Prime Geometry II: Curvature Dynamics of Prime Gaps

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Abstract

Prime gaps are usually treated as weakly dependent random fluctuations of size $\Theta(\log p_n)$, modeled effectively by Cramér's probabilistic framework and the Hardy–Littlewood k-tuple conjectures. In this paper we develop a geometric–dynamical perspective on *curvature* in the prime gap sequence.

For consecutive gaps $g_n = p_{n+1} - p_n$, we define the discrete curvature

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}, \qquad L_n = \chi_n^2,$$

where L_n acts as a local action density. Through computational experiments up to 50,000 primes, we find:

- the total action $S = \sum_n L_n$ for the true prime sequence lies in the bottom $\sim 1\%$ of the distribution formed by random permutations of the same gaps,
- the curvature distribution of primes has dramatically lighter tails and smaller variance than both random permutations and Cramér pseudo-primes,
- sliding-window and multi-scale heatmaps reveal long coherent epochs where gaps predominantly expand or contract,
- return maps (χ_n, χ_{n+1}) and (g_n, g_{n+1}) show tight geometric constraints unique to true primes and absent in both random and Cramér models.

These results indicate that the prime gap sequence behaves not as a memoryless random process, but as a low-curvature dynamical system with multi-scale coherence. This "curvature dynamics" provides a new geometric window into the structure of the primes.

1 Introduction

The statistical behavior of prime gaps has been studied for more than a century, with landmark developments such as:

- Cramér's random model,
- the Hardy–Littlewood prime k-tuple conjectures,
- Montgomery's pair correlation and Dyson's random-matrix analogy.

These perspectives view primes through the lens of *randomness*. In this paper we adopt a complementary viewpoint: a geometric–dynamical description of how consecutive prime gaps *transition* from one to the next.

This continues the "Prime Geometry" framework initiated in PG1, but shifts the focus from algebraic identities to dynamical behavior.

2 Curvature Framework

Let (p_n) be the sequence of primes and

$$g_n = p_{n+1} - p_n$$

its consecutive gaps.

2.1 Discrete curvature

We define curvature by

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}.$$

This quantity measures whether consecutive gaps are opening ($\chi_n > 0$), closing ($\chi_n < 0$), or remaining balanced ($\chi_n \approx 0$).

2.2 Action density

We define the local action

$$L_n = \chi_n^2, \qquad S = \sum_n L_n.$$

Large values of L_n correspond to rapid curvature transitions. A sequence with minimal S exhibits "smooth" curvature flow.

3 Experiment I: Action Minimization

To test whether the prime gaps arrange themselves in a curvature-minimizing way, we compare the total action S_{true} with the distribution of S_{perm} arising from random permutations of the same gaps.

Random permutations preserve all gap frequencies but destroy ordering information.

For 50,000 primes:

$$S_{\text{true}} \approx 3.716 \times 10^4, \qquad \mathbb{E}[S_{\text{perm}}] \approx 3.880 \times 10^4,$$

placing the true action in the bottom $\sim 1\%$ of the permutation distribution. Thus the prime sequence behaves as a *least-action path* through gap space.

