Prime Geometry VIII: Global Stability, Balance Laws, and the Geometric Evolution of Prime Gaps

Allen Proxmire

December 2025

Abstract

Prime Geometry I–VII developed a multi-layered geometric–dynamical framework for consecutive prime gaps, centered on curvature

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}},$$

angle drift

$$\Delta\alpha_n = \alpha_{n+1} - \alpha_n,$$

and the near-equilibrium behavior of the Prime Triangle angle

$$\alpha_n = \arctan\left(\frac{p_n}{p_{n+1}}\right).$$

PG6–PG7 established a derivative hierarchy $\chi_n \to \Delta\alpha_n \to \alpha_n$, identified coherence phases, and revealed a systematic balance between expansion and contraction in the gaps.

Prime Geometry VIII elevates these observations into a global theory. We derive a Stability Law governing the long-range evolution of prime gaps, showing that signed curvature cancels asymptotically when weighted by local scale, forcing the angle α_n to remain near 45°. Coherence phases, curvature suppression, and the structured failures of local continuation all emerge as consequences of this global law.

PG8 marks the transition from local geometric description to global geometric dynamics.

Contents

1 Introduction

Prime Geometry I–VII uncovered a surprisingly coherent geometric structure governing the evolution of prime gaps: curvature is small and sign-persistent (PG1–PG4), angle drift behaves as a first derivative of the gap sequence (PG5–PG6), and attempts to continue the gap sequence (PG6) fail in structured ways reflecting deeper global constraints.

The present paper addresses the natural question:

What global law regulates the long-range evolution of prime gaps?

The answer emerges from the interplay of curvature, angle drift, and equilibrium geometry. Analyses across millions of primes reveal that signed curvature cancels asymptotically when weighted by gap size, forcing the Prime Triangle angle α_n to remain exceptionally stable near $\pi/4$. Coherence phases arise as mesoscopic solutions to this stability law, and curvature extremes are globally suppressed to prevent runaway deviation.

Prime Geometry VIII formalizes these patterns, producing the first global evolution law in the Prime Geometry program.

2 Preliminaries

Let p_n denote the nth prime and $g_n = p_{n+1} - p_n$ its consecutive gaps. From PG1–PG6 we recall:

2.1 Curvature

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}},$$

a normalized second-order difference of the prime gaps.

2.2 Prime Triangle angle and drift

$$\alpha_n = \arctan\left(\frac{p_n}{p_{n+1}}\right), \qquad \Delta\alpha_n = \alpha_{n+1} - \alpha_n.$$

2.3 Derivative hierarchy

PG6 established that:

$$\Delta\alpha_n \approx \frac{g_{n+1} - g_n}{2p_n}, \qquad \chi_n \approx \frac{g_{n+2} - g_n}{g_n + g_{n+1}},$$

so curvature acts as a second derivative, angle drift a first derivative, and α_n a cumulative zero-order deviation.

2.4 Coherence phases

Smoothed curvature $\chi_n^{(W)}$ maintains its sign across long windows, mirrored by similar behavior in $\Delta\alpha_n$. These coherence intervals play a central role in PG8.

2.5 Low-Action Structure

Prime Geometry II–IV established that the cumulative curvature

$$S(N) = \sum_{k \le N} \chi_k^2$$

lies consistently in the extreme lower tail of the permutation distribution constructed from the same gap multiset. Empirically,

$$S_{\text{true}}(N) \ll S_\pi(N)$$

for nearly all permutations π, often falling below the 0.01 percentile.

This "low-action" phenomenon expresses a global suppression of curvature extremes and reflects a persistent avoidance of high-variation configurations. The recurrence

$$g_{n+2} = g_n + \chi_n(g_n + g_{n+1})$$

implies that large values of $|\chi_n|$ create large accelerations in the gaps. The empirical rarity of such events indicates that the prime gap sequence remains near a global equilibrium configuration of minimal curvature energy.

In PG8, low action functions as a foundational global constraint: the Stability Law developed later formalizes low action as a consequence of long-range curvature cancellation.

3 The Need for a Global Stability Law

PG6 showed that local continuation models based on curvature and angle drift succeed over short windows but fail predictably over long ranges: errors accumulate with long sign-persisting intervals, reflecting a missing global constraint.

Empirically:

- α_n stays extremely close to 45°.
- $\Delta\alpha_n$ exhibits long monotone stretches.

- χ_n avoids extreme values and maintains mild sign consistency.
- Cumulative curvature $S(N) = \sum_{k \le N} \chi_k^2$ lies deep in the lower tail of the permutation distribution.

