Pergamon

Appl. Math. Lett. Vol. 11, No. 5, pp. 97–101, 1998

Printed in Great Britain
0893-9659/98 $19.00 + 0.00

PII: S0893-9659(98)00087-1

The Stability of the Shock Profiles of the Burgers' Equation

S. ENGELBERG
Electronics Department, Jerusalem College of Technology—Machon Lev
P.O.B. 16031, Jerusalem 91160, Israel

(Received and accepted July 1997)

Communicated by P. D. Lax

Abstract—In this paper, we study the existence and stability of the shock profiles of the Burgers' equation, $u_t + uu_x = u_{xx}$. We make use of Hopf-Cole transformations to show when such profiles exist, to prove that perturbations of the profiles decay exponentially quickly in an exponentially weighted norm, and to demonstrate that making the weight too large does not generally increase the rate of decay of the perturbation. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Burgers' equation, Shock profile, Nonlinear stability.

We study the decay of perturbations of the viscous shock profiles of the equation:

$$u_t + \left(\frac{u^2}{2}\right)_x = u_{xx}. \tag{1}$$

A viscous shock profile of (1) is a real solution of (1) that is of the form $u(x,t) = u(x - ct)$ and that satisfies $u(x) \to u_+$, $x \to \infty$, and $u(x) \to u_-$, $x \to -\infty$. Here c is the speed with which the profile moves. Equation (1) is a crude model of the evolution of the velocity field, $u(x,t)$, of a fluid with unit density, without external forces acting upon it, but with viscous forces present. If $x(t, x_0)$ is the position of the particle that started at x_0 when $t = 0$, then $\frac{dx(t,x_0)}{dt} = u(x(t,x_0),t)$. If one assumes that the force of the viscosity is equal to u_{xx}, then one finds from Newton's second law that $\frac{d^2(x(t,x_0))}{dt^2} = u_t + uu_x = u_{xx}$.

Since Newton's equations are invariant under Galilean transformations, one hopes that such an invariance exists here as well. If one switches to a coordinate frame that moves with the profile, one looks at the point $x + ct$ at time t. Since one is moving with the profile and since u is the speed of a "particle", the speed that one should see is $u(x + ct, t) - c$. It is easy to see that if $u(x,t)$ satisfies (1), so does $w(x,t) = u(x + ct, t) - c$. In particular, if $u(x - ct)$ solves (1), then $u(x) - c$ solves (1) as well. Thus, any problem about a profile of the from $u(x - ct)$ can be transformed into one about $u(x)$. Therefore, there is no need to consider moving profiles; it is sufficient to consider stationary profiles. That is the only case that we consider.

More general versions of this problem have been studied by Illin and Oleinik [1], Sattinger [2], and Jones, Gardner and Kapitula [3]. The first two papers get exponential decay in an exponentially weighted norm, and the last paper gets algebraic decay in an algebraically weighted norm. Our method is similar to the method employed in [1], but we deal only with (1). We show how to find its stationary viscous shock profiles, and we show how to determine the rate of decay of perturbations of the profiles. We get more precise decay rates and more information about the

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TeX

effects of the weight on the rate of decay of the perturbation than do either [1] or [2]. Our method relies heavily on the precise form of (1)—we make use of the Hopf-Cole transformations [4,5] that relate solutions of (1) to solutions of the heat equation.

The Hopf-Cole transformations are a pair of transformations that take positive solutions of the heat equation into solution of the Burgers' equation and solutions of the Burgers' equation into positive solutions of the heat equation. Suppose one starts with a positive solution of the heat equation, $v(x,t)$. Let $u(x,t) = -2v_x/v$. Then, from the fact that $v_t = v_{xx}$, we find that u satisfies (1).

