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Consensus-Based Control Barrier Function for Swarm*

Manao Machida and Masumi Ichien

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***Abstract*— In swarm control, many robots coordinate their actions in a distributed and decentralized way. We propose a consensus-based control barrier function (CCBF) for a swarm. CCBF restricts the states of the whole distributed system, not just those of the individual robots. The barrier function is approximated by a consensus filter. We prove that CCBF constrains the control inputs for holding the forward invariance of the safety set. Moreover, we applied CCBF to a practical problem and conducted an experiment with actual robots. The results showed that CCBF restricted the states of multiple robots to the safety set. To the best of our knowledge, this is the first CBF that can restrict the state of the whole distributed system with only local communication. CCBF has various applications such as monitoring with a swarm and maintaining the network between a swarm and a base station.**

I. INTRODUCTION

The control barrier function (CBF), proposed by [1], imposes inequality constraints on the control input for holding forward invariance of a safety set. The safety set is represented by a function that provides the constraint of control inputs. [2] proposed a CBF using a quadratic program (QP) for selecting an optimal control input.

In swarm control, many robots coordinate their actions in a distributed and decentralized way [10]. The robots are relatively simple and have only local communication and sensing capabilities. Consensus [11], [12], formation [13], [14], [15] and coverage [16], [17] problems have been studied in swarm control. Swarm control has applications such as waste management [18] and search tasks [19]. Moreover, in swarm control, communication networks need to be maintained [20], [21], [22].

A CBF can be applied to a single robot [3], [4], [5] or to multiple robots [6], [7], [8], [9]. The existing work on CBFs in swarms aims at avoiding collisions [6], [8], [9] and maintaining connectivity [7] between the robots. These barrier functions restrict the distance between pairs of robots and depend on only the local state, that is, each robot's own state and nearby robots' states. Hence, these CBFs for swarms cannot restrict the states of the whole swarm.

In this paper, we propose consensus-based CBF (CCBF), which can restrict the states of swarm robots to a safety set by using only local interactions between the robots. CCBF can incorporate various barrier functions that depend on the state of the whole swarm. Hence, it can restrict not only a single robot's state but also a swarm's one to a safety set. To the best of our knowledge, this is the first CBF that can restrict the states of a whole distributed system with only local communication. CCBF has various applications such as monitoring with a swarm, maintaining the network between a swarm and a base station. In CCBF, the value of the barrier function is approximated by a consensus filter. The consensus filter [23], [24] is a distributed algorithm that allows the nodes of a sensor network to track the average of all of their measurements. We prove that CCBF provides the constraint on the control inputs for holding forward invariance of the safety set. We applied CCBF to a practical problem, i.e., maintaining the network between a swarm and a base station and conducted an experiment with actual robots; we found that CCBF is effective on this practical problem.

Our contributions are as follows:

- We propose CCBF, which can restrict not a single robot's state but the whole swarm's state to a safety set by using only local interactions between the robots.
- We applied CCBF to a practical problem and showed that it is effective for swarm control.

II. PRELIMINARIES

Let $R = \{1, \ldots, n\}$ be a set of n robots. Robot $i \in R$ has a state $x_i \in X_i$, where X_i is the state space of robot i. The state of all of the robots is denoted by $x = [x_1^T, \ldots, x_n^T]^T$, and a combination of state spaces of the robots is denoted by $X = \times_{i \in R} X_i$. We refer to the state of all of the robots as the swarm state.

Let $\mathcal{C} \subset X$ be a nonempty compact safety set and $h : \mathcal{C} \to \mathbf{R}$ be as follows:

$$h(x) > 0 \;\; (x \in \mathrm{Int}(\mathcal{C})) \tag{1}$$
$$h(x) = 0 \;\; (x \in \partial\mathcal{C}) \tag{2}$$
$$\frac{\partial h}{\partial x}(x) \neq 0 \;\; (x \in \partial\mathcal{C}), \tag{3}$$

where $\mathrm{Int}(\mathcal{C})$ is the interior of C, and $\partial C = C \setminus \mathrm{Int}(\mathcal{C})$. We assume that $\mathcal{C} \subset X$ is forward invariant if $x(t) \in \mathcal{C}$ holds for all $t \in [0, \infty)$ when $x(0) \in \mathcal{C}$.

