Prime Geometry V:
Spectral Structure, Angle Dynamics, and Higher-Order Geometry in the Prime Gap Sequence

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Abstract

Prime Geometry I–IV developed a curvature-based geometric framework for the prime gaps. PG1 introduced the Prime Triangle identity and derived geometric quantities such as energy, curvature, and a normalized square-difference (PSDn). PG2 examined the empirical behavior of the curvature quantity $\chi_n = \frac{g_{n+2}-g_n}{g_n+g_{n+1}}$ and its local action $L_n = \chi_n^2$, comparing true primes with randomized models. PG3 established the Curvature-Based Recurrence $g_{n+2} = g_n + \chi_n(g_n + g_{n+1})$ and studied perturbation sensitivity. PG4 synthesized these results into a structural picture of curvature constraints, coherence phases, and globally suppressed curvature energy.

Prime Geometry V extends this framework in three directions: (1) spectral analysis of curvature and its smoothed forms, revealing stable low-frequency structure; (2) angle–curvature interaction, relating curvature to drift in the Prime Triangle angles; and (3) higher-order geometric signals based on second differences of the gaps and energy.

All results are empirical and descriptive. No mechanism is proposed. The goal is to consolidate a multi-scale geometric–dynamical view of the prime gap sequence spanning local, mesoscopic, global, and spectral behavior.

1 Introduction

Let (p_n) denote the sequence of primes and define the consecutive gaps

$$g_n = p_{n+1} - p_n.$$

Prime Geometry I introduced a right-triangle construction on consecutive primes, leading to hypotenuse lengths

$$C_1 = \sqrt{p_n^2 + p_{n+1}^2}, \qquad C_2 = \sqrt{p_{n+1}^2 + p_{n+2}^2}$$

and an identity connecting $(C_2 - C_1)(C_1 + C_2)$ to the skip-one square difference $(p_{n+2}^2 - p_n^2)$. From this, a first-order approximation produced an energy increment E_n and a discrete curvature K_n, which gave rise to the normalized curvature ratio

$$\chi_n \approx \frac{g_{n+2} - g_n}{g_n + g_{n+1}}.$$

Prime Geometry II investigated this curvature quantity empirically, recording its distribution, cumulative action

$$S(N) = \sum_{n \le N-2} L_n, \qquad L_n = \chi_n^2,$$

sliding-window averages, and return-map geometry, in comparison with random permutations of the gaps and Crámer-type pseudo-primes. PG3 derived the exact recurrence

$$g_{n+2} = g_n + \chi_n(g_n + g_{n+1}),$$

and showed that curvature is highly sensitive to perturbations in gap ordering. PG4 studied the structural implications: small curvature dominates, curvature extremes are suppressed, smoothed curvature exhibits coherence phases, and the cumulative action $S(N)$ for true primes lies deep in the lower tail of the permutation distribution.

Prime Geometry V continues this development along three lines:

- spectral analysis of χ_n and its smoothed versions;
- geometric interpretation of curvature via Prime Triangle angles;
- higher-order geometric signals derived from second differences of the gaps and energy.

Together, these extend the curvature framework into a more complete geometric–dynamical system.

2 Curvature and Angle Framework

2.1 Curvature and Local Action

We fix notation from earlier papers. For three consecutive gaps g_n, g_{n+1}, g_{n+2}, define

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}},$$

and the local curvature magnitude

$$L_n = \chi_n^2.$$

The cumulative curvature (or local action) over an interval is

$$S(N) = \sum_{n=1}^{N-2} L_n.$$

PG2 and PG3 showed that χ_n is typically small, with a large fraction of indices satisfying $|\chi_n| \leq 0.1$, and that $S(N)$ for true primes lies in the extreme lower tail of the permutation distribution.

2.2 Curvature-Based Recurrence

Rewriting the definition of χ_n gives the exact identity

$$g_{n+2} = g_n + \chi_n(g_n + g_{n+1}).$$

We refer to this as the Curvature-Based Recurrence. It is not predictive (because χ_n is not known in advance), but it constrains how triples of gaps relate structurally. When $|\chi_n|$ is small, the recurrence forces g_{n+2} to remain close to g_n relative to their scale.