Suppose that one starts with a solution, $u(x,t)$, of (1). Let $U(x,t) = \int_0^x u(\xi,t)\,d\xi$. Defining $V(x,t) = e^{-U(x,t)/2}$, one finds that

$$\begin{aligned} V_t &= V_{xx} + \left(-\frac{(U_x(0,t))^2}{4} + \frac{U_{xx}(0,t)}{2}\right)V \\ &= V_{xx} + \left(-\frac{u^2(0,t)}{4} + \frac{u_x(0,t)}{2}\right)V \equiv V_{xx} + g(t)V. \end{aligned}$$

Letting $v(x,t) = C(t)V(x,t)$, it is clear that $v(x,t)$ satisfies the equation

$$v_t = v_{xx} + (C'(t) + C(t)g(t))V.$$

Picking $C(t)$ to satisfy

$$\frac{C'(t)}{C(t)} + g(t) = 0,$$

one finds that

$$v(x,t) = C(0)e^{-\int_0^t g(\eta)\,d\eta}e^{-\int_0^x u(\xi,t)\,d\xi/2}$$

solves the heat equation. The only part of the solution that has not been determined is the initial value of $C(t)$. Choose $C(0) = 1$. This choice of $C(0)$ forces $v(x,t)$ to always be positive. The pair of formulas $u(x,t) = -2v_x/v$ and $v(x,t) = e^{-\int_0^t g(\eta)\,d\eta}e^{-\int_0^x u(\xi,t)\,d\xi/2}$ are known as the Hopf-Cole transformations.

We now search for $v(x,t)$ that correspond to $u(x)$. Note that since $u(x,t) = u(x)$ is only a function of x, $v(x,t)$ has the form $C(t)X(x)$; $v(x,t)$ is separable. From the fact that v satisfies the heat equation, we see that $X''(x)/X(x) = C'(t)/C(t) = \lambda$. As $X(x)$ is a real-valued function of x, it is clear that $\lambda \in R$.

It is well known that the solutions of $X''(x) = \lambda X(x)$ are $a\sin(\sqrt{-\lambda}x) + b\cos(\sqrt{-\lambda}x)$ if $\lambda < 0$, $a + bx$ if $\lambda = 0$, and $a\sinh(\sqrt{\lambda}x) + b\cosh(\sqrt{\lambda}x)$ if $\lambda > 0$. Considering $-2X'(x)/X(x)$ for the first and second sets of solutions, we find that it is not a continuous function. Thus, it leads to no interesting viscous shock profiles. For the third set, we find that the limit of $-2X'(x)/X(x)$ is $+2\sqrt{\lambda}$ as $x \to -\infty$ and $-2\sqrt{\lambda}$ as $x \to \infty$. Since these are the only possible stationary viscous shock profiles left, this implies that the only possible pairs of values (u_-, u_+) are $(2\sqrt{\lambda}, -2\sqrt{\lambda})$. Furthermore, it is easy to see that $u(x) = -2X'(x)/X(x)$ is continuous for all x if and only if $|b| > |a|$; only for such values of a and b is there no x for which $X(x) = 0$. It is easy to show that $-2X'(x)/X(x) = -2\sqrt{\lambda}\tanh(\sqrt{\lambda}(x + x_0))$. Of course, $C(t) = e^{\lambda t}$ here. The solutions of the heat equation which correspond to the stationary viscous shock profiles are, therefore, $v(x,t) = e^{\lambda t}\cosh(\sqrt{\lambda}(x + x_0))$.

We note that the existence of a family of stationary viscous shock profiles of the form $u(x+x_0)$ is expected. Equation (1) is autonomous; it does not contain explicit x dependence. Hence, if $u(x)$ is a solution of the equation, so is $u(x + x_0)$. For the sake of definiteness, we choose $x_0 = 0$. For this choice of x_0, we find that $X(x) = \cosh(\sqrt{\lambda}x)$.

Note that if we define $w(x,t) = \beta u(\beta x, \beta^2 t)$, then $w(x,t)$ satisfies (1). Thus, to discover how perturbations of $-2\sqrt{\lambda}\tanh(\sqrt{\lambda}x)$ evolve, it is sufficient to look at how perturbation of

$-\tanh(x/2)$ evolve. All of the other cases can be mapped into this one case by the similarity transform. We consider only the unperturbed profile $u_0(x) \equiv -\tanh(x/2)$. We show the following theorem.

