A new approach for optimal control on Navier-Stokes Equations

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Abstract : In this paper, we deal with optimal boundary control for Navier-Stokes problem. We establish the existence of such a control in appropriate functional spaces. Then we study a stabilization problem around a steady state. In view of numerical approximation, we derive rigorously Euler equations satisfied by the control.

1 Introduction

1.1 Setting of the problem

The aim of this paper is to give some directions to solve optimal control problems for Navier-Stokes equations. In optimal control theory, one is interested in minimizing a cost functional involving the kinetic energy and a suitable norm on the control. This kind of problem arise in fluid mechanics when one wants, for example, to increase the performance of a plane by modifying the flow near the wall though non homogeneous boundary conditions (see [BMT], [CFT]). A huge literature is dedicated to the subject treated the theorical point of view (see [B1], [B2], [F], [FGH], [I1] and [I2]). For numerical experiments, to our knowledge, only heuristic methods are available. Most of them are build on linearization technique around a steady state (see [OFZ], [IZ]).

In this paper, we deal with the case where the control lies on a part of the boundary of the domain. The physical case is the study of the flow around an obstacle embedded in an unbounded domain. In this situation, there is no compatible condition on the data. Concerning bounded domain, as we consider an incompressible flow, we have to assume that the mean of the normal component of the control vanishes on the boundary.

The situation is the following. Let Ω be a smooth bounded domain of $\mathbb{R}^2$ and $\partial\Omega$ its boundary. We assume that $\partial\Omega = \Gamma_e \cup \Gamma_c$ with $\Gamma_e \cap \Gamma_c = \emptyset$. Here Γ_e and Γ_c denote respectively the external and interior part of the boundary and for simplicity, we define $\Gamma = \Gamma_e \cup \Gamma_c$.

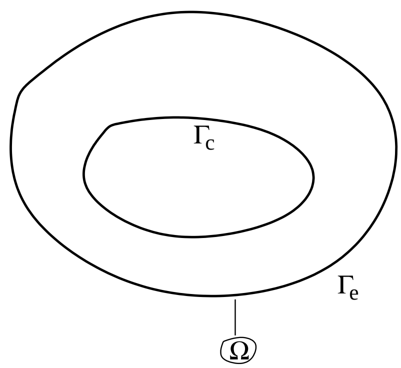

Figure 1: Example of domain Ω.

The control is defined on Γ_c. On Γ_e, we consider Dirichlet conditions.

1.2 Formulation and existence.

Let us give a finite prediction horizon time T over which the control is optimized, a smooth function g modelizing the inflow boundary condition defined on the external boundary Γ_e and a control u_ρ acting on the interior boundary Γ_c. We introduce the following quadratic functional

$$J(u_\rho) = ||u(\tau)||_Y^2 + \alpha ||u_\rho(\tau)||_X^2,$$

where X and Y are two Hilbert spaces in $(t,x) \in [0,T] \times \Omega$ which will be defined in Section 2 and α is a positive regularization parameter. For mathematical analysis, we may assume that $\alpha = 1$ without loss of generality.

The relationship between u and the boundary control u_ρ is given by the

Navier-Stokes equations satisfied for $(t,x) \in [0,T] \times \Omega$

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + (u(t,x)\cdot\nabla)u(t,x) - \dfrac{1}{\mathcal{R}_e}\Delta u(t,x) + \nabla p(t,x) = f(t,x), \\ \mathrm{div}(u)(t,x) = 0, \\ u_{/\Gamma_c}(t,x) = u_\rho(t,x), \;\; u_{/\Gamma_e}(t,x) = g(x), \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where u_0 is a given initial data, f is the volume unit force and $\mathcal{R}_e$ is the Reynolds number. The function u_ρ depends obviously on time t and x whereas for the sake of simplicity g depends only on x (for the existence of such solution see Theorem 1.2).

The first result is concerned with the existence of an optimal control u_ρ^{opt} satisfying

$$J(u_\rho^{opt}) = \inf_{u_\rho \in X} J(u_\rho). \tag{1.2}$$

The Hilbert space X is the set of admissible control. As the Navier-Stokes equations are nonlinear with respect to u_ρ, the functional J is not a convex function. In particular, we are not able to give a uniqueness result for the minimizer. Nevertheless we are able to derive first-order necessary optimality conditions for an optimal control through the derivation of Euler equations. In view of this result we have to study the Frechet derivative of the mapping $u_\rho \longmapsto u$ given by Equation (1.1).

To overcome the loss of uniqueness for the optimal control, in order to provide a constructive efficient method to approach heuristically u_ρ^{opt}, we will introduce a linearized problem upon which we derive some effective results. The first theorem of this paper is the following one.

Theorem 1.1. *Assume that Ω is a C^1 bounded domain of $\mathbb{R}^2$. For any function (u_0, g) given in $H \times H^{\frac{1}{2}}(\Gamma_e)$, where H is given by (1.6), satisfying the compatibility condition (2.7), there exists at least an optimal control u_ρ^{opt} solution to*

$$J(u_\rho^{opt}) = \inf_{u_\rho \in X} J(u_\rho),$$

where X will be described in Section 2.

1.3 Obtention and properties of the optimal control

In this section, we explain briefly our method to deal with the boundary control problem and we present some other results. First of all, in order to deal with Dirichlet boundary conditions on Γ_c, let us introduce the extension $\mathcal{U}_\rho$ of u_ρ solution to the unstationary Stokes problem

$$\begin{cases} \dfrac{\partial \mathcal{U}_\rho}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta \mathcal{U}_\rho + \nabla p = 0, \\ \operatorname{div}(\mathcal{U}_\rho) = 0, \\ \mathcal{U}_{\rho/\Gamma_c} = u_\rho, \ \ \mathcal{U}_{\rho/\Gamma_e} = 0, \ \ \displaystyle\int_{\Gamma_c} \mathcal{U}_\rho \cdot n \, ds = 0. \\ \mathcal{U}_\rho(0) = 0. \end{cases} \tag{1.3}$$

The function $\mathcal{U}_\rho$ belongs to the following Banach space

$$\mathcal{W}^L = \left\{ v \in L^2(0,T;H^1(\Omega)), \ \text{ such that } \frac{\partial v}{\partial t} - \frac{1}{\mathcal{R}_e}\frac{\partial v}{\partial t} + \nabla p = 0, \right.$$
$$\left. \operatorname{div}(v) = 0, \ v(0,\cdot) = 0, \ \ v_{\big|\Gamma e} = 0 \right\},$$

where $\frac{\partial v}{\partial t}$ and $\frac{\partial v}{\partial t}$ are calculated in the sense of distributions in $(0,T) \times \Omega$. Define $v = u - \mathcal{U}_\rho$. Then v is solution to the following Navier-Stokes equations

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v\cdot\nabla)v + (\mathcal{U}_\rho\cdot\nabla)v + (v\cdot\nabla)\mathcal{U}_\rho - \dfrac{1}{\mathcal{R}_e}\Delta v + \nabla p = f - (\mathcal{U}_\rho\cdot\nabla)\mathcal{U}_\rho \\ \operatorname{div}(v) = 0, \\ v_{/\Gamma_c} = 0, \ \ v_{/\Gamma_e} = g, \\ v(0) = u_0. \end{cases} \tag{1.4}$$

Note that in Equation (1.4), $\mathcal{U}_\rho$ is not only a term source since it appears for example in the term $\mathcal{U}_\rho \cdot \nabla v$. Concerning the Cauchy problem (1.4), we recall how to construct a solution for (1.4). Assume that g satisfies the compatibility conditions

$$\begin{cases} \displaystyle\int_{\Gamma_e} g\cdot n \, d\sigma = 0 \\ (u_0 \cdot n)_{\big|\Gamma e} = g \cdot n, \end{cases} \tag{1.5}$$

and introduce the following Banach spaces

$$H = \{v \in L^2(\Omega), \operatorname{div}(v) = 0, \ (v\cdot n)_{|\Gamma_c} = 0\} \tag{1.6}$$

$$V = \{v \in H_0^1(\Omega), \operatorname{div}(v) = 0\}. \tag{1.7}$$

From Theorem 2.4 (page 31) of [T2], one can construct a function $V \in H^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\left\{\begin{array}{l} -\dfrac{1}{\mathcal{R}_e}\Delta V + \nabla p = 0 \text{ in } \Omega \\ \operatorname{div}(V) = 0,\ V_{|\Gamma_e} = g. \end{array}\right. \tag{1.8}$$

Then $w = v - V$ is solution to the following Navier-Stokes equations

$$\left\{\begin{array}{l} \dfrac{\partial w}{\partial t} + \big((w+V)\cdot\nabla\big)(w+V) + (\mathcal{U}_\rho\cdot\nabla)(w+V) + \big((w+V)\cdot\nabla\big)\mathcal{U}_\rho \\ -\frac{1}{\mathcal{R}_e}\Delta w + \nabla p = f - (\mathcal{U}_\rho\cdot\nabla)\mathcal{U}_\rho \\ \operatorname{div}(w) = 0, \\ w_{/\Gamma_c} = 0,\ \ w_{/\Gamma_e} = 0, \\ w(0) = u_0 - V. \end{array}\right. \tag{1.9}$$

Applying Theorem 3.1 and Theorem 3.2 of [T2], one has the following result.

Theorem 1.2. *Assume that $g \in H^{\frac{1}{2}}(\Gamma)$ satisfy (1.5), $f \in L^2(0,T;V')$ and $u_0 \in H$. Then there exists a unique solution $v \in L^\infty(0,T;H)$ to (1.4). Moreover, v is weakly continuous from $[0,T]$ into H.*

Concerning the regularity of the mapping $\mathcal{U}_\rho \longmapsto v$, we prove the following theorem.