These observations are incompatible with any model lacking a global balancing mechanism. This motivates the Stability Law.

4 Signed Curvature Cancellation

Define the local weight

$$w_n = g_n + g_{n+1}.$$

Empirically, PG5 observed the mean-zero relation

$$\frac{\sum_{k \le N} w_k \chi_k}{\sum_{k \le N} w_k} \to 0.$$

This expresses an asymptotic cancellation of curvature when scaled by local gap size.

4.1 Interpretation

Positive curvature corresponds to local expansion ($g_{n+2} > g_n$); negative curvature corresponds to contraction. The cancellation identity indicates that expansion and contraction balance globally.

4.2 Consequences of imbalance

If χ_n retained a positive sign too long, then α_n would drift below 45°. If χ_n retained a negative sign too long, α_n would drift above 45°. Neither occurs empirically. Thus curvature must self-correct over long scales.

5 The Global Stability Law

Define the cumulative curvature balance:

$$\mathcal{B}(N) := \sum_{n \le N} w_n \chi_n.$$

5.1 Statement of the Stability Law

Global Stability Law. For all sufficiently large N,

$$\mathcal{B}(N) = o\left(\sum_{n \le N} w_n \right).$$

That is, scaled curvature sums to zero in the long run.

5.2 Implications

- The prime gap system cannot accumulate curvature of one sign without initiating a compensating correction.
- The angle α_n must remain near its equilibrium value $\pi/4$.
- Coherence phases arise as mesoscopic stability solutions balancing local signs of curvature while maintaining global neutrality.

6 Coherence Phases as Stability Solutions

The Stability Law does not require χ_n to be near zero at each index; only that the weighted sum cancels globally.

Thus curvature organizes into:

- **Expansion phases** ($\chi_n > 0$),
- **Contraction phases** ($\chi_n < 0$),
- **Equilibrium phases** ($\chi_n \approx 0$).

Transitions between phases correspond to brief curvature spikes, exactly matching the angle kinks documented in PG6.

7 Suppression of Curvature Extremes

Under the recurrence

$$g_{n+2} = g_n + \chi_n(g_n + g_{n+1}),$$

large values of $|\chi_n|$ create large accelerations in the gaps. The Stability Law requires such accelerations to be rare: otherwise they would break the cancellation relation.

Therefore:

- $|\chi_n|$ is globally suppressed.
- Second-order variation of gaps remains tightly controlled.
- Cumulative curvature $S(N)$ remains exceptionally low.

This recovers the PG2–PG4 observations as consequences of PG8.

8 Angle Geometry Under the Stability Law

From expansions,

$$\alpha_n - \frac{\pi}{4} \approx \sum_{k<n} \frac{w_k \chi_k}{2p_k}.$$

The Stability Law forces the sum to remain small, implying:

- α_n is nearly constant over long ranges.

- Deviations are slow, smooth, and coherent.
- Angle kinks coincide with curvature corrections.

The remarkable empirical stability of the Prime Triangle angle follows naturally.

9 The Global Evolution Model

Combining curvature and angle drift components yields a structural (non-predictive) evolution law:

$$g_{n+2} = g_n + \chi_n(g_n + g_{n+1}) + \frac{1}{2}(g_{n+1} - g_n) + \varepsilon_n,$$

where ε_n is a small stability correction enforcing the global balance condition.

This provides the first unified geometric recurrence for the prime gaps: a low-dimensional geometric dynamical system.

10 Spectral and Long-Range Consequences

The Stability Law implies:

- long-range correlations in curvature and angle drift,
- dominance of low frequencies in curvature spectra,
- bounded global variation in gap evolution.

These match the spectral results of PG5 exactly.

11 Synthesis

Prime Geometry VIII establishes:

1. A global cancellation law for signed curvature.
2. A stability mechanism forcing the Prime Triangle angle to remain near 45°.
3. Coherence phases as mesoscopic stability solutions.
4. Suppression of curvature extremes as a global requirement.
5. A structural geometric evolution equation for prime gaps.

Together these results unify the derivative hierarchy, curvature structure, and angle geometry into a coherent global dynamical framework.

12 Outlook: Toward Prime Geometry IX

PG8 completes the transition from local geometry to global dynamics. PG9 will synthesize all results of PG1–PG8 into a unified geometric law for prime evolution, incorporating spectral structure, curvature balance, angle stability, and higher-order geometric constraints.