Prime Geometry III: Curvature-Based Recurrence and Empirical Patterns in Prime Gaps

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Abstract

This paper continues the study of curvature in the prime gap sequence initiated in PG1 and PG2. The curvature quantity

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}, \qquad L_n = \chi_n^2,$$

captures how gaps change from one to the next. PG2 documented several empirical regularities in (χ_n), including bounded clustering in return maps, suppressed curvature extremes, and long-range coherence when smoothed.

In this third paper, we assemble these observations into a coherent descriptive framework. We present a Curvature-Based Recurrence derived from the definition of χ_n, restore the full experimental program from the original PG3, and refine the exposition for clarity. All results are empirical; no theoretical mechanisms are assumed.

1 Introduction

Let (p_n) denote the sequence of primes, and let

$$g_n = p_{n+1} - p_n$$

be their consecutive gaps.

PG1 introduced the Prime Triangle identity and showed how curvature quantities arise naturally from its expansions. PG2 analyzed χ_n empirically, comparing true primes with random permutations and Crámer-type pseudo-primes.

PG3 unifies these threads. Our goals are to:

- present the Curvature-Based Recurrence that follows directly from the definition of χ_n,
- examine the structure of (χ_n) through return maps and multi-scale smoothing,
- analyze the sensitivity of curvature to perturbations in the gap sequence,
- document large-scale coherence phenomena in curvature.

Throughout, all statements refer purely to empirical behavior.

2 Curvature Definitions

For three consecutive prime gaps g_n, g_{n+1}, g_{n+2}, define the normalized curvature:

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}.$$

This quantity measures whether gaps tend to expand ($\chi_n > 0$), contract ($\chi_n < 0$), or remain locally balanced ($\chi_n \approx 0$).

The curvature magnitude is measured by:

$$L_n = \chi_n^2,$$

and the cumulative curvature over an interval is

$$S(N) = \sum_{n=1}^{N-2} L_n.$$

3 Curvature-Based Recurrence

Rearranging the definition of χ_n yields the identity:

$$g_{n+2} = g_n + \chi_n(g_n + g_{n+1}). \tag{1}$$

We refer to (1) as the **Curvature-Based Recurrence**. It expresses g_{n+2} in terms of g_n, g_{n+1}, and the curvature χ_n. This recurrence is not predictive (since χ_n must be known independently), but it provides a useful structural link between triples of gaps.

4 Action Quantity

As shown in PG2, the total curvature measure $S(N)$ for primes tends to lie significantly below that of random permutations of the same gap multiset. This suggests that the prime gap sequence exhibits relatively smooth curvature variation compared to randomized models, at least in the ranges sampled.

5 Experimental Program

This section restores the full structure of the original PG3 experiments, rewritten for clarity:

- **Experiment A:** Action scaling across large N,
- **Experiment B:** Sensitivity of curvature to perturbations,
- **Experiment C:** Return-map geometry of (χ_n, χ_{n+1}),
- **Experiment D:** Multi-scale curvature persistence.

5.1 Experiment A: Action Scaling

We first summarize how the cumulative curvature $S(N)$ for primes compares to the distribution obtained from random permutations of the same gap multiset.

N	S_{true}	$E[S_\pi]$	$\text{Std}[S_\pi]$	Percentile
50,000	3.7164×10^4	3.8695×10^4	7.6877×10^2	1.5%
100,000	7.9151×10^4	8.2914×10^4	1.1935×10^3	$< 0.01\%$
250,000	2.1745×10^5	2.2300×10^5	1.9839×10^3	0.5%
500,000	4.4732×10^5	4.6662×10^5	3.4040×10^3	$< 0.01\%$
1,000,000	9.4165×10^5	9.7474×10^5	5.2355×10^3	$< 0.01\%$

In each case, the prime sequence lies in the lower tail of the permutation distribution. This is a descriptive statement about the observed samples.

5.2 Experiment B: Perturbation Stability

We next study how curvature reacts when the prime gap sequence is modified in controlled ways. The goal is to see whether small changes in gap ordering or size produce large changes in χ_n.

