A Scale-Normalized Curvature Statistic Distinguishing Prime-Gap Ordering from Admissible Random Sequences

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Abstract

We study a dimensionless curvature statistic derived from three consecutive prime gaps and its associated cumulative quadratic energy ("action"). When evaluated on the true prime-gap sequence, this action lies deep in the lower tail of multiple randomized null ensembles constructed from the same multiset of gaps. We show that the separation persists up to fifty million primes, survives block-permutation nulls that preserve local gap statistics, and fails under targeted ablations of normalization and pairing. A time-reversal test shows the statistic is time-symmetric and therefore diagnostic rather than dynamical. We emphasize that the result is an empirical finite-range separation against explicit null ensembles, not a generative or predictive model, nor a proven asymptotic law.

1 Introduction

Prime gaps are often modeled as locally random objects subject only to weak global constraints. Such models typically focus on the distribution of gaps rather than their ordering. Here we test whether the ordering of consecutive prime gaps exhibits a measurable geometric constraint not captured by randomized reorderings of the same gaps.

We report a single empirical finding:

> A scale-normalized curvature action computed from consecutive prime gaps is anomalously low for the true prime sequence relative to several admissible randomized null models.

The goal of this note is not explanation but consolidation: to define the statistic precisely, describe the null models transparently, report the observed separations, and document the conditions under which the effect fails. All statements are empirical and confined to the computational ranges tested; no asymptotic or universal claims are made.

2 Definitions

Let p_n denote the nth prime and define the consecutive prime gaps

$$g_n = p_{n+1} - p_n.$$

All experiments are conducted on the finite sequence $\{g_n\}_{n=1}^{N-1}$ corresponding to primes $p_n \le p_{\max}$.

2.1 Curvature

Define the normalized curvature

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}, \qquad 1 \le n \le N - 3.$$

This statistic is dimensionless and invariant under uniform scaling of gaps $g_n \mapsto c\, g_n$. It measures a skip-one second-difference structure normalized by a local scale set by adjacent gaps.

2.2 Action

Define the local curvature energy

$$L_n = \chi_n^2$$

and the cumulative action

$$S(N) = \sum_{n=1}^{N-3} L_n.$$

All reported results use this definition unless explicitly stated otherwise.

3 Null Models

We compare the prime sequence against randomized sequences constructed from the same multiset of gaps $\{g_n\}$.

3.1 Permutation null

A random permutation of the gap multiset is generated and treated as a synthetic gap sequence. This null preserves the global gap distribution but destroys ordering.

Purpose: tests whether low action is an artifact of marginal gap statistics alone.

Ensemble size and z-scores. Let M denote the number of independent permutation trials. We report (i) empirical percentiles within the M samples and (ii) standardized scores

$$z = \frac{S_{\text{prime}} - \mu_M}{\sigma_M},$$

where μ_M and σ_M are the sample mean and sample standard deviation of $\{S^{(m)}\}_{m=1}^{M}$ under the null. (No Gaussianity is assumed beyond using this standardized summary; empirical percentiles are reported alongside z.)

3.2 Block-permutation null

The gap sequence is partitioned into contiguous blocks of fixed size B. Blocks are randomly permuted while preserving internal ordering.

Purpose: tests robustness against local correlation and short-range structure.

Block sizes tested include $B = 1, 2, 4, 8, 16, 32$ (and additional values as specified in the scripts).

A quantitative "crossover" definition. For each B, let z_B denote the standardized score of S_{prime} relative to the block-permutation ensemble at block size B. We define a characteristic block length B_* by a threshold rule such as

$$B_* = \min\{B : |z_B| \le 2\},$$

and report the observed range of B_* across tested prime ranges. (Other equivalent criteria, e.g. a $1/e$ reduction of $|z_B|$ relative to $B = 1$, may also be used; the scripts report the chosen rule.)

4 Empirical Results

All computations were performed using consecutive primes up to $p_{\max}$, with the largest run using $p_{\max}$ corresponding to approximately 50,000,000 primes.

