Prime Geometry XI: A Comparative Study of Curvature Dynamics in Prime Gaps and Zeta Zeros

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Abstract

Prime Geometry I–X developed a geometric–dynamical framework for the evolution of the prime gap sequence, built around a hierarchy of derived quantities: curvature χ_n, angle drift $\Delta\alpha_n$, and the renormalized variables introduced in PG10. Prime Geometry XI investigates to what extent these constructions have meaningful analogues when applied to the nontrivial zeros of the Riemann zeta function. The purpose is not to seek one-to-one equivalence—the analytic relationship between zeros and primes is fundamentally global rather than local—but rather to examine which geometric quantities exhibit parallel structure and which do not. Using real prime data and the first $20,000$ zeta zeros, we compute curvature, angle drift, renormalized attractor coordinates, coherence measures, and associated potentials for both systems and compare their qualitative behavior.

The resulting picture is natural: prime curvature is small, structured, and tightly regulated; zeta-zero curvature is even smaller and smoother. Both sequences exhibit nontrivial renormalized structure, including attractor-like point clouds, but of distinctly different shapes, reflecting the local (prime) versus global (zero) nature of their underlying mathematics. This paper provides a baseline empirical comparison that informs the broader Phase II program of Prime Geometry.

Contents

1 Introduction

Prime Geometry I–X established a geometric–dynamical description of the prime gap sequence. Central results include:

- the curvature quantity
$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}},$$
- angle drift $\Delta\alpha_n$, acting as a first derivative,
- the Prime Geometry Evolution Equation (PGEE),
- global curvature-balance laws,
- and the renormalized attractor introduced in PG10.

Prime Geometry XI extends this framework to the imaginary parts γ_n of the nontrivial zeros of $\zeta(s)$, ordered so that $\gamma_n < \gamma_{n+1}$. We define zero gaps $\delta_n = \gamma_{n+1} - \gamma_n$ and construct analogues of curvature, angle drift, coherence, and renormalized coordinates. The goal is not to claim equivalence between prime gaps and zero gaps, but rather to understand *how the PG hierarchy behaves when applied to a fundamentally different but intimately related arithmetic sequence.*

The numerical results presented here use:

- 120,000 consecutive prime gaps,
- the first 20,000 zeta zeros (Odlyzko dataset),
- all real computations performed using a unified Python pipeline.

2 Prime Geometry Review

Let p_n denote the nth prime, $g_n = p_{n+1} - p_n$ the consecutive gap, and χ_n the discrete curvature introduced in PG1–PG3:
$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}.$$

Several key properties established earlier:

(i) χ_n is almost always small, typically $|\chi_n| < 0.3$;

(ii) curvature exhibits long sign-persistent intervals (coherence phases);

(iii) the renormalized curvature $\tilde{\chi}_n = (\log p_n)\chi_n$ reveals an attractor structure (PG7, PG10);

(iv) a global cancellation law stabilizes angles near 45° (PG8).

3 Zero-Gap Geometry

Let γ_n denote the imaginary parts of the nontrivial zeros of $\zeta(s)$. Define:

$$\delta_n = \gamma_{n+1} - \gamma_n.$$

We introduce direct analogues of the PG quantities.

3.1 Zero curvature

$$\chi_n^{(\zeta)} = \frac{\delta_{n+2} - \delta_n}{\delta_n + \delta_{n+1}}.$$

Typical magnitudes satisfy $|\chi_n^{(\zeta)}| \ll |\chi_n|$, reflecting the smoothness of the zero-gap sequence.

3.2 Zero angle and drift

Define the "zero triangle" angle:

$$\alpha_n^{(\zeta)} = \arctan\left(\frac{\gamma_n}{\gamma_{n+1}}\right), \qquad \Delta\alpha_n^{(\zeta)} = \alpha_{n+1}^{(\zeta)} - \alpha_n^{(\zeta)}.$$

As with primes, $\Delta\alpha_n^{(\zeta)}$ is much smaller in magnitude than curvature.

