Prime Geometry IV: Curvature Constraints, Coherence Phases, and the Structure of Prime Gaps

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Abstract

Prime Geometry IV develops the structural consequences of the curvature quantity

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}},$$

introduced in PG1 and explored empirically in PG2 and PG3. Where PG2 documented statistical regularities in (χ_n) and PG3 developed the Curvature-Based Recurrence, the present paper extracts the organizing principles suggested by the data.

Across all ranges sampled, the empirical distribution of curvature is sharply concentrated near 0, with a substantial fraction of indices satisfying $|\chi_n| < 0.1$. Cumulative curvature $S(N) = \sum \chi_n^2$ for the true primes lies in the extreme lower tail of the permutation distribution (typically within the lowest 1% and often below 0.01%), and smoothed curvature exhibits long intervals of consistent sign across multiple window sizes.

Using these observations, we identify three themes: (1) curvature constraints on second-order variation in the gaps; (2) coherence phases arising from multi-scale sign persistence; and (3) low-action structure expressing global suppression of curvature extremes. All conclusions are descriptive or algebraic; no theoretical mechanism is proposed.

1 Introduction

Let (p_n) be the primes and $g_n = p_{n+1} - p_n$ the gaps. Prime Geometry I introduced the curvature quantity

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}},$$

along with $L_n = \chi_n^2$. Prime Geometry II showed that the distribution of (χ_n) is tightly concentrated, with significantly lighter tails than permutation or Cramér-type models. Prime Geometry III established the exact recurrence

$$g_{n+2} = g_n + \chi_n(g_n + g_{n+1}),$$

and demonstrated the strong sensitivity of curvature to perturbations in the ordering of gaps.

Prime Geometry IV develops the structural implications of these findings. Empirically, the curvature sequence exhibits several striking features:

- A large proportion of indices satisfy $|\chi_n| < 0.1$, indicating frequent near-symmetry in the gaps.

- Smoothed curvature maintains a consistent sign over long intervals for window sizes $W = 500, 2000, 5000$, and 10000, a phenomenon absent from randomized models.
- The cumulative curvature $S(N)$ for true primes is consistently in the lowest tail of the permutation distribution (often below 0.01%).

These empirical anchors motivate the structural analysis developed in the following sections.

2 Curvature Constraints

The curvature quantity measures normalized second-order variation:

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}.$$

Its empirical distribution is sharply concentrated: in all tested ranges, the majority of values lie inside $|\chi_n| < 0.2$, and a surprisingly large fraction satisfy $|\chi_n| < 0.1$. Extreme values are rare and modest in size.

2.1 The Recurrence as a Variation Constraint

The identity

$$g_{n+2} = g_n + \chi_n(g_n + g_{n+1})$$

shows how curvature constrains the rate at which gaps may diverge. Because $|\chi_n|$ is small most of the time, large accelerations between g_n and g_{n+2} are uncommon. Random permutations, by contrast, frequently produce abrupt reversals with large $|\chi_n|$.

2.2 Approximate Symmetry

Small curvature implies local symmetry:

$$g_{n+2} \approx g_n.$$

Since $|\chi_n| < 0.1$ occurs so frequently, approximate symmetry is common in the true sequence. This helps explain the bounded and tightly clustered return maps of (g_n, g_{n+1}) and (χ_n, χ_{n+1}) observed in PG2.

2.3 Envelopes for Gap Variation

From the recurrence,

$$|g_{n+2} - g_n| \leq |\chi_n|(g_n + g_{n+1}).$$

Given that $|\chi_n|$ rarely exceeds 0.4 in the prime data, the second-order variation is strongly bounded except in rare locations where both curvature and gap size align.

2.4 Curvature as a Smoother

The cumulative curvature

$$S(N) = \sum_{n \leq N-2} \chi_n^2$$

for true primes lies consistently below the 1% percentile of the permutation distribution, and often below 0.01%. This indicates global suppression of curvature extremes and systematic avoidance of high-variation patterns.

2.5 Avoidance of Instability

PG3 showed that local perturbations of the gap sequence often produce large distortions in (χ_n), demonstrating that curvature dynamics are sensitive to ordering. The true sequence avoids these unstable regions, staying near a delicate equilibrium.

2.6 Summary of Curvature Constraints

- Small curvature dominates, enforcing local symmetry in gap values.
- Extreme curvature is strongly suppressed, limiting second-order variation.
- Large gap accelerations require simultaneous large gaps and large curvature, an event that is extremely rare.
- The prime gap sequence minimizes curvature energy relative to nearly all permutations, yielding global smoothness.
- The observed sequence avoids instability-triggering curvature patterns, as confirmed by perturbation experiments.