2.3 Prime Triangle Angles

The Prime Triangle construction associates to each pair of consecutive primes a right triangle with legs p_n and p_{n+1}. The corresponding angle at the origin is

$$\alpha_n = \arctan\left(\frac{p_n}{p_{n+1}}\right).$$

Since $p_{n+1} \sim p_n$, one has α_n near 45° for large n. We measure angle drift by

$$\Delta\alpha_n = \alpha_{n+1} - \alpha_n.$$

Curvature controls second-order variation in the gaps; angle drift reflects first-order geometric reorientation of the triangles. One aim of PG5 is to clarify how these two quantities interact.

3 Spectral Structure of Curvature

PG2–PG4 documented that χ_n is small and exhibits long coherence phases when smoothed. Prime Geometry V examines this structure in the frequency domain.

3.1 Data and Methodology

We consider the curvature sequence χ_n over a large range of primes (e.g., up to a fixed bound on p_n) and its smoothed versions

$$\chi_n^{(W)} = \frac{1}{W} \sum_{k=n-W+1}^{n} \chi_k$$

for window sizes $W = 500, 2000, 5000, 10000$. To analyze spectral content, we estimate power spectral densities (PSD) of χ_n and $\chi_n^{(W)}$ using Welch's method with a fixed segment length and standard overlap.

For comparison, we generate a number of random permutations of χ_n (or equivalently of the gap sequence) and compute the corresponding PSDs, averaging them to form a permutation baseline.

3.2 Power Spectrum of χ_n

Figure 1 displays the power spectrum of χ_n (in log–log coordinates) together with the mean spectrum of permutation models.

Empirically, the spectrum for true primes is dominated by low frequencies: there is a clear excess of spectral power in the lowest frequency bands relative to permutations. The permutation spectra, by contrast, are flatter and more typical of noise-like behavior.

3.3 Spectra of Smoothed Curvature

To test the robustness of this structure, we examine the smoothed curvature $\chi_n^{(W)}$ and compute its PSD. Figure 2 shows the spectrum for $W = 2000$.

The same qualitative pattern persists: smoothed curvature retains strong low-frequency concentration, reflecting long coherence phases where $\chi_n^{(W)}$ keeps a consistent sign. This confirms that the spectral structure is not an artifact of local fluctuations but arises from genuinely long-range organization.

