Discrete Structures!

CMPSC 102

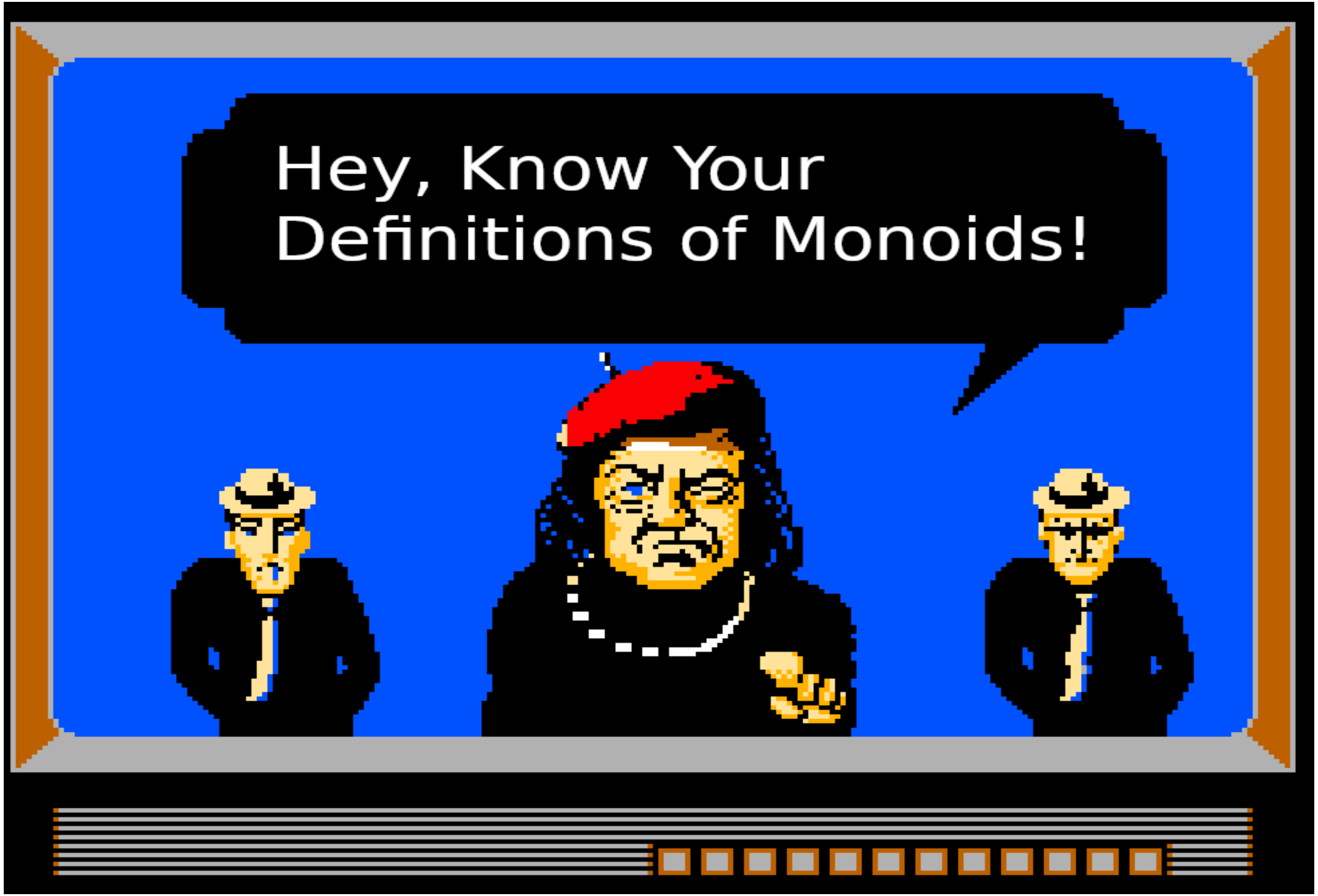
Monoids

Allegheny College

Key Questions and Learning Objectives

- How do I employ the mathematical concepts of sequences, monoids, and lists to implement efficient Python programs that use functions with a clearly specified behavior to perform tasks like finding a name in a file or computing the arithmetic mean of data values?

- To remember and understand some the concept of a monoid, seeing how it connects to practical applications with strings and sequences


Hey, Know Your Definitions of Monoids!