The barrier function is defined as follows:

Definition 1: Consider a nonlinear system:

$$\dot{x} = f(x). \tag{4}$$

A continuously differential h is a zeroing barrier function (ZBF) if there exists $\gamma > 0$ that satisfies the following condition for all $x \in C$.

$$\dot{h}(x) \geq -\gamma h(x). \tag{5}$$

If h is a ZBF, then the safety set $\mathcal{C}$ is forward invariant under the dynamics (4).

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Manao Machida and Masumi Ichien are with Data Science Research Laboratories, NEC Corporation, Kawasaki, Kanagawa, Japan manaomachida@nec.com, m-ichien@nec.com

We use CBF for safety control. Let U be a set of control inputs. Consider an affine control system:

$$\dot{x} = f(x) + g(x)u, \tag{6}$$

where $u \in U$.

The CBF controller for safety control is defined as follow:

Definition 2: Consider the control system (6) and a feedback controller $u = k(x)$. A control input based on the CBF controller is as follows:

$$\begin{aligned} u(x) \in &\arg\min_{u \in U} ||u - k(x)||^2 \\ &s.t.\ \dot{h}(x) \geq -\gamma h(x), \end{aligned} \tag{7}$$

where $\gamma > 0$.
The control input $u(x)$ keeps the state of the robot in the safety set.

The robots have local communication capabilities. Let $G = (R, E)$ be an undirected connected communication graph. Robots $i, j \in R$ can communicate if $(i, j) \in E$ holds. Let $\mathcal{N}_i = \{j \in R | (i, j) \in E\}$ be the set of robots that are adjacent to robot i.

A consensus filter [23], [24] is a distributed algorithm that allows the nodes of a sensor network to track the average of all of their measurements. The high-pass consensus filter is as follows [23]:

$$\begin{aligned} \dot{y}_i &= \sum_{j \in \mathcal{N}_i} (y_j - y_i) + \dot{z}_i \\ y_i(0) &= z_i(0), \end{aligned} \tag{8}$$

where z_i is a signal measured by sensor (robot) i. The consensus filter allows $y_i(t)$ to track $\frac{1}{n}\sum_{j \in R} z_j(t)$. It is important that $\sum_{i \in R} y_i(t) = \sum_{i \in R} z_i(t)$ holds because $\sum_{i \in R} \dot{y}_i = \sum_{i \in R} \dot{z}_i$ holds.

III. Related Work

The existing work on CBFs in swarms aims at avoiding collisions [6], [8], [9] and maintaining connectivity [7] between the robots. In these studies, the CBFs depend on only local information and restrict the distance between any two robots. For example, the CBF (reciprocal barrier function) for collision avoidance between robots $i \in R$ and $j \neq i$ in [6] is as follows:

$$\begin{aligned} h_{ij} &= \frac{(p_i - p_j)^T}{||p_i - p_j||}(v_i - v_j) + \sqrt{4a_{\max}(||p_i - p_j|| - D_s)} \\ B_{ij} &= \frac{1}{h_{ij}} \end{aligned}$$

where p_i, v_i are the position and velocity of robot i, respectively. B_{ij} is a reciprocal barrier function for collision avoidance between robots i and j. $D_s > 0$ is safety distance. Robot i's control input u_i satisfies $||u_i|| \leq a_{\max}$. [6] also described how to design the neighbor set of each robot. They defined the neighbor set of robot i as follows:

$$\mathcal{N}_i = \left\{ j \neq i \left| ||p_i - p_j|| \leq D_s + \frac{1}{4a_{\max}} \left(\left(\frac{4a_{\max}}{\gamma} \right)^{\frac{1}{3}} + 2v_{\max} \right)^2 \right. \right\},$$

where $v_{\max}$ is the velocity limits of the robots, $\gamma > 0$.