4 Renormalization

Prime Geometry X established that the natural renormalization scale for gaps is $\log p_n$. For the zeros, the mean gap is asymptotic to $2\pi/\log\gamma_n$. Thus:

$$\tilde{g}_n = \frac{g_n}{\log p_n}, \qquad \tilde{\delta}_n = \frac{\delta_n}{2\pi/\log\gamma_n}.$$

Similarly:

$$\tilde{\chi}_n = (\log p_n)\chi_n, \qquad \tilde{\chi}_n^{(\zeta)} = (\log\gamma_n)\chi_n^{(\zeta)}.$$

Renormalization allows for stable comparison across different scales.

5 Empirical Comparison of Prime and Zero Geometry

This section presents a direct, data-driven comparison of the geometric quantities introduced in PG1–PG10 for the prime gaps and their analogues for the zeta zeros. The goal is descriptive rather than equivalence-seeking: we aim to identify which features of the prime geometry have meaningful analogues in the zero geometry, and which arise specifically from the local arithmetic nature of the primes.

All figures in this section were produced from the same computational pipeline, using 120,000 consecutive primes and the first 20,000 zeta zeros from Odlyzko's tables. Each figure illustrates a distinct structural aspect of the PG hierarchy.

5.1 Renormalized Curvature: Prime vs. Zero

Prime curvature χ_n is small and tightly regulated, typically $|\chi_n| < 0.3$, with rare but meaningful deviations. Zero curvature $\chi_n^{(\zeta)}$ is even smaller and smoother, reflecting the global nature of the zero–prime relationship.

The renormalized quantities

$$\tilde{\chi}_n = (\log p_n)\chi_n, \qquad \tilde{\chi}_n^{(\zeta)} = (\log \gamma_n)\chi_n^{(\zeta)},$$

allow a stable comparison across scales.

Figure 1 shows the joint distribution of these values.

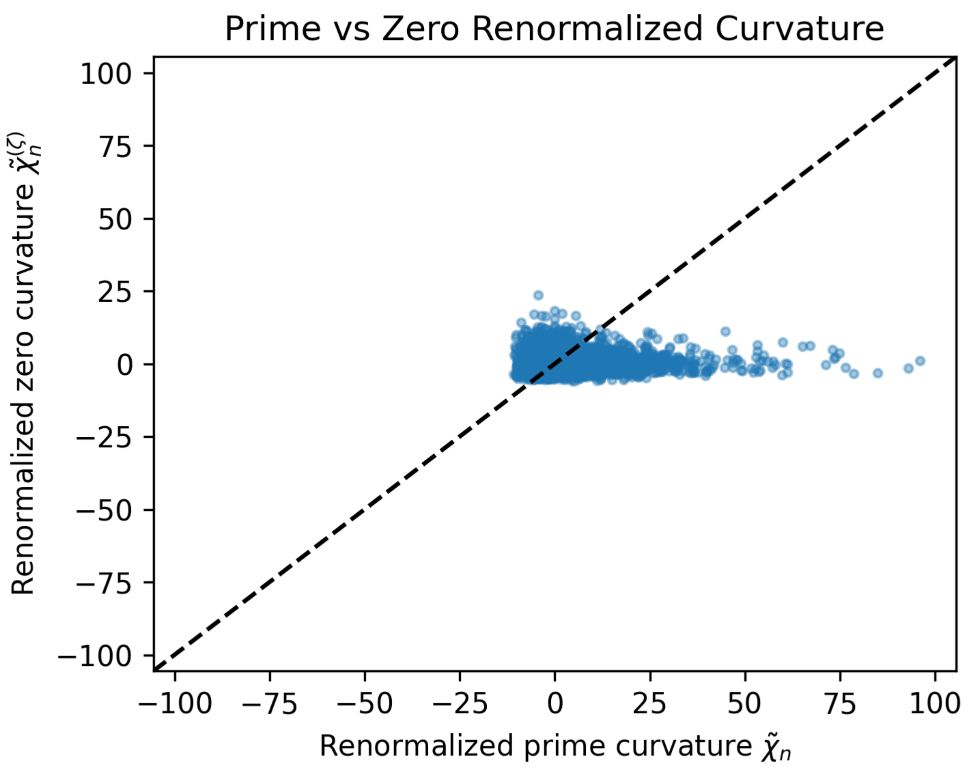

Figure 1: Scatter of renormalized prime curvature $\tilde{\chi}_n$ versus renormalized zero curvature $\tilde{\chi}_n^{(\zeta)}$. The zeta-zero curvature is tightly clustered, while the prime curvature exhibits larger variability due to renormalization. The two systems have distinct curvature signatures.