Monoid Definition

- In Abstract Algebra, a monoid is a **set** equipped with an **associative binary operation** and an **identity element**. For example, the non-negative integers with addition form a monoid, the identity element being 0.

- A monoid is a combination of an object (a,b,c) and an operation (+) that meets the following conditions
  - the operation on two of the objects produces a new object of the same kind
    - int + int = int
  - associative operations
    - (a+b) + c = a + (b+c)
  - a null object e must exist, such that $e + a = a + e = a$
    - $n + 0 = n$

Ref: https://en.wikipedia.org/wiki/Monoid#Definition

Monoids

Associative property: Addition with three integers, **int + int + int = int**

```
a = 10
b = 7
c = 4
print(f"{a+b+c=}")
print(f"{a+b+c==10+7+4}")
```

Identity element: Addition With Null Object, **int + Null = int**

```
a = 10
b = 0
print(f"{a+b=}")
print(f"{a+b==10+0}")
```

Examples of Sequences in Python

- Sequences are commonly found in Python programs!
- Examples of the sequence discrete structure in Python:
    - A string is a sequence of individual characters
    - The range(20) function returns a sequence of numbers
    - Files are sequences of lines containing content
    - Each line in a file is a sequence of individual characters
    - Each individual character is a sequence of numbers
    - Each individual number is a sequence of binary digits
- Do these sequences all have properties in common?
- Can we generalize?

Licensed to Sequence

Make a Sequence

```python
first = "James"
last = "Bond"
print(f"The name is, {last}, {first}-{last}")
```

What is an n-Tuple

```
myStuff = ()
type(myStuff) # is tuple

item_1 = "Omega Watch"
item_2 = "Aston Martin"
item_3 = "Spy Manual"

myStuff = list(myStuff) # conv to list
type(myStuff) # is list

myStuff.append(item_1)
myStuff.append(item_2)
myStuff.append(item_3)

myStuff = tuple(myStuff)
type(myStuff) # is tuple

print(myStuff)
```

What the difference?

What is the difference between sequences and tuples?

Comparing Lists and n-Tuples

- Lists are mutable, Tuples are not

```
# Lists are mutable
my_list = [1, 2, 3]
my_list[0] = 99
print("List after modification:", my_list)
```

An Example *mutable* lists

```
a = [2,3,5,7,11]
print(a)
a[2] = False
print(a)
```

Comparing Lists and n-Tuples

- Lists are mutable, Tuples are not

An error is raised

```
# Tuples are immutable
my_tuple = (1, 2, 3)
# The following line would raise an error:
# 'tuple' object does not support item assignment
my_tuple[0] = 99
```

```
my_tuple = (1,2,3,4,5,6)
print(f"my_tuple : {my_tuple}, {type(my_tuple)}")
my_tuple[2] = False
```

String Concatenations in Python

A sequence of operands to be concatenated.

```python
hello = "hello"
world = "world"
space = " "
message = hello + space + world
print(f"The message is: {message}")
```

- You can concatenate or "glue together" strings
- Can we change orders?

  hello + space + world, space + hello + world,
  or world + space + hello

Reversed String Concatenation

```python
hello = "hello"
world = "world"
space = " "
message = world + space + hello
print(f"The message is: {message}")
```

- What is the **output** of this program segment?
- How does Python **represent** a string in memory?
- What are the different **types** of strings?
- What is an **empty string** in Python?
- How is an empty string different from “ ”?

Licensed to Sequence

Make a Sequence

```python
first = "James"
last = "Bond"
print(f"The name is, {first}, {last}-{first}")
```

- Are concatenated sequences still monoids?

Empty String Concatenation in Python

```python
firstVar = "hello"
secondVar = "world"
empty = "_"
message = secondVar + empty + firstVar
print(f"The message is: {message}")
```

- What is the output of this program segment?
- Why does the order of operations not matter in this case?
- Can we generalize these observations about strings?
- Can we define a general discrete structure with predictable properties?
- If you get confused, revisit what you know about working with str's in Python!

Revisit Empty String Concatenation in Python

```
firstVar = "hello"
secondVar = "world"
empty = "_"
message = secondVar + empty + firstVar
print(f"The message is: {message}")
```

- What is the output of this program segment?
- Why does the order of operations not matter in this case?
- Can we generalize these observations about strings?
- Can we define a general discrete structure with predictable properties?
- If you get confused, revisit what you know about working with str's in Python!