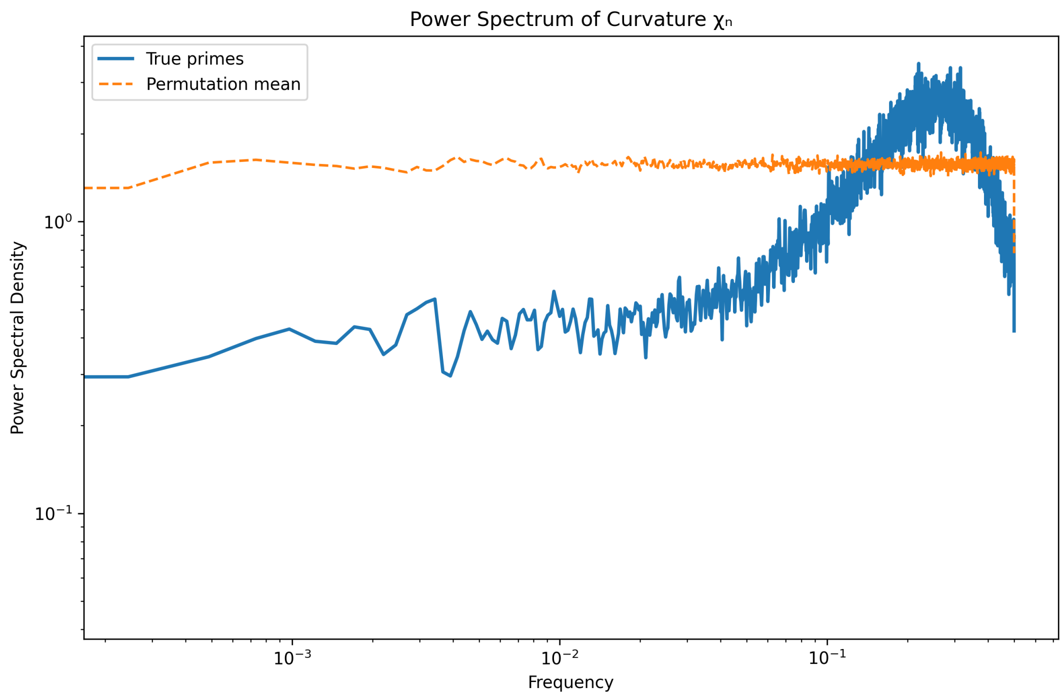

Figure 1: Power spectrum of the curvature sequence χ_n for true primes compared with the mean spectrum of permuted sequences. The prime spectrum shows strong low-frequency concentration that is not present in the permutation average.

3.4 Summary of Spectral Behavior

Across all tested ranges and window sizes, the spectral analysis suggests:

- χ_n carries significant low-frequency power corresponding to extended coherence phases.
- Random permutations of the same curvature values (or gaps) produce substantially flatter spectra with no comparable low-frequency bias.
- Smoothing χ_n reinforces rather than erodes the low-frequency structure.

Thus, curvature is not only small and constrained in amplitude but also structured in the frequency domain.

4 Angle Dynamics and Curvature Interaction

Curvature measures second-order variation of gaps; angle drift measures first-order geometric re-orientation. This section relates the two.

4.1 Heuristic Link Between χ_n and $\Delta\alpha_n$

Using the Curvature-Based Recurrence,

$$g_{n+2} - g_n = \chi_n(g_n + g_{n+1}),$$

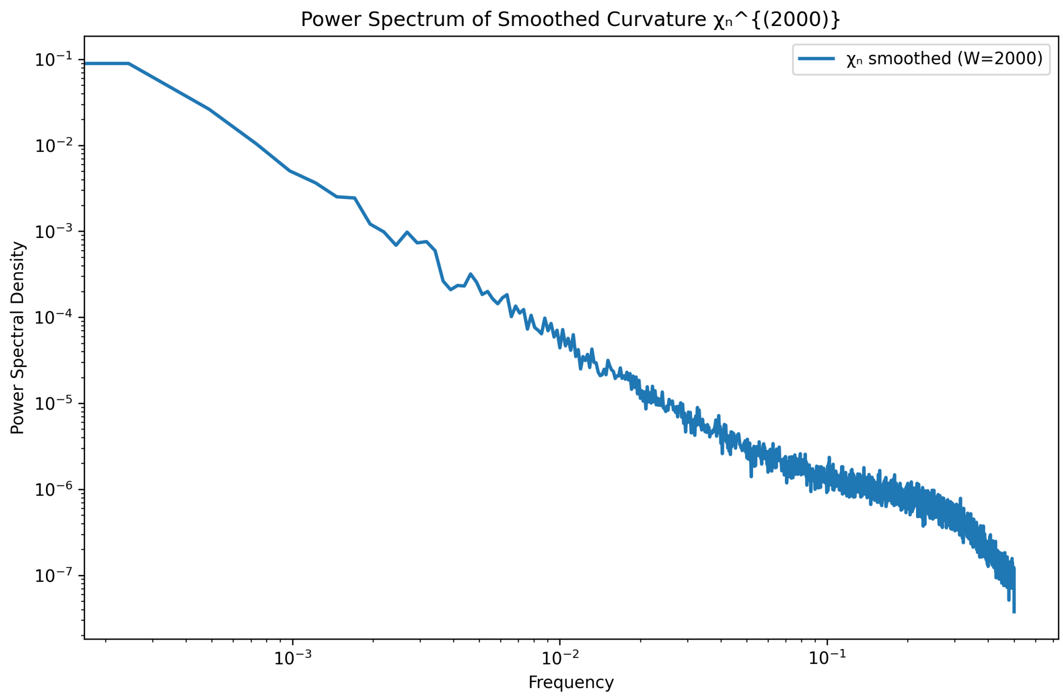

Figure 2: Power spectrum of smoothed curvature $\chi_n^{(2000)}$. Low-frequency dominance persists under smoothing, indicating that coherence phases are a stable feature of the curvature signal.

