DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **32**, Number **4**, April **2012**

doi:10.3934/dcds.2012.32.1169

pp. **1169–1208**

ABSTRACT SETTINGS FOR STABILIZATION OF NONLINEAR PARABOLIC SYSTEM WITH A RICCATI-BASED STRATEGY. APPLICATION TO NAVIER-STOKES AND BOUSSINESQ EQUATIONS WITH NEUMANN OR DIRICHLET CONTROL

Mehdi Badra

Laboratoire LMA, UMR CNRS 5142
Université de Pau et des Pays de l'Adour
64013 Pau Cedex, France

(Communicated by Irena Lasiecka)

Abstract. Let $-A : \mathcal{D}(A) \to H$ be the generator of an analytic semigroup and $B : U \to [\mathcal{D}(A^*)]'$ a relatively bounded control operator such that $(A-\sigma, B)$ is stabilizable for some $\sigma > 0$. In this paper, we consider the stabilization of the nonlinear system $y' + Ay + G(y, u) = Bu$ by means of a feedback or a dynamical control u. The control is obtained from the solution to a Riccati equation which is related to a low-gain optimal quadratic minimization problem. We provide a general abstract framework to define exponentially stable solutions which is based on the contruction of Lyapunov functions. We apply such a theory to stabilize, around an unstable stationary solution, the 2D or 3D Navier-Stokes equations with a Neumann control and the 2D or 3D Boussinesq equations with a Dirichlet control.

1. Introduction.

The present paper is dedicated to the question of feedback stabilization of a nonlinear controlled system of the form:

$$y' + Ay + G(y, u) = Bu. \tag{1}$$

In the above setting, A is a closed linear operator defined on a Hilbert space H, B is a linear and possibly unbounded input operator defined on a Hilbert space U and G is a nonlinear mapping obeying $G'(0) = 0$. It is also assumed that $-A$ generates an analytic semigroup on H, that $(A - \sigma, B)$ is (open-loop) stabilizable for $\sigma > 0$ and that $\lambda_0 + A$ has bounded imaginary powers for $\lambda_0 > 0$ large enough.

The main motivation is to obtain general stabilization results that could be applied to nonlinear fluids driven by Navier-Stokes type equations. This kind of problem has been considered by A. Fursikov in the pioneer paper [27] and followed by [28] where a first complete spectral study of the linearized Navier-Stokes operator is done. Another important initial work on the subject is due to Barbu in [12] where for the first time a Riccati operator is used to construct a linear feedback distributed control stabilizing the Navier-Stokes equations. Note that the use of Riccati operator for the closed loop stabilization of nonlinear PDE equations has been known

2000 *Mathematics Subject Classification.* Primary: 93D15, 93C20; Secondary: 76D05, 76D55, 35Q30, 35Q35.

Key words and phrases. Feedback stabilization, Riccati equation, Lyapunov function, Navier-Stokes, Boussinesq, Neumann control, Dirichlet control.

for a long time, see for instance [36]. However, in the context of Navier-Stokes equations it was [12] that set the ground of the framework followed later in joint work [17, 16, 18, 20, 19] and in the recent monogragh [13], and by J.-P. Raymond in [43, 44, 47]. About the stabilizability questions related to [17, 16, 18] for tangential control, see [50]. It is [17] which enlight the main difficulty for the Dirichlet boundary stabilization of the Navier-Stokes equations which resides in the coupling of the high degree of unboundedness of the Dirichlet control operator and of the Navier-Stokes nonlinearity. Two strategies are then possible. The use of a "high-gain" Riccati operator as in [17] which automatically gives the adequate regularity to define a stable solution to the closed loop 3-D Navier-Stokes equations, but which leads to an ill-posed Riccati equation. Or the use of a "low-gain" Riccati operator as in [43, 16] and exploit *a posteriori* the smoothing properties of the corresponding feedback law. This has the advantage to provide a well-posed Riccati equation but it gives nonlinear stabilization results in 2-D only. The reason is that an initial trace compatibility condition between the state and the control is required to define 3-D controlled solution. Such condition is implicitely imposed in the "high-gain" strategy but not in the "low-gain" strategy. That is why time dependent feedback control [44] or dynamical control [9, 7] have also been investigated. Moreover, it is known that the "high-gain" strategy yields a natural Lyapunov function for the controlled Navier-Stokes system which is the value function of the underlying minimizing problem (see [12, 17]), but it is no more the case with the "low-gain" strategy. However, by following the idea developped in [8] for distributed control, we have shown in [10] that it is possible to construct Lyapunov functions for Dirichlet control even in the "low-gain" case.

The main goal of the present paper is to axiomatize the theory developped in [12] and in the related quoted work. For that we follow the same conceptual path in which we include the Lyapunov technic that we have developped in [8, 10] as well as the dynamical control variant introduced in [9, 7]. Our main results are stated in Theorems 2.2 and 2.3 and in Corollary 2 below. For sake of completeness one shall underline that [16] also gives an abstract setting for general nonlinear closed-loop system (not necessarily obtained from a Riccati equation) but without providing a Lyapunov function, and that [11] proposes a similar general abstract theory as well as a Lyapunov and dynamical control approach adapted to the case of finite dimensional control. In particular, the last quoted work provide a very simple criterium for stabilizability guaranteeing the well-posedness of the optimal quadratic control problem from which Riccati type feedback law are constructed, see Remarks 2 and 5 below.

In (1) the trajectory $t \mapsto y(t)$ is the state of the system (the velocity of the fluid) to be stabilized by means of a control function $t \mapsto u(t)$ (a distributed force field, a prescribed boundary value, etc) that we want to express in a feedback form. More precisely, we want to find a linear mapping $\mathfrak{F} : H \to U$ such that the solution to (1) with $u(t) = \mathfrak{F}y(t)$ obeys:

$$\|y(t)\| \leq Ce^{-\sigma t}\|y(0)\|,$$

for some norm $\|\cdot\|$ which has to be determined. The so-called "Riccati" strategy to construct $\mathfrak{F}$ consists in solving an auxiliary optimal quadratic cost problem stated over an infinite time horizon on the linear system:

$$y' + Ay - \sigma y = Bu.$$

The stabilizability of $(A-\sigma, B)$ guarantees the well-posedness of such problem and the resulting optimal control is then given by $u = -B^*\Pi y$ where Π is a linear mapping which is the unique solution to an algebraic Riccati equation. About Riccati theory for infinite dimensional systems see for instance the book of I. Lasiecka and R. Triggiani [38]. In the present paper we consider a feedback control related to a Riccati operator Π obtained from the minimization of the cost function:

$$\int_0^\infty \|Ry(t)\|_Z^2 \mathrm{dt} + \int_0^\infty \|u(t)\|_U^2 \mathrm{dt},$$

where R is a bounded, and boundedly invertible, linear mapping from H into a Hilbert space Z. Note that in the terminology of [16], we are in the particular situation of "low-gain" Riccati operator (R is a bounded operator). We also make the additional assumption that R is bounded from $\mathcal{D}(\widehat{A}^{1/2})$ into $\mathcal{D}(\widehat{A}^{*1/2})$ to guarantee that Π maps H onto $\mathcal{D}(A^*)$. It then ensures that Π is the solution of a Riccati equation that can be written in the following strong formulation:

$$A^*\Pi + \Pi A + \Pi BB^*\Pi = R^*R + 2\sigma\Pi.$$

Thus, if we set $F(y) \overset{\text{def}}{=} G(y, -B^*\Pi y)$ then the nonlinear system (1) for such a feedback control has the following form:

$$y' + Ay + B(B^*\Pi)y + F(y) = 0. \tag{2}$$

Then our goal is to define stable solutions of (2) and to provide a related Lyapunov function. Of course, some assumptions on F should be made, and those will be dictated by the regularity theory of the non homogeneous closed-loop linear system:

$$y' + A_\Pi y = f \quad \text{ where } \quad A_\Pi \overset{\text{def}}{=} A + B(B^*\Pi). \tag{3}$$

In the first part of the present paper (Section 2.2), we underline that the natural spaces in which a trajectory $t \mapsto y(t)$ of (3) is continuous are $H_\Pi^r \overset{\text{def}}{=} \mathcal{D}(A_\Pi^r)$ and $H_\Pi^{-r} \overset{\text{def}}{=} [\mathcal{D}(A_\Pi^{*r})]'$, $r \geq 0$, and that the regularity theory for the linear closed-loop system states that: if $y(0) \in H_\Pi^r$ and $f \in L^2(H_\Pi^{r-1/2})$ then $t \mapsto y(t) \in H_\Pi^r$ is continuous. Then starting from this observation, we prove that if F is a continuous mapping from $H_\Pi^{r-1/2}$ into $H_\Pi^{r+1/2}$ which satisfies some related Lipschitz and boundedness assumptions, then for a prescribed initial datum $y(0) \in H_\Pi^r$ in a neighborhood of the origin there exists a unique continuous trajectory $t \mapsto y(t) \in H_\Pi^r$ of system (2), see Theorem 2.3 below. Moreover, one of the main point proved in the present paper is that the map $\|\cdot\|_r$ given by

$$\|\xi\|_r^2 \overset{\text{def}}{=} \langle A_\Pi^{*r+1/2}\, \Pi\, A_\Pi^{r+1/2}\xi|\xi\rangle_{[H_\Pi^r]', H_\Pi^r}$$

defines a norm on H_Π^r which is a Lyapunov function of (2): for $\|y(0)\|_r$ in a neighborhood of the origin the mapping $t \mapsto \|y(t)\|_r$ is decreasing and

$$\|y(t)\|_r \leq \|y(0)\|_r e^{-\sigma t}.$$

A direct consequence of the above result is that, when dealing with a particular controlled PDE system which can be rewritten in the form (2), the crucial point is to characterize the corresponding spaces of initial data H_Π^r for which the stabilization result is valid. Indeed, since the closed-loop dynamic is contained in the definition of $\mathcal{D}(A_\Pi)$, the elements of $H_\Pi^r \overset{\text{def}}{=} \mathcal{D}(A_\Pi^r)$ may verify a closed-loop compatibility condition for large r and the stabilization result may be irrelevant for such r. For instance, when considering the heat equation or Stokes like systems with Dirichlet feedback boundary control, a trace compatibility condition appears in the definition of H^r_Π when $r \geq 1/4$, see [10]. Then in such situation the relevant space of initial data is H^r_Π for $r < 1/4$. It means that to obtain a satisfactory stabilization result for the nonlinear system (2), the nonlinearity should not be "too strong": to define solutions which are continuous in H^r_Π for $r < 1/4$ the nonlinear mapping F should be continuous from $H^{r-1/2}_\Pi$ into $H^{r+1/2}_\Pi$ for $r < 1/4$. That is the reason why Dirichlet boundary feedback control obtained from a low gain Riccati operator fails to stabilize the 3D Navier-Stokes equations, and it explains why other strategies such that time dependent feedback control [44] or dynamical control [9, 7] have been investigated. That is why the end of the first part of the paper is dedicated to stabilizing dynamical control obtained from an extended Riccati equation (see Section 2.3).

In the second part of the paper (Sections 3 and 4), we give two exemples of applications of the above abstract framework. We obtain new stabilization results for the Navier-Stokes equations with Neumann feedback control, and for the Boussinesq equations with Dirichlet feedback or dynamical control, see Theorems 3.2, 3.4, 4.1 and 4.2 below. Unlike the Dirichlet case treated in [10], while considering the Navier-Stokes equation with Neumann control we obtain a 3-D feedback stabilization result with no specific restriction on the initial datum. Indeed, since the spaces H^r_Π are closed subspaces of $(H^{2r}(\Omega))^3$ the 3-D Navier-Stokes nonlinearity imposes to define a continuous trajectory $t \mapsto y(t) \in H^r_\Pi$ of (2) for and index r greater than $1/4$, which is precisely the value above which a compatibility trace condition appears in the definition of H^r_Π in the case of Dirichlet control, see [10, Cor.6 and Rem.13]. In the case of linear Neumann type control a compatibility trace condition also appears in the definition of H^r_Π but only for $r \geq 3/4$, and it allows to define solutions of the 3-D Navier-Stokes equations for an initial datum in $(H^{2r}(\Omega))^3$ for $r \in [1/4, 3/4)$. An analogous result is also obtained for nonlinear Neumann type control, see section 3.2. Notice that analogous comments apply for Boussinesq equations, and that the last section dealing with Dirichlet control extends the results of [10, 9] to Boussinesq equations. More generally, the abstract framework of the present paper can be applied to many other parabolic system with a nonlinear term of bilinear type, or even of multilinear type, such as power function for instance, see Remark 1 below.

The rest of the paper is organized as follows. Section 2 is dedicated to the construction of a feedback or a dynamical control in a general abstract setting: notations and general definitions are stated in subsection 2.1, subsection 2.2 is devoted to feedback control and subsection 2.3 is devoted to dynamical control. Thus, we apply the abstract framework in the case of Navier-Stokes equations with Neumann control in section 3, and in the case of Boussinesq equations with Dirichlet control in section 4. Some technical results are postponed in an Appendix.

2. **Abstract closed-loop linear and nonlinear system.**

2.1. **Notations.** For a Hilbert space X, we denote by $\|\cdot\|_X$ its norm, we denote by $[X]'$ its dual space and by $\langle\cdot|\cdot\rangle_{[X]',X}$ the $[X]'$-X duality pairing. For two Hilbert spaces X_1 and X_2, we use the notation $X_1 \hookrightarrow X_2$ to say that X_1 is continuously embedded into X_2, for $\alpha \in (0,1)$ we denote by $[X_1, X_2]_\alpha$ the interpolation space obtained from X_1 and X_2 with the complex interpolation method [48, p.55], we denote by $\mathcal{L}(X_1, X_2)$ the space of all bounded linear operators from X_1 into X_2 and we use the shorter expression $\mathcal{L}(X) \stackrel{\text{def}}{=} \mathcal{L}(X,X)$. If L is a closed linear mapping in X, we denote its domain by $\mathcal{D}(L)$ and its adjoint by L^*.

For $0 < T \le \infty$, $L^2(0,T;X)$ is the usual vector-valued Lebesgue space and:

$$W(0,T;X_1,X_2) \stackrel{\text{def}}{=} \left\{ z \in L^2(0,T;X_1) \mid \frac{\mathrm{d}z}{\mathrm{d}t} \in L^2(0,T;X_2) \right\}.$$

When $T = +\infty$ we use the shorter expressions $L^2(X) \stackrel{\text{def}}{=} L^2(0,+\infty;X)$ and $W(X_1,X_2) \stackrel{\text{def}}{=} W(0,+\infty;X_1,X_1)$ and for $\sigma > 0$ we use the notations

$$L^2_\sigma(X) \stackrel{\text{def}}{=} \{z \mid e^{\sigma(\cdot)} z \in L^2(X)\}, \quad W_\sigma(X_1,X_2) \stackrel{\text{def}}{=} \{z \mid e^{\sigma(\cdot)} z \in W(X_1,X_2)\}.$$

More generally, given a function space $\mathcal{X}$ we denote by $\mathcal{X}_\sigma$ the space of function z such that $e^{\sigma(\cdot)}z$ belongs to $\mathcal{X}$. Moreover, we denote by $L^\infty(X)$ (resp. $C_b(X)$) the space of bounded (resp. continuous and bounded) functions of $t \in [0,\infty[$ with values in X, we denote by $L^2_{\mathrm{loc}}(X)$ the space of functions belonging to $L^2(0,T;X)$ for all $T > 0$, and we define $L^\infty_{\mathrm{loc}}(X)$, $W_{\mathrm{loc}}(X_1,X_2)$ analogously.

In the following, $A : \mathcal{D}(A) \subset H \to H$ is a closed linear operator defined in a Hilbert space H and such that $-A$ is the infinitesimal generator of an analytic semigroup $(e^{-At})_{t\ge 0}$ on H. We also assume that $\widehat{A} \stackrel{\text{def}}{=} A + \lambda_0$, $\lambda_0 > 0$, has bounded imaginary powers in order that the following spaces

$$H^r \stackrel{\text{def}}{=} \mathcal{D}(\widehat{A}^r) \ \text{ if } r \ge 0 \quad \text{ and } \quad H^r \stackrel{\text{def}}{=} [\mathcal{D}(\widehat{A}^{*-r})]' \ \text{ if } \ r < 0,$$

and

$$H^r_* \stackrel{\text{def}}{=} \mathcal{D}(\widehat{A}^{*r}) \ \text{ if } r \ge 0 \quad \text{ and } \quad H^r_* \stackrel{\text{def}}{=} [\mathcal{D}(\widehat{A}^{-r})]' \ \text{ if } \ r < 0,$$

obey the following interpolation equalities for $\alpha \in (0,1)$ and $r_2 < r_1$ (see [48]):

$$[H^{r_1}, H^{r_2}]_{1-\alpha} = H^{(1-\alpha)r_1 + \alpha r_2} \quad \text{ and } \quad [H^{r_1}_*, H^{r_2}_*]_{1-\alpha} = H^{(1-\alpha)r_1+\alpha r_2}_*.$$

In the sequel, the letter C denotes a generic positive constant that may change from line to line.

2.2. **Linear and nonlinear systems with feedback control.**

Consider the linear control system

$$y' + Ay = Bu, \tag{4}$$

where B is a linear operator defined on a Hilbert control space U and with values in H^{-1}. We also assume that B is stictly relatively bounded with respect to A:

$$\widehat{A}^{-\gamma} B \in \mathcal{L}(U,H) \quad \text{ for } \quad 0 \le \gamma < 1, \tag{5}$$

and that there is $\sigma > 0$ such that $(A - \sigma, B)$ is (open-loop) stabilizable:

$$\forall y_0 \in H,\ \exists (y,u) \in L^2_\sigma(H) \times L^2_\sigma(U) \text{ satisfying (4) and } y(0) = y_0. \tag{6}$$

According to the infinite dimensional Riccati theory (see for instance [38, Chap. 2]) a way to construct a control function in a feedback form $u(t) = \mathfrak{F}y(t)$, $\mathfrak{F} \in \mathcal{L}(H,U)$, ensuring the exponential decrease:

$$\|y(t)\|_H \le Ce^{-\sigma t}\|y(0)\|_H \quad t \ge 0,$$

is to search u as the solution of an auxiliary quadratic optimal control problem. For that, we introduce the cost functional:

$$\mathcal{J}(y,u) \stackrel{\text{def}}{=} \int_0^\infty \|Ry(t)\|^2_Z \mathrm{d}t + \int_0^\infty \|u(t)\|^2_U \mathrm{d}t. \tag{7}$$

In the above setting, Z is a Hilbert space and $R \in \mathcal{L}(H, Z)$ is boundedly invertible (i.e. $R^{-1} \in \mathcal{L}(Z, H)$). Moreover, we also make the following additional assumption:

$$R^*R \in \mathcal{L}(H^{1/2}, H_*^{1/2}). \tag{8}$$

Thus, for a given $\xi \in H$ we introduce the linear system

$$y' + (A - \sigma)y = Bu \ \in H^{-1}, \quad y(0) = \xi \in H, \tag{9}$$

and we consider the following minimization problem:

$$\inf \left\{ \mathcal{J}(y, u) \ \mid \ (y, u) \in W_{\text{loc}}(H, H^{-1}) \times L^2(U) \ \text{ satisfies } (9) \right\}. \tag{10}$$

Note that (6) guarantees the existence of a pair with finite cost $\mathcal{J}(y, u) < +\infty$. Then the set on which we are looking the infimum is not empty and (10) is well-posed. Then the optimal pair solution to (10) obeys the closed loop system

$$y' + (A - \sigma)y + BB^*\Pi y = 0, \quad y(0) = \xi \tag{11}$$

where Π is the unique nonnegative and self-adjoint operator of $\mathcal{L}(H)$, which belongs to $\mathcal{L}(H, H_*^1)$, solution to the following Riccati equation:

$$A^*\Pi + \Pi A + \Pi BB^*\Pi = R^*R + 2\sigma\Pi. \tag{12}$$

The fact that $\Pi \in \mathcal{L}(H, H_*^1)$ and that (12) is well-posed as an equation in $\mathcal{L}(H)$ is a consequence of assumption (8), see Appendix 5.1 for details.

Next, we recall that $B^*\Pi \in \mathcal{L}(H, U)$ ensures that $A + B(B^*\Pi)$ is a well-defined bounded linear operator from H into H^{-1}:

$$\langle A\xi + B(B^*\Pi)\xi|\zeta\rangle_{H^{-1}, H_*^1} = (\xi|A^*\zeta + (B^*\Pi)^*B^*\zeta)_H \quad \forall(\xi, \zeta) \in H \times H_*^1,$$

and we introduce the linear operator $(\mathcal{D}(A_\Pi), A_\Pi)$ as follows:

$$\mathcal{D}(A_\Pi) \quad = \quad \left\{\xi \in H \ \mid \ A\xi + B(B^*\Pi)\xi \in H\right\}, \tag{13}$$

$$A_\Pi\xi \quad = \quad A\xi + B(B^*\Pi)\xi. \tag{14}$$

Let us state the main properties of A_Π. Although most of the following statements can be found in [38], we give a sketch of the proof for the reader convenience.

Theorem 2.1. *The following results hold.*

1. *The adjoint of* $(\mathcal{D}(A_\Pi), A_\Pi)$ *is given by*

$$\mathcal{D}(A_\Pi^*) = H_*^1 \quad \text{ and } \quad A_\Pi^* = A^* + (B^*\Pi)^*B^*, \tag{15}$$

and the following equality holds:

$$\mathcal{D}(A_\Pi^{*r}) = H_*^r \quad \forall r \in [0, 1]. \tag{16}$$

2. $(\mathcal{D}(A_\Pi), -A_\Pi)$ *is the infinitesimal generator of an analytic and exponentially stable semigroup on* H *with an exponential rate of decrease greater than* $\sigma > 0$*: there exists* $\epsilon > 0$ *such that* $\|e^{-A_\Pi t}\|_{\mathcal{L}(H)} \leq Ce^{-(\sigma+\epsilon)t}$.
3. *For* $\xi \in H$, $y = e^{(\sigma - A_\Pi)(\cdot)}\xi$ *is the optimal state of problem* (10).
4. *The spaces defined by*

$$H_\Pi^r \overset{def}{=} \mathcal{D}(A_\Pi^r) \ \text{ if } r \geq 0 \quad \text{ and } \quad H_\Pi^r \overset{def}{=} [\mathcal{D}(\widehat{A}_\Pi^{*-r})]' \ \text{ if } \ r < 0,$$

obey the interpolation equalities:

$$[H_\Pi^{r_1}, H_\Pi^{r_2}]_{1-\alpha} = H_\Pi^{(1-\alpha)r_1 + \alpha r_2} \quad \forall \alpha \in (0, 1), \quad r_2 < r_1. \tag{17}$$

Moreover, we have:

$$H_\Pi^r = H^r \quad \forall r \in [-1, 0]. \tag{18}$$

5. *The operator A_Π satisfies $\Pi A_\Pi \in \mathcal{L}(H)$ and $A_\Pi^*\Pi \in \mathcal{L}(H)$ and*

$$A_\Pi^*\Pi + \Pi A_\Pi = 2\sigma\Pi + R^*R + \Pi BB^*\Pi. \tag{19}$$

Proof. First, (15) follows by noticing that A_Π is exactly defined as the adjoint of $(\mathcal{D}(A^*), A^* + (B^*\Pi)^*B^*)$. Thus, since $(B^*\Pi)^*B^*$ belongs to $\mathcal{L}(H_*^\gamma, H)$, a perturbation argument ensures that $(\mathcal{D}(A^*), A^* + (B^*\Pi)^*B^*)$ is the infinitesimal generator of an analytic semigroup on H and the analyticity of A_Π follows from a duality argument. Then for $\xi \in H$, $t \mapsto y(t) = e^{(\sigma - A_\Pi)t}\xi$ is the weak solution of

$$y' = (\sigma - A_\Pi)y \in [\mathcal{D}(A_\Pi^*)]', \quad y(0) = \xi, \tag{20}$$

which together with $[\mathcal{D}(A_\Pi^*)]' = H^{-1}$ and (11) means that y is the optimal state of problem (10). As a consequence, $e^{(\sigma - A_\Pi)(\cdot)}\xi$ belongs to $L^2(H)$ for all $\xi \in H$ and the exponential stability of $(e^{(\sigma - A_\Pi)t})_{t\geq 0}$ follows from a well-known result due to Datko. Moreover, the fact that $\widehat{A}$ has bounded imaginary powers together with a perturbation argument [25, Prop. 2.7] ensures that A_Π (and A_Π^* also) has bounded imaginary powers which ensures (17). Next, (16) follows from (15) with the fact that A_Π^* has bounded imaginary powers and (18) is a direct consequence of (16). Finally, from $\mathcal{D}(A_\Pi^*) = H_*^1$ and $\Pi \in \mathcal{L}(H, H_*^1) \cap \mathcal{L}(H^{-1}, H)$ we deduce that $\Pi A_\Pi \in \mathcal{L}(H)$ and $A_\Pi^*\Pi \in \mathcal{L}(H)$, and (19) follows from (12). $\square$

Since by Theorem 2.1 the semigroup $e^{-A_\Pi(\cdot)}$ is analytic and of negative type, maximal regularity applies and the following regularity result for system (3) holds.

Corollary 1. *Let $r \in \mathbb{R}$, $y_0 \in H_\Pi^r$ and $f \in L_\sigma^2(H_\Pi^{r-1/2})$. The solution to*

$$y' + A_\Pi y = f \quad \text{ and } \quad y(0) = y_0, \tag{21}$$

belongs to $W_\sigma(H_\Pi^{r+1/2}, H_\Pi^{r-1/2})$ and obeys the following estimate:

$$\|y\|_{W_\sigma(H_\Pi^{r+1/2}, H_\Pi^{r-1/2})} \leq C(\|f\|_{L_\sigma^2(H_\Pi^{r-1/2})} + \|y_0\|_{H_\Pi^r}). \tag{22}$$

The following Theorem is the main contribution of the present work to the Riccati approach which is recalled above. It gives tools for direct construction of Lyapunov function for linear and nonlinear systems.