Characterizing String Concatenations

- Define S to be the set of all possible strings
- What properties of S are always true?
  - For $s_1, s_2 \in S$ and the concatenation operator "+", $s_1 + s_2 \in S$
  - For $s_1, s_2, s_3 \in S$, "+" **is associative**: $(s_1 + s_2) + s_3 = s_1 + (s_2 + s_3)$
  - For $s_1, s_2, \in S$, "+" **is not commutative**: $(s_1 + s_2) \neq s_2 + s_1$
  - For $s_1, s_2, \in S$, if $s_1 = s_2$ or $s_1 = \epsilon$, then "+" **is commutative**
- These properties of strings help us to **generalize** and **understand** their behavior!
- The **monoid** discrete structure generalizes data that "behaves like strings"

Properties (of real numbers)

Said in a different way from previous slide

Property	Addition	Multiplication
Commutative	$a + b = b + a$	$a \cdot b = b \cdot a$
Associative	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Distributive	$a \cdot (b + c) = a \cdot b + a \cdot c$	$a \cdot (b + c) = a \cdot b + a \cdot c$
Identity	$a + 0 = a$	$a \cdot 1 = a$
Inverse	$a + (-a) = 0$	$a \cdot \frac{1}{a} = 1$

- Remember that strings do not behave like numbers when using these properties.

Properties of Strings and Integers

String

- Concatenation through the use of the + operator
- Identity: exists in the ”” string
  - "this " + " " = " this "
  - len("this" + "")
- Concatenation is associative but **is not** commutative

Integers

- Two integers separated by an + operator creates another integer.
- Addition of integers is the associative property.
- Identity: exists as a 0
  - $n + 0 = n$
- Concatenation is associative and commutative

Monoid Classes :: __init__

File: sandbox/base_permutations.py

```
#!/usr/bin/env python3
# -*- coding: utf-8 -*-

class Monoid:
        def __init__(self, null, typeify, operator):
                # __init__ allows class variables to be definded
                # when the class is initiated
                self.null = null
                self.typeify = typeify
                self.operator = operator
```

- Sets up the class in terms of object's variables

Monoid Classes :: __call__

File: sandbox/base_permutations.py

```
def __call__(self, *args):
        # __call__ method enables classes for which
        # the instances behave like functions and
        # can be called as such
        result = self.null
        for arg in args:
                arg = self.typeify(arg)
                result = self.operator(result, arg)
        return result
```

- Sets up ability for the class to be *called* as a function to simplify programming

Monoid Classes :: __call__

File: sandbox/base_permutations.py

```
# Function for adding numbers
def add(a, b):
        return a + b

number_monoid = Monoid(0, int, add)

print(number_monoid(1, 2, 3, 4)) # Output: 10
print(number_monoid(5)) # Output: 5
print(number_monoid()) # Output: 0 (identity element)
```

Main Function :: cartesian_prod()

File: sandbox/base_permutations.py

```
def cartesian_prod(a_list, b_list):
        print(f"my a_list and my b_list : {a_list} and {b_list}")
        # input()
        c = []
        for a in a_list:
                for b in b_list:
                        c.append(a + b)
        return c
```

- Function to Calculate Cartesian product

Command

File: sandbox/base_permutations.py

```
cartesian_product_monoid = Monoid([’’],
                                  lambda x: x,
                                  cartesian_prod)
# define class

print(cartesian_prod(["A", "B"], ["1", "2"]))
```

- Command to initiate class and pass in list variables for permutation calculation

Command

File: sandbox/base_permutations.py

```
base_list = ['A','C','G','T']

print("Length 2 cartesian products")
permutations_list = cartesian_product_monoid(base_list, base_list)
print(f"\t [+] Length 2 Permutations_list = {permutations_list}")
print(f"\t [+] Number of permutations : {len(permutations_list)}")
```

- Prepare the list of characters
- Call cartesian product monoid(), assign all results to permutations_list for length 2 products

Command

File: sandbox/base_permutations.py

```
print("Length 4 cartesian products")
permutations_list = cartesian_product_monoid(base_list, base_list, base_list, base_list)
print(f"\t [+] Length 4 Permutations_list = {permutations_list}")
print(f"\t [+] Number of permutations : {len(permutations_list)}")
```

- Prepare the list of characters
- Call cartesian product monoid(), assign all results to permutations_list for length 4 products