Theorem 1.3. *The map : $\mathcal{U}_\rho \in \mathcal{W}^L \cap H^\sigma(0,T;L^2) \longmapsto v$ where v is the solution to (2.6) is Frechet differentiable. Furthermore, for all $h \in \mathcal{W}^L \cap H^\sigma(0,T;L^2)$, $z = v'(\mathcal{U}_\rho)\cdot h$ is the solution to*

$$\left\{\begin{array}{l} \dfrac{\partial z}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta z + v\cdot\nabla z + z\cdot\nabla v + \mathcal{U}_\rho\cdot\nabla z + z\cdot\nabla\mathcal{U}_\rho \\ +h\cdot\nabla v + v\cdot\nabla h + \mathcal{U}_\rho\cdot\nabla h + h\cdot\nabla\mathcal{U}_\rho + \nabla p = 0, \\ \operatorname{div}(z) = 0, \\ z_{/\Gamma} = 0, \\ z(0) = 0. \end{array}\right.$$

In addition, we also establish the weak form of the first order optimality condition for $\mathcal{U}_\rho^{opt}$.

Corollary 1.1. *Under the assumptions of Theorem 1.1, the extension $\mathcal{U}_\rho^{opt}$ defined by (1.3) of u_ρ^{opt} satisfies : for all $h \in \mathcal{W}_\sigma^{NL}$*

$$(z, \mathcal{U}_\rho^{opt})_X + ((\mathcal{U}_\rho^{opt}, h))_Y = 0,$$

where z solves the equation

$$\begin{cases} \dfrac{\partial z}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta z + v \cdot \nabla z + z \cdot \nabla v + \mathcal{U}_\rho^{opt} \cdot \nabla z + z \cdot \nabla \mathcal{U}_\rho^{opt} \\ +h \cdot \nabla v + v \cdot \nabla h + \mathcal{U}_\rho^{opt} \cdot \nabla h + h \cdot \nabla \mathcal{U}_\rho^{opt} + \nabla \pi = 0, \\ \operatorname{div}(z) = 0, \\ z_{/\Gamma} = 0, \\ z(0) = 0. \end{cases}$$

and $(\cdot,\cdot)_X$ (resp. $((\cdot,\cdot))_Y$) denotes the scalar product of X (resp. Y) which are defined in Section 2 .

Concerning the linear version of the optimal boundary control problem, we prove the following theorem.

Theorem 1.4. *Assume that Ω is a C^1 bounded domain of $\mathbb{R}^2$. For any function (u_0, g) given in $H \times H^{\frac{1}{2}}(\Gamma_e)$ satisfying the compatibility condition* (2.7)*, there exists a unique optimal control u_ρ^{opt} solution to*

$$J_L(u_\rho^{opt}) = \inf_{u_\rho \in \mathcal{W}_c^L} J_L(u_\rho),$$

where $\mathcal{W}_c^L$ is defined in section 2 *and J_L is one of the suitable functional fitted to the linearized version of the optimal control problem.*

Finally, we give the Euler-Lagrange equation for the optimal control solution to the linearized version of the optimal control problem.

Theorem 1.5. *Let $\mathcal{U}_\rho^{opt}$ be the optimal control of Theorem* 1.4*. Then, for all $h \in \mathcal{W}^L$,*

$$((u + \mathcal{U}_\rho^{opt}, h))_Y = 0,$$

where u is the unique solution of $a(u,v) = (\ell, v)$ for all v in $\mathcal{W}^L$ with $\ell = \mathcal{L}\mathcal{U}_\rho^{opt} - \bar{v} \cdot \nabla p - {}^t(\nabla p)\bar{v}$ (see Section 5 for the definition of a and $\mathcal{L}$).

Notations : In this paper, for $1 \leq p < +\infty$ and $s > 0$, L^p is the usual Lebesgue spaces. For simplicity, the usual norm on L^p is denoted by $\|\cdot\|_p$. For $s \geq 0$, $H^s(\Omega)$ the usual Sobolev space define by interpolation (see [LM]). Let $\mathcal{X}$ be a Banach space. We denote by $L^p((0,T);\mathcal{X})$ $(1 \leq p < +\infty)$ the space of functions $m \;:\; (0,T) \longrightarrow \mathcal{X}$ such that m is measurable and

$$\|m\|_{L^p((0,T);\mathcal{X})} = \left(\int_0^T \|m(t)\|_{\mathcal{X}}^p dt\right)^{\frac{1}{p}} < +\infty.$$

Let $r \geq 0$ and $s \geq 0$. Denote $\mathcal{Q} = (0,T) \times \Omega$ and

$$H^{r,s}(\mathcal{Q}) = L^2(0,T;H^r(\Omega)) \cap H^s(0,T;L^2(\Omega))$$

endowed with the norm

$$\|u\|_{H^{r,s}(\mathcal{Q})} = (\|u\|^2_{L^2(0,T;H^r)} + \|u\|^2_{H^s(0,T;L^2)})^{\frac{1}{2}}.$$

In the same way we set

$$H^{r,s}((0,T) \times \Gamma_c) = L^2(0,T;H^r(\Gamma_c)) \cap H^s(0,T;L^2(\Gamma_c))$$

We also define

$$L^2_0(\Omega) = L^2(\Omega)/\mathbb{R},$$

and

$$W^{-1,\infty}(0,T;L^2_0(\Omega)) = \left\{ u \in \mathcal{D}'(\mathcal{Q}), u = \sum_{|\alpha|\leq 1} \partial^\alpha f_\alpha,\ f_\alpha \in L^\infty(0,T;L^2_0(\Omega)) \right\}$$

endowed with the norm

$$\|u\|_{W^{-1,\infty}(0,T;L^2_0(\Omega))} = \inf \left(\sup_{|\alpha|\leq 1} \|f_\alpha\|_{L^\infty(0,T;L^2_0(\Omega))} \right).$$

Furthermore, we denote by $(\cdot,\cdot)$ (resp. $((\cdot,\cdot))$) the usual scalar product on $L^2(\Omega)$ (resp. $H^1(\Omega)$). Throughout this paper, C will denote a generic constant which may change from one line to another.

The paper is organized as follows. In section 2, we introduce the spaces fitted to the boundary control problem. Section 3 is devoted to the proof of Theorem 1.1 concerning the existence of a solution to optimal boundary control problem. In section 4, we prove the Frechet differentiability of $\mathcal{U}_\rho \longmapsto v$ where v is solution to (2.6). (see Theorem 1.3). Section 5 deals with the linear version of the optimal control problem (see Theorem 1.4).

2 Functional Spaces for boundary control

In this section, we present the functional approach fitted to the problem introduced in Section 1.2. As usual, when we considere Dirichlet boundary problem, we have to build an extension of the boundary condition. This is achieved using appropriate trace theorems. We first recall the definition of $\mathcal{W}^L$:

$$\mathcal{W}^L = \Big\{ v \in L^2(0,T;H^1),\ \frac{\partial v}{\partial t} - \frac{1}{\mathcal{R}_e}\Delta v + \nabla p = 0,$$
$$\mathrm{div}(v) = 0,\ v(0,\cdot) = 0,\ \ v_{\big|\Gamma e} = 0 \Big\}.$$

Then we have

Lemma 2.1. *The space $\mathcal{W}^L$ is a closed subset of $L^2(0,T;H^1)$ for the induced topology.*

Proof.
• We first note that every $v \in \mathcal{W}^L$ satisfies $\dfrac{\partial v}{\partial t} \in L^2(0,T;V')$ Indeed, for any ϕ in $L^2(0,T;V)$, we have the following estimate :

$$\left|\left\langle \frac{\partial v}{\partial t}, \phi \right\rangle\right| \leq \frac{1}{\mathcal{R}_e}\|\phi\|_{L^2(0,T;V)}\|v\|_{L^2(0,T;H^1)}, \tag{2.1}$$

where $\langle \cdot,\cdot \rangle$ denotes the duality product between V and $V^{'}$. Hence, the functional space $\mathcal{W}^L$ is continuously embedded in $\mathcal{C}^0(0,T;V')$. We can notice that since $\mathcal{W}^L \not\subset L^2(0,T,V)$, we do not have $\mathcal{W}^L \subset C([0,T;[V,V^{'}]_{\frac{1}{2}})$ where $[\cdot,\cdot]_{\frac{1}{2}}$ denotes the interpolation space of order $\frac{1}{2}$ between V and $V^{'}$.
• Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{W}^L$ which converges to v in $\mathcal{W}^L$ for the $L^2(0,T;H^1)$-topology. By continuity of divergence and trace operators, it is obvious that :

$$v_{\big|\Gamma e} = 0,$$
$$\mathrm{div}(v) = 0.$$

We now have to check that v is solution of Stokes equation in the distribution sens

$$\frac{\partial v}{\partial t} - \frac{1}{\mathcal{R}_e}\Delta v - \nabla p = 0. \tag{2.2}$$