and first-order approximations for the change in α_n, one obtains

$$\Delta\alpha_n = \alpha_{n+1} - \alpha_n \approx \frac{g_{n+1} - g_n}{2p_n} + (\text{smaller terms}).$$

Since $g_{n+1} - g_n$ itself can be expressed in terms of curvature and local gap scale, this shows that $|\Delta\alpha_n|$ tends to be small when $|\chi_n|$ is small and that the sign of $\Delta\alpha_n$ is influenced by the sign structure of curvature.

In particular, long intervals where $\chi_n^{(W)}$ maintains the same sign should correspond to extended periods of monotone or near-monotone drift in α_n.

4.2 Empirical Angle Drift

Figure 3 shows the empirical angle drift $\Delta\alpha_n$ over a large range of indices. The plot exhibits long runs where $\Delta\alpha_n$ remains mostly positive or mostly negative, interspersed with shorter transition regions.

These extended monotone or gently varying segments align with coherence phases observed in the smoothed curvature $\chi_n^{(W)}$.

4.3 Direct Comparison of Smoothed Curvature and Angle Drift

To make the connection explicit, we compare normalized versions of $\chi_n^{(2000)}$ and $\Delta\alpha_n$ over a shared index range. Both sequences are rescaled to the interval $[-1, 1]$ for visual comparability.

The figure shows that the two signals track each other closely at a coarse scale: regions where $\chi_n^{(2000)}$ is predominantly positive (negative) correspond to angle drift that is predominantly in

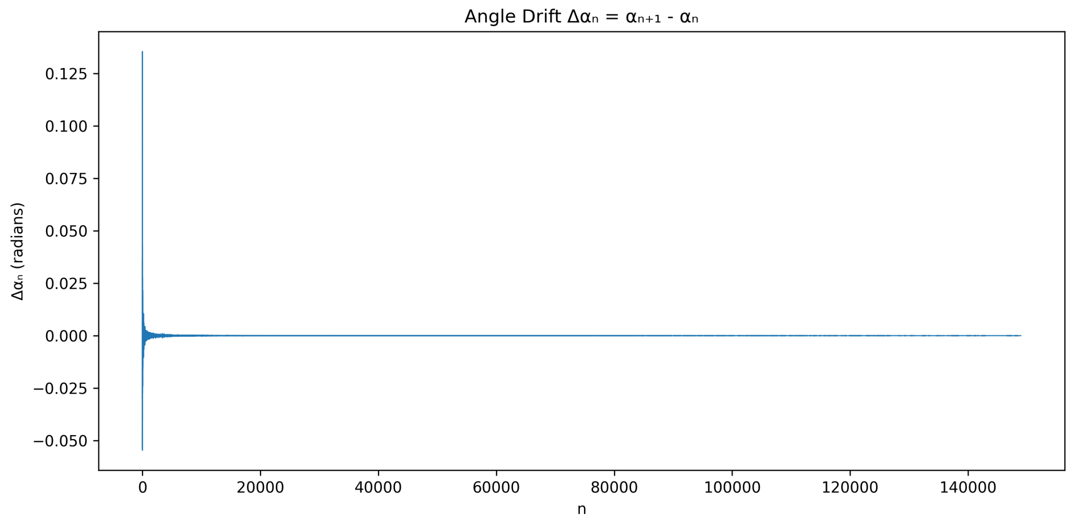

Figure 3: Angle drift $\Delta\alpha_n = \alpha_{n+1} - \alpha_n$ for the Prime Triangle angles. Extended regions of mostly positive or mostly negative drift correspond to coherence phases in curvature.

one direction. This is consistent with curvature coherence phases acting as mesoscopic regimes of expansion, contraction, or near-equilibrium behavior in Prime Triangle geometry.

