The Alpha Ceiling and Twin Prime Distribution

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Abstract

We investigate a geometric formulation of prime distribution based on prime triangles and their associated alpha angle. A proved inequality shows that a non-twin prime triangle can exceed the alpha angle of a twin only after a computable threshold, which for the minimal gap occurs beyond $2p_n$. We call this bound the alpha ceiling. Thus, any violation requires a prime beyond the threshold. Computations up to 5×10^9 show no such violations: the ceiling remains unbroken. We conclude with two conjectures: that twin primes set local α-ceilings, and that the gap between consecutive twin primes never exceeds the smaller twin—yielding a simple uniform bound on twin-prime gaps.

1. Definitions

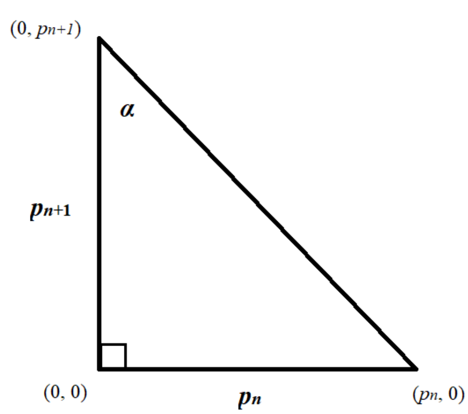
Consecutive Primes: Two primes (p_n, p_{n+1}) with no primes between them.

Twin Primes: Consecutive primes with gap 2, i.e. $(p_n, p_n + 2)$.

Prime Triangle: For consecutive primes (p_n, p_{n+1}), define the right triangle with vertices (0, 0), $(p_n, 0)$, $(0, p_{n+1})$. As shown in Figure 1.

Alpha Angle (α): The angle at $(0, p_{n+1})$, opposite the leg of length p_n of a prime triangle: $\alpha = \arctan(p_n / p_{n+1})$. As shown in Figure 1.

Figure 1: The Prime Triangle associated with consecutive primes (p_n, p_{n+1}).



2. Lemma: The Alpha Ceiling Inequality

Lemma.

Let $(p_n, p_n + 2)$ be a twin prime pair. For any consecutive prime pair $(p_k, p_k + g_k)$ after it, if $\alpha_k > \alpha_n$, then

$$p_k > (p_n)(g_k)/2.$$

Corollary (Contrapositive).

Let (p_n, p_{n+1}) be a twin prime pair and suppose $p_n < p_k < 2p_n$. Then the consecutive-prime angle satisfies

$$\alpha_k \leq \alpha_n.$$

In words: any non-twin consecutive prime pair occurring strictly between a twin p_n and its double cannot exceed the α-ceiling set by that twin.

Proof.

Since arctan is increasing, $\alpha_k > \alpha_n$ if $(p_k / (p_k + g_k)) > (p_n / (p_n + 2))$. Cross-multiplying and simplifying yields $2p_k > (p_n)(g_k)$, or equivalently $p_k > (p_n)(g_k)/2$.

Figure 2 illustrates how the twin prime α sets the ceiling.

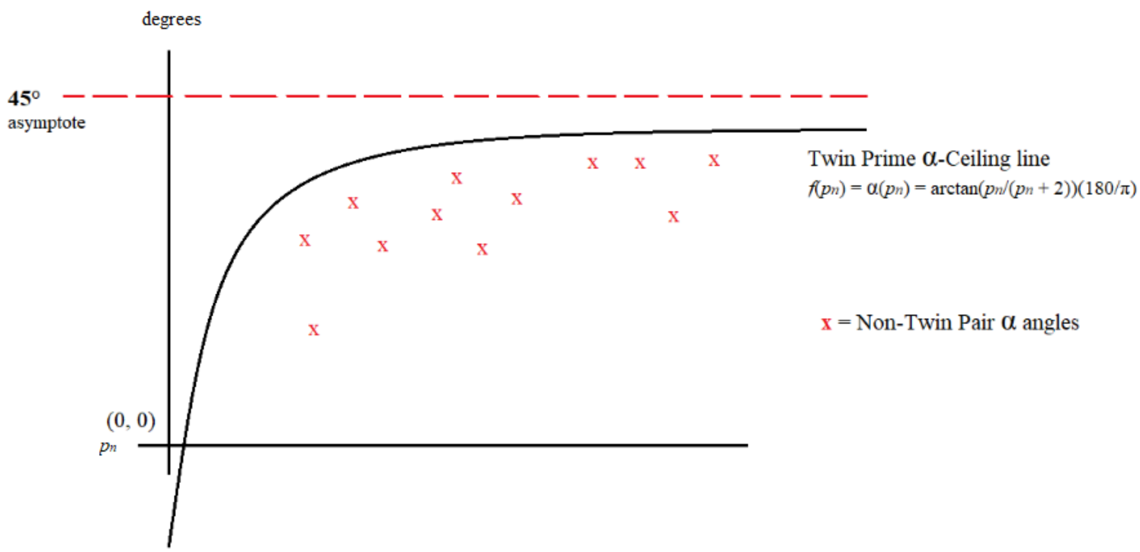
Examples:

• If $g_k = 4$, violation requires $p_k > 2p_n$.

• If $g_k = 6$, violation requires $p_k > 3p_n$.

• Larger gaps push the threshold higher.

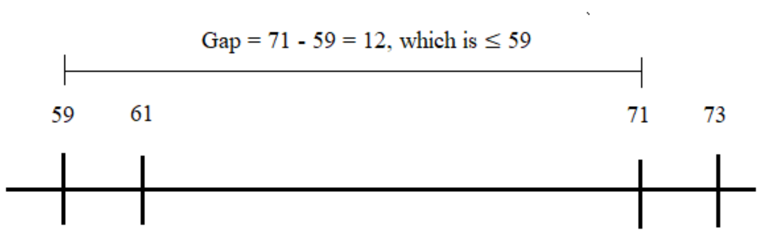
Figure 2: The α-ceiling: twin prime α's set the maximum until a threshold is passed



3. Computational Evidence and Numeric Implications

The violation thresholds from the α-ceiling lemma suggest a natural bound on gaps between consecutive twin primes. Specifically, for any twin prime pair $(p_n, p_n + 2)$, the earliest possible α-ceiling violation for a non-twin gap $g \geq 2$ occurs at $p_k > (p_n)(g)/2$. Applying this pattern to the minimal gap $g = 2$ implies that the next twin prime should occur **before a gap exceeding p_n**. Figure 3 illustrates the twin prime gap and condition.

Figure 3: Example of the twin-to-twin gap condition, $p_m - p_n \leq p_n$.



Computations up to 5×10^9 confirm the following:

- No α-ceiling violations were observed.
- For all consecutive twin prime pairs $(p_n, p_n + 2)$ and $(p_m, p_m + 2)$, the gap $p_m - p_n$ does not exceed p_n.

These results support the numeric consequence suggested by the α-ceiling framework.

4. Conjectures

Conjecture 1 − Twin Prime α-Ceiling Conjecture

For every twin prime pair $(p_n, p_n + 2)$ and the next twin pair $(p_m, p_m + 2)$, all consecutive primes (p_k, p_{k+1}) with $n < k < m$ satisfy

$$\alpha_k \le \alpha_n.$$

Equivalently, twin primes set local α-ceilings that remain unbroken until the next twin.

Logical gap from Lemma to Conjecture 1.

The lemma (and its corollary) prove only a local restriction: no consecutive prime pair with base less than $2p_n$ can exceed α_n. Conjecture 1 should therefore be read as a conjectural extrapolation of the lemma, supported by extensive computation but not deduced from it.

Conjecture 2 − Twin Prime Gap Bound Conjecture

For any twin prime pair $(p_n, p_n + 2)$ with $p_n > 5$, let $(p_m, p_m + 2)$ be the next twin prime pair. Then

$$p_m - p_n \le p_n.$$

Equivalently, the distance between consecutive twin primes never exceeds the smaller twin's first prime for $p_n > 5$.

This conjecture asserts a uniform bound on twin-prime gaps, tighter than any bound currently known.

5. Discussion

• The α-ceiling lemma provides a clear inequality: violation requires $p_k > (p_n)(g_k)/2$.

• Applying this pattern to the minimal gap $g = 2$ leads naturally to the bound expressed in Conjecture 2.

• While these conjectures do not prove the infinitude of twin primes, they offer a geometric and numeric framework connecting prime triangles, α-angles, and twin-prime distribution.

• Computations up to 5×10^9 support both conjectures: α-ceilings remain unbroken, and twin-prime gaps never exceed the previous twin's first prime.

References

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