Let ϕ be in $\mathscr{C}^1(0,T;V)$, such that $\phi(T) = 0$. Then each v_n satisfies the weak formulation :

$$-\int_0^T \int_\Omega v_n(\tau)\frac{\partial \phi(\tau)}{\partial t} d\tau + \frac{1}{\mathcal{R}_e}\int_0^T \int_\Omega (\nabla v_n : \nabla \phi)\, dx d\tau = 0, \tag{2.3}$$

where " : " denotes the contract product of two second order tensor, i.e. if $\mathcal{T} = [\mathcal{T}_{ij}]$ and $\mathcal{S} = [\mathcal{S}_{ij}]$ are two second order tensors, $\mathcal{T} : \mathcal{S} = [\mathcal{T}_{ij}\mathcal{S}_{ij}]$. Since $(v_n)_{n\in\mathbb{N}}$ converges to v in $L^2(0,T;H^1(\Omega))$, we can perform the limit as n goes to $+\infty$ in equation (2.3) to obtain, $\forall \phi \in \mathcal{C}^0(0,T;V)$,

$$\mathrm{div}\phi = 0,\ -\int_0^T \int_\Omega v(\tau)\frac{\partial \phi(\tau)}{\partial t} d\tau + \frac{1}{R_e}\int_0^T \int_\Omega (\nabla v : \nabla \phi)\, dx d\tau = 0.$$

Then by de Rham Theorem's (see [BF], [T2] for more details), there exists a distribution $p \in W^{-1,\infty}(0,T;L^2_0(\Omega))$ such that

$$\frac{\partial v}{\partial t} - \frac{1}{\mathcal{R}_e}\Delta v + \nabla p = 0.$$

• Moreover, by estimate (2.1), $\left(\dfrac{\partial v_n}{\partial t}\right)_{n\in\mathbb{N}}$ converges weakly up to a subsequence to $\dfrac{\partial v}{\partial t}$ in $L^2(0,T;V')$. It follows that this subsequence $(v_{n_k})_{k\in\mathbb{N}}$ converge to v in $\mathscr{C}^0(0,T;V')$, hence $v(0) = 0$.

□

We notice that the functions of $\mathcal{W}^L$ have traces on Γ_c in $L^2(0,T;H^{\frac{1}{2}}(\Gamma_c))$, as the maps

$$\begin{array}{rccc} \mathscr{R}_{\Gamma_c} : & L^2(0,T;H^1(\Omega)) & \longrightarrow & L^2(0,T;H^{\frac{1}{2}}(\Gamma_c)) \\ & v & \longrightarrow & \mathscr{R}_{\Gamma_c}(v) = v_{\big|\Gamma c} \end{array}$$

is continuous.

Lemma 2.2. *The mapping $\mathscr{R}_{\Gamma_c}$ from $\mathcal{W}^L$ to $L^2(0,T;H^{\frac{1}{2}}(\Gamma_c))$ defined by $\mathscr{R}_{\Gamma_c}(v) = v_{|\Gamma_c}$ is injective.*

Proof. Let (v_1, v_2) two functions of $\mathcal{W}^L$ vanishing on Γ_e such that $\mathscr{R}_{\Gamma_c}(v_1) = \mathscr{R}_{\Gamma_c}(v_2)$ and denote $v = v_2 - v_1$. As v belongs to $L^2(0,T;V)$ and satisfies an homogeneous Stokes problem with zero initial value, by uniqueness of weak solution, we have $v = 0$.

□

In the sequel, we denote $\mathcal{W}^L_c$ the space of the restrictions to Γ_c of the functions in $\mathcal{W}^L$

$$\mathcal{W}^L_c = \mathscr{R}_{\Gamma_c}(\mathcal{W}^L) = \left\{ w_{\big|\Gamma_c}, w \in \mathcal{W}^L \right\}.$$

Lemma 2.3. *The space $\mathcal{W}_c^L$ endowed with the norm:*

$$\|\gamma\|_{\mathcal{W}_c^L} = \|v\|_{L^2(0,T;H^1)}$$

where v is the unique function of $\mathcal{W}^L$ such that $\mathscr{R}_{\Gamma_c}(v) = \gamma$ is a Hilbert space.

Proof. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of $\mathcal{W}_c^L$. For each $n \in \mathbb{N}$, define v_n such that $\gamma_n = \mathscr{R}_{\Gamma_c}(v_n)$. Then $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of $L^2(0,T;H^1)$ and the result follows from the continuity of $\mathcal{R}_{\Gamma_c}$.

Remark 2.1.

- *The set $\mathcal{W}_c^L$ is nonempty. Indeed*

$$\Big\{\gamma \in H^{\frac{1}{2},1}((0,T)\times\Gamma_c),\ \int_{\Gamma_c}\gamma\cdot\nu d\sigma = 0\Big\} \subset \mathcal{W}_c^L$$

  (see Theorem 5.2 in [M]).

- *We have to notice here that we do not know that the temporal derivatives of a function of $\mathcal{W}^L$ belongs to a space with values in $H^{-1}(\Omega)$. It means that in a certain way, there is no control on time evolution for the functions belonging to $\mathcal{W}_c^L$. This state of fact makes us think that this space is not adapted for solving optimal boundary control problem for the Navier-Stokes equations. Anyway, as we will see in the sequel, it will be a suitable one for the study of the linearized Navier-Stokes Equations.*

Definition 2.1. *For the study of the linearized version of the optimal control problem, the space of admissible control is $\mathcal{W}_c^L$.*

In order to handle the optimal control problem for nonlinear Navier-Stokes equations, we introduce a new functional space. We first recall the following (see [T2])

Definition 2.2. *Let $0 < \sigma < 1$. For $v \in \mathcal{W}^L$, we denote $D_t^\sigma(v)$ the fractional derivative of order σ with respect to time t*

$$D_t^\sigma(v)(t) = \mathcal{F}^{-1}\Big(i\tau^\sigma\mathcal{F}\big(v(\cdot)(\tau)\big)\Big)(t),$$

where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}$. We then define

$$H^\sigma(\mathbb{R}, L^2(\Omega)) = \left\{v \in .../D_t^\sigma(v) \in L^2(\mathbb{R};L^2(\Omega))\right\}.$$

Finally, $H^\sigma(0,T;L^2(\Omega))$ will denote the set of the restriction of functions in $H^\sigma(\mathbb{R},L^2(\Omega))$ to $[0,T]$.

Definition 2.3. *Define*

$$\mathcal{W}_{\sigma}^{NL} = \mathcal{W}^{L} \cap H^{\sigma}(0,T;L^2(\Omega))$$

equipped with the norm $\|v\|_{\mathcal{W}_{\sigma}^{NL}} = \left(\|v\|^2_{L^2(0,T;H^1(\Omega))} + \|D_t^{\sigma} v\|^2_{L^2(0,T;L^2(\Omega))}\right)^{\frac{1}{2}}$. *Finally, denote* $\mathcal{W}_{\sigma,c}^{NL}$ *the trace space associated through the map* $\mathscr{R}_{|\Gamma_c}$. *The space* $\mathcal{W}_{\sigma,c}^{NL}$ *is the set of admissible control for the nonlinear optimal boundary control problem.*

Remark 2.2.

- *The space* $\mathcal{W}_{\sigma}^{NL}$ *is not empty. Indeed, the stationary solutions to the Stokes Equation defining the space* $\mathcal{W}^{L}$ *belongs to* $\mathcal{W}_{\sigma}^{NL}$.
- *The space* $\mathcal{W}_{\sigma}^{NL}$ *is a closed subspace of* $\mathcal{W}^{L}$ *and so this is a Hilbert space.*
- *In boundary control theory, it is classical (see [M]) to consider control in* $H^{1,1}((0,T)\times\Gamma_c)$ *which is more regular than* $\mathcal{W}_{\sigma,c}^{NL}$.
- *The trace space above devoted to the study of the nonlinear case is more general an less regular than the one used in [FGH]. Furthermore, in the linear case where we consider the problem of stabilization around a steady state, the cost functional does not involve time derivatives of the control.*

The bounded sets of $\mathcal{W}_{\sigma}^{NL}$ are weakly relatively compact in $L^2(0,T;L^2(\Omega))$ as it is prove in the following lemma.

Lemma 2.4. *For all* $0<\sigma<1$, *the embedding of* $\mathcal{W}_{\sigma}^{NL}$ *endowed with the norm* $\|v\|_{\mathcal{W}_{\sigma}^{NL}}$ *into* $L^2(0,T;L^2(\Omega))$ *is compact.*

Proof. We recall (see [S2]):

Lemma 2.5 (Aubin-Lions-Simon)**.** *Let* B_1, B_0, B_{-1} *be three Banach spaces such that* $B_1 \hookrightarrow B_0 \hookrightarrow B_{-1}$ *with compact embedding from* B_1 *into* B_0. *For* $0 < s \leq 1$, *denote*

$$W_s = \left\{ v \in L^2(0,T;B_1),\ \sup_{0<h<1}\left(\frac{1}{h^s}\big(\tau_h(\tilde{v}) - \tilde{v}\big)\right) \in L^2(\mathbb{R};B_{-1}) \right\},$$

where $\tau_h(v)(t) = v(t+h)$.

Then the embedding $W_s \hookrightarrow L^2(0,T;B_0)$ *is compact.*

Let us notice that for $0 < \sigma < 1$, $\mathcal{W}_\sigma^{NL} \subset W_\sigma$ with $B_1 = H^1(\Omega)$ and $B_0 = B_{-1} = L^2(\Omega)$ from which the result follows.

□

We also recall the following classical result which is a direct consequence of Corollary 31 of [S1].

