Scale-Persistent Geometric and Ordering Structure in the Prime Gap Sequence

Allen Proxmire

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Abstract

We present empirical evidence that the sequence of consecutive prime gaps occupies a region of configuration space not reproduced by standard admissible random reorderings. Two independent probes are examined: (i) a scale-normalized curvature statistic defined on triples of consecutive gaps, and (ii) a rank-based ordering statistic measuring lagged correlations in the gap sequence. Both probes exhibit statistically significant deviations from permutation and block-permutation null ensembles that preserve local gap structure. These deviations persist over the tested prime ranges and strengthen monotonically as the range increases from 10^6 to 5×10^6. Moreover, the lagwise deviation profiles of the two probes are positively correlated under identical null models, suggesting sensitivity to a shared ordering structure. No dynamical law, generative mechanism, or asymptotic claim is proposed; the results establish only finite-range, empirical deviations relative to explicit null ensembles.

1 Introduction

The statistical structure of prime gaps is traditionally studied through the distribution of individual gaps or normalized gap sizes. While such approaches successfully capture first-order properties, they largely treat the ordering of gaps as random beyond trivial local constraints. In this work, we investigate whether ordering and geometric relationships among consecutive gaps exhibit structure that survives admissible randomization.

We adopt a strictly empirical approach. Two independent statistics are defined on the prime gap sequence and compared against randomized null ensembles designed to preserve gap admissibility and short-range structure. Our goal is not explanation, but discrimination: to test whether the observed ordering of prime gaps differs reproducibly from such nulls over the tested computational ranges.

The central question addressed here is therefore limited in scope:

> Do consecutive prime gaps exhibit empirically detectable geometric or ordering structure that is not reproduced by admissible random reorderings?

All claims are finite-range and ensemble-relative.

2 Definitions and Null Models

2.1 Prime gaps

Let p_n denote the nth prime and define the consecutive prime gaps

$$g_n = p_{n+1} - p_n.$$

All experiments are conducted on the finite sequence $\{g_n\}$ truncated at $p_n \leq p_{\max}$.

2.2 Curvature statistic

We define a scale-normalized curvature statistic on triples of consecutive gaps by

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}, \qquad 1 \leq n \leq N - 3.$$

This quantity measures a signed second-difference structure normalized by local gap scale. It is dimensionless and invariant under uniform rescaling $g_n \mapsto c\,g_n$.

Aggregate curvature behavior is summarized using statistics derived from $\{\chi_n\}$, including quadratic energies and lagged dependence measures.

2.3 Ordering dependence (Spearman statistic)

To probe ordering structure directly, we compute lagged Spearman rank correlations. For a sequence x_n, define

$$\rho_x(k) = \mathrm{Spearman}(x_n, x_{n+k}), \qquad k \geq 2,$$

where lag $k = 1$ is excluded due to known local admissibility effects that induce trivial anticorrelations.

This statistic is applied both to the gap sequence $x_n = g_n$ and to the curvature sequence $x_n = \chi_n$.

2.4 Null ensembles and Z-scores

Two admissible null ensembles are used:

1. **Permutation null**: the gap multiset $\{g_n\}$ is randomly permuted.

2. **Block-permutation null**: the gap sequence is partitioned into contiguous blocks of fixed length B (default $B = 32$), which are permuted as units.

Block permutation preserves short-range structure while destroying longer-range ordering. All significance statements are reported relative to this more conservative null.

For any statistic T, Z-scores are computed as

$$Z = \frac{T_{\mathrm{obs}} - \mu_{\mathrm{null}}}{\sigma_{\mathrm{null}}},$$

where μ_{null} and σ_{null} are the sample mean and standard deviation over M independent null realizations. Ensemble sizes M and block sizes B are reported explicitly by the accompanying code.

Because multiple lags are scanned, individual Z-threshold exceedances are interpreted descriptively. Summary quantities (fraction of significant lags, longest run, aggregate excess) are used as comparative diagnostics rather than formal hypothesis tests.

3 Empirical Results

3.1 Curvature extremality

Curvature-derived quadratic energies computed from $\{\chi_n\}$ are systematically lower for the true prime sequence than for both permutation and block-permutation nulls. This separation persists across all tested values of $p_{\max}$.

We therefore treat the curvature statistic as a robust empirical discriminator relative to these null ensembles, without implying invariance in a mathematical or asymptotic sense.

3.2 Ordering dependence of prime gaps

The ordering statistic $\rho_g(k)$ exhibits positive dependence across a wide range of lags. Relative to block-permutation nulls, deviations persist up to the maximum tested lag $k = 150$ and increase monotonically with $p_{\max}$.

A descriptive coherence summary for block size $B = 32$ and threshold $Z > 2$ is shown below:

$p_{\max}$	# gaps	last sig. lag	frac. sig.	longest run	excess
10^6	78,497	148	0.597	10	102.47
2×10^6	148,932	147	0.758	14	157.39
5×10^6	348,512	150	0.879	87	271.15

These trends indicate increasing deviation from null expectations over the tested range, but do not by themselves establish a scaling law.

3.3 Cross-statistic comparison

To assess whether the curvature statistic and the ordering statistic respond to similar structure, we compare their lagwise Z-score profiles:

$$Z_g(k), \quad Z_\chi(k), \qquad k = 2, \ldots, 150.$$

Across the tested ranges, these profiles exhibit positive Pearson and Spearman correlations on the order of 0.5–0.6. This correlation is consistent with the interpretation that both statistics are sensitive to overlapping features of gap ordering, though alternative explanations (e.g. shared sensitivity to slow trends or broad spectral structure) cannot be excluded on the basis of these data alone.

4 Interpretation and Limits

Taken together, the results provide finite-range empirical evidence that the ordering of prime gaps differs from that produced by admissible random reorderings, in ways detectable by both geometric and rank-based statistics.

We emphasize the limits of these conclusions:

- The results are empirical and ensemble-relative.

- No asymptotic, dynamical, or generative claims are made.

- Terms such as "constraint" and "structure" refer only to observed deviations over the tested ranges.

In particular, these findings do not imply an equation of motion, a variational principle, or any direct connection to zeta zeros or the Riemann Hypothesis.

5   Reproducibility

All experiments were performed using deterministic Python pipelines. Prime generation, null ensemble construction, Z-score computation, and summary diagnostics are fully reproducible using the archived scripts. Parameters reported by the code include $p_{\max}$, ensemble sizes M, block sizes B, and all derived statistics.