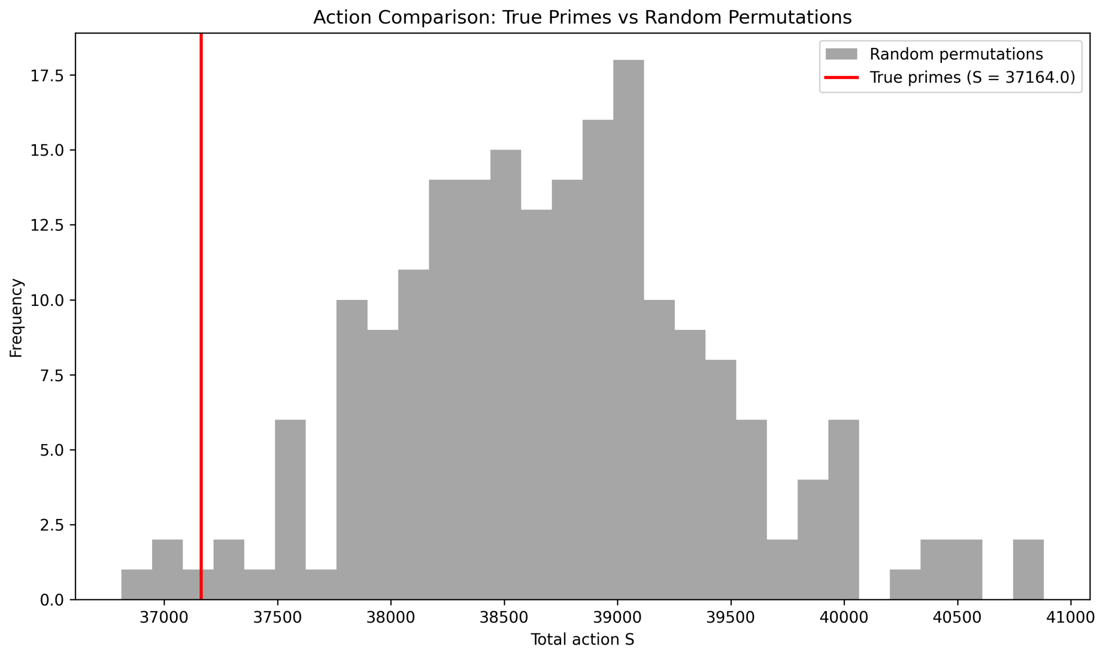

Figure 1: **Action comparison.** The total action S of the prime sequence lies far below the bulk of the distribution from random permutations, indicating significantly suppressed curvature transitions.

4 Experiment II: Curvature Distribution

We compare the empirical distribution of curvature values χ_n for:

- true primes,
- random permutations of gaps,
- Cramér pseudo-primes.

True primes exhibit:

- a sharply concentrated peak,
- lighter tails,
- maximum $|\chi_n|$ around 14,

while random permutations reach larger extremes (~ 27), and Cramér gaps produce χ_n values exceeding 50.

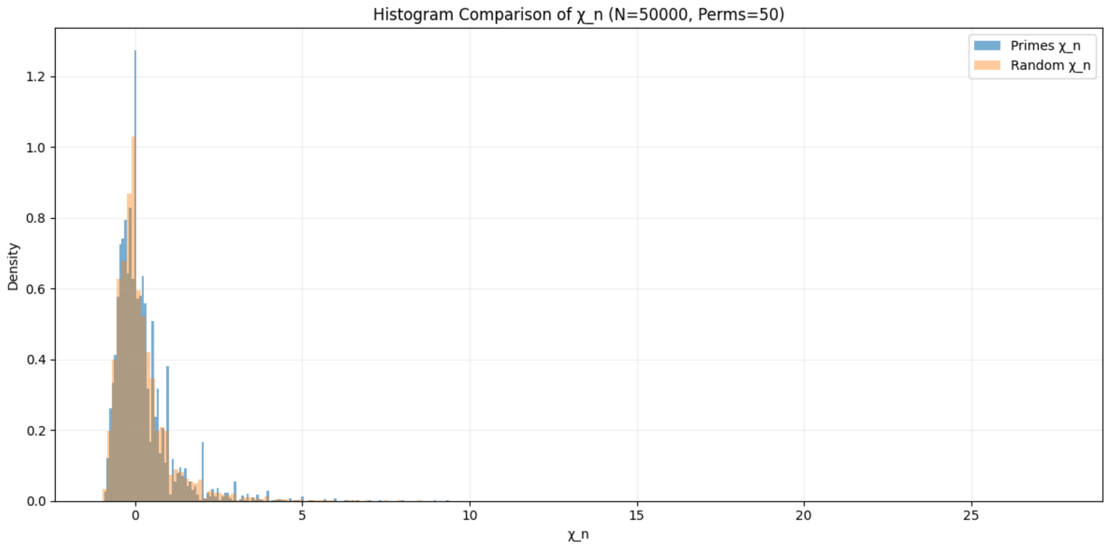

Figure 2: **Distribution of curvature χ_n.** True primes have dramatically lighter tails and a narrower distribution than either random permutations or Cramér pseudo-primes, indicating suppressed curvature fluctuations.

