ADVANCES IN MATHEMATICS 22, 312–355 (1976)

On the Stability of Waves of Nonlinear Parabolic Systems

D. H. SATTINGER*

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

DEDICATED TO THE MEMORY OF PROFESSOR NORMAN LEVINSON

1. INTRODUCTION

Consider the nonlinear parabolic system

$$\frac{\partial u_i}{\partial t} = D_i \frac{\partial^2 u_i}{\partial x^2} - f_i\left(u_1, \ldots, u_n, \frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_n}{\partial x}\right), \tag{1.1}$$

where the functions f_i are smooth functions of their arguments and the D_i are positive numbers. System (1.1) may be written in the abbreviated form

$$u_t = Au + f(u, u_x), \tag{1.2}$$

where

$$u = (u_1, \ldots, u_n), \qquad Au = \left(D_1 \frac{\partial^2 u_1}{\partial x^2}, \ldots\right), \qquad \text{and} \qquad u_x = \left(\frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_n}{\partial x}\right).$$

This paper is concerned with the *stability* of a special class of solutions of (1.2), namely, traveling wave solutions of the form

$$u(x, t) = \varphi(x + ct), \tag{1.3}$$

where c is a constant. Wave solutions of the type (1.3) arise in numerous problems of physical interest; for example, propagation of nerve impulses [1, 6, 7, 12, 13]; propagation of favorable genes [1, 2, 23]; shock waves [3, 9. 11]; and flame propagation [16–20]. The reader is referred to the excellent article [1] by Cohen for further discussion and examples.

* The preparation of this research was supported in part by NSF Grant No. GP 37512.

If one looks for solutions of (1.2) which are perturbations of a traveling wave, that is, for solutions of the form

$$u(x, t) = \varphi(x + ct) + v(x, t)$$

and transforms to a moving coordinate frame $\xi = x + ct$, he then obtains an equation of the following type for v.

$$v_t = Lv + R(v), \tag{1.4}$$

where L is a second-order linear differential operator and R is a nonlinear first-order operator. (The precise forms of L and R will be derived in the course of our investigation.) One easily sees that $\varphi'(\xi)$ is a null function of L:

$$L\varphi' = 0$$

so that 0 is in general a member of $\sigma(L)$, the spectrum of L. This fact is an immediate consequence of the invariance of equations (1.2) under the group of translations $x \to x + \gamma$. Since 0 belongs to $\sigma(L)$ one cannot expect to get asymptotic stability of the waves in a strict sense; that is, if the initial data are of the form

$$u(x, 0) = \varphi(x) + \epsilon u_0(x),$$

then in general one cannot expect that

$$[u(\xi, t) - \varphi(\xi)] \to 0 \qquad \text{as} \quad t \to \infty.$$

Rather, one should expect that at best

$$[u(\xi, t) - \varphi(\xi + \gamma)] \to 0 \qquad \text{as} \quad t \to \infty, \tag{1.5}$$

for some suitably chosen phase shift γ.

The situation in the present case is analogous to the phenomenon of orbital stability of periodic motions of an autonomous system. If $u(t)$ is a periodic solution of the system of ordinary differential equations

$$\dot{x} = f(x), \tag{1.6}$$

then a small perturbation of the form $u(0) + \epsilon\,\delta u$ tends, as $t \to \infty$, not to $u(t)$ but to $u(t + \delta)$ for some suitable phase shift δ. Such stability is called *orbital stability* [10] and is connected with the fact that (1.6) is invariant under the group of time translations $t \to t + \delta$.

In this paper we prove a general stability theorem (Theorem 4.1) to the effect that φ is asymptotically stable to small perturbations (in the sense of (1.5)) provided that $\sigma(L)$ consists of an isolated eigenvalue at the origin together with some set lying interior to a parabola in the left half-plane. This statement will be made more precise in later sections, after an appropriate norm has been introduced.

A stability result of this type has been proved in an elegant manner by Evans [6] for a special class of systems which occur in the theory of nerve axon impulses. In Evans' case the nonlinear term f is independent of u_x while some of the diffusion coefficients D_i are zero. Our approach is quite different from Evans', and we proceed by reducing the problem to an application of the implicit function theorem on a Banach space. Our theory also differs from Evans' in the introduction of weighted norms.

The stability theorem is proved in Sections 2–4. In Section 5 we analyze the resolvent transformation $(\lambda - L)^{-1}$ in the case of a single equation

$$u_t = u_{xx} + f(u, u_x) \tag{1.7}$$

(u, f scalars). Under many circumstances of physical interest, (1.7) admits wave solutions φ which are monotone and tend to finite limits as $\xi \to \pm\infty$. The stability of such waves can be demonstrated in a number of cases by use of a maximum principle argument in the spectral analysis of L. This argument, which seems first to have been advanced by Gelfand [20], runs briefly as follows. Since $\varphi' > 0$ and $L\varphi' = 0$, L has a positive null function. Using a well-known maximum principle argument [21], and giving due care to the asymptotic behavior of solutions of $(L - \lambda)\psi = 0$, we can show that no eigenvalue of L has positive real part.

One consequence of our analysis is the systematic, advantageous introduction of appropriate weighted norms. Thus, instead of considering the perturbation equation (1.4) on the Banach space L_∞ with norm

$$\| u \|_0 = \sup_x | u(x)|$$

we consider (1.4) on a Banach space $\mathscr{B}_{w,0}$ with a weighted norm

$$\| u \|_{w,0} = \sup_x | w(x) u(x)|,$$

where $w(x) \geqslant 1$ is an appropriately chosen weight function. When considered on such a weighted space the operator L may have much "nicer" spectral properties than it would have on L_∞.

Our theory is applied to a number of well-known examples in Section 6. One of the most interesting examples is the case of "KPP" waves [2]. Kolmogorov, Petrovsky, and Piskunov [2] investigated traveling wave solutions of the equation

$$u_t = u_{xx} + u(1 - u);$$

and found that this equation admits a family of waves $\varphi(x + ct, c)$ for $| c | \geqslant 2$. A complete spectral analysis of the associated linear operator L can be given in this case. If L is considered as an operator on the Banach space L_∞ it has a continuous family of eigenvalues distributed over the interval $-1 < \lambda < 1$. However, if the weight function

$$w(\xi) = 1 + e^{-c\xi/2}$$

is introduced, then $\sigma(L)$ is contained in the region

$$\left\{\mathrm{Re}\sqrt{\lambda + \left(1 + \frac{c^2}{4}\right)} \leqslant \frac{c}{2}\right\} \cup \left\{-1 \leqslant \lambda \leqslant 1 - \frac{c^2}{4}\right\}.$$

The region

$$\left\{\mathrm{Re}\sqrt{\lambda + \left(1 + \frac{c^2}{4}\right)} \leqslant \frac{c}{2}\right\}$$

is bounded by a parabola which extends to infinity in the left half-plane and which crosses the real axis at $\lambda = -1$. In the case $c = 2$ the continuous spectrum of L extends along the interval $[-1, 0]$.

Waves for Burgers' equation

$$u_t + uu_x = u_{xx}$$

are also analyzed (see [3, 5]). The appropriate weight function in this case turns out to be

$$w(\xi) = \cosh \frac{c\,|\,\xi\,|}{2}.$$

In that case a nice stability theory for the waves can be developed. The spectrum of L consists of a continuous spectrum on the interval $-\infty < \lambda < -c^2/4$ together with an isolated eigenvalue at the origin.

The author would like to acknowledge a number of useful conversations with his colleagues, Professors H. Weinberger and D. Aronson, with respect to the subject matter of this paper. Special appreciation is

accorded D. Aronson for his interesting seminar lectures on the subject of stability of diffusion waves, which did much to stimulate the author's interest in the subject.

2. Linear Stability Theory of Waves

Consider the nonlinear parabolic system

$$\begin{aligned} u_t &= Au + f(u, u_x), \\ u(x, 0) &= u_0(x), \end{aligned} \tag{2.1}$$

where $u = (u^1, \ldots, u^n)$, A is a second-order elliptic operator with constant coefficients, and f is a vector valued function of $u^1, \ldots, u^n, \partial u^1/\partial x, \ldots, \partial u^n/\partial x$. We assume f is at least C^4 in all its arguments. Suppose that (2.1) admits solutions of the form

$$u(x, t, c) = \varphi(\xi, c), \qquad \xi = x + ct, \tag{2.2}$$

for some value, or values, of the wave speed c. Since (2.1) is autonomous (f and A are independent of x and t), the wave φ in (2.2) gives rise, for each c, to a one parameter family of waves

$$\varphi(\xi + \gamma, c). \tag{2.3}$$

Let $\mathscr{B}$ denote the class of all continuous bounded vector valued functions on $\{-\infty < x < \infty\}$. The family (2.3) can be regarded as a one-dimensional manifold $\mathscr{M}$ in $\mathscr{B}$, and we are interested in the *stability* of $\mathscr{M}$. Let $w(x)$ be a (smooth) positive weight function, $w(x) \geqslant 1$, and denote by $\| \; \|_{w,0}$ the norm

$$\| u \|_{w,0} = \sup_x \sup_{1 \leqslant i \leqslant n} | u_i w(x) |.$$

Define $\| \; \|_{w,j}$ by

$$\| u \|_{w,j} = \| u \|_{w,0} + \| u_x \|_{w,0} + \cdots + \left\| \frac{d^j u}{dx^j} \right\|_{w,0},$$

and let $\mathscr{B}_{w,j}$ be the Banach space of functions on $\{-\infty < x < \infty\}$ with finite $\| \; \|_{w,j}$ norm. We assume that the waves φ in (2.3) belong to $\mathscr{B}_{w,1}$. The distance of a function u in $\mathscr{B}_{w,1}$ from $\mathscr{M}$ may be defined by

$$\rho(u, \mathscr{M}) = \inf_\gamma \| u(x) - \varphi(x + \gamma, c) \|_{w,1},$$

where γ ranges over $-\infty < \gamma < \infty$.

DEFINITION 2.1. *The wave φ is asymptotically stable in the norm $\| \|_{w,1}$ if there is a $\delta > 0$ such that*

$$\lim_{t\to\infty} \rho(u, \varphi) = 0 \tag{2.4}$$

whenever

$$\rho(u_0, \varphi) < \delta.$$

(Here $u(x, t)$ is the solution of (2.1) *with initial data u_0.)*

Condition (2.4) certainly holds if there exist γ such that

$$\lim_{t\to\infty} \| u(x, t) - \varphi(x + ct + \gamma, c)\| = 0.$$

Let the initial conditions be given in the form

$$u(x, 0) = \varphi(x, c) + \epsilon u_0(x),$$

and let us suppose the solution to (2.1) be written in the form

$$u(x, t) = \varphi(x + ct + \gamma, c) + \epsilon v(x, t, \epsilon),$$

where $\| v \|_{w,1} \to 0$ as $t \to \infty$ and $\gamma(\epsilon) = \epsilon h(\epsilon)$. We derive an equation for the perturbation v as follows. Letting $\xi = x + ct$ we get, in this moving coordinate system,

$$\begin{aligned} v_t &= Av - cv' + f_\varphi v + f_{\varphi'} v_\xi + \epsilon R(v, v'), \\ v(\xi, 0, \epsilon) &= y_0(\xi) + \frac{\varphi(\xi, c) - \varphi(\xi + \gamma, c)}{\epsilon}, \end{aligned} \tag{2.5}$$

where

$$R(\xi, v, v', \epsilon) = \epsilon^{-2}\{f(\varphi + \epsilon v, \varphi' + \epsilon v') - f(\varphi, \varphi') - ef_\varphi v - ef_{\varphi'} v'\} \tag{2.6}$$

and f_φ, $f_{\varphi'}$ denote the matrices

$$\frac{\partial f_i}{\partial \varphi^j} \quad \text{and} \quad \frac{\partial f_i}{\partial \varphi_\xi{}^j},$$

respectively. All quantities above are evaluated at $\xi + \epsilon h$. Primes denote differentiation with respect to ξ.

In the next section we show that R is a smooth operator and remains bounded as $\epsilon \to 0$. Letting $\epsilon \to 0$ in (2.5) we get, formally,

$$\begin{aligned} v_t &= Lv, \\ v(\xi, 0) &= u_0(\xi) - h(0)\varphi', \end{aligned} \tag{2.7}$$

where the linear operator L is given by

$$Lv = Av - cv' + f_{\varphi'}v' + f_{\varphi}v. \tag{2.8}$$

Equations (2.7) constitute the linearized stability problem for the perturbations.

Let us note that formally 0 belongs to $\sigma(L)$. In fact, in the moving coordinate system $\xi = x + ct$ we have the following equation for φ.

$$0 = A\varphi + \epsilon\varphi' + f(\varphi, \varphi'). \tag{2.9}$$

Differentiating this equation with respect to ξ we get

$$0 = A\varphi' - c(\varphi')' + f_{\varphi}\varphi' + f_{\varphi'}\varphi'',$$
$$0 = L\varphi'.$$

If 0 is an isolated point in $\sigma(L)$ and if the rest of $\sigma(L)$ lies strictly in the left half-plane then the solution of (2.7) tends asymptotically, as $t \to \infty$, to

$$\begin{aligned} Pv(\xi, 0) &= Pu_0 - hP\varphi' \\ &= Pu_0 - h\varphi', \end{aligned}$$

where P is the projection onto the null space of L. Now 0 being a simple eigenvalue of L, P has the form

$$Pu = \langle u, e^* \rangle \varphi',$$

where e^* is an element of the dual space $\mathscr{B}_{w,0}^*$ such that $\langle \varphi', e^* \rangle = 1$. Therefore the solution v of (2.7) tends to zero as $t \to \infty$ iff

$$h - \langle u_0, e^* \rangle = 0, \tag{2.10}$$

and this condition uniquely determines h.

