Discrete Structures: CMPSC 102

Oliver BONHAM-CARTER

Fall 2019
Week 6

- Water is *blue*, the shirt is also *blue*, therefore this shirt is made up of water?

Logic is ...

a **truth-preserving** **system** of **inference**

- **Truth-preserving**: If the initial statements are True, the inferred statements will also be true
- **System**: A set of mechanistic transformations which are based on syntax alone
- **Inference**: The process of deriving (inferring) new statements from old statements (i.e., finding conclusions based on previous observations)

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Prepositional Logic

- A *proposition* is a statement that is either *true* or *false*
- Every proposition is true or false, but its truth value (true or false) may be unknown
- Examples:
  - You are in Discrete Structures (CMPSC102) $\rightarrow$ True
  - Today is Sunday $\rightarrow$ False
  - $1 == 2 \rightarrow$ False
  - It is currently raining in Paris $\rightarrow$ who can say?
  - In Alice's pocket, she has exactly 58 cents $\rightarrow$ who can say?

Different from philosophical assessments of truth...

- Philosopher Ludwig Wittgenstein observed, structures have spatial locations, but facts do not
- *The Eiffel Tower could be moved from Paris to Rome, but there is a fact that it is currently in Paris (and nowhere else)*
- "Truth" is something of a system of beliefs (i.e., I believe that it is in Paris, because I have seen it there)

Statements

- A proposition statement:
  - Denoted by a capital letter (i.e., "A")
  - The negation of a proposition statement
    - $\sim A$: "**not** A"
  - Two proposition statements joined by a *connective*
    - $A \wedge B$: "A **and** B"
    - $A \vee B$: "A **or** B"
  - If a connective joins complex statements, parenthesis are added
    - $A \wedge (B \vee C)$: "A and (B or C)"

Prepositional Statements

- $A =$ "It is raining"
  - $\sim A$: "not A"
  - "It is **not** raining"
- Composition
  - $A =$ "Peanut butter"
  - $B =$ "Jelly"
  - $A \wedge B$: "A and B"
  - A sandwich composed of: *peanut butter* **and** *jelly*
- Either one or the other
  - $A =$ "I will wear a white t-shirt"
  - $B =$ "I will wear a blue t-shirt"
  - I will wear a t-shirt: $A \vee B$: "A or B"
  - I will wear the white **or** the blue t-shirt (this is still true!)

Propositional Statements

- Negation of proposition A is $\sim A$
- A: It is sunny.
- $\sim A$: It is not sunny.
- A: Newton often drank tea.
- $\sim A$: Newton did not often drink tea.
- A: I am from the planet Zogitron.
- $\sim A$: I am not from the planet Zogitron. (regretfully)

Truth Tables

- Compound propositional statements are built out of simple statements using logical operations: negations, conjugations (using **AND**'s) and disjunctions (using **OR**'s)
- Determined by *Truth Tables*
  - A table of a combinations of truths using the connectives
  - Truth tables define the truth value of a connective for every possible truth value of its terms

Basic Truth Table

Let A be a propositional statement ...

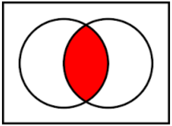
A	$\sim A$
True	False
False	True

In Python

```
True # True
not True # False
not False # True

A = True
not(A) # False
```

Logical AND (and its Conjunction)



- Conjunction of A and B is: $A \wedge B$
  - A: We are learning logical expressions
  - B: We are learning Python code for logical expressions
  - $A \wedge B$: We are learning logical expressions and python code for logical expressions

In Python

```
A = True
B = True
A & B # True
A and B # True
```

Logical AND (and its Conjunction)

- Conjunction of A and B is: $A \wedge B$
  - A: I like art
  - B: I like this work of art
  - $A \wedge \sim B$: I like art but (**AND**) I do not like this work of art

In Python

```
A = True
B = True
A and not(B) # False
```

Remember

We are discussing only the **Logical Truth** of the statements. The *meaning* of the combined statements is human insight.