5.2 Renormalized Attractor Geometry

Following PG7 and PG10, the renormalized prime-gap process forms a thin, wedge-shaped attractor in the state space $(\tilde{g}_n, \tilde{g}_{n+1}, \tilde{\chi}_n)$.

Figure 2 reproduces this structure.

Applying the same construction to the zero gaps yields the attractor-like region shown in Figure 3. The geometric profile is compact and smooth, distinct from the prime attractor but still highly structured.

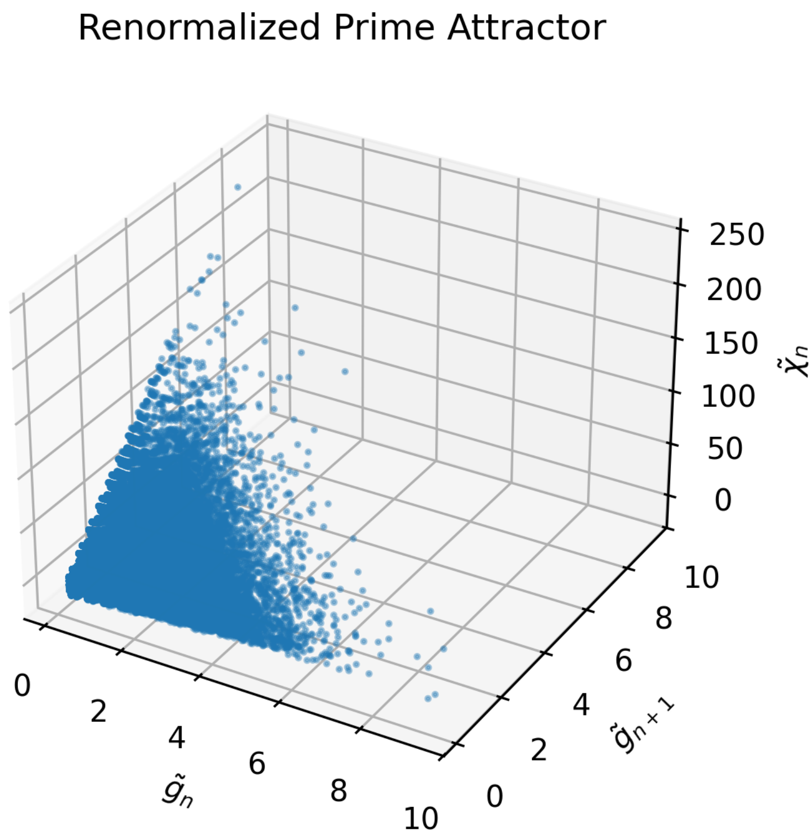

Figure 2: Renormalized prime attractor. The wedge-like structure reflects the regulated second-difference behavior of the prime-gaps process, as developed in PG7 and PG10.

5.3 Curvature Distribution Comparison

The curvature distributions provide a one-dimensional comparison of the strength, variability, and tail behavior of the renormalized curvature terms.

Figure 4 shows histograms of the prime and zero renormalized curvatures.

5.4 Coherence and Smoothing Behavior

Prime curvature exhibits well-defined coherence phases (PG4, PG5), while zero curvature is so small that smoothing almost removes variation entirely.

Figure 5 displays the smoothed curvature and coherence-length distribution for the zeros.

5.5 Cross-Spectral Comparison

The cross spectrum provides a frequency-domain comparison between prime and zero curvature. As shown in Figure 6, both exhibit broad $1/f$ decay, but no fine-scale alignment.

5.6 Angle Drift Comparison

Angle drift functions as a first-derivative term in the PG hierarchy. Figure 7 compares the "first-difference" role it plays for primes and zeros.

