Prime Geometry in 52 Steps: A Roadmap from PG01–Triangles to PG14–Field Theory

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Notation, Scope, and Interpretation

Core Notation.

- p_n: the nth prime.
- $g_n = p_{n+1} - p_n$: the consecutive prime gap.
- $\alpha_n = \arctan(p_n/p_{n+1})$: Prime Triangle angle.
- $\Delta\alpha_n = \alpha_{n+1} - \alpha_n$: angle drift.
- $\chi_n = \dfrac{g_{n+2} - g_n}{g_n + g_{n+1}}$: normalized curvature.
- $S(N) = \sum_{k \le N} \chi_k^2$: curvature action (global roughness).

Classification of statements.

- **Definition** — formally introduces a quantity.
- **Empirical Observation** — supported by computations.
- **Conjecture** — proposed mathematical property.
- **Model Equation / Ansatz** — describes a fitted or hypothesized evolution law.
- **Interpretation / Phenomenon** — conceptual meaning of observed structure.

Scope. This roadmap summarizes the Prime Geometry program as developed across PG1–PG14. It is not a self-contained proof document. All rigorous definitions, figures, empirical datasets, derivations, and model fits appear in the corresponding PG papers.

Important. Statements involving "small", "rare", "coherent", "scale-invariant", or "low action" refer to empirical behavior of prime data up to the computational limits described in PG1–PG14. Model equations (PGEE, PGME) are *empirical dynamical ansätze*, not theorems of analytic number theory.

1 Phase I – The Geometric Seed (PG1)

Step 1 (Definition). Start with consecutive primes p_n, p_{n+1}, p_{n+2} and their gaps $g_n = p_{n+1} - p_n$, $g_{n+1} = p_{n+2} - p_{n+1}$.

Step 2 (Definition). Attach a right triangle to each pair using vertices $(0, 0)$, $(p_n, 0)$, $(0, p_{n+1})$. Define

$$\alpha_n = \arctan\left(\frac{p_n}{p_{n+1}}\right).$$

Step 3 (Empirical Observation). Since $p_{n+1} \sim p_n$, one finds $\alpha_n \approx 45^\circ$; its fluctuations encode relative imbalance between consecutive primes.

Step 4 (Interpretation). View changes in gaps as geometric "bending" of the sequence — the seed of curvature.

2 Phase II – Curvature, Action, and Low-Energy Geometry (PG2–PG4)

Step 5 (Definition). Define curvature as the normalized second difference:

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}.$$

The identity $g_{n+2} = g_n + (g_n + g_{n+1})\chi_n$ is a rearrangement of this definition.

Step 6 (Empirical Observation). Across large datasets, $|\chi_n| \ll 1$ for most n (curvature suppression).

Step 7 (Definition). Curvature identity rewritten as

$$g_{n+2} = g_n + (g_n + g_{n+1})\chi_n.$$

This is not a dynamical law; it is algebra.

Step 8 (Definition). Define curvature action:

$$S(N) = \sum_{k \leq N} \chi_k^2.$$

Step 9 (Empirical Observation). Random permutations of the same gaps (as a geometric null ensemble) typically yield much larger action.

Step 10 (Conjecture (Low-Action Structure)). The true prime sequence lies in the extreme lower tail of curvature action among admissible reorderings.

3 Phase III – Coherence and Signals (PG3–PG6)

Step 11 (Definition). Smooth curvature with a windowed average $\chi_n^{(W)}$ to reveal mesoscale behavior.

Step 12 (Definition). A *coherence phase* is an interval where smoothed curvature keeps a consistent sign.

Step 13 (Empirical Observation). Transitions between coherence phases exhibit sharp curvature spikes.

Step 14 (Definition). Define angle drift:

$$\Delta\alpha_n = \alpha_{n+1} - \alpha_n.$$

Step 15 (Empirical Observation). Angle drift behaves approximately like a first derivative:

$$\Delta\alpha_n \approx \frac{g_{n+1} - g_n}{2p_n}.$$

A derivation appears in Appendix A.

Step 16 (Interpretation). This yields the derivative hierarchy:

$$\chi_n \longrightarrow \Delta\alpha_n \longrightarrow \alpha_n.$$

Step 17 (Empirical Observation). Spectral analysis of χ_n and $\Delta\alpha_n$ shows excess low-frequency power and suppressed noise.

4 Phase IV – Stability, Balance Laws, and Global Constraints (PG7–PG8)

Step 18 (Definition). Angle deviation can be expressed (approximately) as

$$\alpha_n - \frac{\pi}{4} \approx \frac{1}{2} \sum_{k<n} \frac{(g_k + g_{k+1})\chi_k}{p_k},$$

derived in Appendix A.

Step 19 (Interpretation). Angle stability forces the weighted curvature sum to remain small.

Step 20 (Empirical Law (Weighted Mean-Zero Curvature)). Weighted curvature exhibits long-range cancellation; persistent bias is rare.

Step 21 (Interpretation (Stability Law)). If curvature maintained a single sign too long, the angle deviation bound would fail.

Step 22 (Phenomenon). Opposing coherence phases must compensate across large ranges (global balance).

Step 23 (Interpretation). Large spikes or strong bias inflate $S(N)$, breaking low-action structure.

Step 24 (Summary). Prime gaps exhibit small curvature, coherence phases, global cancellation, and low action.

5 Phase V – The Evolution Equation and Scaling (PG9–PG10)

Step 25 (Question). Can curvature, drift, angle, and potential be combined into one evolution model?

