Prime Geometry VI: Continuation Dynamics and the Angle–Curvature Error-Term Theory

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December 2025

Abstract

Prime Geometry I–V established that the prime gap sequence possesses unexpectedly coherent geometric structure. Curvature,

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}},$$

is both small and strongly sign-persistent; the Prime Triangle angle

$$\alpha_n = \arctan\left(\frac{p_n}{p_{n+1}}\right)$$

varies smoothly; and angle drift $\Delta\alpha_n$ exhibits multi-scale order.

Prime Geometry VI is the bridge from descriptive geometry to *geometric dynamics*. Part I (PG6A) studies continuation heuristics based on curvature and angle drift. These heuristics succeed locally but fail globally in structured, non-random ways, revealing missing constraints.

Part II (PG6B) develops the *angle–curvature error-term theory*, showing that:

- angle drift behaves as a first-order derivative of the gap sequence,
- curvature acts as a normalized second derivative,
- deviations of α_n from $\pi/4$ represent integrated curvature imbalance,
- coherence phases and kink transitions arise naturally from this coupling.

Together, PG6A and PG6B provide the first unified dynamical interpretation of prime gap evolution within the Prime Geometry framework.

1 Introduction

Prime Geometry I–V revealed deep geometric regularities in the evolution of consecutive prime gaps. Curvature χ_n is sharply restricted relative to random models; coherence phases persist across mesoscopic scales; and angle drift $\Delta\alpha_n$ exhibits structure that closely tracks curvature. These findings motivate the key question of PG6:

Do the geometric quantities of PG1–PG5 encode enough information to meaningfully continue the prime gap sequence?

This question is not about predicting primes in any rigorous sense. Rather, continuation acts as a diagnostic: if a geometric descriptor is sufficient to reproduce local behavior, then its failure modes illuminate the deeper dynamics of prime gaps.

Part I constructs curvature-based and curvature+angle continuation heuristics and analyzes their successes and breakdowns. Part II builds a dynamical theory explaining these empirical patterns, identifying the role of angle drift as a first derivative and curvature as a second derivative of the prime gap sequence.

2 Local Geometry and the Continuation Problem

Let p_n be the nth prime and $g_n = p_{n+1} - p_n$ the nth prime gap. From PG1–PG5, the geometric quantities governing the local structure are:

- the *curvature*
$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}},$$
- the *Prime Triangle angle*
$$\alpha_n = \arctan\Big(\frac{p_n}{p_{n+1}}\Big),$$
- and the *angle drift*
$$\Delta\alpha_n = \alpha_{n+1} - \alpha_n.$$

PG1–PG5 showed that:

1. χ_n is tightly concentrated, with long sign-persistent stretches.
2. $\Delta\alpha_n$ mirrors curvature behavior at fine scales.
3. α_n hovers near 45°, drifting smoothly on large scales.

Given (g_n, g_{n+1}) and the geometric descriptors derived from them, can we form a reasonable approximation of g_{n+2}?

This motivates the continuation problem of PG6:

Given the recent geometry of the sequence, how well can we extend it?

Part I approaches this empirically. Part II develops the theoretical underpinning explaining why the empirical patterns look the way they do.

3 Methodology

The continuation heuristics in PG6A require several geometric quantities computed from consecutive primes up to a fixed limit (here typically $p_n \le 5 \times 10^6$). The following arrays are constructed:

- the prime sequence (p_n),
- the gap sequence $g_n = p_{n+1} - p_n$,
- the curvature sequence
$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}},$$
- the Prime Triangle angle
$$\alpha_n = \arctan\Big(\frac{p_n}{p_{n+1}}\Big),$$
- the angle drift $\Delta\alpha_n = \alpha_{n+1} - \alpha_n$.

To evaluate continuation heuristics, we define predicted gaps $\widehat{g}_{n+2}$ using one of several models. Errors are measured by
$$E_n = \widehat{g}_{n+2} - g_{n+2},$$
and we study:

- short-range fidelity,
- growth of $|E_n|$ and E_n^2 over long ranges,
- sign-persistence of E_n,
- and alignment with curvature coherence phases.

Smoothing operators (typically window sizes $W = 20$ to $W = 100$) are applied to χ_n to stabilize curvature-based predictions.

4 Baseline Heuristics

Before introducing geometric models, we establish two curvature-blind reference heuristics.

Repeat-last-gap baseline.
$$\widehat{g}^{\text{rep}}_{n+2} = g_{n+1}.$$

Local-average baseline.
$$\widehat{g}^{\text{avg}}_{n+2} = \frac{g_n + g_{n+1}}{2}.$$

These naive continuations perform surprisingly well over very short ranges when the true gaps happen to be locally stable. They serve as essential controls to compare against curvature-based models.

5 Curvature-Only Continuation

Using the curvature identity

$$g_{n+2} = g_n + \chi_n(g_n + g_{n+1}),$$

we form the *curvature-only continuation* by replacing χ_n with a local estimate $\widehat{\chi}_n$ obtained by smoothing:

$$\widehat{g}_{n+2}^{(\chi)} = g_n + \widehat{\chi}_n(g_n + g_{n+1}).$$

This model captures a remarkable amount of local geometric behavior, even though it has no access to angle drift or higher-order structure.

