Algorithmic Complexity Continued

Goals

- Cover concept of doubling experiments
- Analyze common algorithms
- Code Exploration

Doubling Experiment

Doubling Experiment

Definition

- a thought experiment, or actual experiment, where input is doubled, and complexity behavior is characterized empirically!

Example

- say an algorithm has an unknown complexity
- when n is 10,000, the timer reports 3 seconds
- when n is doubled to 20,000, the timer reports 6 seconds
- when n is double again to 40,000 the time reports 12 seconds
- the ratio for the outputs was 6/3 = 2 and 12/6 = 2
- therefore, for each doubling, the output time scales linearly → O(n)

Doubling Experiment

Definition

- a thought experiment, or actual experiment, where input is doubled, and complexity behavior is characterized empirically!

Example 2

- say an algorithm has an unknown complexity
- when n is 10,000, the timer reports 3 seconds
- when n is doubled to 20,000, the timer reports 12 seconds
- when n is double again to 40,000 the time reports 48 seconds
- the ratio for the outputs was 12/3 = 4 and 48/12 = 4
- therefore, for each doubling, the output time scales by $2^2 \rightarrow O(n^2)$

Doubling Experiment: Linear

Double the size of the program's input

14.98 seconds

31.45 seconds

Doubling ratio is approximately 2

Likely worst-case time complexity is O(n)

Doubling Experiment: Quadratic

Double the size of the program's input

12.63 seconds	51.48 seconds

Doubling ratio is approximately 4

Likely worst-case time complexity is O(n^2)

Doubling Experiment: Cubic

Double the size of the program's input

11.23 seconds

89.72 seconds

Doubling ratio is approximately 8

Likely worst-case time complexity is O(n^3)

Critical Thinking

linear: $n \rightarrow 2n$ everything doubles

quadratic: $n^2 \rightarrow (2n)^2 = 2^2 * n^2$ everything scaled by factor of 4

logarithmic: $\log_2(n) \rightarrow \log_2(2n) = \log_2(2) + \log_2(n) = 1 + \log_2(n)$ shifting ratios

Work out 3 doubling ratios starting with $n = 4$ going through an $O(\log_2(n))$ algorithm

Common Algorithms & Complexity

Basic Loop

```
counter = 0
for i in range(n):
    # do a constant number of things
    counter += 1
    print(counter)
```

- input was n
- counter goes up to n
- algorithm is O(n)

Linear Search

```
def linear_search(L, x):
    for e in L:
        if e == x:
            return True
    return False
```

This is a linear algorithm, because in the worst case, every number will have to be examined once!

Complexity analysis is "**worst-case**" analysis

Basic Double Loop

```
counter = 0
for i in range(n):
    for j in range(n):
        # do a constant number of things
        counter += 1
        print(counter)
```

- input was n
- but counter goes up to n^2
- algorithm is $O(n^2)$

Square Root Algorithms (exhaustive, bisection)

Exhaustive search

- $x*epsilon^2$ times through the loop…technically **linear** but $epsilon^2$ dominates

Bisection Search

- approx $\mathbf{log_2(x)}$ times through the loop

For x = 100, epsilon = 0.0001,
Exhaustive search → 10 billion steps
Bisection search → approx 10 steps

```
def square_root_exhaustive(x, epsilon):
    """Assumes x and epsilon are positive floats & epsilon < 1
       Returns a y such that y*y is within epsilon of x"""
    step = epsilon**2
    ans = 0.0
    while abs(ans**2 - x) >= epsilon and ans*ans <= x:
         ans += step
    if ans*ans > x:
       raise ValueError
    return ans
```

```
def square_root_bi(x, epsilon):
    """Assumes x and epsilon are positive floats & epsilon < 1
       Returns a y such that y*y is within epsilon of x"""
    low = 0.0
    high = max(1.0, x)
    ans = (high + low)/2.0
    while abs(ans**2 - x) >= epsilon:
       if ans**2 < x:
          low = ans
       else:
          high = ans
       ans = (high + low)/2.0
    return ans
```

Powerset

Definition:

- for a given set, the powerset is a set with all possible subsets
- order does not matter

Example:

- find the powerset of {A,B,C,D,E,F}
- {{}, {A}, {B}, {A,B}, {C}, {A,C}, {B,C}, {A,B,C},{D}, {A,D}, {B,D}, {A,B,D}, {C,D}, {A,C,D}, {B,C,D}, {A,B,C,D}, {E}, {A,E}, {B,E}, {A,B,E}, {C,E}, {A,C,E}, {B,C,E}, {A,B,C,E},{D,E}, {A,D,E}, {B,D,E}, {A,B,D,E}, {C,D,E}, {A,C,D,E}, {B,C,D,E}, {A,B,C,D,E}, {F}, {A,F}, {B,F}, {A,B,F}, {C,F}, {A,C,F}, {B,C,F}, {A,B,C,F},{D,F}, {A,D,F}, {B,D,F}, {A,B,D,F}, {C,D,F}, {A,C,D,F}, {B,C,D,F}, {A,B,C,D,F}, {E,F}, {A,E,F}, {B,E,F}, {A,B,E,F}, {C,E,F}, {A,C,E,F}, {B,C,E,F}, {A,B,C,E,F},{D,E,F}, {A,D,E,F}, {B,D,E,F}, {A,B,D,E,F}, {C,D,E,F}, {A,C,D,E,F}, {B,C,D,E,F}, {A,B,C,D,E,F}}
- **Growth of output is exponentially related to input size: $O(2^n)$**

Explore Code