In their method, the swarm can avoid collisions between the robots by using B_{ij} only if $j \in \mathcal{N}_i$ holds. Hence, their CBFs may depend on only local states, that is, the robot i's state and one of the other robots near robot i. These CBFs are useful in the distributed system. However, how to use a CBF that depends on the whole state in a distributed system is an open problem.

In our method, we consider a CBF that depends on the whole swarm's state. CCBF has various applications besides avoiding collisions. In section V, we describe an application of CCBF.

IV. Consensus-Based Control Barrier Function

Let us consider the following ZBF.

$$B(x) = \sum_{m \in M} \Phi_m\Big(\sum_{i \in R} \phi_{im}(x_i)\Big) \tag{9}$$

where $\Phi_m : \mathbf{R}^{l_m} \to \mathbf{R}$ and $\phi_{im} : X_i \to \mathbf{R}^{l_m}$ are differentiable, l_m is an arbitrary natural number, and $M = \{1, \ldots, \mathcal{M}\}$. $\mathcal{M}$ is an arbitrary integer. This representation includes the simplest form of the Kolmogorov–Arnold representation theorem [25], [26], so that various functions can be represented by (9).

Now let us show how to use our representation (9).

Example 1: Consider a safety set based on the task success probability. Let $p_i(x_i)$ be the probability that robot $i \in R$ having state $x_i \in X_i$ completes the task. A ZBF is as follows:

$$B(x) = 1 - (1 - p_1(x_1))(1 - p_2(x_2)) - q. \tag{10}$$

If a swarm's state is in the safety set corresponding to this ZBF, then the swarm completes the task with a $q \geq 0$ or more probability.

This ZBF is represented by (9) as follows:

$$\begin{aligned} B(x) =& 1 - q + \frac{1}{2}\left\{(1 - p_1(x_1)) + (1 - p_2(x_2))\right\}^2 \\ &- \frac{1}{2}\left\{(1 - p_1(x_1))^2 + (1 - p_2(x_2))^2\right\}, \end{aligned} \tag{11}$$

that is,

$$\begin{aligned} \Phi_1(c) &= 1 - q + \frac{1}{2}c^2 \\ \phi_{11}(x_1) &= 1 - p_1(x_1),\ \phi_{21}(x_2) = 1 - p_2(x_2) \\ \Phi_2(c) &= -\frac{1}{2}c \\ \phi_{12}(x_1) &= (1 - p_1(x_1))^2,\ \phi_{22}(x_2) = (1 - p_2(x_2))^2 \end{aligned} \tag{12}$$

where $c \in \mathbf{R}$, $M = \{1, 2\}$, $l_1 = l_2 = 1$.

The time derivative of ZBF (9) is

$$\dot{B}(x) = \sum_{i \in R} \sum_{m \in M} \nabla\Phi_m\Big(\sum_{j \in R} \phi_{jm}(x_j)\Big)^T \frac{\partial \phi_{im}}{\partial x_i}(x_i)^T \dot{x}_i, \tag{13}$$

where $\nabla\Phi_m$ is the gradient of Φ_m. If each robot $i \in R$ in a swarm satisfies the following condition,

$$\sum_{m \in M} \nabla\Phi_m\Big(\sum_{j \in R} \phi_{jm}(x_j)\Big)^T \frac{\partial \phi_{im}}{\partial x_i}(x_i)^T \dot{x}_i \geq -\gamma B(x) \tag{14}$$