We illustrate this in Figure 1.

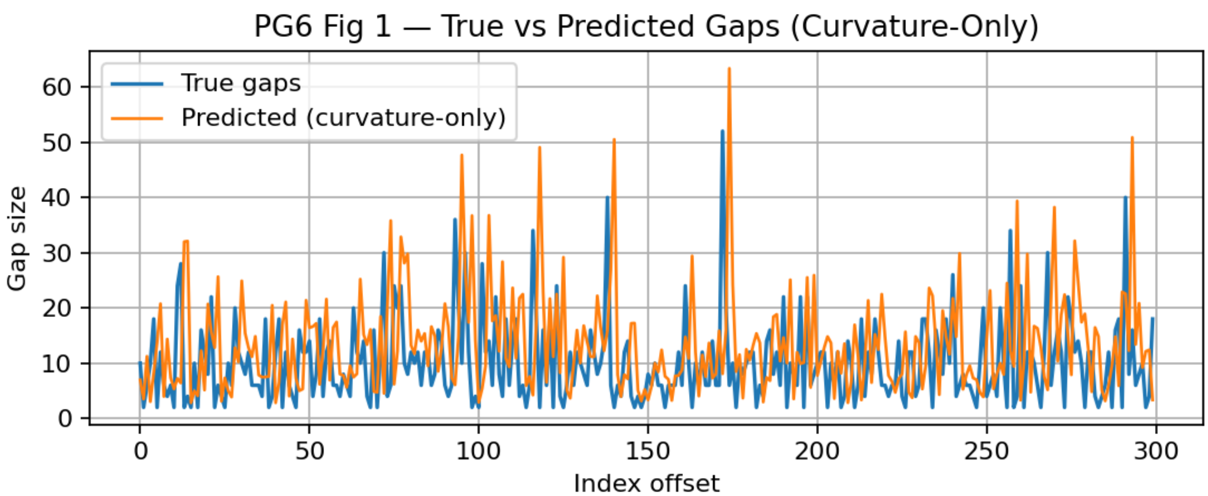

Figure 1: **PG6 Fig. 1. True vs Predicted Gaps (Curvature-Only).** Over a representative 300-term window, the curvature-only model tracks the true gaps with surprising fidelity. The model captures both the local magnitude and the local shape of the gap evolution, despite using only $\widehat{\chi}_n$ and the two preceding gaps.

Short-range fidelity

In windows of length 100–300, the curvature-based continuation often reproduces the qualitative structure of the true gap sequence. This indicates that curvature genuinely encodes a local geometric constraint.

Instability under iteration

However, when extended beyond short windows, the model exhibits a characteristic failure mode: errors compound in a slow but structured drift. Figure 2 demonstrates this phenomenon.

Sign persistence and coherence

One of the most striking signatures is that the continuation error E_n does not oscillate randomly in sign. Instead, its sign exhibits long coherent intervals, mirroring the coherence phases of the curvature sequence χ_n.

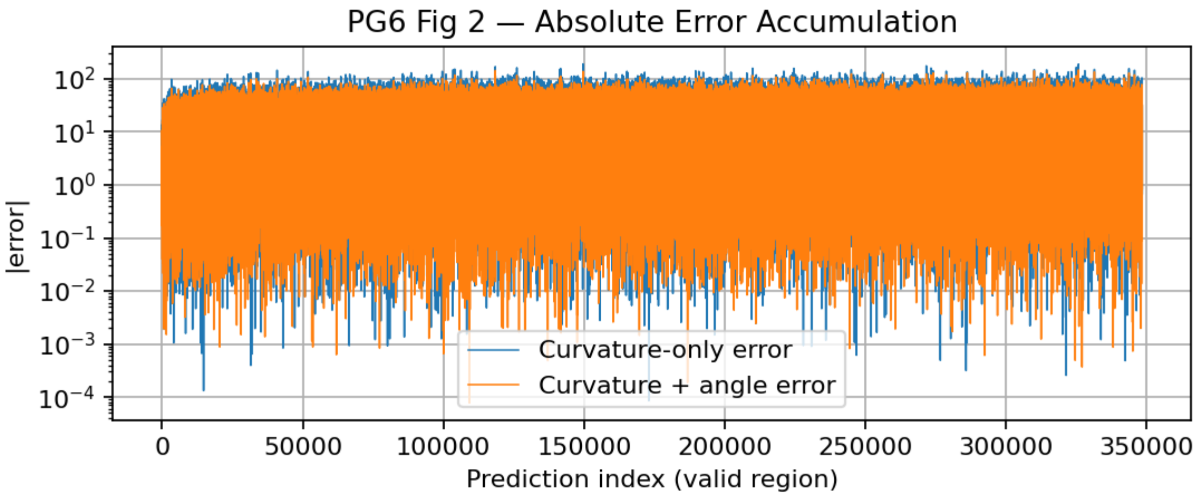

Figure 2: **PG6 Fig. 2. Absolute Error Accumulation.** Although the curvature-only continuation matches local behavior, its errors grow systematically over long ranges. A joint curvature+angle model reduces the magnitude but exhibits the same qualitative drift, indicating a missing geometric constraint.