Logical AND (and its Conjunction)

Table for Conjunction

A	B	$A \wedge B$
False	False	False
False	True	False
True	False	False
True	True	True

```
A = True
B = True
not(A) and not(B) # False
not(A) and B # False
A and not(B) # False
A and B # True
```

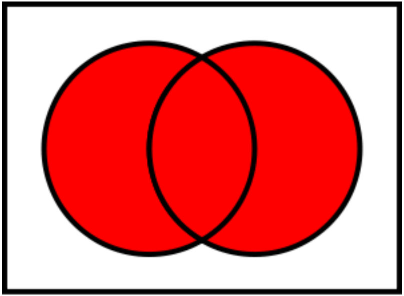
Logical OR (and its Disjunction)

- Also known as the inclusive OR
- Conjunction of A or B is: $A \vee B$
  - A: Today is sunny
  - B: It is Monday
  - $A \vee B$: Today is sunny or it is Monday
- This statement is true if any of the following hold:
  - Today is sunny
  - It is Monday
  - Both
- Otherwise it is false

In Python

```
A = True
B = False
A or B # True
A | B # True
```

Logical OR (and its Disjunction)



Logical OR (and its Disjunction)

Truth Table for Disjunction (Inclusive OR)

A	B	$A \vee B$
False	False	False
False	True	True
True	False	True
True	True	True

```
A = True
B = True
not(A) or not(B) # False
not(A) or B # False
A or not(B) # False
A or B # True
```

Exclusive OR

- The inclusive OR is true if either or both arguments are true
- The exclusive OR is true if both arguments are opposite truths

Truth Table for Exclusive OR ($\otimes$) (*either or*)

A	B	A $\otimes$ B
False	False	False
False	True	True
True	False	True
True	True	False

Exclusive OR

In Python

```
A = False
B = False
not(A and not(B)) # True

A = False
B = True
not(A and not(B)) # True

A = True
B = False
not(A and not(B)) # False

A = True
B = True
not(A and not(B)) # True
```

Implication $\rightarrow$

- The conditional implication connective is $\rightarrow$
  - $A \rightarrow B$: A implies B
  - *If A is true, then B is also true such that the statement* $A \rightarrow B$ *is False only when A is True and B is False*
- Logically equivalent to: $\sim (A \wedge \sim B) == (\sim A \vee B)$

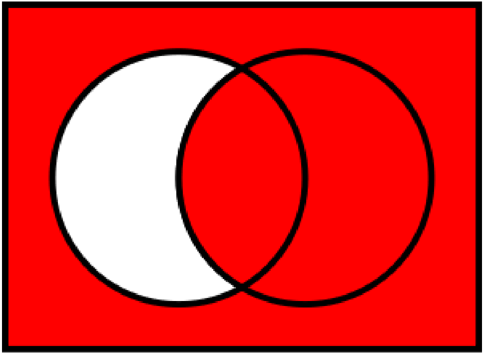
Truth Table for the Conditional Implication

A	B	A $\rightarrow$ B
False	False	True
False	True	True
True	False	False
True	True	True

Meaning of Conditional Implication ($\rightarrow$)

- A: Homework is due
- B: It is Monday
- $A \rightarrow B$
  - If homework is due, then it must be Monday.
- Can we also conclude...?
  - If it is Monday, then a homework is due.

Venn Diagram of Implication $\rightarrow$



Truth Table for the Conditional Implication

```
A = False
B = False
not(A) or B # True

A = False
B = True
not(A) or B # True

A = True
B = False
not(A) or B # False

A = True
B = True
not(A) or B # True
```

Implication $\leftrightarrow$

- The conditional implication connective is $\leftrightarrow$
  - $A \leftrightarrow B$: "A if and only if B",
- Logically equivalent to: $(A \rightarrow B) \wedge (B \rightarrow A) == (A \wedge B) \vee (\sim A \wedge \sim B)$

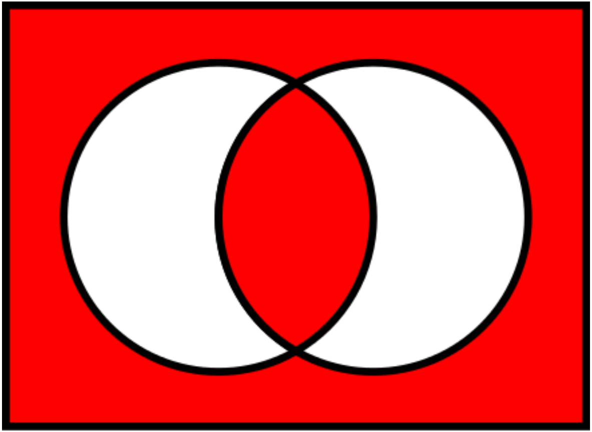
Truth Table for the Conditional Implication

A	B	A $\leftrightarrow$ B
False	False	True
False	True	False
True	False	False
True	True	True

Meaning of Conditional Implication ($\leftrightarrow$)

- A: You can buy the shirt
- B: You have enough money
- $A \leftrightarrow B$
  - You can buy the shirt, if and only if, you have enough money (and *vice versa*)
- Can we also conclude...?
  - If you have enough money, then you can buy the shirt.