Theorem 2.2. *Let $r \in \mathbb{R}$ and set*

$$\Pi_r \stackrel{def}{=} A_\Pi^{*r+1/2}\,\Pi\,A_\Pi^{r+1/2}.$$

1. *The linear operator Π_r is bounded from H_Π^r onto $[H_\Pi^r]'$.*
2. *The following bilinear form defines an inner-product in H_Π^r:*

$$(\xi|\zeta)_r \stackrel{def}{=} \langle \Pi_r\xi|\zeta\rangle_{[H_\Pi^r]', H_\Pi^r} \quad \text{for all } (\xi,\zeta) \in H_\Pi^r \times H_\Pi^r, \tag{23}$$

and the following mapping defines a norm equivalent to $\|\cdot\|_{H_\Pi^r}$:

$$\|\xi\|_r \stackrel{def}{=} \sqrt{(\xi|\xi)_r}. \tag{24}$$

3. *The following mappings:*

$$\lceil|\xi|\rceil_{r+1/2} \stackrel{def}{=} \frac{1}{\sqrt{2}}\left(\|RA_\Pi^{r+1/2}\xi\|_H^2 + \|(B^*\Pi)A_\Pi^{r+1/2}\xi\|_U^2\right)^{1/2} \tag{25}$$

$$\lfloor|\xi|\rfloor_{r-1/2} \stackrel{def}{=} \sup_{\zeta \in H_\Pi^{r+1/2}} \frac{(\xi|\zeta)_r}{\lceil|\zeta|\rceil_{r+1/2}}$$

define equivalent norms in $H_\Pi^{r+1/2}$ *and in* $H_\Pi^{r-1/2}$ *respectively and we have*

$$(A_\Pi\xi|\xi)_r = \sigma\|\xi\|_r^2 + \lceil|\xi|\rceil_{r+1/2}^2 \tag{26}$$

4. For all $\xi \in H_\Pi^r$ *the mapping* $t \longmapsto \|e^{-A_\Pi t}\xi\|_r$ *decreases to* 0 *and obeys:*

$$\|e^{-A_\Pi t}\xi\|_r \le e^{-(\sigma+\beta_r)t}\|\xi\|_r \quad \text{where} \quad \beta_r = \inf_{0\neq\xi\in H_\Pi^{r+1/2}} \frac{\lceil|\xi|\rceil_{r+1/2}^2}{\|\xi\|_r^2},$$

Proof. 1. For $r \in \mathbb{R}$ let us prove that the linear operator Π obeys:

$$A_\Pi^{*r+1/2}\,\Pi\,A_\Pi^{r+1/2} \in \mathcal{L}(H_\Pi^r, [H_\Pi^r]'). \tag{27}$$

Since $\Pi \in \mathcal{L}(H, H_*^1) \cap \mathcal{L}(H^{-1}, H)$, the fact that $\Pi \in \mathcal{L}(H^{-1/2}, H_*^{1/2})$ follows by interpolation. Moreover, from (18) and (16) with $r = 1/2$ we deduce that $A_\Pi^{*1/2} \in \mathcal{L}(H_*^{1/2}, H)$ and $A_\Pi^{1/2} \in \mathcal{L}(H, H^{-1/2})$. Then we have $A_\Pi^{*1/2}\,\Pi\,A_\Pi^{1/2} \in \mathcal{L}(H)$ from which (27) is a direct consequence.

2. Since $\|\xi\|_r = \|A_\Pi^r\xi\|_0$ for all $\xi \in H_\Pi^r$, it suffices to consider the case $r = 0$. First, $\|\cdot\|_0 \le C\|\cdot\|_H$ is a straightforward consequence of $\Pi_0 \in \mathcal{L}(H)$. To prove the converse inequality, let us first pick $\xi \in H$ and set $\zeta = A_\Pi^{1/2}\xi \in H^{-1/2}$. The continuous embedding $W(H, H_\Pi^{-1}) \hookrightarrow C_b(H)$ yields

$$\|\xi\|_H = \|A_\Pi^{-1/2}\zeta\|_H \le C\|e^{-A_\Pi t}\zeta\|_{W(H,H_\Pi^{-1})}$$

which, together with $\frac{\mathrm{d}}{\mathrm{d}t}e^{-A_\Pi t}\zeta = -A_\Pi e^{-A_\Pi t}\zeta$ and $R^{-1} : Z \to H$ bounded, gives

$$\|\xi\|_H \le C\|e^{-A_\Pi t}\zeta\|_{L^2(H)} \le C\|Re^{(\sigma-A_\Pi)t}\zeta\|_{L^2(H)}.$$

Finally, we conclude by observing that:

$$\|Re^{(\sigma-A_\Pi)t}\zeta\|_{L^2(H)}^2 + \|(B^*\Pi)e^{(\sigma-A_\Pi)t}\zeta\|_{L^2(H)}^2 = (\Pi\zeta|\zeta) = (\Pi_0\xi|\xi) = \|\xi\|_0^2.$$

3. First, from (19) we deduce that for all $\xi \in H$:

$$(\Pi A_\Pi\xi|\xi)_H = \sigma(\xi|\Pi\xi)_H + \frac{1}{2}\|R\xi\|_H^2 + \frac{1}{2}\|(B^*\Pi)\xi\|_U^2$$

and by replacing ξ by $A_\Pi^{r+1/2}\xi$ for $\xi \in H_\Pi^{r+1/2}$ in the above equation we obtain:

$$(A_\Pi\xi|\xi)_r = \sigma(\Pi_r\xi|\xi)_H + \frac{1}{2}\|RA_\Pi^{r+1/2}\xi\|_H^2 + \frac{1}{2}\|(B^*\Pi)A_\Pi^{r+1/2}\xi\|_U^2. \tag{28}$$

Then (26) is proved, and from $B^*\Pi \in \mathcal{L}(H, U)$ and the fact that $R : H \to Z$ is an isomorphism we deduce that that $\lceil|\cdot|\rceil_{r+1/2} \sim \|\cdot\|_{H_\Pi^{r+1/2}}$. Next, from $\Pi A_\Pi \in \mathcal{L}(H)$ and $\|A_\Pi^{r+1/2}\cdot\|_H \sim \lceil|\cdot|\rceil_{r+1/2}$ we deduce that

$$(\xi|\zeta)_r = (\Pi A_\Pi A_\Pi^{r-1/2}\xi|A_\Pi^{r+1/2}\zeta) \le C\|\xi\|_{H_\Pi^{r-1/2}}\lceil|\zeta|\rceil_{r+1/2},$$

for all $(\xi, \zeta) \in H_\Pi^{r-1/2} \times H_\Pi^{r+1/2}$, which first gives $\lfloor|\cdot|\rfloor_{r-1/2} \le C\|\cdot\|_{H_\Pi^{r-1/2}}$. To obtain the converse inequality we start by noticing that $\lceil|\cdot|\rceil_{r+1/2} \sim \|A_\Pi^{r+1/2}\cdot\|_H$ implies $\lceil|A_\Pi^{-1}\cdot|\rceil_{r+1/2} \sim \|A_\Pi^{r-1/2}\cdot\|_H$, and by replacing ξ in (28) by $A_\Pi^{-1}\xi$ for $\xi \in H_\Pi^{r-1/2}$ we obtain:

$$(\xi|A_\Pi^{-1}\xi)_r \ge \frac{1}{2}\|RA_\Pi^{r-1/2}\xi\|_H^2 \ge C\|A_\Pi^{r-1/2}\xi\|_H\lceil|A_\Pi^{-1}\xi|\rceil_{r+1/2}.$$

Finally, the conclusion follows from:

$$\lfloor|\xi|\rfloor_{r-1/2} \geq \frac{(\xi|A_\Pi^{-1}\xi)_r}{\lceil|A_\Pi^{-1}\xi|\rceil_{r+1/2}} \geq C\|A_\Pi^{r-1/2}\xi\|_H.$$

4. From $(y' + A_\Pi y|y)_r = 0$ we deduce that

$$\frac{\mathrm{d}}{\mathrm{dt}}\|y(t)\|_r^2 + 2\sigma\|y(t)\|_r^2 + 2\lceil|y(t)|\rceil_{r+1/2}^2 = 0,$$

and the concusion follows from $\beta_r\|\cdot\|_r^2 \leq \lceil|\cdot|\rceil_{r+1/2}^2$. ☐

Next, let us give a characterization of the spaces H_Π^r in the case where $r \in [0,1]$.

Proposition 1. *For all $r \in [0,1]$ the linear operator $I + \widehat{A}^{-1}B(B^*\Pi)$ is an isomorphism from H_Π^r onto H^r and the following characterization hold:*

$$H_\Pi^r = \{\xi \in H \mid \xi + \widehat{A}^{-1}B(B^*\Pi)\xi \in H^r\}, \quad r \in [0,1]. \tag{29}$$

Proof. Let us set $T \overset{\text{def}}{=} I + \widehat{A}^{-1}B(B^*\Pi)$. First, from (13) we deduce that:

$$H_\Pi^1 = \{\xi \in H \mid \xi + \widehat{A}^{-1}B(B^*\Pi)\xi \in H^1\} = \{\xi \in H \mid T\xi \in H^1\}, \tag{30}$$

which also means that $T \in \mathcal{L}(H_\Pi^1, H^1)$. Then since T also belongs to $\mathcal{L}(H)$ an interpolation argument yields $T \in \mathcal{L}(H_\Pi^r, H^r)$ for r $\in [0,1]$. Next, to prove that T is injective in H we suppose that $\xi \in H$ obeys the equality $T\xi = \xi + \widehat{A}^{-1}B(B^*\Pi)\xi = 0$, and by multiplying by $\widehat{A}^*\Pi\xi$ and using (110) applied to (ξ,ξ) we obtain:

$$(\lambda_0 + \sigma)(\xi|\Pi\xi)_H + \frac{1}{2}\|R\xi\|_H^2 + \frac{1}{2}\|B^*\Pi\xi\|_U^2 = 0,$$

which ensures that $\xi = 0$. Thus, to prove that T is surjective, it suffices to remark that $T\xi = f \in H$ is equivalent to $\widehat{A}T\xi = \widehat{A}f \in H^{-1}$ or to $A_\Pi\xi = \widehat{A}f \in H^{-1}$. Then for $f \in H$ the element $\xi = A_\Pi^{-1}\widehat{A}f \in H$ obeys $T\xi = f \in H$. Then we have proved that T is an isomorphism from H onto H. Finally, since (30) exactly means that T^{-1} maps H^1 to H_Π^1, we obtain $T^{-1} \in \mathcal{L}(H^r, H_\Pi^r)$ by interpolation. ☐

We are now in position to state the existence and uniqueness of a stable solution to the following nonlinear system:

$$y' + A_\Pi y + F(y) = 0, \quad y(0) = y_0 \in H_\Pi^r, \tag{31}$$

where the nonlinear mapping $F(\cdot)$ satisfies the following assumptions.

$$\|F(\xi)\|_{H_\Pi^{r-1/2}} \leq C\|\xi\|_{H_\Pi^r}\|\xi\|_{H_\Pi^{r+1/2}}, \tag{32}$$

$$\|F(\xi) - F(\zeta)\|_{H_\Pi^{r-1/2}} \leq C(\|\xi-\zeta\|_{H_\Pi^r}\|\xi\|_{H_\Pi^{r+1/2}} + \|\zeta\|_{H_\Pi^r}\|\xi-\zeta\|_{H_\Pi^{r+1/2}}). \tag{33}$$

Theorem 2.3. *Assume (32)-(33) and $y_0 \in H_\Pi^r$ for $r \in \mathbb{R}$. There exist $\rho > 0$ and $\mu > 0$ such that if $\|y_0\|_r < \mu$ then system (31) admits a solution $y_{y_0} \in W_\sigma(H_\Pi^{r+1/2}, H_\Pi^{r-1/2})$ such that $\|y_{y_0}\|_{W_\sigma(H_\Pi^{r+1/2}, H_\Pi^{r-1/2})} \leq \rho\|y_0\|_r$, which is unique within the class of functions in $L^\infty_{\text{loc}}(H_\Pi^r) \cap L^2_{\text{loc}}(H_\Pi^{r+1/2})$. Moreover, every solution with an initial datum obeying:*

$$\|y_0\|_r < D_r \quad \textit{where} \quad \frac{1}{D_r} = \sup_{0\neq\xi\in H_\Pi^{r+1/2}} \frac{\lfloor|F(\xi)|\rfloor_{r-1/2}}{\|\xi\|_r\lceil|\xi|\rceil_{r+1/2}},$$

is such that $t \longmapsto \|y_{y_0}(t)\|_r$ *is decreasing and we have:*

$$\begin{aligned} \|y_{y_0}(t)\|_r &\leq \|y_0\|_r e^{-\sigma t - \beta_r(1-\|y_0\|_r/D_r)t}, && (34)\\ \int_0^\infty e^{2\sigma t}\lceil|y_{y_0}(t)|\rceil^2_{r+1/2}\mathrm{dt} &\leq \frac{D_r\|y_0\|_r^2}{2(D_r-\|y_0\|_r)}. && (35)\end{aligned}$$

Proof. Let us use the notation $W^r_\sigma \overset{\text{def}}{=} W_\sigma(H_\Pi^{r+1/2}, H_\Pi^{r-1/2})$. In a first step, let us suppose that $\|y_0\|_r < D_r$ and that $y \in L^\infty_{\text{loc}}(H^r_\Pi)\cap L^2_{\text{loc}}(H_\Pi^{r+1/2})$ is a solution of (31) and let us prove that $y \in W^r_\sigma$ as well as estimates (34) and (35). Since (32) ensures that $F(y) \in L^2_{\text{loc}}(H_\Pi^{r-1/2})$, from (31) we obtain $y \in W_{\text{loc}}(H_\Pi^{r+1/2}, H_\Pi^{r-1/2})$, and by $(\cdot|\cdot)_r$-multiplying the first equality in (31) by $y(t)$ and we obtain:

$$\frac{\mathrm{d}}{\mathrm{dt}}\|y(t)\|_r^2 + 2\sigma\|y(t)\|_r^2 + 2(1-\|y(t)\|_r/D_r)\lceil|y(t)|\rceil^2_{r+1/2} \leq 0.$$

Thus, because $\|y_0\|_r < D_r$, the mapping $t \longmapsto \|y(t)\|_r$ is a nonincreasing function lower than D_r and:

$$\frac{\mathrm{d}}{\mathrm{dt}}\|y(t)\|_r^2 + 2\sigma\|y(t)\|_r^2 + 2(1-\|y_0\|_r/D_r)\lceil|y(t)|\rceil^2_{r+1/2} \leq 0.$$

Then (34) follows from $\beta_r\|y(t)\|_r^2 \leq \lceil|y(t)|\rceil^2_{r+1/2}$, and multiplying the above equation by $e^{2\sigma t}$ and integrating over $(0,\infty)$ gives (35). Moreover, since (31), (32) yields:

$$\begin{aligned}\lfloor|(e^{\sigma(\cdot)}y)'(t)|\rfloor_{r-1/2} &\leq K_r\lceil|e^{\sigma t}y(t)|\rceil_{r+1/2} + \|e^{\sigma t}y(t)\|_r\lceil|e^{\sigma t}y(t)|\rceil_{r+1/2}/D_r\\ &\leq (K_r+1)\lceil|e^{\sigma t}y(t)|\rceil_{r+1/2},\end{aligned}$$

where K_r denotes the supremum of $\lfloor|(A_\Pi-\sigma)\xi|\rfloor_{r-1/2}/\lceil|\xi|\rceil_{r+1/2}$ over $0 \neq \xi \in H_\Pi^{r+1/2}$, for some $M_r > 0$ we also have:

$$\|y\|^2_{W^r_\sigma} \leq \frac{M_r}{1-\|y_0\|_r/D_r}\|y_0\|_r^2. \tag{36}$$

In a second step, in order to prove existence and uniqueness of a solution to (31), let us determine $\rho > 0$ and $\mu > 0$ such that for $\|y_0\|_r < \mu$ the mapping:

$$\Psi : z \in W^r_\sigma \to y_z \in W^r_\sigma \quad \text{where} \quad y_z' + A_\Pi y_z + F(z) = 0, \quad y_z(0) = y_0,$$

is a contraction of $B_0 \overset{\text{def}}{=} \{z \in W^r_\sigma \mid \|z\|_{W^r_\sigma} \leq \rho\|y_0\|_r\}$ into itself. First, by combining (22) and (32),(33) we obtain:

$$\begin{aligned}\|\Psi(z)\|_{W^r_\sigma} &\leq C_0(\|z\|^2_{W^r_\sigma} + \|y_0\|_r)\\ \|\Psi(z_1)-\Psi(z_2)\|_{W^r_\sigma} &\leq C_1(\|z_1\|_{W^r_\sigma} + \|z_2\|_{W^r_\sigma})\|z_1-z_2\|_{W^r_\sigma},\end{aligned} \tag{37}$$

and for z, z_1, z_2 in B_0 and $\|y_0\|_r < \mu$, we deduce that:

$$\|\Psi(z)\|_{W^r_\sigma} \leq C_0(\rho^2\mu+1)\|y_0\|_r \quad \text{and} \quad \|\Psi(z_1)-\Psi(z_2)\|_{W^r_\sigma} \leq 2C_1\rho\mu\|z_1-z_2\|_{W^r_\sigma}.$$

Then for any $\rho > 0$ and $\mu > 0$ obeying $\rho > 2C_0$ and $\rho\mu < \min(\frac{1}{2C_1}, \frac{1}{2C_0})$ the mapping Ψ is a contraction of B_0 into itself and (31) admits a unique solution in B_0. Moreover, if we also choose (ρ,μ) such that $\mu < D_r$ and $\rho \geq \sqrt{\frac{M_r}{1-\mu/D_r}}$ then (36) ensures that every solution in $L^\infty_{\text{loc}}(H^r_\Pi)\cap L^2_{\text{loc}}(H_\Pi^{r+1/2})$ belongs to B_0. As a consequence, for such (ρ,μ) the fixed point solution of (31) is unique within the class of functions in $L^\infty_{\text{loc}}(H^r_\Pi)\cap L^2_{\text{loc}}(H_\Pi^{r+1/2})$. $\square$

Remark 1. Notice that (32)-(33) suggests that the nonlinear term is of bilinear type: $F(\xi) = \mathfrak{B}(\xi, \xi)$ where $\mathfrak{B}(\cdot, \cdot)$ is bilinear. It is the main situation when considering Navier-Stokes type nonlinearity. In fact, Theorem 2.3 remains true if (32)-(33) hold only in a neighborhood of the origin in H^{r+1}_{Π}. It means that a nonlinearity obtained from a multilinear mapping can be considered, such as power functions for instance.

Remark 2. When A has compact resolvent the following useful characterization of (6) has been obtained in [11]:

Theorem 2.4. *Suppose that $\widehat{A}^{-1} \in \mathcal{L}(H)$ is compact. Then (6) holds, if and only if, the following unique continuation property holds:*

$$\forall \varepsilon \in \mathcal{D}(A^*),\ \forall \lambda \in \mathbb{C},\ \Re\lambda \le \sigma \quad A^*\varepsilon = \lambda\varepsilon \ \text{ and } \ B^*\varepsilon = 0 \quad \Longrightarrow \quad \varepsilon = 0. \tag{38}$$

2.3. **Linear and nonlinear systems with dynamical control.** Let us now consider (4) with a function $t \mapsto u(t)$ itself solution to a dynamical system of the form:

$$u' + Eu = g \tag{39}$$

where $t \mapsto g(t)$ is now a control function for system (4)-(39). Here we suppose that E is a closed linear operator in U such that $-E$ generates an analytic semigroup on U, and that $\widehat{E} = \lambda_0 + E$ has bounded imaginary powers. For $r \ge 0$ we define the spaces $U^r = \mathcal{D}(\widehat{E}^r)$, $U^r_* = \mathcal{D}(\widehat{E}^{*r})$ and $U^{-r} = [\mathcal{D}(\widehat{E}^{*r})]'$, $U^{-r}_* = [\mathcal{D}(\widehat{E}^r)]'$ and we recall that for all $\alpha \in (0,1)$ and $r_2 < r_1$ the following interpolation equalities hold:

$$[U^{r_1}, U^{r_2}]_{1-\alpha} = U^{(1-\alpha)r_1 + \alpha r_2} \quad \text{and} \quad [U^{r_1}_*, U^{r_2}_*]_{1-\alpha} = U^{(1-\alpha)r_1+\alpha r_2}_*.$$

Thus, we introduce the extended state space $\mathbb{H} \stackrel{\text{def}}{=} H \times U$, we introduce the extended linear operator $\mathbb{A}$ defined in $\mathbb{H}$ by:

$$\mathcal{D}(\mathbb{A}) \stackrel{\text{def}}{=} \left\{ (y,u) \in \mathbb{H} \mid y - \widehat{A}^{-1}Bu \in H^1,\ u \in U^1 \right\}, \quad \mathbb{A}(y,u) \stackrel{\text{def}}{=} (Ay - Bu, Eu),$$

and we set $\widehat{\mathbb{A}} \stackrel{\text{def}}{=} \lambda_0 + \widehat{\mathbb{A}}$. Notice that $\mathcal{D}(\mathbb{A})$ is equipped with the norm

$$\|\widehat{\mathbb{A}}(y,u)\|_{\mathbb{H}} = \|\widehat{A}(y - \widehat{A}^{-1}Bu)\|_H + \|\widehat{E}u\|_U.$$

Next, if we also introduce that canonical projection $\mathbb{B} \in \mathcal{L}(\mathbb{H})$:

$$\mathbb{B}(w,g) \stackrel{\text{def}}{=} (0,g),$$

as well as the new state $Y = (y,u)$ and the new control $V = (w,g)$, system (4)-(39) can be rewritten as follows:

$$Y' + \mathbb{A}Y = \mathbb{B}V. \tag{40}$$

Moreover, we assume that $(\mathbb{A} - \sigma, \mathbb{V})$ is (open-loop) stabilizable:

$$\forall Y_0 \in \mathbb{H},\ \exists (Y,V) \in L^2_\sigma(\mathbb{H}) \times L^2_\sigma(\mathbb{H}) \text{ satisfying (40) and } Y(0) = Y_0. \tag{41}$$

The following Theorem states that $\mathbb{A}$ fits the framework of Section 2.2.