Lemma 2.6. *For all $\frac{1}{2} < \sigma < 1$ and $2 \leq q \leq +\infty$, $\mathcal{W}_\sigma^{NL} \subset L^q(0, T; L^2)$.*

We are now able to make more precise the formulation of the optimal boundary control problem. The control u_ρ will be taken in the space $\mathcal{W}_{\sigma,c}^{NL}$ for $\frac{1}{2} < \sigma < 1$. By definition of $\mathcal{W}_\sigma^{NL}$, it is obvious that for any $u_\rho \in \mathcal{W}_{\sigma,c}^{NL}$, there exists a unique solution $\mathcal{U}_\rho \in \mathcal{W}_\sigma^{NL}$ of Stokes equations (see Lemma 2.2)

$$\begin{cases} \dfrac{\partial \mathcal{U}_\rho}{\partial t} - \dfrac{1}{\mathcal{R}_e} \Delta \mathcal{U}_\rho + \nabla p = 0, \\ \operatorname{div}(\mathcal{U}_\rho) = 0, \\ \mathcal{U}_{\rho/\Gamma_c} = u_\rho, \ \mathcal{U}_{\rho/\Gamma_e} = 0, \ \displaystyle\int_{\Gamma_c} \mathcal{U}_\rho \cdot n \, ds = 0, \\ \mathcal{U}_\rho(0) = 0. \end{cases} \tag{2.4}$$

We introduce the quadratic functional J as follows

$$J(u_\rho) = \int_0^T ||u(\tau)||_2^2 d\tau + ||u_\rho||^2_{\mathcal{W}_{\sigma,c}^{NL}}.$$

From the above considerations, it is more convenient to work with the extension $\mathcal{U}_\rho$ given by (2.4) instead of the boundary control u_ρ. Following this idea, the problem (1.2) is reduced to find $\mathcal{U}_\rho^{opt}$ solution of

$$\mathcal{J}(\mathcal{U}_\rho^{opt}) = \inf_{\mathcal{U}_\rho \in \mathcal{W}_\sigma^{NL}} \mathcal{J}(\mathcal{U}_\rho), \tag{2.5}$$

where $\mathcal{J}$ is defined by

$$\mathcal{J}(\mathcal{U}_\rho^{opt}) = \int_0^T ||v(\tau)||_2^2 d\tau + ||\mathcal{U}_\rho||^2_{\mathcal{W}_\sigma^{NL}}.$$

and $v = u - \mathcal{U}_\rho$ is solution to

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla) v + (\mathcal{U}_\rho \cdot \nabla) v + (v \cdot \nabla) \mathcal{U}_\rho - \dfrac{1}{\mathcal{R}_e} \Delta v + \nabla p = f - (\mathcal{U}_\rho \cdot \nabla) \mathcal{U}_\rho \\ \operatorname{div}(v) = 0, \\ v_{/\Gamma_c} = 0, \ \ v_{/\Gamma_e} = g, \\ v(0) = u_0. \end{cases} \tag{2.6}$$

The data (u_0, g) have to be taken in $H \times H^{\frac{1}{2}}(\Gamma_e)$. Moreover, we recall that (u_0, g) has to satisfy the compatibility conditions :

$$\begin{cases} \displaystyle\int_{\Gamma_e} g \cdot n d\sigma = 0 \\ (u_0 \cdot n)\Big|_{\Gamma e} = g \cdot n. \end{cases}$$

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The proof is based on a compacity argument. Let $(u_\rho^n)_{n\in\mathbb{N}} \in \mathcal{W}_{\sigma,c}^{NL}$ be a minimizing sequence for problem (1.2). According to Section 2, we introduce for each $n \in \mathbb{N}$ the extension $\mathcal{U}_\rho^n \in \mathcal{W}_\sigma^{NL}$ of u_ρ^n given by (1.3). Then it is clear that $(\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ is a minimizing sequence for problem (2.5) which means that if we denote

$$\mu = \inf_{\mathcal{U}_\rho \in \mathcal{W}_\sigma^{NL}} \mathcal{J}(\mathcal{U}_\rho),$$

then

$$\lim_{n\to+\infty} \mathcal{J}(\mathcal{U}_\rho^n) = \mu. \tag{3.1}$$

The main point is to show the lower semi-continuity of $\mathcal{J}$ for the weak topology. Recall here that

$$\mathcal{J}(\mathcal{U}_\rho^n) = \int_0^T ||v^n(\tau)||_2^2 d\tau + ||\mathcal{U}_\rho^n||^2_{\mathcal{W}_\sigma^{NL}},$$

where $v^n \in L^2(0, T; V)$ is the solution to

$$\begin{cases} \dfrac{\partial v^n}{\partial t} + (v^n \cdot \nabla)v^n + (\mathcal{U}_\rho^n \cdot \nabla)v^n + (v^n \cdot \nabla)\mathcal{U}_\rho^n - \dfrac{1}{\mathcal{R}_e}\Delta v^n + \nabla p \\ = f - (\mathcal{U}_\rho^n \cdot \nabla)\mathcal{U}_\rho^n \\ \operatorname{div}(v^n) = 0, \\ v^n_{/\Gamma_c} = 0, \ \ v^n_{/\Gamma_e} = g, \\ v^n(0, x) = u_0(x), \end{cases} \tag{3.2}$$

First, since $(\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ satisfies (3.1), the sequence $(\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{W}_\sigma^{NL}$. Since $\mathcal{W}_\sigma^{NL}$ is a Hilbert space, there exists $\mathcal{U}^* \in \mathcal{W}_\sigma^{NL}$ such that up to a subsequence $\mathcal{U}_\rho^n$ weakly converges to $\mathcal{U}^*$ in $\mathcal{W}_\sigma^{NL}$. We deduce that

$$\|\mathcal{U}^*\|_{\mathcal{W}_\sigma^{NL}} \leq \liminf_{n\to+\infty} \|\mathcal{U}_\rho^n\|_{\mathcal{W}_\sigma^{NL}}.$$

Moreover, by Lemma 2.4, we obtain that

$$\mathcal{U}_\rho^n \longrightarrow \mathcal{U}^* \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{3.3}$$

In order to conclude that $\mathcal{U}^*$ is solution to (2.5), we have to show that the sequence $(v^n)_{n\in\mathbb{N}}$ strongly converges to v^* in $L^2(0,T;L^2(\Omega))$ where v^* is linked to $\mathcal{U}^*$ by equation (2.6). For that purpose, we have to perform the limit $n \longrightarrow +\infty$ in Equation (3.2). As usual, it is easier to work with homogeneous Dirichlet boundary conditions. We then introduce an extension of the function g as follows. Recall that $g \in H^{\frac{1}{2}}(\Gamma_e)$, $\int_{\Gamma_e} g \cdot n = 0$ and that g is independent of t. Let G satisfies the following Stokes equation

$$\begin{cases} -\Delta G + \nabla p = 0, \\ \mathrm{div}(G) = 0, \\ G_{/\Gamma_c} = 0, \ \ G_{/\Gamma_e} = g. \end{cases} \tag{3.4}$$

Denote for each $n \in \mathbb{N}$ $w^n = v^n - G$. Then $w^n \in L^2(0,T;V)$ is solution to

$$\begin{cases} \dfrac{\partial w^n}{\partial t} + w^n \cdot \nabla w^n + (\mathcal{U}_\rho^n + G) \cdot \nabla w^n + w^n \cdot \nabla(\mathcal{U}_\rho^n + G) \\ -\frac{1}{\mathcal{R}_e} \Delta w^n + \nabla \pi = k_\rho^n, \\ \mathrm{div}(w^n) = 0, \\ w^n(0,x) = u_0(x) - G(x), \\ w^n_{/\Gamma_c} = 0, \ \ w^n_{/\Gamma_e} = 0, \end{cases} \tag{3.5}$$

where $k_\rho^n = -(G\cdot\nabla G + \mathcal{U}_\rho^n \cdot \nabla G + G\cdot\nabla\mathcal{U}_\rho^n + \mathcal{U}_\rho^n \cdot \nabla\mathcal{U}_\rho^n)$. We want to investigate the limit $n \to +\infty$ in Equation (3.5). By Lemma 2.6, the sequence $(\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2)$. We first deal with the term k_ρ^n. We proceed as follows. Let $1 < r < 2$ a real which will be chosen later and define $p = \frac{2r}{2-r}$. By Hölder's inequality,

$$\|\mathcal{U}_\rho^n \cdot \nabla\mathcal{U}_\rho^n\|_r \leq C\|\nabla\mathcal{U}_\rho^n\|_2 \|\mathcal{U}_\rho^n\|_p. \tag{3.6}$$

We recall now the classical Gagliardo-Nirenberg inequality.

Lemma 3.1. *For $2 \leq p \leq +\infty$, and all function $\phi \in H^1(\Omega)$,*

$$\|\phi\|_p \leq C\|\phi\|_2^{\frac{2}{p}} \|\nabla\phi\|_2^{\frac{p-2}{p}}.$$

By Lemma 3.1,

$$\|\mathcal{U}_\rho^n\|_p \leq C\|\mathcal{U}_\rho^n\|_2^{\frac{2-r}{r}} \|\nabla\mathcal{U}_\rho^n\|_2^{\frac{2(r-1)}{r}}. \tag{3.7}$$

Then using (3.6), (3.7), we derive

$$\|\mathcal{U}_\rho^n \cdot \nabla \mathcal{U}_\rho^n\|_r \le C\|\nabla \mathcal{U}_\rho^n\|_2^{\frac{3r-2}{r}} \|\mathcal{U}_\rho^n\|_2^{\frac{2-r}{r}}. \tag{3.8}$$

Chosing $r = \frac{6}{5}$, we deduce that the sequence $(\mathcal{U}_\rho^n \cdot \nabla \mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ is bounded in $L^{\frac{3}{2}}(0,T;L^{\frac{6}{5}}(\Omega))$. Then there exists a function $l \in L^{\frac{3}{2}}(0,T;L^{\frac{6}{5}}(\Omega))$ such that

$$\mathcal{U}_\rho^n \cdot \nabla \mathcal{U}_\rho^n \rightharpoonup l \text{ weakly in } L^{\frac{3}{2}}(0,T;L^{\frac{6}{5}}(\Omega)). \tag{3.9}$$

Moreover, from (3.3), we deduce that

$$\mathcal{U}_\rho^n \cdot \nabla \mathcal{U}_\rho^n \rightharpoonup \mathcal{U}^* \cdot \nabla \mathcal{U}^* \text{ weakly in } L^1(0,T;L^1(\Omega)). \tag{3.10}$$

By unicity of the limit in $\mathcal{D}'(]0,T[\times\Omega)$ and using (3.9) and (3.10) we obtain $l = \mathcal{U}^* \cdot \nabla \mathcal{U}^*$.

