The Alpha Ceiling and Twin Prime Distribution

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Abstract

We investigate a geometric formulation of prime distribution based on prime triangles and their associated alpha angle. A central inequality shows that between successive twin prime pairs, the alpha angle of any intervening non-twin prime triangle cannot exceed the alpha angle of the preceding twin. We call this bound the "alpha ceiling". Any violation would require a prime beyond a computable threshold. Bertrand's theorem guarantees the existence of a prime before this threshold, giving the primes an "opportunity" not to break the alpha ceiling. Computations up to 5 x 10^9 support that this opportunity is always realized. We conclude with a conjecture that twin primes set local α-ceilings throughout the prime sequence.

1. Definitions

Consecutive Primes: Two primes (p_n, p_{n+1}) with no primes between them.

Twin Primes: Consecutive primes with gap 2, i.e. $(p_n, p_n + 2)$.

Prime Triangle: For consecutive primes (p_n, p_{n+1}), define the right triangle with vertices (0, 0), $(p_n, 0)$, $(0, p_{n+1})$.

Alpha Angle (α): The angle at $(0, p_{n+1})$, opposite the leg of length p_n: $\alpha = \arctan(p_n / p_{n+1})$.

2. Lemma: The Alpha Ceiling Inequality

Lemma.
Let $(p_n, p_n + 2)$ be a twin prime pair. For any consecutive prime pair $(p_k, p_k + g_k)$ after it, if $\alpha_k > \alpha_n$, then $p_k > (p_n)(g_k)/2$.

Proof.
Since arctan is increasing, $\alpha_k > \alpha_n$ if $(p_k / (p_k + g_k)) > (p_n / (p_n + 2))$. Cross-multiplying and simplifying yields $2p_k > (p_n)(g_k)$, or equivalently $p_k > (p_n)(g_k)/2$.

Examples:

- If $g_k = 4$, violation requires $p_k > 2p_n$.

• If $g_k = 6$, violation requires $p_k > 3p_n$.

• Larger gaps push the threshold higher.

3. Remark: Bertrand's Buffer Zone

By Bertrand–Chebyshev's Theorem, for any $n > 1$ there exists a prime between n and $2n$. In particular, there is always a prime between p_n (the base of a twin prime pair) and $2p_n$.

Since the earliest possible α-ceiling violation for non-twin gaps $g \geq 4$ occurs only at $p_k > 2p_n$, Bertrand's theorem guarantees at least one intervening prime before violation is possible.

This gives the primes an "opportunity" not to break the ceiling: between twins, there must exist at least one prime below the violation threshold. Empirically, this opportunity always results in alpha remaining below the ceiling until the next twin.

4. Computational Evidence

Computations up to $5 \text{ x } 10^9$ confirm the following:

• No α-ceiling violations were observed.

• For every consecutive prime pair (p_k, p_{k+1}) between twin primes,

$\alpha_k \leq \alpha_n$, i.e., the local α-ceiling remains unbroken.

Only twin primes set new α-records in all checked ranges.

These results provide strong empirical support for the Twin Prime α-Ceiling Conjecture, demonstrating that the ceiling holds at least up to one billion.

5. Conjecture

Twin Prime α-Ceiling Conjecture.

For every twin prime pair $(p_n, p_n + 2)$ and the next twin pair $(p_m, p_m + 2)$, all consecutive primes (p_k, p_{k+1}) with $n < k < m$ satisfy $\alpha_k \leq \alpha_n$.

Equivalently, twin primes set local α-ceilings that remain unbroken until the next twin.

6. Discussion

• The Lemma provides a clean inequality: violation requires $p_k > (p_n)(g)/2$.

• Bertrand’s theorem ensures that before this threshold is reached, at least one prime must appear — giving the structure a buffer zone.

• Computations show that this buffer is always sufficient: no violation occurs before the next twin.

Thus, while not a proof of the infinitude of twin primes, this framework offers a novel geometric perspective and a rigorous necessary condition for any potential counterexample.

References

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