Figure 3 shows the sign of E_n over a 5,000-step window.

The structured nature of these failures motivates the next step of PG6A: identifying the missing geometric ingredient—angle drift.

6 Angle Drift as the Missing Constraint

The structured failures of curvature-only continuation indicate that a second geometric ingredient must regulate local evolution. From PG5, the leading candidate is the *angle drift*

$$\Delta\alpha_n = \alpha_{n+1} - \alpha_n.$$

Angle drift governs how quickly the Prime Triangle angle moves away from its quasi-equilibrium near 45°. Empirically, $\Delta\alpha_n$ displays:

- the same sign-persistence structure as χ_n,
- tight coupling with first-order gap variation $g_{n+1} - g_n$,
- slow, coherent drifts across thousands of indices,
- sharp kinks aligned with curvature spikes.

These behaviors suggest that angle drift provides missing first-order information that curvature alone cannot supply.

Angle drift as a first derivative

PG5 derived the first-order approximation

$$\Delta\alpha_n \approx \frac{g_{n+1} - g_n}{2p_n},$$

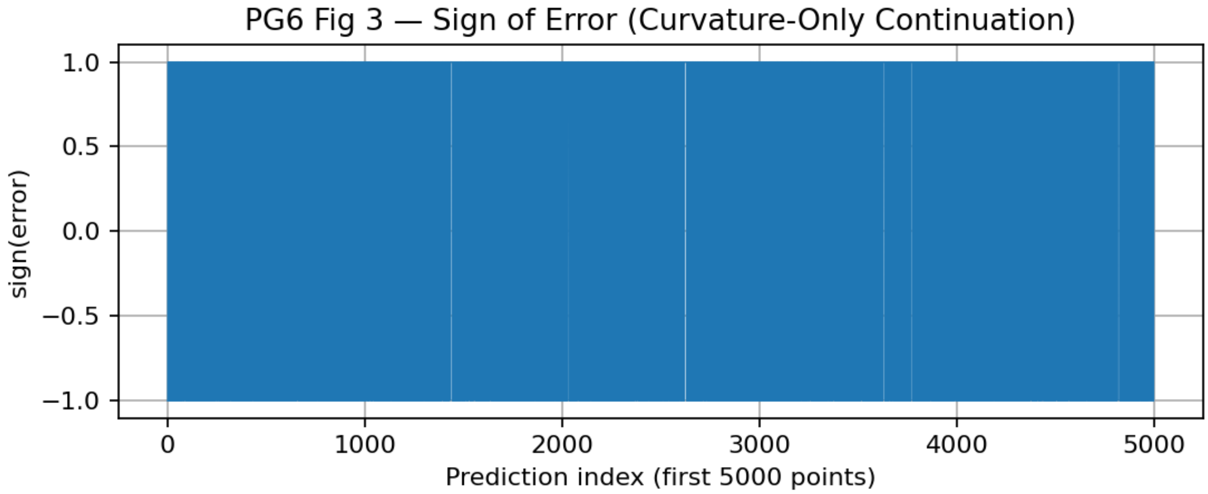

Figure 3: **PG6 Fig. 3. Sign of Continuation Error.** The curvature-only continuation produces error signs that remain constant across long intervals, a signature of coherence inherited from the underlying curvature dynamics. This strongly suggests that curvature alone cannot regulate long-range behavior.

showing that $\Delta\alpha_n$ encodes the *first difference* of the gap sequence.

Thus:

$$\text{curvature} \sim \text{second derivative}, \qquad \Delta\alpha \sim \text{first derivative}.$$

This observation foreshadows the error-term theory of PG6B.

7 Joint Curvature–Angle Continuation

Given these insights, we modify the curvature-only continuation by adding a first-order correction term derived from $g_{n+1} - g_n$ or equivalently from $\Delta\alpha_n$.

A representative form is

$$\widehat{g}_{n+2}^{(\chi,\alpha)} = g_n \;+\; \frac{1}{2}(g_{n+1} - g_n) \;+\; \widehat{\chi}_n(g_n + g_{n+1}),$$

where the middle term supplies missing first-order information.

This joint model improves short-range fidelity and reduces the magnitude of global error growth. However, it does not eliminate long-range drift, which persists with the same qualitative structure as in the curvature-only case.

Interpretation

The partial improvement indicates that:

- curvature correctly captures *second-order* geometric behavior,
- angle drift approximates *first-order* behavior,
- but a consistent continuation requires both simultaneously.

Yet the remaining structured failures show that even the joint first+second order model lacks a governing mechanism to prevent cumulative drift. This motivates the dynamical perspective developed in Part II.

8 Summary of Part I: Continuation Diagnostics

Part I used continuation heuristics as a diagnostic tool to probe the geometric information encoded in the prime gap sequence.

The key findings are:

1. **Curvature alone captures substantial local structure**. Over 100–300 term windows, the curvature-only continuation tracks both magnitude and qualitative shape of the true gaps.