This procedure gives us the solution v and the phase shift γ to first order in ϵ. Our aim, in the next two sections, is to extend this analysis to the full nonlinear problem (viz., $\epsilon \neq 0$).

3. Preliminary Analysis

In this section we establish a number of technical results which will be needed in the course of our investigations. Below φ denotes the wave solution and w denotes an arbitrary (smooth) weight function.

LEMMA 3.1. *Suppose $\varphi \in \mathscr{B}_{w,4}$. Then the initial data in (2.5) can be written in the form*

$$v(\xi, 0, \epsilon) = u_0 - h\varphi' + \epsilon g(\epsilon, \xi, h), \tag{3.1}$$

where $g(\epsilon, \xi, h)$ is a Frechet differentiable mapping from R^2 to $\mathscr{B}_{w,1}$; and g is uniformly bounded as $\epsilon \to 0$ if h remains bounded.

Proof. By comparing (3.1) with (2.5) we see that

$$g(\epsilon, \xi, h) = (1/\epsilon^2)\{\varphi(\xi) - \varphi(\xi + \epsilon h) + \varphi'(\xi)\, \epsilon h\}.$$

By Taylor's theorem this can be written

$$g(\epsilon, \xi, h) = -h^2 \int_0^1 \tau\varphi''(\xi + \epsilon h(1 - \tau)\, d\tau.$$

Assuming $\varphi \in \mathscr{B}_{w,3}$ then $g \in \mathscr{B}_{w,1}$ for each fixed h and ϵ and clearly is uniformly bounded as $\epsilon \to 0$. The Frechet derivative of g with respect to h, when g is considered as a mapping $h \to \mathscr{B}_{w,1}$ is the linear operator

$$\delta h \to (\partial g/\partial h)\, \delta h = \left(-2h \int_0^1 (1 - \tau)\, \varphi''(\xi - \tau\epsilon h)\, d\tau\right) \delta h$$
$$+ \left(h^2 \epsilon \int_0^1 \tau(1 - \tau)\, \varphi'''(\xi - \tau\epsilon h)\, d\tau\right) \delta h.$$

If $\varphi \in \mathscr{B}_{w,4}$ then $\partial g/\partial h$ is a bounded transformation from R to $\mathscr{B}_{w,1}$. The details of the verification are left to the reader.

LEMMA 3.2. *Let $f \in C^2$ and let $\varphi \in \mathscr{B}_{w,2}$. Then the linear operator*

$$Cv = f_\varphi \,|_{\xi+\gamma}\, v + f_{\varphi'} \,|_{\xi+\gamma}\, v',$$

where $f_\varphi \,|_{\xi+\gamma} = f_\varphi(\varphi(\xi + \epsilon h), \varphi'(\xi + \epsilon h))$, etc., takes the form

$$Cv = f_\varphi \,|_\xi\, v + f_{\varphi'} \,|_\xi\, v_\xi + \epsilon Bv,$$

where B is a bounded transformation from $\mathscr{B}_{w,1}$ to $\mathscr{B}_{w,0}$ which is uniformly bounded as $\epsilon \to 0$.

Proof. We again apply Taylor's theorem, this time to functions of two variables. Thus

$$f_\varphi(\varphi(\xi + \epsilon h), \varphi'(\xi + \epsilon h)) = f_\varphi(\varphi(\xi), \varphi'(\xi)) + \epsilon h \int_0^1 (f_{\varphi\varphi}\varphi' + f_{\varphi\varphi'}\varphi'')\, d\tau,$$

where the arguments in the integrand are evaluated at the point $\xi + \epsilon\tau h$. The second expression, namely, that arising from $f_{\varphi'}v'$, may be represented in a similar manner. Thus B is certainly a uniformly bounded operator from $\mathscr{B}_{w,1}$ to $\mathscr{B}_{w,0}$ if f is in C^2 and $\varphi \in \mathscr{B}_{w,2}$.

LEMMA 3.3. *The remainder term R in Eqs.* (2.5) *is a Frechet differentiable map from $\mathscr{B}_{w,1} \times R$ to $\mathscr{B}_{w,0}$ provided $f \in C^3$.*

Proof. We first note that with our choice of norms, the spaces $\mathscr{B}_{w,j}$ are Banach algebras, that is,

$$\| uv \|_{w,j} \leqslant \| u \|_{w,j} \| v \|_{w,j}$$

for any j. Applying Taylor's theorem as in Lemma 3.1, we get

$$R(\xi, v, v', \epsilon) = -\int_0^1 (\tau - 1)[f_{\varphi\varphi}v^2 + 2f_{\varphi\varphi'}vv' + f_{\varphi'\varphi'}(v')^2]\, d\tau, \qquad (3.2)$$

where $f_{\varphi\varphi}$, etc., are evaluated at $(\varphi(\xi + \gamma\tau) + \epsilon\tau v(\xi),\ \varphi'(\xi + \gamma\tau) + \epsilon\tau v^1(\xi))$. Then, since

$$\int_0^1 (\tau - 1) f_{\varphi\varphi}\, d\tau, \qquad \int_0^1 f_{\varphi\varphi'}(\tau - 1)\, d\tau, \qquad \int_0^1 f_{\varphi'\varphi'}(\tau - 1)\, d\tau$$

are uniformly bounded in ξ provided v, v' are bounded, the estimate

$$\| R(\xi, v, v', \epsilon)\|_{w,0} \leqslant C \| v \|_{w,1}^2$$

follows easily. To investigate the differentiability of R we must differentiate (3.2) with respect to v and v'. The reader may do this and will see that the resulting partial derivatives are uniformly bounded provided f is in C^3 and $v \in \mathscr{B}_{w,1}$.

We assume that L has an isolated eigenvalue at the origin. This means that the resolvent transformation $(\lambda - L)^{-1}$ has a simple pole at the origin. Let

$$P = \frac{1}{2\pi i} \int_C (\lambda - L)^{-1}\, d\lambda,$$

where C is a circle enclosing the origin and no other points in $\sigma(L)$. Then P is a projection onto the null space of L (see [23, p. 418]). Furthermore, $PL = LP = 0$, and P has the form

$$Pu = \langle u, e^* \rangle\, \varphi',$$

where e^* is an element of the dual space $\mathscr{B}^*_{w,0}$. Since $PLu = 0$,

$$\langle Lu, e^* \rangle = 0 \tag{3.3}$$

for all $u \in C_0^\infty(\mathbb{R})$. The linear functional $u \to \langle u, e^* \rangle$ is easily seen to be a distribution in the sense of Schwarz. Since the coefficients of L are smooth functions, L has a well-defined adjoint, and (3.3) is a statement of the fact that

$$L^* e^* = 0 \tag{3.4}$$

in the sense of distributions. Since (3.4) is a system of ordinary differential equations, however, e^* can be identified with a smooth function φ^* which satisfies $L^*\varphi^* = 0$ in the classical sense.

In one special class of wave problems we may determine φ exactly, namely, when the equations are of the form

$$u_t = u_{xx} + f(u),$$

with f, u scalars. The wave $\varphi(\xi)$ satisfies the ordinary differential equation

$$\varphi'' - c\varphi' + f(\varphi) = 0$$

and the operators L and L^* are

$$Lw = w'' - cw' + f'(\varphi)w$$

$$L^*v = v'' + cv' + f'(\varphi)v.$$

Now $L\varphi' = 0$ and, putting $e^* = e^{-c\xi}\varphi'(\xi)$ we see that $L^*e^* = 0$, so the adjoint eigenfunction is $e^{-c\xi}\varphi'(\xi)$. The functional in (2.10) in this case is therefore

$$\langle u_0, e^* \rangle = \int_{-\infty}^{\infty} u_0(\xi)\, e^{-c\xi}\varphi'(\xi)\, d\xi.$$

LEMMA 3.4. *Let the operator L given by* (2.8) *satisfy the following hypotheses* (*considered as a transformation on $\mathscr{B}_{w,0}$*).

(i) *L has an isolated simple eigenvalue at the origin while the remainder of its spectrum lies in the parabolic region $\mathscr{P} = \{y^2 + a + x < 0\}$* ($a > 0$) *in the left half-plane.*

(ii) *The resolvent transformation* $(\lambda - L)^{-1}$ *has the following asymptotic behavior. Given any* $\delta > 0$ *there is a constant* $c(\delta)$ *such that*

$$\|(\lambda - L)^{-1} u\|_{w,1} \leqslant \frac{c(\delta)}{\mathscr{F}|\lambda|} \| u \|_{w,0}$$

for all λ *in* $|\arg \lambda| \leqslant \pi - \delta$ *and exterior to* $\mathscr{P}$.

Let $Q = I - P$ *and let* u *satisfy the initial value problem*

$$\begin{aligned} u_t &= Lu + Qh, \\ u(0) &= 0, \end{aligned} \tag{3.5}$$

for $h(t) \in \mathscr{B}_{w,0}$ *for* $t \geqslant 0$. *Then for any* ω, $0 < \omega < a$, *there is a constant* $c(\omega)$ *such that*

$$\| u(t)\|_{w,1} \leqslant c(\omega) \int_0^t \frac{e^{-\omega s}}{\sqrt{s}} \| h(t - s)\|_{w,0} \, ds. \tag{3.6}$$

Proof. Let e^{-tL} denote the semigroup generated by the operator L. The solution of (3.5) can be represented in the form

$$u(t) = \int_0^t e^{-(t-s)L} Qh(s) \, dx.$$

By assumption (ii) the operator $e^{-tL}Q$ can be shown to have the representation

$$e^{-tL}Q = \frac{1}{2\pi i} \int_C e^{-s\lambda} (\lambda - L)^{-1} \, d\lambda, \tag{3.7}$$

where C is any contour which lies strictly in the left half-plane, (thus avoiding the origin) and tends to infinity along rays in the left half-plane (see Fig. 3.1). (This is a standard result from the operational calculus of analytic semigroups; a proof is given in Appendix A.) By the estimate (ii) there exists a constant C such that

$$\begin{aligned} \| e^{-tL} Qh \|_{w,1} &\leqslant \frac{1}{2\pi} \int_C | e^{-t\lambda} | \, \|(\lambda - L)^{-1} Qh \|_{w,1} \, | d\lambda | \\ &\leqslant C \int \frac{| e^{-t\lambda} |}{\sqrt{|\lambda|}} \| h \|_{w,0} \, | d\lambda |. \end{aligned}$$

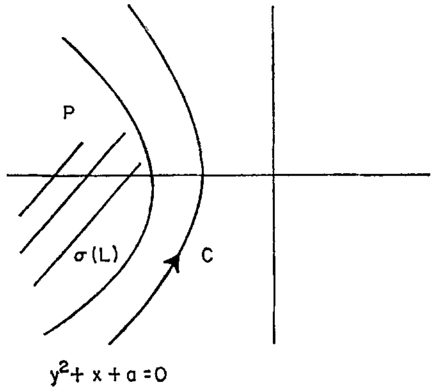

FIGURE 3.1

(Note that Q is a bounded transformation on $\mathscr{B}_{w,1}$ since $\mathscr{P}$ is bounded and $Q = I - P$.) We now choose the contour C to consist of the two rays $\lambda = -\omega + \rho e^{\pm i\gamma}$, where $0 < \omega < a$, $0 < \rho < \infty$, and $\pi/2 < \gamma < \pi$ is chosen so that the rays do not enter the region $\mathscr{P}$. Then

$$\int_C \frac{|e^{-t\lambda}|}{|\lambda|^{1/2}} |d\lambda| = 2\int_0^\infty \frac{e^{-\omega t} e^{\rho t \cos\gamma}}{[(\rho\cos\gamma - \omega)^2 + \rho^2 \sin^2\gamma]^{1/4}} d\rho$$

$$\leqslant 2\int_0^\infty \frac{e^{-\omega t} e^{\rho t \cos\gamma}}{(\omega^2 + \rho^2 \sin^2\gamma)^{1/4}} d\rho$$

$$\leqslant 2^{5/4}\int_0^\infty \frac{e^{-\omega t} e^{\rho t \cos\gamma}}{(\omega + \rho\sin\gamma)^{1/2}} d\rho$$

$$\leqslant 2^{7/4} e^{-\omega t}\int_0^\infty \frac{e^{\rho t \cos\gamma}}{\sqrt{\omega} + \sqrt{\rho\sin\gamma}} d\rho.$$

Let

$$I(x) = \int_0^\infty \frac{e^{-|x|\rho}}{\sqrt{\omega} + \sqrt{\rho\sin\gamma}} d\rho.$$

Then

$$I(x)\,|x|^{1/2} \leqslant \int_0^\infty \frac{e^{-|x|\rho}}{\sqrt{\rho\sin\gamma\,|x|}} d(\rho\,|x|) = \frac{1}{\sqrt{\sin\gamma}}\int_0^\infty \frac{e^{-\tau}\,d\tau}{\sqrt{\tau}}.$$

Therefore

$$\int_C \frac{|e^{-t\lambda}|\,|d\lambda|}{\sqrt{|\lambda|}} \leqslant C\frac{e^{-\omega t}}{\sqrt{t}},$$

and

$$\begin{aligned}\| u(t)\|_{w,1} &\leqslant \int_0^t \| e^{-(t-s)L}Qh(s)\|_{w,1}\, ds \\ &\leqslant C\int_0^t \frac{e^{-\omega(t-s)}}{\sqrt{t-s}} \| h(s)\|_{w,0}\, ds \\ &= C\int_0^t \frac{e^{-\omega s}}{\sqrt{s}} \| h(t-s)\|_{w,0}\, ds.\end{aligned}$$

We also obtain the estimate

$$\| e^{-tL}Qh \|_{w,1} \leqslant Ce^{-\omega t}/\sqrt{t} \| h(t)\|_{w,0} .$$

We now introduce the following norms on functions defined on the half-space $\{-\infty < x < \infty, t > 0\}$: For $\omega > 0$,

$$\| u \|_{w,j,\omega} = \sup_{t<0} e^{\omega t} \| u(\cdot, t)\|_{w,j} .$$

We denote the corresponding Banach spaces of continuously differentiable functions by $\mathscr{E}^{\omega}_{w,j}$. Let $Q\mathscr{B}_{w,j}$ and $Q\mathscr{E}^{\omega}_{w,j}$ denote the Banach subspaces of functions u for which $Pu = 0$ and $Pu(t) = 0$ for all $t \geqslant 0$, respectively. Thus

$$\begin{aligned}Q\mathscr{B}_{w,j} &= \{u: u \in \mathscr{B}_{w,j} , \langle u, e^*\rangle = 0\}, \\ Q\mathscr{E}^{\omega}_{w,j} &= \{u: u \in \mathscr{E}^{\omega}_{w,j} , \langle u(t), e^*\rangle = 0 \text{ for } t \geqslant 0\}.\end{aligned}$$

The solution of Eq. (3.5) defines a transformation $h \to u$ from $\mathscr{E}^{\omega}_{w,0}$ to $Q\mathscr{E}^{\omega}_{w,1}$ for $\omega < a$. Let us denote this transformation by K.

