \mathbf{y}_0\in\mathbf{V}^\varepsilon_n(\Omega) \quad \text{ with } 0\le\varepsilon<1/2. \tag{6.3}$$

Lemma 6.2. *Let us suppose that* (6.3) *is satisfied. Then, equation (6.2) admits a unique solution* $\mathbf{y}$ *in* $L^2(0,\infty;\mathbf{V}^0_n(\Omega))$ *which obeys*

$$\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))} \le C\big(|\mathbf{y}_0|_{\mathbf{V}^\varepsilon_n(\Omega)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}\big).$$

Proof. Let us split equation (6.2) as follows:

$$\begin{aligned}
&\mathbf{y}_\alpha' = A_\alpha \mathbf{y}_\alpha - QB\widetilde{M}\widetilde{M}^* B^* Q^* \Pi \mathbf{y}_\alpha + Q\mathbf{f} \quad \text{in } (0,\infty),\\
&\mathbf{y}_\alpha(0) = Q\mathbf{y}_0\\
&\mathbf{y}_{\alpha^-}' = A_{\alpha^-}\mathbf{y}_{\alpha^-} - (I-Q)B\widetilde{M}\widetilde{M}^* B^* Q^* \Pi \mathbf{y}_\alpha + (I-Q)\mathbf{f} \quad \text{in } (0,\infty),\\
&\mathbf{y}_{\alpha^-}(0) = (I-Q)\mathbf{y}_0.
\end{aligned} \tag{6.4}$$

We consider the first equation of this system. We notice that it can be written in the form

$$\mathbf{y}_\alpha{}' = A_\pi \mathbf{y}_\alpha + Q\mathbf{f}, \qquad \mathbf{y}_\alpha(0) = Q\mathbf{y}_0, \tag{6.5}$$

where the operator A_π is defined in section 5. Due to Remark 5.3, the solution to equation (6.5) obeys

$$|\mathbf{y}_\alpha(t)|_{\mathbf{Y}_\alpha} \le C(e^{-\beta t}|Q\mathbf{y}_0|_{\mathbf{Y}_\alpha} + \int_0^t e^{-\beta(t-\tau)}|Q\mathbf{f}(\tau)|_{\mathbf{Y}_\alpha} d\tau),$$

for some $\beta > 0$. It follows that

$$\|\mathbf{y}_\alpha\|_{L^2(0,\infty;\mathbf{Y}_\alpha)} \le C\big(|\mathbf{y}_0|_{\mathbf{V}_n^\varepsilon(\Omega)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}\big). \tag{6.6}$$

Let us consider the second equation of system (6.4). We can remark that $B\widetilde{M}\widetilde{M}^* B^* Q^* \Pi \mathbf{y}_\alpha$ belongs to $L^2(0,\infty;(D(A^*))')$. Finally, with Proposition 3.4 we have

$$\tilde{\mathbf{f}} = -(I-Q)B\widetilde{M}\widetilde{M}^* B^* Q^* \Pi \mathbf{y}_\alpha + (I-Q)\mathbf{f} \in L^2(0,\infty;(D(A_{\alpha^-}^*))').$$

Since $A - \lambda_0 I$ generates an analytic semigroup on $\mathbf{V}_n^0(\Omega)$, the operator A_{α^-}, with domain $D(A_{\alpha^-})$ in $\mathbf{Y}_{\alpha^-}$, generates an analytic semigroup on $\mathbf{Y}_{\alpha^-}$. From [21, Proposition 2.2], A_{α^-} satisfies the spectrum determined growth assumption on $\mathbf{Y}_{\alpha^-}$. Then, A_{α^-} is of negative type, since

$$\sup \operatorname{Re} \sigma(A_{\alpha^-}) < 0.$$

We can notice that $(I-Q)\mathbf{y}_0$ belongs to $\mathbf{Y}_{\alpha^-} \subset [(D(A_{\alpha^-}^*))', \mathbf{Y}_{\alpha^-}]_{1/2}$. Using Lemma 6.1 with $X = \mathbf{Y}_{\alpha^-}$ and $\mathcal{A} = A_{\alpha^-}$, the solution $\mathbf{y}_{\alpha^-}$ belongs to $L^2(0,\infty;\mathbf{Y}_{\alpha^-})$ and we have