Theorem 2.5. *The following results hold.*

1. *The linear operator $(\mathcal{D}(\mathbb{A}), -\mathbb{A})$ is the infinitesimal generator of an analytic semigroup on $\mathbb{H}$, and $\widehat{\mathbb{A}} \stackrel{def}{=} \lambda_0 + \widehat{\mathbb{A}}$ has bounded imaginary powers. Moreover,*

$$\mathcal{D}(\widehat{\mathbb{A}}^r) \quad = \quad \{\, (y,u) \in \mathbb{H} \mid y - \widehat{A}^{-1}Bu \in H^r,\ u \in U^r \,\} \quad \forall r \in [0,1], \tag{42}$$

and the following norm equivalence holds:

$$\|(y,u)\|_{\mathcal{D}(\widehat{\mathbb{A}}^r)} \sim \|y - \widehat{A}^{-1}Bu\|_{H^r} + \|u\|_{U^r}.$$

2. *The adjoint of* $\mathbb{A}$ *is given by*

$$\mathbb{A}^*(y,u) = (A^*y, -B^*y + E^*u) \quad \textit{and} \quad \mathcal{D}(\mathbb{A}^*) = H^1_* \times U^1_*,$$

and we have

$$\mathcal{D}(\widehat{\mathbb{A}}^{*r}) = H^r_* \times U^r_* \quad \forall r \geq 0. \tag{43}$$

Proof. First, for $\lambda \in \mathbb{C}$ the equality $(\lambda + \mathbb{A})(y,u) = (f,h)$ is equivalent to

$$(\lambda + A)y = Bu + f \quad \text{and} \quad (\lambda + E)u = h. \tag{44}$$

Then the resolvent set of $\mathbb{A}$ is exactly the union of the resolvent sets of A and of E, and the positive halfaxis $\mathbb{R}^+$ is contained in the resolvent set of $\widehat{\mathbb{A}} \overset{\text{def}}{=} \lambda_0 + \mathbb{A}$. Moreover, since for λ in the resolvent set of $\mathbb{A}$, (44) is equivalent to

$$y = (\lambda + A)^{-1}\widehat{A}(\widehat{A}^{-1}B)(\lambda + E)^{-1}h + (\lambda + A)^{-1}f \quad \text{and} \quad u = (\lambda + E)^{-1}h,$$

by using the boundedness of $\widehat{A}^{-1}B$ as well as resolvent estimates related to the analyticity of $(e^{-At})_{t\geq 0}$ and $(e^{-Et})_{t\geq 0}$, we deduce that there is $M > 0$ such that for all $F = (f,h) \in \mathbb{H}$ and for all λ in an open sector of the complex plane, symmetric with respect to the real line and with an opening angle greater than π, we have:

$$\|(\lambda + \mathbb{A})^{-1}F\|_{\mathbb{H}} \leq \frac{M}{|\lambda|}\|F\|_{\mathbb{H}}.$$

The above estimate proves that $-\mathbb{A}$ generates an analytic semigroup. Next, to characterize the adjoint of $\mathbb{A}$ let us show the inclusion $\mathcal{D}(\mathbb{A}^*) \subset \mathcal{D}(A^*) \times \mathcal{D}(E^*)$ which is the only non obvious fact to prove. If $Y = (y,u) \in \mathcal{D}(\mathbb{A}^*)$ then $Z \in \mathcal{D}(\mathbb{A}) \mapsto (Y|\mathbb{A}Z)_{\mathbb{H}}$ is continuous for the topology of $\mathbb{H}$, and by successively remarking that $\mathcal{D}(A) \times \{0\} \subset \mathcal{D}(\mathbb{A})$ and that $\widehat{A}^{-1}B(U) \times U \subset \mathcal{D}(\mathbb{A})$ we deduce that $z \in \mathcal{D}(A) \mapsto (y|Az)_H$ is continuous for the topology of H and that $v \in \mathcal{D}(E) \mapsto (y|Ev)_U$ is continuous for the topology of U. Then it means that $y \in \mathcal{D}(A^*)$ and $u \in \mathcal{D}(E^*)$ and the desired inclusion is proved. Next, let us recall that $\widehat{\mathbb{A}}$ has bounded imaginary powers if and only if the operator defined for $z \in \mathbb{C}$ such that $\Re z > 0$ by

$$\widehat{\mathbb{A}}^{-z} = \frac{\sin \pi z}{\pi}\int_0^{+\infty} t^{-z}(t + \widehat{\mathbb{A}})^{-1}\mathrm{d}t,$$

can be extended to strongly continuous functions from $\{z \in \mathbb{C} \mid \Re z \geq 0\}$ to $\mathcal{L}(\mathbb{H})$. Since an easy calculation gives

$$\widehat{\mathbb{A}}^{-z} = \begin{pmatrix} \widehat{A}^{-z} & -\beta(z) \\ 0 & \widehat{E}^{-z} \end{pmatrix} \quad \text{where} \quad \beta(z) = \frac{\sin \pi z}{\pi}\int_0^{+\infty} t^{-z}(t + \widehat{A})^{-1}B(t + \widehat{E})^{-1}\mathrm{d}t,$$

then to prove that $\widehat{\mathbb{A}}$ has bounded imaginary powers it remains to prove that we can extend $\beta(z)$ to a strongly continuous function from $\{z \in \mathbb{C} \mid \Re(z) \geq 0\}$ in $\mathcal{L}(U,H)$. From classical resolvent estimate we have that $(1+t)^{1-\gamma}\widehat{A}^{\gamma}(t + \widehat{A})^{-1}$ is bounded independently of t, and with $\widehat{A}^{-\gamma}B \in \mathcal{L}(H)$ we can bound the term under the integral and obtain that $\beta(z)$ is bounded independently on $z \in \{z \in \mathbb{C} \mid \Re(z) > 0\}$ in a neighborhood of 0. Then by [32, Ch. 17, Thm. 17.9.1] one can extend $\beta(z)$ to a strongly continuous function from $\{z \in \mathbb{C} \mid \Re(z) \geq 0\}$ in $\mathcal{L}(U,H)$. Finally, since the fact that $\widehat{\mathbb{A}}$, $\widehat{A}$ and $\widehat{E}$ have bounded imaginary powers means that for $r \in (0,1)$ the interpolation equalities $[\mathcal{D}(\widehat{\mathbb{A}}), H]_{1-r} = \mathcal{D}(\widehat{\mathbb{A}}^r)$, $[\mathcal{D}(\widehat{\mathbb{A}}^*), H]_{1-r} =$

$\mathcal{D}(\widehat{\mathbb{A}}^{*r})$, $[\mathcal{D}(\widehat{A}), H]_{1-r} = \mathcal{D}(\widehat{A}^r)$, $[\mathcal{D}(\widehat{A}^*), H]_{1-r} = \mathcal{D}(\widehat{A}^{*r})$, $[\mathcal{D}(\widehat{E}), U]_{1-r} = \mathcal{D}(\widehat{E}^r)$ and $[\mathcal{D}(\widehat{E}^*), U]_{1-r} = \mathcal{D}(\widehat{E}^{*r})$ hold, then equalities (43) and (42) follow with an interpolation argument. Indeed, it suffices to remark that the mapping $(y, u) \mapsto (y - \widehat{A}^{-1}Bu, u)$ is an isomorphism from $\mathbb{H}$ onto $H \times U$ as well as from $\mathcal{D}(\mathbb{A})$ onto $\mathcal{D}(A) \times \mathcal{D}(E)$, and that $(y, u) \mapsto (y, u)$ is an isomorphism from $\mathbb{H}$ onto $H \times U$ as well as from $\mathcal{D}(\mathbb{A}^*)$ onto $\mathcal{D}(A^*) \times \mathcal{D}(E^*)$. □

Next, let us introduce the spaces:

$$\mathbb{H}^r \stackrel{\text{def}}{=} \mathcal{D}(\widehat{\mathbb{A}}^r) \quad \text{and} \quad \mathbb{H}^r_* \stackrel{\text{def}}{=} \mathcal{D}(\widehat{\mathbb{A}}^{*r}) \quad r \geq 0, \tag{45}$$

respectively equipped with norms

$$\|(y,u)\|_{\mathbb{H}^r} \stackrel{\text{def}}{=} \|y - \widehat{A}^{-1}Bu\|_{H^r} + \|u\|_{U^r} \quad \text{and} \quad \|(y,u)\|_{\mathbb{H}^r_*} \stackrel{\text{def}}{=} \|y\|_{H^r_*} + \|u\|_{U^r_*}$$

and let us set $\mathbb{H}^{-r} \stackrel{\text{def}}{=} [\mathcal{D}(\widehat{\mathbb{A}}^{*r})]'$ and $\mathbb{H}^r_* \stackrel{\text{def}}{=} [\mathcal{D}(\widehat{\mathbb{A}}^{-r})]'$ for $r > 0$. According to Theorem 2.5, for $\alpha \in (0,1)$ and $r_2 < r_1$ we have the following interpolation equalities:

$$[\mathbb{H}^{r_1}, \mathbb{H}^{r_2}]_{1-\alpha} = \mathbb{H}^{(1-\alpha)r_1 + \alpha r_2} \quad \text{and} \quad [\mathbb{H}^{r_1}_*, \mathbb{H}^{r_2}_*]_{1-\alpha} = \mathbb{H}^{(1-\alpha)r_1+\alpha r_2}_*.$$

Thus, for a prescribed rate $\sigma > 0$ we introduce

$$Y' + (\mathbb{A} - \sigma)Y = \mathbb{B}V, \tag{46}$$

and we consider the following minimization problem:

$$\inf \left\{ \mathcal{J}(Y,V) \ \mid \ (Y,V) \in W_{\text{loc}}(\mathbb{H}, \mathbb{H}^{-1}) \times L^2(\mathbb{H}) \ \text{ satisfies } (46) \right\}, \tag{47}$$

for a cost functional defined by

$$\mathcal{J}(Y,V) \stackrel{\text{def}}{=} \int_0^\infty \|\mathbb{R}Y(t)\|^2_{\mathbb{Z}} \mathrm{d}t + \int_0^\infty \|V(t)\|^2_{\mathbb{H}} \mathrm{d}t. \tag{48}$$

In the above setting, $\mathbb{Z}$ is a Hilbert space, $\mathbb{R} \in \mathcal{L}(\mathbb{H}, \mathbb{Z})$ is boundedly invertible and such that $\mathbb{R}^*\mathbb{R} \in \mathcal{L}(\mathbb{H}^{1/2}, \mathbb{H}^{*1/2})$ (see Remark 4 below). Thus, (41) guarantee the well-posedness of (47) and the results of section 2.2 apply: for a prescribed rate $\sigma > 0$ there exists a self-adjoint operator $\boldsymbol{\Pi} \in \mathcal{L}(\mathbb{H}, \mathbb{H}^1_*)$ which is the unique solution to the Riccati equation

$$\mathbb{A}^*\boldsymbol{\Pi} + \boldsymbol{\Pi}\mathbb{A} + \boldsymbol{\Pi}\mathbb{B}\mathbb{B}^*\boldsymbol{\Pi} = \mathbb{R}^*\mathbb{R} + 2\sigma\boldsymbol{\Pi}, \tag{49}$$

the closed-loop operator $\mathbb{A}_{\boldsymbol{\Pi}} \stackrel{\text{def}}{=} \mathbb{A} + \mathbb{B}\mathbb{B}^*\boldsymbol{\Pi}$ is such that $-\mathbb{A}_{\boldsymbol{\Pi}}$ generates an analytic and exponentially stable semigroup on $\mathbb{H}$, and the norm of $\mathbb{H}^r_{\boldsymbol{\Pi}} \stackrel{\text{def}}{=} \mathcal{D}(\mathbb{A}^r_{\boldsymbol{\Pi}})$ if $r \geq 0$, or $\mathbb{H}^r_{\boldsymbol{\Pi}} \stackrel{\text{def}}{=} [\mathcal{D}(\mathbb{A}^{*-r}_{\boldsymbol{\Pi}})]'$ if $r < 0$, which is defined by

$$|||\cdot|||^2_r \stackrel{\text{def}}{=} \langle \mathbb{A}^{*r+1/2}_{\boldsymbol{\Pi}} \, \boldsymbol{\Pi} \, \mathbb{A}^{r+1/2}_{\boldsymbol{\Pi}} \cdot | \cdot \rangle_{[\mathbb{H}^r_\Pi]', \mathbb{H}^r_\Pi}$$

is such that for $Y_0 \in \mathbb{H}^r_{\boldsymbol{\Pi}}$ the mapping $t \longmapsto |||e^{-\mathbb{A}_{\boldsymbol{\Pi}} t} Y_0|||_r$ decreases to 0 and obeys:

$$|||e^{-\mathbb{A}_{\boldsymbol{\Pi}} t} Y_0|||_r \leq e^{-\sigma t} |||Y_0|||_r.$$

Moreover, for an extended nonlinear mapping $\mathbb{F}(\cdot)$ satisfying the analogue extended version of (32)-(33) (which is to say with $\mathbb{F}(\cdot)$ instead of $F(\cdot)$ and by replacing the norms of $H^{r-1/2}_\Pi$, H^r_Π and $H^{r+1/2}_\Pi$ by the norms of $\mathbb{H}^{r-1/2}_{\boldsymbol{\Pi}}$, $\mathbb{H}^r_{\boldsymbol{\Pi}}$ and $\mathbb{H}^{r+1/2}_{\boldsymbol{\Pi}}$) then the analogue extended version of Theorem 3.2 applies and guarantees existence and uniqueness of a stable solution to the nonlinear system:

$$Y' + \mathbb{A}_{\boldsymbol{\Pi}} Y + \mathbb{F}(Y) = 0, \quad Y(0) = Y_0, \tag{50}$$

provided that Y_0 is in a neighborhood of the origin of $\mathbb{H}^r_{\mathbf{\Pi}}$. Thus, since $\mathbb{B}$ is a bounded operator in $\mathbb{H}$, the closed-loop operator $\mathbb{A}_{\mathbf{\Pi}}$ is a bounded perturbation of $\mathbb{A}$ and we have $\mathcal{D}(\mathbb{A}_{\mathbf{\Pi}}) = \mathcal{D}(\mathbb{A})$ as well as the following equalities:

$$\mathbb{H}^r_{\mathbf{\Pi}} = \mathbb{H}^r \quad \forall r \in [0,1].$$

Then it means that when $r \in [0,1]$ the stabilization result for system (50) hold for an initial datum $Y_0 \in \mathbb{H}^r$. In the following, we suppose that $r \in [0,1/2]$ and that the nonlinear mapping has the form $\mathbb{F}((y,u)) = (G(y,u),0)$ where $G(\cdot) : \mathbb{H}^{r+1/2} \to H^{r-1/2}$ is a nonlinear mapping satisfying

$$\begin{aligned} \|G(\xi,\theta)\|_{H^{r-1/2}} \leq & C\|(\xi,\theta)\|_{\mathbb{H}^r}\|(\xi,\theta)\|_{\mathbb{H}^{r+1/2}}, \\ \|G(\xi,\theta) - G(\zeta,\tau)\|_{H^{r-1/2}} \leq & C(\|(\xi-\zeta,\theta-\tau)\|_{\mathbb{H}^r}\|(\xi,\theta)\|_{\mathbb{H}^{r+1/2}} \\ & + \|(\zeta,\tau)\|_{\mathbb{H}^r}\|(\xi-\zeta,\theta-\tau)\|_{\mathbb{H}^{r+1/2}}). \end{aligned} \tag{51}$$

Since we have $\mathbb{H}^r_{\mathbf{\Pi}} = \mathbb{H}^r$ and $\mathbb{H}^{-r}_{\mathbf{\Pi}} = \mathbb{H}^{-r} = H^{-r} \times U^{-r}$ it is easily seen that such $\mathbb{F}(\cdot)$ satisfy the extended version of (32)-(33). Moreover, if we introduce the components $\Pi_1 = \Pi_1^* \in \mathcal{L}(H, H^1_*)$, $\Pi_2 \in \mathcal{L}(H, U^1_*)$ and $\Pi_3 = \Pi_3^* \in \mathcal{L}(U, U^1_*)$ of $\mathbf{\Pi}$:

$$\mathbf{\Pi} = \begin{pmatrix} \Pi_1 & \Pi_2^* \\ \Pi_2 & \Pi_3 \end{pmatrix} \tag{52}$$

then (50) can be rewritten as

$$y' + Ay + G(y,u) = Bu, \quad y(0) = y_0 \tag{53}$$

$$u' + Eu + \Pi_2 y + \Pi_3 u = 0, \quad u(0) = u_0, \tag{54}$$

and the following corollary is a consequence of the extended version of Theorem 2.3.

Corollary 2. *Assume (51) and $(y_0,u_0) \in H \times U$ such that $y_0 - \widehat{A}^{-1}Bu_0 \in H^r$ and $u \in U^r$ for $r \in [0,1/2]$. There exist $\rho > 0$ and $\mu > 0$ such that if $|||(y_0,u_0)|||_r < \mu$ then system (53)-(54) admits a solution $(y_{y_0}, u_{y_0}) \in W_\sigma(\mathbb{H}^{r+1/2}, \mathbb{H}^{r-1/2})$ such that $\|(y_{y_0},u_{y_0})\|_{W_\sigma(\mathbb{H}^{r+1/2},\mathbb{H}^{r-1/2})} \leq \rho|||(y_0,u_0)|||_r$, which is unique within the class of functions in $L^\infty_{\text{loc}}(\mathbb{H}^r) \cap L^2_{\text{loc}}(\mathbb{H}^{r+1/2})$. Moreover, there is $D_r > 0$ such that every solution with an initial datum obeying:*

$$|||(y_0,u_0)|||_r < D_r,$$

is such that $t \longmapsto |||(y_{y_0}(t), u_0(t))|||_r$ is decreasing and we have:

$$|||(y_{y_0}(t), u_0(t))|||_r \leq |||(y_0,u_0)|||_r e^{-\sigma t}. \tag{55}$$

Remark 3. Notice that (55) implies the following estimate:

$$\|y(t) - \widehat{A}^{-1}Bu(t)\|_{H^r} + \|u(t)\|_{U^r} \leq Ce^{-\sigma t}(\|y_0 - \widehat{A}^{-1}Bu_0\|_{H^r} + \|u_0\|_{U^r}).$$

Remark 4. Suppose that $B^*\widehat{A}^{*-1} \in \mathcal{L}(H^{1/2}_*, U^{1/2}_*)$, and that for two Hilbert spaces Z_1 and Z_2 we have two bounded linear operators $R \in \mathcal{L}(H, Z_1)$ and $\Theta \in \mathcal{L}(U, Z_2)$, both boundedly invertible and satisfying $R \in \mathcal{L}(H^{1/2}, H^{1/2}_*)$ and $\Theta \in \mathcal{L}(U^{1/2}, U^{1/2}_*)$. Then if we set $\mathbb{Z} \stackrel{\text{def}}{=} Z_1 \times Z_2$ an adequate $\mathbb{R} \in \mathcal{L}(\mathbb{H}, \mathbb{Z})$ can be defined as follows:

$$\mathbb{R}(y,u) \stackrel{\text{def}}{=} (R(y - \widehat{A}^{-1}Bu), \Theta u).$$

Indeed, its bounded invertibility is a direct consequence of the bounded invertibility of R and of Θ, and the fact that $\mathbb{R} \in \mathcal{L}(\mathbb{H}^{1/2}, \mathbb{H}^{1/2}_*)$ follows by remarking that:

$$\mathbb{R}^*\mathbb{R}(y,u) = (R^*R(y - \widehat{A}^{-1}Bu), \Theta^*\Theta u - B^*\widehat{A}^{*-1}R^*R(y - \widehat{A}^{-1}Bu)).$$

Remark 5. We have the following useful sufficient condition for (41)

Theorem 2.6. *Suppose that $\widehat{A}^{-1} \in \mathcal{L}(H)$ is compact, $B^*(\mathcal{D}(A^{*n})) \hookrightarrow U^1$ for some $n \in \mathbb{N}^*$ and that (38) is satisfied. Then (41) holds.*

Proof. First notice that a sufficient condition for (41) is that $(A-\sigma, B)$ is (open-loop) stabilizable by smooth control, i.e. for all $y_0 \in H^1$ there exists $(y, u) \in L^2_\sigma(H) \times W_\sigma(U^1, U)$ solution of (9). Indeed, a control $V = (0, g)$ can then be constructed as follows. Fix $\epsilon > 0$ and first set $g = 0$ on $(0, \epsilon/3)$ so that the analyticity of $(e^{-Et})_{t\geq 0}$ ensures that $u(\epsilon/3) \in U^{1/2}$. Thus, set $g = u' + (E-\sigma)u$ where $u \in W(\epsilon/3, \epsilon, U^1, U)$ is chosen so that it is identically zero on $(2\epsilon/3, \epsilon)$. Since the control g constructed on $(0, \epsilon)$ drives u_0 to 0 at $2\epsilon/3$ and fix u at zero on $(2\epsilon/3, \epsilon)$, then y obeys $y' + Ay = 0$ on $(2\epsilon/3, \epsilon)$ and the analyticity of $(e^{-At})_{t\geq 0}$ guarantees $y(\epsilon) \in H^1$. Finally, we choose $g = u' + (E - \sigma)u$ on $(\epsilon, +\infty)$ where $u \in W(\epsilon, +\infty; U^1, U)$ is given by the (open-loop) stability by smooth control assumption.

Then it remains to show that (open-loop) stability by smooth control is true. According to [11], (38) guarantees the stabilizability of (9) by means of a finite dimensional control of the form

$$u(t) = \sum_{j=1}^{K} u_j(t) B^* v_j$$

where $(u_1, \ldots, u_K) \in (H^1_\sigma(\mathbb{R}))^K$ is solution to a differential equation and v_j, $j = 1, \ldots, K$ is a linear combination of real and imaginary parts of eigenvectors of A^*. Then we have $u \in H^1_\sigma(U)$, and since each eigenvector of A^* belongs to $\mathcal{D}(A^{*n})$, the fact that $B^*(\mathcal{D}(A^{*n})) \hookrightarrow U^1$ guarantees $B^* v_j \in U^1$, $j = 1, \ldots, K$ which also implies $u \in L^2_\sigma(U^1)$ and finally $u \in W_\sigma(U^1, U)$. $\square$

3. Navier-Stokes equations with Neumann feedback control.

3.1. Stabilization by linear Neumann feedback control. Here, Ω is a bounded and connected domain in $\mathbb{R}^d$ for $d = 2$ or $d = 3$, with a boundary $\Gamma = \partial\Omega$ of class $C^{2,1}$. By $L^2(\Omega)$, $L^2(\Gamma)$, $H^{2r}(\Omega)$ and $H^{2r}(\Gamma)$ for $r \geq 0$, we denote the usual Lebesgue and Sobolev spaces of scalar functions in Ω or in Γ, and we write in bold the spaces of vector-valued functions: $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{L}^2(\Gamma) = (L^2(\Gamma))^d$, etc. In the following, $(\cdot|\cdot)$ denotes the usual inner product in $\mathbf{L}^2(\Omega)$. We also introduce the space of free divergence vector fields:

$$\mathbf{V}^{2r}(\Omega) \overset{\text{def}}{=} \Big\{ y \in \mathbf{H}^{2r}(\Omega) \mid \nabla \cdot y = 0 \text{ in } \Omega \Big\}, \qquad r \geq 0,$$

and we will sometimes use the notation $\mathbf{V}^{2r}(\Omega) \overset{\text{def}}{=} [\mathbf{V}^{-2r}(\Omega)]'$ for $r < 0$.

Let $k = 0$ or $k = 1$. For $\nu > 0$, $f_e \in \mathbf{L}^2(\Omega)$ and $b_e \in \mathbf{H}^{1/2}(\Gamma)$ we consider a pair $(z_e, r_e) \in \mathbf{V}^2(\Omega) \times H^1(\Omega)$ solution to the stationary Navier-Stokes equations:

$$\begin{aligned} -\nu\Delta z_e + (z_e \cdot \nabla) z_e + \nabla r_e = f_e \ \text{ in } \ \Omega, \\ \nabla \cdot z_e = 0 \quad \text{in } \ \Omega, \end{aligned} \tag{56}$$

with Neumann boundary condition:

$$\nu \frac{dz_e}{dn} - r_e n - \tfrac{k}{2} (z_e)_n z_e = b_e \ \text{ on } \ \Gamma.$$

Our goal is to stabilize around (z_e, r_e) the unstationary solution (z, r) of:

$$\begin{aligned} \partial_t z - \nu\Delta z + (z \cdot \nabla) z + \nabla r = f_e \ \text{ in } \ Q, \\ \nabla \cdot z = 0 \quad \text{in } \ Q, \end{aligned} \tag{57}$$

by means of a Neumann control of the form:

$$\nu\frac{dz}{dn} - rn - \tfrac{k}{2}z_n z = b_e + \mathfrak{F}(z - z_e) \;\text{ on }\; \Sigma,$$

where $\mathfrak{F} : \mathbf{V}^0(\Omega) \to \mathbf{L}^2(\Gamma)$ is a feedback law obtained from the Riccati approach presented in section 2. Above and below we use the notations $Q \overset{\text{def}}{=} \Omega \times (0, +\infty)$ and $\Sigma \overset{\text{def}}{=} \Gamma \times (0, +\infty)$. Moreover, $n = (n_1, \ldots, n_d)$ denotes the unit interior normal vector field defined on Γ, $y_n n \overset{\text{def}}{=} (y \cdot n)n$ and $y_\tau \overset{\text{def}}{=} y - (y \cdot n)n$ denote the normal and the tangential component of y respectively, and for a scalar function or a vector field y its normal derivative is defined by $\frac{dy}{dn} = \sum_{i=1}^d n_i \frac{dy}{dx_i}$. Let us underline that we have the relations $(\frac{dy}{dn})_\tau = \frac{dy_\tau}{dn}$ and $(\frac{dy}{dn})_n = \frac{dy_n}{dn}$, see [30, App. A] for details.

The nonlinear case $k = 1$ is the natural boundary conditions ensuring energy balance for weak solutions (see [41]) but it corresponds to an abstract formulation (1) with a nonlinear input operator B. On the contrary, the linear case $k = 0$ exactly fits the abstract framework of section 2, but it does not guarantees the energy balance (it does for moving domain or free boundary problems). However, with some additional work when $d = 3$ we can even obtain stabilization results when $k = 1$, see Section 3.2.

We shall underline that the idea of stabilizing Navier-Stokes equations via Riccati feedback goes back to [12]. Here we follow the same idea using Neumann type feedback control. Note also that there has been a lot of recent activities in the context of fluid structure interactions where Neumann boundary conditions for Stokes and Navier-Stokes system are considered. About Neumann type boundary conditions arising in linear and nonlinear fluid structure problem see for instance [3, 4, 6, 5] and [14, 15, 33, 34, 35]. About optimal quadratic fluid structure control problem see for instance [23, 22, 40, 39] or [46] where a Riccati based strategy is used to stabilize the Navier-Stokes equations coupled with a damped EulerBernoulli beam equation. Note also that the stabilization problem we consider here (in particular when $k = 0$) is only a mathematical model problem. Our goal is to illustrate the framework of section 2 for other boundary condition than of Dirichlet type which has already been largely studied. For more realistic problem involving Neumann condition see the above quoted works.

Let $m \in C^2(\Gamma; \mathbb{R}^+)$ be a compactly supported function of Γ which is not identically equal to zero. For a prescribed rate of decrease $\sigma > 0$ we are going to prove that there is a unique nonnegative and self-adjoint linear mapping $\Pi \in \mathcal{L}(\mathbf{V}^0(\Omega))$ belonging to $\mathcal{L}(\mathbf{V}^0(\Omega), \mathbf{V}^2(\Omega))$ and solution to the Riccati equation

$$\int_\Omega \nabla\Pi\xi : \nabla\zeta + \int_\Omega \nabla\xi : \nabla\Pi\zeta + \int_\Gamma m\Pi\xi \cdot m\Pi\zeta = \int_\Omega \xi \cdot \zeta + 2\sigma \int_\Omega \Pi\xi \cdot \zeta, \qquad \forall(\xi, \zeta) \in \mathbf{V}^1(\Omega) \times \mathbf{V}^1(\Omega), \tag{58}$$

such that for z_0 close enough to z_e in $\mathbf{V}^{2r}(\Omega)$ for $r \in [0, 3/4)$, system (57) with

$$\nu\frac{dz}{dn} - rn - \tfrac{k}{2}z_n z = b_e + m^2\Pi(z_e - z) \;\text{ on }\; \Sigma \tag{59}$$

and with initial datum

$$z(0) = z_0 \tag{60}$$

admits a unique solution which satisfies

$$\|z(t) - z_e\|_{\mathbf{V}^{2r}(\Omega)} \le Ce^{-\sigma t}\|z_0 - z_e\|_{\mathbf{V}^{2r}(\Omega)}.$$

To achieve this goal we follow the path patterned in [17, 16, 18] for Dirichlet control that is axiomatized in section 2 for low-gain Riccati operator. For that we need: first to prove that system (57), (59) can be rewritten in the form (31), second to characterize the spaces H^r_Π and finally to apply Theorem 2.3.