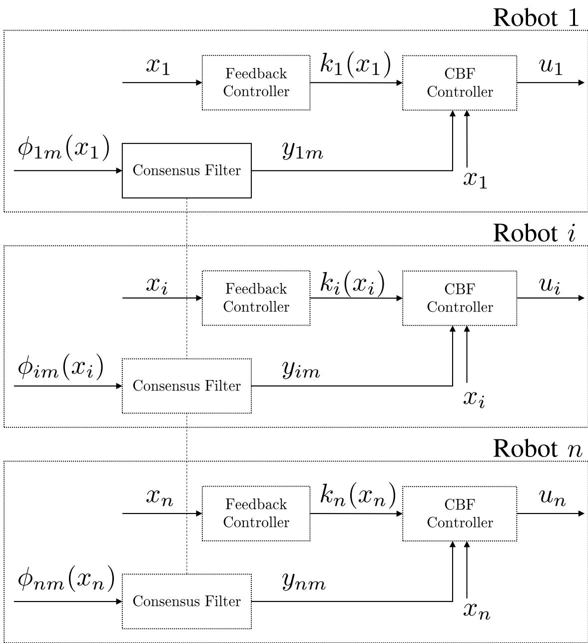

Fig. 1: CCBF controller

then $\dot{B}(x) \geq n\gamma B(x)$ holds. However, $\sum_{j\in R}\phi_{jm}(x_j)$ depends on the whole swarm's state $x \in X$.

Figure 1 shows a CCBF controller, where u_i is the control input of robot $i \in R$.

In CCBF, each robot approximates $\sum_{j\in R}\phi_{jm}(x_j)\,(m \in M)$ by using a consensus filter. Robot $i \in R$ measures $\phi_{im}(x_i)\,(m \in M)$, and the consensus filter allows the robots to track the average of their measurements, that is, $\frac{1}{n}\sum_{j\in R}\phi_{jm}(x_j)$. The information transmitted between robots i and $j \in \mathcal{N}_i$ is $[y_{i1},\dots,y_{i\mathcal{M}}]$ and $[y_{j1},\dots,y_{j\mathcal{M}}]$. Hence, each robot $i \in R$ approximates $\sum_{j\in R}\phi_{jm}(x_j)$ using only local information: x_i, $[y_{i1},\dots,y_{i\mathcal{M}}]$, and $[y_{j1},\dots,y_{j\mathcal{M}}]\,(\forall j \in \mathcal{N}_i)$.

CCBF is as follows:

- Consensus Filter

$$\begin{aligned} \dot{y}_{im} &= k_L \sum_{j\in\mathcal{N}_i}(y_{jm}-y_{im}) + \dot{\phi}_{im}(x_i) \\ y_{im}(0) &= \phi_{im}(x_i(0)), \end{aligned} \tag{15}$$

  where $k_L > 0$ is the gain of the consensus.

- CBF Controller

$$\begin{aligned} &u_i(x_i) \in \arg\min_{u_i} \|u_i - k_i(x_i)\|^2 \\ &s.t. \sum_{m\in M}\nabla\Phi_m(ny_{im})^T\dot{\phi}_{im}(x_i) \geq -\gamma\sum_{m\in M}\Phi_m(ny_{im}) \end{aligned} \tag{16}$$

Let $\mathcal{B} = \{y \in \mathbf{R}^{n\mathcal{M}} \,|\, \sum_{i\in R}\sum_{m\in M}\Phi_m(ny_{im}) \geq 0\}$. The following is a theorem on the forward invariance of the safety set.

Theorem 1: If Φ_m is concave for all $m \in M$, then $\mathcal{C}\times\mathcal{B}$ is forward invariant under CCBF (15), (16).

The proof of this theorem is shown in Appendix A. Theorem 1 shows that CCBF restricts the swarm's state $x \in X$ to a safety set $\mathcal{C}$. The forward invariant set $\mathcal{C}\times\mathcal{B}$ is similar to $\mathcal{C}\times\mathbf{R}^{n\mathcal{M}}$ after a sufficient time passes, because $\sum_{m\in M}\Phi_m(ny_{im})$ tracks $B(x)$. This means that the swarm's state can reach almost all states in $\mathcal{C}$.

Let $\mathcal{B}_+ = \{y \in \mathbf{R}^{n\mathcal{M}} | \forall i \in R, \Phi_m(ny_{im}) \geq 0\}$. When B is represented by $B(x) = \Phi(\sum_{i\in R}\phi_i(x_i))$ and $\phi_i : X_i \to \mathbf{R}$, the following theorem holds.

Theorem 2: Let $B(x) = \Phi(\sum_{i\in R}\phi_i(x_i))$ and $\phi_i : X_i \to \mathbf{R}$. If Φ is monotone, then $\mathcal{C}\times\mathcal{B}_+$ is forward invariant under CCBF (15), (16).