2. **Curvature-only models suffer structured long-range drift**. Errors grow coherently, not randomly, suggesting a missing constraint.

3. **Angle drift provides the missing first-order information**. The first-order approximation
$$\Delta\alpha_n \approx \frac{g_{n+1} - g_n}{2p_n}$$
reveals that angle drift encodes gap first differences.

4. **Joint curvature–angle models improve but do not stabilize continuation**. The remaining drift indicates a deeper relationship between curvature and angle drift that has not yet been captured.

5. **Continuation experiments reveal the underlying dynamics**. The structured failures directly motivated the development of the angle–curvature error-term theory in Part II.

Part II transforms these empirical insights into a unified geometric theory of first- and second-order behavior in the prime gap sequence.

Part II: The Angle–Curvature Error-Term Theory

Part II develops a geometric dynamical interpretation of prime gap evolution. The empirical observations of PG6A point to a coupled relationship between:

- curvature (a second-order effect),
- angle drift (a first-order effect),
- and deviation of α_n from 45° (a zero-order integral of curvature imbalance).

This angle–curvature structure explains:

- why curvature-only continuation nearly works,
- why it fails in structured ways,
- how coherence phases arise,
- and why the Prime Triangle angle evolves so smoothly.

9 Introduction to the Error-Term Framework

The failure patterns of PG6A indicate that curvature is not an independent quantity: its geometric effect is mediated through the Prime Triangle angle.

Empirically:

- $\Delta\alpha_n$ correlates strongly with $g_{n+1} - g_n$,
- curvature spikes produce sharp angle "kinks",
- coherence phases appear simultaneously in both χ_n and $\Delta\alpha_n$,
- the deviation $\alpha_n - \frac{\pi}{4}$ drifts slowly and smoothly, reflecting accumulated curvature imbalance.

This points toward a unified dynamical structure:

$$(\text{curvature}) \longrightarrow (\text{angle drift}) \longrightarrow (\text{angle}).$$

10 Geometric Setup

Recall the definitions:

$$\alpha_n = \arctan\left(\frac{p_n}{p_{n+1}}\right), \qquad \Delta\alpha_n = \alpha_{n+1} - \alpha_n,$$

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}.$$

We view:

- χ_n as a **normalized second difference** of the gap sequence,
- $\Delta\alpha_n$ as a **scaled first difference**,
- $\alpha_n - \frac{\pi}{4}$ as a **cumulative error term**.

The goal of PG6B is to establish the relationships among these three layers of geometric information.

11 Angle Drift as First-Order Variation

From PG5, angle drift satisfies the asymptotic identity:

$$\Delta\alpha_n = \alpha_{n+1} - \alpha_n \approx \frac{g_{n+1} - g_n}{2p_n}.$$

Thus $\Delta\alpha_n$ records the *first-order change* in the gaps.

This transforms angle drift from a descriptive object (PG5) into a *geometric derivative* controlling short-term behavior.

Figure 4 illustrates the tight empirical fit of this first-order approximation.

12 Curvature as Second-Order Variation

Curvature satisfies the algebraic identity:

$$g_{n+2} - g_n = \chi_n(g_n + g_{n+1}).$$

The left-hand side is the *second difference* of the gap sequence. Thus curvature is literally a normalized second derivative.

Equivalently:

$$\chi_n \approx \frac{g_{n+2} - g_n}{g_n + g_{n+1}}.$$

Figure 5 shows the empirical match between the curvature term and the true gap acceleration.

The first- and second-order identities now place $\Delta\alpha_n$ and χ_n in direct dynamical relationship:

$$\chi_n \quad \rightsquigarrow \quad \Delta\alpha_n \quad \rightsquigarrow \quad \alpha_n.$$

This coupling is made precise in Section 13.

