

# The Alpha Ceiling and Twin Prime Distribution

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## Abstract

We investigate a geometric formulation of prime distribution based on prime triangles and their associated alpha angle. A proved inequality shows that a non-twin prime triangle can exceed the alpha angle of a twin only after a computable threshold, which for the minimal gap occurs beyond  $2p_n$ . We call this bound the alpha ceiling. Thus, any violation requires a prime beyond the threshold. Computations up to  $5 \times 10^9$  show no such violations: the ceiling remains unbroken. We conclude with two conjectures: that twin primes set local  $\alpha$ -ceilings, and that the gap between consecutive twin primes never exceeds the smaller twin—yielding a simple uniform bound on twin-prime gaps.

## 1. Definitions

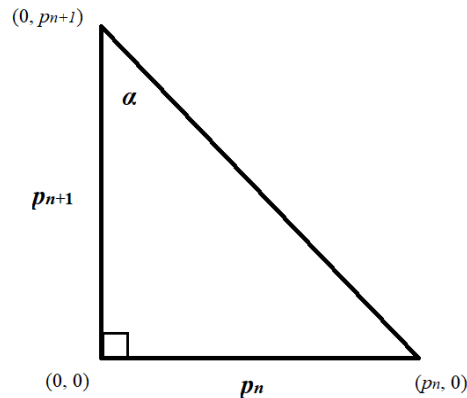
**Consecutive Primes:** Two primes  $(p_n, p_{n+1})$  with no primes between them.

**Twin Primes:** Consecutive primes with gap 2, i.e.  $(p_n, p_n + 2)$ .

**Prime Triangle:** For consecutive primes  $(p_n, p_{n+1})$ , define the right triangle with vertices  $(0, 0)$ ,  $(p_n, 0)$ ,  $(0, p_{n+1})$ . As shown in Figure 1.

**Alpha Angle ( $\alpha$ ):** The angle at  $(0, p_{n+1})$ , opposite the leg of length  $p_n$  of a prime triangle:  $\alpha = \arctan(p_n / p_{n+1})$ . As shown in Figure 1.

**Figure 1:** The Prime Triangle associated with consecutive primes  $(p_n, p_{n+1})$ .



## 2. Lemma: The Alpha Ceiling Inequality

### Lemma.

Let  $(p_n, p_n + 2)$  be a twin prime pair. For any consecutive prime pair  $(p_k, p_k + g_k)$  after it, if  $\alpha_k > \alpha_n$ , then

$$p_k > (p_n)(g_k)/2.$$

### Corollary (Contrapositive).

Let  $(p_n, p_{n+1})$  be a twin prime pair and suppose  $p_n < p_k < 2p_n$ . Then the consecutive-prime angle satisfies

$$\alpha_k \leq \alpha_n.$$

In words: any non-twin consecutive prime pair occurring strictly between a twin  $p_n$  and its double cannot exceed the  $\alpha$ -ceiling set by that twin.

### Proof.

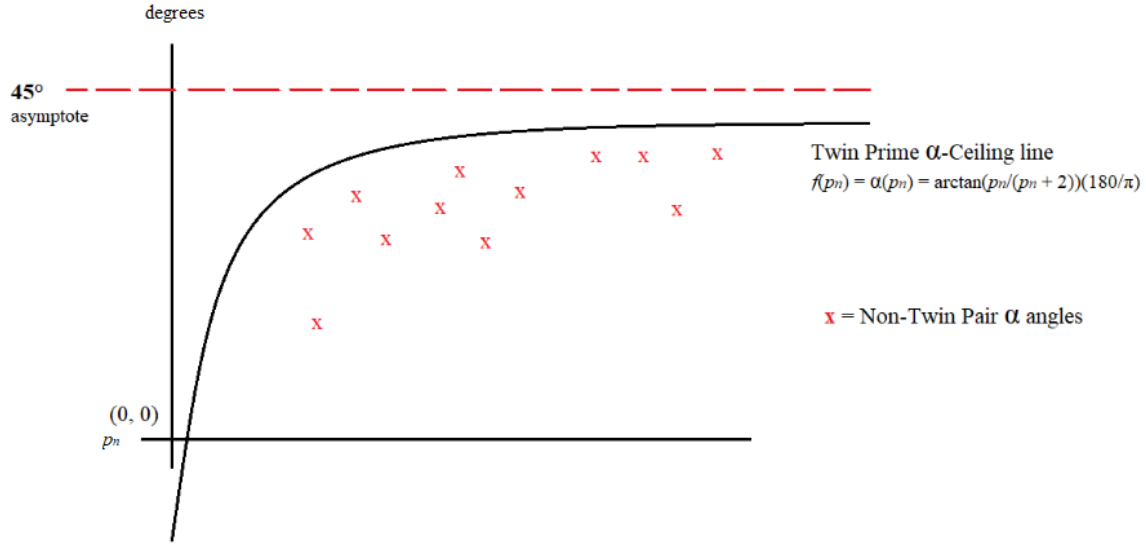
Since arctan is increasing,  $\alpha_k > \alpha_n$  if  $(p_k / (p_k + g_k)) > (p_n / (p_n + 2))$ . Cross-multiplying and simplifying yields  $2p_k > (p_n)(g_k)$ , or equivalently  $p_k > (p_n)(g_k)/2$ .