The proof of this theorem is shown in Appendix B. Theorem 2 shows that CCBF restricts a swarm's state to the safety set even if the ZBF is monotone and not concave in the special case $B(x) = \Phi(\sum_{i\in R}\phi_i(x_i))$ and $\phi_i : X_i \to \mathbf{R}$.

V. Example

Let us introduce an application of CCBF: maintaining a network between a swarm and a base station.

This problem is considered in [27]. In [27], some robots must be proximal to the base station to receive instructions. In this example, we introduce an "OR" constraint and a constraint on the probability. We can easily apply these constraints to other problems. For example, the monitoring task with a swarm in which at least one robot keeps monitoring the object can be represented by the "OR" constraint.

A. "OR" constraint

First, let us consider a simple communication model. Robot $i \in R$ can communicate with the base station if $x_i \in \mathcal{C}_i = \{x_i \in X_i | \, \|x_i - o\| \leq r\}$ holds, where x_i, o are positions of robot i and the base station, respectively. If at least one robot can communicate with the base station, then the whole swarm can receive messages from the base station through a multi-hop network in the swarm. Hence, the swarm in the safety set $\mathcal{C} = \{x \in X | \exists i \in R, x_i \in \mathcal{C}_i\}$ maintains communication with the base station. This constraint is an "OR" constraint, that is, it requires that at least one robot satisfies the constraint for itself. Let $h(x_i) = r^2 - \|x_i - o\|^2$. The ZBF (9) is as follows:

$$B(x) = \sum_{i\in R}\phi_i(x_i) \tag{17}$$

$$\Phi(c) = c \tag{18}$$

$$\phi_i(x_i) = \begin{cases} h(x_i) & (x_i \in \mathcal{C}_i) \\ 0 & (x_i \in X_i \setminus \mathcal{C}_i) \end{cases} \tag{19}$$

ϕ_i is not differentiable on $x_i \in \partial\mathcal{C}_i$, so we use the following d_ϕ instead of $\frac{\partial \phi_i}{\partial x_i}$ to restrict the control input.

$$d_\phi(x_i) = \begin{cases} \frac{\partial h}{\partial x_i}(x_i) & (x_i \in \mathcal{C}_i) \\ 0 & (x_i \in X_i \setminus \mathcal{C}_i) \end{cases} \tag{20}$$

The constraint (16) using d_ϕ is as follows:

$$d_\phi(x_i)^T\dot{x}_i \geq -\gamma\Phi(ny_i) \tag{21}$$

This CCBF restricts the swarm's state to the safety set because of Theorem 1.

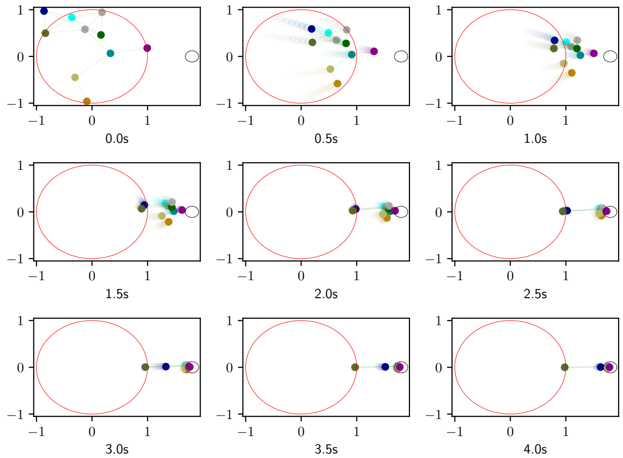

Fig. 2: State of swarm under CCBF with $\gamma = k_L = 5$ for maintaining network between swarm and base station. Each robot in a red circle can communicate with the base station. The black circle represents the destination of the robots; small circles represent positions of the robots; and green lines show the communication network in the swarm.