THEOREM 1. *If $u(x,0) = u_0(x) + w_0(x)$, $0 < \alpha \leq 1/2$, and $w_0(x)$ satisfies:*

1. $\int_{-\infty}^{\infty} w_0(x)\,dx = 0$,
2. $|w_0(x)/2| \leq Ce^{-\alpha|x|}$,

then $w(x,t) = u(x,t) - u_0(x)$ and $|w(x,t)| \leq De^{(-\alpha+\alpha^2)t}e^{-\alpha|x|}$.

Property 1 is not really a restriction on the type of data that tends asymptotically to a "tanh profile". If the integral of $w_0(x)$ is not zero, then it is easy to show that there exists a unique δ such that $\int_{-\infty}^{\infty}(w_0(x) + u_0(x) - u_0(x+\delta))\,dx = 0$. One can consider $w_0(x) + u_0(x) - u_0(x+\delta)$ a perturbation of $u_0(x+\delta)$. Since (1) is translation invariant, Theorem 1 implies that $u(x,t)$ solution tends to $u_0(x+\delta)$ exponentially quickly as $t \to \infty$. Additionally, Property 2 is not a restriction on the size of $w_0(x)$—the constant C is an arbitrarily large positive constant. Property 2 only requires that $w_0(x)$ decay like $e^{-\alpha|x|}$.

To prove Theorem 1, we consider what effect the perturbation of the initial data of $u(x,t)$ has on the initial data of the heat equation satisfied by v. We know that $v(x,0) = e^{-\int_0^x u(\xi,0)\,d\xi/2}$. Here, $u(x,0) = u_0(x) + w_0(x)$. When integrated, we find that $v(x,0) = \cosh(x/2)e^{-\int_0^x w_0(\xi)\,d\xi/2}$. Since $\int_{-\infty}^{\infty} w_0(\xi)\,d\xi = 0$, we find that $\int_0^{\infty} w_0(\xi)\,d\xi = \int_0^{-\infty} w_0(\xi)\,d\xi = 2E$ for some constant E. For positive x, we find that $v(x,0) = e^{-E}\cosh(x/2)e^{\int_x^{\infty} w_0(\xi)\,d\xi/2}$, and for negative x, we find that $v(x,0) = e^{-E}\cosh(x/2)e^{\int_x^{-\infty} w_0(\xi)\,d\xi/2}$. It is a simple calculus exercise to show that if $|x| < \ln 2$, then $|e^x - 1| < \kappa|x|$, where $\kappa = \ln(2e)$. From Property 2, it is clear that both $|\int_x^{\infty} w_0(\xi)\,d\xi/2|$, $x \geq 0$, and $|\int_x^{-\infty} w_0(\xi)\,d\xi/2|$, $x \leq 0$, are bounded by $(C/\alpha)e^{-\alpha|x|}$. We find that for all sufficiently large $|x|$:

$$\left|z(x,0) \equiv v(x,0) - e^{-E}\cosh\left(\frac{x}{2}\right)\right| \leq \kappa\left(\frac{C}{\alpha}\right)e^{-E}\cosh\left(\frac{x}{2}\right)e^{-\alpha|x|}.$$

From the definition of $\cosh(x)$, it is clear that for all $\nu > 0$, $\cosh(\nu x) \leq e^{\nu|x|}/2 + 1/2 \leq e^{\nu|x|}$ and that $e^{\nu|x|} \leq 2\cosh(\nu x)$. Therefore, $\cosh(x/2)e^{-\alpha|x|} \leq e^{(1/2-\alpha)|x|} \leq 2\cosh((1/2-\alpha)x)$. We conclude that for sufficiently large $|x|$:

$$|z(x,0)| \leq \kappa 2\left(\frac{C}{\alpha}\right)e^{-E}\cosh\left(\left(\frac{1}{2}-\alpha\right)x\right).$$

