The Alpha Ceiling and Twin Prime Distribution

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Abstract

We investigate a geometric formulation of prime distribution based on prime triangles and their associated alpha angle. A central inequality shows that between successive twin prime pairs, the alpha angle of any intervening non-twin prime triangle cannot exceed the alpha angle of the preceding twin. We call this bound the "alpha ceiling". Any violation would require a prime beyond a computable threshold. Bertrand's Theorem guarantees the existence of a prime before this threshold, giving the primes an "opportunity" not to break the alpha ceiling. Computations up to 5 x 10^9 support that this opportunity is always realized. We conclude with conjectures that twin primes set local α-ceilings and bounds on twin-prime gaps throughout the prime sequence.

1. Definitions

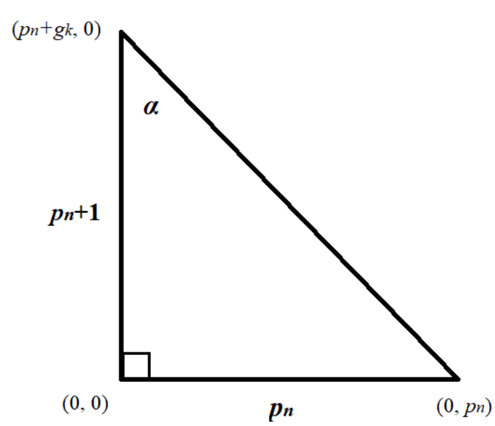
Consecutive Primes: Two primes (p_n, p_{n+1}) with no primes between them.

Twin Primes: Consecutive primes with gap 2, i.e. $(p_n, p_n + 2)$.

Prime Triangle: For consecutive primes (p_n, p_{n+1}), define the right triangle with vertices (0, 0), $(p_n, 0)$, $(0, p_{n+1})$. As shown in Figure 1.

Alpha Angle (α): The angle at $(0, p_{n+1})$, opposite the leg of length p_n of a prime triangle: $\alpha = \arctan(p_n / p_{n+1})$. As shown in Figure 1.

Figure 1: The Prime Triangle associated with consecutive primes (p_n, p_{n+1}).



2. Lemma: The Alpha Ceiling Inequality

Lemma.

Let $(p_n, p_n + 2)$ be a twin prime pair. For any consecutive prime pair $(p_k, p_k + g_k)$ after it, if $\alpha_k > \alpha_n$, then

$$p_k > (p_n)(g_k)/2.$$

Proof.

Since arctan is increasing, $\alpha_k > \alpha_n$ if $(p_k / (p_k + g_k)) > (p_n / (p_n + 2))$. Cross-multiplying and simplifying yields $2p_k > (p_n)(g_k)$, or equivalently $p_k > (p_n)(g_k)/2$.

Figure 2 illustrates how the twin prime α sets the ceiling.

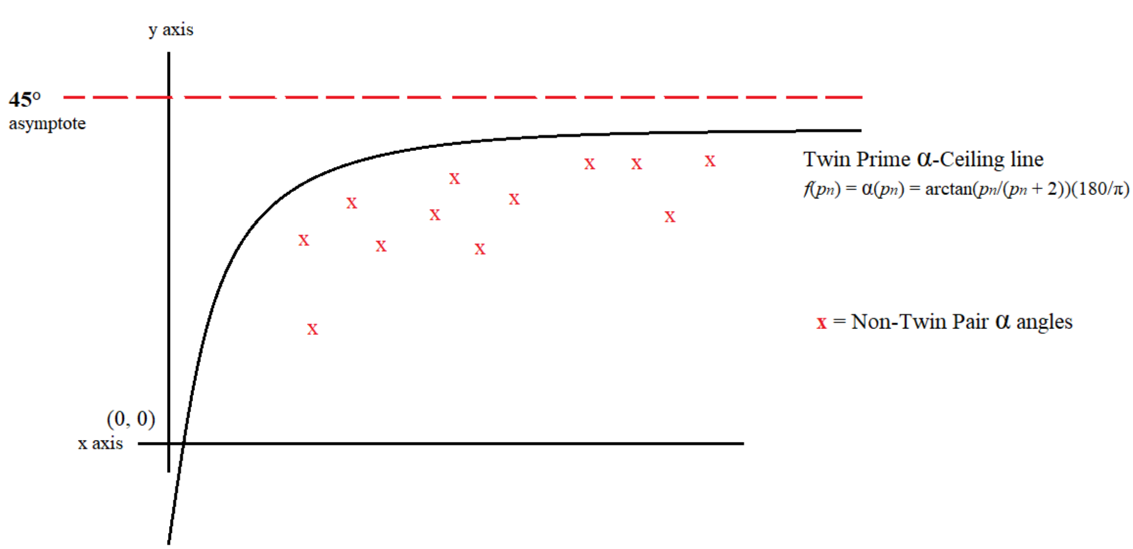
Examples:

• If $g_k = 4$, violation requires $p_k > 2p_n$.