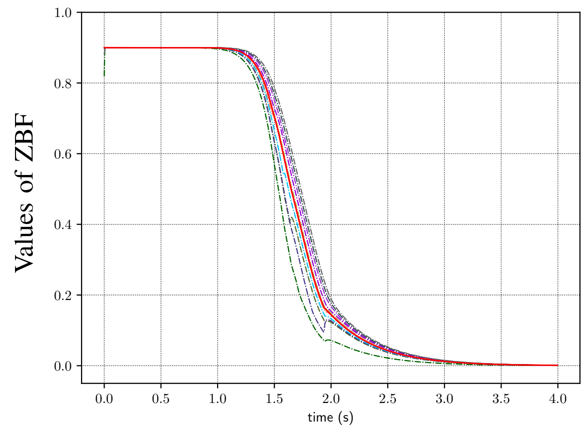

Fig. 3: Comparison of true ZBF value and approximate ZBF values under CCBF with $\gamma = 5$, $k_L = 20$, $q = 0.1$. The solid red line shows the true ZBF value $B(x(t))$, and each dash-dotted line shows an approximate ZBF value $\Phi(ny_i(t))$ of $i \in R$.

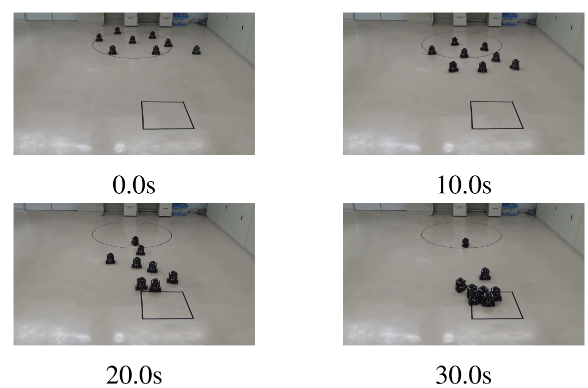

Fig. 4: State of swarm in experiment on CCBF with eight robots. A red circle shows a area where at least one robot needs to stay; and a black square represents a destination of robots.

Figure 2 shows a numerical simulation of CCBF for maintaining the network between the swarm and the base station. All robots in the swarm head for their destination, represented by the black circle. Then, one robot stays in $\mathcal{C}_i$ represented by a red circle, and the others reach their destination. This means that CCBF maintains a network between the swarm and the base station while allowing the swarm to move as much as possible.

B. Constraint on Probability

Now let us consider a more complex communication model. Here, we assume that $p(x_i)$ is the success rate of communication between robot i and the base station. The safety set $\mathcal{C}$ is the set of states in which the success rate of communication between the swarm and the base station is not less than $q > 0$.

$$\mathcal{C} = \{x \in X | (1 - \Pi_{i \in R}(1 - p(x_i))) \geq q\} \tag{22}$$

The ZBF for $\mathcal{C}$ is as follows:

$$B(x) = (1 - \Pi_{i \in R}(1 - p(x_i))) - q \tag{23}$$

This function is represented by

$$\Phi(c) = 1 - e^c - q \tag{24}$$

$$\phi_i(x_i) = \log(1 - p(x_i)). \tag{25}$$

This CCBF restricts the swarm's state to the safety set because of Theorem 2.

Figure 3 shows a numerical simulation of CCBF for the success rate of communication between the swarm and the base station. The settings of the simulation are the same as in Fig.2 except for the value of k_L. In Fig. 3, the approximate values $\Phi(ny_i)$ for each $i \in R$ track the true ZBF value $B(x)$. It shows that the consensus filter in our method is useful to approximate true ZBF value. Moreover, $\min_{i \in R} \Phi(ny_i(t)) \leq B(x(t))$ holds for all $t \geq 0$. This is an important property in the proof of Theorem 2. This CCBF satisfies $B(x(t)) \geq \min_{i \in R} \Phi(ny_i(t)) \geq 0$ for $t \geq 0$, that is, it restricts the swarm's state to the safety set.