The other terms of k_ρ^n can be treated in a similar way to obtain

$$k_\rho^n \rightharpoonup (G\cdot\nabla G + \mathcal{U}^*\cdot\nabla G + G\cdot\nabla\mathcal{U}^* + \mathcal{U}^*\cdot\nabla\mathcal{U}^*) \text{ weakly in } L^{\frac{4}{3}}(0,T;L^{\frac{6}{5}}(\Omega)).$$

We now deal with the left-hand side of Equation (3.5). We begin by usual energy estimates on Navier-Stokes equations. Taking the inner product in $\mathbb{R}^2$ of equation (3.5) with w^n and integrating in space over Ω gives using the fact that $\int_\Omega (w^n \cdot \nabla w^n)\cdot w^n dx = 0$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|w^n\|_2^2) + \frac{1}{\mathcal{R}_e}\|\nabla w^n\|_2^2 \le \left|\int_\Omega k_\rho^n \cdot w^n dx\right| + \left|\int_\Omega ((G+\mathcal{U}_\rho^n)\cdot\nabla w^n)\cdot w^n dx\right| \\ + \left|\int_\Omega (w^n \cdot \nabla(G+\mathcal{U}_\rho^n))\cdot w^n dx\right| \end{aligned}\tag{3.11}$$

First of all, since $(\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2)$, it is obvious by Lemma 3.1 that this sequence is bounded in $L^4(0,T;L^4)$. We estimate separetly each term of the right-hand-side of inequality (3.11). We begin with $\left|\int_\Omega k_\rho^n \cdot w^n dx\right|$. Since $w^n = 0$ on $\partial\Omega$ an integration by parts gives

$$\left|\int_\Omega (\mathcal{U}_\rho^n \cdot \nabla \mathcal{U}_\rho^n)\cdot w^n dx\right| = \left|\int_\Omega (\mathcal{U}_\rho^n \cdot \nabla w^n)\cdot \mathcal{U}_\rho^n dx\right|$$

Then, by $L^4 - L^2 - L^4$ estimate

$$\begin{aligned}\left|\int_\Omega (\mathcal{U}_\rho^n \cdot \nabla \mathcal{U}_\rho^n)\cdot w^n dx\right| &\le \|\mathcal{U}_\rho^n\|_4^2 \|\nabla w^n\|_2 \\ &\le C_\varepsilon \|\mathcal{U}_\rho^n\|_4^4 + \varepsilon\|\nabla w^n\|_2^2\end{aligned}$$

where ε will be chosen later. Treating the other terms of k_ρ^n in the same way, one obtains

$$\left|\int_\Omega k_\rho^n \cdot w^n dx\right| \leq f_1(t) + 4\varepsilon \|\nabla w^n\|_2^2, \tag{3.12}$$

where $f_1(t)$ is a function in $L^1(0,T)$ since $(\mathcal{U}_\rho^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2)$ and $L^2(0,T;H^1)$. For the second term, we make an $L^4 - L^2 - L^4$ estimate to obtain

$$\begin{aligned}\left|\int_\Omega ((G+\mathcal{U}_\rho^n)\cdot\nabla w^n)\cdot w^n dx\right| &\leq C\|w^n\|_4\|\nabla w^n\|_2\|G+\mathcal{U}_\rho^n\|_4 \\ &\leq C\|w^n\|_2^{\frac{1}{2}}\|\nabla w^n\|_2^{\frac{3}{2}}\|G+\mathcal{U}_\rho^n\|_4 \\ &\leq C_\varepsilon\|G+\mathcal{U}_\rho^n\|_4^4\|w^n\|_2^2 + \varepsilon\|\nabla w^n\|_2^2, \end{aligned} \tag{3.13}$$

by Lemma 3.1 and Hölder's inequality. Again integration by parts gives for the last term

$$\begin{aligned}\left|\int_\Omega (w^n\cdot\nabla(G+\mathcal{U}_\rho^n))\cdot w^n dx\right| &= \left|\int_\Omega ((G+\mathcal{U}_\rho^n)\cdot\nabla w^n)\cdot w^n dx\right| \\ &\leq C_\varepsilon\|G+\mathcal{U}_\rho^n\|_4^4\|w^n\|_2^2 + \varepsilon\|\nabla w^n\|_2^2. \end{aligned} \tag{3.14}$$

Collecting (3.12), (3.13) and (3.14) we have

$$\frac{1}{2}\frac{d}{dt}(\|w^n\|_2^2) + \frac{1}{\mathcal{R}_e}\|\nabla w^n\|_2^2 \leq f_1(t) + C_\varepsilon\|G+\mathcal{U}_\rho^n\|_4^4\|w^n\|_2^2 + 6\varepsilon\|\nabla w^n\|_2^2. \tag{3.15}$$

Thus chosing $6\varepsilon \leq \frac{1}{2\mathcal{R}_e}$, we obtain

$$\frac{d}{dt}(\|w^n\|_2^2) + \frac{1}{2\mathcal{R}_e}\|\nabla w^n\|_2^2 \leq f_1(t) + C_\varepsilon\|G+\mathcal{U}_\rho^n\|_4^4\|w^n\|_2^2. \tag{3.16}$$

We then conclude by Gronwall's lemmas (see [BF])) that

- w^n is bounded in $L^\infty(0,T;L^2(\Omega))$, (3.17)
- ∇w^n is bounded in $L^2(0,T;L^2(\Omega))$. (3.18)

From (3.17) and (3.18), we can deduce by classical arguments (see [BF]), that

$$\frac{\partial w_n}{\partial t} \text{ is bounded in } L^2(0,T;V'),$$

where V' is the dual space of V. Then using the same arguments as above, one can show that there exists $w \in L^2(0,T;V)$ and a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ still denoted by $(w_n)_{n\in\mathbb{N}}$ such that

$$\begin{aligned}
&w_n \longrightarrow w \text{ strongly in } L^2(0,T;L^2(\Omega)),\\
&\nabla w_n \rightharpoonup \nabla w \text{ weakly in } L^2(0,T;L^2(\Omega)),\\
&w_n \cdot \nabla w_n \rightharpoonup w \cdot \nabla w \text{ weakly in } L^{\frac{4}{3}}(0,T;L^{\frac{6}{5}}(\Omega)).
\end{aligned}$$

Taking the limit $n \to +\infty$ into equation (3.5), we obtain

$$\left\{\begin{array}{l}
\dfrac{\partial w}{\partial t} + w\cdot\nabla w + (\mathcal{U}^* + G)\cdot\nabla w + w\cdot\nabla(\mathcal{U}^* + G) - \dfrac{1}{\mathcal{R}_e}\Delta w + \nabla\pi = k^*,\\
\operatorname{div}(w) = 0,\\
w_{/\Gamma_c} = 0, \;\; w_{/\Gamma_e} = 0,\\
w(0,x) = u_0(x) - G(x),
\end{array}\right. \tag{3.19}$$

where $k^* = -(G\cdot\nabla G + \mathcal{U}^*\cdot\nabla G + G\cdot\nabla\mathcal{U}^* + \mathcal{U}^*\cdot\nabla\mathcal{U}^*)$. This shows that $\mathcal{U}^*$ is a solution of the minimization problem (2.5). $\square$

4 Frechet Derivability of $\mathcal{U}_\rho \mapsto v$.

In this section, we prove that the solution v of equation (2.6) can be differentiate with respect to $\mathcal{U}_\rho$ (see [CFT] for more details). Let $\mathcal{U}_\rho$ and h be two functions of $\mathcal{W}_\sigma^{NL}$ Denote by v^h the solution of equation (2.6) with $\mathcal{U}_\rho + h$ and v the solution of (2.6) with $\mathcal{U}_\rho$. Introduce $w = v^h - v$. By a direct computation, one can prove that w is solution to

$$\left\{\begin{array}{l}
\dfrac{\partial w}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta w + w\cdot\nabla w + w\cdot\nabla v + v\cdot\nabla w + w\cdot\nabla h + h\cdot\nabla w + \mathcal{U}_\rho\cdot\nabla w\\
+w\cdot\nabla\mathcal{U}_\rho + h\cdot\nabla v + v\cdot\nabla h + h\cdot\nabla\mathcal{U}_\rho + \mathcal{U}_\rho\cdot\nabla h + h\cdot\nabla h + \nabla\pi = 0,\\
\operatorname{div}(w) = 0,\\
w_{/\Gamma} = 0,\\
w(0,x) = 0.
\end{array}\right. \tag{4.1}$$