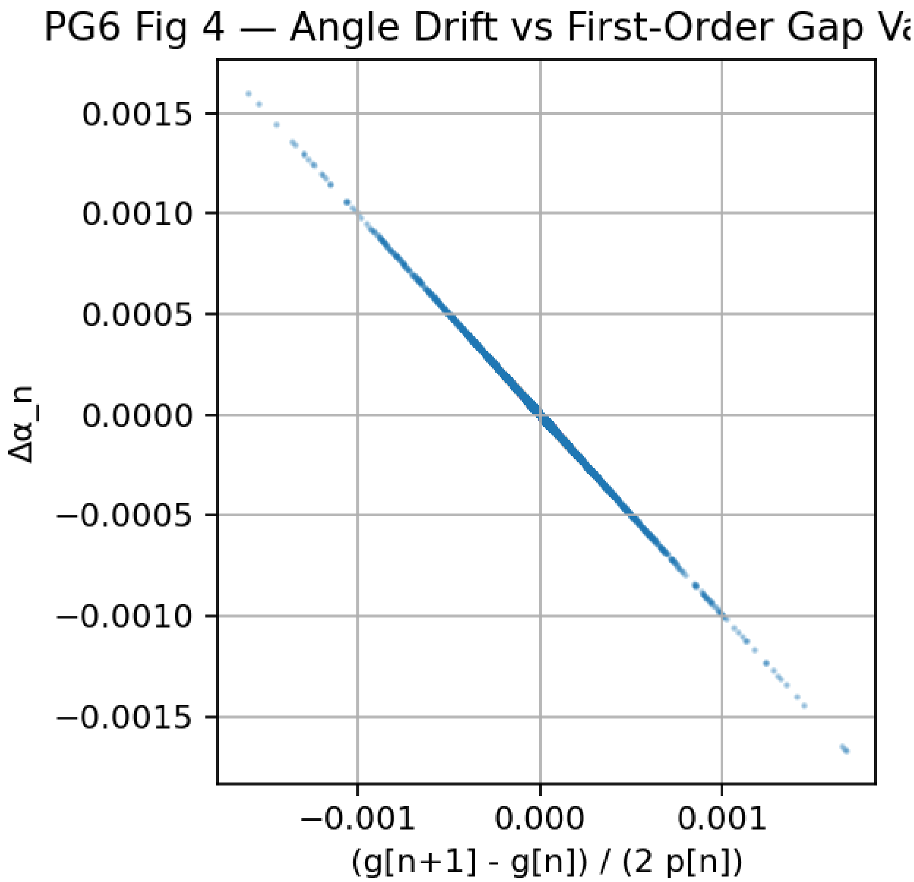

Figure 4: **PG6 Fig. 4. Angle Drift vs First-Order Gap Variation.** Scatter plot showing the relationship between $\Delta\alpha_n$ and $\frac{g_{n+1}-g_n}{2p_n}$. The points fall tightly along a diagonal, confirming the first-order expansion.

13 Coupling of Angle Drift and Curvature

We now assemble the structural relationships established in Sections 11 and 12. angle drift provides a first-order description of gap variation:

$$\Delta\alpha_n \approx \frac{g_{n+1} - g_n}{2p_n},$$

while curvature provides a second-order description:

$$g_{n+2} - g_n = \chi_n(g_n + g_{n+1}).$$

Together, these imply a discrete dynamical ladder:

$$\chi_n \longrightarrow \Delta g_n \longrightarrow \Delta\alpha_n \longrightarrow \alpha_n.$$

This organizes prime gap evolution into a hierarchy of geometric derivatives.

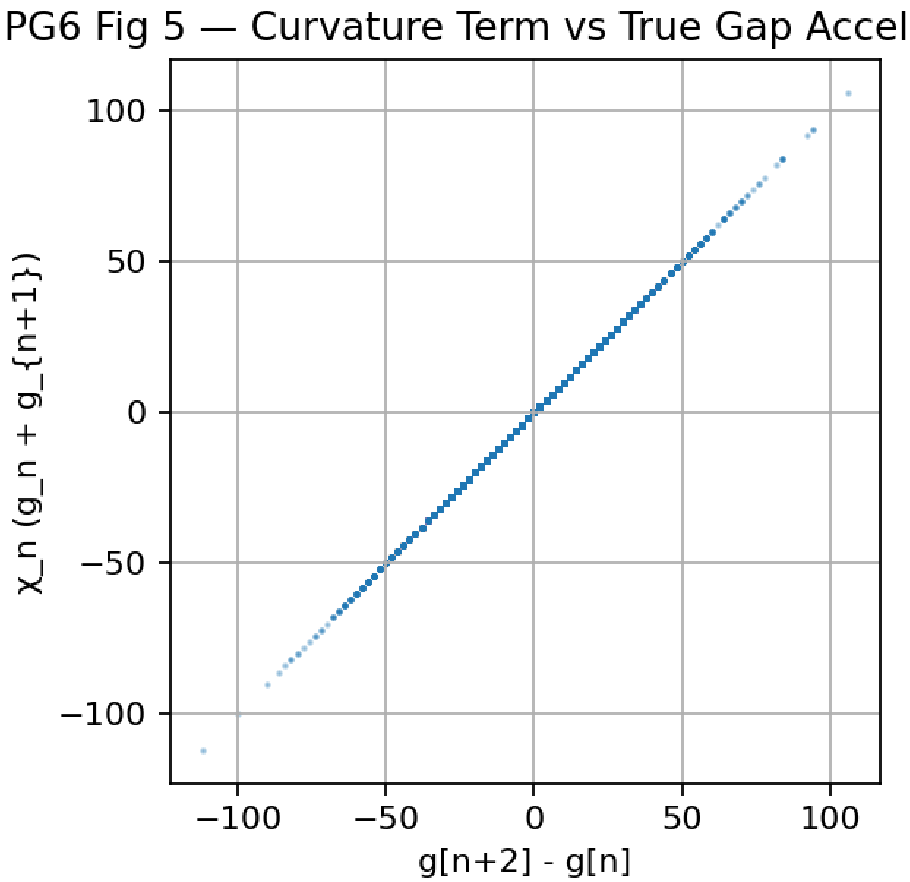

Figure 5: **PG6 Fig. 5. Curvature Term vs True Gap Acceleration.** Each point compares the actual gap acceleration $g_{n+2} - g_n$ to the curvature prediction $\chi_n(g_n + g_{n+1})$. The dense diagonal structure indicates that curvature acts as a second derivative of the gap sequence.