LEMMA 3.5. *The transformation K is a bounded transformation from $\mathscr{E}^{\omega}_{w,0}$ to $Q\mathscr{E}^{\omega}_{w,1}$ for any ω, $0 < \omega < a$.*

Proof. By Lemma 3.4 we have

$$\| u(t)\|_{w,1} \leqslant C(\omega')\int_0^t \frac{e^{\omega' s}}{\sqrt{s}} \| h(t-s)\|_{w,0}\, ds,$$

where $0 < \omega < \omega' < a$. Since $h \in \mathscr{E}^{\omega}_{w,0}$, $\| h(s)\|_{w,0} \leqslant \| h \|_{w,0,\omega} e^{-\omega s}$, so

$$\begin{aligned}\| u(t)\|_{w,1} &\leqslant C(\omega') \| h \|_{w,0,\omega} \int_0^t \frac{e^{-\omega' s}}{\sqrt{s}} e^{-\omega(t-s)}\, ds \\ &\leqslant C(\omega') \| h \|_{w,0,\omega} e^{-\omega t} \int_0^\infty \frac{e^{(\omega-\omega')s}}{\sqrt{s}}\, ds \\ &\leqslant \text{const.} \| h \|_{w,0,\omega} e^{-\omega t},\end{aligned}$$

where the constant depends on $(\omega - \omega')$.

It remains to show that $Pu(t) \equiv 0$; however,

$$Pu(t) = P \int_0^t e^{-(t-s)L} Qh(s)\, ds = \int_0^t e^{-(t-s)L} PQh\, ds = 0$$

since $PQ = 0$.

4. The Stability Theorems

THEOREM 4.1. *Let the operator L satisfy the conditions of Lemma 3.4, and assume $f \in C^3$ and $\varphi(\xi) \in \mathscr{B}_{w,4}$. Let $u(x, t)$ satisfy the initial value problem (2.1) with initial data of the form*

$$u(x, 0) = \varphi(x) + \epsilon u_0(x),$$

where $u_0 \in \mathscr{B}_{w,1}$. Let $\omega < a$. Then for sufficiently small ϵ there exists a C^1 function $\gamma(\epsilon)$ and a constant $K(\omega)$ such that

$$\| u(\xi, t) - \varphi(\xi + \gamma(\epsilon))\|_{w,1} \leqslant Ke^{-\omega t}, \qquad t \geqslant 0.$$

The function $\gamma(\epsilon)$ is of the form $\gamma = \epsilon h(\epsilon)$, where h is continuous and tends to a finite limit as $\epsilon \to 0$, namely,

$$h(0) = \langle u_0 , e^* \rangle,$$

where e^* is the adjoint eigenfunction discussed in the previous section.

Proof. We introduce a moving coordinate frame $\xi = x + ct$ and linearize Eqs. (2.1) about the wave $\varphi(\xi + \gamma(\epsilon))$, where γ is to be determined. As in Section 2, we get

$$v_t = Lv + \epsilon Bv + \epsilon R(\xi, v, v', \epsilon), \tag{4.1}$$

$$v(\xi, 0, \epsilon) = u_0(\xi) - h\varphi' + \epsilon g(\epsilon, h, \xi), \tag{4.2}$$

where the prime denotes differentiation with respect to ξ, and $\gamma(\epsilon) = \epsilon h(\epsilon)$. The operator B in (4.1) is the operator introduced in Lemma 3.2. We now write

$$v(t) = Pv + Qv = p(t)\, \varphi' + \zeta(t), \tag{4.3}$$

where P, Q are the projections introduced in Section 3 and $p(t) = \langle v(t), e^*\rangle$. Substituting (4.3) into (4.1) we get

$$\zeta_t = L\zeta + \epsilon QBv + \epsilon QR(\xi, v, v'),$$
$$\dot{p} = \epsilon\langle Bv, e^*\rangle + \epsilon\langle R(\xi, v, v'), e^*\rangle.$$

Operating on (4.2) with the projection P we get

$$p(0) = \langle u_0, e^*\rangle - h + \epsilon\langle g(\epsilon, \xi, h), e^*\rangle,$$

since e^* is normalized so that $\langle\varphi', e^*\rangle = 1$. Inverting the equation for ζ we get, in the notation of Lemma 3.5,

$$\zeta = \epsilon K[QBv + QR(v, v')] + e^{-tL}Q[u_0 + \epsilon g(\epsilon, \xi, h)], \tag{4.4}$$

since $\zeta(0) = Q[u_0 + g(\epsilon, \xi, h)]$. On the other hand,

$$p(t) = \epsilon\int_0^t \langle Bv, e^*\rangle + \langle R(v, v'), e^*\rangle\, ds + p(0).$$

Our intention being to construct a solution of (4.1)–(4.2) which tends to zero as $t \to \infty$, we set

$$\begin{aligned} 0 &= \epsilon\int_0^\infty \langle Bv, e^*\rangle + \langle R(v, v'), e^*\rangle\, ds + p(0) \\ &= \epsilon\int_0^\infty \langle Bv + R(v, v'), e^*\rangle\, ds + \langle u_0, e^*\rangle - h + \epsilon\langle g(\epsilon, \xi, h), e^*\rangle. \end{aligned} \tag{4.5}$$

If this condition is satisfied, then

$$p(t) = -\epsilon\int_t^\infty \langle Bv + R(v, v'), e^*\rangle\, ds. \tag{4.6}$$

Equations (4.4), (4.5), (4.6) may be collected as follows. We define

$$\mathscr{F}_1(\zeta, p, h)\epsilon) = \zeta - \epsilon KQ[Bv + R(v, v')] - e^{tL}Q[u_0 + \epsilon g(\epsilon, \xi, h)],$$

$$\mathscr{F}_2(\zeta, p, h)\epsilon) = p(t) + \epsilon\int_t^\infty \langle Bv + R(v, v'), e^*\rangle\, ds,$$

$$\mathscr{F}_3(\zeta, p, h)\epsilon) = h - \langle u_0, e^*\rangle - \epsilon\langle g(\epsilon, \xi, h), e^*\rangle - \epsilon\int_0^\infty \langle Bv + R(v, v'), e^*\rangle\, ds.$$

Now setting $\mathscr{F}(\zeta, p, h, \epsilon) = (\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3)$, Eqs. (4.4)–(4.6) may be written in the compact form

$$\mathscr{F}(\zeta, p, h, \epsilon) = 0. \tag{4.7}$$

We wish to construct solutions of (4.7), $\zeta(\epsilon)$, $p(\epsilon)$, $h(\epsilon)$, for small ϵ, where $\zeta \in Q\mathscr{E}^{\omega}_{w,1}$, $p \in \mathbb{R}^{\omega}$, and $h \in \mathbb{R}$. (We define $\mathbb{R}^{\omega}$ to be the Banach space of continuous functions on $0 \leqslant t < \infty$ with the norm

$$\| p \|_{\omega} = \sup_{t=0} e^{\omega t} \, | p(t) | .)$$

When $\epsilon = 0$ we may take

$$\begin{aligned} h &= h_0 = \langle u_0 , e^* \rangle, \\ p &= p_0(t) = 0, \\ \zeta &= (t) = e^{-tL} Q u_0 . \end{aligned}$$

Because of our assumptions on the spectrum of L, $\zeta_0 \in Q\mathscr{E}^{\omega}_{w,1}$.

We now claim that the mapping $\mathscr{F}$, by virtue of Lemmas 3.1–3.4, is a Frechet differentiable mapping from the Banach space

$$Q\mathscr{E}^{\omega}_{w,1} \times R^{\omega} \times R$$

to itself. For example, let us check the transformation

$$(p, \zeta) \to \int_t^{\infty} \langle Bv, e^* \rangle \, ds.$$

Since $v = p(t)\varphi' + \zeta$ we have

$$\| v(\cdot, t) \|_{w,1} \leqslant | p(t) | \, \| \varphi \|_{w,1} + \| \zeta(\cdot, t) \|_{w,1} .$$

On the other hand,

$$\| Bv \|_{w,0} \leqslant \text{const.} \, \| v(\cdot, t) \|_{w,1} ,$$

so

$$\begin{aligned} |\langle Bv, e^* \rangle| &\leqslant \text{const.} \, \| v(\cdot, t) \|_{w,1} \\ &\leqslant \text{const.} (| p(t) | + \| \zeta(\cdot, t) \|_{w,1}) \\ &\leqslant \text{const.} \, e^{-\omega t} (\| p \|_{\omega} + \| \zeta \|_{w,1,\omega}). \end{aligned}$$

Hence

$$\begin{aligned} \left| \int_j^{\infty} \langle Bv, e^* \rangle \, ds \right| &\leqslant \text{const.} \left(\int_t^{\infty} e^{-\omega s} \, ds \right) (\| p \|_{\omega} + \| \zeta \|_{w,1,\omega}) \\ &\leqslant \text{const.} \frac{e^{-\omega t}}{\omega} (\| p \|_{\omega} + \| \zeta \|_{w,1,\omega}). \end{aligned}$$

The other terms may be treated in a similar manner. We note that R is a differentiable mapping from $\mathscr{E}^{\omega}_{w,1}$ to $\mathscr{E}^{\omega}_{w,0}$; Q is a bounded transformation from $\mathscr{B}_{w,j}$ to $\mathscr{B}_{w,j}$; K is a bounded mapping from $Q\mathscr{E}^{\omega}_{w,0}$ to $Q\mathscr{E}^{\omega}_{w,1}$; and a nonlinear functional of the form $v \to \langle Z(v), e^* \rangle$ is a differentiable mapping from $\mathscr{B}_{w,1}$ to $\mathbb{R}$ whenever Z is a differentiable mapping from $\mathscr{B}_{w,1}$ to $\mathscr{B}_{w,0}$. The details of verifying the assertion that $\mathscr{F}$ is a Frechet differentiable mapping are largely straightforward, and are left to the reader.

We may thus apply the implicit function theorem to the mapping $\mathscr{F}$. To this end, we compute the Frechet derivative of $\mathscr{F}$ at $(\zeta_0, p_0, h_0, 0)$:

$$\mathscr{F}' |_{(\zeta_0, p_0, h_0, 0)} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and this operator is clearly invertible. Therefore, by the implicit function theorem in a Banach space [14, Chap. III]), there exists a vector $(\zeta(\epsilon), p(\epsilon), h(\epsilon))$ in $\mathscr{E}^{\omega}_{w,1} \times \mathbb{R}^{\omega} \times R$, which is continuously differentiable in ϵ, such that

$$\mathscr{F}(\zeta(\epsilon), p(\epsilon), h(\epsilon), \epsilon) \equiv 0.$$

Remarks. (1) The argument of Theorem 4.1 does not yield functions p and ξ with a high degree of regularity. In particular, it is not apparent from the proof that $\dot{p}(t)$ or $\partial\xi/\partial t$ are continuous functions of t. The regularity of v in the variables x and t may, however, be inferred from the classical regularity arguments for solutions of parabolic systems.

(2) If stronger assumptions are made on the regularity of f, correspondingly stronger, regularity results are obtained for the dependence of ξ, p, and h on ϵ. In particular, we have

COROLLARY 4.2. *If the function f is analytic in $u_1, \ldots, u_n$, $\partial u_1/\partial x, \ldots$, $\partial u_n/\partial x$ in some (complex) neighborhood of the range of values of $\varphi(\xi)$ and $\varphi'(\xi)$ $(-\infty < \xi < \infty)$ then the functions ζ, p, h of Theorem* 4.1 *are analytic in ϵ.*

Proof. If f is analytic then the mapping $\mathscr{F}$ constructed in Theorem 4.1 is differentiable in the complex Banach space of complex valued functions ζ, p, h, that is, $\mathscr{F}$ is an analytic mapping of the complex Banach space

$$Q\mathscr{E}^{\omega}_{w,1} \times \mathbb{C}^{\omega} \times \mathbb{C}$$

to itself. The resulting functions ζ, p, h are differentiable functions of the complex variable ϵ, hence are analytic (see [14, Chap. III]).