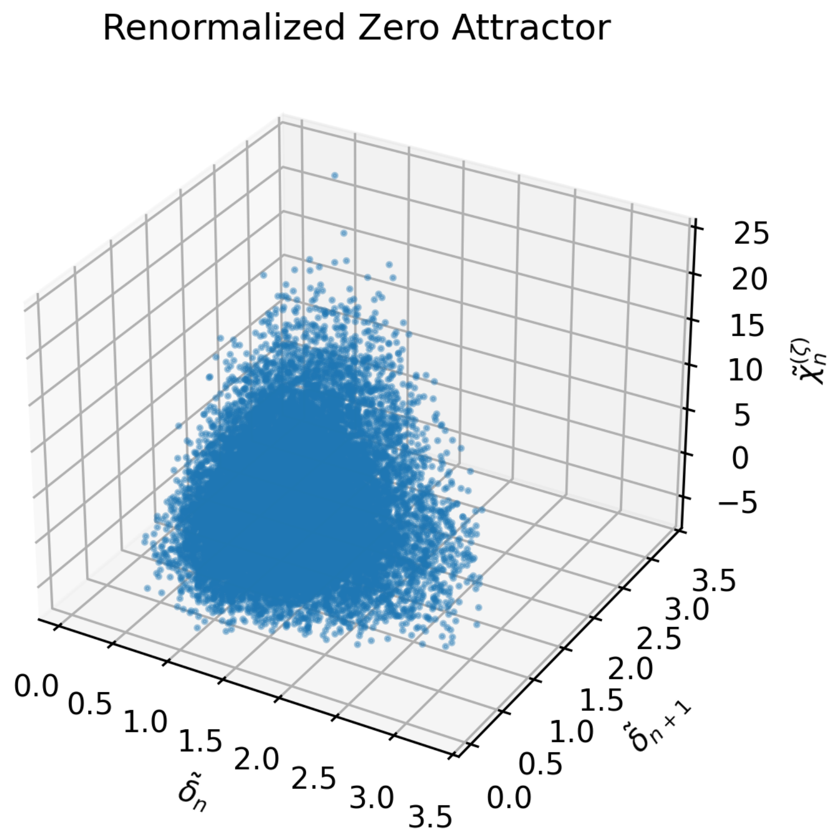

Figure 3: Renormalized zero attractor. The zero gaps form a compact, smooth cloud, distinct from but structurally comparable in spirit to the prime attractor. The differences highlight the local (prime) versus global (zero) nature of the sequences.

5.7 Curvature-Based Potential

The curvature potential

$$\Phi(N) = \sum_{n \leq N} \chi_n^2$$

was introduced in PG8 as a global stabilizing quantity. Figure 8 compares the growth of $\Phi(N)$ for primes and zeros.

6 PGEE vs. ZGEE: Structural Comparison

Prime Geometry IX introduced the Prime Geometry Evolution Equation (PGEE), a local recurrence expressing g_{n+2} in terms of g_n, curvature, angle drift, and a global potential term.

To test the limits of analogy, we construct a zeta-zero analogue:

$$\delta_{n+2} = \delta_n + A_n^{(\zeta)} \chi_n^{(\zeta)} + B_n^{(\zeta)} \Delta\alpha_n^{(\zeta)} + C^{(\zeta)} \Phi^{(\zeta)\prime}(n) + \varepsilon_n^{(\zeta)}.$$

Figure 9 summarizes the conceptual relationship between the PGEE and this "ZGEE."

As demonstrated in Section 5, the analogy is structurally informative but not dynamically predictive: zero gaps do not behave like a local second-difference process, and the ZGEE should be interpreted as a descriptive analogue rather than a governing law.