We then introduce the expected derivative of $v(\mathcal{U}_\rho)$, namely $z = v'(\mathcal{U}_\rho)(h)$, solution to

$$\left\{\begin{array}{l}
\dfrac{\partial z}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta z + v\cdot\nabla z + z\cdot\nabla v + \mathcal{U}_\rho\cdot\nabla z + z\cdot\nabla\mathcal{U}_\rho\\
+h\cdot\nabla v + v\cdot\nabla h + \mathcal{U}_\rho\cdot\nabla h + h\cdot\nabla\mathcal{U}_\rho + \nabla\pi = 0,\\
\operatorname{div}(z) = 0,\\
z_{/\Gamma} = 0,\\
z(0) = 0.
\end{array}\right. \tag{4.2}$$

Finally, denote $y = w - z$. We claim that $\|y\|_{H^1(\Omega)} = o(\|h\|_{H^1(\Omega)})$. It is then natural to make some estimates on the equation satisfied by y which can be written as follows :

$$\left\{\begin{array}{l} \dfrac{\partial y}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta y + (v + \mathcal{U}_\rho)\cdot\nabla y + y\cdot\nabla(v+\mathcal{U}_\rho) + \nabla\pi = -N(w,h), \\ \operatorname{div}(y) = 0, \\ y_{/\Gamma} = 0, \\ y(0) = 0, \end{array}\right. \tag{4.3}$$

where $N(w,h) = w\cdot\nabla w + h\cdot\nabla w + w\cdot\nabla h + h\cdot\nabla h$. Multiplying Equation (4.3) by y and integrating over Ω, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\|y\|_2^2) + \frac{1}{\mathcal{R}_e}\|\nabla y\|_2^2 \leq & \left|\int_\Omega \{(y\cdot\nabla v)y + (v\cdot\nabla y)y + (\mathcal{U}_\rho\cdot\nabla y)y + (y\cdot\nabla\mathcal{U}_\rho)y\}\,dx\right| \\ & + \left|\int_\Omega N(h,w)ydx\right|, \\ & \leq \|y\|_4^2\|\nabla v\|_2 + \|v\|_4\|\nabla y\|_2\|y\|_4 + \|\mathcal{U}_\rho\|_4\|\nabla y\|_2\|y\|_4 \\ & + \|y\|_4^2\|\nabla\mathcal{U}_\rho\|_2 + \left|\int_\Omega N(h,w)ydx\right|, \end{aligned} \tag{4.4}$$

by usual $L^4 - L^4 - L^2$-estimate. For the last term of Inequality (4.4), we estimate separetly each components. Since $w = 0$ on Γ, an integration by parts gives

$$\left|\int_\Omega (w\cdot\nabla w)ydx\right| = \left|\int_\Omega (w\cdot\nabla y)wdx\right| \leq \|w\|_4^2\|\nabla y\|_2, \tag{4.5}$$

using again $L^4 - L^4 - L^2$-estimate. With the same arguments, one can prove

$$\left|\int_\Omega (h\cdot\nabla w)ydx\right| = \left|\int_\Omega (h\cdot\nabla y)wdx\right| \leq \|h\|_4\|\nabla y\|_2\|w\|_4, \tag{4.6}$$

$$\left|\int_\Omega (w\cdot\nabla h)ydx\right| = \left|\int_\Omega (w\cdot\nabla y)hdx\right| \leq \|h\|_4\|\nabla y\|_2\|w\|_4, \tag{4.7}$$

$$\left|\int_\Omega (h\cdot\nabla h)ydx\right| = \left|\int_\Omega (h\cdot\nabla y)hdx\right| \leq \|h\|_4^2\|\nabla y\|_2. \tag{4.8}$$

Collecting (4.4), (4.5), (4.6), (4.7) and (4.8), we derive using Lemma 3.1 and Young Inequality

$$\frac{1}{2}\frac{d}{dt}\|y\|_2^2+\frac{1}{\mathcal{R}_e}\|\nabla y\|_2^2 \leq \frac{1}{2\mathcal{R}_e}\|\nabla y\|_2^2 + C\left\{(\|\nabla v\|_2+\|\nabla \mathcal{U}_\rho\|_2)^2\|y\|_2^2\right. \\ \left. +(\|v\|_4+\|\mathcal{U}_\rho\|_4)^4\|y\|_2^2+\|w\|_4^4+\|h\|_4^4\right\}. \quad (4.9)$$

Multiplying Inequality (4.9) by 2, we get

$$\frac{d}{dt}\|y\|_2^2+\frac{1}{\mathcal{R}_e}\|\nabla y\|_2^2 \leq f(t)\|y\|_2^2+C(\|w\|_4^4+\|h\|_4^4), \quad (4.10)$$

where $f(t)=C(\|\nabla v\|_2+\|\nabla \mathcal{U}_\rho\|_2)^2+(\|v\|_4+\|\mathcal{U}_\rho\|_4)^4$ is a function in $L^1_{loc}(\mathbb{R}_+)$. Recalling that $y(0)=0$, the Gronwall inequality provides

$$\|y(t)\|_2^2 \leq C\int_0^t e^{\int_s^t f(\tau)d\tau}\left(\|w(s)\|_4^4+\|h(s)\|_4^4\right)ds, \quad (4.11)$$

$$\int_0^t \|\nabla y(s)\|_2^2 ds \leq C\int_0^t \left(\|w(s)\|_4^4+\|h(s)\|_4^4\right)ds \\ + C\int_0^t f(r)\left\{\int_0^r e^{\int_s^r f(\tau)d\tau}\left(\|w\|_4^4+\|h\|_4^4\right)ds\right\}dr. \quad (4.12)$$

The only remaining thing to do is to estimate $\|w\|_{L^4(0,T;L^4)}$. Starting from (4.1), we derive, using the fact that $\int_\Omega (w\cdot\nabla w)\cdot w dx=0$, the following classical energy estimates :

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2+\frac{1}{\mathcal{R}_e}\|\nabla w\|_2^2 \quad (4.13)$$

$$\leq C\left(\|\nabla v\|_2+\|\nabla \mathcal{U}_\rho\|_2+\|h\|_2\right)\|w\|_4^2 \\ + C\left(\|v\|_4+\|\mathcal{U}_\rho\|_4+\|h\|_4\right)\|\nabla w\|_2\|w\|_4 \\ + C\left\{\|\nabla v\|_2\|h\|_4+\|v\|_4\|\nabla h\|_2+\|\nabla \mathcal{U}_\rho\|_2\|h\|_4\right. \\ \left. +\|\mathcal{U}_\rho\|_4\|\nabla h\|_2+\|h\|_4\|\nabla h\|_2\right\}\|w\|_4 \quad (4.14)$$

By Lemma 3.1, we deduce from (4.13) and Young Inequality

$$\frac{d}{dt}\|w\|_2^2+\frac{1}{2\mathcal{R}_e}\|\nabla w\|_2^2 \leq m(t)\|w\|_2^2+C(\|h\|_2^2+\|\nabla h\|_2^2), \quad (4.15)$$

where $m(t)$ is a function depending only on the H^1-norm of v, h and $\mathcal{U}_\rho$. Again, applying the Gronwall's Lemma and using Lemma 3.1, we get

$$\|w\|_{L^4(0,T;L^4)} \leq C\|h\|_{L^2(0,T;H^1)}.$$

In conclusion, it is clear from (4.11) and (4.12) that

$$\|y\|_{L^2(0,T;H^1)} \leq C\|h\|^2_{L^2(0,T;H^1)}.$$

This proves the derivability of v with respect to $\mathcal{U}_\rho$ and that the derivative $v^{'}(\mathcal{U}_\rho) \cdot h$ satisfies equation (4.2).

□

Proof of Corollary 1.1. This is a direct consequence of the above computation. Let $\mathcal{U}_\rho^{opt}$ be given by Theorem 1.1 Then one has $J^{'}(\mathcal{U}_\rho^{opt}) \cdot h = 0$ for all $h \in \mathcal{W}_\sigma^{NL}$.

□

5 Stabilization result : the linear case.

In this section, we want to characterize the optimal control $\mathcal{U}_\rho$ given by Theorem 5.1. This problem is well-known when one deals with ODE (see [T1]), but few has been done in the context of PDE's.