In some instances L may not have an eigenvalue at 0 (see Sect. 6), for the reason that $\varphi' \notin \mathscr{B}_{w,0}$. In this case, assuming $\sigma(L)$ is strictly contained in some region $y^2 + x + a < 0$, the wave $\varphi(\xi)$ is asymptotically stable in the strict sense, that is, there is no need to introduce the wave shift γ. Specifically, the following is the case.

THEOREM 4.3. *Let L and f satisfy the conditions of Theorem 4.1, except that $0 \notin \sigma(L)$, and let $u(x, t)$ satisfy (2.1) with initial data of the form*

$$u(x, 0) = \varphi(x) + \epsilon u_0(x),$$

Then there is a constant $K > 0$ such that

$$\| u(\xi, t) - \varphi(\xi)\|_{w,1} \leqslant Ke^{-\omega t}, \qquad t \to \infty,$$

for any ω, $0 < \omega < a$.

Proof. The proof is a simplified version of that of Theorem 4.1. The modifications are left to the reader.

5. CONSTRUCTION OF THE RESOLVENT

In the proof of the stability theorem certain assumptions were made about the asymptotic behavior of the resolvent transformation $(\lambda - L)^{-1}$, as well as about the spectrum of L. In this section we construct the resolvent operator $(\lambda - L)^{-1}$ and investigate the spectral properties of L for the case of a single equation

$$u_t = u_{xx} + f(u, u_x)$$

(u, f scalar functions). The operator L in this case has the form

$$Lu = u'' - 2bu' + qu,$$

where the prime denotes $d/d\xi$, and

$$2b = c - f_{u'}(\varphi, \varphi'),$$

$$q(\xi) = f_u(\varphi, \varphi').$$

Let us consider the resolvent equation

$$u'' - 2bu' + qu - \lambda u = g(\xi). \tag{5.1}$$

We make the transformation

$$u(\xi) = v(\xi)\, e^{B(\xi)},$$

where

$$B(\xi) = \int_0^\xi b(s)\, ds.$$

We then get the following equation for v.

$$(M - \lambda)v = e^{-B}g, \tag{5.2}$$

where

$$Mv = v'' + [b' - b^2 + q]v. \tag{5.3}$$

When M is defined as in (5.3) we have

$$L = e^B M e^{-B},$$

where e^B denotes the operation of multiplication by the function $e^{B(\xi)}$. Consequently,

$$(\lambda - L)^{-1} = e^B(\lambda - M)^{-1}e^{-B}. \tag{5.4}$$

We now let

$$p(\xi) = b'(\xi) - b^2(\xi) + q(\xi),$$

$$p_\pm = \lim_{\xi\to\pm\infty} p(\xi),$$

$$q_\pm = \lim_{\xi\to\pm\infty} q(\xi),$$

$$\gamma_\pm(\lambda) = \sqrt{\lambda - p_\pm}\,,$$

where the symbol $\sqrt{}$ denotes that branch of $\sqrt{z}$ which is positive when z is real and positive. The functions $\gamma_\pm(\lambda)$ are single valued and analytic in the plane cut along the negative real axis from $-\infty$ to $p_\pm$. In the future, we shall denote these regions by $\{|\arg \lambda - p_\pm| < \pi\}$.

Assuming that φ' and φ'' tend to zero as $\xi \to \pm\infty$, we see that $b'(\xi) \to 0$ as $\xi \to \pm\infty$. Then

$$p_\pm = q_\pm - (b_\pm)^2,$$

where $b_\pm = \lim_{\xi\to\pm\infty} b(\xi)$.

LEMMA 5.1. *Suppose that*

$$\int_0^\infty | p(\xi) - p_+ | \, d\xi, \qquad \int_{-\infty}^0 | p(\xi) - p_- | \, d\xi < +\infty.$$

Then the homogeneous equation

$$(M - \lambda)\varphi = 0$$

has a system of solutions φ_1, φ_2, ψ_1, *and* $\psi_2(\xi, \lambda)$ *which have the following asymptotic properties.*

$$\varphi_1 = e^{-\gamma_+(\lambda)\xi}\left[1 + O\left(\frac{1}{\sqrt{|\lambda|}}\right)\right], \qquad \xi \to +\infty,$$

$$\varphi_1' = e^{-\gamma_+(\lambda)\xi}[-\gamma_+(\lambda) + O(1)], \qquad \xi \to \infty,$$

$$\varphi_2 = e^{\gamma_+(\lambda)\xi}\left[1 + O\left(\frac{1}{\sqrt{|\lambda|}}\right)\right], \qquad \xi \to +\infty,$$

$$\varphi_2' = e^{\gamma_+(\lambda)\xi}[\gamma_+(\lambda) + O(1)], \qquad \xi \to +\infty,$$

$$\psi_1 = e^{\gamma_-(\lambda)\xi}\left[1 + O\left(\frac{1}{\sqrt{|\lambda|}}\right)\right], \qquad \xi \to -\infty,$$

$$\psi_1' = e^{\gamma_-(\lambda)\xi}[\gamma_-(\lambda) + O(1)], \qquad \xi \to -\infty,$$

$$\psi_2 = e^{-\gamma_-(\lambda)\xi}\left[1 + O\left(\frac{1}{\sqrt{|\lambda|}}\right)\right], \qquad \xi \to -\infty,$$

$$\psi_2' = e^{-\gamma_-(\lambda)\xi}[-\gamma_-(\lambda) + O(1)], \qquad \xi \to -\infty.$$

Furthermore, φ_1, φ_2 *(respectively,* ψ_1, ψ_2*) are single-valued analytic functions of* λ *in the plane cut along the real axis from* $-\infty$ *to* p_+ *(respectively, from* $-\infty$ *to* p_-*).*

Proof. The results of Lemma 5.1 are standard conclusions in the theory of asymptotic behavior of solutions of ordinary differential equations. For completeness, we indicate their proof below.

To construct φ_1, we look for a solution of $(M - \lambda)\varphi = 0$ in the form $\varphi_1 = z_1 e^{-\gamma_+(\lambda)\xi}$. Note that $\operatorname{Re} \gamma + (\lambda) > 0$ when $| \arg \lambda - p_+ | < \pi$. We get the following equation for z_1.

$$z_1'' - 2\gamma_+ z_1 + \hat{p} z_1 = 0,$$

where $\hat{p} = p - p_+$. This equation may be integrated from ξ to ∞ to obtain

$$z'(\xi) = \int_\xi^\infty e^{2\gamma_+(\xi - s)} \hat{p}(s)\, z(s)\, ds, \tag{5.5}$$

$$z(\xi) = 1 + \int_\xi^\infty \hat{p} \left(\frac{1 - e^{2\gamma_+(\xi - s)}}{2\gamma_+} \right) z(s)\, ds. \tag{5.6}$$

In deriving (5.5), (5.6) we assumed that $z' \to 0$ and $z \to 1$ as $\xi \to \infty$. Since $\int_0^\infty |\hat{p}|\, ds < +\infty$, Eq. (5.6) may be integrated by successive approximations. The estimates for φ_1 and φ_1' then follow readily from (5.5), (5.6).

To construct φ_2, set $\varphi_2 = z_2 e^{\gamma_+(\lambda)\xi}$; the following equation is then obtained for z_2.

$$z_2'' + 2\gamma_+ z_2' + \hat{p} z_2 = 0.$$

Assuming $z_2'(0) = 0$ we derive the following integral equation for z_2.

$$z_2' = -\int_0^\xi e^{-2\gamma_+(\xi - s)} \hat{p} z_2\, ds, \tag{5.7}$$

$$z_2(\xi) = 1 + \frac{1}{2\gamma_+} \int_0^\xi \hat{p} z (e^{-2\gamma_+(\xi - s)} - 1)\, ds. \tag{5.8}$$

The integral equation (5.8) may be solved by successive approximations. Since $\operatorname{Re} \gamma_+ > 0$ the kernel in (5.8) is dominated by $|\hat{p}(s)|$, which is integrable on $(0, \infty)$. The estimates for φ_2 and φ_2' then follow from (5.7) and (5.8).

The treatment of the asymptotic behavior of ψ_1 and ψ_2 as $\xi \to -\infty$ is entirely similar.

Since (5.6) and (5.8) are solved by successive approximations, the solutions $z_1(\xi, \gamma_+)$, $z_2(\xi, \gamma_+)$ are analytic functions of γ_+ in the punctured complex plane $\gamma_+ \neq 0$. The function $\gamma_+(\lambda)$ is in turn an analytic function of λ in $\{|\arg \lambda \ - p_+| < \pi\}$, that is, in the plane cut from $-\infty$ to p_+ along the negative real axis. The same remarks apply to the solutions ψ_1 and ψ_2.

LEMMA 5.2. *The Wronskian*

$$W(\lambda) = \varphi\psi' - \psi\varphi',$$

where φ and ψ are any two solutions of $(M - \lambda)\varphi = 0$, is independent of ξ and is an analytic function of λ in the complex plane cut from $-\infty$ to $\max\{p_+, p_-\}$. *Furthermore, the functions φ_i, ψ_i of Lemma* 5.1 *have the following representations*

$$\varphi_i(\xi, \lambda) = A_i(\lambda)\,\psi_1(\xi, \lambda) + B_i(\lambda)\,\psi_2(\xi, \lambda), \qquad \xi \to -\infty, \tag{5.9}$$

$$\psi_i(\xi, \lambda) = C_i(\lambda)\,\varphi_1(\xi, \lambda) + D_i(\lambda)\,\varphi_2(\xi, \lambda), \qquad \xi \to +\infty, \tag{5.10}$$

where the coefficients A_i, B_i, C_i, D_i are analytic functions of λ in the same cut plane and are $O(1)$ *as $\lambda \to \infty$ in the sector $|\arg \lambda| \leqslant \pi - \delta$ for any* $\delta > 0$.

Proof. The statement concerning $W(\lambda)$ is a well-known fact from the theory of ordinary differential equations. Since φ_1 and φ_2 form a basis of solutions for $\xi > 0$, (5.9) is justified; and (5.8) is justified for the corresponding reason. The coefficients A_i and B_i may be determined by solving the following system of equations

$$\varphi_i = A_i\psi_1 + B_i\psi_2,$$

$$\varphi_i' = A_i\psi_1' + B_i\psi_2'.$$

We get

$$A_i = \frac{\psi_2'\varphi_i - \psi_2\varphi_i'}{W(\lambda)},$$

where

$$W(\lambda) = \psi_1\psi_2' - \psi_2\psi_1',$$

and a similar expression is obtained for $B_i(\lambda)$. From Lemma 5.1 we see that

$$W(\lambda) = -2\gamma_-(\lambda) + O(1)$$

as $\lambda \to \infty$ in the cut plane, whereas, for example,

$$\psi_2'\varphi_1\,\psi_2\varphi_1' = \gamma_+(\lambda) - \gamma_-(\lambda) + O(1).$$

Thus

$$A_1(\lambda) = \frac{\gamma_-(\lambda) - \gamma_+(\lambda)}{2\gamma_-(\lambda)} + O\left(\frac{1}{\sqrt{|\lambda|}}\right).$$

Similar asymptotic expressions can be developed for the other coefficients A_i, B_i, C_i, and D_i.

By an *eigenfunction* of M we mean a bounded nontrivial solution φ of $M\varphi = \lambda\varphi$ for some value of λ, any such number λ then being called an *eigenvalue* of M. By Lemmas 5.1 and 5.2 any solution of $(M - \lambda)\varphi = 0$ behaves asymptotically as $\xi \to \infty$ like

$$Ae^{\gamma_+(\lambda)\xi} + Be^{-\gamma_+(\lambda)\xi};$$

so φ will be bounded only if

$$A = 0 \qquad \text{or} \qquad \operatorname{Re} \gamma_+(\lambda) = 0.$$

The second condition, however, is impossible if λ is excluded from the negative ray $-\infty < \lambda < p_+$. A similar result holds as $\xi \to -\infty$; and so we see that all bounded solutions of $M\varphi = \lambda\varphi$ in fact decay exponentially as $\xi \to \pm\infty$ whenever

$$|\arg(\lambda - \bar{p})| < \pi \qquad (\bar{p} = \max(p_+, p_-)).$$

All eigenvalues of M must therefore in fact be real. For, defining

$$(u, v) = \int_{-\infty}^{\infty} u(\xi)\, \bar{v}(\xi)\, d\xi,$$

we have

$$\lambda(\varphi, \varphi) = (M\varphi, \varphi) = (\varphi, M\varphi) = (\varphi, \lambda\varphi) = \bar{\lambda}(\varphi, \varphi),$$

hence $(\lambda - \bar{\lambda})(\varphi, \varphi) = 0$ and $\lambda = \bar{\lambda}$.