VI. Experimental Results

We conducted an experiment on a robot operating system (ROS) and Turtlebot3 Burger. Turtlebot3 Burger is a non-holonomic robot, and the control input of robot $i \in R$ represents $u_i = (u_{v_i}, u_{w_i})^T \in \mathbf{R}^2$, where $(v_i, w_i)^T = u_i$, and v_i and w_i are the velocity and angular velocity of robot i, respectively. In addition, $u_{v_i} \in [0.0, 0.1]$ and $u_{w_i} \in [-1.0, 1.0]$ hold.

Figure 4 shows the swarm state with eight robots. Note that the experiment is described in Section V-A. The robots in the red circle can communicate with the base station. All robots head for their destination represented by the black square.

Let o, x_i be positions of destination and robot i, respectively. Moreover, we define a set of safety transfer vectors $\mathcal{W}$ and a set of velocities $\mathcal{W}_v$ as follows: any $z_i \in \mathcal{W}$ satisfies

$$\sum_{m \in M} \nabla \Phi_m(ny_{im})^T \frac{\partial \phi_{im}}{\partial x_i}(x_i)^T z_i \geq -\gamma \sum_{m \in M} \Phi_m(ny_{im}) \tag{26}$$

and any $u_{v_i} \in \mathcal{W}_v$ satisfies

$$\begin{aligned} &\sum_{m \in M} \nabla \Phi_m(ny_{im})^T \frac{\partial \phi_{im}}{\partial x_i}(x_i)^T \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{bmatrix} u_{v_i} \\ &\geq -\gamma \sum_{m \in M} \Phi_m(ny_{im}), \end{aligned} \tag{27}$$

where θ_i is the orientation of robot i. We calculated the control input as follows:

$$\begin{aligned} k_i(x_i) &= k_u(o - x_i) && (28)\\ z_i &\in \arg\min_{z_i' \in \mathcal{W}} ||z_i' - k_i(x_i)|| && (29)\\ u_{v_i}^* &= [\cos(\theta_i)\, sin(\theta_i)]\, z_i && (30)\\ u_{w_i} &= -k_w \angle \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} z_i && (31)\\ u_{v_i} &\in \arg\min_{u_{v_i}' \in \mathcal{W}_v} \left|\left| u_{v_i}' - u_{v_i}^* \right|\right|, && (32) \end{aligned}$$

where $\angle abc\,(a, b, c \in \mathbf{R}^2)$ represents an angle in a circular measure formed by vector $a - b$ and $c - b$, and $k_u, k_w > 0$ is the gain. Each robot communicates with three other robots, that is, $|\mathcal{N}_i| = 3\,(\forall i \in R)$. In Figure 4, one robot stays in the red circle; this means CCBF restricted the eight robots to the safety set with only local communication.

VII. Conclusion

We proposed a consensus-based control barrier function (CCBF) for a swarm. CCBF can restrict the states of a whole distributed system to a safety set with only local communication in the system. Our method restricts not only the states of the individual robots; it also restricts the state of the whole swarm to a safety set. In CCBF, the value of a barrier function is approximated by a consensus filter. We proved that CCBF provides the constraint of control inputs for holding forward invariance of the safety set. We also applied CCBF to the practical problem and conducted the experiment with actual robots. The results showed that CCBF can restrict the states of the multiple robots to a safety set.

In this paper, we showed how to calculate the inequality constraints on the basis of a CBF for the distributed system. However, we did not discuss whether there exists a control input $u \in U$ that satisfies the inequality constraints on the basis of CCBF. Thus, one possible direction for future work is to study a class of consensus-based ZBFs which satisfy the following condition: for all $(x, y) \in (C, \mathcal{B})$, there exists $u \in U$ such that $\dot{\Phi}(ny_i) \geq -\gamma \Phi(ny_i)$ holds.