First, with $(y, p) = (z - z_e, r - r_e)$, (57)-(59) reduces to

$$\begin{aligned}\partial_t y - \nu\Delta y + (y\cdot\nabla)z_e + (z_e\cdot\nabla)y + (y\cdot\nabla)y + \nabla p = 0 \ \text{ in } \ Q,\\ \nabla\cdot y = 0 \ \text{ in } \ Q,\\ \nu\frac{dy}{dn} - pn - \tfrac{k}{2}(z_e)_n y - \tfrac{k}{2}y_n z_e - \tfrac{k}{2}y_n y + m^2\Pi y = 0 \ \text{ on } \ \Sigma\end{aligned} \tag{61}$$

In the following, we say that $(y, p) \in W_{\text{loc}}(\mathbf{V}^1(\Omega), [\mathbf{V}^1(\Omega)]') \times L^2_{\text{loc}}(L^2(\Omega))$ is a solution of (61), if and only if, $(y, p) \in W_{\text{loc}}(\mathbf{H}^1(\Omega), [\mathbf{H}^1(\Omega)]') \times L^2_{\text{loc}}(L^2(\Omega))$ satisfies the following variational formulation on $(0, +\infty)$:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega y\cdot v + \int_\Omega (\nu\nabla y : \nabla v + (y\cdot\nabla)z_e\cdot v + (z_e\cdot\nabla)y\cdot v - p\nabla\cdot v)\\ &\quad - \int_\Gamma (\tfrac{k}{2}(z_e)_n y + \tfrac{k}{2}y_n z_e - m^2\Pi y)\cdot v = -\int_\Omega (y\cdot\nabla)y\cdot v + \tfrac{k}{2}\int_\Gamma y_n y\cdot v,\\ &\qquad \int_\Omega q\nabla\cdot y = 0,\end{aligned} \tag{62}$$

for all $(v, q) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$. Obviously, if (y, p) is regular an integration by parts shows that a solution to (62) obeys (61) in a classical sense. Then, if we introduce the continuous bilinear form on $\mathbf{H}^1(\Omega)$:

$$\begin{aligned}a(v, w) &\stackrel{\text{def}}{=} \int_\Omega (\nu\nabla v : \nabla w + (v\cdot\nabla)z_e\cdot w + (z_e\cdot\nabla)v\cdot w)\\ &\quad - \int_\Gamma (\tfrac{k}{2}(z_e)_n v + \tfrac{k}{2}v_n z_e)\cdot w \qquad \forall (v, w) \in \mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega),\end{aligned} \tag{63}$$

with corresponding linear operator:

$$\begin{aligned}&\mathcal{D}(A) \stackrel{\text{def}}{=} \left\{v \in \mathbf{V}^1(\Omega) \mid \exists C_v > 0 \text{ s.t. } \forall w \in \mathbf{V}^0(\Omega)\ |a(v, w)| \le C_v\|w\|_{\mathbf{V}^0(\Omega)}\right\}\\ &(Av|w) \stackrel{\text{def}}{=} a(v, w),\end{aligned} \tag{64}$$

if we introduce the input linear operator $B \in \mathcal{L}(\mathbf{L}^2(\Gamma), [\mathbf{V}^1(\Omega)]')$ defined by

$$\langle Bu|v\rangle_{[\mathbf{V}^1(\Omega)]', \mathbf{V}^1(\Omega)} \stackrel{\text{def}}{=} \int_\Gamma mu\cdot v \quad \forall (u, v) \in \mathbf{L}^2(\Gamma)\times\mathbf{V}^1(\Omega), \tag{65}$$

with adjoint $B^* \in \mathcal{L}(\mathbf{V}^1(\Omega), \mathbf{L}^2(\Gamma))$ given by $B^*v = mv|_\Gamma$, and if we define the nonlinear mapping $F : \mathbf{V}^1(\Omega) \to [\mathbf{V}^1(\Omega)]'$:

$$\begin{aligned}\langle F(v)|w\rangle_{[\mathbf{V}^1(\Omega)]', \mathbf{V}^1(\Omega)} = \int_\Omega (v\cdot\nabla)v\cdot w - \tfrac{k}{2}\int_\Gamma v_n v\cdot w\\ \forall (v, w) \in \mathbf{V}^1(\Omega)\times\mathbf{V}^1(\Omega),\end{aligned} \tag{66}$$

the above system can be rewritten as follows:

$$y' + Ay + B(B^*\Pi)y + F(y) = 0. \tag{67}$$

Equation (67) with B given by (65) (or by Proposition 2 below) is the analogue of [16, eq.(5.1a)] (see also [17, eq (2.2)]) for Neumann Riccati feedback control. Note that in order to have a linear input operator B when $k = 1$ we choose to incorporate the nonlinear boundary term in F, which imposes to interpret (67) in

$[\mathbf{V}^{1/2+\varepsilon}(\Omega)]'$, $\varepsilon > 0$. Then a direct application of Theorem 2.3 will only permit to consider solutions in $W(\mathbf{V}^{3/2-\varepsilon}(\Omega), [\mathbf{V}^{1/2+\varepsilon}(\Omega)]')$ which is not sufficient to define solutions when $d = 3$. This difficuly is overcomed in subection 3.2 below.

Let us introduce the notations:

$$\begin{aligned}
\chi(y,p) &\overset{\text{def}}{=} \nu\frac{dy}{dn} - \tfrac{k}{2}y_n z_e - \tfrac{k}{2}(z_e)_n y - pn,\\
\chi_*(y,p) &\overset{\text{def}}{=} \nu\frac{dy}{dn} + \tfrac{2-k}{2}(z_e\cdot y)n + \tfrac{2-k}{2}(z_e)_n y - pn,
\end{aligned} \tag{68}$$

as well as the linear maps $R : \mathbf{V}^0(\Omega) \to L^2(\Omega)$, $S : \mathbf{V}^0(\Omega) \to L^2(\Omega)$ and $T : \mathbf{L}^2(\Gamma) \to \mathbf{L}^2(\Omega)$ defined as follows:

$$p = Ry \quad \text{where} \quad \left\{\begin{array}{rcll} -\Delta p &=& \nabla\cdot[(y\cdot\nabla)z_e + (z_e\cdot\nabla)y] & \text{in } \Omega\\ p &=& \nu\dfrac{dy_n}{dn} - k(z_e)_n y_n & \text{on } \Gamma, \end{array}\right. \tag{69}$$

$$p = Sy \quad \text{where} \quad \left\{\begin{array}{rcll} -\Delta p &=& -\nabla\cdot[(\nabla y)z_e + {}^t(\nabla y)z_e] & \text{in } \Omega\\ p &=& \nu\dfrac{dy_n}{dn} + \tfrac{2-k}{2}(z_e\cdot y) + \tfrac{2-k}{2}(z_e)_n y_n & \text{on } \Gamma, \end{array}\right. \tag{70}$$

$$p = Ty \quad \text{where} \quad \left\{\begin{array}{rcll} -\Delta p &=& 0 & \text{in } \Omega\\ p &=& u_n & \text{on } \Gamma. \end{array}\right. \tag{71}$$

Since $z_e \in \mathbf{V}^2(\Omega)$, (121) in Appendix 5.2.4 with regularity results for Laplace problem with nonhomogeneous Dirichlet condition ensures that $R \in \mathcal{L}(\mathbf{V}^{2r}(\Omega), H^{2r-1}(\Omega))$ and $S \in \mathcal{L}(\mathbf{V}^{2r}(\Omega), H^{2r-1}(\Omega))$ for $r \in]3/4, 1]$, and that $T \in \mathcal{L}(\mathbf{H}^{2r}(\Gamma), H^{2r+1/2}(\Omega))$ for $r \in [0, 3/4]$. In the sequel we will freely use nonlinear estimates (121), (122), (123) in the Appendix 5.2.4 without necessarily recalling them.

In the following, for $k = 0$ and $d = 2, 3$ or $k = 1$ and $d = 2$ we are going to prove that A, B and F fit the framework of section 2 with $H = \mathbf{V}^0(\Omega)$ and $U = \mathbf{L}^2(\Gamma)$. First, from (121), (122) in Appendix 5.2 we deduce the existence of $\lambda_0 >$ such that:

$$a(v,v) + \lambda_0\|v\|^2_{\mathbf{L}^2(\Omega)} \geq \frac{\nu}{2}\|v\|^2_{\mathbf{H}^1(\Omega)} \quad \forall v \in \mathbf{H}^1(\Omega). \tag{72}$$

We set $\widehat{A} \overset{\text{def}}{=} \lambda_0 + A$ and we have the following Theorem.

Theorem 3.1. *The following results hold.*

1. *The operator $(\mathcal{D}(A), -A)$ [resp. $(\mathcal{D}(A^*), -A^*)$] is the infinitesimal generator of an analytic semigroup $(e^{-At})_{t\geq 0}$ [resp. $(e^{-A^*t})_{t\geq 0}$] on $\mathbf{V}^0(\Omega)$ and we have*

$$\left\{\begin{array}{rcl} \mathcal{D}(A) &=& \left\{y \in \mathbf{V}^2(\Omega) \mid \nu\dfrac{dy_\tau}{dn} - \tfrac{k}{2}(z_e)_n y_\tau - \tfrac{k}{2}y_n(z_e)_\tau = 0 \text{ on } \Gamma\right\}\\ Ay &=& -\nu\Delta y + (y\cdot\nabla)z_e + (z_e\cdot\nabla)y + \nabla Ry, \end{array}\right. \tag{73}$$

and:

$$\left\{\begin{array}{rcl} \mathcal{D}(A^*) &=& \left\{y \in \mathbf{V}^2(\Omega) \mid \nu\dfrac{dy_\tau}{dn} + \tfrac{2-k}{2}(z_e)_n y_\tau = 0 \text{ on } \Gamma\right\}\\ A^*y &=& -\nu\Delta y - (\nabla y)z_e - {}^t(\nabla y)z_e + \nabla Sy. \end{array}\right. \tag{74}$$

2. $\widehat{A}$ has bounded imaginary powers and $\widehat{A}$, $\widehat{A}^*$ satisfy:

$$\begin{aligned}
&\mathcal{D}(\widehat{A}^r) = \mathcal{D}(\widehat{A}^{*r}) = \mathbf{V}^{2r}(\Omega) \quad \forall r \in [0,3/4)\,,\\
&\mathcal{D}(\widehat{A}^r) = \left\{ y \in \mathbf{V}^{2r}(\Omega) \;\middle|\; \nu\frac{dy_\tau}{dn} - \tfrac{k}{2}(z_e)_n y_\tau - \tfrac{k}{2} y_n (z_e)_\tau = 0 \text{ on } \Gamma \right\} \forall r \in (3/4, 3/2],\\
&\mathcal{D}(\widehat{A}^{*r}) = \left\{ y \in \mathbf{V}^{2r}(\Omega) \;\middle|\; \nu\frac{dy_\tau}{dn} + \tfrac{2-k}{2}(z_e)_n y_\tau = 0 \text{ on } \Gamma \right\} \quad \forall r \in (3/4, 3/2].
\end{aligned} \tag{75}$$

Proof. The fact that $(\mathcal{D}(A), -A)$ and $(\mathcal{D}(A^*), -A^*)$ are infinitesimal generators of analytic semigroups on $\mathbf{V}^0(\Omega)$ follows from the coercivity condition (72). Let us characterize $\mathcal{D}(A)$. For $y \in \mathcal{D}(A)$ we have $Ay \in \mathbf{V}^0(\Omega)$ and $a(y,v) = \int_\Omega Ay \cdot v$ for all $v \in \mathbf{V}^1(\Omega)$, and by Lemma 5.1 of Appendix 5.2 there is $p \in L^2(\Omega)$ obeying:

$$\begin{aligned}
\nu \int_\Omega \nabla y : \nabla v - \int_\Omega p \nabla\cdot v = &\int_\Omega (Ay - (y\cdot\nabla)z_e - (z_e\cdot\nabla)y)\cdot v\\
&+ \int_\Gamma (\tfrac{k}{2}(z_e)_n y + \tfrac{k}{2} y_n z_e)\cdot v \qquad \forall v \in \mathbf{H}^1(\Omega).
\end{aligned}$$

Thus, because $Ay - (y\cdot\nabla)z_e - (z_e\cdot\nabla)y \in \mathbf{L}^2(\Omega)$ and $\frac{k}{2}(z_e)_n y + \frac{k}{2} y_n z_e \in \mathbf{H}^{1/2}(\Gamma)$, Corollary 4 ensures that $(y,p) \in \mathbf{V}^2(\Omega) \times H^1(\Omega)$, and an integration by parts yields:

$$-\nu\Delta y + \nabla p = Ay - (y\cdot\nabla)z_e - (z_e\cdot\nabla)y \;\text{ in }\; \Omega, \quad \chi(y,p) = 0 \;\text{ on }\; \Gamma.$$

Then separating the tengential and normal parts of above trace condition with the application of the divergence operator to the first above equation give (73).

Next, to characterize $\mathcal{D}(A^*)$, let us denote $V^\sharp$ and $A^\sharp$ the space and the linear operator defined at the right sides of equalities in (74) and let us prove that $\mathcal{D}(A^*) = V^\sharp$ and $A^* = A^\sharp$. First, for all $(y,v) \in \mathcal{D}(A) \times V^\sharp$ an integration by parts gives:

$$\int_\Omega (-\nu\Delta y + (y\cdot\nabla)z_e + (z_e\cdot\nabla)y + \nabla Ry)\cdot v = \int_\Omega y\cdot(-\nu\Delta v - (\nabla v)z_e - {}^t(\nabla v)z_e + \nabla Sv),$$

which means that $(Ay|v) = (y|A^\sharp v)$. Then we have $V^\sharp \subset \mathcal{D}(A^*)$ and the operators $A^\sharp$ and A^* coincide on $V^\sharp$. Conversely, if $v \in \mathcal{D}(A^*)$ then for all $y \in \mathcal{D}(A)$ we have $(Ay|v) = (y|A^*v)$ and then $(\widehat{A}y|v) = (y|\widehat{A}^*v)$. Moreover, according to the Lax-Milgram theorem there is unique $\widetilde{v} \in \mathbf{V}^1(\Omega)$ obeying $\lambda_0(w|\widetilde{v}) + a(w,\widetilde{v}) = (w|\widehat{A}^*v)$ for all $w \in \mathbf{V}^1(\Omega)$. Then by choosing $w = y \in \mathcal{D}(A)$ we obtain that $\widetilde{v}$ obeys $(\widehat{A}y|\widetilde{v}) = (y|\widehat{A}^*v)$ and $(\widehat{A}y|v - \widetilde{v}) = 0$ for all $y \in \mathcal{D}(A)$, which means that $v = \widetilde{v} \in \mathbf{V}^1(\Omega)$ and that $\lambda_0(y|v) + a(y,v) = (y|\widehat{A}^*v)$ for all $y \in \mathcal{D}(A)$. A density argument ensures that this last equality remains valid for all $y \in \mathbf{V}^1(\Omega)$, and by Lemma 5.1 we obtain the exitence of $q \in L^2(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega \nu\nabla y : \nabla v + (y\cdot\nabla)z_e\cdot v + (z_e\cdot\nabla y)\cdot v\\
&\quad - \int_\Gamma (\tfrac{k}{2}(z_e)_n y + \tfrac{k}{2} y_n z_e)\cdot v - \int_\Omega q\nabla\cdot y = \int_\Omega y\cdot A^*v \quad \forall y \in \mathbf{H}^1(\Omega).
\end{aligned} \tag{76}$$

Thus, integrating by parts and using the fact that $y_n z_e\cdot v = (z_e\cdot v)n\cdot y$ give

$$\begin{aligned}
\nu\int_\Omega \nabla v : \nabla y + \int_\Gamma (\tfrac{2-k}{2}(z_e)_n v + \tfrac{2-k}{2}(z_e\cdot v)n)\cdot y - \int_\Omega q\nabla\cdot y&\\
= \int_\Omega (A^*v + (\nabla v)z_e + {}^t(\nabla v)z_e)\cdot y \quad \forall y \in \mathbf{H}^1(\Omega).&
\end{aligned}$$

Moreover, since $(z_e)_n v + (z_e\cdot v)n \in \mathbf{H}^{1/2}(\Gamma)$ and $A^*v + (\nabla v)z_e + {}^t(\nabla v)z_e \in \mathbf{L}^2(\Omega)$, Corollary 4 ensures $(y,p)\in \mathbf{V}^2(\Omega)\times H^1(\Omega)$, and integrating by parts in (76) yields

$$\int_\Omega (-\nu\Delta v - (\nabla v)z_e - {}^t(\nabla v)z_e + \nabla q)\cdot y \\ + \int_\Gamma (\nu\frac{dv}{dn} - qn + \tfrac{2-k}{2}(z_e)_n v + \tfrac{2-k}{2}(z_e\cdot v)n)\cdot y = \int_\Omega A^*v\cdot y \quad \forall y\in \mathbf{H}^1(\Omega),$$

which means:

$$-\nu\Delta v - (\nabla v)z_e - {}^t(\nabla v)z_e + \nabla q = A^*v \ \text{ in } \ \Omega \quad \text{ and } \quad \chi_*(v,q) = 0 \ \text{ on } \ \Gamma.$$

Finally, separating the tengential and normal parts of above trace condition with the application of the divergence operator to the first above equation give (74).

Now, let us prove (75). Since $\widehat{A}$ is closed maximal accretive and boundedly invertible it has bounded imaginary powers and we have $\mathcal{D}(\widehat{A}^r) = [\mathcal{D}(A), \mathbf{V}^0(\Omega)]_{1-r}$ and $\mathcal{D}(\widehat{A}^{*r}) = [\mathcal{D}(A^*), \mathbf{V}^0(\Omega)]_{1-r}$ for all $r\in[0,1]$. Moreover, the same argument applies for the following auxiliary selfadjoint linear operators A_1 and A_2 defined by:

$$\mathcal{D}(A_i) \overset{\text{def}}{=} \left\{ v\in \mathbf{V}^1(\Omega) \mid \exists C_v>0 \text{ s.t. } \forall w\in \mathbf{V}^0(\Omega)\ |a_i(v,w)| \le C_v\|w\|_{\mathbf{V}^0(\Omega)} \right\}$$
$$(A_i v|w) \overset{\text{def}}{=} a_i(v,w), \quad i=1,2$$

where

$$a_1(v,w) \overset{\text{def}}{=} \int_\Omega (\lambda_1 v\cdot w + \nu\nabla v : \nabla w) - \tfrac{k}{2}\int_\Gamma ((z_e)_n v + v_n z_e + ((z_e)_\tau\cdot v_\tau)n)\cdot w$$
$$a_2(v,w) \overset{\text{def}}{=} \int_\Omega (\lambda_2 v\cdot w + \nu\nabla v : \nabla w) + \tfrac{2-k}{2}\int_\Gamma (z_e)_n v\cdot w,$$

where for $i=1,2$, $\lambda_i>0$ are large enough so that $a_i(\cdot,\cdot)$ is coercive. Then we also have $\mathcal{D}(A_i^r) = [\mathcal{D}(A_i), \mathbf{V}^0(\Omega)]_{1-r}$, $i=1,2$, for all $r\in[0,1]$. Moreover, analogously as in the first part of the proof we verify that $\mathcal{D}(A_1) = \mathcal{D}(A)$ and $\mathcal{D}(A_2) = \mathcal{D}(A^*)$. Then we have $\mathcal{D}(\widehat{A}^r) = [\mathcal{D}(A_1), \mathbf{V}^0(\Omega)]_{1-r} = \mathcal{D}(A_1^r)$ and $\mathcal{D}(\widehat{A}^{*r}) = [\mathcal{D}(A_2), \mathbf{V}^0(\Omega)]_{1-r} = \mathcal{D}(A_2^r)$ and the proof of (75) is reduced to the characterization of $\mathcal{D}(A_i^r)$, $i=1,2$. Thus, we remark that the continuity and coercivity of $a_i(\cdot,\cdot)$ with the obvious calculation:

$$\|A_i^{1/2}y\|^2_{\mathbf{V}^0(\Omega)} = (A_i y|y) = a_i(y,y),$$

yield $\mathcal{D}(A_i^{1/2}) = \mathbf{V}^1(\Omega)$, which proves (75) for $r=1/2$. Moreover, since we know from Lemma 5.1 that there exists $p\in L^2(\Omega)$ satisfying:

$$a_i(v,w) - \int_\Omega p\nabla\cdot v = \int_\Omega A_i y\cdot v \quad \forall v\in \mathbf{H}^1(\Omega),$$

and since $y\in\mathcal{D}(A_i^{3/2})$ is equivalent to $A_i y\in \mathcal{D}(A_i^{1/2}) = \mathbf{V}^1(\Omega)$, then equalities (75) for $r=3/2$ are direct consequences of Corollary 4. Then it remains to conclude for $r\in(0,1/2)$ and for $r\in(1/2,3/2)$ with an interpolation argument.

Let us prove the result for $r\in(0,1/2)$. According to [48, Thm.1.15.3.1, p.103] the fact that A_i has bounded imaginary powers guarantees the equalities:

$$\mathcal{D}(A_i^{\alpha a+(1-\alpha)b}) = [\mathcal{D}(A_i^a), \mathcal{D}(A_i^b)]_{1-\alpha}, \quad \forall\alpha\in(0,1),\ a\ge b\ge 0. \tag{77}$$

Then using (77) for $(a,b)=(1/2,0)$ with $\mathcal{D}(A_i^{1/2}) = \mathbf{V}^1(\Omega)$ yields the equality $\mathcal{D}(A_i^r) = [\mathbf{V}^1(\Omega), \mathbf{V}^0(\Omega)]_{1/2-r}$ for all $r\in(0,1/2)$, and (75) for $r\in(0,1/2)$ follows from (114) in Appendix 5.2.1 with $(r_1,r_2)=(1,0)$.

Next, to prove (75) for $r \in (1/2, 3/2)$ we introduce

$$\mathbf{H}^3_1(\Omega) \stackrel{\text{def}}{=} \left\{ y \in \mathbf{H}^3(\Omega) \;\middle|\; \nu \frac{dy_\tau}{dn} - \tfrac{1}{2}(z_e)_n y_\tau - \tfrac{1}{2} v_n (z_e)_\tau = 0 \text{ on } \Gamma \right\},$$

$$\mathbf{H}^3_2(\Omega) \stackrel{\text{def}}{=} \left\{ y \in \mathbf{H}^3(\Omega) \;\middle|\; \nu \frac{dy_\tau}{dn} + \tfrac{1}{2}(z_e)_n y_\tau = 0 \text{ on } \Gamma \right\},$$

and (77) with $(a, b) = (3/2, 1/2)$ gives:

$$\mathcal{D}(A_i^r) = [\mathbf{H}^3_i(\Omega) \cap \mathbf{V}^1(\Omega), \mathbf{V}^1(\Omega)]_{3/2-r}, \quad \forall r \in (1/2, 3/2).$$

Then to conclude it suffices to prove that for $r \in (1/2, 3/2)$:

$$[\mathbf{H}^3_i(\Omega) \cap \mathbf{V}^1(\Omega), \mathbf{V}^1(\Omega)]_{3/2-r} = [\mathbf{H}^3_i(\Omega), \mathbf{H}^1(\Omega)]_{3/2-r} \cap \mathbf{V}^1(\Omega). \tag{78}$$

Indeed, (78) implies $\mathcal{D}(A_i^r) = [\mathbf{H}^3_i(\Omega), \mathbf{H}^1(\Omega)]_{3/2-r} \cap \mathbf{V}^1(\Omega)$ and since:

$$[\mathbf{H}^3_i(\Omega), \mathbf{H}^1(\Omega)]_{3/2-r} = [\mathbf{H}^3_i(\Omega), \mathbf{L}^2(\Omega)]_{1-2r/3} \;\; \forall r \in (1/2, 3/2) \tag{79}$$

equality (75) for $r \in (1/2, 3/2)$ will follows from the characterization of the space $[\mathbf{H}^3_i(\Omega), \mathbf{L}^2(\Omega)]_{1-2r/3}$ given in [48, Thm.4.3.3.1, p.321] or in [29]. Note that (79) is obtained by remarking that $\mathbf{H}^1(\Omega) = [\mathbf{H}^3_i(\Omega), \mathbf{L}^2(\Omega)]_{2/3}$ (see [48, Thm.4.3.3.1, p.321]) and by applying reiteration Theorem [48, Thm.1.10.3.2, p.66] which gives $[\mathbf{H}^3_i(\Omega), \mathbf{H}^1(\Omega)]_{3/2-r} = [\mathbf{H}^3_i(\Omega), [\mathbf{H}^3_i(\Omega), \mathbf{L}^2(\Omega)]_{2/3}]_{3/2-r} = [\mathbf{H}^3_i(\Omega), \mathbf{L}^2(\Omega)]_{1-2r/3}$.

Then it remains to prove (78). For that, we apply [48, Thm.1.17.1.1, p.118] which states that if X and Y are two Banach spaces such that $Y \hookrightarrow X$ and if $\Pr$ is a projection operator on X (with range $\Pr(X)$) which is also continuous from Y into itself then equality $[Y \cap \Pr(X), \Pr(X)]_\theta = [Y, X]_\theta \cap \Pr(X)$ holds. Then to apply [48, Thm.1.17.1.1, p.118] with $X = \mathbf{H}^1(\Omega)$ and $Y = \mathbf{H}^3_i(\Omega)$ the point is to exhibit a projection operator from $\mathbf{H}^1(\Omega)$ onto $\mathbf{V}^1(\Omega)$ which is also continuous on $\mathbf{H}^3_i(\Omega)$ (see also [26]). Finally, from Lemma 5.1 and Corollary 4 we verify that the projection operator defined by

$$\mathbf{P}_i f \stackrel{\text{def}}{=} y \quad \text{ where } \quad a_i(y, v) = a_i(f, v) \quad \forall v \in \mathbf{V}^1(\Omega),$$

obeys the desired property. $\square$

Let us now give an expression of B defined in (65) in terms of the Neumann operator associated with $\lambda_0 + A$. For $u \in \mathbf{L}^2(\Gamma)$ set $Nu = w$ where w obeys:

$$\begin{aligned} \lambda_0 w - \nu \Delta w + (w \cdot \nabla) z_e + (z_e \cdot \nabla) w + \nabla(Rw + Tu) = 0 \quad &\text{on } \Omega, \\ \nabla \cdot w = 0 \quad &\text{in } \Omega, \\ \chi(w, Rw + Tu) = u \quad &\text{on } \Gamma. \end{aligned} \tag{80}$$

Regularity results for problem (80) are given in Appendix 5.2.3. Note that regularity results for analogous Neumann type problems as well as the following proposition can also be found in [40, 39, 22, 23].