Since $z(x,0)$ is bounded on any finite interval, and since $\cosh(\nu x) \geq 1$, it is clear that there exists $F > 0$ such that

$$|z(x,0)| \leq F\cosh\left(\left(\frac{1}{2}-\alpha\right)x\right). \tag{2}$$

We know that $v(x,t)$ satisfies the heat equations. We also know that the solution of the heat equation that corresponds to the initial data $\cosh(x/2)$ is $e^{t/4}\cosh(x/2)$. Thus, we know that $v(x,t) - e^{-E}e^{t/4}\cosh(x/2)$ satisfies the heat equation as well. It is well known (see, for example, [6, Chapter 7, Section 1]) that solutions of the heat equation, $\zeta(x,t)$, that are either nonnegative or that satisfy a bound of the form $|\zeta(x,t)| \leq M^{a|x|^2}$ can be written as $\zeta(x,t) = \int_{-\infty}^{\infty} K(x-y,t)\zeta(y,0)\,dy$ where $K(x,t)$ is the heat kernel. That is, the solution is the convolution of the heat kernel and the initial data. This formulation allows us to show that if $\zeta(x,t)$ and $\Delta(x,t)$ satisfy the heat equation and if $0 \leq \zeta(x,0) \leq \Delta(x,0)$, then $\zeta(x,t) \leq \Delta(x,t)$. It also allows us to show that if $|\zeta(x,t)|$ is bounded by $Me^{a|x|^2}$, $\zeta(x,t)$ and $\Delta(x,t)$ both satisfy the heat equation, and $|\zeta(x,0)| \leq \Delta(x,0)$, then $|\zeta(x,t)| \leq \Delta(x,t)$.

The function $v(x,t)$ has already been shown to be positive. Also, the solution of the heat equation with initial data $e^{-E}\cosh(x/2) + F\cosh((1/2-\alpha)x)$ is $e^{-E}e^{t/4}\cosh(x/2) + Fe^{(1/2-\alpha)^2 t}$

$\cosh((1/2-\alpha)x)$. Since we have shown that $v(x,0) \leq \cosh(x/2)+F\cosh((1/2-\alpha)x)$, we find that $v(x,t) \leq e^{t/4}\cosh(x/2) + Fe^{(1/2-\alpha)^2 t}\cosh((1/2-\alpha)x)$; we see that $v(x,t)$ is actually bounded by $Me^{a|x|^2}$. Defining $z(x,t) \equiv v(x,t) - e^{-E}e^{t/4}\cosh(x/2)$, we can use (2) and the second type of maximum principle to conclude that

$$|z(x,t)| \leq F\cosh\left(\left(\frac{1}{2}-\alpha\right)x\right)e^{(1/2-\alpha)^2 t}. \tag{3}$$

Because we are interested in $-2v_x(x,t)/v(x,t)$, we must find a lower bound for $v(x,t)$. From the definition of $v(x,0)$, we see that there exists a $\mu > 0$ such that for all x, $v(x,0) > \mu$. As μ is a positive solution of the heat equation, this holds true for all t. Also, from (3) we find that $v(x,t) \geq e^{-E}\cosh(x/2)e^{t/4} - F\cosh((1/2-\alpha)x)e^{(1/2-\alpha)^2 t} \geq e^{-E}\cosh(x/2)e^{t/4}/2 \equiv G\cosh(x/2)e^{t/4}$ for all sufficiently large t.

Finally, we note that because we are interested in $u(x,t) = -2v_x(x,t)/v(x,t)$, we need to know about the behavior of $v_x(x,t)$ and $z_x(x,t)$. Since $v(x,t)$ can be written as the convolution of the heat kernel with $v(x,0)$, it is clear that $v_x(x,t)$ can be written as the convolution of the heat kernel with its initial data—with $v_x(x,0)$—as long as the convolution integral converges uniformly. If we can show that $v_x(x,0)$ is bounded by $Me^{m|x|}$, then the integral will converge uniformly. In addition, we then find that $v_x(x,t)$ satisfies the heat equation and the maximum principle. (Clearly, $v_x(x,t)$ *ought* to satisfy the heat equation. The heat equation is a linear constant coefficient PDE.) Thus, we will find that $z_x(x,t) = v_x(x,t) - e^{-E}\sinh(x/2)e^{t/4}/2$ satisfies the heat equation as well. From the definition of $v(x,0)$, we find that