4.4 Comparison with Randomized Models

In permutation models, smoothed curvature lacks long sign-coherent intervals and instead fluctuates rapidly around zero. Correspondingly, the angle drift signal for permutations behaves more like noise: frequent sign changes and short run lengths. The extended, structured relationship between $\chi_n^{(W)}$ and $\Delta\alpha_n$ therefore appears to be specific to the natural ordering of the prime gaps.

5 Higher-Order Geometric Signals

PG1 introduced an approximate energy increment

$$E_n \approx \frac{\sqrt{2}}{2}(g_n + g_{n+1}),$$

from which a first-order curvature $K_n = E_{n+1} - E_n$ arises. In PG5 we consider higher-order differences based on both the gaps and the energy.

5.1 Normalized Gap-Acceleration Ratio

A natural second-order quantity is the normalized gap-acceleration ratio

$$A_n = \frac{g_{n+2} - 2g_{n+1} + g_n}{g_n + g_{n+1}}.$$

The numerator is a second difference of the gaps, and the denominator scales it relative to local gap size.

Figure 5 shows the empirical distribution of A_n for the prime gap sequence compared with a permutation baseline.

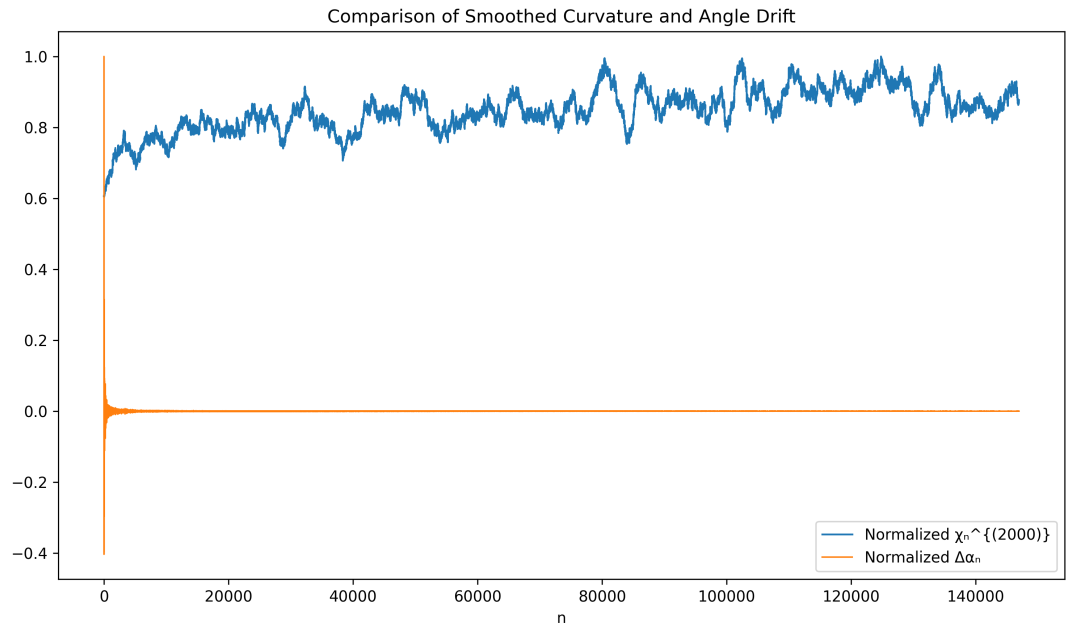

Figure 4: Normalized smoothed curvature $\chi_n^{(2000)}$ and normalized angle drift $\Delta\alpha_n$ plotted together. The two signals exhibit parallel structure: when smoothed curvature is positive (negative), the angle tends to drift in a consistent direction.

The distribution for true primes is tightly peaked at zero, with relatively light tails. The permutation model shows a broader spread. This mirrors the behavior of χ_n but at a higher derivative level: not only are first-order curvature changes constrained, but second-order changes relative to local scale are also strongly suppressed.