$$\begin{aligned}
\|\mathbf{y}_{\alpha^-}\|_{L^2(0,\infty;\mathbf{V}_n^0(\Omega))} &\le C\big(|(I-Q)\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)} + \|\tilde{\mathbf{f}}\|_{L^2(0,\infty;(D(A_{\alpha^-}^*))')}\big)\\
&\le C\big(|\mathbf{y}_0|_{\mathbf{V}_n^\varepsilon(\Omega)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))} + \|\mathbf{y}_\alpha\|_{L^2(0,\infty;\mathbf{Y}_\alpha)}\big).
\end{aligned}$$

Using estimate (6.6) on $\mathbf{y}_\alpha$, we have

$$\|\mathbf{y}_{\alpha^-}\|_{L^2(0,\infty;\mathbf{V}_n^0(\Omega))} \le C\big(|\mathbf{y}_0|_{\mathbf{V}_n^\varepsilon(\Omega)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}\big),$$

and the proof is complete. □

Corollary 6.1. *Let us consider the solution $\mathbf{y}_\alpha$ of (6.5). The control*

$$\mathbf{u} = \sum_{i=1}^{n_c} v_i \zeta_i, \qquad \text{with } \mathbf{v} = (v_1, \cdots, v_{n_c}) = -\widetilde{M}^* B^* Q^* \Pi \mathbf{y}_\alpha$$

belongs to $\mathbf{V}^{2,1}(\Sigma_\infty)$ and

$$\|\mathbf{u}\|_{\mathbf{V}^{2,1}(\Sigma_\infty)} \le C(|\mathbf{y}_0|_{\mathbf{V}_n^\varepsilon(\Omega)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}).$$

Proof. We have already proved in Lemma 6.2 that the solution to equation

$$\mathbf{y}_\alpha{}' = A_\pi \mathbf{y}_\alpha + Q\mathbf{f}, \qquad \mathbf{y}_\alpha(0) = Q\mathbf{y}_0,$$

belongs to $L^2(0,\infty;\mathbf{Y}_\alpha)$. Since $A_\pi \in \mathcal{L}(\mathbf{Y}_\alpha)$, we clearly obtain $A_\pi \mathbf{y}_\alpha \in L^2(0,\infty;\mathbf{Y}_\alpha)$. Moreover, $Q\mathbf{f}$ belongs to $L^2(0,\infty;\mathbf{Y}_\alpha)$ since $\mathbf{f} \in L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))$ and $Q \in \mathcal{L}((D(A^*))', \mathbf{Y}_\alpha)$. Then, we can conclude that $\mathbf{y}_\alpha \in H^1(0,\infty;\mathbf{Y}_\alpha)$, and we have

$$\|\mathbf{y}_\alpha\|_{H^1(0,\infty;\mathbf{Y}_\alpha)} \le C(\|\mathbf{y}_\alpha\|_{L^2(0,\infty;\mathbf{Y}_\alpha)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}).$$

Moreover, $\mathbf{v}$ belongs to $H^1(0,\infty;\mathbb{R}^{n_c})$ since $\widetilde{M}^* B^* Q^* \Pi$ is a linear continuous operator from $H^1(0,\infty;\mathbf{Y}_\alpha)$ to $H^1(0,\infty;\mathbb{R}^{n_c})$. Thus, we have

$$\|\mathbf{v}\|_{H^1(0,\infty;\mathbb{R}^{n_c})} \le C(\|\mathbf{y}_\alpha\|_{L^2(0,\infty;\mathbf{Y}_\alpha)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}).$$