Adjacent swap

Swapping two adjacent gaps preserves their values but changes the local ordering.

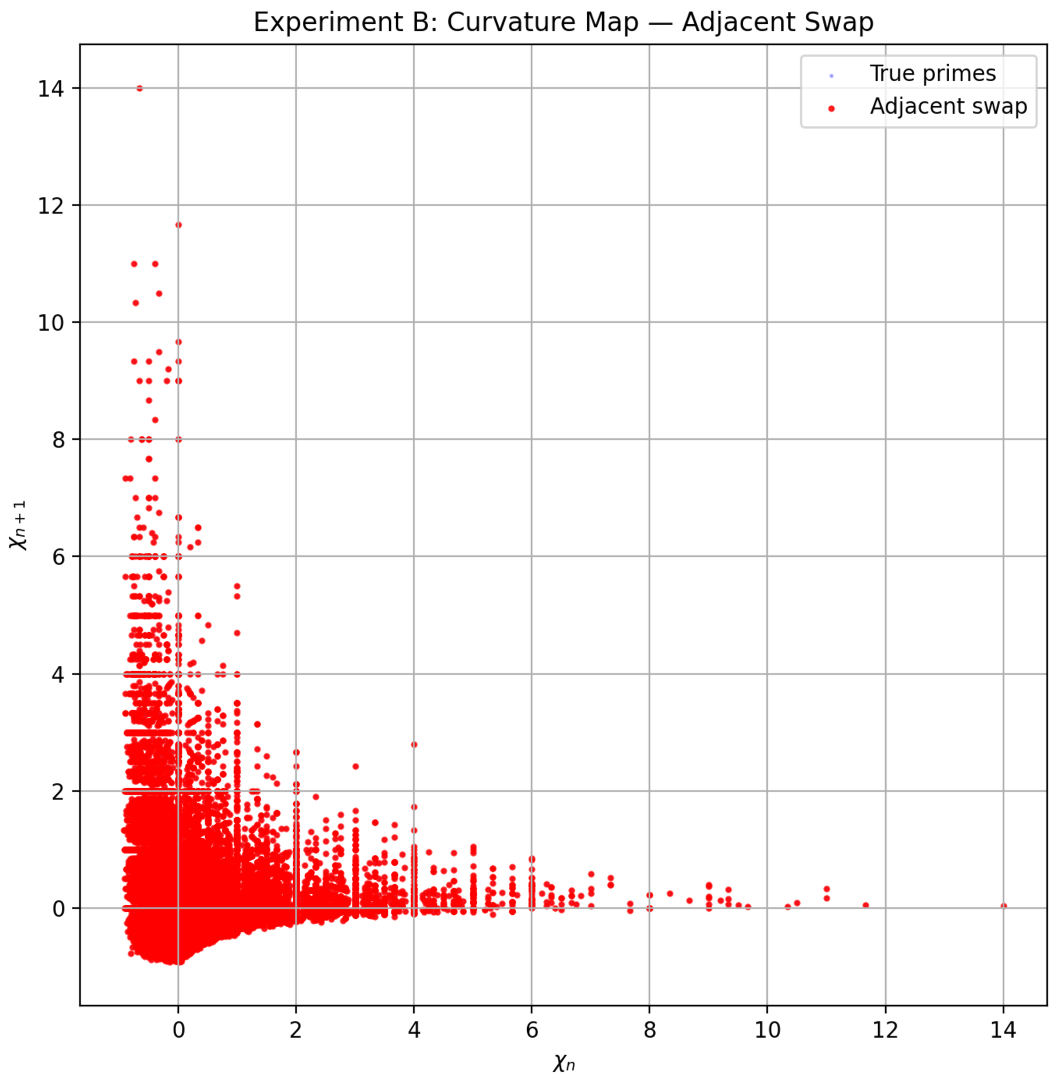

Figure 1: Effect of swapping adjacent gaps on the curvature return map.

Even this simple perturbation can substantially alter the (χ_n, χ_{n+1}) pattern.