13.1 Coherence Alignment

Empirically, both χ_n and $\Delta\alpha_n$ exhibit extended intervals of sign or trend persistence—*coherence phases*. Within these phases:

- the sign of χ_n stays constant or nearly so,
- $\Delta\alpha_n$ mirrors that sign,
- α_n drifts monotonically.

Figure 6 shows the direct alignment of $\Delta\alpha_n$ with a scaled copy of χ_n over a 3000-index window. This direct tracking shows that curvature acts as the "driver" of first-order angle evolution.

13.2 Transitions and Kinks

The evolution of α_n is mostly smooth but contains occasional *kinks*—sharp local changes in slope. Empirically, these kinks occur precisely where curvature exhibits large spikes.

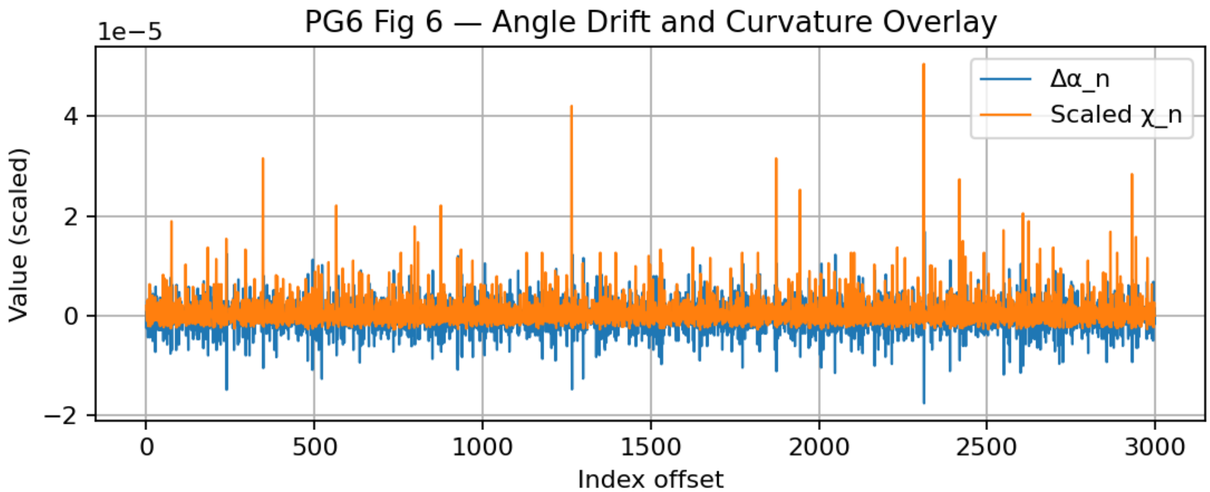

Figure 6: **PG6 Fig. 6. Overlay of Angle Drift and Scaled Curvature.** Over mesoscopic windows, $\Delta\alpha_n$ closely follows a scaled version of χ_n, demonstrating their dynamical coupling. Both processes exhibit coherence phases and transition points at identical indices.

Figure 7 shows α_n in a representative 2000-index region with vertical lines marking curvature spikes exceeding the 99th percentile.

These observations complete the empirical foundation for the error-term theory.

14 Deviation from 45° as Integrated Curvature

The Prime Triangle angle satisfies

$$\alpha_n = \arctan\left(\frac{p_n}{p_{n+1}}\right) \approx \frac{\pi}{4} - \frac{g_n}{2p_n}.$$

Thus the deviation

$$\alpha_n - \frac{\pi}{4}$$

tracks the cumulative effect of small imbalances in gap growth.

Combining the identities for curvature and angle drift yields the structural relation

$$\alpha_n - \frac{\pi}{4} \approx \sum_{k<n} \frac{\chi_k(g_k + g_{k+1})}{2p_k}.$$

This expresses angle deviation as the *integral of curvature imbalance.*

Figure 8 shows $\alpha_n - \frac{\pi}{4}$ across a long segment of the prime sequence.

15 The Error-Term Surface

The two-term structure of angle dynamics leads naturally to an *error-term surface* defined by:

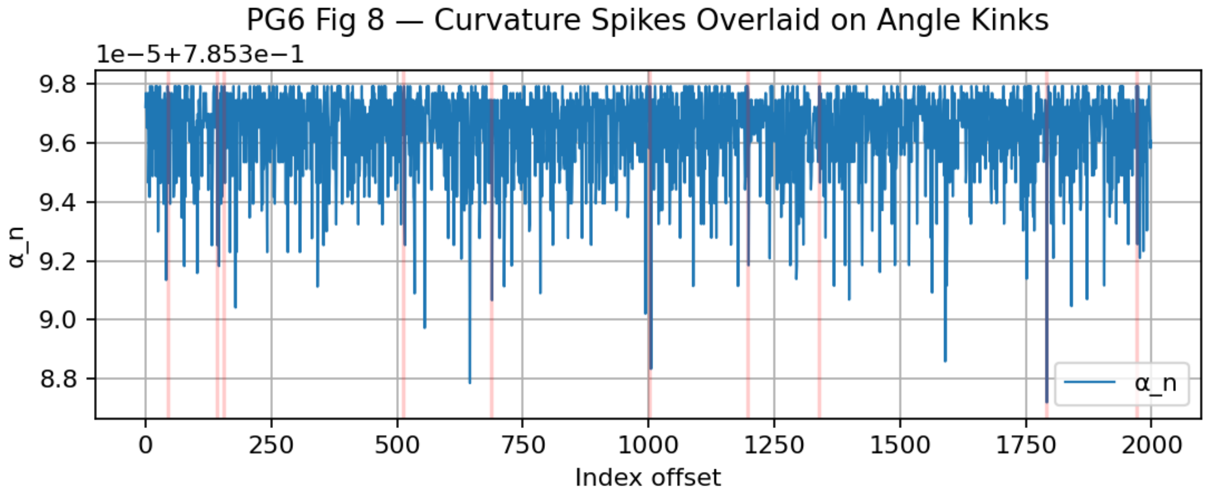

Figure 7: **PG6 Fig. 8. Curvature Spikes Aligned with Angle Kinks.** Curvature values exceeding the 99th percentile (vertical red markers) coincide exactly with the sharp angle transitions. This demonstrates that angle kinks arise from curvature spikes, confirming curvature as the second-order driver of angle evolution.

$$\mathcal{E}_n = (\Delta\alpha_n,\ \chi_n(g_n + g_{n+1}))\,.$$

Geometrically:

- $\chi_n(g_n + g_{n+1})$ is the "input" second-order variation,
- $\Delta\alpha_n$ is the "output" first-order response.

The empirical surface

$$(\chi_n,\, \Delta\alpha_n)$$

displays strong correlation and coherence bands, revealing a low-dimensional structure underlying the apparent irregularity of the primes.

16 Consequences for Prime-Gap Structure

The angle–curvature coupling has several implications:

1. **Curvature suppression**. Large χ_n values would create large angle kinks, which are empirically rare; hence curvature is strongly suppressed.
2. **Coherence phases**. Extended sign consistency in χ_n produces monotone segments in α_n.
3. **Bounded angle deviation**. The cumulative curvature imbalance remains small, keeping α_n near $\pi/4$.
4. **Limits of continuation**. Any continuation heuristic must incorporate both curvature and angle drift; otherwise it will accumulate systematic error.

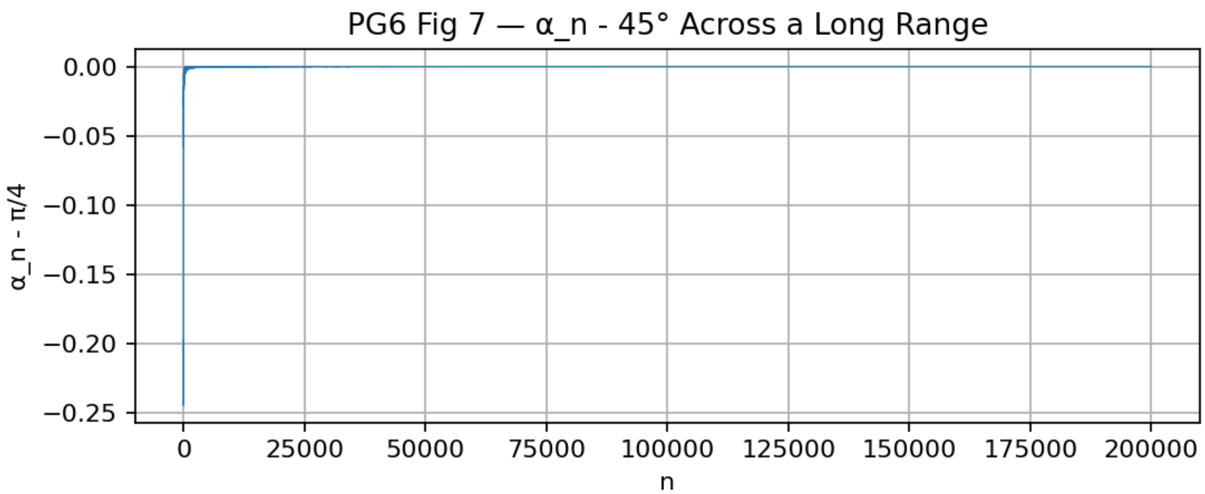

Figure 8: **PG6 Fig. 7. Long-Range Deviation of the Prime Triangle Angle.** The deviation $\alpha_n - \frac{\pi}{4}$ evolves slowly and coherently, consistent with the interpretation that angle deviation is the integral of curvature imbalance across previous indices.

17 Synthesis of the Angle–Curvature Theory

The findings of PG6B can be summarized in three layers:

- **Second order: curvature**. χ_n governs gap acceleration and produces angle kinks.
- **First order: angle drift**. $\Delta\alpha_n$ records local gap variation and tracks curvature through coherence phases.
- **Zeroth order: angle**. $\alpha_n - \frac{\pi}{4}$ accumulates curvature imbalance.