Figure 2 illustrates how the twin prime  $\alpha$  sets the ceiling.

### Examples:

- If  $g_k = 4$ , violation requires  $p_k > 2p_n$ .
- If  $g_k = 6$ , violation requires  $p_k > 3p_n$ .
- Larger gaps push the threshold higher.

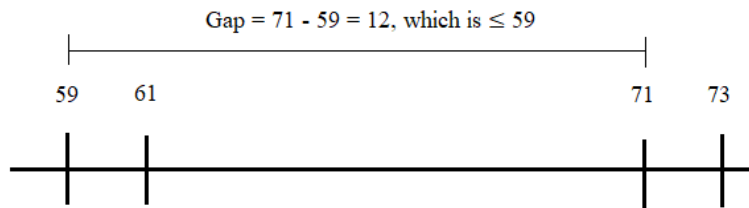
**Figure 2:** The  $\alpha$ -ceiling: twin prime  $\alpha$ 's set the maximum until a threshold is passed



### 3. Computational Evidence and Numeric Implications

The violation thresholds from the  $\alpha$ -ceiling lemma suggest a natural bound on gaps between consecutive twin primes. Specifically, for any twin prime pair  $(p_n, p_n + 2)$ , the earliest possible  $\alpha$ -ceiling violation for a non-twin gap  $g > 2$  occurs at  $p_k > (p_n)(g)/2$ . Applying this pattern to the minimal gap  $g = 2$  implies that the next twin prime should occur **before a gap exceeding  $p_n$** . Figure 3 illustrates the twin prime gap and condition.

**Figure 3:** Example of the twin-to-twin gap condition,  $p_m - p_n \leq p_n$ .



Computations up to  $5 \times 10^9$  confirm the following:

- No  $\alpha$ -ceiling violations were observed.
- For all consecutive twin prime pairs  $(p_n, p_n + 2)$  and  $(p_m, p_m + 2)$ , the gap  $p_m - p_n$  does not exceed  $p_n$ .

These results support the numeric consequence suggested by the  $\alpha$ -ceiling framework.

## 4. Conjectures

### Conjecture 1 – Twin Prime $\alpha$ -Ceiling Conjecture

For every twin prime pair  $(p_n, p_n + 2)$  and the next twin pair  $(p_m, p_m + 2)$ , all consecutive primes  $(p_k, p_{k+1})$  with  $n < k < m$  satisfy

$$\alpha_k \leq \alpha_n.$$

Equivalently, twin primes set local  $\alpha$ -ceilings that remain unbroken until the next twin.

### Logical gap from Lemma to Conjecture 1.

The lemma (and its corollary) prove only a local restriction: no consecutive prime pair with base less than  $2p_n$  can exceed  $\alpha_n$ . Conjecture 1 should therefore be read as a conjectural extrapolation of the lemma, supported by extensive computation but not deduced from it.

### Conjecture 2 – Twin Prime Gap Bound Conjecture

For any twin prime pair  $(p_n, p_n + 2)$  with  $p_n > 5$ , let  $(p_m, p_m + 2)$  be the next twin prime pair. Then

$$p_m - p_n \leq p_n.$$

Equivalently, the distance between consecutive twin primes never exceeds the smaller twin's first prime for  $p_n > 5$ .

This conjecture asserts a uniform bound on twin-prime gaps, tighter than any bound currently known.

## 5. Discussion

- The  $\alpha$ -ceiling lemma provides a clear inequality: violation requires  $p_k > (p_n)(g_k)/2$ .
- Applying this pattern to the minimal gap  $g = 2$  leads naturally to the bound expressed in Conjecture 2.
- While these conjectures do not prove the infinitude of twin primes, they offer a geometric and numeric framework connecting prime triangles,  $\alpha$ -angles, and twin-prime distribution.
- Computations up to  $5 \times 10^9$  support both conjectures:  $\alpha$ -ceilings remain unbroken, and twin-prime gaps never exceed the previous twin's first prime.

## References

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