5 Experiment III: Sliding-Window Curvature

Sliding-window averages of L_n reveal structured behavior. For a window size $W = 1000$, the prime sequence exhibits alternating valleys and peaks of curvature that persist across thousands of indices.

Random permutations, which share the same gap multiset, do not show this structure.

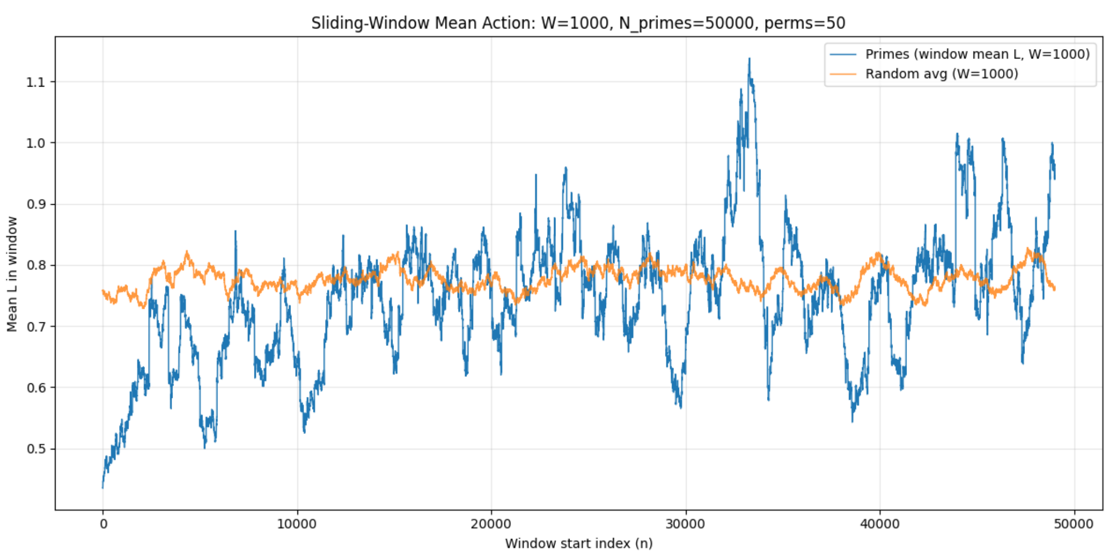

Figure 3: **Sliding-window mean L_n.** Primes show long coherent intervals of high or low curvature, while random permutations flatten into noise.

6 Experiment IV: Multi-Scale Heatmaps

To probe structure across scales, we compute moving averages over window sizes:

$$W \in \{200, 500, 1000, 2000, 5000\}.$$

6.1 Mean action

The heatmap of L_n shows persistent vertical bands—regions of low or high curvature that survive coarse-graining across all scales.

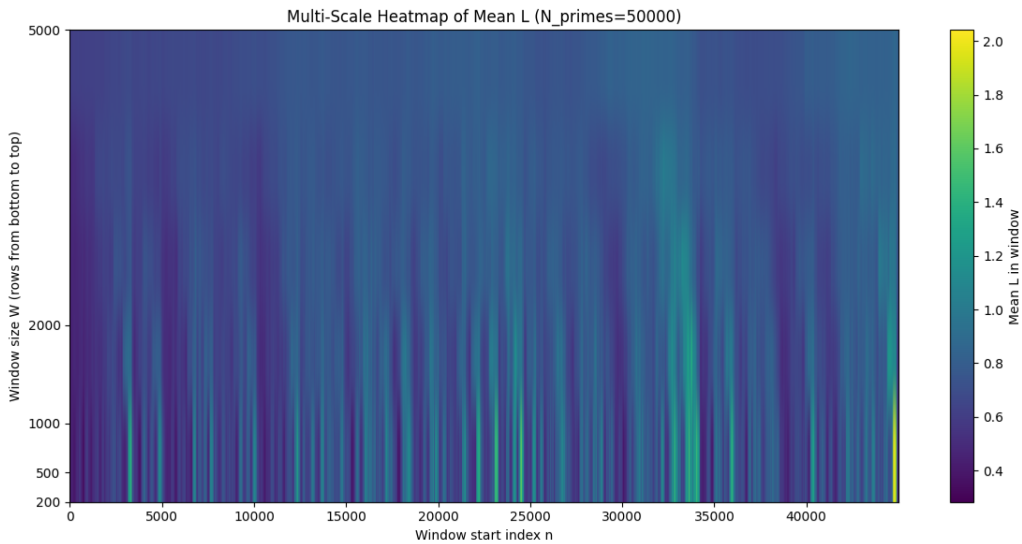

Figure 4: **Multi-scale heatmap of mean L_n.** Vertical bands persisting across window sizes indicate stable curvature structure across scales.