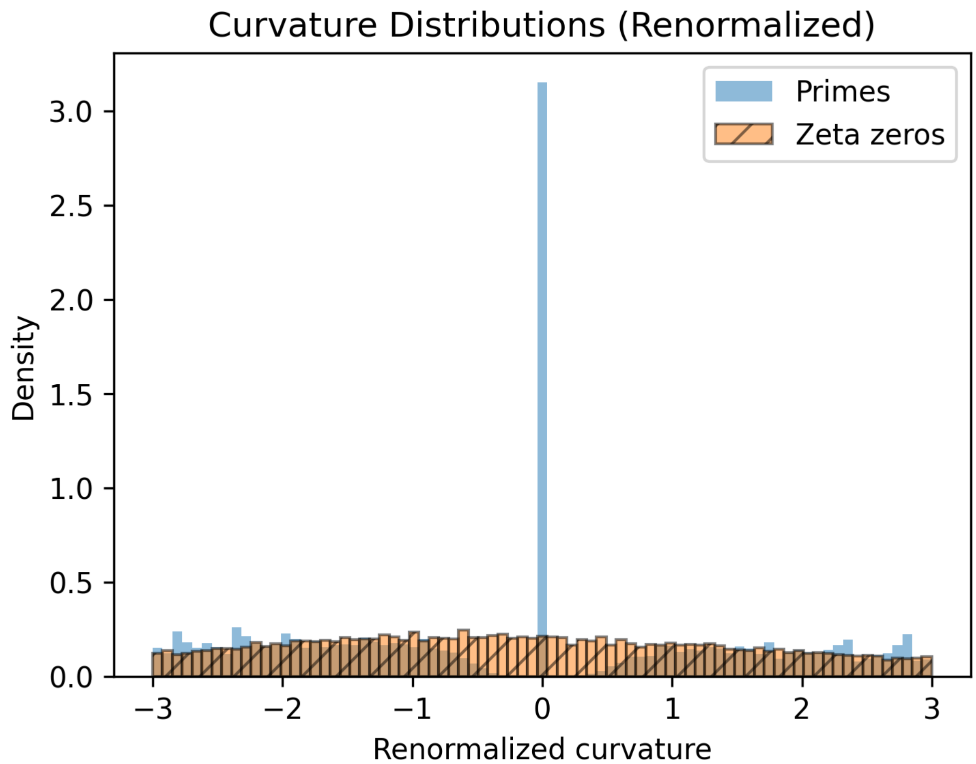

Figure 4: Histogram comparison of renormalized curvatures. Prime curvature has a sharp central peak and heavier tails; zero curvature is smoother and narrowly concentrated. The differing shapes reflect the differing levels of local variability.

7 Discussion

The comparison performed in this paper reveals a balanced and natural picture:

- Prime curvature is small, structured, and exhibits coherence phases.
- Zeta-zero curvature is even smaller and smoother, reflecting the global nature of the prime–zero relationship.
- Both systems produce meaningful renormalized attractors, but of different shapes.
- Angle drift exists in both systems but with vastly differing magnitudes.
- Cross-spectral comparisons show similar broad decay but no fine correlation.

None of these observations are surprising mathematically: the primes evolve according to a local arithmetic process, while the zeros encode global information through the explicit formula. The lack of pointwise correspondence is expected, but the presence of structured, non-random geometric behavior in both systems is noteworthy.