Since Ω is an open subset of class C^4, the space $\mathcal{U}$ is included in $H^{5/2}(\Gamma)$. Then, we have proved that $\mathbf{u} \in \mathbf{V}^{2,1}(\Sigma_\infty)$ and that

$$\|\mathbf{u}\|_{\mathbf{V}^{2,1}(\Sigma_\infty)} \le C(\|\mathbf{y}_\alpha\|_{L^2(0,\infty;\mathbf{Y}_\alpha)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}),$$

and with estimate (6.6), the proof is complete. □

Theorem 6.1. *Let us suppose that* (6.3) *is satisfied. Equation (6.2) admits a unique solution* $\mathbf{y}$ *in the space* $\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)$ *which obeys*

$$\|\mathbf{y}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)} \le C_1\big(|\mathbf{y}_0|_{\mathbf{V}^{\varepsilon}_n(\Omega)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}\big).$$

Proof. From Lemma 6.2, we know that

$$\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))} \le C\big(|\mathbf{y}_0|_{\mathbf{V}^{\varepsilon}_n(\Omega)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}\big).$$

Let us set $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$, where $\mathbf{y}_1$ is solution to

$$\mathbf{y}_1' = (A-\lambda_0)\mathbf{y}_1 + BM\mathbf{u} \quad \text{in } (0,\infty), \quad \mathbf{y}(0) = 0, \tag{6.7}$$

and $\mathbf{y}_2$ is solution to

$$\mathbf{y}_2' = (A-\lambda_0)\mathbf{y}_2 + \lambda_0\mathbf{y} + \mathbf{f} \quad \text{in } (0,\infty), \quad \mathbf{y}(0) = \mathbf{y}_0. \tag{6.8}$$

Due to [18, Lemma 8.3], since $\mathbf{u}$ belongs to $\mathbf{V}^{2,1}(\Sigma_\infty)$, $\mathbf{y}_1$ belongs to $\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)$ for $0 \le \varepsilon < 1/2$, and we have

$$\|\mathbf{y}_1\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon}(Q_\infty)} \le C\|\mathbf{u}\|_{\mathbf{V}^{2,1}(\Sigma_\infty)}.$$

Let us consider the equation on $\mathbf{y}_2$. We can check that for $0 \le \varepsilon < 1/2$, we have:

$$\mathbf{V}^{\varepsilon}_n(\Omega) = [[D(A^*), \mathbf{V}^0_n(\Omega)]'_{1/2}, [\mathbf{V}^0_n(\Omega), D(A)]_{1/2}]_{(1+\varepsilon)/2}.$$

Furthermore,

$$\lambda_0\mathbf{y} + \mathbf{f} \in L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega)) = L^2(0,\infty;[D(A^*), \mathbf{V}^0_n(\Omega)]'_{(1+\varepsilon)/2}).$$

By using an interpolation result, with Lemma 6.1 and [8, chapter 3], it follows that:

$$\mathbf{y}_2 \in L^2(0,\infty;[\mathbf{V}^0_n(\Omega), D(A)]_{(1+\varepsilon)/2}) \cap H^1(0,\infty;[D(A^*), \mathbf{V}^0_n(\Omega)]'_{(1+\varepsilon)/2}).$$

As $[\mathbf{V}_n^0(\Omega), D(A)]_{(1+\varepsilon)/2} \subset \mathbf{V}^{1+\varepsilon}(\Omega)$, we clearly obtain that $\mathbf{y}_2 \in L^2(0,\infty; \mathbf{V}^{1+\varepsilon}(\Omega))$. By interpolation, $\mathbf{y}_2$ belongs to $H^{1/2+\varepsilon/2}(0,\infty; \mathbf{V}_n^0(\Omega))$ and obeys:

$$\|\mathbf{y}_2\|_{V^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)} \le C(|\mathbf{y}_0|_{\mathbf{V}_n^\varepsilon(\Omega)} + \|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}_n^0(\Omega))} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}). \tag{6.9}$$

The solution $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$ belongs to $\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)$ for $0 \le \varepsilon < 1/2$, and we have

$$\|\mathbf{y}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)} \le C(|\mathbf{y}_0|_{\mathbf{V}_n^\varepsilon(\Omega)} + \|\mathbf{u}\|_{\mathbf{V}^{2,1}(\Sigma_\infty)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}).$$