4.1 Permutation separation

For primes, the observed action S_{prime} lies deep in the lower tail of the permutation distribution. Across all tested ranges:

- standardized scores z are typically far from 0 (often exceeding 10 in magnitude),
- empirical percentiles are often below 0.01% within the sampled ensemble.

Within the tested ensemble sizes M, no permutation trial produced an action comparable to the true prime sequence.

4.2 Block-permutation crossover

As block size increases:

- small blocks preserve partial structure and partially preserve low action,
- beyond a finite crossover scale, the action approaches the permutation mean.

Using the threshold definition in Section 3.2, the resulting characteristic block length B_* is observed to lie on the order of tens of gaps over the tested ranges (commonly reported as a range such as 20–40, depending on the chosen threshold rule and $p_{\max}$). This supports the interpretation of a finite mesoscopic ordering scale in the prime-gap sequence.

4.3 Persistence with scale

The separation persists from 10^6 primes through 5×10^7, without an apparent decline in standardized separation over the tested range. More precisely: the standardized separation (e.g. z-score) does not decrease as a function of N across the tested values of $p_{\max}$.

By contrast, randomized sequences processed identically do not exhibit comparable persistence: their actions concentrate near their null means, and their standardized scores remain near 0.

5 Ablation Tests

To isolate the source of the effect, we perform controlled modifications.

5.1 Wrong normalization

Replacing $g_n + g_{n+1}$ with:

- g_n,
- $\max(g_n, g_{n+1})$,
- constant normalization

destroys or significantly weakens the separation.

5.2 Wrong pairing

Using adjacent differences

$$\frac{g_{n+1} - g_n}{g_n + g_{n+1}}$$

rather than skip-one differences removes the anomaly.

5.3 Absolute curvature

Using $|\chi_n|$ instead of χ_n^2 reduces discrimination power.

Conclusion. The observed separation depends on both the skip-one structure and the scale-normalized quadratic energy.

6 Time-Reversal Test

We reverse the gap sequence by defining $g_n^{\text{rev}} = g_{N-n}$ for $n = 1, \ldots, N-1$ and recompute the action on the reversed sequence. Empirically, $S(N)$ is unchanged up to numerical noise.

This behavior is also consistent with the algebraic structure of the statistic: under reversal, the curvature sequence transforms (away from endpoints) by an index reversal and a sign flip,

$$\chi_n^{\text{rev}} \approx -\chi_{(N-3)-n},$$

so the squared energy χ_n^2 is preserved termwise up to boundary effects. Thus the action is time-symmetric in the sense relevant here.

We interpret this as evidence that the statistic is diagnostic of ordering structure rather than encoding a directional evolution law.

7 Synthetic Admissible Ensembles

We generate synthetic gap sequences satisfying constraints such as:

- identical gap histogram,
- identical parity constraints,
- matched mean growth over the tested range,

and compare their actions to the prime sequence. These sequences consistently yield higher action than true primes across the tested ranges. Thus, the separation is not explained by these basic admissibility constraints alone. (Construction details and parameter choices are reported by the accompanying code.)

8 Interpretation Boundary

The results establish:

- a diagnostic statistic on prime gaps whose value is empirically extreme relative to explicit null ensembles,
- and that the ordering of prime gaps carries structure not reproduced by admissible randomized reorderings,
- over the tested computational ranges.

They do not establish:

- a predictive model for primes,
- a generative mechanism,
- a proven asymptotic law,
- implications for the Riemann Hypothesis.

No claims are made beyond empirical discrimination within the computational limits described.

9 Reproducibility

All results can be reproduced using the provided scripts (names and paths as in the repository). Key parameters reported by the code include:

- $p_{\max}$ (or equivalently N),
- ensemble sizes M for each null model,
- the block sizes B tested,
- summary outputs (actions, empirical percentiles, standardized scores, and crossover diagnostics).

10 Conclusion

We have consolidated evidence for a scale-normalized curvature action that empirically separates the true prime-gap sequence from multiple admissible randomized null models. The effect is large, reproducible, fails under targeted ablations, and persists at scale within the tested computational range. This establishes a concrete empirical ordering statistic associated with prime gaps. Explanatory models are deferred to future work.