LEMMA 5.3. *We consider M as an operator on the Banach space L_∞. The eigenvalues of M in the interval $\lambda > \bar{p}$ ($\bar{p} = \max(p_-, p_+)$) are confined to the interval $\bar{p} < \lambda < \| p \|_\infty$, and form a discrete set of points on the real axis which can cluster only at $\lambda = \bar{p}$. M has a continuum of eigenvalues in the semiinfinite interval $-\infty < \lambda - \bar{p}$.*

Proof. The Wronskian

$$W(\lambda) = \varphi_1\psi_1' - \varphi_1'\psi_1$$

is an analytic function of λ in the cut plane $|\arg(\lambda - \bar{p})| < \pi$; and the eigenvalues of M are the zeroes of W, hence are discrete for $\operatorname{Re} \lambda > \bar{p}$. When $\lambda > \| p \|_\infty$, φ must satisfy

$$\varphi'' + (p - \lambda)\varphi = 0$$

and

$$\varphi \to 0 \qquad \text{as} \quad |\xi| \to \infty,$$

while $(p - \lambda) < 0$ everywhere. This situation, however, is precluded by the maximum principle. In fact, at a positive maximum of φ we have $\varphi'' \leqslant 0$ and $(p - \lambda)\varphi < 0$, so that the differential equation for φ cannot hold. Similarly, φ cannot have a negative minimum; yet φ must tend to zero as $\xi \to \pm\infty$.

We now show that $(M - \lambda)\psi = 0$ has a bounded solution on $-\infty < \xi < \infty$ whenever $\lambda < \bar{p}$. If $p_- < \lambda < p_+$, then there is one exponentially growing and one decaying solution as $\xi \to -\infty$. As $\xi \to +\infty$ the solutions are oscillatory, but bounded. Therefore, choosing ψ to be that solution which decays exponentially as $\xi \to -\infty$ we see that ψ must be a linear combination of oscillatory solutions as $\xi \to +\infty$. When $\lambda < p_-$ we get solutions which are oscillatory at both $\xi = \pm\infty$. A similar situation obtains if $p_+ < p_-$.

LEMMA 5.4. *As $\lambda \to \infty$ in the sector $|\arg \lambda| < \pi - \delta$ for any $\delta > 0$, the resolvent transformation $(\lambda - M)^{-1}$ satisfies the following estimates*

$$\|(\lambda - M)^{-1} g\|_0 \leqslant \frac{C}{|\lambda|} \|g\|_0, \tag{5.11}$$

$$\|(\lambda - M)^{-1} g\|_1 \leqslant \frac{C}{\sqrt{|\lambda|}} \|g\|_0, \tag{5.12}$$

where C depends on δ, and

$$\|g\|_0 = \sup_\xi |g(\xi)|$$

$$\|g\|_1 = \sup_\xi |g'(\xi)| + \sup_\xi |g(\xi)|.$$

Proof. To obtain the estimates (5.11) and (5.12) we must estimate the integrals

$$\int_{-\infty}^{\infty} |G(x, y, \lambda)|\, dy \tag{5.13}$$

and

$$\int_{-\infty}^{\infty} |G_x(x, y, \lambda)|\, dy, \tag{5.14}$$

respectively, where $G(x, y, \lambda)$ is the Green's function for the operator $(M - \lambda)$. Now

$$G(x, y, \lambda) = \frac{1}{W(\lambda)} \begin{cases} \psi_1(y, \lambda)\, \varphi_1(x, \lambda), & x > y, \\ \varphi_1(y, \lambda)\, \psi_1(x, \lambda), & x < y, \end{cases}$$

where

$$W(\lambda) = \psi_1'\varphi_1 - \psi_1\varphi_1'.$$

From Lemma 5.1, $W(\lambda)$ has the asymptotic behavior

$$\begin{aligned} W(\lambda) &= -(\sqrt{\lambda - p_-} + \sqrt{\lambda - p_+}) + O(1) \\ &= -(\gamma^+(\lambda) + \gamma^-(\lambda)) + O(1) \end{aligned}$$

as $\lambda \to \infty$ in the cut plane $\{| \arg \lambda | < \pi - \delta\}$.

We first estimate (5.13); we have

$$\begin{aligned} &\int_{-\infty}^{\infty} | G(x, y, \lambda)| \, dy \\ &\quad = \frac{1}{| W(\lambda)|} \left[\int_{-\infty}^{x} | \varphi_1(x, \lambda)| \, | \psi_1(y, \lambda)| \, dy + \int_{x}^{\infty} | \psi_1(x, \lambda)| \, | \varphi_1(y, \lambda)| \, dy\right]. \end{aligned}$$

Let us suppose $x > 0$ and estimate the first term:

$$\int_{-\infty}^{x} | \varphi_1(x, \lambda)| \, | \psi_1(y, \lambda)| \, dy \leqslant e^{-\mathrm{Re}\gamma_+(\lambda)x} \left[\int_{-\infty}^{0} | \psi_1(y, \lambda)| \, dy + \int_{0}^{x} | \psi_1(y, \lambda)| \, dy\right].$$

The first integral is bounded by

$$\begin{aligned} \int_{-\infty}^{0} | \psi_1(y, \lambda)| \, dy &\leqslant \int_{-\infty}^{0} e^{\mathrm{Re}\gamma_-(\lambda)y} \left(1 + O\frac{1}{\sqrt{| \lambda |}}\right) dy \\ &\leqslant \frac{1}{\mathrm{Re}\, \gamma_-(\lambda)} \left(1 + O\left(\frac{1}{\gamma_-}\right)\right) \leqslant \frac{C(\delta)}{| \gamma_-(\lambda)|}. \end{aligned}$$

The second integral is estimated as follows.

$$\begin{aligned} \int_{0}^{x} | \psi_1(y, \lambda)| \, dy &\leqslant | A | \int_{0}^{x} e^{\mathrm{Re}\gamma_+(\lambda)y} \, dy \\ &= | A | \frac{e^{\mathrm{Re}\gamma_+(\lambda)x} - 1}{\mathrm{Re}\, \gamma_+(\lambda)}, \end{aligned}$$

where A is one of the coefficients in Lemma 5.2. (We have omitted the second term in ψ_2 from (5.10) since this term is clearly bounded.) Therefore

$$\int_{-\infty}^{x} | \varphi_1(x, \lambda)| \, | \psi_1(y, \lambda)| \, dy \leqslant \frac{C(\delta)}{| \gamma_-(\lambda)|} + \frac{C'(\delta)}{| \gamma_+(\lambda)|} \leqslant \frac{C''(\delta)}{\sqrt{| \lambda |}}$$

as $\lambda \to \infty$ in $| \arg \lambda | < \pi - \delta$. The estimate of the other term proceeds in much the same way, as well as the estimate in case $x < 0$. The factor

$| W(\lambda)|^{-1}$, as well as the factor $| \lambda |^{-1/2}$ from the integral, combine to give the asymptotic behavior $| \lambda |^{-1}$ in (5.11).

To obtain the estimate of (5.14) we simply note that differentiation of G with respect to x brings down a factor of order $\sqrt{| \lambda |}$.

THEOREM 5.5. *Suppose $\bar{p} \leqslant 0$ so that the spectrum of M is discrete in $\lambda > 0$. If there exists a positive function φ such that $M\varphi = 0$ then M has no positive eigenvalues.*

Proof. By hypothesis φ satisfies the differential equation

$$\varphi'' + p\varphi = 0.$$

Suppose ψ is a bounded solution of $(M - \lambda)\psi = 0$. Then

$$\begin{aligned} \psi &= O(e^{-\gamma_+\xi}), \qquad \xi \to +\infty, \\ &= O(e^{\gamma_-\xi}), \qquad \xi \to -\infty, \end{aligned}$$

where $\gamma_\pm(\lambda) = \sqrt{\lambda - p_\pm}$. If φ tends to infinity as $\xi \to \infty$, then

$$u = \psi/\varphi$$

tends to zero as $\xi \to \infty$. On the other hand, if φ tends to zero as $\xi \to \infty$, then $\varphi = 0(e^{-\gamma_+(0)\xi})$ as $\xi \to \infty$ and

$$u(\xi) = O(e^{(-\gamma_+(\lambda)+\gamma_+(0))\xi}) \qquad \text{as} \quad \xi \to \infty.$$

Since $\lambda > 0$, $u \to 0$ as $\xi \to \infty$. The same is true as $\xi \to -\infty$. Substituting $\psi = \varphi u$ into the equation for ψ we obtain

$$\begin{gathered} u'' + \frac{2\varphi'}{\varphi} u' - \lambda u = 0, \\ u \to 0 \qquad \text{as} \quad \xi \to \pm\infty. \end{gathered} \tag{5.15}$$

Now, however, we may apply the same maximum principle argument that we applied in the proof of Lemma 5.3 to conclude that u must be identically zero.

We now turn to an analysis of the resolvent transformation $(\lambda - L)^{-1}$ on the Banach space $\mathscr{B}_{w,0}$ weighted by the function

$$w(\xi) = 1 + e^{-B(\xi)}. \tag{5.16}$$

By $\sigma(L)$ we mean the spectrum of L relative to the Banach space $\mathscr{B}_{w,0}$. We assume that

$$\lim_{\xi \to \pm\infty} 2b(\xi) = \kappa^\pm \tag{5.17}$$

and that

$$\int_0^\infty \left| b(\xi) - \frac{\kappa^+}{2} \right| d\xi < +\infty, \qquad \int_{-\infty}^0 \left| b(\xi) - \frac{\kappa^-}{2} \right| d\xi < +\infty. \tag{5.18}$$

Condition (5.18) is satisfied as $\xi \to +\infty$, for example, if $(\varphi, \varphi') \to (1, 0)$ as $\xi \to \infty$ and $(\varphi - 1) = 0(e^{-\gamma\xi})$, $\varphi' = 0(e^{-\gamma\xi})$ for some $\gamma > 0$.

Let $\mathscr{P}^+$, $\mathscr{P}^-$ denote the two regions

$$\mathscr{P}^\pm = \left\{\lambda: \operatorname{Re} \sqrt{\lambda - p_\pm} > \frac{|\kappa^\pm|}{2}\right\}.$$

$\mathscr{P}^+$ and $\mathscr{P}^-$ are the regions exterior to the parabolas

$$\rho_\pm = \frac{(\kappa^\pm)^2}{2(1 + \cos\theta)}, \tag{5.19}$$

where $\lambda - p_\pm = \rho_\pm e^{i\theta}$, $-\pi < \theta < \pi$. The parabolas (5.19) meet the real axis at $\lambda = q^\pm$ and extend to infinity in the left half-plane. In the following, M is considered as an operator on L_∞.

THEOREM 5.6. *Under the assumptions on b and q of Lemma 5.1, and in (5.17), (5.18), we may draw the following conclusions about the resolvent operator $(\lambda - L)^{-1}$ considered as a transformation on $\mathscr{B}_{w,0}$.*

(I) *If $\kappa^+ > 0$ and $\kappa^- > 0$, then the resolvent sets of L and M coincide in the region $\mathscr{P}^+$.*

(II) *If $\kappa^+ < 0$ and $\kappa^- < 0$, then the resolvent sets coincide in $\mathscr{P}^-$.*

(III) *If $\kappa^+ < 0$ and $\kappa^- > 0$, then the resolvent sets coincide everywhere.*

(IV) *If $\kappa^+ > 0$ and $\kappa^- < 0$, then the resolvent sets coincide in $\mathscr{P}^- \cap \mathscr{P}^+$.*

Furthermore, the resolvent $(\lambda - L)^{-1}$ satisfies the following asymptotic estimates as $|\lambda| \to \infty$ in $|\arg\lambda| < \pi - \delta$.

$$\|(\lambda - L)^{-1} u\|_{w,0} \leqslant \frac{C(\delta)}{|\lambda|} \| u \|_{w,0}, \tag{5.20}$$

$$\|(\lambda - L)^{-1} u\|_{w,1} \leqslant \frac{C(\delta)}{\sqrt{|\lambda|}} \| u \|_{w,0}. \tag{5.21}$$

These estimates hold as $\lambda \to \infty$ in $P^+ \cap P^-$.