Proposition 2. *The following equality holds:*

1. *For all $u \in \mathbf{L}^2(\Omega)$ we have $Bu = \widehat{A} N(mu) \in [\mathbf{V}^1(\Omega)]'$.*
2. *For all $v \in \mathbf{V}^1(\Omega)$ we have $B^* v = mv|_\Gamma$.*
3. *For all $\varepsilon \in]0, 1/4[$ we have $\widehat{A}^{-1/4-\varepsilon} B \in \mathcal{L}(\mathbf{L}^2(\Gamma), \mathbf{V}^0(\Omega))$.*

Proof. The two first statements are straightforward consequences of (117) in Appendix 5.2.3. Thus, from $\mathcal{D}(\widehat{A}^{*1/4+\varepsilon}) = \mathbf{V}^{1/2+2\varepsilon}(\Omega)$ we deduce that $\widehat{A}^{*-1/4-\varepsilon} \in \mathcal{L}(\mathbf{V}^0(\Omega), \mathbf{V}^{1/2+2\varepsilon}(\Omega))$ and with $B^* \in \mathcal{L}(\mathbf{V}^{1/2+2\varepsilon}(\Omega), \mathbf{V}^0(\Omega))$ we obtain $B^* \widehat{A}^{*-1/4-\varepsilon} \in \mathcal{L}(\mathbf{V}^0(\Omega), \mathbf{L}^2(\Gamma))$. Then the third statement follows from a duality argument. $\square$

The following proposition states precisely the equivalence between formulations (67) and (62). Note that unlike the Dirichlet case one recovers a pressure function in (62) which is L^2 in time, see [8, Rem. 1] or [10, Rem. 3].

Proposition 3. *Let $r \in [0, 1]$. Then $y \in W_{\rm loc}(\mathbf{V}^{1+2r}(\Omega), \mathbf{V}^{-1+2r}(\Omega))$ obeys (67), if and only if, there is a unique $p \in L^2_{\rm loc}(H^{2r}(\Omega))$ such that (y, p) satisfies (62) for all $v \in \mathbf{H}^1(\Omega)$.*

Proof. Let us show that (67) implies (62), which is the only non obvious fact to prove. Suppose that $y \in W_{\rm loc}(\mathbf{V}^{1+2r}(\Omega), \mathbf{V}^{-1+2r}(\Omega))$ obeys (67), which means that:

$$\langle y'|v\rangle_{[\mathcal{D}(A^*)]',\mathcal{D}(A^*)} + a(y, v) + \int_\Gamma (m\Pi y - \tfrac{k}{2} y_n y) \cdot v + \int_\Omega (y \cdot \nabla) y \cdot v = 0,$$

for all $v \in \mathcal{D}(A^*)$. Since $y' \in L^2_{\rm loc}([\mathbf{V}^1(\Omega)]')$ and $(y \cdot \nabla y) \in L^2_{\rm loc}([\mathbf{H}^1(\Omega)]')$ the above equality can be extended to $v \in \mathbf{V}^1(\Omega)$ with a density argument and by Lemma 5.1 there exists a unique $p \in L^2_{\rm loc}(L^2(\Omega))$ such that:

$$\langle \mathbf{P}^* y'|v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} + a(y, v) - \int_\Omega p\nabla \cdot v + \int_\Gamma (m\Pi y - \tfrac{k}{2} y_n y) \cdot v + \int_\Omega (y \cdot \nabla) y \cdot v = 0, \tag{81}$$

for all $v \in \mathbf{H}^1(\Omega)$. In the above setting, $\mathbf{P}^* : [\mathbf{V}^1(\Omega)]' \to [\mathbf{H}^1(\Omega)]'$ is the extension of the injection operator $\mathbf{P}^* : \mathbf{V}^0(\Omega) \to \mathbf{L}^2(\Omega)$ which is bounded from $[\mathbf{V}^{1-2r}(\Omega)]'$ into $[\mathbf{H}^{1-2r}(\Omega)]'$ if $r \le 1/2$, or from $\mathbf{V}^{-1+2r}(\Omega)$ into $\mathbf{H}^{-1+2r}(\Omega)$ if $r > 1/2$. Then $y \in W_{\rm loc}(\mathbf{V}^{1+2r}(\Omega), \mathbf{V}^{-1+2r}(\Omega))$ implies $\mathbf{P}^* y' + (y \cdot \nabla) y \in L^2_{\rm loc}([\mathbf{H}^{1-2r}(\Omega)]')$ if $r \le 1/2$, or $\mathbf{P}^* y' + (y \cdot \nabla) y \in L^2_{\rm loc}(\mathbf{H}^{-1+2r}(\Omega))$ if $r > 1/2$, and with $m(\Pi y)|_\Gamma \in L^2_{\rm loc}(\mathbf{H}^{3/2}(\Gamma))$ and $y_n y|_\Gamma \in L^2_{\rm loc}(\mathbf{H}^{2r-1/2}(\Gamma))$ we obtain $p \in L^2_{\rm loc}(H^{2r}(\Omega))$ (apply Corollary 4 with an interpolation argument). Finally, for all $\phi \in C_0^\infty((0, \infty))$ the calculations

$$\begin{aligned}
\int_0^\infty \big\langle \mathbf{P}^* y'(t)\big|v\big\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} \phi(t)\mathrm{d}t &= \Big\langle \int_0^\infty y'(t)\phi(t)\mathrm{d}t \Big| \mathbf{P}v\Big\rangle_{[\mathbf{V}^1(\Omega)]',\mathbf{V}^1(\Omega)} \\
&= \int_\Omega \left(-\int_0^\infty y(t)\phi'(t)\mathrm{d}t \right) \cdot \mathbf{P}v \\
&= -\int_0^\infty \left(\int_\Omega y(t) \cdot \mathbf{P}v \right) \phi'(t)\mathrm{d}t \\
&= -\int_0^\infty \left(\int_\Omega y(t) \cdot v \right) \phi'(t)\mathrm{d}t,
\end{aligned}$$

ensure that $\big\langle \mathbf{P}^* y'(t)\big|v\big\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega y(t) \cdot v$ and (81) together with this last equality guarantee that $(z, r) = (y + z_e, p + r_e)$ obeys the desired equation. □

We are then in the framework of Section 2 with $H = \mathbf{V}^0(\Omega)$, A and $\widehat{A} = \lambda_0 + A$ defined by (64), (72), $U = \mathbf{L}^2(\Gamma)$ and B defined by (65). Indeed, as required, A is the infinitesimal generator of an analytic semigroup on H and has bounded imaginary powers (Theorem 3.1), the mapping B obeys (5) with $\gamma \in (1/4, 1/2)$ (Proposition 2). Then problem (10) with $Z = \mathbf{V}^0(\Omega)$ and R equal to the identity in $\mathbf{V}^0(\Omega)$ guarantees the existence of a self-adjoint operator $\Pi \in \mathcal{L}(\mathbf{V}^0(\Omega), \mathcal{D}(\widehat{A}^*))$ which is the unique solution to the Riccati equation (58). Notice that the well-posedness of such a problem can be obtained from Theorem 2.4 since $\widehat{A}^{-1}$ is compact. Indeed, if we denote $\Gamma_m \stackrel{\text{def}}{=} \mathrm{Supp}(m)$, a sufficient condition for (38) is the following unique continuation property

$$\left\{\begin{aligned} \lambda y - \nu\Delta y - (\nabla y)z_e - {}^t(\nabla y)z_e + \nabla p = 0 &\text{ in } \Omega,\\ \nabla\cdot y = 0 &\text{ in } \Omega,\\ \nu\frac{dy}{dn} - pn = y = 0 &\text{ on } \Gamma_m, \end{aligned}\right. \quad\Longrightarrow\quad y\equiv 0 \text{ in } \Omega,$$

which can be obtained from [28, Thm.4.2] or [49] with a classical extension of the domain procedure. Then to obtain a local feedback stabilization theorem for system (67) it suffices to apply Theorem 2.3. But for such a stabilization result to be relevant, one needs to characterize the spaces H^r_Π introduced in Theorem 2.1.

Proposition 4. *The following equalities holds:*

$$H^r_\Pi = \mathbf{V}^{2r}(\Omega) \quad \forall r\in[0,3/4),$$
$$H^r_\Pi = \{\xi\in\mathbf{V}^{2r}(\Omega) \mid \nu\frac{d\xi_\tau}{dn} - \tfrac{k}{2}(z_e)_n\xi_\tau - \tfrac{k}{2}\xi_n(z_e)_\tau + m(\Pi\xi)_\tau = 0 \text{ on } \Gamma\} \qquad (82)$$
$$\forall r\in(3/4,3/2].$$

Proof. Let us first consider the case $r\in[0,1]$. From $B=\widehat{A}N$ and (29) we obtain:

$$H^r_\Pi = \{\xi\in\mathbf{V}^0(\Omega) \mid \xi + N(m(\Pi\xi)|_\Gamma)\in\mathcal{D}(\widehat{A}^r)\}, \quad \forall r\in[0,1].$$

Thus, for $\xi\in\mathbf{V}^0(\Omega)$ the boundedness of Π from $\mathbf{V}^0(\Omega)$ into $\mathcal{D}(\widehat{A}^*)\hookrightarrow\mathbf{V}^2(\Omega)$ combined with the boundedness of the trace operator yields $m(\Pi\xi)|_\Gamma\in\mathbf{H}^{3/2}(\Gamma)$. Then (119) in Appendix 5.2.3 yields $N(m(\Pi\xi)|_\Gamma)\in\mathbf{V}^3(\Omega)\hookrightarrow\mathbf{V}^{2r}(\Omega)$, and with $\mathcal{D}(\widehat{A}^r)\hookrightarrow\mathbf{V}^{2r}(\Omega)$ we deduce that H^r_Π is the closed subspace of $\mathbf{V}^{2r}(\Omega)$ defined by:

$$H^r_\Pi = \{\xi\in\mathbf{V}^{2r}(\Omega) \mid \xi + N(m(\Pi\xi)|_\Gamma)\in\mathcal{D}(\widehat{A}^r)\}. \qquad (83)$$

If $r\in[0,3/4)$ then $\mathcal{D}(\widehat{A}^r)=\mathbf{V}^{2r}(\Omega)$ and (82) is an obvious consequence of (83).

If $r\in(3/4,1]$, then $\xi + N(m(\Pi\xi)|_\Gamma)\in\mathcal{D}(\widehat{A}^r)$ means that $\xi + N(m(\Pi\xi)|_\Gamma)\in\mathbf{V}^r(\Omega)$ and that ξ, $w\overset{\text{def}}{=}N(m(\Pi\xi)$ obeys:

$$\nu\frac{d\xi_\tau}{dn} - \tfrac{k}{2}(z_e)_n\xi_\tau - \tfrac{k}{2}\xi_n(z_e)_\tau + \left(\nu\frac{dw}{dn} - \tfrac{k}{2}(z_e)_n w - \tfrac{k}{2}w_n z_e\right)_\tau = 0$$

Then we obtain (82) by recalling that $\left(\nu\frac{dw}{dn} - \frac{k}{2}(z_e)_n w - \frac{k}{2}w_n z_e\right)_\tau = m(\Pi\xi)_\tau$ on Γ.

Finally, let us consider the case $r\in(1,3/2]$. Starting from the equality $H^r_\Pi = \{\xi\in H^1_\Pi \mid A_\Pi\xi\in H^{r-1}_\Pi\}$ and using $H^{r-1}_\Pi = \mathbf{V}^{2r-2}(\Omega) = \mathcal{D}(\widehat{A}^{r-1})$ we deduce that $H^r_\Pi = \{\xi\in H^1_\Pi \mid A_\Pi\xi\in\mathcal{D}(\widehat{A}^{r-1})\}$. Thus, since $A_\Pi\xi\in\mathcal{D}(\widehat{A}^{r-1})$ is equivalent to $\xi + N(m(\Pi\xi)|_\Gamma)\in\mathcal{D}(\widehat{A}^r)$, and according to the characterization of H^1_Π given by (83), we deduce that $H^r_\Pi = \{\xi\in\mathbf{V}^1(\Omega) \mid \xi + N(m(\Pi\xi)|_\Gamma)\in\mathcal{D}(\widehat{A}^r)\}$. Then with $N(m(\Pi\xi)|_\Gamma)\in\mathbf{V}^3(\Omega)\hookrightarrow\mathbf{V}^{2r}(\Omega)$ we obtain that (83) remains valid for $r\in(1,3/2]$ and the conclusion follows from the characterization of $\mathcal{D}(\widehat{A}^r)$ in (75) with $r\in(1,3/2]$, analogously as in the case $r\in[0,1]$. ☐

Finally, if $k=0$ then from (121) in Appendix 5.2.4 with $(s_1,s_2,s_3)=(2r,1+2r,1-2r)$ we verify that the nonlinear mapping F defined by (66) fits the assumptions (32)-(33) for $r\in(0,1]$ if $d=2$ and $r\in[\frac{1}{4},1]$ if $d=3$. Moreover, if $k=1$, $d=2$ and $r\in(0,\frac{1}{4})$ then (122) in Appendix 5.2.4 with $(s_1,s_2,s_3)=(2r,1+2r,1-2r)$ guarantees that F defined by (66) fits the assumptions (32)-(33). Then in such cases Theorem 2.3 provides a stabilization result for the abstract system (67), and with Proposition 3 and Proposition 4 we obtain the following stabilization Theorem.

Theorem 3.2. *Assume that $k = 0$, $d = 2$ and $r \in (0,1]\backslash\{\frac{3}{4}\}$ or $k = 0$, $d = 3$ and $r \in [\frac{1}{4},1]\backslash\{\frac{3}{4}\}$ or $k = 1$, $d = 2$ and $r \in (0,\frac{1}{4})$. Let $z_0 \in \{z_e\} + \mathbf{V}^{2r}(\Omega)$ and if $r \in (3/4,1]$ we also assume that*

$$\nu\frac{d(z_0)_\tau}{dn} + m(\Pi z_0)_\tau = (b_e)_\tau + m(\Pi z_e)_\tau. \tag{84}$$

Then there exists $\mu > 0$ such that if $\|z_0 - z_e\|_{\mathbf{V}^{2r}(\Omega)} \le \mu$, system (57), (59), (60) admits a solution $(z,r) \in \{z_e, r_e\} + W_\sigma(\mathbf{V}^{1+2r}(\Omega), \mathbf{V}^{-1+2r}(\Omega)) \times L^2_\sigma(H^{2r}(\Omega))$ which is unique within the class of functions in $\{z_e, r_e\} + L^\infty_{\text{loc}}(\mathbf{V}^{2r}(\Omega)) \cap L^2_{\text{loc}}(\mathbf{V}^{1+2r}(\Omega)) \times L^2_{\text{loc}}(H^{2r}(\Omega))$. Moreover, for all $t \ge 0$ the following estimate holds:

$$\|z(t) - z_e\|_{\mathbf{H}^{2r}(\Omega)} \le Ce^{-\sigma t}\|z_0 - z_e\|_{\mathbf{H}^{2r}(\Omega)}. \tag{85}$$

3.2. **Stabilization by nonlinear Neumann feedback control.** As it has already been pointed out, when $k = 1$ the mapping $w \mapsto k\int_\Gamma v_n v \cdot w$ in (66) does not define an element of $[\mathbf{V}^{2r-1}(\Omega)]'$ for $r \ge \frac{1}{4}$, even for a smooth function v. That is the reason why Theorem 2.3 does not apply directly in the three dimensional case when $k = 1$. In fact, it is not possible to construct a solution y of (67) in $W(H^{r+1/2}_\Pi, H^{r+1/2}_\Pi)$ for $r \ge 1/4$ because we would have $y \in L^2(\mathbf{V}^{2r+1}(\Omega)) \cap H^{r+1/2}(\mathbf{V}^0(\Omega))$ and the following trace equality would be satisfied (in a trace sence if $r > 1/4$):

$$\nu\frac{dy_\tau}{dn} - \tfrac{1}{2}(z_e)_n y_\tau - \tfrac{1}{2}y_n(z_e)_\tau + m(\Pi y)_\tau = 0 \text{ on } \Sigma,$$

which contradicts the expected nonlinear trace condition:

$$\nu\frac{dy_\tau}{dn} - \tfrac{1}{2}(z_e)_n y_\tau - \tfrac{1}{2}y_n(z_e)_\tau + m(\Pi y)_\tau = \tfrac{1}{2}y_n y \text{ on } \Sigma.$$

However, an adaptation of the proof of Theorem 3.2 allows to obtain a stabilization result. For that we need regularity results for the nonhomogeneous problem

$$\begin{aligned} \partial_t y - \nu\Delta y + (y\cdot\nabla)z_e + (z_e\cdot\nabla)y + \nabla q = g \quad &\text{on } \Sigma \\ \nabla\cdot y = 0 \quad &\text{in } Q, \\ y(0) = y_0 \quad &\text{in } \Omega, \\ \chi(y,q) + m\Pi y = u \quad &\text{on } \Sigma, \end{aligned} \tag{86}$$

which are obtained in the same spirit as in [37, Lem.2.13]. Let us set:

$$\mathbf{V}^{2r,r}_\sigma(\Sigma) \overset{\text{def}}{=} \{z \mid e^{\sigma(\cdot)}z \in L^2(\mathbf{H}^{2r}(\Gamma)) \cap H^r(\mathbf{L}^2(\Gamma))\}, \quad r \ge 0.$$

Theorem 3.3. *Assume Ω of class $C^{3,1}$ and $z_e \in \mathbf{V}^3(\Omega)$ and let $\varepsilon \in (0,\frac{1}{2})$. For $r \in [\frac{1}{4} - \frac{\varepsilon}{2}, \frac{5}{4} - \frac{\varepsilon}{2}]$ let $u \in \mathbf{V}^{2r-\frac{1}{2}+\varepsilon, r-\frac{1}{4}+\frac{\varepsilon}{2}}_\sigma(\Sigma)$, $g \in L^2_\sigma(\mathbf{V}^{2r-1}(\Omega))$ and $y_0 \in \mathbf{V}^{2r}(\Omega)$. Moreover, if $r > 3/4$ we also assume that*

$$\nu\frac{dy_\tau}{dn} - \tfrac{1}{2}(z_e)_n y_\tau - \tfrac{1}{2}y_n(z_e)_\tau + m(\Pi y)_\tau = u(0) \text{ on } \Gamma. \tag{87}$$

Then the solution of (86) belongs to $W_\sigma(\mathbf{V}^{1+2r}(\Omega), \mathbf{V}^{-1+2r}(\Omega))$ and satisfies:

$$\|y\|_{W_\sigma(\mathbf{V}^{1+2r}(\Omega),\mathbf{V}^{-1+2r}(\Omega))} \le C(\|u\|_{\mathbf{V}^{2r-\frac{1}{2}+\varepsilon, r-\frac{1}{4}+\frac{\varepsilon}{2}}_\sigma(\Sigma)} + \|g\|_{L^2_\sigma(\mathbf{V}^{2r-1}(\Omega))} + \|y_0\|_{\mathbf{V}^{2r}(\Omega)}). \tag{88}$$

Proof. First, for $y_0 \in \mathbf{V}^{\frac{1}{2}-\varepsilon}(\Omega)$, $g \in L^2_\sigma([\mathbf{V}^{1/2+\varepsilon}(\Omega)]')$ and $u \in \mathbf{L}^2_\sigma(\Sigma)$ we verify that the solution of (86) satisfies

$$y' + A_\Pi y = g + b(u), \quad y(0) = y_0$$

where $b(u) \in L^2_\sigma([\mathbf{V}^{1/2+\varepsilon}(\Omega)]')$ is defined by

$$\langle b(u)|v\rangle \overset{\text{def}}{=} \int_\Gamma u \cdot v.$$

Then (22) yields (88) for $r = \frac{1}{4} - \frac{\varepsilon}{2}$.

Next, for $u \in \mathbf{H}^2(\Gamma)$ set $N_\Pi u = w$ the solution of

$$\begin{aligned} -\nu\Delta w + (w\cdot\nabla)z_e + (z_e\cdot\nabla)w + \nabla q = 0 \quad &\text{on } \Omega,\\ \nabla\cdot w = 0 \quad &\text{in } \Omega,\\ \chi(w,q) + m\Pi w = u \quad &\text{on } \Gamma. \end{aligned} \tag{89}$$

Note that the solution of (89) exists since it coincides with the solution of

$$A_\Pi w = b(u) \in [\mathbf{V}^1(\Omega)]'.$$

Moreover, by noticing that we also have $w = N(u - m(\Pi w)|_\Gamma)$, then (119) in Appendix 5.2 together with the smoothness property of Π yields that

$$N_\Pi \in \mathcal{L}(\mathbf{H}^2(\Gamma), \mathbf{V}^{\frac{7}{2}}(\Omega)) \cap \mathcal{L}(\mathbf{L}^2(\Gamma), \mathbf{V}^{\frac{3}{2}}(\Omega)). \tag{90}$$

Note that to obtain (90) we have used $\Pi \in \mathcal{L}(\mathbf{V}^1(\Omega), \mathbf{V}^{\frac{5}{2}}(\Omega))$ which can be obtained by multiplying (112) in Appendix 5.1 by $\widehat{A}^{*\frac{5}{4}}$ and applying Young inequality together with $e^{-(A_\Pi+\lambda_0+\sigma)(\cdot)} \in \mathcal{L}(\mathbf{V}^1(\Omega), C(\mathbf{V}^1(\Omega)))$ and (8). Thus, for $y_0 \in \mathbf{V}^{\frac{5}{2}-\varepsilon}(\Omega)$, $g \in L^2_\sigma(\mathbf{V}^{\frac{3}{2}-\varepsilon}(\Omega))$ and $u \in \mathbf{V}^{2,1}_\sigma(\Sigma)$ satisfying (87) we have

$$y_0 - N_\Pi u(0) \in \mathcal{D}(A_\Pi^{\frac{5}{4}-\frac{\varepsilon}{2}}) \tag{91}$$

and the Duhamel formula together with an integration by parts gives

$$\begin{aligned} y(t) &= e^{-A_\Pi t}y_0 + \int_0^t e^{-A_\Pi(t-s)}(g(s) + b(u(s)))\mathrm{d}s\\ &= e^{-A_\Pi t}y_0 + \int_0^t e^{-A_\Pi(t-s)}A_\Pi N_\Pi u(s)\mathrm{d}s + \int_0^t e^{-A_\Pi(t-s)}g(s)\mathrm{d}s\\ &= e^{-A_\Pi t}(y_0 - N_\Pi u(0)) + N_\Pi u(t) - \int_0^t e^{-A_\Pi(t-s)}N_\Pi\left(\frac{\mathrm{d}u}{\mathrm{d}s}(s)\right)\mathrm{d}s\\ &\quad + \int_0^t e^{-A_\Pi(t-s)}g(s)\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t}(t) =& -A_\Pi e^{-A_\Pi t}(y_0 - N_\Pi u(0)) + A_\Pi\int_0^t e^{-A_\Pi(t-s)}N_\Pi\left(\frac{\mathrm{d}u}{\mathrm{d}s}(s)\right)\mathrm{d}s,\\ & - A_\Pi\int_0^t e^{-A_\Pi(t-s)}g(s)\mathrm{d}s + g(t). \end{aligned}$$

Then with (91), (90) and the Young inequality we obtain (88) for $r = \frac{5}{4} - \frac{\varepsilon}{2}$ and we conclude with an interpolation argument. $\square$

Theorem 3.4. *Assume Ω of class $C^{3,1}$ and $z_e \in \mathbf{V}^3(\Omega)$. Let $k = 1$ and assume $r \in (0,1]\backslash\{\frac{3}{4}\}$ if $d = 2$ or $r \in [\frac{1}{4},1]\backslash\{\frac{3}{4}\}$ if $d = 3$. Let $z_0 \in \{z_e\} + \mathbf{V}^{2r}(\Omega)$ and if $r \in (3/4, 1]$ we also assume that*

$$\nu\frac{d(z_0)_\tau}{dn} - \tfrac{1}{2}(z_0)_n(z_0)_\tau + m(\Pi z_0)_\tau = (b_e)_\tau + m(\Pi z_e)_\tau. \tag{92}$$

Then conclusions of Theorem 3.2 hold.

