Prime Geometry I:
The Prime Triangle Identity and Derived Geometric Quantities

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Abstract

This note presents an elementary identity satisfied by three consecutive primes and develops a set of geometric quantities derived from it: an energy term, a discrete curvature, a normalized curvature ratio, and a square-difference quantity (PSDn). These definitions form the foundation for the empirical work of PG2 and the structural analysis of PG3. The purpose here is expository: to introduce the constructions cleanly and consistently.

1 Introduction

Let (p_n) denote the sequence of primes. For three consecutive primes $p_n < p_{n+1} < p_{n+2}$, define the right–triangle hypotenuse lengths

$$C_1 = \sqrt{(p_n)^2 + (p_{n+1})^2}, \qquad C_2 = \sqrt{(p_{n+1})^2 + (p_{n+2})^2}.$$

These "Prime Triangles" satisfy a simple algebraic identity relating the difference and the sum of C_1 and C_2 to the skip-one square difference $(p_{n+2})^2 - (p_n)^2$. The identity motivates several derived quantities used in later notes.

2 Prime Triangle Identity

A direct computation gives the exact identity

$$(C_2 - C_1)(C_1 + C_2) = (p_{n+2})^2 - (p_n)^2. \tag{1}$$

Let the consecutive gaps be

$$g_n = p_{n+1} - p_n, \qquad g_{n+1} = p_{n+2} - p_{n+1},$$

and define the skip-one gap

$$G_n = g_n + g_{n+1} = p_{n+2} - p_n.$$

Since

$$(p_{n+2})^2 - (p_n)^2 = (p_{n+2} - p_n)(p_{n+2} + p_n) = G_n(2p_n + G_n),$$

identity (1) becomes

$$(C_2 - C_1)(C_1 + C_2) = G_n(2p_n + G_n).$$

3 Energy

Define the (geometric) energy transition between consecutive Prime Triangles:

$$E_n := C_2 - C_1.$$

Using expansions with $g_n, g_{n+1} \ll p_n$,

$$E_n \approx \frac{\sqrt{2}}{2}(g_n + g_{n+1}) = \frac{\sqrt{2}}{2}G_n.$$

A refinement yields

$$\frac{E_n}{G_n} \approx \frac{\sqrt{2}}{2} - \frac{g_n - g_{n+1}}{2\sqrt{2}(2p_n + G_n)}.$$

These are simply first-order approximations for later reference.

4 The PSD Quantity

Because identity (1) rewrites the skip-one square difference in terms of C_1 and C_2, it is convenient to define the normalized *Prime Square-Difference quantity (PSDn)*:

$$\text{PSDn} := \frac{(C_2 - C_1)(C_1 + C_2)}{12}.$$

The factor 12 is chosen so that subsequent expressions simplify. From the algebra above,

$$\text{PSDn} = \frac{G_n(2p_n + G_n)}{12}.$$

A first-order expansion gives

$$\text{PSDn} \approx \frac{p_n G_n}{6},$$

hence

$$\frac{\text{PSDn}}{p_n G_n} = \frac{1}{6} + \frac{G_n}{12p_n}.$$

Nothing further is assumed about PSDn here; it is simply a convenient normalization of the square jump.

5 Discrete Curvature of the Energy

Define the discrete curvature (second difference of the energy):

$$K_n := E_{n+1} - E_n.$$

Using the earlier approximations,

$$E_n \approx \frac{\sqrt{2}}{2}(g_n + g_{n+1}), \qquad E_{n+1} \approx \frac{\sqrt{2}}{2}(g_{n+1} + g_{n+2}),$$

so

$$K_n \approx \frac{\sqrt{2}}{2}(g_{n+2} - g_n).$$

6 Normalized Shape Curvature

Normalize curvature by energy:

$$\chi_n := \frac{K_n}{E_n}.$$

Substituting the first-order forms yields

$$\chi_n \approx \frac{g_{n+2} - g_n}{g_n + g_{n+1}}.$$

This dimensionless ratio will be the central quantity in PG2 and PG3.

7 Local Curvature Measure

For bookkeeping define

$$L_n := \chi_n^2.$$

A cumulative quantity is then

$$S(N) = \sum_{n=1}^{N-2} L_n.$$

In this note these are merely definitions; PG2 studies their empirical behavior.

8 Summary

Identity (1),

$$(C_2 - C_1)(C_1 + C_2) = (p_{n+2})^2 - (p_n)^2,$$

generates several useful derived quantities:

$$E_n = C_2 - C_1, \qquad K_n = E_{n+1} - E_n, \qquad \chi_n = \frac{K_n}{E_n}, \qquad L_n = \chi_n^2,$$

and the square-difference normalization

$$\text{PSDn} = \frac{(C_2 - C_1)(C_1 + C_2)}{12}.$$

This document introduces these quantities cleanly so that PG2 and PG3 can build on a consistent foundation.