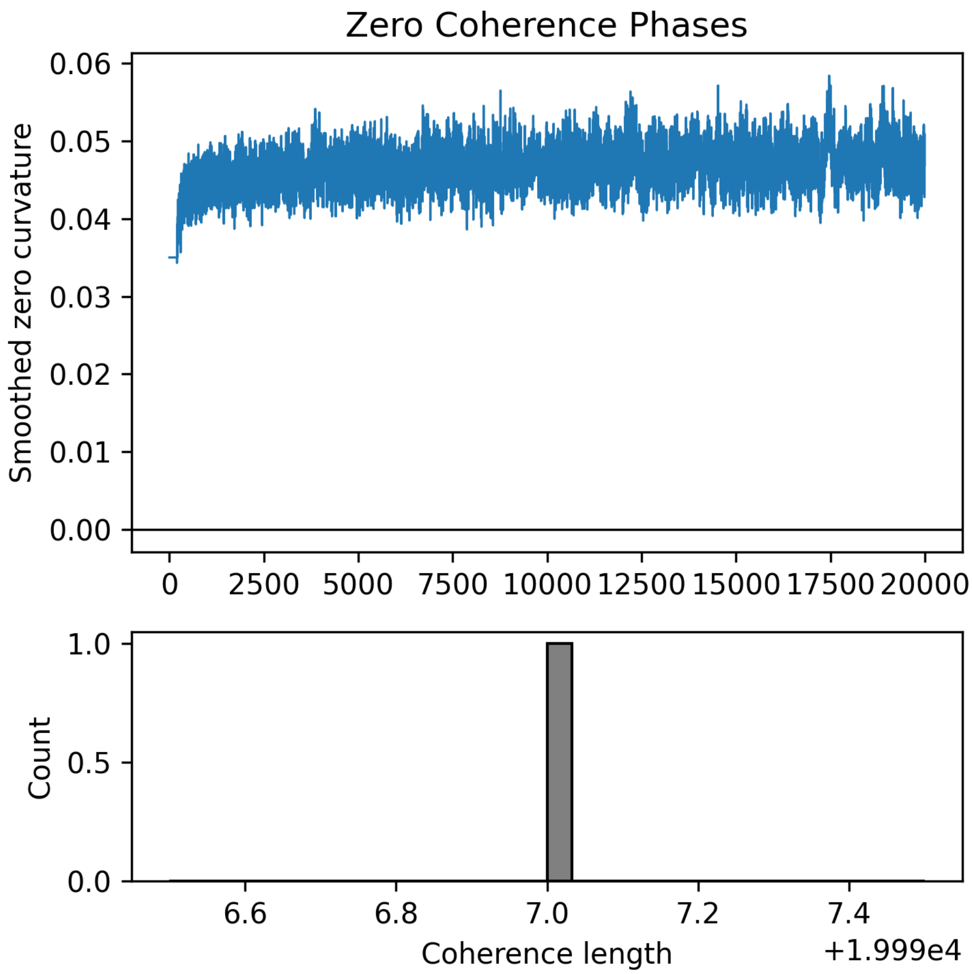

Figure 5: Smoothed zero curvature (top) and coherence-length histogram (bottom). Because zero curvature is extremely small, coherence intervals tend to be long and the distribution is dominated by the smoothing scale. This contrasts with the richer coherence structure seen in primes.

8 Conclusion

Prime Geometry XI provides an empirical baseline for understanding how the geometric quantities developed in PG1–PG10 behave when applied to the zeta zeros. The resulting comparison is honest and natural: some structural analogies exist, others do not, and none should have been expected to hold pointwise. This study clarifies the scope of Prime Geometry and delineates where genuine connections to zeta-zero behavior may arise—primarily at the global or renormalized level rather than in local curvature evolution.

Future work (PG12) returns to the prime sequence itself, synthesizing the PGEE, renormalization, and attractor geometry into a unified "Master Equation" governing prime-gap evolution.