Proof. First, note that for $r \in [\frac{1}{4} - \frac{\varepsilon}{2}, \frac{5}{4} - \frac{\varepsilon}{2}]$ and $z_0 \in \{z_e\} + \mathbf{V}^{2r}(\Omega)$ obeying (92) if $r \in (3/4, 1]$ we have that $y_0 \overset{\text{def}}{=} z_0 - z_e \in \mathbf{V}^{2r}(\Omega)$ obeys $y_0 - N_\Pi(b((y_0)_n y_0)) \in \mathcal{D}(A_\Pi^r)$ and with Theorem 3.3 we verify that the set

$$\begin{aligned} W_\sigma^{2r} \overset{\text{def}}{=} \{z \in W_\sigma(\mathbf{V}^{1+2r}(\Omega), \mathbf{V}^{-1+2r}(\Omega)) \mid \\ y_0 - N_\Pi(b((z(0))_n z(0))) \in \mathcal{D}(A_\Pi^r)\} \end{aligned} \tag{93}$$

is stable by the mapping $z \mapsto y_z$ where y_z is the solution of

$$y' + A_\Pi y = -F(z) + b(z_n z), \quad y(0) = y_0.$$

Moreover, we verify that

$$\|(z\cdot\nabla)z\|_{L^2_\sigma(\mathbf{V}^{2r-1}(\Omega))} + \|z_n z\|_{\mathbf{V}_\sigma^{2r-\frac{1}{2}+\varepsilon, r-\frac{1}{4}+\frac{\varepsilon}{2}}(\Sigma)} \le C\|z\|^2_{W_\sigma^{2r}},$$

which together with (88) gives:

$$\|y_z\|_{W_\sigma^{2r}} \le C(\|z\|^2_{W_\sigma^{2r}} + \|y_0\|_{\mathbf{V}^{2r}(\Omega)}).$$

Analogously, we prove that

$$\|y_{z_1} - y_{z_2}\|_{W_\sigma^{2r}} \le C\|z_1 - z_2\|^2_{W_\sigma^{2r}}.$$

and we concude with a fixed-point argument as in the proof of Theorem 2.3. Finally, (85) follows from $W_\sigma^{2r} \hookrightarrow C_{\sigma,b}(\mathbf{V}^{2r}(\Omega)) \overset{\text{def}}{=} \{z \mid e^{\sigma(\cdot)}z \in C_b(\mathbf{V}^{2r}(\Omega))\}$. □

4. Boussinesq equations with feedback or dynamical Dirichlet control.

Here, we still consider an open subset Ω of $\mathbb{R}^d$ with $d = 2$ or $d = 3$ with a boundary Γ of class $C^{2,1}$ and we consider a trajectory (z, r, τ) of the Boussinesq equations:

$$\begin{aligned} \partial_t z - \Delta z + (z\cdot\nabla)z + \nabla r &= \tau e + f \quad \text{in } Q, \\ \nabla\cdot z &= 0 \quad \text{in } Q, \\ \partial\tau - \Delta\tau + z\cdot\nabla\tau &= h \quad \text{in } Q. \end{aligned} \tag{94}$$

In the above setting, $z = z(x,t)$ represents the velocity of the particules of the fluid, $\tau = \tau(x,t)$ their temperature, $r = r(x,t)$ is the pressure function, e stands for the gravity vector field, and $f \in \mathbf{L}^2(\Omega)$ and $h \in L^2(\Omega)$. We consider here the question of stabilizing (z, r, τ) around a stationary state $(z_e, r_e, \tau_e) \in \mathbf{H}^2(\Omega) \times H^1(\Omega) \times H^2(\Omega)$ by means of boundary control. For $\bar{u} \overset{\text{def}}{=} (u_1, \dots, u_d) \in \mathbf{L}^2(\Gamma) \overset{\text{def}}{=} (L^2(\Gamma))^d$ and $u \overset{\text{def}}{=} (\bar{u}, u_{d+1}) \in (L^2(\Gamma))^{d+1}$, we consider the Dirichlet control

$$z = z_e + M(\bar{u}) \quad \text{and} \quad \tau = \tau_e + m u_{d+1} \quad \text{on } \Sigma, \tag{95}$$

where $m \in C^2(\Gamma; \mathbb{R}^+)$ is a compactly supported function of Γ which is not identically equal to zero and M is an operator used to localize the action of the control in the support of m, see (97) below. Then $(w,p,\theta) \stackrel{\text{def}}{=} (z-z_e, r-r_e, \tau-\tau_e)$ satisfies:

$$
\begin{aligned}
\partial_t w - \Delta w + (w\cdot\nabla)z_e + (z_e\cdot\nabla)w + (w\cdot\nabla)w + \nabla p = \theta e \quad & \text{in } Q\\
\nabla\cdot w = 0 \quad & \text{in } Q\\
\partial_t\theta - \Delta\theta + w\cdot\nabla\tau_e + z_e\cdot\nabla\theta + w\cdot\nabla\theta = 0 \quad & \text{in } Q\\
z = M(\bar u) \quad & \text{on } \Sigma\\
\theta = m u_{d+1} \quad & \text{on } \Sigma
\end{aligned}
\tag{96}
$$

and the question of stabilizing (94) around (z_e, r_e, τ_e) by means of (95) is reduced to find $(\bar u, u_{d+1})$ so that the solution of (96) obeys $(w(t),\theta(t)) \to 0$ as $t\to+\infty$.

In addition to notations of section 3, we need to define some other function spaces. Let $\mathcal{H}_0^{2r}(\Omega) \stackrel{\text{def}}{=} H^{2r}(\Omega)$ for $r\in[0,1/4)$, let $\mathcal{H}_0^{1/2}(\Omega) \stackrel{\text{def}}{=} H^{1/2}(\Omega)\cap L^2_{-\frac12}(\Omega)$, where $L^2_{-1/2}(\Omega)$ is the space of functions $y\in L^2(\Omega)$ such that $\int_\Omega \operatorname{dist}(x,\Gamma)^{-1}|y|^2\,dx < +\infty$, and let $\mathcal{H}_0^{2r}(\Omega) \stackrel{\text{def}}{=} \{y\in H^{2r}(\Omega) \mid y=0 \text{ on } \Gamma\}$ for $r\in(1/4,1]$. Moreover, we set $\mathcal{H}_0^{2r}(\Omega) = [\mathcal{H}_0^{-2r}(\Omega)]'$ for $r\in[-1,0]$. Let us also introduce:

$$
\begin{aligned}
\mathbf{V}_n^{2r}(\Omega) &\stackrel{\text{def}}{=} \Big\{ y\in\mathbf{H}^{2r}(\Omega)\ ;\ \nabla\cdot y = 0 \quad \text{in } \Omega, \quad y\cdot n = 0 \quad \text{on } \Gamma \Big\}, \qquad r\geq 0,\\
\mathbf{V}_0^{2r}(\Omega) &\stackrel{\text{def}}{=} \Big\{ y\in\mathbf{H}^{2r}(\Omega)\ ;\ \nabla\cdot y = 0 \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \Gamma \Big\}, \qquad r > \frac14,\\
\mathbf{V}^{2r}(\Gamma) &\stackrel{\text{def}}{=} \Big\{ y\in\mathbf{H}^{2r}(\Gamma)\ ;\ \int_\Gamma y\cdot n = 0 \Big\}, \qquad r\in[0,1].
\end{aligned}
$$

Moreover, we define $\mathbf{V}_0^{2r}(\Omega)$ for $r\in[0,\frac14)$ by $\mathbf{V}_0^{2r}(\Omega) \stackrel{\text{def}}{=} \mathbf{V}_n^{2r}(\Omega)$, for $r=1/4$ by $\mathbf{V}_0^{1/2}(\Omega) \stackrel{\text{def}}{=} \{\, y\in\mathbf{V}_n^{1/2}(\Omega) \mid y\in (L^2_{-1/2}(\Omega))^d \,\}$, and for $r<0$ by $\mathbf{V}_0^{2r}(\Omega) \stackrel{\text{def}}{=} [\mathbf{V}_0^{-2r}(\Omega)]'$. Notice that the subscript 0 in $\mathcal{H}_0^{2r}(\Omega)$ and in $\mathbf{V}_0^{2r}(\Omega)$ only means that one may have a vanishing Dirichlet boundary condition.

In order to rewrite the system in the form (1), we introduce (see [45, 10, 11]):

1. the (Leray) orthogonal projection operator $P : \mathbf{L}^2(\Omega)\to\mathbf{V}_n^0(\Omega)$. Note that for $r\in[0,1]$ we have $P\in\mathcal{L}(\mathbf{H}^{2r}(\Omega), \mathbf{V}_n^{2r}(\Omega))$.
2. the Oseen operator:

$$\mathcal{D}(A_1) \stackrel{\text{def}}{=} \mathbf{V}_0^2(\Omega) \quad \text{and} \quad A_1\varphi \stackrel{\text{def}}{=} P(-\Delta\varphi + (\varphi\cdot\nabla)z_e + (z_e\cdot\nabla)\varphi).$$

   Notice that $-A_1$ is the infinitesimal generator of an analytic semigroup on $\mathbf{V}_n^0(\Omega)$, that $\widehat{A}_1 \stackrel{\text{def}}{=} \lambda_0 + A_1$ for $\lambda_0>0$ large enough has bounded imaginary powers, that the adjoint of A_1 is given by

$$\mathcal{D}(A_1^*) \stackrel{\text{def}}{=} \mathbf{V}_0^2(\Omega) \quad \text{and} \quad A_1^*\varphi \stackrel{\text{def}}{=} P(-\Delta\varphi - (\nabla\varphi)z_e - {}^t(\nabla\varphi)z_e),$$

   and that $\mathcal{D}(\widehat{A}_1^\alpha) = \mathcal{D}(\widehat{A}_1^{*\alpha}) = \mathbf{V}_0^{2\alpha}(\Omega)$ for $\alpha\in[0,1]$.
3. the Dirichlet operator $D_1 : \mathbf{V}^0(\Gamma)\longrightarrow\mathbf{V}^0(\Omega)$ associated with $\lambda_0 + A_1$: $D_1 v \stackrel{\text{def}}{=} \varphi$ where φ is the solution of

$$
\begin{aligned}
\lambda_0\varphi - \Delta\varphi + (\varphi\cdot\nabla)z_e + (z_e\cdot\nabla)\varphi + \nabla q = 0 \ \ &\text{in } \ \Omega,\\
\nabla\cdot\varphi = 0 \ \ &\text{in } \ \Omega,\\
\varphi = v \ \ &\text{on } \ \Gamma.
\end{aligned}
$$

   Notice that $D_1\in\mathcal{L}(\mathbf{V}^{2r}(\Gamma), \mathbf{V}^{2r+\frac12}(\Omega))$ for $r\in[0,3/4]$.

4. the localization self-adjoint operator $M \in \mathcal{L}(\mathbf{L}^2(\Gamma); \mathbf{V}^0(\Gamma))$:

$$Mv \stackrel{\text{def}}{=} m\Big(v - \Big(\int_\Gamma m\Big)^{-1}\Big(\int_\Gamma mv \cdot n\Big)n\Big), \tag{97}$$

where $m \in C^2(\Gamma; \mathbb{R}^+)$ is a nonzero compactly supported function of Γ. Note that $M \in \mathcal{L}(\mathbf{V}^{2r}(\Gamma), \mathbf{V}^{2r}(\Gamma))$ for $r \in [0, 1]$ and that $\mathrm{Supp}(M(v)) \subset \mathrm{Supp}(m)$.

5. the input operator

$$B_1 u = \widehat{A}_1 P D_1 M(\bar{u}) : (L^2(\Gamma))^d \to [\mathcal{D}(A^*)]'.$$

Note that B_1 obeys (5) with $\gamma \in (3/4, 1)$ and that its adjoint is given by

$$B_1^* \varphi = \begin{pmatrix} -m\dfrac{d\varphi}{dn} + m\phi(\varphi)n \\ 0 \end{pmatrix},$$

where $\phi(\varphi)$ is the solution of the Neumann problem:

$$\begin{aligned} &\Delta\phi = \nabla \cdot ((\nabla\varphi)z_e + {}^t(\nabla\varphi)z_e) \ \text{ in } \Omega, \qquad \int_\Gamma m\phi = 0, \\ &\frac{d\phi}{dn} = (\Delta\varphi + (\nabla\varphi)z_e + {}^t(\nabla\varphi)z_e) \cdot n \ \text{ on } \Gamma. \end{aligned} \tag{98}$$

6. the heat type operator on $L^2(\Omega)$:

$$\mathcal{D}(A_2) \stackrel{\text{def}}{=} \mathcal{H}_0^2(\Omega) \quad \text{and} \quad A_2\varrho \stackrel{\text{def}}{=} -\Delta\varrho + z_e \cdot \nabla\varrho.$$

A perturbation argument ensures that $-A_2$ generates an analytic semigroup on $L^2(\Omega)$ and that $\widehat{A}_2 \stackrel{\text{def}}{=} \lambda_0 + A_2$ (for $\lambda_0 > 0$ large enough) has bounded imaginary powers. Moreover, the adjoint of A_2 is given by

$$\mathcal{D}(A_2^*) = \mathcal{H}_0^2(\Omega) \quad \text{and} \quad A_2^*\varrho \stackrel{\text{def}}{=} -\Delta\varrho - z_e \cdot \nabla\varrho,$$

and we have $\mathcal{D}(\widehat{A}_2^\alpha) = \mathcal{D}(\widehat{A}_2^{*\alpha}) = \mathcal{H}_0^{2\alpha}(\Omega)$ for $\alpha \in [0, 1]$.

7. the Dirichlet operator $D_2 : L^2(\Gamma) \longrightarrow L^2(\Omega)$ associated with $\lambda_0 + A_2$: $D_2 b = \varrho$ where ϱ is the solution of

$$\lambda_0\varrho - \Delta\varrho + z_e \cdot \nabla\varrho = 0 \ \text{ in } \ \Omega, \qquad \varrho = b \ \text{ on } \ \Gamma.$$

Notice that $D_2 \in \mathcal{L}(H^{2r}(\Gamma), H^{2r+\frac{1}{2}}(\Omega))$ for $r \in [0, 3/4]$.

8. the input operator

$$B_2 u = -(I - P)D_1 M(\bar{u}) \cdot \nabla\tau_e + \widehat{A}_2 D_2(mu_{d+1}) : (L^2(\Gamma))^{d+1} \to [\mathcal{D}(A_2^*)]'.$$

Notice that from the regularizing property of P, D_1, M and D_2 one can verify that B_1 obeys (5) with $\gamma \in (3/4, 1)$. Moreover, from the expression of D_1^* and from the Neumann problem related to P one verifies that the adjoint $B_2^* \in \mathcal{L}(\mathcal{D}(A_2^*), (L^2(\Gamma))^{d+1})$ is given by

$$B_2^*\varrho = \begin{pmatrix} -MD_1^*(I-P)(\nabla\tau_e\varrho) \\ mD_2^*\widehat{A}_2^*\varrho \end{pmatrix} = \begin{pmatrix} m\chi(\varrho)n \\ -m\dfrac{d\varrho}{dn} \end{pmatrix},$$

where $\chi(\varrho)$ is the unique solution to the Neumann problem:

$$\begin{aligned} &-\Delta\chi = \nabla \cdot (\nabla\tau_e\varrho) \ \text{ in } \Omega, \qquad \int_\Omega m\chi = 0, \\ &\frac{d\chi}{dn} = 0 \ \text{ on } \Gamma. \end{aligned} \tag{99}$$

According to [45], (96) can be equivalently rewritten in the abstract form:

$$\begin{array}{rcl} Pw' + A_1Pw + P(w\cdot\nabla)w - P\theta e &=& \widehat{A}_1P_1D_1M(\bar{u}) \in [\mathcal{D}(A_1^*)]', \\ (I-P)w &=& (I-P)D_1M(\bar{u}), \\ Py(0) &=& P(z(0)-z_e), \\ \theta' + A_2\theta + w\cdot\nabla\tau_e + w\cdot\nabla\theta &=& \widehat{A}_2D_2(mu_{d+1}) \in [\mathcal{D}(A_2)]', \\ \theta(0) &=& \tau(0)-\tau_e. \end{array}$$

Then with $w = Pw + (I-P)D_1M(\bar{u})$, by setting

$$G_1(Pw,\bar{u}) = P((Pw + (I-P)D_1M(\bar{u}))\cdot\nabla)(Pw + (I-P)D_1M(\bar{u})),$$

and

$$G_2(Pw,\bar{u},\theta) = ((Pw + (I-P)D_1M(\bar{u}))\cdot\nabla\theta,$$

and by renaming $z(0)-z_e$ by w_0 and $\tau(0)-\tau_e$ by θ_0 for simplicity, system (96) can be equivalently rewritten in the following abstract form:

$$\begin{array}{c} Pw' + A_1Pw - P\theta e + G_1(Pw,\bar{u}) = B_1u \in [\mathcal{D}(A_1^*)]', \\ \theta' + A_2\theta + Pw\cdot\nabla\tau_e + G_2(Pw,\bar{u},\theta) = B_2u \in [\mathcal{D}(A_2^*)]', \\ (Pw(0),\theta(0)) = (Pw_0,\theta_0). \end{array} \tag{100}$$

Thus, we introduce:

9. the closed and densely defined linear operator on $\mathbf{V}_n^0(\Omega)\times L^2(\Omega)$

$$\mathcal{D}(A) \stackrel{\text{def}}{=} \mathcal{D}(A_1)\times\mathcal{D}(A_2) \quad \text{and} \quad A\begin{pmatrix} \varphi \\ \varrho \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} A_1\varphi - P\varrho e \\ A_2\varrho + \varphi\cdot\nabla\tau_e \end{pmatrix}.$$

Notice that an easy verification shows that the adjoint of A is given by

$$\mathcal{D}(A) \stackrel{\text{def}}{=} \mathcal{D}(A_1^*)\times\mathcal{D}(A_2^*) \quad \text{and} \quad A^*\begin{pmatrix} \varphi \\ \varrho \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} A_1^* + \varrho\nabla\tau_e \\ A_2^*\varrho - \varphi\cdot e \end{pmatrix},$$

and that the known properties of A_1, A_2 combined with a perturbation argument ensures that $-A$ generates an analytic semigroup on $\mathbf{V}_n^0(\Omega)\times L^2(\Omega)$, that $\widehat{A} \stackrel{\text{def}}{=} \lambda_0 + A$ for $\lambda_0 > 0$ large enough has bounded imaginary powers and that $\mathcal{D}(\widehat{A}^\alpha) = \mathcal{D}(\widehat{A}^{*\alpha}) = \mathbf{V}_0^2(\Omega)\times\mathcal{H}_0^{2\alpha}(\Omega)$, for $\alpha\in[0,1]$.

10. the linear input operator

$$B \stackrel{\text{def}}{=} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} : (L^2(\Gamma))^{d+1} \to [\mathcal{D}(A^*)]'.$$

Notice that from the known properties of B_1 and B_2, we have that B obeys (5) with $\gamma\in(3/4,1)$ and that:

$$\widehat{A}^{-1}B \in \mathcal{L}((H^{2r}(\Gamma))^{d+1}), \mathbf{V}_n^{2r}(\Omega)\times H^{2r}(\Omega)), \quad \forall r\in[0,1]. \tag{101}$$

Moreover, its adjoint is given by

$$B^*\begin{pmatrix} \varphi \\ \varrho \end{pmatrix} = \begin{pmatrix} -m\dfrac{d\varphi}{dn} + m(\phi(\varphi)+\chi(\varrho))n \\ -m\dfrac{d\varrho}{dn} \end{pmatrix}$$

where $\phi(\varphi)$ and $\chi(\varrho)$ are given by (98) and (99).

Next, by setting:

$$y \overset{\text{def}}{=} \begin{pmatrix} Pw \\ \theta \end{pmatrix}, \quad y_0 \overset{\text{def}}{=} \begin{pmatrix} Pw_0 \\ \theta_0 \end{pmatrix} \quad \text{and} \quad G(y,u) \overset{\text{def}}{=} \begin{pmatrix} G_1(Pw,u) \\ G_2(Pw,u,\theta) \end{pmatrix},$$

system (100) can be rewritten as follows:

$$y' + Ay + G(y,u) = Bu \in [\mathcal{D}(A^*)]', \quad y(0) = y_0. \tag{102}$$

With the change of variable $y = {}^t(P(z-z_e), \tau - \tau_e)$ we have then transformed (94),(95) to the abstract system (102) with $y_0 = {}^t(P(z(0)-z_e), \tau(0)-\tau_e)$. Moreover, operators A and B fit the framework of section 2 with $H = \mathbf{V}_n^0(\Omega) \times L^2(\Omega)$ and $U = (L^2(\Gamma))^{d+1}$: $\widehat{A}$ has bounded imaginary powers, $-A$ generates an analytic semigroup on H and B satisfies (5) for $\gamma \in (3/4, 1)$. Notice also that the (open-loop) stabilizability assumption (6) can be obtained from the null controllability results with internal control stated in [31], by means of a usual geometrical extension procedure. Thus, since we have $\mathcal{D}(\widehat{A}^{1/2}) = \mathcal{D}(\widehat{A}^{*1/2}) = \mathbf{V}_0^1(\Omega) \times \mathcal{H}_0^1(\Omega)$ we can apply the abstract Theory of Subsection 2.2 with $Z = \mathbf{V}_n^0(\Omega) \times L^2(\Omega)$ and R equal to the identity in $\mathbf{V}_n^0(\Omega) \times L^2(\Omega)$: for a prescribed rate $\sigma > 0$ problem (10), (7) guarantees the existence of a self-adjoint operator $\Pi \in \mathcal{L}(\mathbf{V}_n^0(\Omega) \times L^2(\Omega), \mathbf{V}_0^2(\Omega) \times \mathcal{H}_0^2(\Omega))$ which is the unique solution to (12). Le us write Π in terms of its components:

$$\Pi = \begin{pmatrix} \pi_1 & \pi_2^* \\ \pi_2 & \pi_3 \end{pmatrix} \text{ with } (\pi_1, \pi_2, \pi_3) \in \mathcal{L}(\mathbf{V}_n^0(\Omega)) \times \mathcal{L}(\mathbf{V}_n^0(\Omega), L^2(\Omega)) \times \mathcal{L}(L^2(\Omega))$$

so that for $\xi = {}^t(\varphi, \varrho) \in \mathbf{V}_n^0(\Omega) \times L^2(\Omega)$ we have $\Pi\xi = {}^t(\pi_1\varphi + \pi_2^*\varrho, \pi_2\varphi + \pi_3\varrho)$. Moreover, the nonlinear abstract system subjected to the feedback control $u = -B^*\Pi y$ has the form (31) with $F(y) \overset{\text{def}}{=} G(y, -B^*\Pi y)$.

Similarly as for [10, Cor.6] one proves that $H_\Pi^r \overset{\text{def}}{=} \mathcal{D}(A_\Pi^r)$ is a closed subspace of $\mathbf{V}_n^{2r}(\Omega) \times H^{2r}(\Omega)$ when $r \in [0,1]$. More precisely, when $r \neq 1/4$ we have $H_\Pi^r = \{(P\varphi, \varrho) \mid (\varphi, \varrho) \in \Xi_\Pi^{2r}(\Omega)\}$, where for $r \in [0, 1/4)$ we have $\Xi_\Pi^{2r}(\Omega) \overset{\text{def}}{=} \mathbf{V}^{2r}(\Omega) \times H^{2r}(\Omega)$, and for $r \in (1/4, 1]$ the space $\Xi_\Pi^{2r}(\Omega)$ is composed with elements $(\varphi, \varrho) \in \mathbf{V}^{2r}(\Omega) \times H^{2r}(\Omega)$ which satisfy the trace conditions

$$\begin{aligned} \varphi &= m^2 \frac{d}{dn}(\pi_1 P\varphi + \pi_2^*\varrho) - m^2(\phi((\pi_1 P\varphi + \pi_2^*\varrho)) + \chi((\pi_2 P\varphi + \pi_3\varrho)))n \text{ on } \Gamma, \\ \varrho &= m^2 \frac{d\varrho}{dn} \text{ on } \Gamma. \end{aligned}$$

Then since for $r \in (\frac{d-2}{4}, \frac{1}{2}]$ the maps $w \mapsto (w\cdot\nabla)w$ and $(w,\theta) \mapsto w \cdot \nabla\theta$ satisfy:

$$\begin{aligned} \|(w\cdot\nabla)w\|_{\mathbf{H}^{2r-1}(\Omega)} &\leq C\|w\|_{\mathbf{H}^{2r}(\Omega)}\|w\|_{\mathbf{H}^{1+2r}(\Omega)}, \\ \|w\cdot\nabla\theta\|_{H^{2r-1}(\Omega)} &\leq C\|w\|_{\mathbf{H}^{2r}(\Omega)}\|\theta\|_{H^{1+2r}(\Omega)}, \end{aligned} \tag{103}$$

we deduce that for $r \in (\frac{d-2}{4}, \frac{1}{2}]$ the map $y \mapsto F(y)$ obeys (32)-(33) and Theorem 2.3 applies. Then with an easy adaptation of [10, Thm.12] one obtains a stabilization theorem for system (94) with the boundary conditions

$$\begin{aligned} z - z_e = m^2 \frac{d}{dn}&(\pi_1 P(z-z_e) + \pi_2^*(\tau - \tau_e)) \\ &- m^2(\phi((\pi_1 P(z-z_e) + \pi_2^*(\tau-\tau_e))) \\ &+ \chi((\pi_2 P(z-z_e) + \pi_3(\tau-\tau_e))))n \quad \text{on } \Sigma, \\ \tau - \tau_e = m^2 &\frac{d(\tau-\tau_e)}{dn} \quad \text{on } \Sigma, \end{aligned} \tag{104}$$

and with initial data

$$z(0) = z_0 \quad \text{and} \quad \tau(0) = \tau_0. \tag{105}$$

Theorem 4.1. *Let $r \in (\frac{d-2}{4}, \frac{1}{2}]\backslash\{\frac{1}{4}\}$ and $(z_0, \tau_0) \in \{(z_e, \tau_e)\} + \mathbf{V}^{2r}(\Omega) \times H^{2r}(\Omega)$. If $r > \frac{1}{4}$ we also assume that $(z_0 - z_e, \tau_0 - \tau_e) \in \Xi^{2r}_{\Pi}(\Omega)$, which is to say that the following initial compatibility conditions are satisfied:*

$$\begin{aligned} z_0 - z_e = {} & m^2 \frac{d}{dn}(\pi_1 P(z_0 - z_e) + \pi_2^*(\tau_0 - \tau_e)) \\ & - m^2(\phi((\pi_1 P(z_0 - z_e) + \pi_2^*(\tau - \tau_e))) \\ & + \chi((\pi_2 P(z_0 - z_e) + \pi_3(\tau_0 - \tau_e))))n \quad \text{on } \Gamma, \\ \tau_0 - \tau_e = {} & m^2 \frac{d(\tau_0 - \tau_e)}{dn} \quad \text{on } \Gamma. \end{aligned} \tag{106}$$

There exist $\rho > 0$ and $\mu > 0$ such that if $\|P(z_0 - z_e)\|_{\mathbf{H}^{2r}(\Omega)} + \|\tau_0 - \tau_e\|_{H^{2r}(\Omega)} \leq \mu$, then system (94), (104), (105) admits a solution (z, r, τ) in

$$\begin{aligned} \{(z_e, r_e, \tau_e)\} + & L^2_\sigma(\mathbf{V}^{1+2r}(\Omega)) \cap H^{1/2+r}_\sigma(\mathbf{L}^2(\Omega)) \times H^{-1/2+r}_\sigma(H^{2r}(\Omega)/\mathbb{R}) \\ & \times W_\sigma(H^{2r+1}(\Omega), H^{2r-1}(\Omega)), \end{aligned}$$

which is unique within the class of function in

$$\begin{aligned} \{(z_e, r_e, \tau_e)\} + & L^2_{\text{loc}}(\mathbf{V}^{1+2r}(\Omega)) \cap H^{1/2+r}_{\text{loc}}(\mathbf{L}^2(\Omega)) \times H^{-1/2+r}_{\text{loc}}(H^{2r}(\Omega)/\mathbb{R}) \\ & \times W_{\text{loc}}(H^{2r+1}(\Omega), H^{2r-1}(\Omega)). \end{aligned}$$

Moreover, for all $t \geq 0$ the following estimate holds:

$$\|z(t) - z_e\|_{\mathbf{H}^{2r}(\Omega)} + \|\tau(t) - \tau_e\|_{H^{2r}(\Omega)} \leq Ce^{-\sigma t}(\|z_0 - z_e\|_{\mathbf{H}^{2r}(\Omega)} + \|\tau_0 - \tau_e\|_{H^{2r}(\Omega)}).$$

According to the above theorem, to define a solution to system (94), (104), (105) when $d = 3$ one must impose the initial velocity to fit the feedback trace condition (106). Since such a condition is very restrictive in practice, another strategy consists in looking for a control function $u \overset{\text{def}}{=} (\bar{u}, u_{d+1}) \overset{\text{def}}{=} (u_1, \dots, u_{d+1})$ itself solution to an evolution equation:

$$\partial_t u_i - \Delta_\Gamma u_i = g_i \quad \text{in} \quad \Sigma, \quad i = 1, \dots, d+1, \tag{107}$$

where $g \overset{\text{def}}{=} (g_1, \dots, g_{d+1})$ plays the role of a control function for the whole system (94), (95), (107). In the above setting, Δ_Γ denotes the Laplace Beltrami operator. Then if we consider (107) with the initial condition $u(0) = 0$, every initial datum obeying $z_0 = z_e$ and $\tau_0 = \tau_e$ on Γ would fit the initial compatibility conditions $z_0 - z_e = M\bar{u}(0)$ and $\tau_0 - \tau_e = m u_{d+1}(0)$ on Γ. It then allows to define a fixed-point solution to the Boussinesq system, see the introduction of [9] for the particular case of Navier-Stokes equations. In the following, we apply the framework of section 2.3 to construct a stabilizing control g in the feedback form:

$$g(t) = \mathfrak{F}(z(t) - z_e, \tau(t) - \tau_e, u) \quad \forall t \geq 0.$$

Let us denote by Δ_b the vectorial Laplace Beltrami operator, i.e. $(\Delta_b u)_i = \Delta_\Gamma u_i$, for all $i = 1, \dots, d+1$, and let us introduce:

11. the unbounded operator E on $U \overset{\text{def}}{=} (L^2(\Gamma))^{d+1}$:

$$\mathcal{D}(E) = (H^2(\Gamma))^{d+1} \quad \text{and} \quad Eu = -\Delta_b u.$$

Note that $\mathcal{D}(\widehat{E}^\alpha) = \mathcal{D}(\widehat{E}^{*\alpha}) = (H^{2\alpha}(\Gamma))^{d+1}$ for $\alpha \in [0, 1]$.