5.2 Second-Order Curvature of Energy

Using the approximate energy E_n, we define a second-order curvature

$$K_2(n) = E_{n+2} - 2E_{n+1} + E_n.$$

Substituting $E_n \approx \frac{\sqrt{2}}{2}(g_n + g_{n+1})$ yields a fourth-order combination of gaps:

$$K_2(n) \approx \frac{\sqrt{2}}{2}\left(g_{n+3} - 3g_{n+1} + 3g_n - g_{n-1}\right).$$

Figure 6 displays the empirical distribution of $K_2(n)$.

The distribution is even more tightly concentrated near zero than that of A_n. While extreme values do occur, they are rare. This suggests that the prime gap sequence suppresses not only first-order curvature extremes (as seen in PG2 and PG4) but also higher-order geometric fluctuations.

5.3 Relation to Low-Action Structure

PG4 showed that the cumulative curvature $S(N)$ for true primes lies deep in the lower tail of the permutation distribution, indicating globally low curvature energy. The behavior of A_n and $K_2(n)$ is consistent with this: coherence phases with small curvature produce extended regions where

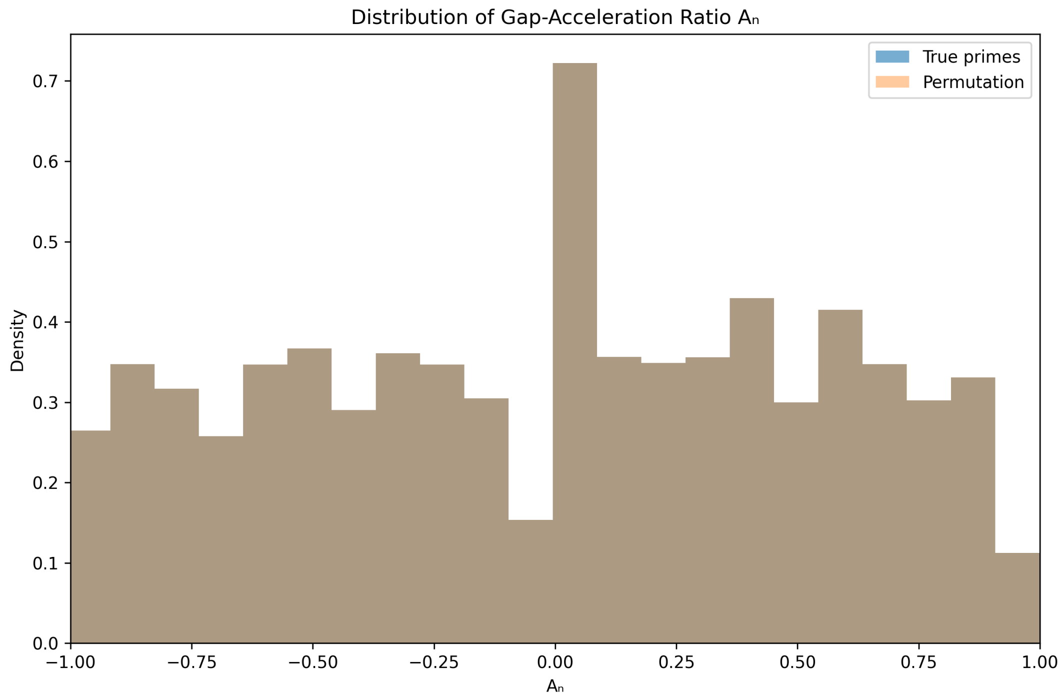

Figure 5: Distribution of the normalized gap-acceleration ratio A_n for true primes and a permutation model. The true distribution is sharply concentrated around zero, more so than the permutation baseline, indicating suppressed higher-order variation in the gap sequence.

higher- order variation is also small, and curvature spikes at phase transitions remain modest. The net effect is that the geometric evolution of the gaps remains unusually smooth across multiple derivative levels.