Proof. Under our assumptions on b,

$$B(\xi) \sim \frac{\kappa^{\pm}}{2}\,\xi + \text{const.}, \qquad \xi \to \pm\infty,$$

hence

$$e^{B(\xi)} \sim \text{const.}\, e^{(\kappa^{\pm}/2)\xi}, \qquad \xi \to \pm\infty.$$

Assuming $u \in \mathscr{B}_{w,0}$ we estimate

$$\|(\lambda - L)^{-1}u\|_{w,0}\,.$$

Since $(\lambda - L)^{-1}u = e^{B}(\lambda - M)^{-1}e^{-B}u$ by (5.4), and since $e^{-B}u \in L_\infty$, we have, by Lemma 5.4,

$$\| e^{-B}(\lambda - L)^{-1}u\|_0 \leqslant \frac{\text{const.}}{|\lambda|}\| e^{-B}u\|_0\,, \qquad \lambda \to \infty, \tag{5.22}$$

where $\|\ \|_0$ is the sup norm given in Lemma 5.4. It remains to estimate

$$\|(\lambda - L)^{-1}u\|_0 = \| e^{B}(\lambda - M)^{-1}e^{-B}u\|_0\,.$$

If we are in case (I), $e^B \to +\infty$ as $\xi \to +\infty$, while $e^B \to 0$ as $\xi \to -\infty$, so it is only necessary to estimate

$$| e^{(\kappa^+/2)\xi}(\lambda - M)^{-1}e^{-B}u| \leqslant e^{(\kappa^+/2)\xi}\int_{-\infty}^{\infty} |G(\xi, y, \lambda)|\, e^{-B(y)}\, |u(y)|\, dy \tag{5.23}$$

as $\xi \to +\infty$. For $\xi > 0$ the above integral is dominated by

$$C_1 \frac{e^{(\kappa^+\xi)/2}}{|W(\lambda)|}\Big[\int_{-\infty}^{\xi} |\varphi_1(\xi, \lambda)|\, |\psi_1(y, \lambda)|\, e^{-B(y)}\, |u(y)|\, dy$$
$$+ \int_{\xi}^{\infty} |\psi_1(\xi, \lambda)|\, |\varphi_1(y, \lambda)|\, e^{-B(y)}\, |u(y)|\, dy\Big]. \tag{5.24}$$

Now

$$\int_{-\infty}^{\xi} |\varphi_1(\xi, \lambda)|\, |\psi_1(y, \lambda)|\, e^{-B(y)}\, |u(y)|\, dy$$
$$\leqslant C_2 e^{-\mathrm{Re}\gamma_+(\lambda)\xi}\Big[\int_{-\infty}^{0} e^{\mathrm{Re}\gamma_-(\lambda)y}\, dy\, \| u\|_{w,0} + C_3\int_0^{\xi} e^{\mathrm{Re}\gamma_+(\lambda)y} e^{(-\kappa^+y)/2}\, dy\, \| u\|_{w,0}\Big]$$
$$\leqslant C_4 e^{-\mathrm{Re}\gamma_+(\lambda)\xi}\left[\frac{1}{\mathrm{Re}\,\gamma_-(\lambda)} + \frac{e^{(\mathrm{Re}\gamma_+(\lambda) - (\kappa^+/2))\xi}}{\mathrm{Re}\,\gamma_+(\lambda) - (\kappa^+/2)}\right]\| u\|_{w,0}\,.$$

The second term in (5.24) is dominated by

$$C_5 e^{\mathrm{Re}\,\gamma_+(\lambda)\xi} \int_\xi^\infty e^{-\mathrm{Re}\,\gamma_+(\lambda)y} e^{-(\kappa^+ y)/2}\, dy \,\|\, u \,\|_{w,0}$$

$$\leqslant C_5 e^{\mathrm{Re}\,\gamma_+(\lambda)\xi} \left[\frac{e^{-\mathrm{Re}\,\gamma_+(\lambda)+(\kappa^+/2))\xi}}{\mathrm{Re}\,\gamma_+(\lambda) + (\kappa^+/2)}\right] \|\, u \,\|_{w,0}\,.$$

Thus, the integral (5.23) is dominated by

$$\frac{C_6}{|\, W(\lambda)|}\left[\frac{e^{((\kappa_+/2)-\mathrm{Re}\,\gamma_+(\lambda))\xi}}{\mathrm{Re}\,\gamma_-(\lambda)} + \frac{1 - e^{(\kappa^+/2-\mathrm{Re}\,\gamma_+(\lambda))\xi}}{\mathrm{Re}\,\gamma_+(\lambda) - (\kappa^+/2)} + \frac{1}{\mathrm{Re}\,\gamma_+(\lambda) + (\kappa^+/2)}\right] \|\, u \,\|_{w,0}\,. \tag{5.25}$$

The expression (5.25) is uniformly bounded by

$$\frac{C_7}{|\,\lambda\,|} \|\, u \,\|_{w,0}$$

as $\xi \to +\infty$ and $|\,\lambda\,| \to \infty$ in $|\arg \lambda\,| \leqslant \pi - \delta$, provided that $\lambda \in \mathscr{P}^+$.

To obtain the estimate (5.21) we first note that, since B' is bounded, it suffices to estimate the integral

$$\int_{-\infty}^\infty e^{B(\xi)}\, |\, G_\xi(\xi, y, \lambda)|\, e^{-B(y)}\, |\, u(y)|\, dy$$

as $\xi \to +\infty$. The estimate (5.21) is then obtained in the same manner as that for (5.20) after noting that differentiation of G with respect to ξ brings down factors of order $0(|\,\gamma_\pm(\lambda)|)$.

Case (II) may be reduced to case (I) by making the transformation $\xi \to -\xi$, $c \to -c$ in the operator L.

Case (III) is immediate; for in this case $e^{-B} \to \infty$ as $\xi \to \pm\infty$, and

$$\|(\lambda - L)^{-1} u \,\|_0 \leqslant \text{const.}\, \|\, e^{-B}(\lambda - L)^{-1} u \,\|_0$$

$$\leqslant \frac{\text{const.}}{|\,\lambda\,|} \|\, u \,\|_{w,0}$$

by (5.22).

The proof of Case (IV) runs like that of case (I). Since $e^B \to \infty$ as $\xi \to \pm\infty$, it is necessary to estimate the integral

$$\int_{-\infty}^\infty e^{B(\xi)}\, |\, G(\xi, y, \lambda)|\, e^{-B(y)}\, |\, u(y)|\, dy$$

as $\xi \to \pm\infty$. To obtain the appropriate estimate as $\xi \to +\infty$ one must require that $\lambda \in \mathscr{P}^+$, while to obtain the appropriate estimate as $\xi \to -\infty$ one must demand that $\lambda \in \mathscr{P}^-$.

COROLLARY 5.7. *The conditions on L in the stability theorem (see Lemma* 3.4) *are satisfied if* $\varphi' > 0$ *and*

(I) $q_+ < 0$ *and* $q_- < (\kappa^-)^2/4$ *in case* I;

(II) $q_- < 0$ *and* $q_+ < (\kappa^+)^2/4$ *in case* II;

(III) $q_- < (\kappa^-)^2/4$ *and* $q_+ < (\kappa^+)^2/4$ *in case* (III);

(IV) $q_+ < 0$ *and* $q_- < 0$ *in Case* (IV).

Proof. The statements follow from Theorem 5.6 and the relations

$$p_\pm = q_\pm - \frac{(\kappa^\pm)^2}{4}.$$

In order for the spectrum of L to satisfy the conditions of Lemma 3.4 it is necessary for the regions $\mathscr{P}^+$, $\mathscr{P}^-$, $\mathscr{P}^+ \cap \mathscr{P}^-$ to lie strictly in the left half-plane and $\bar{p} = \max\{p_-, p_+\} < 0$.

6. KPP WAVES

In their classic paper [2], Kolmogorov *et al.* discussed the stability of traveling waves for the simple parabolic equation

$$u_t = u_{xx} + f(u), \tag{6.1}$$

where f has the general shape shown in Fig. 6.1. Such equations arise as simple models for the propagation of favorable genes [24]. It is assumed that $f(u) \geqslant 0$ for $0 \leqslant u \leqslant 1$; $f'(0) = \alpha > 0$; $f'(1) = -\beta \leqslant 0$; and that $f(u)$ is concave. Setting $\xi = x + ct$ and looking for solutions of (6.1) of the form $u(x, t) = \varphi(\xi)$, we get the following ordinary differential equation for φ

$$\varphi'' - c\varphi' + f(\varphi) = 0. \tag{6.2}$$

There is no loss in generality in assuming $c > 0$. Equation (6.2) may be written as a first-order system

$$\begin{aligned} \varphi' &= p, \\ p' &= cp - f(\varphi). \end{aligned} \tag{6.3}$$

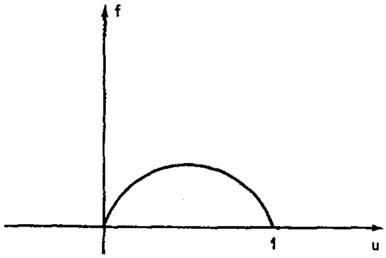

FIGURE 6.1

Since, in (6.1), the solution u represents the proportion of a population possessing a certain genetic structure, physical considerations require that $0 \leqslant u \leqslant 1$. We are therefore interested in solutions of (6.2) which lie between 0 and 1.

THEOREM 6.1 (Kolmogorov *et. al.* [2]). *For each value of c, $c \geqslant 2\sqrt{\alpha}$, system* (6.3) *has a solution* $(\varphi(\xi, c), p(\xi, c))$ *such that*

(i) $0 \leqslant \varphi \leqslant 1, p \geqslant 0,$

(ii) $(\varphi, p) \to (0, 0)$ as $\xi \to -\infty$, $(\varphi, p) \to (1, 0)$ as $\xi \to +\infty$.

Although the proof of Theorem 6.1 was given in [2], we shall reproduce it here, not only for the sake of completeness, but, more importantly, because a detailed knowledge of the asymptotic properties of φ will be important in our discussion of stability.

Proof. The points (0, 0) and (1, 0) are singular points of the system (6.3). The linearization of (6.3) about (0, 0) is

$$\frac{d}{d\xi}\begin{pmatrix}\varphi \\ p\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -\alpha & c\end{pmatrix}\begin{pmatrix}\varphi \\ p\end{pmatrix}.$$

The eigenvalues at (0, 0) are thus given by the roots of

$$\lambda^2 - c\lambda + \alpha = 0,$$

namely,

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4\alpha}}{2}, \qquad \lambda_2 = \frac{c + \sqrt{c^2 - 4\alpha}}{2}. \tag{6.4}$$

Corresponding to each root λ_i we have a solution of the form

$$\varphi = e^{\lambda_i \xi}, \qquad p = \varphi' = \lambda_i e^{\lambda_i \xi},$$

except in the case $c^2 = 4\alpha$, when the second solution is of the form

$$\varphi = O(\xi e^{\lambda_i \xi}) \qquad \text{as} \quad \xi \to -\infty.$$

If $c^2 - 4\alpha < 0$ then the eigenvalues are complex, and $(0, 0)$ is a stable spiral as $\xi \to -\infty$ (assuming $c > 0$). The solutions of (6.2) are in that case not positive as $\xi \to -\infty$, and we must therefore require the roots to be real, hence $c^2 \geqslant 4\alpha$. In that case, $(0, 0)$ is an unstable node.

Similarly, linearizing the system (6.3) about $(1, 0)$ we find

$$\frac{d}{d\xi}\begin{pmatrix} w \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \beta & c \end{pmatrix}\begin{pmatrix} w \\ p \end{pmatrix},$$

where $\varphi = 1 + w$. The asymptotic behavior of solutions as $\xi \to \infty$ is thus $w = e^{\gamma\xi}$, $p = \gamma e^{\gamma\xi}$, where

$$\gamma^2 - c\gamma - \beta = 0, \qquad \gamma_1 = \frac{c + \sqrt{c^2 + 4\beta}}{2}, \qquad \gamma_2 = \frac{c - \sqrt{c^2 + 4\beta}}{2}.$$

The eigenvalues γ_1 and γ_2 are of opposite signs, since $\gamma_1\gamma_2 = -\beta < 0$. Thus $(1, 0)$ is a saddle.

Let us show there exists a trajectory $\{\varphi, p\}$ connecting $(0, 0)$ to $(1, 0)$. Consider the diagram in Fig. 6.2. The line $0B$ is the straight line $p = \lambda_1\varphi$, where λ_1 is the smaller root in (6.4). We denote the vector field (6.3) by $\vec{v}(\varphi, p) = (p, cp - f(\varphi))$. On the line AB ($A = (1, 0)$), $\vec{v} = (p, cp)$, so all trajectories cross AB moving upward and out of the triangle $0AB$. On the line $0A$ the trajectories cross vertically and downward, since $\vec{v} = (0, -f(\varphi))$ on $0A$. Furthermore, the trajectories slope to the right for $p > 0$ and to the left for $p < 0$, since $\varphi' = p > 0$ for $p > 0$ and $\varphi' = p < 0$ for $p < 0$.

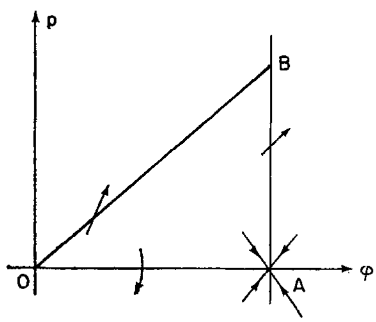

FIGURE 6.2

Finally, on the line $0B$ ($p = \lambda_1\varphi$), we have

$$\frac{dp}{d\varphi} = \frac{cp - f(\varphi)}{p} = c - \frac{f(\varphi)}{p}.$$

Now by our assumption that f is concave, $f(u) \leqslant \alpha u$, so

$$f(\varphi) = \alpha\varphi + g(\varphi),$$

where $g(\varphi) = o(\varphi)$ as $\varphi \to 0$, and $g(\varphi) < 0$. Therefore, on $p = \lambda_1\varphi$,

$$\begin{aligned}\left.\frac{dp}{d\varphi}\right|_{p=\lambda_1\varphi} &= c - \frac{\alpha\varphi + g(\varphi)}{\lambda_1\varphi} \\ &= c - \frac{\alpha}{\lambda_1} - \frac{1}{\lambda_1}\frac{g(\varphi)}{\varphi} \\ &> c - \frac{\alpha}{\lambda_1} = \lambda_1 .\end{aligned}$$

Thus, at each point on $0B$, the slope of the trajectory is greater than the slope of $0B$.

Now consider the trajectory which tends to $(1, 0)$ as $\xi \to \infty$ from within $0AB$. This trajectory is tangent to the line $p = \gamma_2\varphi$, where $\gamma_2 = (c - \sqrt{c^2 + 4\beta})/2 < 0$. Tracing this trajectory backward we see that, since its tangent can never become vertical in the interior of $0AB$, it can never exit the region $0AB$. Therefore, as $\xi \to -\infty$, it must tend to the origin.