$$v_x(x,0) = \frac{e^{-E}\sinh(x/2)}{2e^{-E}\sinh(x/2)}\left(e^{\int_x^{\pm\infty} w_0(\xi)\,d\xi/2} - 1\right) + \cosh\left(\frac{x}{2}\right)\left(-\frac{w_0(x)}{2}\right)e^{-\int_0^x w_0(\xi)\,d\xi/2}$$

for $\pm x \geq 0$. We see that $v_x(x,0)$ is indeed exponentially bounded. Using the same type of estimates that we used for $v(x,t)$, and making use of Property 2 again, we find that $|v_x(x,t) - e^{-E}\sinh(x/2)e^{t/4}/2| \leq H\cosh((1/2-\alpha)x)e^{(1/2-\alpha)^2 t}$.

We would like to bound $w(x,t) = u(x,t) - u_0(x) = v_x/v - (v_0)_x/v_0$. Rewriting this expression, we find that

$$\begin{aligned}\left|\frac{v_x}{v} - \frac{(v_0)_x}{v_0}\right| &= \left|\frac{((v_x-(v_0)_x)v_0 + (v_0)_x(v_0-v))}{(vv_0)}\right| \\ &\leq \left|\frac{v_x-(v_0)_x}{v}\right| + \left|u_0\frac{v_0-v}{v}\right| \\ &\leq \frac{H\cosh((1/2-\alpha)x)e^{(1/2-\alpha)^2 t}}{Ge^{t/4}\cosh(x/2)} \\ &\quad + \frac{F\cosh((1/2-\alpha)x)e^{(1/2-\alpha)^2 t}}{Ge^{t/4}\cosh(x/2)}.\end{aligned}$$

Clearly the perturbation is bounded by $De^{(-\alpha+\alpha^2)t}e^{-\alpha|x|}$. We find that the perturbation decays exponentially fast both in time and in space.

We note that all of our results are valid only for $0 < \alpha \leq 1/2$. What happens if the perturbation decays faster than $e^{-|x|/2}$? We find that there is generally only a very marginal gain. In fact, we find that the following theorem holds.

THEOREM 2. *If* $\alpha \geq 1/2$, $\int_{-\infty}^{\infty} w_0(x)\,dx = 0$, $e^{\alpha|x|}w_0(x)$, $e^{\alpha|x|}\int_x^{\infty} w_0(\xi)\,d\xi$, $e^{\alpha|x|}\int_x^{-\infty} w_0(\xi)\,d\xi \in L^1(R)$, *and if all of these function tend to* 0 *as* $|x| \to \infty$, *then*

$$|v(x,t) - u_0(x)| \leq C\frac{e^{-|x|/2}}{\sqrt{t}\,e^{t/4}}. \tag{4}$$

We note that if there exists an $\alpha > 1/2$ such that $e^{\alpha|x|}w_0(x)$ is bounded, then the decay conditions above are certainly met.

The proof of this result is quite similar to the proof of our previous result. One looks at the perturbation of the initial value of $v(x,t)$. The conditions above are sufficient to guarantee that $z(x,0)$ and $z_x(x,0)$ are both in L^1. From the standard theory of solutions of the heat equation, this shows that $|z(x,t)|$, $|z_x(x,t)| \leq J/\sqrt{t}$. An immediate consequence of this is that $|w(x,t)| \leq Ke^{-|x|/2}/(\sqrt{t}\,e^{t/4})$.