3 Coherence Phases

Define the smoothed curvature

$$\chi_n^{(W)} = \frac{1}{W} \sum_{k=n-W+1}^{n} \chi_k.$$

Across all tested ranges, $\chi_n^{(W)}$ exhibits extended intervals of predominantly positive, negative, or near-zero sign. We refer to these intervals as **coherence phases**.

3.1 Definition and Interpretation

Coherence phases represent sustained regimes in which curvature consistently favors expansion ($\chi_n^{(W)} > 0$), contraction ($\chi_n^{(W)} < 0$), or equilibrium-like symmetry ($\chi_n^{(W)} \approx 0$). These regimes reveal mesoscopic organization in the gap sequence that is not captured by un-smoothed curvature alone.

3.2 Equilibrium Interpretation

The Prime Triangle angle $\alpha_n = \arctan(p_n/p_{n+1})$ is close to 45° when consecutive primes are comparable in size. Since small curvature implies $g_{n+2} \approx g_n$, it also corresponds to slow drift in these angles. During coherence phases where χ_n is near zero, the angles remain close to their equilibrium orientation, reflecting balanced local behavior in the gaps.

3.3 Persistence Across Scales

For window sizes $W = 500, 2000, 5000,$ and 10000, the sign of $\chi_n^{(W)}$ remains stable over long regions. Coherence persists under coarse-graining: larger windows reveal the same large-scale structure. Permutation models show no comparable persistence; their smoothed curvature oscillates rapidly and lacks coherent organization.

3.4 Phase Transitions

Transitions between coherence phases occur when local gap conditions change enough to alter the sign or magnitude of $\chi_n^{(W)}$. These transitions appear as short intervals marked by modest curvature spikes. Curvature patterns that would trigger sustained instability are avoided. They do not propagate instability; the system settles quickly into a new coherent regime.

3.5 Comparison with Random Models

Random permutations of the gap multiset do not display coherence phases. Their smoothed curvature behaves like noise: it fluctuates around zero with short run lengths and no stable sign structure. The extended coherence intervals seen in the true primes exceed random expectations by orders of magnitude, showing that coherence is inherent to the natural ordering of the gaps.

3.6 Structural Interpretation

Coherence phases unify the local constraints of Section 2 with the global behavior of Section 4. They reflect extended regions of controlled drift in the gap values, alternating with brief, mild transitions. Taken together, coherence phases are the dynamical footprint of the prime sequence's tendency to maintain a stable geometric orientation—close to 45°, with controlled drift.

4 Low-Action Structure

The cumulative curvature

$$S(N) = \sum_{n \le N-2} \chi_n^2$$

provides a global measure of geometric roughness in the prime gap sequence. Each term χ_n^2 quantifies how much the gap configuration bends at a given index, and the sum records the total variation accumulated over a long interval. Curvature spikes contribute disproportionately to $S(N)$, while extended regions of small curvature contribute very little.

4.1 Magnitude of the Effect

Across all ranges tested in PG2 and PG3, the value of $S(N)$ for the true primes lies in the extreme lower tail of the permutation distribution of the same gap multiset: typically below the lowest 1% and often below 0.01%. Random permutations routinely exhibit far larger curvature energy due to abrupt reversals in gap size, frequent large curvature values, and rapid alternation between expansion and contraction. The true sequence is therefore not merely smoother—it is exceptionally so when compared to the overwhelming majority of reorderings.

4.2 Local Behavior and Global Consequences

Sections 2 and 3 showed that a substantial fraction of indices satisfy $|\chi_n| < 0.1$ and that smoothed curvature produces long coherence phases of consistent sign. Low action is the global expression of these two phenomena. Long coherence phases create broad intervals in which curvature contributes almost nothing to $S(N)$, and the relatively rare phase transitions produce only modest curvature spikes. Because the recurrence relation constrains second-order variation, these transitions do not propagate instability and remain localized.

Thus

$$S(N) = (\text{small contributions over long intervals}) + (\text{moderate contributions at transitions}),$$

where the first term dominates due to coherence and the second is limited by bounded curvature.

4.3 Structural Asymmetry with Random Models

Random permutations destroy the structure responsible for low action. They exhibit:

- rapid oscillation of curvature around zero,
- short run lengths with no coherent sign persistence,
- widely varying magnitudes of χ_n,
- frequent curvature spikes.