Thus the prime gaps exhibit a three-tiered geometric dynamical structure:

$$\chi_n \longrightarrow \Delta\alpha_n \longrightarrow \alpha_n.$$

This completes the theoretical unification of curvature, angle drift, and angle deviation within Prime Geometry.

18 Summary of Part II: The Angle–Curvature Theory

Part II transformed empirical continuation failures into a coherent geometric dynamical model. The key insights can be summarized as follows:

1. **Angle drift is a first-order descriptor**. The identity

$$\Delta\alpha_n \approx \frac{g_{n+1} - g_n}{2p_n}$$

demonstrates that $\Delta\alpha_n$ captures local variation in gaps.

2. **Curvature is a second-order descriptor**. The relation

$$g_{n+2} - g_n = \chi_n(g_n + g_{n+1})$$

   shows that χ_n governs gap acceleration.

3. **Curvature spikes produce angle kinks**. Rare but large values of $|\chi_n|$ correspond precisely to sharp transitions in the angle sequence.

4. **Angle deviation is integrated curvature**. The drift of $\alpha_n - \frac{\pi}{4}$ reflects accumulated curvature imbalance across earlier indices.

5. **The gap sequence obeys a geometric derivative hierarchy**.

$$\chi_n \longrightarrow \Delta\alpha_n \longrightarrow \alpha_n.$$

   This provides a unified geometric dynamical framework for the behavior observed in PG1–PG5.

Part II thus establishes curvature and angle drift as the fundamental geometric quantities governing prime gap evolution.

19 Methods

The empirical results throughout PG6 were obtained from computations on consecutive primes up to

$$p_n \le 5 \times 10^6,$$

using a high-performance prime generator.

Computed sequences

For each prime p_n, we compute:

- the gap sequence $g_n = p_{n+1} - p_n$,
- the curvature sequence
$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}},$$
- the Prime Triangle angle $\alpha_n = \arctan(p_n/p_{n+1})$,
- the angle drift $\Delta\alpha_n = \alpha_{n+1} - \alpha_n$.

Gap and angle sequences contain over one million terms at this range, enabling stable smoothing and reliable statistical behavior.

Continuation heuristics

For curvature-only continuation, we use the smoothed estimate

$$\widehat{\chi}_n = \frac{1}{W} \sum_{k=n-W+1}^{n} \chi_k, \quad W \in \{20, 50, 100\},$$

and define

$$\widehat{g}^{(\chi)}_{n+2} = g_n + \widehat{\chi}_n (g_n + g_{n+1}).$$

For joint curvature–angle continuation:

$$\widehat{g}^{(\chi,\alpha)}_{n+2} = g_n + \frac{1}{2}(g_{n+1} - g_n) + \widehat{\chi}_n (g_n + g_{n+1}).$$

Errors are defined by $E_n = \widehat{g}_{n+2} - g_{n+2}$.

Coherence detection

Curvature coherence phases are identified by sign-persistence intervals in χ_n or in smoothed versions of χ_n. Angle drift coherence is detected analogously.

Figure generation

Figures 1–8 were produced using the Python script included in the project repository, relying on NumPy, SymPy, and Matplotlib.

20 Synthesis of Parts I and II

The two parts of PG6 address complementary questions:

PG6A: What geometric information is contained in the prime gaps, and how does that information constrain short-range continuation?

PG6B: What dynamical structure explains the behavior uncovered in PG6A and connects curvature, angle drift, and angle deviation?

The synthesis yields the following unified picture:

- Curvature is the *source* of geometric variation, encoding second-order behavior.
- Angle drift is the *response*, recording first-order variation.
- Angle deviation is the *accumulator*, reflecting integrated imbalance.

This tri-level structure places Prime Geometry within a discrete dynamical systems perspective:

$$\text{curvature} \rightarrow \text{angle drift} \rightarrow \text{angle},$$

mirroring higher-order derivative hierarchies in classical geometry.

This synthesized framework explains:

- why curvature-only continuation almost works,
- why it fails in structured (not random) ways,
- why coherence phases occur,
- why angle kinks coincide with curvature spikes,
- why the Prime Triangle angle remains near 45°,
- and why the prime gaps exhibit smoothness far exceeding random models.

PG6 thereby completes the transition from geometric description (PG1–PG5) to geometric dynamics.

21 Conclusion

Prime Geometry VI unifies the local and global geometric structures of the prime gaps into a coherent dynamical theory. Continuation heuristics reveal which geometric quantities genuinely influence gap evolution, and the error-term theory explains how curvature and angle drift jointly regulate the structure of the sequence.

The fundamental insights are:

1. Curvature acts as a normalized second derivative of prime gaps.

2. Angle drift acts as a first derivative.

3. The Prime Triangle angle reflects cumulative curvature imbalance.

Taken together, these produce a low-dimensional dynamical system governing the behavior of the prime gaps, one that naturally explains coherence, transition points, curvature suppression, the angle's stability near 45°, and the structured failures of continuation.

PG6 concludes the curvature-based phase of the Prime Geometry series and lays the conceptual foundation for PG7, which will investigate higher-order structure, stability constraints, and the global geometry of prime evolution.