It is easy to show that the time decay rate $1/(e^t\sqrt{t})$ is not an artifact of the proof above; it is the best possible decay rate. An example shows this. Let the perturbation $w_0(x)$ be the perturbation which gives initial data $v(x,0) = \cosh(x/2) + e^{-x^2/(4t_0)}/\sqrt{4(\pi t_0)}$. Lemma 1 (below) shows that such a $w_0(x)$ exists. The perturbation here is precisely a "snapshot" of the heat kernel. Thus, we know that $v(x,t) = \cosh(x/2)e^{t/4} + e^{-x^2/(4(t-t_0))}/\sqrt{4(\pi(t-t_0))}$, and we find that the perturbation decays at the rate $1/\sqrt{t}$. Using the arguments above, it is clear that the overall rate of decay is $1/(e^{t/4}\sqrt{t})$. It is easy to show that the perturbation $w_0(x)$ satisfies the L^1 condition for *all* $\alpha \geq 0$. Thus, one cannot generally get a decay rate better than $1/(e^{t/4}\sqrt{t})$ by requiring that one's perturbations die at a rate faster than $e^{-|x|/2}$.

Finally, we prove the following.

LEMMA 1. *Suppose that* $v(x,0) = \cosh(x/2) + \Xi(x)$, $\Xi(x) > -\cosh(x/2)$, *and that there exists an* $\alpha > 0$ *such that* $|\Xi(x)|, |\Xi'(x)| \leq Ce^{(1/2-\alpha)|x|}$. *Then there exists a* $u(x,0) = u_0(x) + w_0(x)$ *and* $\int_{-\infty}^{\infty} w_0(x)\,dx = 0$, $|w_0(x)| \leq Ce^{-\alpha|x|}$ *such that* $u(x,0) = -2v_x(x,0)/v(x,0)$.

We prove this by finding $w_0(x)$ and checking that it has all of the properties claimed for it. Clearly, if such a $w_0(x)$ exists, it must satisfy the relation

$$\cosh\left(\frac{x}{2}\right)e^{-\int_0^x w_0(\xi)\,d\xi/2} = \cosh\left(\frac{x}{2}\right) + \Xi(x).$$

Solving for $w_0(x)$, we find that

$$w_0(x) = -2\left(\ln\left(1 + \frac{\Xi(x)}{\cosh(x/2)}\right)\right)'.$$

As $\Xi(x) > -\cosh(x/2)$, $w_0(x)$ is well defined. Furthermore, we find that

$$\int_a^b w_0(\xi)\,d\xi = -2\left(\ln\left(1 + \frac{\Xi(a)}{\cosh(a/2)}\right) - \ln\left(1 + \frac{\Xi(b)}{\cosh(b/2)}\right)\right).$$

Thus, as $\alpha > 0$, $\int_{-\infty}^{\infty} w_0(x)\,dx = 0$. Finally, we find that

$$w_0(x) = \frac{-2}{1 + \Xi(x)/\cosh(x/2)}\left(\frac{\Xi'(x)\cosh(x/2) - \Xi(x)\sinh(x/2)/2}{\cosh^2(x/2)}\right).$$

Clearly, the first term is of order 1, and the second term is of order $e^{-\alpha|x|}$. This completes the proof of the lemma.

REFERENCES

1. A.M. Illin and O.A. Oleinik, Behavior of the solution of the Cauchy problem for certain quasilinear equations for unbounded increase in time, *Amer. Math. Soc. Translations*, Ser. 2, Vol. 42, pp. 19–23, Amer. Math. Soc., Providence, RI, (1964).
2. D.H. Sattinger, Stability of waves of nonlinear parabolic systems, *Adv. Math.* **22**, 312–355 (1976).
3. C.K.R.T. Jones, R. Gardner and T. Kapitula, Stability of traveling waves for non-convex scalar viscous conservation laws, *Comm. Pure Appl. Math.* **46**, 505–526 (1993).
4. E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Comm. Pure Appl. Math.* **3**, 201–230 (1950).
5. J.D. Cole, On a quasilinear parabolic equation occuring in aerodynamics, *Q. Appl. Math.* **9**, 225–236 (1951).
6. F. John, *Partial Differential Equations*, Fourth edition, Springer-Verlag, New York, (1982).