During the course of the proof of Theorem 6.1 we have established the following result, which will be important to us in our stability analysis of φ.

COROLLARY 6.2. *Let φ denote the wave whose existence was demonstrated in Theorem* 6.1. *Then when $c^2 > 4\alpha$*

$$\begin{aligned}\varphi &\sim 1 - e^{\gamma_2\xi}, &\quad \xi &\to +\infty,\\ &\sim e^{\lambda_1\xi}, & \xi &\to -\infty,\\ \varphi' &\sim \gamma_2 e^{\gamma_2\xi}, & \xi &\to +\infty,\\ &\sim \lambda_1 e^{\lambda_1\xi}, & \xi &\to -\infty,\end{aligned}$$

where

$$\gamma_2(c) = \frac{c - \sqrt{c^2 + 4\beta}}{2} < 0,$$

$$\lambda_1(c) = \frac{c - \sqrt{c^2 - 4\alpha}}{2} > 0.$$

We have been unable to determine the asymptotic behavior of φ when $c^2 = 4\alpha$ as $\xi \to -\infty$. The root at (0, 0) is a double one, so the solutions of the linearized system have the behavior $e^{\sqrt{\alpha}\xi}$ and $\xi e^{\sqrt{\alpha}\xi}$. Let us denote by (H) the hypothesis

$$\varphi(\xi) = O(e^{\sqrt{\alpha}\xi}), \qquad \xi \to -\infty. \tag{H}$$

Since

$$\frac{dp}{d\varphi} = c - \frac{f(\varphi)}{p} \leqslant c,$$

$p \leqslant c\varphi$ and $\varphi' = 0(e^{\sqrt{\alpha}\xi})$ as well, if (H) holds. We discuss below some of the consequences of the validity or nonvalidity of (H).

Let us now consider the spectral analysis of the associated linear operator

$$Lu = u'' - cu' + f'(\varphi)u.$$

Let ψ be a solution to the homogeneous equation

$$(L - \lambda)\psi = 0. \tag{6.5}$$

As $\xi \to -\infty$, ψ is asymptotic to the solutions of

$$w'' - cw' + (\alpha - \lambda)w = 0 \tag{6.6}$$

since $f'(\varphi) \to \alpha$ as $\xi \to -\infty$. Thus, $\psi \sim e^{\gamma\xi}$ as $\xi \to -\infty$, where

$$\gamma^2 - c\gamma + (\alpha - \lambda) = 0.$$

If the roots of this equation are denoted by γ_1, γ_2, then

$$\gamma_1 + \gamma_2 = c,$$

$$\gamma_1\gamma_2 = (\alpha - \lambda).$$

If λ is real and $\lambda < \alpha$, then $\gamma_1\gamma_2 > 0$ so either γ_1 and γ_2 are both positive (since $\gamma_1 + \gamma_2 = c > 0$), or γ_1 and γ_2 are complex conjugates of each other with positive real parts. In either case, all solutions of (6.6), hence of (6.5), decay exponentially as $\xi \to -\infty$ when $\lambda < \alpha$. On the other hand, as $\xi \to +\infty$ the solutions of (6.5) are asymptotic to $e^{\gamma\xi}$, where this time

$$\gamma^2 - c\gamma - (\beta + \lambda) = 0,$$
$$\gamma_1\gamma_2 = -(\beta + \lambda),$$
$$\gamma_1 + \gamma_2 = c.$$

If $\lambda > -\beta$, then one solution grows and one decays as $\xi \to +\infty$.

Therefore, for all λ in the interval $-\beta < \lambda < \alpha$, Eq (6.5) *has a solution $\psi(\xi, \lambda)$ which is uniformly bounded on $-\infty < \xi < \infty$ and tends to zero exponentially as $\xi \to \pm\infty$.* For, we may choose ψ to be that solution which decays exponentially as $\xi \to +\infty$. Then, for $\xi \to -\infty$, ψ is asymptotic to a linear combination of the two solutions of (6.6) (each of which decays exponentially as $\xi \to -\infty$).

Since α and β are positive, this means that the operator L, considered as an operator on the Banach space L_∞, has a continuum of eigenvalues in the interval $(-\beta, \alpha)$. In order to obtain any kind of stability properties of the KPP waves it is therefore necessary to introduce a weight function w; and in this case, the appropriate weight function is

$$w(\xi) = 1 + e^{-(c/2)\xi} \tag{6.7}$$

(c may be positive or negative here).

THEOREM 6.3. *The conditions of the stability theorem (Theorem 4.3) are satisfied if $c^2 > 4\alpha$ and w is given by* (6.7). *The spectrum of L (considered as an operator on $\mathscr{B}_{w,0}$) is contained in the set*

$$\{\lambda\colon \operatorname{Re} \sqrt{\lambda + \beta + (c^2/4)} \leqslant c/2\} \cup \{\lambda = x\colon -\beta \leqslant x \leqslant \alpha - (c^2/4)\}.$$

The resolvent $(\lambda - L)^{-1}$ is analytic at the origin. Thus, the KPP *waves are stable in the norm with weight function* (6.7) *whenever $c^2 > 4\alpha$. When $c^2 = 4\alpha$ the continuous spectrum of L extends all the way to the origin $\lambda = 0$; and $\varphi' \in \mathscr{B}_{w,0}$ iff* (H) *holds.*

Proof. Theorem 6.3 is an application of Theorem 5.6 and Corollary 5.7. Since $\kappa^+ = \kappa^- = c$ we are in case (I) if $c > 0$ and case (II) if $c < 0$. Taking the case $c > 0$ we see that

$$q_+ = -\beta, \qquad p_+ = -\beta - (c^2/4),$$
$$q_- = \alpha, \qquad p_- = \alpha - (c^2/4).$$

The parabola $\partial\mathscr{P}^+$ crosses the real axis at $\lambda = -\beta$. When $c^2 > 4\alpha$, $p_- < 0$ so the spectrum is discrete in a neighborhood of the origin. We know that $L\varphi' = 0$ and $\varphi' > 0$ so by Theorem 5.5 there are no eigenvalues of L in the right half-plane. Let us show that $\varphi' \notin \mathscr{B}_{w,0}$. From Corollary 6.2,

$$\varphi' = O\left(\exp\left\{\frac{c - \sqrt{c^2 - 4\alpha}}{2}\,\xi\right\}\right) \qquad \text{as} \quad \xi \to -\infty$$

$(c^2 > 4\alpha)$. Therefore

$$e^{-c\xi/2}\varphi' \sim \text{const.}\, e^{\frac{-\sqrt{c^2 - 4\alpha}}{2}\xi}$$

and $e^{-c\xi/2}\varphi'$ grows exponentially as $\xi \to -\infty$ if $c^2 > 4\alpha$.

In the case $c^2 = 4\alpha$ we have

$$q_+ = -\beta, \qquad p_- = \alpha - (c^2/4) = 0$$

so that the continuous spectrum of M, hence of L, occupies the interval $-\beta \leqslant \lambda \leqslant 0$, by Lemma 5.3. Furthermore $\varphi' \in \mathscr{B}_{w,0}$ if and only if (H) holds. In fact, if $\varphi = 0(e^{\sqrt{\alpha}\xi})$ then $\varphi' = 0(e^{\sqrt{\alpha}\xi})$ as well, and

$$w\varphi' = O(e^{-(c/2)\xi}e^{(c/2)\xi}) = O(1) \qquad \text{as} \quad \xi \to -\infty.$$

It is easily seen that φ' decays exponentially as $\xi \to +\infty$, so $\varphi' \in \mathscr{B}_{w,0}$. On the other hand, if $\varphi' = 0(\xi e^{\sqrt{\alpha}\xi})$, then

$$w\varphi' = 0(\xi e^{-(c/2)\xi}e^{(c/2)\xi}) = 0(\xi) \qquad \text{as} \quad \xi \to -\infty.$$

Remarks. The Banach space $\mathscr{B}_{w,0}$ with weight function w given by (6.7), is "relatively small" when $c^2 > 4\alpha$. For one thing, a wave $\varphi(\xi, c)$ is an infinite distance from any translate in the norm $\|\ \|_{w,0}$:

$$\|\varphi(\xi, c) - \varphi(\xi + \gamma, c)\|_{w,0} = +\infty$$

for any γ. Furthermore, any initial data $u_0(\xi)$ which is identically zero as $\xi \to -\infty$ and identically one as $\xi \to +\infty$ is an infinite distance from any wave $\varphi(\xi, c)$, $c^2 > 4\alpha$. This fact is an immediate consequence of Corollary 6.2.

Kolmogorov *et al.* [2] proved that the solution of the Cauchy problem (6.1) with initial data

$$\begin{aligned} u_0(\xi) &= 1, \qquad \xi \geqslant 0,\\ &= 0, \qquad \xi < 0, \end{aligned}$$

is asymptotic, in some weak sense, to the wave $\varphi(\xi, 2\sqrt{\alpha})$. There is a feeling among a number of researchers that the wave of speed $2\sqrt{\alpha}$ may be the asymptotic limit of the solution of the initial value problem when the initial data belongs to an appropriate class. The asymptotic stability of the wave $\varphi(\xi, 2\sqrt{\alpha})$ is, however, a delicate one, since the continuous spectrum of L extends all the way to $\lambda = 0$. Moreover, initial data of the class discussed in the preceeding paragraph is an infinite distance from $\varphi(\xi, 2\sqrt{\alpha})$ unless (H) holds.

One might hope to eliminate some part of the continuous spectrum of L in the neighborhood of the origin by the introduction of a more restrictive norm; but the price of using a more restrictive norm (assuming one can be found) would be a smaller class of allowable initial perturbations.

7. Burgers' Equation

Burgers' equation,

$$u_t + uu_x = u_{xx}, \tag{7.1}$$

has received considerable attention in the literature [3, 5, 8] (see also [9] for a discussion of shocks obtained as limits of wave solutions as the viscosity tends to zero.) In contrast to the relatively weak stability possessed by the KPP waves, Eq. (7.1) admits wave solutions which exhibit a strong stability.

The wave solutions of (7.1) are known explicitly to be

$$\begin{aligned} u(x, t) &= \varphi(\xi), \qquad \xi = x + ct, \\ \varphi(\xi) &= -c(1 + \tanh \tfrac{1}{2}c\xi). \end{aligned} \tag{7.2}$$

The stability of these waves has previously been treated by Peletier [3] and Oleinik and Kruzhkov [8].

The linearized operator L takes the form

$$Lv = v'' - (\varphi + c)v' - \varphi' v.$$

Now

$$\begin{aligned} \varphi + c = -c \tanh \tfrac{1}{2}c\xi \quad &-c, \qquad \xi \to \infty, \\ &\sim c, \qquad \xi \to -\infty, \end{aligned}$$

and

$$\varphi' = -\frac{c^2}{2}\operatorname{sech}^2 \frac{c\xi}{2},$$

so

$$\kappa^{+} = -c, \qquad \kappa^{-} = c,$$
$$q^{\pm} = 0,$$
$$p^{\pm} = \frac{c^2}{4} - .$$

We are thus in case (III) so that the spectrum of L coincides with that of M. Let us calculate B and M:

$$b(\xi) = \tfrac{1}{2}(\varphi(\xi) + c) = -\frac{c}{2}\tanh\frac{c\xi}{2}.$$

Thus

$$e^{B(\xi)} = \operatorname{sech}\frac{c\xi}{2} \sim e^{-c|\xi|/2} \qquad \text{as} \quad |\xi| \to \infty,$$

and the weight function w may be taken to be

$$w(\xi) = \cosh\frac{c\xi}{2}.$$

The operator M is

$$Mv = v'' + pv$$

where

$$\begin{aligned} p &= \frac{1}{2}\varphi' - \frac{(\varphi + c)^2}{4} \\ &= -\frac{c^2}{4}\left[\operatorname{sech}^2\frac{c\xi}{2} + \tanh^2\frac{c\xi}{2}\right] \\ &= -\frac{c^2}{4}. \end{aligned}$$

Now

$$w\varphi' = \cosh\frac{c\xi}{2}\varphi' = -\frac{c^2}{2}\operatorname{sech}\frac{c\xi}{2},$$

so $w\varphi'$ is bounded. Therefore $\varphi' \in \mathscr{B}_{w,0}$ and L, hence M, has an isolated eigenvalue at the origin. The spectrum of L consists of a continuous spectrum on the interval $-\infty < \lambda < -c^2/4$ plus an isolated eigenvalue at the origin.

Thus, Burgers' waves are stable.

8. Huxley's Equation

Huxley's equation is (6.1) with $f(u)$ given by

$$f(u) = u(1 - u)(u - a),$$

where $0 < a < \frac{1}{2}$. This equation admits traveling wave solutions φ which are monotone, that is, φ' is of one sign (see [1, 4]). Huxley found the exact solution

$$\varphi(\xi) = \frac{1}{1 + e^{-\xi/\sqrt{2}}}, \qquad \xi = x + ct, \qquad c = \sqrt{2}\left(\frac{1}{2} - a\right).$$

For this wave we easily calculate

$$\kappa^+ = \kappa^- = c > 0$$

while

$$q^+ = -(1 - a),$$
$$q^- = -a,$$

and

$$\varphi' > 0.$$

We are therefore in case (I) and the wave is stable according to Theorem 5.6. The weight function is again (6.7).