Due to Corollary 6.1, the proof is complete. □

6.2 Stabilization of the two dimensional Navier-Stokes equations

Theorem 6.2. *For all $0 \le \varepsilon < 1/2$, there exist $\mu_0 > 0$ and a nondecreasing function η from $\mathbb{R}^+$ into itself, such that if $\mu \in (0, \mu_0)$ and $|\mathbf{y}_0|_{\mathbf{V}_n^\varepsilon(\Omega)} \le \eta(\mu)$, then equation (6.1) admits a unique solution in the set*

$$D_\mu = \Big\{\mathbf{y} \in \mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty) \mid \|\mathbf{y}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)} \le \mu\Big\}.$$

Moreover $(I-P)\mathbf{y}$ belongs to $H^{1/2+\varepsilon/2}(0,\infty; \mathbf{V}^{1/2}(\Omega)) \cap L^2(0,\infty; \mathbf{V}^{1+\varepsilon}(\Omega))$.

From Theorem 6.2, the solution of (1.3) obeys

$$\|e^{\alpha(\cdot)}\mathbf{y}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)} \le \mu.$$

It remains to show Theorem 6.2. For that, we will need few lemmas.

Lemma 6.3. *If $\mathbf{z}$ belongs to $\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)$ with $0 \le \varepsilon < 1/2$, let us denote by $\mathbf{f} = PF(\mathbf{z})$ then*

$$\|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))} \le C_2\|\mathbf{z}\|^2_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q)}.$$

Proof. This proof can be adapted from [18, Lemma 6.4]. □

Lemma 6.4. *The mapping PF is locally Lipshitz continuous from $\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)$ into the space $L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))$. More precisely, we have*

$$\begin{aligned}&\|PF(\mathbf{z}_1) - PF(\mathbf{z}_2)\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}\\ &\le C_2(\|\mathbf{z}_1\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)} + \|\mathbf{z}_2\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)})\|\mathbf{z}_1 - \mathbf{z}_2\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)}.\end{aligned}$$

for all $\mathbf{z}_1$ and $\mathbf{z}_2 \in \mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)$.

Proof. See [18, Lemma 6.5]. □

Lemma 6.5. *If $\mathbf{y}$ belongs to $\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)$, for some $0 \le \varepsilon < 1/2$, then*

$$\|(I-P)D_A M\mathbf{u}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)} \le C_3\|\mathbf{y}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)}$$

with

$$\mathbf{u} = \sum_{i=1}^{n_c} v_i\zeta_i, \qquad \text{with } \mathbf{v} = (v_1, \cdots, v_{n_c}) = -\widetilde{M}^* B^* Q^* \Pi Q P\mathbf{y}.$$

Proof. Clearly, $P\mathbf{y}$ belongs to $H^{1/2+\varepsilon/2}(0,\infty;\mathbf{V}^0_n(\Omega))$. From Remark (3.5), $MB^*Q^*\Pi QP\mathbf{y}$ belongs to $H^{1/2+\varepsilon/2}(0,\infty;\mathcal{U})$. Finally, we have proved that $\mathbf{v}$ belongs to $H^{1/2+\varepsilon/2}(0,\infty;\mathbb{R}^{n_c})$ and it follows that

$$\|(I-P)D_AM\mathbf{u}\|_{H^{1/2+\varepsilon/2}(0,\infty;\mathbf{V}^0_n(\Omega))} \leq C_3\|\mathbf{y}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)}.$$

Let us show the second estimate. Obviously, $\mathbf{v}$ belongs to $L^2(0,\infty;\mathbb{R}^{n_c})$, and since Ω is an open subset of class C^4, we have already proved that $\mathcal{U}\subset H^{5/2}(\Gamma)$. It clearly follows that $\mathbf{u}$ belongs to $L^2(0,\infty;H^{5/2}(\Gamma))$ and that $(I-P)D_AM\mathbf{u}\in L^2(0,\infty;H^{1+\varepsilon}(\Omega))$. Thus, we have

$$\|(I-P)D_AM\mathbf{u}\|_{L^2(0,\infty;H^{1+\varepsilon}(\Omega))} \leq C_3\|\mathbf{y}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q_\infty)},$$

and the second estimate is proved. □

Proof. **Proof of Theorem 6.2.** The proof follows the lines of [18, Theorem 6.1]. □

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