• If $g_k = 6$, violation requires $p_k > 3p_n$.

• Larger gaps push the threshold higher.

Figure 2: Alpha Ceiling Inequality Visualization



3. Remark: Bertrand's Theorem

By Bertrand's Theorem, for any $n > 1$ there exists a prime between n and $2n$. In particular, there is always a prime between p_n (the base of a twin prime pair) and $2p_n$.

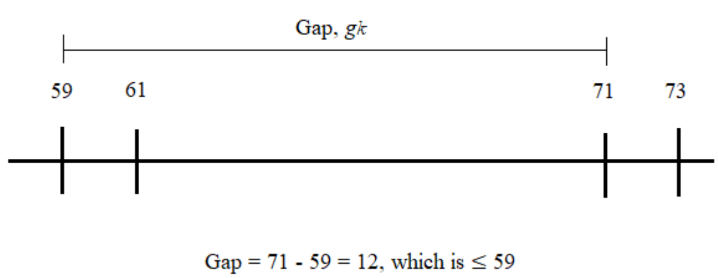
Since the earliest possible α-ceiling violation for non-twin gaps $g \geq 4$ occurs at $p_k > 2p_n$, Bertrand's theorem guarantees at least one intervening prime before violation is possible.

This gives the primes an "opportunity" not to break the ceiling. Empirically, this opportunity, when realized, always results in alpha remaining below the ceiling until the next twin.

4. Computational Evidence and Numeric Implications

The violation thresholds from the α-ceiling lemma suggest a natural bound on gaps between consecutive twin primes. Specifically, for any twin prime pair $(p_n, p_n + 2)$, the earliest possible α-ceiling violation for a non-twin gap $g \geq 2$ occurs at $p_k > (p_n)(g)/2$. Applying this pattern to the minimal gap $g = 2$ implies that the next twin prime should occur **before a gap exceeding p_n**. Figure 3 illustrates the twin prime gap and condition.

Figure 3: Example of the twin-to-twin gap condition, $p_m - p_n \leq p_n$.



Computations up to 5×10^9 confirm the following:

- No α-ceiling violations were observed.
- For all consecutive twin prime pairs $(p_n, p_n + 2)$ and $(p_m, p_m + 2)$, the gap $p_m - p_n$ does not exceed p_n.

These results support the numeric consequence suggested by the α-ceiling framework.

5. Conjectures

Conjecture 1 − Twin Prime α-Ceiling Conjecture

For every twin prime pair $(p_n, p_n + 2)$ and the next twin pair $(p_m, p_m + 2)$, all consecutive primes (p_k, p_{k+1}) with $n < k < m$ satisfy

$$\alpha_k \le \alpha_n.$$

Equivalently, twin primes set local α-ceilings that remain unbroken until the next twin.

Conjecture 2 − Twin Prime Gap Bound Conjecture

For any twin prime pair $(p_n, p_n + 2)$ with $p_n > 5$, let $(p_m, p_m + 2)$ be the next twin prime pair. Then

$$p_m - p_n \le p_n \ .$$

Equivalently, the distance between consecutive twin primes never exceeds the smaller twin's first prime for $p_n > 5$.

6. Discussion

• The α-ceiling lemma provides a clear inequality: violation requires $pk > (pn)(g)/2$.

• Bertrand's Theorem ensures that a prime always appears before extreme violation thresholds, providing an "opportunity" to preserve the α-ceiling.

• Applying this pattern to the minimal gap $g = 2$ leads naturally to the bound expressed in Conjecture 2.

• While these conjectures do not prove the infinitude of twin primes, they offer a **geometric and numeric framework** connecting prime triangles, α-angles, and twin-prime distribution.

• Computations up to 5×10^9 support both conjectures: α-ceilings remain unbroken, and twin-prime gaps never exceed the previous twin's first prime.

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