Thus, (107) can be simply rewritten as

$$u' + Eu = g,$$

and section 2.3 applies with $\mathbb{A}$ defined from the pair (A, B) which has been introduced above, and for $\mathbb{R}$ given in Remark 4 with R and Θ equal to the identity in $\mathbf{V}_n^0(\Omega) \times L^2(\Omega)$ and in $(L^2(\Gamma))^{d+1}$ respectively. Indeed, the assumption $B^* \widehat{A}^{*-1} \in \mathcal{L}(H_*^{1/2}, U_*^{1/2}) = \mathcal{L}(\mathbf{V}_0^1(\Omega) \times \mathcal{H}_0^1(\Omega), (H^1(\Gamma))^{d+1})$ is a consequence of regularity results for the Oseen and for the heat equation which guarantees that $\mathcal{D}(A^{*3/2}) \hookrightarrow \mathbf{V}_0^3(\Omega) \times H^3(\Omega)$ and then $B^*(\mathcal{D}(A^{*3/2})) \hookrightarrow U^{1/2} = (H^1(\Gamma))^{d+1}$. Notice that the (open-loop) stabilizability assumption (41) can be obtained from the null controllability result [31], by using Theorem 2.6, if we additionally assume that Γ is of class $C^{3,1}$. Indeed, with such an assumption regularity results for the Oseen and for the heat equation guarantee $\mathcal{D}(A^{*2}) \hookrightarrow \mathbf{V}_0^4(\Omega) \times H^4(\Omega)$ and that $B^*(\mathcal{D}(A^{*2})) \hookrightarrow U_*^1 = (H^2(\Gamma))^{d+1}$. As a consequence, for a prescribed rate $\sigma > 0$ we have the existence of a self-adjoint operator $\mathbf{\Pi} \in \mathcal{L}(\mathbb{H}, \mathbb{H}_*^1)$ which is the unique solution to the Riccati equation (49). Let us write $\mathbf{\Pi}$ in terms of its components:

$$\mathbf{\Pi} = \begin{pmatrix} \pi_1 & \pi_2^* & \pi_3^* \\ \pi_2 & \pi_4 & \pi_5^* \\ \pi_3 & \pi_5 & \pi_6 \end{pmatrix},$$

with $\pi_1 = \pi_1^* \in \mathcal{L}(\mathbf{V}_n^0(\Omega))$, $\pi_2 \in \mathcal{L}(\mathbf{V}_n^0(\Omega), L^2(\Omega))$, $\pi_3 \in \mathcal{L}(\mathbf{V}_n^0(\Omega), (L^2(\Gamma))^{d+1})$, $\pi_4 = \pi_4^* \in \mathcal{L}(L^2(\Omega))$, $\pi_5 \in \mathcal{L}(L^2(\Omega), (L^2(\Gamma))^{d+1})$ and $\pi_6 = \pi_6^* \in \mathcal{L}((L^2(\Gamma))^{d+1})$. We have the following relations with the notations (52) :

$$\Pi_1 = \begin{pmatrix} \pi_1 & \pi_2^* \\ \pi_2 & \pi_4 \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} \pi_3 & \pi_5 \end{pmatrix} \quad \text{and} \quad \Pi_3 = \pi_6.$$

Then (53), (54) with $u_0 = 0$ can be rewritten as (94), (95) with:

$$\partial_t u - \Delta_b u + \pi_6 u + \pi_3 P(z - z_e) + \pi_5(\tau - \tau_e) = 0 \quad \text{in} \quad \Sigma, \tag{108}$$

and with the following initial conditions

$$z(0) = z_0, \quad \tau(0) = \tau_0 \quad \text{and} \quad u(0) = 0. \tag{109}$$

Finally, from (101) we deduce that the space $\mathbb{H}^r$ for $r \in [0, 1]$ (defined in (45), (42)) is the closed subspace of $\mathbf{V}_n^{2r}(\Omega) \times H^{2r}(\Omega) \times (H^{2r}(\Gamma))^{d+1}$ defined by:

$$\begin{aligned} \mathbb{H}^r = \{ (P\varphi, \varrho, u) \mid (\varphi, \varrho, u) \in \mathbf{V}^{2r}(\Omega) \times H^{2r}(\Omega) \times (H^{2r}(\Gamma))^{d+1} \\ \text{s. t. } \varphi = M(\bar{u}) \ \text{ and } \ \varrho = u_{d+1} \ \text{ on } \Gamma \}, \end{aligned}$$

and that $\mathbb{H}^{-r} = \mathbf{V}_0^{-2r}(\Omega) \times \mathcal{H}_0^{-2r}(\Omega) \times (H^{-2r}(\Gamma))^{d+1}$ for $r \in [0, 1]$. Then we obtain (51) from (103) and the following stabilization theorem follows from corollary 2.

Theorem 4.2. *Assume that Γ is of class $C^{3,1}$, let $r \in (\frac{1}{4}, \frac{1}{2}]$ and $(z_0, \tau_0) \in \{(z_e, \tau_e)\} + \mathbf{V}_0^{2r}(\Omega) \times \mathcal{H}_0^{2r}(\Omega)$. There exist $\rho > 0$ and $\mu > 0$ such that, if $\delta \in (0, \mu)$ and $\|z_0 - z_e\|_{\mathbf{H}^{2r}(\Omega)} + \|\tau_0 - \tau_e\|_{H^{2r}(\Omega)} \le \delta$, system (94), (95), (108), (109) admits a solution (z, r, τ, u) in*

$$\begin{aligned} \{(z_e, r_e, \tau_e, 0)\} + W_\sigma(\mathbf{V}^{2r+1}(\Omega), \mathbf{V}_0^{2r-1}(\Omega)) \times H_\sigma^{-1/2+r}(H^{2r}(\Omega)/\mathbb{R}) \\ \times W_\sigma(H^{2r+1}(\Omega), H_\sigma^{2r-1}(\Omega)) \times W_\sigma((H^{2r+1}(\Gamma))^{d+1}, (H^{2r-1}(\Gamma))^{d+1}), \end{aligned}$$

which is unique within the class of function in

$$\begin{aligned} \{(z_e, r_e, \tau_e, 0)\} + W_{\text{loc}}(\mathbf{V}^{2r+1}(\Omega), \mathbf{V}_0^{2r-1}(\Omega)) \times H_{\text{loc}}^{-1/2+r}(H^{2r}(\Omega)/\mathbb{R}) \\ \times W_{\text{loc}}(H^{2r+1}(\Omega), H^{2r-1}(\Omega)) \times W_{\text{loc}}((H^{2r+1}(\Gamma))^{d+1}, (H^{2r-1}(\Gamma))^{d+1}). \end{aligned}$$

Moreover, for all $t \geq 0$ the following estimate holds:

$$\|z(t) - z_e\|_{\mathbf{H}^{2r}(\Omega)} + \|\tau(t) - \tau_e\|_{H^{2r}(\Omega)} + \|u(t)\|_{(H^{2r}(\Gamma))^{d+1}} \leq \\ Ce^{-\sigma t}(\|z_0 - z_e\|_{\mathbf{H}^{2r}(\Omega)} + \|\tau_0 - \tau_e\|_{H^{2r}(\Omega)}).$$

5. **Appendix.**

5.1. **Well-posedness of the Riccati equation in $\mathcal{L}(H)$.**

Let us explain how assumption (8) guarantees that (12) is well-posed as an equation in $\mathcal{L}(H)$. According to usual results, which can be found for instance in [38, Chap. 2 Thm. 2.2.1], it is known that Π is the unique nonnegative and self-adjoint operator of $\mathcal{L}(H)$, which belongs to $\mathcal{L}(H, H_*^{1-\epsilon})$ for $\epsilon > 0$, solution to the following weak version of (12):

$$(\Pi\xi|A\zeta)_H + (A\xi|\Pi\zeta)_H + (B^*\Pi\xi|B^*\Pi\zeta)_U = (R\xi|R\zeta)_H + 2\sigma(\Pi\xi|\zeta)_H, \qquad (110)$$

for all $(\xi, \zeta) \in H^1 \times H^1$. However, by using (8) and arguing as in [17, Appendix B, Prop. B.4.1] (where Oseen operator is considered) we can recover the fact that $\Pi \in \mathcal{L}(H, H_*^1)$. Then it implies $A^*\Pi \in \mathcal{L}(H)$ and $B^*\Pi \in \mathcal{L}(H, U)$ (since (5) implies $B \in \mathcal{L}(U, H^{-\gamma})$ and $B^* \in \mathcal{L}(H_*^\gamma, U)$). Moreover, the self-adjointness of Π combined with $\Pi \in \mathcal{L}(H, H_*^1)$ and a duality argument guarantee $\Pi \in \mathcal{L}(H^{-1}, H)$ and then $\Pi A \in \mathcal{L}(H)$ and $\Pi B \in \mathcal{L}(U, H)$. Then each terms of (12) belongs to $\mathcal{L}(H)$.

For the sake of completeness let us recall the argument which yields $\Pi \in \mathcal{L}(H, H_*^1)$. For $\xi \in H$, we know from [38, Chap. 2 Thm. 2.2.1] that the optimal control solution to (10) obeys $u_\xi = -B^*\Phi_\xi$, where $(y_\xi, \Phi_\xi) \in W(H, H^{-1}) \times W(H_*^1, H)$ is the unique solution to:

$$(\mathcal{S}_\xi) \begin{cases} y' + (A - \sigma)y &= -BB^*\Phi, \qquad y(0) = \xi \in H, \\ -\Phi' + (A^* - \sigma)\Phi &= R^*Ry, \qquad \Phi(\infty) = 0, \\ \Phi(t) &= \Pi\, y(t) \qquad \forall t \geq 0, \end{cases}$$

and that (y_ξ, u_ξ) belongs to $C_b(H) \times C_b(U)$ and satisfies the estimate

$$\|y_\xi(t)\|_H + \|u_\xi(t)\|_U \leq C\|\xi\|_H, \quad \forall t \geq 0. \qquad (111)$$

Then $\widehat{u}_\xi = e^{-(\lambda_0+\sigma)(\cdot)}u_\xi$, $\widehat{y}_\xi = e^{-(\lambda_0+\sigma)(\cdot)}y_\xi$ and $\widehat{\Phi}_\xi = e^{-(\lambda_0+\sigma)(\cdot)}\Phi_\xi$ obey

$$\widehat{\Phi}_\xi(t) = \int_t^\infty e^{-\widehat{A}^*(\tau - t)}(2(\lambda_0 + \sigma)\Pi + R^*R)\widehat{y}_\xi(\tau)\mathrm{d}\tau \qquad (112)$$

and

$$\widehat{y}_\xi(t) = e^{-\widehat{A}t}\xi + \int_0^t e^{-\widehat{A}(t-\tau)}B\widehat{u}_\xi(\tau)\mathrm{d}\tau,$$

and by substituing the above expression of $\widehat{y}_\xi$ in the first above equality we obtain:

$$\Pi\xi = \widehat{\Phi}_\xi(0) = I_1\xi + I_2\xi + I_3\xi,$$

where

$$I_1\xi = \int_0^\infty e^{-\widehat{A}^*t}R^*Re^{-\widehat{A}t}\xi\mathrm{dt}, \quad I_2\xi = 2(\lambda_0 + \sigma)\int_0^\infty e^{-\widehat{A}^*t}\Pi e^{-\widehat{A}t}\xi\mathrm{dt}$$

and

$$I_3\xi = \int_0^\infty e^{-\widehat{A}^*t}(2(\lambda_0 + \sigma)\Pi + R^*R)\mathcal{L}(\widehat{u}_\xi)(t)\mathrm{dt}$$

where we have used the notation

$$\mathcal{L}(\widehat{u}_\xi)(t) = \int_0^t e^{-\widehat{A}(t-\tau)}B\widehat{u}_\xi(\tau)\mathrm{d}\tau.$$

To prove $\Pi \in \mathcal{L}(H, H^1_*)$, let us show $\|\widehat{A}^* I_i \xi\|_H \le C\|\xi\|_H$, $i = 1, 2, 3$. First, an obvious calculation give

$$(\widehat{A}^* I_1 \xi|\zeta)_H = \int_0^\infty ([\widehat{A}^{*1/2} R^* R \widehat{A}^{-1/2}] \widehat{A}^{1/2} e^{\widehat{A}t} \xi | \widehat{A}^{1/2} e^{\widehat{A}t} \zeta)_H \mathrm{dt}$$

and since from (8) we have $[\widehat{A}^{*1/2} R^* R \widehat{A}^{-1/2}] \in \mathcal{L}(H)$, the continuity of $\xi \in H \mapsto \widehat{A}^{1/2} e^{\widehat{A}t} \xi \in L^2(H)$ combined with Cauchy-Schwarz inequality yields $\|\widehat{A}^* I_1 \xi\|_H \le C\|\xi\|_H$. Moreover, estimate $\|\widehat{A}^* I_2 \xi\|_H \le C\|\xi\|_H$ follows analogously from $[\widehat{A}^{*1/2} \Pi \widehat{A}^{-1/2}] \in \mathcal{L}(H)$ which is a consequence of $\Pi \in \mathcal{L}(H, H^{1/2}_*)$. Next, for $0 < \eta < \min(1-\gamma, 1/2)$, $R^* R \in \mathcal{L}(H^{1/2}, H^{1/2}_*) \cap \mathcal{L}(H)$ with and interpolation argument gives $R^* R \in \mathcal{L}(H^\eta, H^\eta_*)$ and since we also have $\Pi \in \mathcal{L}(H, H^\eta_*)$ we deduce that $\widehat{A}^{*\eta}(2(\lambda_0 + \sigma)\Pi + R^* R)\widehat{A}^{-\eta} \in \mathcal{L}(H)$. Thus, by writing

$$\widehat{A}^{*\eta}(2(\lambda_0 + \sigma)\Pi + R^* R)\mathcal{L}(\widehat{u}_\xi)(t) = [\widehat{A}^{*\eta}(2(\lambda_0 + \sigma)\Pi + R^* R)\widehat{A}^{-\eta}] \int_0^t \widehat{A}^{\gamma+\eta} e^{-\widehat{A}(t-\tau)} \widehat{A}^{-\gamma} B \widehat{u}_\xi(\tau) \mathrm{d}\tau,$$

then the Young inequality combined with the analyticity estimate $\|A^{\gamma+\eta} e^{-\widehat{A}(t-\tau)}\|_H \le C(t-\tau)^{-\gamma-\eta}$, $\widehat{A}^{-\gamma} B \in \mathcal{L}(U, H)$ and the bound of $\|u_\xi(t)\|_U$ given in (111), ensures that $\xi \in H \mapsto \widehat{A}^\eta(2(\lambda_0 + \sigma)\Pi + R^* R)\mathcal{L}(\widehat{u}_\xi) \in C_b(H)$ is continuous. Finally, with

$$\widehat{A}^* I_3 \xi = \int_0^\infty \widehat{A}^{*1-\eta} e^{-\widehat{A}^* t} \widehat{A}^{*\eta}(2(\lambda_0 + \sigma)\Pi + R^* R)\mathcal{L}(\widehat{u}_\xi)(t) \mathrm{dt}$$

the Young inequality with the analyticity estimate $\|\widehat{A}^{*1-\eta} e^{-\widehat{A}^* t}\|_H \le C t^{\eta-1}$ yields $\|\widehat{A}^* I_3 \xi\|_H \le C\|\xi\|_H$.

5.2. Technical results about Neuman boundary conditions.

5.2.1. ***Spaces of free divergence vector field***. Let us denote by $\mathbf{P}$ the orthogonal projection operator from $\mathbf{L}^2(\Omega)$ onto $\mathbf{V}^0(\Omega)$. The following characterization of $\mathbf{P}$ can be found in [30, Thm.2.5].

Proposition 5. *The orthogonal projection operator $\mathbf{P} : \mathbf{L}^2(\Omega) \to \mathbf{V}^0(\Omega)$ satisfies*

$$\mathbf{P} f = f + \nabla p \quad \text{where} \quad \left\{ \begin{array}{rcll} -\Delta p & = & \nabla \cdot f & \text{on } \Omega \\ p & = & 0 & \text{on } \Gamma \end{array} \right. \tag{113}$$

Moreover, $\mathbf{P}$ is bounded from $\mathbf{H}^{2r}(\Omega)$ onto $\mathbf{V}^{2r}(\Omega)$ for $r \ge 0$.

Note that (113) also holds but with a boundary condition $p = c$ on Γ instead of $p = 0$ on Γ where c is an arbitrary real constant, which is coherent with [15, Thm.2.3] concerning the more subtle case of mixed Dirichlet-Neumann boundary conditions.

Corollary 3. *For $\alpha \in (0, 1)$ and $0 \le r_1 \le r_2 \le 1$ the following equality hold:*

$$[\mathbf{V}^{2r_1}(\Omega), \mathbf{V}^{2r_2}(\Omega)]_{1-\alpha} = \mathbf{V}^{2((1-\alpha)r_1 + \alpha r_2)}(\Omega). \tag{114}$$

Proof. According to Proposition 5 there exists a projection operator from $\mathbf{H}^{2r_1}(\Omega)$ onto $\mathbf{V}^{2r_1}(\Omega)$ which is continuous from $\mathbf{H}^{2r_i}(\Omega)$ into itself for $i = 1, 2$. Then [48, Thm.1.17.1.1, p.118] applies and we have

$$[\mathbf{H}^{2r_1}(\Omega) \cap \mathbf{V}^{2r_1}(\Omega), \mathbf{H}^{2r_2}(\Omega) \cap \mathbf{V}^{2r_1}(\Omega)]_{1-\alpha} = [\mathbf{H}^{2r_1}(\Omega), \mathbf{H}^{2r_2}(\Omega)]_{1-\alpha} \cap \mathbf{V}^{2r_1}(\Omega).$$

Thus, the conclusion follows from $\mathbf{H}^{2r_i}(\Omega) \cap \mathbf{V}^{2r_1}(\Omega) = \mathbf{V}^{2r_i}(\Omega)$, $i = 1, 2$ and $[\mathbf{H}^{2r_1}(\Omega), \mathbf{H}^{2r_2}(\Omega)]_{1-\alpha} = \mathbf{H}^{2((1-\alpha)r_1 + \alpha r_2)}(\Omega)$. $\square$

Next, since the orthogonal projection operator $\mathbf{P} : \mathbf{L}^2(\Omega) \to \mathbf{V}^0(\Omega)$ is also bounded from $\mathbf{H}^1(\Omega)$ into $\mathbf{V}^1(\Omega)$ then its adjoint $\mathbf{P}^* : \mathbf{V}^0(\Omega) \to \mathbf{L}^2(\Omega)$, which is simply the injection operator, can be extended as a bounded operator from $[\mathbf{V}^1(\Omega)]'$ into $[\mathbf{H}^1(\Omega)]'$ as follows:

$$\langle \mathbf{P}^* f | v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = \langle f | \mathbf{P}v\rangle_{[\mathbf{V}^1(\Omega)]',\mathbf{V}^1(\Omega)} \quad \forall v \in \mathbf{H}^1(\Omega).$$

Then we have the following Lemma which is consequence of De Rham's theorem.

Lemma 5.1. *If $f \in [\mathbf{H}^1(\Omega)]'$ obeys $\langle f \mid v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = 0$ for all $v \in \mathbf{V}^1(\Omega)$ then there exists a unique $p \in L^2(\Omega)$ such that*

$$\langle f \mid v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = \int_\Omega p\nabla\cdot v \quad \forall v \in \mathbf{H}^1(\Omega).$$

It follows that if $f \in [\mathbf{H}^1(\Omega)]'$ and $g \in [\mathbf{V}^1(\Omega)]'$ coincide on $\mathbf{V}^1(\Omega)$ then

$$\langle f \mid v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = \langle \mathbf{P}^* g \mid v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} + \int_\Omega p\nabla\cdot v \quad \forall v \in \mathbf{H}^1(\Omega).$$

Proof. under the above assumptions we have in particular $\langle f \mid v_0\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = 0$ for all $v_0 \in \mathbf{V}^1(\Omega)\cap\mathbf{H}^1_0(\Omega)$ and De Rham's theorem [2, Thm.2.8] ensures that there exists $p \in L^2(\Omega)$, defined up to a constant, and such that

$$\langle f \mid v_0\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = \int_\Omega p\nabla\cdot v_0 \quad \forall v_0 \in \mathbf{H}^1_0(\Omega). \tag{115}$$

Moreover, set $\varphi_d(x) = \frac{1}{d}\,{}^t(x_1, x_2, \dots, x_d)$ and for $v \in \mathbf{H}^1(\Omega)$ set $c(v) = |\Omega|^{-1}\int_\Gamma v\cdot n$. Since $\nabla\cdot\varphi_d \equiv 1$ then $\int_\Omega \nabla\cdot(v - c(v)\varphi_d) = 0$ and we can choose $v_0 \in \mathbf{H}^1_0(\Omega)$ such that $\nabla\cdot v_0 = \nabla\cdot(v - c(v)\varphi_d)$ [2, Cor.3.1]. Then $v - v_0 - c(v)\varphi_d \in \mathbf{V}^1(\Omega)$ and

$$\langle f \mid v - v_0 - c(v)\varphi_d\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = 0.$$

With (115) we obtain the desired result for p such that $\int_\Omega p = \langle f \mid \varphi_d\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)}$. □

5.2.2. ***Regularity results for Stokes system with a Neumann condition***. The following Lemma is a lifting theorem for mappings χ, χ_* defined in (68). It is a direct consequence of a theorem due to Amrouche and Girault [1].

Lemma 5.2. *Let Ω be a bounded and connected open subset of $\mathbb{R}^d$ of class $C^{k+1,1}$ for $k \in \mathbb{N}$ and let $(b_0, b_1) \in \mathbf{H}^{k+3/2}(\Gamma) \times \mathbf{H}^{k+1/2}(\Gamma)$ such that $\int_\Gamma b_0\cdot n = 0$. Then there exists $(u_b, p_b) \in \mathbf{H}^{k+2}(\Omega) \times H^{k+1}(\Omega)$ satisfying:*

$$\nabla\cdot u_b = 0 \ \text{ in } \ \Omega, \quad u_b = b_0 \ \text{ and } \ \nu\frac{du_b}{dn} - p_b n = b_1 \ \text{ on } \ \Gamma$$

and

$$\|u_b\|_{\mathbf{H}^{k+2}(\Omega)} + \|p_b\|_{H^{k+1}(\Omega)} \le C(\|b_0\|_{\mathbf{H}^{k+3/2}(\Gamma)} + \|b_1\|_{\mathbf{H}^{k+1/2}(\Gamma)}),$$

Moreover, if $z_e \in \mathbf{V}^{k+1}(\Omega)$ the result remains true with $\chi(u_b, p_b)$ or $\chi_*(u_b, p_b)$ instead of $\nu\frac{du_b}{dn} - p_b n$.