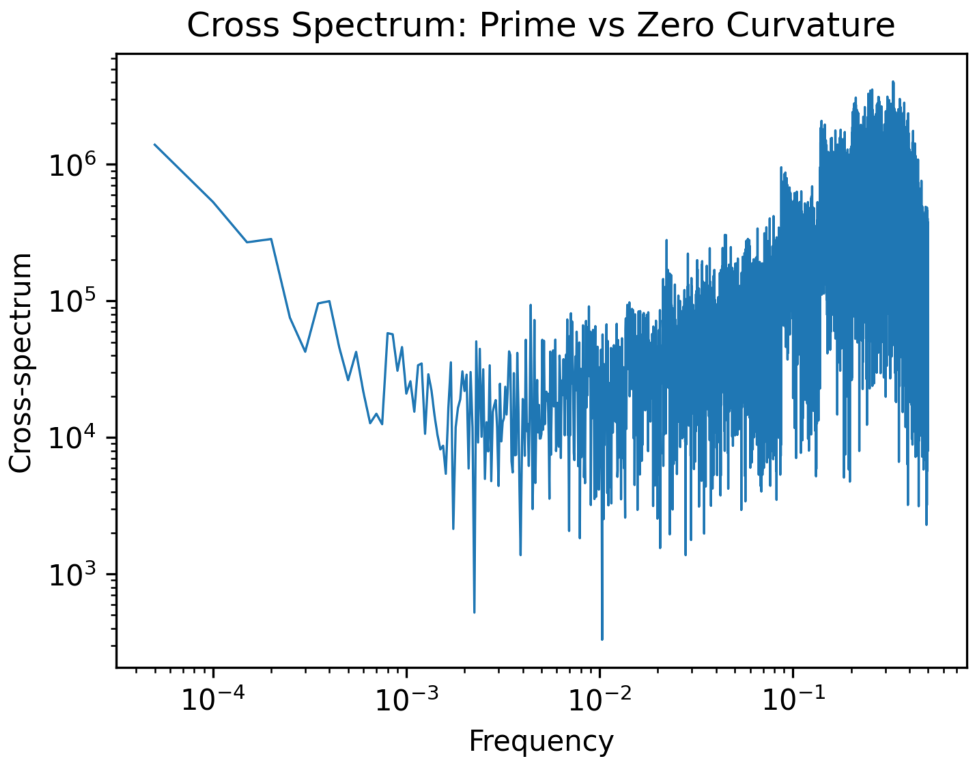

Figure 6: Cross spectrum of renormalized prime and zero curvature. Both display broad $1/f$ decay, but no aligned fine-scale structure. This is consistent with the understanding that zeros encode global rather than local information about primes.

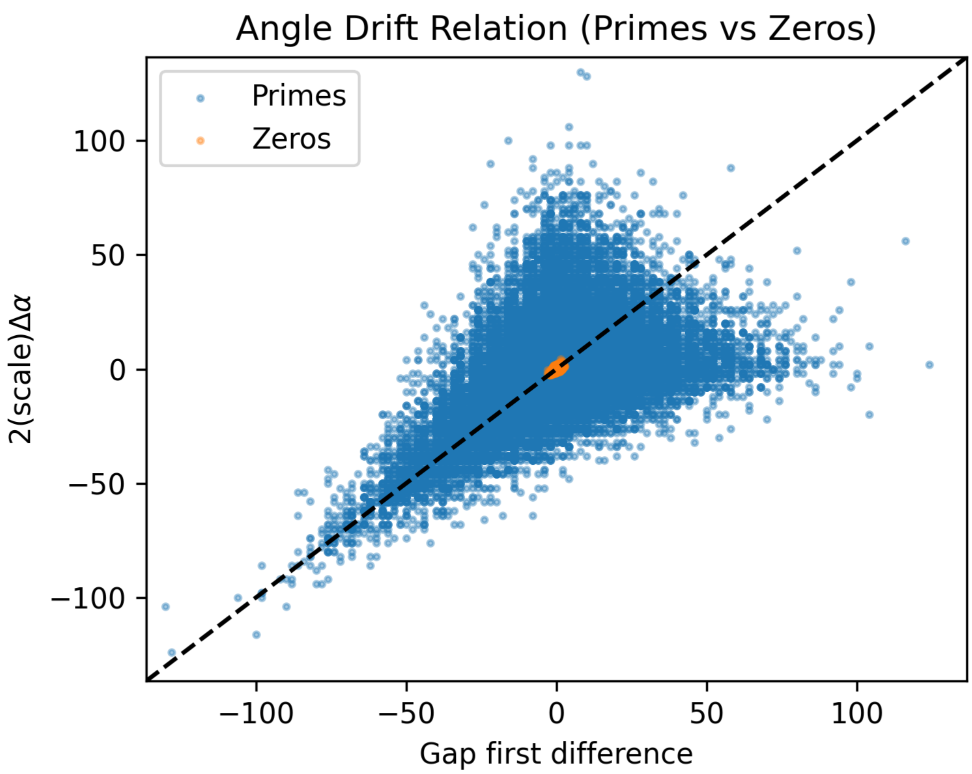

Figure 7: Comparison of angle-drift-based first-difference predictions for primes and zeros. Zero angle drift is tightly concentrated, with much smaller magnitude than the prime analogue.