Venn Diagram of Implication $\leftrightarrow$



Implication $\leftrightarrow$

Truth Table for the Conditional Implication

```
A = False
B = False
((A and B) or (not(A) and not(B))) # True

A = False
B = True
((A and B) or (not(A) and not(B))) # False

A = True
B = False
((A and B) or (not(A) and not(B))) # False

A = True
B = True
((A and B) or (not(A) and not(B))) # True
```

Consider this ...

Try coding these...

- Use Python to test the following conditions:
  - $\sim A \vee B$, for A = True and B = False
  - $\sim A \vee \sim B$, for A = True and B = False
  - $\sim A \rightarrow B$, for A = True and B = False

Compound Truth tables

A	B	~A	$A \vee B$	$(\sim A) \wedge (A \vee B)$
0	0	1	0	0
0	1	1	1	1
1	0	0	1	0
1	1	0	1	0

Legend

- AND is denoted by : $\wedge$
- OR is denoted by : $\vee$
- Contradiction is denoted by : $\sim$
- Equivalency is denoted by : $\equiv$

Tautologies and Contradictions

- A **tautology** is a compound proposition that is always true.
- A **contradiction** is a compound proposition that is always false.
- A **contingency** is neither a tautology nor a contradiction.
- A compound proposition is satisfiable if there is at least one assignment of truth values to the variables that makes the statement true.

Tautologies and Contradictions

A	~A	$A \vee \sim A$	$A \wedge \sim A$
0	1	1	0
1	0	1	0

- $A \vee \sim A$: **Tautology**: Statement is always true no matter value of A.
- $A \wedge \sim A$: **Contradiction**: Statement is always false, no matter the value of A.

Tautologies and Contradictions

- Famous song for which the title is a contradiction: *With or Without You*, by U2 (1987)

Logical Equivalence

- Two compound propositions, A and B, are logically equivalent if A $\leftrightarrow$ B is a tautology.
- Notation: $A \equiv B$
- De Morgans Laws:
  - $\sim (A \wedge B) \equiv \ \sim A \vee \sim B$
  - $\sim (A \vee B) \equiv \ \sim A \wedge \sim B$
  - How do we know this or prove this claim? A truth table!

Prove: $\sim(A \wedge B) \equiv \sim A \vee \sim B$

A	B	∼A	∼B	$(A \wedge B)$	$\sim(A \wedge B)$	$\sim A \vee \sim B$
0	0					
0	1					
1	0					
1	1					

How to complete the table??

Use a Truth Table to Make a Proof

Prove: $\sim (A \wedge B) \equiv \ \sim A \vee \sim B$

A	B	∼A	∼B	$(A \wedge B)$	$\sim (A \wedge B)$	$\sim A \vee \sim B$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

Like this!

Use a Truth Table to Make a Proof

Prove: $\sim(A \vee B) \equiv \sim A \wedge \sim B$

A	B	∼A	∼B	$(A \vee B)$	$\sim(A \vee B)$	$\sim A \wedge \sim B$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

Use a Truth Table to Make a Proof

Prove: $\sim(A \leftrightarrow B) \equiv (A \leftrightarrow \sim B)$

A	B	∼B	$A \leftrightarrow B$	$\sim(A \leftrightarrow B)$	$A \leftrightarrow \sim B$
0	0	1	1	0	0
0	1	0	0	1	1
1	0	1	0	1	1
1	1	0	1	0	0

Truth Table for the Conditional Implication

```
A = False
B = False
((A and B) or (not(A) and not(B))) # True
```

Consider this ...

Show that: $(A \land B) \rightarrow (A \lor B)$

A	B	$(A \land B)$	$(A \lor B)$	$(A \land B) \rightarrow (A \lor B))$
1				
1				
0				
0				

Is the statement $[(A \land B) \rightarrow (A \lor B)]$ a tautology or a contradiction.