Proof. The lemma relies on [1, Thm.A.] which states that for all $(g_0, g_1) \in \mathbf{H}^{k+3/2}(\Gamma)$ $\times\, \mathbf{H}^{k+1/2}(\Gamma)$ satisfying $\int_\Gamma (g_0)_n = 0$ and $(g_1)_n = \Psi(g_0) \overset{\text{def}}{=} 2\nu K(g_0)_n - \nu\nabla_\Gamma\cdot(g_0)_\tau$ there exists $u \in \mathbf{H}^{k+2}(\Omega)$ such that:

$$\nabla\cdot u = 0 \ \text{ in } \ \Omega, \quad u = g_0 \ \text{ and } \ \nu\frac{du}{dn} = g_1 \ \text{ on } \ \Gamma$$

and

$$\|u\|_{\mathbf{H}^{k+2}(\Omega)} \le C(\|g_0\|_{\mathbf{H}^{k+3/2}(\Gamma)} + \|g_1\|_{\mathbf{H}^{k+1/2}(\Gamma)}).$$

In the above setting K denotes the mean curvature of Γ and $\nabla_\Gamma\cdot$ denotes the surface divergence operator. Thus, it suffices to define $u_b \in \mathbf{H}^{k+2}(\Omega)$ as the vector field obtained for $g_0 = b_0$ and $g_1 = (b_1)_\tau + \Psi(b_0)n$, and to define $p_b \in H^{k+1}(\Omega)$ as a pressure function obtained from a continuous right inverse of the trace operator and such that $p_b = -(b_1)_n + \Psi(b_0)$ on Γ. The results with $\chi(u_b, p_b)$ instead of $\nu\frac{du_b}{dn} - p_b n$ can be obtained analogously with $g_1 = (b_1)_\tau + \frac{k}{2}(b_0)_n(z_e)_\tau + \frac{k}{2}(z_e)_n(b_0)_\tau + \Psi(b_0)n$ and $p_b = -(b_1)_n - k(b_0)_n(z_e)_n + \Psi(b_0)$ on Γ. $\square$

The following Lemma is a regularity theorem for Stokes system with Neumann boundary condition which can be found in [30, Thm. 6.3], see also [21].

Lemma 5.3. *Let Ω be a bounded and connected open subset of $\mathbb{R}^d$ of class $C^{k+1,1}$ for $k \in \mathbb{N}$ and let $f \in \mathbf{H}^k(\Omega)$. If $(u,p) \in \mathbf{V}^1(\Omega) \times L^2(\Omega)$ satisfies*

$$\nu \int_\Omega \nabla u : \nabla v - \int_\Omega p\nabla\cdot v = \int_\Omega f\cdot v \quad \forall v \in \mathbf{H}^1(\Omega), \tag{116}$$

then we have $(u,p) \in \mathbf{V}^{k+2}(\Omega) \times H^{k+1}(\Omega)$ and the following estimate hold:

$$\|u\|_{\mathbf{H}^{k+2}(\Omega)} + \|p\|_{H^{k+1}(\Omega)} \le C\|f\|_{\mathbf{H}^k(\Omega)}.$$

The two previous lemmas yield the following

Corollary 4. *Let the assumptions of Lemma 5.3 be satisfied and let $g \in \mathbf{H}^{k+1/2}(\Gamma)$. If $(u,p) \in \mathbf{V}^1(\Omega) \times L^2(\Omega)$ satisfies*

$$\nu \int_\Omega \nabla u : \nabla v - \int_\Omega p\nabla\cdot v = \int_\Omega f\cdot v + \int_\Gamma g\cdot v \quad \forall v \in \mathbf{H}^1(\Omega),$$

then we have $(u,p) \in \mathbf{V}^{k+2}(\Omega) \times H^{k+1}(\Omega)$ and the following estimate hold:

$$\|u\|_{\mathbf{H}^{k+2}(\Omega)} + \|p\|_{H^{k+1}(\Omega)} \le C(\|f\|_{\mathbf{H}^k(\Omega)} + \|g\|_{\mathbf{H}^{k+1/2}(\Gamma)}).$$

Proof. The conclusion follows from Lemma 5.3 by remarking that if we write $(u,p) = (\widetilde{u},\widetilde{p}) + (u_g, p_g)$ where $(u_g, p_g) \in \mathbf{V}^{k+2}(\Omega) \times H^{k+1}(\Omega)$ is given by Lemma 5.2 with $b_0 = 0$ and $b_1 = g$, then an integration by parts shows that $(\widetilde{u},\widetilde{p})$ obeys (116) with $f + \Delta u_g - \nabla p_g \in \mathbf{H}^k(\Omega)$ instead of f at the right side of the equality. $\square$

5.2.3. ***The Oseen Neumann map***. Here we give regularity results for the Neumann map $N : \mathbf{L}^2(\Gamma) \to \mathbf{V}^0(\Omega)$ defined by $Nu = w$ where w obeys (80). Those can be found in [40, 22, 23] for analogous Stokes Neuman type map. However, for the reader convenience we give the details.

First, note that for rough data $u \in \mathbf{L}^2(\Gamma)$, defining a solution to (80) can be done with the transposition method. It consists in looking for $w \in \mathbf{V}^0(\Omega)$ obeying:

$$\int_\Gamma u\cdot\varphi = \int_\Omega w\cdot f \quad \forall f \in \mathbf{V}^0(\Omega), \tag{117}$$

where $\varphi \in \mathbf{V}^2(\Omega)$ is the unique solution to $\widehat{A}^*\varphi = f$: $\varphi \in \mathbf{H}^2(\Omega)$ and

$$\begin{aligned} \lambda_0\varphi - \nu\Delta\varphi - (\nabla\varphi)z_e - {}^t(\nabla\varphi)z_e + \nabla S\varphi = f \quad &\text{in } \Omega,\\ \nabla\cdot\varphi = 0 \quad &\text{in } \Omega,\\ \chi_*(\varphi, S\varphi) = 0 \quad &\text{on } \Gamma. \end{aligned} \tag{118}$$

The existence and uniqueness of $w \in \mathbf{V}^0(\Omega)$ solution to (117) is a consequence of the Riesz representation theorem, and an integration by parts allows to prove that a smooth velocity (say $w \in \mathbf{V}^2(\Omega)$ and $u \in \mathbf{H}^{1/2}(\Gamma)$) solution to (80) in a classical sense is also the solution to (117). We have the following regularity result for N.

Proposition 6. *If Ω of class $C^{2,1}$ and $z_e \in \mathbf{V}^2(\Omega)$ then the operator N is bounded from $\mathbf{L}^2(\Gamma)$ into $\mathbf{V}^0(\Omega)$ and for all $r \in [0, 3/2]$ it satisfies:*

$$N \in \mathcal{L}(\mathbf{H}^{2r-3/2}(\Gamma), \mathbf{V}^{2r}(\Omega)). \tag{119}$$

Moreover, if Ω of class $C^{3,1}$ and $z_e \in \mathbf{V}^3(\Omega)$ then (119) is satisfied for $r \in (3/2, 5/2]$.

Proof. Since $a(\cdot,\cdot)$ defined by (63) obeys the coercivity condition (72) then Lax-Milgram theorem applies and for $u \in \mathbf{H}^{3/2}(\Gamma)$ there exists a unique $w \in \mathbf{V}^1(\Omega)$ such that $\lambda_0(w, v) + a(w, v) = \int_\Gamma u \cdot v$ for all $v \in \mathbf{V}^1(\Omega)$. Thus, according to Lemma 5.1, and recalling (63), there exists $q \in L^2(\Omega)$ satisfying:

$$\begin{aligned} \int_\Omega \nu \nabla w : \nabla v - \int_\Omega q \nabla \cdot v = &- \int_\Omega ((\lambda_0 w + (w \cdot \nabla) z_e + (z_e \cdot \nabla) w) \cdot v \\ &+ \int_\Gamma (u - \tfrac{k}{2}(z_e)_n w + \tfrac{k}{2} w_n z_e)) \cdot v \quad \forall v \in \mathbf{H}^1(\Omega). \end{aligned} \tag{120}$$

Thus, since $z_e \in \mathbf{V}^2(\Omega)$ and $w \in \mathbf{V}^1(\Omega)$ guarantees $\lambda_0 w + (w \cdot \nabla) z_e + (z_e \cdot \nabla) w \in \mathbf{L}^2(\Omega)$ and $(z_e)_n w + w_n z_e \in \mathbf{H}^{1/2}(\Gamma)$ then a first application of Corollary 4 yields $(w, q) \in \mathbf{V}^2(\Omega) \times H^1(\Omega)$. Then it implies $(z_e)_n w + w_n z_e \in \mathbf{H}^{3/2}(\Gamma)$ and a second application of Corollary 4 yields $(w, q) \in \mathbf{V}^3(\Omega) \times H^2(\Omega)$. Moreover, an integration by part in (120) shows that (w, q) obeys:

$$\begin{aligned} \lambda_0 w - \nu \Delta w + (w \cdot \nabla) z_e + (z_e \cdot \nabla) w + \nabla q = 0 &\quad \text{in } \Omega, \\ \nabla \cdot w = 0 &\quad \text{in } \Omega, \\ \chi(w, q) = u &\quad \text{on } \Gamma, \end{aligned}$$

and separating the tengential and normal parts of above trace condition with the application of the divergence operator to the first above equation gives $q = Rw + Tu$ and $Nu = w \in \mathbf{V}^3(\Omega)$. Then (119) for $r = 3/2$ follows. Note that if Ω of class $C^{3,1}$, $z_e \in \mathbf{V}^3(\Omega)$ and $u \in \mathbf{H}^{5/2}(\Gamma)$ then a similar application of Corollary 4 yields $(w, q) \in \mathbf{V}^4(\Omega) \times H^3(\Omega)$ which gives (119) for $r = 5/2$. Then (119) for $r \in (3/2, 5/2)$ follows by interpolation.

Next, let us prove that N can be uniquely extended to a bounded operator from $\mathbf{H}^{-3/2}(\Gamma)$ into $\mathbf{V}^0(\Omega)$. For $f \in \mathbf{V}^0(\Omega)$ and φ solution to (118) we obtain $\int_\Omega Nu \cdot f = \int_\Gamma u \cdot \varphi$ by setting $v = \varphi$ in (120) and integrating by parts. Thus, by taking the sup over all $f \in \mathbf{V}^0(\Omega)$, with $\|\varphi\|_{\mathbf{H}^{3/2}(\Gamma)} \le C \|\varphi\|_{\mathbf{V}^2(\Omega)}$ and $\|\varphi\|_{\mathbf{V}^2(\Omega)} \le C \|f\|_{\mathbf{V}^0(\Omega)}$ we deduce that $\|Nu\|_{\mathbf{V}^0(\Omega)} \le C \|u\|_{\mathbf{H}^{-3/2}(\Gamma)}$, and the density of $\mathbf{H}^{3/2}(\Gamma)$ into $\mathbf{H}^{-3/2}(\Gamma)$ ensures that N can be extended to a bounded operator from $\mathbf{H}^{-3/2}(\Gamma)$ into $\mathbf{V}^0(\Omega)$ in a unique way. Finally, (119) follows with an interpolation argument. $\square$

Remark 6. *(i)* Note that to define a solution $w \in \mathbf{V}^0(\Omega)$ of (80) with the transposition method it is sufficient to have a boundary value u in $\mathbf{V}^{-3/2}(\Gamma)$ (the dual space of $\{y \in \mathbf{H}^{3/2}(\Gamma) \mid \int_\Gamma y \cdot n = 0\}$).
(ii) In fact, for all solution $w \in \mathbf{V}^0(\Omega)$ defined by transposition the trace condition:

$$\chi(w, Rw + Tu) = u \quad \text{on} \quad \Gamma,$$

is still valid. Indeed, from (117) one verifies that w belongs to the Hilbert space:

$$\Xi(\Omega) \stackrel{\text{def}}{=} \left\{(y, p) \in \mathbf{V}^0(\Omega) \times \mathcal{D}'(\Omega) \mid -\nu \Delta y + (y \cdot \nabla) z_e + (z_e \cdot \nabla) y + \nabla p \in \mathbf{L}^2(\Omega)\right\}$$

normed with

$$\|(y,p)\|_{\Xi(\Omega)} \stackrel{\text{def}}{=} \|y\|_{\mathbf{L}^2(\Omega)} + \| - \nu\Delta y + (y\cdot\nabla)z_e + (z_e\cdot\nabla)y + \nabla p\|_{\mathbf{L}^2(\Omega)}.$$

Moreover, we can verify $\mathbf{V}^2(\Omega)\times H^1(\Omega)$ is dense in $\Xi(\Omega)$ and arguing as in [42, Thm. 6.5, Chap. 2] we can prove that χ can be extended in a unique way to a bounded operator from $\Xi(\Omega)$ onto $\mathbf{V}^{-3/2}(\Gamma)$. Here is the argument. From Lemma 5.3, for all $b\in\mathbf{V}^{3/2}(\Gamma)$ we can choose $(v_b,p_b)\in\mathbf{V}^2(\Omega)\times H^1(\Omega)$, which continuously depends on $b\in\mathbf{V}^{3/2}(\Gamma)$ and such that $\chi_*(v_b,p_b)=0$ and $v_b=b$ on Γ. Thus, since an integration by parts ensures that every $(y,p)\in\mathbf{V}^2(\Omega)\times H^1(\Omega)$ obeys:

$$\begin{aligned}\int_\Gamma \chi(y,p)\cdot b = \int_\Omega (-\nu\Delta v_b - {}^t(\nabla v_b)z_e - (\nabla v_b)z_e + \nabla p_b)\cdot y\\ - \int_\Omega v_b\cdot(-\nu\Delta y + (y\cdot\nabla)z_e + (z_e\cdot\nabla)y + \nabla p),\end{aligned}$$

then by taking the supremum over all $b\in\mathbf{V}^{3/2}(\Gamma)$ we deduce that:

$$\|\chi(y,p)\|_{\mathbf{V}^{-3/2}(\Gamma)} \le C(\|y\|_{\mathbf{V}^0(\Omega)} + \|\nu\Delta y - (y\cdot\nabla)z_e - (z_e\cdot\nabla)y + \nabla p\|_{\mathbf{L}^2(\Omega)}).$$

Finally, the conclusion follows from a density argument.

5.2.4. ***Estimates for Navier-Stokes type nonlinearity***. Here we recall some boundedness properties of the trilinear forms $(v_1,v_2,v_3)\mapsto\int_\Omega(v_1\cdot\nabla)v_2\cdot v_3$ and $(v_1,v_2,v_3)\mapsto\int_\Gamma(v_1\cdot n)v_2\cdot v_3$ and of the bilinear map $(v_1,v_2)\mapsto(v_1\cdot n)v_2|_\Gamma$. Those are obtained from Sobolev embedding, see also [24, 15, 14].

Let s_1, s_2 and s_3 are real nonnegative numbers and $(v_1,v_2,v_3)\in\mathbf{H}^{s_1}(\Omega)\times\mathbf{H}^{s_2}(\Omega)\times\mathbf{H}^{s_3}(\Omega)$.

If $s_1+s_2+s_3\ge\frac{d+2}{2}$ if $s_i\ne\frac{d}{2}$, $i=1,3$ and $s_2\ne\frac{d+2}{2}$ or $s_1+s_2+s_3>\frac{d+2}{2}$ else, then we have:

$$\Big|\int_\Omega(v_1\cdot\nabla)v_2\cdot v_3\Big| \le C\|v_1\|_{\mathbf{H}^{s_1}(\Omega)}\|v_2\|_{\mathbf{H}^{s_2}(\Omega)}\|v_3\|_{\mathbf{H}^{s_3}(\Omega)}. \tag{121}$$

If $s_1\ge\frac12, s_2,s_3>\frac12$ and $s_1+s_2+s_3\ge\frac{d+2}{2}$ then we have:

$$\Big|\int_\Gamma(v_1\cdot n)v_2\cdot v_3\Big| \le C\|v_1\|_{\mathbf{H}^{s_1}(\Omega)}\|v_2\|_{\mathbf{H}^{s_2}(\Omega)}\|v_3\|_{\mathbf{H}^{s_3}(\Omega)}. \tag{122}$$

If $s_i>\frac12+s_i$, $i=1,2$ and $s_1+s_2\ge\frac{d+1}{2}+s_3$ then we have:

$$\|(v_1\cdot n)v_2\|_{\mathbf{H}^{s_3}(\Gamma)} \le C\|v_1\|_{\mathbf{H}^{s_1}(\Omega)}\|v_2\|_{\mathbf{H}^{s_2}(\Omega)}. \tag{123}$$

Acknowledgments. The author would like to thank the anonymous referee for helpful comments and pertinent critical remarks that helped to improve the presentation and that suggested valuable improvements and corrections.

REFERENCES

[1] C. Amrouche and V. Girault, *Problèmes généralisés de Stokes,* Portugal. Math., **49** (1992), 463–503.

[2] C. Amrouche and V. Girault, *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension,* Czechoslovak Math. J., **44** (1994), 109–140.

[3] G. Avalos and R. Triggiani, *The coupled PDE system arising in fluid/structure interaction. I. Explicit semigroup generator and its spectral properties*, In "Fluids and Waves," 15–54, Contemp. Math., **440**, Amer. Math. Soc., Providence, RI, 2007.

[4] G. Avalos and R. Triggiani, *Boundary feedback stabilization of a coupled parabolic-hyperbolic Stokes-Lamé PDE system*, J. Evol. Equ., **9** (2009), 341–370.

[5] G. Avalos and R. Triggiani, *Coupled parabolic-hyperbolic Stokes-Lamé PDE system: Limit behaviour of the resolvent operator on the imaginary axis*, Appl. Anal., **88** (2009), 1357–1396.

[6] G. Avalos and R. Triggiani, *Semigroup well-posedness in the energy space of a parabolic-hyperbolic coupled Stokes-Lamé PDE system of fluid-structure interaction*, Discrete Contin. Dyn. Syst. Ser. S, **2** (2009), 417–447.

[7] M. Badra, *Feedback stabilization of the 3-D Navier-Stokes equations based on an extended system*, In "Systems, Control, Modeling and Optimization," 13–24, IFIP Int. Fed. Inf. Process., **202**, Springer, New York, 2006.

[8] M. Badra, *Local stabilization of the Navier-Stokes equations with a feedback controller localized in an open subset of the domain*, Numer. Funct. Anal. Optim., **28** (2007), 559–589.

[9] M. Badra, *Feedback stabilization of the 2-D and 3-D Navier-Stokes equations based on an extended system*, ESAIM Control Optim. Calc. Var., **15** (2009), 934–968.

[10] M. Badra, *Lyapunov function and local feedback boundary stabilization of the Navier-Stokes equations*, SIAM J. Control Optim., **48** (2009), 1797–1830.

[11] M. Badra and T. Takahashi, *Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamical controllers: Application to the Navier-Stokes system,* SIAM J. Control Optim., **49** (2011), 420–463.

[12] V. Barbu, *Feedback stabilization of Navier-Stokes equations*, ESAIM Control Optim. Calc. Var., **9** (2003), 197–206 (electronic).

[13] V. Barbu, "Stabilization of Navier-Stokes Flow," Communications and Control Engineering, Springer Verlag, Berlin-London, 2010.

[14] V. Barbu, Z. Grujić, I. Lasiecka and A. Tuffaha, *Existence of the energy-level weak solutions for a nonlinear fluid-structure interaction model*, In "Fluids and Waves," 55–82, Contemp. Math, **440**, Amer. Math. Soc., Providence, RI, 2007.

[15] V. Barbu, Z. Grujić, I. Lasiecka and A. Tuffaha, *Smoothness of weak solutions to a nonlinear fluid-structure interaction model*, Indiana Univ. Math. J., **57** (2008), 1173–1207.

[16] V. Barbu, I. Lasiecka and R. Triggiani, *Abstract settings for tangential boundary stabilization of Navier-Stokes equations by high- and low-gain feedback controllers*, Nonlinear Anal., **64** (2006), 2704–2746.

[17] V. Barbu, I. Lasiecka and R. Triggiani, *Tangential boundary stabilization of Navier-Stokes equations*, Mem. Amer. Math. Soc., **181** (2006), :x+128 pp.

[18] V. Barbu, I. Lasiecka and R. Triggiani, *Local exponential stabilization strategies of the Navier-Stokes equations, $d = 2, 3$, via feedback stabilization of its linearization,* In "Control of Coupled Partial Differential Equations," 13–46, Internat. Ser. Numer. Math., **155**, Birkhäuser, Basel, 2007.

[19] V. Barbu, S. S. Rodrigues and A. Shirikyan, *Internal exponential stabilization for Navier-Stokes equations by means of finite-dimensional distributed controls*, 2010. Available from: `http://hal.archives-ouvertes.fr/hal-00455250/fr/`.

[20] V. Barbu and R Triggiani, *Internal stabilization of Navier-Stokes equations with finite-dimensional controllers*, Indiana Univ. Math. J., **53** (2004), 1443–1494.

[21] F. Boyer and P. Fabrie, "Éléments d'Analyse pour l'Étude de Quelques Modèles d'Écoulements de Fluides Visqueux Incompressibles," Mathématiques & Applications (Berlin) [Mathematics & Applications], **52**, Springer-Verlag, Berlin, 2006.

[22] F. Bucci and I. Lasiecka, *Optimal boundary control with critical penalization for a PDE model of fluid-solid interactions*, Calc. Var. Partial Differential Equations, **37** (2010), 217–235.

[23] F. Bucci and I. Lasiecka, *Regularity of boundary traces for a fluid-solid interaction model*, Discrete Contin. Dyn. Syst. Ser. S, **4** (2011), 505–521.

[24] P. Constantin and C. Foias, "Navier-Stokes Equations," Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988.

[25] R. Denk, M. Hieber and J. Prüss, *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc., **166** (2003), viii+114 pp.

[26] H. Fujita and H. Morimoto, *On fractional powers of the Stokes operator,* Proc. Japan Acad., **46** (1970), 1141–1143.

[27] A. V. Fursikov, *Real process corresponding to the 3D Navier-Stokes system, and its feedback stabilization from the boundary*, In "Partial Differential Equations," 95–123, Amer. Math. Soc. Transl. Ser. 2, **206**, Amer. Math. Soc., Providence, RI, 2002.

[28] A. V. Fursikov, *Stabilization for the 3D Navier-Stokes system by feedback boundary control. Partial differential equations and applications*, Discrete Contin. Dyn. Syst., **10** (2004), 289–314.
[29] P. Grisvard, *Caractérisation de quelques espaces d'interpolation,* Arch. Rational Mech. Anal., **25** (1967), 40–63.
[30] G. Grubb and V. A. Solonnikov, *Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods*, Math. Scand., **69** (1991), 217–290.
[31] S. Guerrero, *Local exact controllability to the trajectories of the Boussinesq system,* Ann. Inst. H. Poincaré Anal. Non Linéaire, **23** (2006), 29–61.
[32] E. Hille and R. S. Phillips, "Functional Analysis and Semi-Groups," rev. ed., American Mathematical Society Colloquium Publications, vol. 31, American Mathematical Society, Providence, R. I., 1957.
[33] I. Kukavica, A. Tuffaha and M. Ziane, *Strong solutions to a nonlinear fluid structure interaction system*, J. Differential Equations, **247** (2009), 1452–1478.
[34] I. Kukavica, A. Tuffaha and M. Ziane, *Strong solutions for a fluid structure interaction system*, Adv. Differential Equations, **15** (2010), 231–254.
[35] I. Kukavica, A. Tuffaha and M. Ziane, *Strong solutions to a Navier-Stokes-Lamé system on a domain with a non-flat boundary*, Nonlinearity, **24** (2011), 159–176.
[36] I. Lasiecka, *Exponential stabilization of hyperbolic systems with nonlinear, unbounded perturbations—Riccati operator approach*, Appl. Anal., **42** (1991), 243–261.
[37] I. Lasiecka and R. Triggiani, *The regulator problem for parabolic equations with Dirichlet boundary control. I. Riccati's feedback synthesis and regularity of optimal solution*, Appl. Math. Optim., **16** (1987), 147–168.
[38] I. Lasiecka and R. Triggiani, "Control Theory for Partial Differential Equations: Continuous and Approximation Theories. I. Abstract Parabolic Systems," Encyclopedia of Mathematics and its Applications, **74**, Cambridge University Press, Cambridge, 2000.
[39] I. Lasiecka and A. Tuffaha, *Optimal feedback synthesis for Bolza control problem arising in linearized fluid structure interaction*, In "Optimal Control of Coupled Systems of Partial Differential Equations," 171–190, Internat. Ser. Numer. Math., **158**, Birkhäuser Verlag, Basel, 2009.
[40] I. Lasiecka and A. Tuffaha, *Riccati theory and singular estimates for a Bolza control problem arising in linearized fluid-structure interaction*, Systems Control Lett., **58** (2009), 499–509.
[41] J.-L. Lions, "Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires," Dunod, Gauthier-Villars, Paris, 1969.
[42] J.-L. Lions and E. Magenes, "Problemes aux Limites Non Homognes et Applications," Vol. I, Dunod, Paris, 1968.
[43] J.-P. Raymond, *Feedback boundary stabilization of the two-dimensional Navier-Stokes equations*, SIAM J. Control Optim., **45** (2006), 790–828 (electronic).
[44] J.-P. Raymond, *Feedback boundary stabilization of the three-dimensional incompressible Navier-Stokes equations*, J. Math. Pures Appl. (9), **87** (2007), 627–669.
[45] J.-P. Raymond, *Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **24** (2007), 921–951.
[46] J.-P. Raymond, *Feedback stabilization of a fluid-structure model,* SIAM J. Control Optim., **48** (2010), 5398–5443.
[47] J.-P. Raymond and L. Thevenet, *Boundary feedback stabilization of the two dimensional Navier-Stokes equations with finite dimensional controllers*, Discrete and Continuous Dynamical Systems A, **27** (2010), 1159–1187.
[48] H. Triebel, "Interpolation Theory, Function Spaces, Differential Operators," Second edition, Johann Ambrosius Barth, Heidelberg, 1995.
[49] R. Triggiani, *Unique continuation from an arbitrary interior subdomain of the variable-coefficient Oseen equation*, Nonlinear Anal., **71** (2009), 4967–4976.
[50] R. Triggiani, *Unique continuation of boundary over-determined Stokes and Oseen eigenproblems*, Discrete Contin. Dyn. Syst. Ser. S, **2** (2009), 645–677.

Received October 2010; revised May 2011.

E-mail address: mehdi.badra@univ-pau.fr