5.1 Formulation and context

Let $\bar{v}$ be a steady state of the Navier-Stokes equations, i.e. $\bar{v}$ is a solution in $H^2(\Omega)$ to

$$\begin{cases} -\dfrac{1}{\mathcal{R}_e}\Delta\bar{v} + \bar{v}\cdot\nabla\bar{v} + \nabla\pi = 0, \\ \operatorname{div}(\bar{v}) = 0 \\ \bar{v}_{/\Gamma_c} = 0, \ \ \bar{v}_{/\Gamma_e} = g, \end{cases} \tag{5.1}$$

where g belongs to $H^{\frac{3}{2}}(\Gamma_e)$. We denote by v the solution to the Navier-Stokes equation (1.1) with boundary conditions

$$v_{/\Gamma_c} = u_\rho, \ \ v_{/\Gamma_e} = g,$$

where u_ρ belongs to $\mathcal{W}_c^L$. Then $w = v - \bar{v}$ is solution to the following equation

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta w + w\cdot\nabla w + \bar{v}\cdot\nabla w + w\cdot\nabla\bar{v} + \nabla\pi = 0 \\ \operatorname{div}(w) = 0, \\ w_{/\Gamma_e} = 0, \ \ w_{/\Gamma_c} = u_\rho, \\ w(0) = 0. \end{cases} \tag{5.2}$$

For practical applications, we hope that w will be small so we introduce the linearized version of Equation (5.2) around 0 which reads as follows

$$\begin{cases} \dfrac{\partial w_L}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta w_L + \bar{v}\cdot\nabla w_L + w_L\cdot\nabla\bar{v} + \nabla\pi = 0 \\ \mathrm{div}(w_L) = 0. \\ w_{L/\Gamma_e} = 0, \;\; w_{L/\Gamma_c} = u_\rho, \\ w_L(0) = 0. \end{cases} \tag{5.3}$$

Since Equation (5.3) is linear in u_ρ, it is clear that the solution w_L depends linearly of the boundary value u_ρ. So there exists a linear operator $\mathcal{L}_1$ from $\mathcal{W}_c^L$ to $\mathcal{W}^L$ such that $w_L = \mathcal{L}_1(u_\rho)$. Let $\mathcal{U}_\rho$ be the extension of u_ρ given by the weak solution of

$$\begin{cases} \dfrac{\partial \mathcal{U}_\rho}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta \mathcal{U}_\rho + \nabla\pi = 0, \\ \mathrm{div}(\mathcal{U}_\rho) = 0, \\ \mathcal{U}_{\rho/\Gamma_c} = u_\rho, \;\; \mathcal{U}_{\rho/\Gamma_e} = 0, \\ \mathcal{U}_\rho(0) = 0, \end{cases} \tag{5.4}$$

Again, since Equation (5.4) is linear in $\mathcal{U}_\rho$, there exists a linear continuous operator $\mathcal{L}_2$ from $\mathcal{W}_c^L$ to $\mathcal{W}^L$ such that $\mathcal{U}_\rho = \mathcal{L}_2(u_\rho)$. By Lemma 2.2, it is obvious that $\mathcal{L}_2$ is injective and surjective by construction. Then $\mathcal{L}_2$ is one-to-one and the operator $\mathcal{L}_2^{-1}$ is also linear continuous. It follows that w_L depends linearly of $\mathcal{U}_\rho$ in the following sense : define $\mathcal{L} = \mathcal{L}_1\mathcal{L}_2^{-1}$, then $\mathcal{L}$ is continuous and

$$w_L = \mathcal{L}(\mathcal{U}_\rho). \tag{5.5}$$

Finally, define $\tilde{w} = w_L - \mathcal{U}_\rho$ to obtain

$$\begin{cases} \dfrac{\partial \tilde{w}}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta \tilde{w} + \bar{v}\cdot\nabla(\tilde{w} + \mathcal{U}_\rho) + (\tilde{w} + \mathcal{U}_\rho)\cdot\nabla\bar{v} + \nabla\pi = 0 \\ \mathrm{div}(\tilde{w}) = 0, \\ \tilde{w}_{/\Gamma_e} = 0, \;\; \tilde{w}_{/\Gamma_c} = 0, \\ \tilde{w}(0) = 0. \end{cases} \tag{5.6}$$

Remark that $\tilde{w}$ is a linear function in $\mathcal{U}_\rho$. As mentionned in the Introduction, this case is much simpler than the nonlinear one. Indeed, the introduction of $\mathcal{W}_{\sigma,c}^{NL}$ is not useful. Estimates on time derivatives $\mathcal{U}_\rho$ are not necessary to perform the limit in Navier Stokes equation since nonlinear terms have been cancelled. Weak convergence are sufficient for that purpose. For example, one can solve the optimal control problem in $\mathcal{W}_c^L$ as it is proved in Theorem 5.1. First, introduce the cost functional

$$\mathcal{J}_L(\mathcal{U}_\rho) = \frac{1}{2}\int_0^T (||\tilde{w} + \mathcal{U}_\rho||_2^2 + ||\mathcal{U}_\rho||_{H^1}^2)dt.$$

Theorem 5.1. *Assume that* Ω *is a* C^1 *bounded domain of* $\mathbb{R}^2$*. For any function* (u_0, g) *given in* $H(\Omega) \times H^{\frac{3}{2}}(\Gamma_e)$ *satisfying the compatibility condition* (2.7)*, there exists a* ***unique*** *optimal control* $\mathcal{U}_\rho^{opt}$ *solution to*

$$\mathcal{J}_L(\mathcal{U}_\rho^{opt}) = \inf_{\mathcal{U}_\rho \in \mathcal{W}^L} \mathcal{J}_L(\mathcal{U}_\rho). \tag{5.7}$$

Moreover $\mathcal{U}_\rho^{opt}$ *satisfies the Euler-Lagrange equations*

$$\mathcal{J}_L^{'}(\mathcal{U}_\rho^{opt}) \cdot h = \int_0^T \big\{(z + h, \tilde{w} + \mathcal{U}_\rho^{opt})_{L^2} + ((\mathcal{U}_\rho^{opt}, h))_{H^1}\big\}dt = 0, \tag{5.8}$$

for all h *solution to*

$$\begin{cases} \dfrac{\partial h}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta h + \nabla\pi = 0, \\ \mathrm{div}(h) = 0, \\ h_{/\Gamma_c} = k, h_{/\Gamma_e} = 0, \\ h(0) = 0, \end{cases} \tag{5.9}$$

where k *is an arbitrary function given on the boundary where the control is given. The function* z *arising in Equation* (5.8) *is equal to* $\frac{\partial \tilde{w}}{\partial \mathcal{U}_\rho}(\mathcal{U}_\rho^{opt}) \cdot h$ *and satisfies the following equation*

$$\begin{cases} \dfrac{\partial z}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta z + \bar{v} \cdot \nabla(z + h) + (z + h) \cdot \nabla\bar{v} + \nabla\pi = 0, \\ \mathrm{div}(z) = 0, \\ z_{/\Gamma_e} = 0, \;\; z_{/\Gamma_c} = 0, \\ z(0) = 0. \end{cases} \tag{5.10}$$

Remark 5.1.

- *As* $\tilde{w}$ *is a linear function of* $\mathcal{U}_\rho$*,* z *is independent of the optimal control* $\mathcal{U}_\rho^{opt}$*.*

- *Note that since* $u_\rho \in \mathcal{W}_c^L$*, we need a bound on* $\|u_\rho\|_{L^2(\Gamma)}$*. Then the extension* $\mathcal{U}_\rho$ *of* u_ρ *has to be taken in* $H^1(\Omega)$*.*

- *For physical applications, it is usual to replace $\|\mathcal{U}_\rho\|^2_{H^1}$ by $\|rot(\mathcal{U}_\rho)\|^2_2$. Since we deal with functions which vanish on a part of the boundary $\partial\Omega$ and as $\mathcal{U}_\rho$ is a free divergence vector field, the H^1-norm and $\|rot(\cdot)\|_2$ are equivalent. Thus Theorem 5.1 is also true if we replace in $\mathcal{J}_L$ the H^1-norm by $\|rot(\cdot)\|_2$.*

- *In practice, in order to stabilize stationary solutions we have to minimize the vorticity which is exactly rot(v) (see [BMT]).*

Proof. Let $(\mathcal{U}^n_\rho)_{n\in\mathbb{N}}$ be a minimizing sequence for problem (5.7). Then $(\mathcal{U}^n_\rho)_{n\in\mathbb{N}}$ is bounded in the reflexive Banach space $L^2(0,T;H^1(\Omega))$. Thus, there exists $\mathcal{U}^*_\rho \in L^2(0,T;H^1(\Omega))$ such that $\mathcal{U}^n_\rho$ converges weakly to $\mathcal{U}^*$ in $L^2(0,T;H^1(\Omega))$. Combining with usual energy estimates on Navier Stokes equation, it is obvious that $\mathcal{U}^*$ is solution to problem (5.7). A straightforward calculation gives

$$\mathcal{J}^{'}(\mathcal{U}^{opt}_\rho)\cdot h = \int_0^T \Big\{(z+h,\tilde{w}+\mathcal{U}^{opt}_\rho)_{L^2} + ((\mathcal{U}^{opt}_\rho,h))_{H^1}\Big\}dt = 0.$$

Assume now that there exists two optimal control $\mathcal{U}^{opt,1}_\rho$ and $\mathcal{U}^{opt,2}_\rho$. Denote $\tilde{w}^1$ and $\tilde{w}^2$ the solutions of Equation (5.6) corresponding to $\mathcal{U}^{opt,1}_\rho$ and $\mathcal{U}^{opt,2}_\rho$. By (5.8), we can write for all $h\in\mathcal{W}^L$ and $i=1,2$,

$$\int_0^T \Big\{(z+h,\tilde{w}^i+\mathcal{U}^{opt,i}_\rho)_{L^2} + ((\mathcal{U}^{opt,i}_\rho,h))_{H^1}\Big\}dt = 0, \tag{5.11}$$

where

$$z = \frac{\partial\tilde{w}^1}{\partial\mathcal{U}_\rho}(\mathcal{U}^{opt,1}_\rho)\cdot h = \frac{\partial\tilde{w}^2}{\partial\mathcal{U}_\rho}(\mathcal{U}^{opt,2}_\rho)\cdot h$$

(see Remark 5.1). Since $\mathcal{U}^{opt,1}_\rho$ and $\mathcal{U}^{opt,2}_\rho$ satisfy Equation (5.4), then $h = \mathcal{U}^{opt,1}_\rho - \mathcal{U}^{opt,2}_\rho$ satisfies Equation (5.9) with $k = u^1_\rho - u^2_\rho$. Substracting Equations (5.11) for $i=1,2$ and taking $h = \mathcal{U}^{opt,1}_\rho - \mathcal{U}^{opt,2}_\rho$, we obtain

$$\int_0^T \Big\{\|h\|^2_2 + ((z+h,\tilde{w}^1-\tilde{w}^2+h))_{H^1}\Big\}dt = 0. \tag{5.12}$$