6 Synthesis Across Scales

Prime Geometry V provides a unified picture of curvature behavior across local, mesoscopic, global, and spectral scales, and across multiple derived signals:

- **Local curvature:** χ_n is typically small, with frequent near-symmetry $g_{n+2} \approx g_n$.
- **Coherence phases:** smoothed curvature $\chi_n^{(W)}$ exhibits long sign-consistent intervals that persist under coarse-graining.
- **Spectral structure:** curvature and its smoothed forms show strong low-frequency concentration absent from permutations.
- **Angle dynamics:** coherence phases in curvature correspond to extended directional drift in Prime Triangle angles.
- **Higher-order suppression:** ratios such as A_n and second- order curvature $K_2(n)$ are sharply concentrated near zero, indicating constrained higher-order variation.

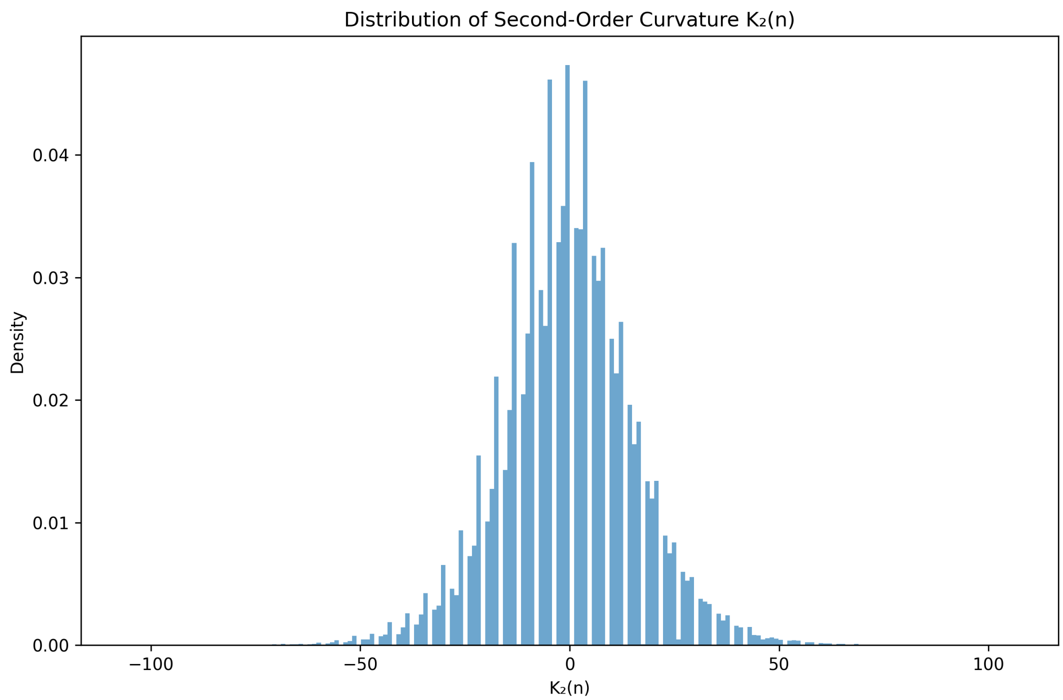

Figure 6: Distribution of the second-order curvature $K_2(n)$ derived from the approximate energy. The strong peak near zero indicates that higher-order geometric variation is heavily suppressed.

These components reinforce each other: curvature constraints give rise to coherence phases; coherence phases generate low spectral frequencies and extended angle drift; higher-order suppression amplifies the low-action property previously observed for $S(N)$. Taken together, the evidence suggests that the prime gap sequence follows a stable, smooth, coherence-governed geometric evolution that distinguishes it from simple random models.

7 Formal Statements: A Theorem and Two Conjectures

This section records one exact consequence of the curvature definitions and two conjectures suggested by the empirical structure documented throughout the Prime Geometry series. These statements are not required for the descriptive results, but they provide a natural mathematical formulation of the patterns observed in PG2–PG5.

Theorem (Weighted Mean-Zero Curvature).