6.2 Mean curvature

The sign-coherence of χ_n is even more striking. Large-scale red/blue regions correspond to epochs in which the gaps persistently expand or contract.

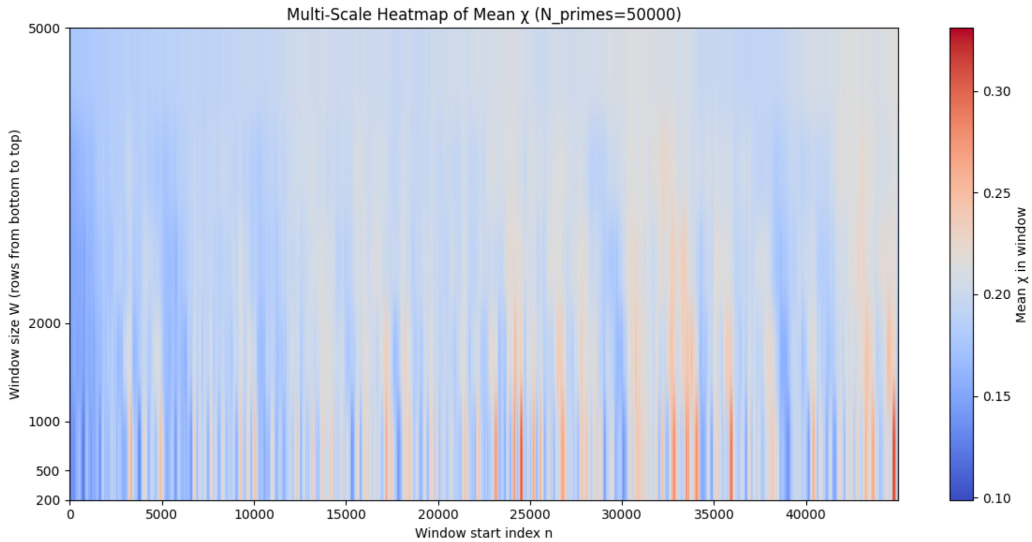

Figure 5: **Multi-scale heatmap of mean χ_n.** Red/blue regions show long periods of consistent gap expansion or contraction—directional phases absent in random or Cramér models.

7 Experiment V: Return-Map Geometry

Return maps encode how the prime gap system evolves from one step to the next.

7.1 Gaps

The gap map (g_n, g_{n+1}) reveals a wedge-shaped region with strong geometric constraints, unique to true primes.