Using (5.9) and (5.10), we derive the equation satisfies by $z+h$

$$\begin{cases} \dfrac{\partial(z+h)}{\partial t} - \dfrac{1}{\mathcal{R}_e}\Delta(z+h) + \bar{v}\cdot\nabla(z+h) + (z+h)\cdot\nabla\bar{v} + \nabla\pi = 0 \\ \mathrm{div}(z) = 0, \\ z_{/\Gamma_e} = 0, \;\; z_{/\Gamma_c} = 0, \\ z(0) = 0. \end{cases} \tag{5.13}$$

By a direct calculation on Equation (5.6) and (5.9), one shows that $\tilde{w}^1-\tilde{w}^2+h$ satisfies also (5.13). Since (5.13) is linear, it is clear that $\tilde{w}^1-\tilde{w}^2+h = z+h$. Then (5.12) gives

$$\int_0^T \left(\|h\|_2^2 + \|z+h\|_{H^1}^2\right) = 0,$$

and then $\mathcal{U}_\rho^{opt,1} = \mathcal{U}_\rho^{opt,2}$. □

5.2 Euler-Lagrange equations for optimal control.

In this section, we make more precise the Euler-Lagrange equation satisfied by the optimal control of Theorem 5.1. we prove the following theorem.

Theorem 5.2. *Let $\mathcal{U}_\rho^{opt}$ be the optimal control of Theorem* 5.1*. Then, for all $h \in \mathcal{W}^L$,*

$$\int_0^T ((u+\mathcal{U}_\rho^{opt}, h))_{H^1} dt = 0,$$

where u is the unique solution of $a(u,v) = (\ell, v)$ for all v in $\mathcal{W}^L$ with $\ell = \mathcal{L}\mathcal{U}_\rho^{opt} - \bar{v}\cdot\nabla p - {}^t(\nabla p)\bar{v}$ and a is the bilinear form defined by (5.19)*.*

We begin with the following definition :

Definition 5.1. *Let A be the following unbounded operator from V to V' define by*

$$\forall p \in V, \;\; < A(r), p >= \int_\Omega \Big(\frac{1}{\mathcal{R}_e}\nabla r\cdot\nabla p + (\bar{v}\cdot\nabla r)\cdot p + (r\cdot\nabla\bar{v})\cdot p\Big).$$

We compute the adjoint of the operator A as follows.

Lemma 5.1. *The adjoint of A is the operator define by*

$$\forall r \in V, \;\; < {}^tA(p), r >= \int_\Omega \Big(\frac{1}{\mathcal{R}_e}\nabla p\cdot\nabla r - (\bar{v}\cdot\nabla p)\cdot r + ({}^t(\nabla\bar{v})\cdot p)\cdot r\Big).$$

Proof. Let r and p be in V. We separate each term of A. Using the fact that $\mathrm{div}(\bar{v}) = 0$, the regularity $H^2(\Omega)$ for the strong stationary solution $\bar{v}$ and integration by parts, we have

$$\begin{aligned}
\int_\Omega (\bar{v}\cdot\nabla r)\cdot p = \sum_{i,j=1}^2 \int_\Omega (\bar{v}_j \frac{\partial r_i}{\partial x_j})p_i &= \sum_{i,j=1}^2 \int_\Omega \frac{\partial(\bar{v}_j r_i)}{\partial x_j} p_i \\
&= -\sum_{i,j=1}^2 \int_\Omega \bar{v}_j r_i \frac{\partial p_i}{\partial x_j} + \sum_{i,j=1}^2 \int_\Gamma \bar{v}_j r_i p_i n_j \\
&= -\int_\Omega (\bar{v}\cdot\nabla p)\cdot r,
\end{aligned}$$

since $p = 0$ on Γ. In the same way, we get

$$\int_\Omega (r \cdot \nabla \bar{v}) \cdot p = \sum_{i,j=1}^{2} \int_\Omega r_j p_i \frac{\partial \bar{v}_i}{\partial x_j} = \int_\Omega \big({}^t(\nabla \bar{v}) p\big) \cdot r.$$

Next, let us introduce the adjoint problem

$$\begin{cases} -\dfrac{dp}{dt}(t) + {}^t A p(t) = \tilde{w} + \mathcal{U}_\rho, \\ \operatorname{div}(p) = 0, \\ p(T) = 0, \\ p_{/\Gamma} = 0. \end{cases} \tag{5.14}$$

In order to derive an equation for $\mathcal{U}_\rho^{opt}$, we first recall that $\mathcal{U}_\rho^{opt}$ satisifes

$$\mathcal{J}_L'(\mathcal{U}_\rho^{opt}) \cdot h = \int_0^T \Big\{ (z + h, \tilde{w} + \mathcal{U}_\rho^{opt})_{L^2} + ((\mathcal{U}_\rho^{opt}, h))_{H^1} \Big\} dt = 0, \tag{5.15}$$

where h and z satisfies respectively (5.9) and (5.10). As $w_L = \tilde{w} + \mathcal{U}_\rho^{opt}$, the first term is easy computed:

$$\int_0^T \Big\{ (z + h, \tilde{w} + \mathcal{U}_\rho^{opt})_{L^2} \Big\} dt = \int_0^T (z, w_L)_{L^2} dt + \int_0^T (h, w_L)_{L^2} dt.$$

Using Equation (5.14) we have

$$\begin{aligned} \int_0^T (z, w_L) dt &= \int_0^T (z, -p' + {}^t A p)_{L^2} dt \\ &= \int_0^T (z' + Az, p)_{L^2} dt - (z(T), p(T))_{L^2} + (z(0), p(0))_{L^2}. \end{aligned}$$

Since $z(0) = 0$, $P(T) = 0$ and as z solves Equation (5.10), we obtain

$$\begin{aligned} \int_0^T (z, w_L)_{L^2} dt &= -\int_0^T (\bar{v} \cdot \nabla h + h \cdot \nabla \bar{v}, p)_{L^2} dt \\ &= -\int_0^T (\bar{v} \cdot \nabla p + {}^t(\nabla \bar{v}) p, h)_{L^2} dt. \end{aligned} \tag{5.16}$$

Hence,

$$\int_0^T \Big\{ (z + h, \tilde{w} + \mathcal{U}_\rho^{opt})_{L^2} \Big\} dt = -\int_0^T (\bar{v} \cdot \nabla p + {}^t(\nabla \bar{v}) p - w_L, h)_{L^2} dt.$$

Collecting (5.15) and (5.16) we derive that

$$\int_0^T \Big\{(w_L - \bar{v}\cdot\nabla p - {}^t(\nabla p)\bar{v}, h)_{L^2} + ((\mathcal{U}_\rho^{opt}, h))_{H^1}\Big\}dt = 0. \tag{5.17}$$

Using the linearity of the map : $\mathcal{U}_\rho^{opt} \mapsto w_L$ through the operator $\mathcal{L}$, we finally obtain

$$\int_0^T \Big\{(\mathcal{L}\mathcal{U}_\rho^{opt} - \bar{v}\cdot\nabla p - {}^t(\nabla p)\bar{v}, h)_{L^2} + ((\mathcal{U}_\rho^{opt}, h))_{H^1}\Big\}dt = 0. \tag{5.18}$$

In this term appears two inner product in different Hilbert spaces. So in view to give a better interpretation of the above formula, we introduce the bilinear continuous and coercive form

$$\begin{aligned} a : \mathcal{W}^L \times \mathcal{W}^L &\longrightarrow \mathbb{R} \\ u, v \qquad &\longmapsto \int_\Omega (uv + \nabla u\cdot\nabla v)_{L^2} = ((u,v))_{H^1}, \end{aligned} \tag{5.19}$$

and define $\ell = \mathcal{L}\mathcal{U}_\rho^{opt} - \bar{v}\cdot\nabla p - {}^t(\nabla p)\bar{v}$. It is obvious that $\ell \in L^2(\Omega)$ and that the fonction

$$h \longmapsto (\ell, h)_{L^2}$$

is linear continuous from $\mathcal{W}^L$ to $\mathbb{R}$. Then by Lax-Milgram theorem, there exists a unique $u \in \mathcal{W}^L$ such that for all $h \in \mathcal{W}^L$, $a(u,h) = (\ell,h)_{L^2}$. So the optimal control $\mathcal{U}_\rho^{opt}$ is characterized by

$$\int_0^T ((u + \mathcal{U}_\rho^{opt}, h))_{H^1} dt = 0,$$

for all h in $\mathcal{W}^L$ where u is the unique solution of $a(u,v) = (\ell, v)_{L^2}$ for all v in $\mathcal{W}^L$ with $\ell = \mathcal{L}\mathcal{U}_\rho^{opt} - \bar{v}\cdot\nabla p - {}^t(\nabla p)\bar{v}$. □

Remark 5.2. *Even in the linear case, we do not know how to replace, in the functional $\mathcal{J}_L$, the H^1-norm of $\mathcal{U}_\rho$ by the L^2-norm.*

Conclusion. In this paper, we have introduce a new set of functional spaces to handle optimal boundary control for Navier-Stokes equations. In this context, we establish the existence of an optimal control. In the linear case, we consider the problem of stabilization around a steady state. In this situation, the optimal control is unique. Furthermore, we bring to the fore the Euler equations that characterize this solution. This type of equations are very useful for numerical approach. We postponed this study to a future work.

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