The true primes, by contrast, maintain extended equilibrium-like regions with small curvature and transitions that are brief and mild. This produces an enormous disparity in cumulative curvature between the two sequences.

4.4 Interpretive Remarks

Low action is not a mechanism and is not presented as a physical or variational principle. It is an empirical structural property: among all reorderings of the same multiset of gaps, the natural prime order accumulates exceptionally little curvature. The phenomenon is a direct consequence of the curvature constraints and coherence phases documented throughout the sequence.

4.5 Unified Perspective

Curvature constraints limit local variation, coherence phases organize mesoscopic behavior, and low action expresses the resulting smoothness at a global scale. Together, these features indicate that the prime gap sequence evolves in a stable, structured, and unusually smooth geometric manner. This unifies the local, intermediate, and global observations into a single curvature-based framework.

5 Synthesis and Outlook

The curvature framework developed in PG1–PG4 reveals that the prime gap sequence is governed by a coherent geometric structure across local, mesoscopic, and global scales. The combined effects of curvature constraints, coherence phases, and low-action behavior distinguish the true primes sharply from randomized models. This section synthesizes the main findings and outlines directions for future work.

5.1 Curvature as the Organizing Variable

The normalized curvature

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}$$

captures the relative second-order change in the prime gaps in a scale-invariant manner. Across all computations in PG2 and PG3, the curvature sequence exhibits several consistent features:

- **Frequent near-symmetry:** a large fraction of indices satisfy $|\chi_n| < 0.1$.
- **Suppression of extremes:** large curvature values are significantly rarer than in permutation or Cramér-type models.
- **Strong recurrence constraints:** the identity $g_{n+2} = g_n + \chi_n(g_n + g_{n+1})$ limits the rate of second-order variation.
- **Geometric meaning:** small curvature corresponds to slow drift in the Prime Triangle angles, keeping the system near its equilibrium geometry.

Thus curvature functions not merely as a statistic, but as a structural signal governing the evolution of the gaps.

5.2 Coherence Across Scales

Smoothing curvature over windows $W = 500, 2000, 5000$, and 10000 reveals long coherence phases in which $\chi_n^{(W)}$ maintains a consistent sign. These phases reflect extended expansion, contraction, or equilibrium-like regimes in the gaps and persist under coarse-graining.

Random permutations of the same gap multiset exhibit no analogous structure: their smoothed curvature oscillates rapidly with short run lengths and no mesoscopic organization. Coherence phases therefore represent a genuine property of the natural prime order rather than a consequence of the gap multiset.

5.3 Global Interpretation: Low Action and Suppressed Roughness

The cumulative curvature

$$S(N) = \sum_{n=1}^{N-2} \chi_n^2$$

captures the global geometric roughness of the prime gap sequence. Across all ranges tested, the true primes fall into the lowest 1% of the permutation distribution of $S(N)$, and frequently into the lowest 0.01%.

This extreme separation arises from the structural properties identified earlier:

- Coherence phases contribute minimal curvature energy.
- Phase transitions introduce only modest curvature spikes.
- Recurrence constraints prevent runaway second-order variation.

Low action is thus the macroscopic expression of the same organization revealed locally and mesoscopically.

5.4 Directions for Future Development

Several natural extensions follow from the curvature framework:

- **Spectral analysis of curvature:** Fourier or wavelet analysis of (χ_n) may reveal frequency-domain patterns linked to coherence phases.

- **Angle–curvature dynamics:** relating χ_n to the Prime Triangle angles may deepen the geometric understanding of equilibrium behavior.
- **PSD–curvature connections:** joint analysis of PSD quantities and curvature may uncover higher-order structural regularities.
- **Higher-order geometric signals:** extending curvature to longer difference operators or alternative normalizations may expose additional structure.
- **Refined comparison models:** exploring renewal processes or locally correlated random-gap models may help isolate features unique to the true primes.

These directions continue the central theme of PG4: the prime gap sequence displays a stable, smoothly evolving geometric organization that distinguishes it from natural random analogues.

6 Conclusion

Prime Geometry IV presents a consolidated geometric-dynamical account of curvature in the prime gaps. The sequence exhibits constrained curvature, coherent organization across scales, and globally minimized curvature energy. These observations complete the curvature-focused portion of the Prime Geometry program and provide a foundation for future geometric and spectral investigations.