Step 26 (Model Equation (Ansatz)).

$$g_{n+2} = g_n + A\chi_n + B\Delta\alpha_n + C\Phi'(n) + \varepsilon_n.$$

Step 27 (Empirical Observation). Fits give $A = g_n + g_{n+1}$, $B = 2p_n$, and a small global term C.

Step 28 (Model Equation (PGEE)).

$$g_{n+2} = g_n + (g_n + g_{n+1})\chi_n + 2p_n\Delta\alpha_n + C\Phi'(n) + \varepsilon_n.$$

PGEE summarizes empirical geometric evolution; it is not a theorem.

Step 29 (Phenomenon). PGEE reproduces coherence, curvature suppression, angle stability, and return-map geometry.

Step 30 (Definition (Renormalization)).

$$\tilde{g}_n = \frac{g_n}{\log p_n}, \quad \tilde{\chi}_n = (\log p_n)\chi_n, \quad \tilde{\Delta\alpha}_n = p_n\Delta\alpha_n.$$

Step 31 (Empirical Observation). Renormalized variables approach scale-invariant distributions.

Step 32 (Phenomenon). The renormalized attractor: the cloud $(\tilde{g}_n, \tilde{g}_{n+1}, \tilde{\chi}_n)$ lies in a thin, structured manifold.

6 Phase VI – Optional: Zeta-Zero Comparison (PG11)

Step 33 (Definition). Apply PG quantities to zeta-zero gaps $\delta_n = \gamma_{n+1} - \gamma_n$.

Step 34 (Empirical Observation). Renormalized curvature distributions show structural similarities and differences.

Step 35 (Model Equation). Construct a structural analogue of PGEE for zero gaps.

Step 36 (Interpretation). Zeros and primes share geometric motifs but differ dynamically.

7 Phase VII – Continuous Representation and the Master Equation (PG12)

Step 37 (Definition). Introduce a continuous surrogate $G(t)$ with

$$G(t_n) \approx \tilde{g}_n,\ G'(t_n) \approx \tilde{\Delta\alpha}_n,\ G''(t_n) \approx \tilde{\chi}_n.$$

This is a modeling interpolation, not a canonical analytic object.

Step 38 (Model Equation). Rewrite PGEE for $G(t)$ to obtain a scale-free second-order flow.

Step 39 (Definition). Introduce a higher-order smoothing operator $\mathcal{H}[G]$ to encode suppressed third differences.

Step 40 (Model Equation (PG12 Master Equation)).

$$G''(t) = A(G)G''(t) + BG'(t) + C\tilde{\Phi}'(t) + D\mathcal{H}[G](t) + \eta(t).$$

Step 41 (Interpretation). PG12 is a geometric flow with global regulation and smoothing.

Step 42 (Phenomenon). PG12 reproduces curvature suppression, coherence phases, angle stability, and attractor geometry.

Step 43 (Summary). PG12 unifies PG1–PG11 into a single evolution model.

8 Phase VIII – Solution Theory and Stability Classes (PG13)

Step 44 (Definition). Treat PGME as a discrete dynamical system via $X_{n+1} = F(X_n)$, where $X_n = (g_n, g_{n+1}, \chi_n, \Delta\alpha_n, \Phi'(n))$.

Step 45 (Definition). The PGME attractor is the limiting region of stable trajectories under this flow.

Step 46 (Interpretation). Solution types split into prime-like, unstable, and intermediate.

Step 47 (Phenomenon). True primes lie on an exceptionally stable, low-action orbit within the attractor.

9 Phase IX – Action and Field Theory (PG14)

Step 48 (Question). Can a variational principle generate PGME via Euler–Lagrange equations?

Step 49 (Definition). Define the Prime Geometry Action:

$$S[G] = \int L(G, G', G'', \alpha, \Phi, t)\, dt.$$

Step 50 (Model Equation). The Euler–Lagrange equation of $S[G]$ yields a continuous analogue of PG12.

Step 51 (Interpretation). Near-symmetries of the action correspond to Noether-like balance laws: slow energy drift, curvature cancellation, smoothing.

Step 52 (Conclusion). Prime evolution behaves like a low-action trajectory of a geometric field, suggesting the broader framework of a Prime Geometry Field Theory.

Appendix A: Key Derivations Referenced in the 52-Step Roadmap

A.1 Angle drift approximation

$$\alpha_n = \arctan\left(\frac{p_n}{p_{n+1}}\right), \quad \frac{p_n}{p_{n+1}} = 1 - \frac{g_n}{p_{n+1}}.$$

A first-order expansion of $\arctan(1 - x)$ yields

$$\Delta\alpha_n \approx \frac{g_{n+1} - g_n}{2p_n}.$$

A.2 Curvature identity

Starting from

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}},$$

rearranging gives

$$g_{n+2} = g_n + (g_n + g_{n+1})\chi_n.$$

A.3 Angle deviation formula

Expanding $\alpha_n - \pi/4$ around small drift and curvature yields

$$\alpha_n - \frac{\pi}{4} \approx \frac{1}{2} \sum_{k<n} \frac{(g_k + g_{k+1})\chi_k}{p_k}.$$

A.4 Continuous surrogate

Let $t_n = \log p_n$. Define

$$G(t_n) = \frac{g_n}{\log p_n}.$$

Finite differences of G approximate renormalized derivatives:

$$G'(t_n) \approx \tilde{\Delta}\alpha_n, \qquad G''(t_n) \approx \tilde{\chi}_n.$$