Appendix

A. Proof of Theorem 1

The inequality,

$$\sum_{i \in R} \frac{1}{n} \Phi_m(ny_{im}) \leq \Phi_m(\sum_{i \in R} y_{im}) = \Phi_m(\sum_{i \in R} \phi_{im}(x_i))$$

holds because $\Phi_m\,(m \in M)$ is concave and because of the properties of the consensus filter $\sum_{i \in R} y_{im} = \sum_{i \in R} \phi_{im}(x_i)$. Let $V(y) = \sum_{m \in M} \sum_{i \in R} \Phi_m(ny_{im})$. Obviously, $V(y) \leq nB(x)$ holds. The time derivative of V in CCBF (15) and (16) is

$$\begin{aligned} \dot{V}(y) &= \sum_{m \in M} \sum_{i \in R} \nabla \Phi_m(ny_{im})^T n\dot{y}_{im} \\ &= -nk_L \sum_{m \in M} \sum_{i \in R} \sum_{j \in \mathcal{N}_i} \nabla \Phi_m(ny_{im})^T (y_{im} - y_{jm}) \\ &\quad + n \sum_{i \in R} \sum_{m \in M} \nabla \Phi_m(ny_{im})^T \dot{\phi}_{jm}(x_i) \\ &\geq -nk_L \sum_{m \in M} \sum_{i \in R} \sum_{j \in \mathcal{N}_i} \nabla \Phi_m(ny_{im})^T (y_{im} - y_{jm}) \\ &\quad - n\gamma V(y). \end{aligned}$$

The following condition holds for the first term.

$$\begin{aligned} W_m &= \sum_{i \in R} \sum_{j \in \mathcal{N}_i} \nabla \Phi_m(ny_{im})^T (y_{im} - y_{jm}) \\ &= \sum_{(i,j) \in E} \frac{1}{2} (\nabla \Phi_m(ny_{im}) - \nabla \Phi_m(ny_{jm}))^T (y_{im} - y_{jm}) \end{aligned}$$

Moreover, $(\nabla \Phi_m(a) - \nabla \Phi_m(b))^T (a - b) \leq 0$ holds because Φ_m is concave. Thus, $\dot{V}(y) \geq -n\gamma V(y)$ holds.

If $(x(0), y(0)) \in \mathcal{C} \times \mathcal{B}$ holds then $B(x(t)) \geq \frac{1}{n} V(y(t)) \geq 0$ holds for all $t \in [0, \infty)$. Hence, $\mathcal{C} \times \mathcal{B}$ is forward invariant.

B. Proof of Theorem 2

We will show that $\dot{\Phi}(ny_i) \geq -n\gamma \Phi(ny_i)$ holds for all $i \in \arg\min_{j \in R} \Phi(ny_j)$. Let $i \in \arg\min_{j \in R} \Phi(ny_j)$. The time derivative of Φ in CCBF (15) and (16) is

$$\begin{aligned} \dot{\Phi}(ny_i) &= \nabla \Phi(ny_i)^T n\dot{y}_i \\ &= \nabla \Phi(ny_i)^T nk_L \sum_{j \in \mathcal{N}_i} (y_j - y_i) + \nabla \Phi(ny_i)^T n\dot{\phi}_i(x_i). \end{aligned}$$

The first term is larger than 0 because 1) if Φ is monotonically increasing, then $\nabla \Phi(ny_i) \geq 0$ holds and $y_j \geq y_i$ holds for all $j \in R$, or 2) if Φ is monotonically decreasing, then $\nabla \Phi(ny_i) \leq 0$ holds and $y_j \leq y_i$ holds for all $j \in R$. Thus, the following condition holds.

$$\dot{\Phi}(ny_i) \geq n \nabla \Phi(ny_i)^T \dot{\phi}_i(x_i) \geq -n\gamma \Phi(ny_i)$$

If $y(0) \in \mathcal{B}_+$, then $\min_{i \in R} \Phi(ny_i(t)) \geq 0$ holds for all $t \in [0, \infty)$. $\min_{i \in R} \Phi(ny_i) \leq \Phi(\sum_{i \in R} \phi_i(x_i))$ holds because Φ is monotonic and there exists $j, k \in R$ such that $ny_j \leq \sum_{i \in R} \phi_i(x_i) \leq ny_k$ holds. Hence, if $(x(0), y(0)) \in \mathcal{C} \times \mathcal{B}_+$, then $\Phi(\sum_{i \in R} \phi_i(x_i(t))) \geq 0$ holds for all $t \in [0, \infty)$.

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