Scale-Stable Geometric and Ordering Constraints in the Prime Gap Sequence

Allen Proxmire

December 2025

Abstract

We present empirical evidence that the sequence of consecutive prime gaps occupies a constrained region of configuration space not captured by standard admissible random models. Two independent probes are examined: (i) a scale-normalized geometric curvature functional defined on triples of consecutive gaps, and (ii) a rank-based ordering dependence statistic measuring lagged correlations in the gap sequence. Both probes exhibit statistically significant deviations from permutation and block-permutation null ensembles that preserve local gap structure. These deviations persist across mesoscopic scales and strengthen as the prime range increases from 10^6 to 5×10^6. Moreover, the lagwise deviation profiles of the two probes are strongly correlated under identical null models, indicating that they detect the same underlying empirically observable ordering constraint on the prime gap configuration space. No dynamical law or generative mechanism is proposed; the results establish only the existence and coherence of a nontrivial structural constraint over the tested computational ranges, not a proven asymptotic law.

1 Introduction

The statistical structure of prime gaps is traditionally studied through the distribution of individual gaps or normalized gap sizes. While such approaches successfully capture first-order properties, they largely treat the ordering of gaps as random beyond trivial local constraints. In this work, we investigate whether the ordering and geometric relationships among consecutive gaps exhibit structure that survives admissible randomization.

Rather than proposing a generative model or dynamical law, we adopt a strictly empirical approach. We define two independent invariants on the prime gap sequence and compare their behavior to carefully chosen null ensembles that preserve gap admissibility and short-range structure. Our goal is not to explain the origin of any observed structure, but to determine whether such structure exists, is reproducible, and is coherent across distinct measurements.

The central question addressed here is therefore limited in scope:

> Does the prime gap sequence exhibit nontrivial geometric or ordering constraints that are not reproduced by admissible random reorderings?

All claims are empirical and confined to the tested computational ranges.

2 Definitions and Null Models

2.1 Prime gaps

Let p_n denote the nth prime and define the prime gaps

$$g_n = p_{n+1} - p_n.$$

All experiments are conducted on the sequence $\{g_n\}$ truncated at primes $p \leq p_{\max}$.

2.2 Curvature invariant

We define a scale-normalized curvature functional on triples of consecutive gaps by

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}.$$

This quantity measures signed second-difference structure normalized by local scale. Aggregate curvature behavior is summarized through statistics derived from $\{\chi_n\}$, including action-like sums and ordering dependence under lag.

2.3 Ordering dependence (Spearman statistic)

To probe ordering structure, we compute lagged Spearman rank correlations. For a sequence x_n, define

$$\rho_x(k) = \text{Spearman}(x_n, x_{n+k}), \qquad k \geq 2,$$

where lag $k = 1$ is excluded due to known trivial anticorrelations arising from local admissibility constraints.

This statistic is applied both to the gap sequence $x_n = g_n$ and to the curvature sequence $x_n = \chi_n$.

2.4 Null models

Two admissible null ensembles are employed:

1. **Permutation null**: the gap sequence is randomly permuted.
2. **Block-permutation null**: the gap sequence is partitioned into contiguous blocks of fixed length B (here $B = 32$), which are then randomly permuted.

The block-permutation null preserves short-range gap structure while destroying longer-range ordering. All reported significance is measured relative to this more conservative null.

3 Empirical Results

3.1 Curvature extremality

The curvature sequence $\{\chi_n\}$ exhibits aggregate behavior that is atypical relative to both permutation and block-permutation nulls. In particular, curvature-derived action measures are systematically lower for the prime sequence than for admissible random reorderings, and this extremality persists across the tested prime ranges.

This establishes the curvature functional as a genuine empirical geometric invariant of the prime gap sequence.

3.2 Ordering dependence of prime gaps

The gap ordering statistic $\rho_g(k)$ shows statistically significant positive dependence across a wide range of lags. When compared against block-permutation nulls, the deviation persists up to the maximum tested lag $k = 150$ and strengthens with increasing $p_{\max}$.

A coherence summary for block size $B = 32$ and significance threshold $Z > 2$ is shown below:

$p_{\max}$	# gaps	last sig. lag	frac. sig.	longest run	excess
10^6	78,497	148	0.597	10	102.47
2×10^6	148,932	147	0.758	14	157.39
5×10^6	348,512	150	0.879	87	271.15

The increasing fraction and coherence of significant lags indicate a scale-stable, mesoscopic ordering constraint. By contrast, randomized sequences processed identically do not exhibit such growth and rapidly converge to null expectations.

3.3 Cross-statistic coherence

To test whether the curvature invariant and gap ordering statistic detect the same underlying structure, we compare their lagwise deviation profiles. For each lag k, Z-scores relative to the block-permutation null are computed:

$$Z_g(k), \quad Z_\chi(k).$$

Across lags $k = 2, \ldots, 150$, these profiles are strongly correlated:

$p_{\max}$	Pearson corr.	Spearman corr.
2×10^6	0.594	0.618
5×10^6	0.530	0.614

The stability of this correlation across scale demonstrates that the geometric curvature invariant and the ordering-based statistic respond to the same empirically observable constraint on the prime gap configuration space.

4 Interpretation and Limits

The results presented here establish the existence of a nontrivial structural constraint on the prime gap sequence that is:

- detectable by independent geometric and ordering-based probes,
- robust under admissible block-permutation nulls,
- persistent across mesoscopic scales,
- and increasingly coherent with larger prime ranges.

No claim is made regarding causality, dynamics, or generative laws. In particular, these findings do not imply the existence of an equation of motion, a variational principle, asymptotic universality, or any direct connection to zeta zeros or the Riemann Hypothesis. The results are diagnostic, not explanatory, and confined to the tested computational ranges.

5 Reproducibility

All experiments were conducted using pure-Python scripts with explicit parameter control. Prime generation, null ensembles, and statistical summaries are fully reproducible. Scripts and data products are archived in the accompanying repository.