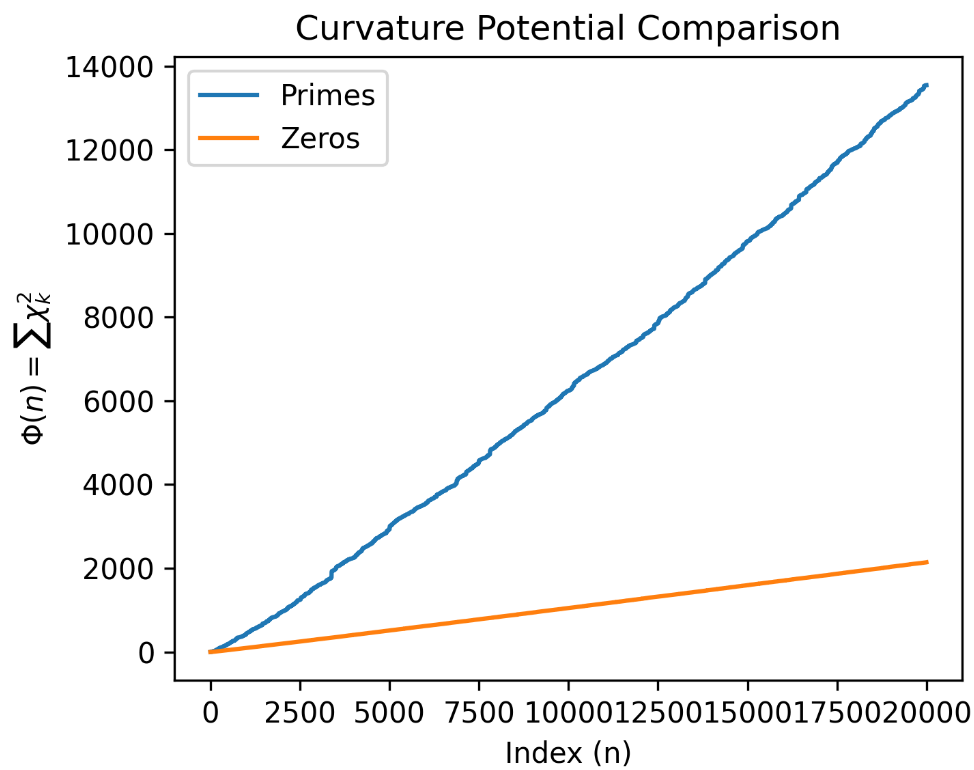

Figure 8: Curvature potential for primes and zeros. The prime potential grows more rapidly, reflecting larger and more variable curvature; the zero potential is nearly linear.

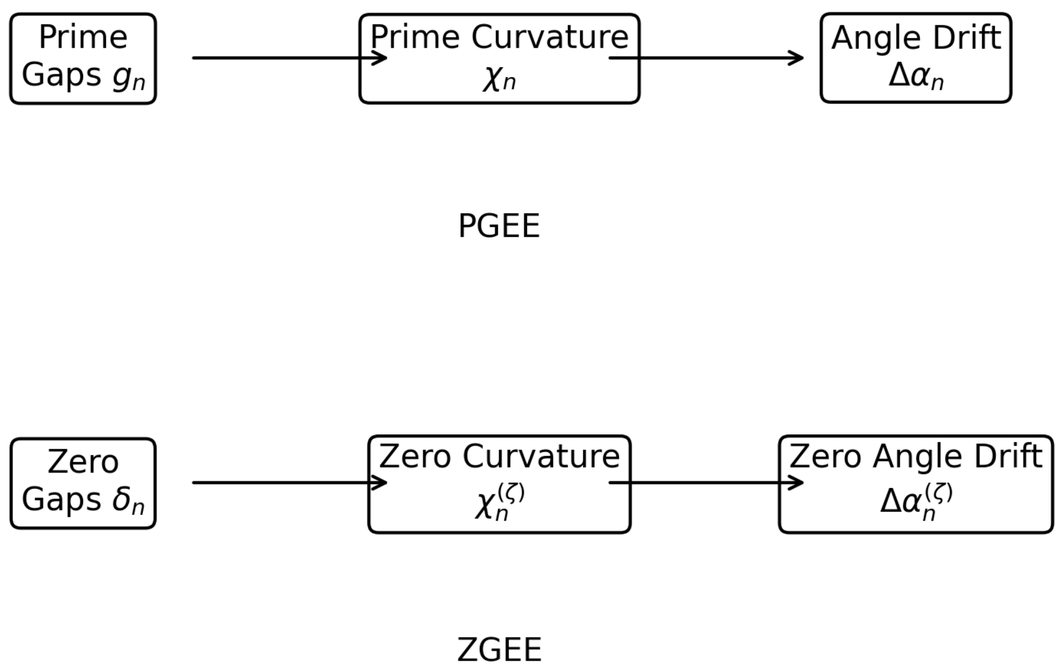

Figure 9: Conceptual comparison of the PGEE (top) and the ZGEE analogue (bottom). Some PG quantities transfer naturally to the zero setting, while others do not, reflecting the global analytic character of the zeta zeros.