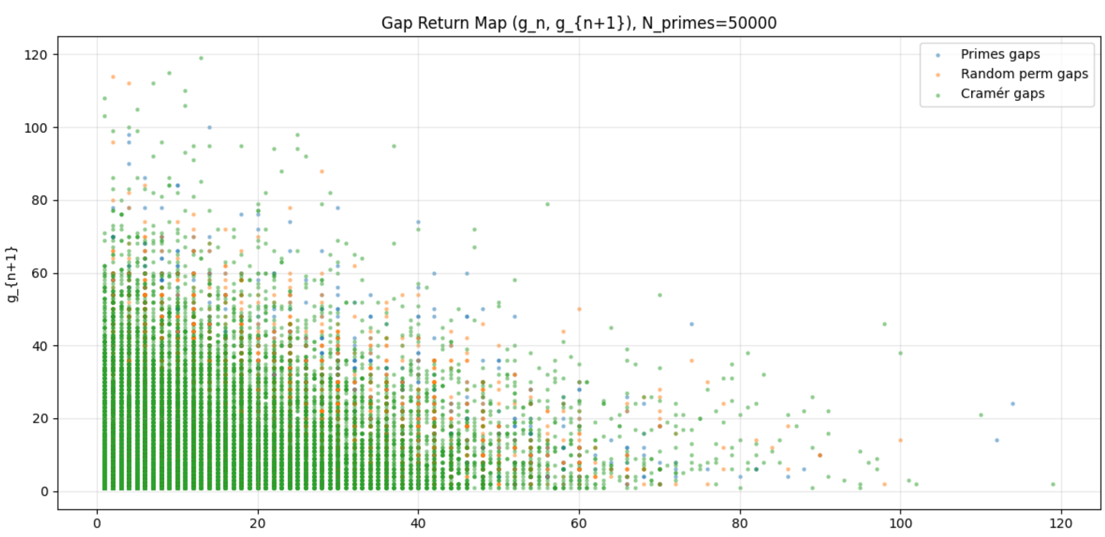

Figure 6: **Gap return map (g_n, g_{n+1}).** True primes occupy a narrow geometric region with clear boundaries. Random permutations and Cramér gaps populate the plane much more uniformly.

7.2 Curvature

The return map (χ_n, χ_{n+1}) contains the strongest evidence of a dynamical system. True primes form a compact cluster near $(0, 0)$, bounded and highly structured. Random permutations are more diffuse, while Cramér gaps explode into large chaotic curvature.

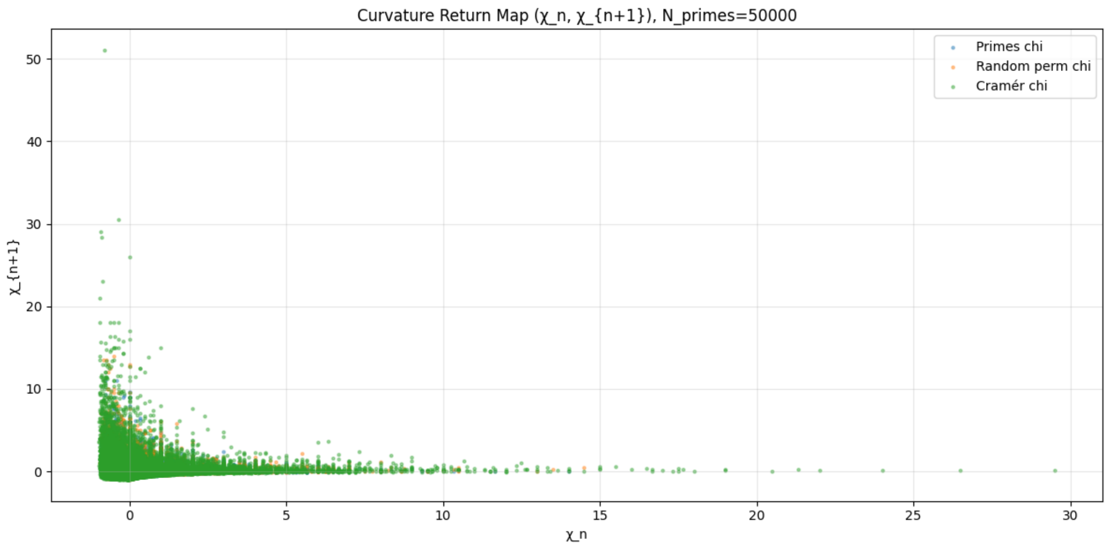

Figure 7: **Curvature return map** (χ_n, χ_{n+1})**.** Prime curvature evolves within a tightly constrained attractor-like region, unlike random permutations or Cramér pseudo-primes, which produce large, unstructured curvature transitions.

8 Synthesis

Across all experiments we observe a unified phenomenon:

> **Prime gaps evolve through a low-curvature, self-constrained dynamical process with multi-scale coherence and geometric regularity.**

No existing random model reproduces the combination of:

- least-action behavior,
- suppressed curvature distribution,
- long coherent directional phases,
- multi-scale heatmap structure,
- constrained return-map geometry.

Curvature dynamics thus offers a new geometric lens for interpreting prime structure, complementing classical analytic and probabilistic frameworks.

Acknowledgments

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