Let us show that $\varphi' \in \mathscr{B}_{w,0}$. Since $f'(0) = -a$, φ, hence φ', are asymptotic as $\xi \to -\infty$ to $e^{\gamma\xi}$, where

$$\gamma^2 - c\gamma - a = 0.$$

The roots of this equation are real and of opposite sign. Let

$$\gamma_1 = \frac{c + \sqrt{c^2 + 4a}}{2}$$

be the positive root; then $\varphi' = 0(e^{\gamma_1\xi})$ as $\xi \to -\infty$, and

$$w\varphi' = e^{-c\xi/2}\varphi' = O\left(e^{\frac{\sqrt{c^2 + 4a}}{2}\xi}\right) \qquad \text{as} \quad \xi \to -\infty.$$

Since there is no problem as $\xi \to +\infty$, $\varphi' \in \mathscr{B}_{w,0}$.

The traveling wave solution is stable in the $L\infty$ norm (one may take the weight function $w \equiv 1$). This will be shown in a later paper.

9. The Stability Problem for the General Single Equation

Consider the general scalar equation (1.7); the associated wave $u(x, t) = \varphi(\xi)$, $\xi = x + ct$, satisfies the ordinary differential equation

$$\varphi'' - c\varphi' + f(\varphi, \varphi') = 0. \tag{9.1}$$

Suppose that $f(0, 0) = f(1, 0) = 0$, so that $(0, 0)$ and $(1, 0)$ in the phase plane are equilibrium points (there is no loss of generality in placing two such points at $(0, 0)$ and $(1, 0)$); and suppose (9.1) admits a solution $\varphi(\xi)$ such that (φ, φ') passes from $(0, 0)$ to $(1, 0)$ in such a way that

(i) $\varphi > 0$, $\varphi' > 0$,

(ii) $\varphi \to 0$ as $\xi \to -\infty$; $\varphi \to 1$ as $\xi \to +\infty$,

(iii) $\varphi' \to 0$ as $|\xi| \to \infty$.

We ask: Under what conditions is the wave φ stable to small perturbations relative to the weighted norm $\| \ \|_{w,0}$? From Section 5 we know that

$$b = \frac{c}{2} - f_{u'}(\varphi, \varphi'),$$

$$q = f_u(\varphi, \varphi'),$$

so that

$$\begin{aligned} \kappa^+ &= c - f_{u'}(1, 0), \\ \kappa^- &= c - f_{u'}(0, 0), \\ q_+ &= f_u(1, 0), \\ q_- &= f_u(0, 0). \end{aligned} \tag{9.2}$$

If we linearize (9.1) about $(1, 0)$ we get

$$\varphi'' + (f_{u'}(1, 0) - c)\varphi' + f_u(1, 0)\varphi = 0; \tag{9.3}$$

while if (9.1) is linearized about $(0, 0)$ we get

$$\varphi'' + (f_{u'}(0, 0) - c)\varphi' + f_u(0, 0)\varphi = 0. \tag{9.4}$$

An equilibrium point $(1, 0)$ (or $(0, 0)$) is called a *saddle* if the characteristic exponents of (9.3) (or (9.4)) have opposite signs. An equilibrium point is called a *stable node* (for $\xi \to +\infty$) if the characteristic exponents are both real and negative, and an *unstable node* if they are both real and positive.

In view of the relations (9.2) the characteristic exponents of (9.3) are roots of the quadratic equation

$$\gamma^2 - \kappa^+\gamma + q_+ = 0 \tag{9.5}$$

while for (9.4) we get

$$\gamma^2 - \kappa^-\gamma + q_- = 0. \tag{9.6}$$

Referring first to (9.5) we see that the roots γ_1, γ_2 satisfy

$$\gamma_1\gamma_2 = q_+,$$
$$\gamma_1 + \gamma_2 = \kappa^+.$$

Thus $(1, 0)$ is a *saddle* if $q_+ < 0$; a *stable node* if $0 < q_+ < (\kappa_+/2)^2$ and $\kappa_+ < 0$; and an *unstable node* $0 < q_+ < (\kappa_+/2)^2$ and $\kappa_+ > 0$. Similarly, $(0, 0)$ is a saddle if $q_- < 0$; a stable node if $0 < q_- < (\kappa_-/2)^2$ and $\kappa_- < 0$; and an unstable node if $0 < q_- < (\kappa_-/2)^2$ and $\kappa_- > 0$.

Since by assumption the hypothesized wave is a trajectory going from $(0, 0)$ to $(1, 0)$, we see that $(1, 0)$ must be either a saddle (S) or a stable node (N), while $(0, 0)$ must either be a saddle (S) or an unstable node, which we may denote in the obvious way by SS, SN, NS, and NN.

(SS) Here we must have $q_+ < 0$ and $q_- < 0$. In this case the wave is stable in the $L\infty$ norm.

(SN) Here $q_- < 0$, while $q_+ > 0$ and $\kappa^+ < 0$. We are therefore either in case (II) or (III), and we get stability by Corollary 5.7.

(NS) Here $q_- > 0$ and $\kappa^- > 0$ while $q_+ < 0$ and we are in case (I) or (III) and we get stability by Corollary 5.7.

(NN) In this case $0 < q_- < (\kappa_-/2)^2$ and $0 < q_+ < (\kappa_+/2)^2$ while $\kappa_+ < 0$ and $\kappa_- > 0$. This is case (III) again, and the wave is stable.

Note that (IV) does not occur for any of these cases. Case (II) would replace case (I) if we made the transformation $c \to -c$, $x \to -x$ in the original equation.

Summarizing,

THEOREM 9.1. *Let the scalar equation* (1.7) *admit a transition wave* $\varphi(\xi)$ *which is positive, increasing, and which interconnects Saddles and Nodes. Then* φ *is stable in the appropriate norm. In the case* (SS) *the wave is stable in the uniform norm.*

In [25] somewhat sharper stability results are stated. Let the weight functions $w_\pm$ be given as follows.

$$w_+(\xi) = 1 + \exp\{\pm\tfrac{1}{2} \mid \kappa^\pm \mid \xi\}$$

Then suitable weight functions for stability are given in the table below.

(0, 0) \ (1,0)	S	N
S	1	w_+
N	w_-	$w_+ + w_-$

We see that the weight function grows exponentially in the direction of a node. The weight function may be identically one if both endpoints are saddles.

The results stated above are somewhat sharper than those proved in this paper. We thank Professor D. G. Aronson for pointing out to us that such sharper results might be available. A proof of these results will be given in a future paper.

If one of the equilibrium points is a node then a weighted norm must be introduced in order to obtain a stability result as in Theorem 4.1. For, let us suppose that (1, 0) is a stable node, that is, that $q_+ \geqslant 0$ and $\kappa_+ \geqslant 0$. We consider the spectral analysis of the associated operator L on the Banach space $L\infty$. Let ψ satisfy $(L - \lambda)\psi = 0$. As $\xi \to \pm\infty$ ψ is asymptotic to the solutions of the following equations.

$$w'' - \kappa^+ w' + (q_+ - \lambda)w = 0, \qquad \xi \to +\infty, \tag{9.7}$$

$$w'' - \kappa^- w' + (q_- - \lambda)w = 0, \qquad \xi \to -\infty. \tag{9.8}$$

If $\lambda < q_+$ then (since $\kappa^+ < 0$) both solutions of (9.7) decay exponentially as $\xi \to +\infty$. At $\xi = -\infty$ there are two possibilities: Either $q_- < 0$ in which case one solution tends to zero exponentially as $\xi \to -\infty$; or $q_- \geqslant 0$ and $\kappa_- > 0$ in which case at least one solution decays as $\xi \to -\infty$. In either case, for $\lambda < q_+$ we may construct solutions of $(L - \lambda)\psi = 0$ which are bounded on the entire real line. We choose one of the solutions which decays as $\xi \to -\infty$ and continue it to $\xi = +\infty$. As $\xi \to +\infty$ it is asymptotic to a linear combination of the two solutions of (9.7), both of which decay.

Therefore, if $q_+ > 0$, L has a continuum of eigenvalues extending into the right half-plane. According to Evans' instability result [6], the wave solution must be unstable in the $L\infty$ norm. If $q_+ = 0$ we have a continuum of eigenvalues extending to the origin; presumeably this means that the wave function φ is not exponentially stable (although the perturbation may perhaps decay algebraically).

Appendix A

We have [22, p. 627]

$$(\lambda - L)^{-1} = \int_0^\infty e^{-\lambda t} e^{-tL}\, dt,$$

hence for any $u \in \mathscr{B}_{w,0}$, $v^* \in \mathscr{B}^*_{w,0}$,

$$\langle (\lambda - L)^{-1} Qu, v^* \rangle = \int_0^\infty e^{-\lambda t} \langle e^{-tL} Qu, v^* \rangle\, dt.$$

The left-hand quantity is analytic in the region exterior to the region $\mathscr{P} = \{y^2 + x + a < 0\}$. It is also regular at the origin since Q projects onto the complement of the null space of L. The function on the left also decays as $|\lambda| \to \infty$ along rays in the left half-plane; so by the classical Laplace inversion formula for functions,

$$\langle e^{-tL} Qu, v^* \rangle = \frac{1}{2\pi i} \int_C e^{\lambda t} \langle (\lambda - L)^{-1}, v^* \rangle\, d\lambda, \tag{A.1}$$

where C is any contour of the type described in Lemma 3.4. Since (A.1) holds for any $v^* \in \mathscr{B}^*_{w,0}$, and since the contour integral is the strong limit in $\mathscr{B}_{w,0}$ of a sequence of finite sums we have

$$e^{-tL} Qu = \frac{1}{2\pi i} \int_C e^{\lambda t} (\lambda - L)^{-1} u\, d\lambda.$$

References

1. H. Cohen, "Nonlinear Diffusion Problems," IBM Technical Report.
2. A. Kolmogorov, I. Petrovsky, and N. Piskunov, Étude de l'équation de la chaleur avec croissance... et son application à un problème biologique, *Bull. Univ. État Moscou* **6** (1937), 1–25.

3. L. A. PELETIER, Asymptotic stability of traveling waves, *in* "Instability of Continuous Systems," pp. 418–422, IUTAM Symposium, 1969.
4. H. P. MCKEAN, Nagumo's equation, *Advances in Math.* **4** (1970), 209–223.
5. P. LAX, "Nonlinear Partial Differential Equations of Evolution," pp. 831–840, Congrès International des Mathematiciens, Nice, September 1970.
6. J. W. EVANS, Nerve axon equations, I: Linear approximations, *Indiana Univ. Math. J.* **21** (1972), 877–885.
7. A. L. HODGKIN AND A. F. HUXLEY, A quantitative description of membrane current and its application to conduction and excitation in nerves, *J. Physiol.* **117** (1952), 500–544.
8. O. A. OLEINIK AND S. N. KRUZHKOV, Russian Math. Surveys **16** (1961), 105ff.
9. C. C. CONLEY AND J. SMOLLER, Shock Waves as limits of progressive wave solutions of higher order equations, *Comm. Pure Appl. Math.* **24** (1971).
10. E. A. CODDINGTON AND N. LEVINSON, "Ordinary Differential Equations," McGraw-Hill, New York, 1955.
11. A. JEFFREY AND T. KAKUTANI, Stability of the Burgers shock wave and the Korteweg-de Vries soliton, *Indiana Univ. Math. J.* **20** (1970), 463–469.
12. J. W. EVANS, Nerve axon equations, II: Stability at rest, *Indiana Univ. Math. J.* **22** (1972), 75–90.
13. J. W. EVANS, Nerve axon equations, III: Stability of the nerve impulse, *Indiana Univ. Math. J.* **22** (1972), 577–593.
14. D. H. SATTINGER, "Topics in Stability and Bifurcation Theory," Lecture Notes in Mathematics, No. 309, Springer-Verlag, New York/Berlin.
15. M. H. PROTTER AND H. F. WEINBERGER, "Maximum Principles in Differential Equations," Prentice-Hall, Englewood Cliffs, N. J., 1967.
16. JA. I. KANEL, On the stability of solutions of the Cauchy problem for equations occurring in the theory of combustion, *Mat. Sb.* **59** (1962), 245–288.
17. JA. I. KANEL, On the stability of solutions of the equations of the theory of combustion for finite initial functions, *Mat. Sb.* **65** (1964), 398–413.
18. JA. I. KANEL, "The behavior of solutions of the Cauchy problem when the time tends to infinity, in the case of quasilinear equations arising in the theory of combustion, *DAN SSSR* 132 (1960), 268–271.
19. JA. I. KANEL, Some problems involving burning-theory equations, *DAN SSSR* **136** (1961), 277–280.
20. I. M. GELFAND, Some problems in the theory of quasilinear equations, *Amer. Math. Soc. Transl.* **29** (1963), 295–381.
21. M. H. PROTTER AND H. F. WEINBERGER, On the spectrum of general second order operators, *Bull. Amer. Math. Soc.* **72** (1966), 251–255.
22. N. DUNFORD AND J. T. SCHWARTZ, "Linear Operators," Vol. I, Interscience, New York, 1958.
23. F. RIESZ AND B. SZ.-NAGY, "Functional Analysis," Ungar, New York, 1955.
24. F. HOPPENSTEAD, "Mathematical Theories of Population," New York University Lecture Notes.
25. D. H. SATTINGER, "Stability of Traveling Waves of Nonlinear Parabolic Equations, Seventh International Conference on Nonlinear Oscillations, East Berlin, 1975.