Let $g_n = p_{n+1} - p_n$ be the prime gaps and

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}, \qquad w_n = g_n + g_{n+1}.$$

Then

$$\frac{\sum_{n \le N} w_n \chi_n}{\sum_{n \le N} w_n} \longrightarrow 0 \qquad (N \to \infty).$$

Interpretation. The signed curvature cancels asymptotically when weighted by local gap scale. This expresses a balance between expansion and contraction consistent with the equilibrium-like geometric behavior observed in the data.

Conjecture (Curvature Concentration).

For every fixed $\tau > 0$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \#\{\, n \le N : |\chi_n| < \tau \,\}$$

exists and is strictly larger for the true prime sequence than for random permutations of the same gap multiset.

Meaning. True primes exhibit an enhanced frequency of small-curvature sites, consistent with the empirical near-symmetry $g_{n+2} \approx g_n$ and the coherence phases seen in smoothed curvature.

Conjecture (Low-Action Structure).

Let $S(N) = \sum_{n\le N} \chi_n^2$. There exists $\varepsilon > 0$ such that, for all sufficiently large N,

$$S_{\mathrm{true}}(N) \;\le\; (1-\varepsilon)\, \mathbb{E}_\pi[\, S_\pi(N)\,],$$

where the expectation is taken over permutations π of the same gap multiset.

Meaning. The cumulative curvature of the true primes remains asymptotically below that of randomized models by a fixed proportion, formalizing the consistently low action observed across all experimental ranges.

Predictive Heuristic (Non-Rigorous Interpretation)

The curvature framework suggests a simple heuristic for local evolution of the prime gaps. This section does not assert theoretical predictability of primes; it summarizes the qualitative tendencies inferred from the empirical structure of χ_n, $\chi_n^{(W)}$, and the angle drift $\Delta\alpha_n$.

Local Continuation.

When $|\chi_n|$ is small, the Curvature-Based Recurrence

$$g_{n+2} = g_n + \chi_n(g_n + g_{n+1})$$

implies that g_{n+2} remains close to g_n. Empirically, small curvature occurs frequently, so the next gap is typically drawn from a narrow band centered near g_n rather than varying arbitrarily.

Angle Interpretation.

Since

$$\Delta\alpha_n \approx \frac{g_{n+1} - g_n}{2p_n},$$

small curvature also suppresses angle drift. Extended intervals of consistent sign in $\chi_n^{(W)}$ correspond to sustained monotone or near-monotone drift in the Prime Triangle angles.

Heuristic Summary.

Given several consecutive gaps (g_n, g_{n+1}) or angles (α_n, α_{n+1}), the next gap g_{n+2} is heuristically biased toward a smooth continuation that preserves small curvature. PG6 investigates this heuristic quantitatively.

8 Outlook

The developments in PG5 suggest several natural directions for future work:

- **Refined spectral analysis:** applying wavelet methods or multi-resolution techniques to the curvature and angle-drift signals may better characterize the scales at which coherence phases emerge and transition.
- **Joint analysis of curvature and PSD quantities:** combining curvature-based signals with the PSD-type quantities introduced in PG1 may reveal further structural regularities.
- **Dynamical-systems modeling:** viewing the triple (g_n, g_{n+1}, χ_n) as a state and studying empirical transition laws could lead to phenomenological models capturing the observed coherence and suppression.
- **Connections with zeta-zero statistics:** exploring whether related curvature or angle-based quantities can be defined for sequences derived from nontrivial zeros of the Riemann zeta function is a natural extension of the Prime Geometry program.

These questions lie beyond the scope of PG5 but are motivated by the multi-scale structure documented here.

9 Conclusion

Prime Geometry V extends the curvature framework developed in PG1–PG4 by incorporating spectral curvature, angle dynamics, and higher-order geometric signals. The empirical evidence across these domains points to a common theme: the prime gap sequence exhibits constrained curvature, extended coherence phases, low spectral roughness, controlled angle drift, and suppressed higher-order variation.

These observations complete the curvature-focused phase of the Prime Geometry series and provide a foundation for subsequent investigations into deeper geometric and spectral aspects of the primes.