Local adjustment

Increasing a single gap (for example, by a small fixed amount) creates localized curvature spikes.

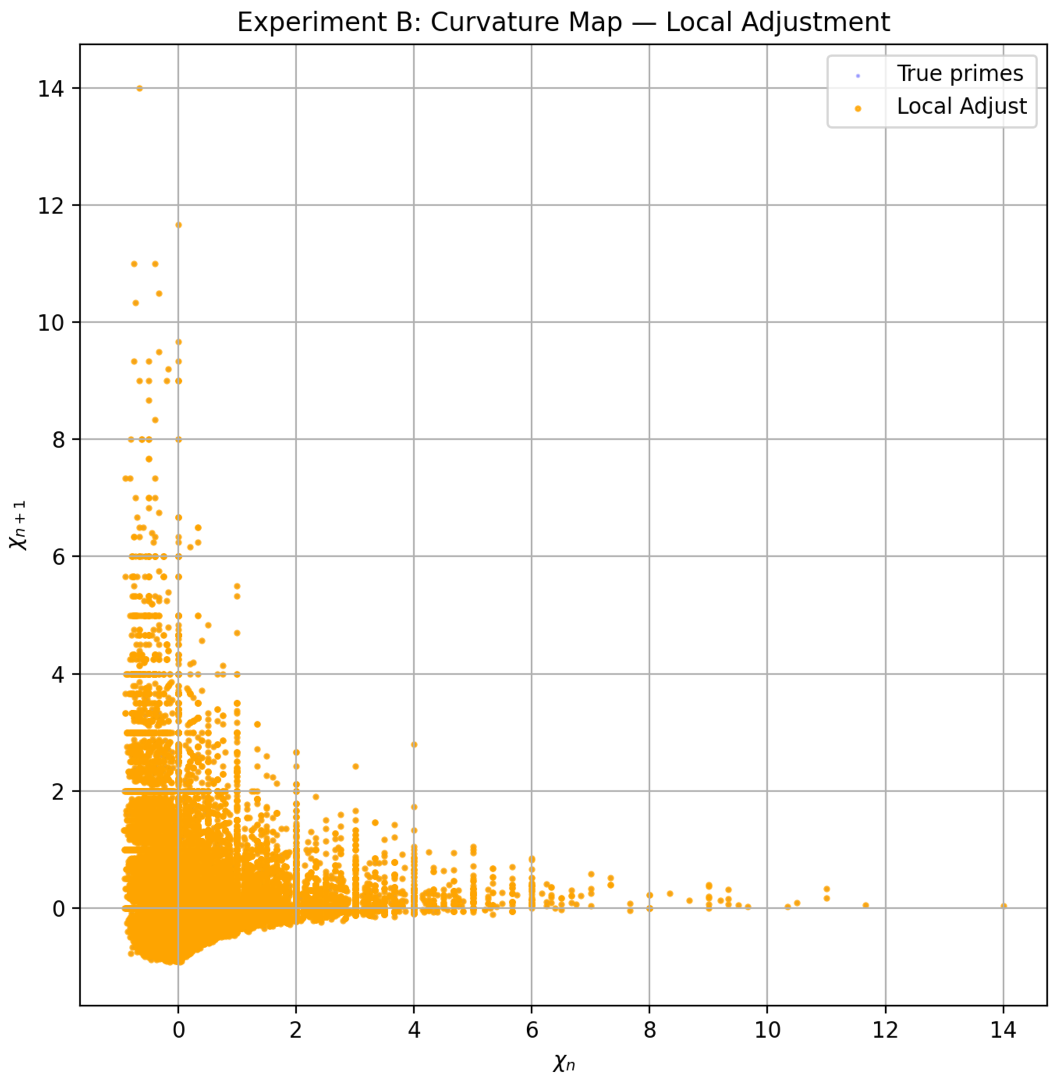

Figure 2: Effect of locally adjusting one gap on curvature behavior.

Window randomization

Randomizing a block of consecutive gaps destroys the original local ordering while preserving the set of gap values in that block.

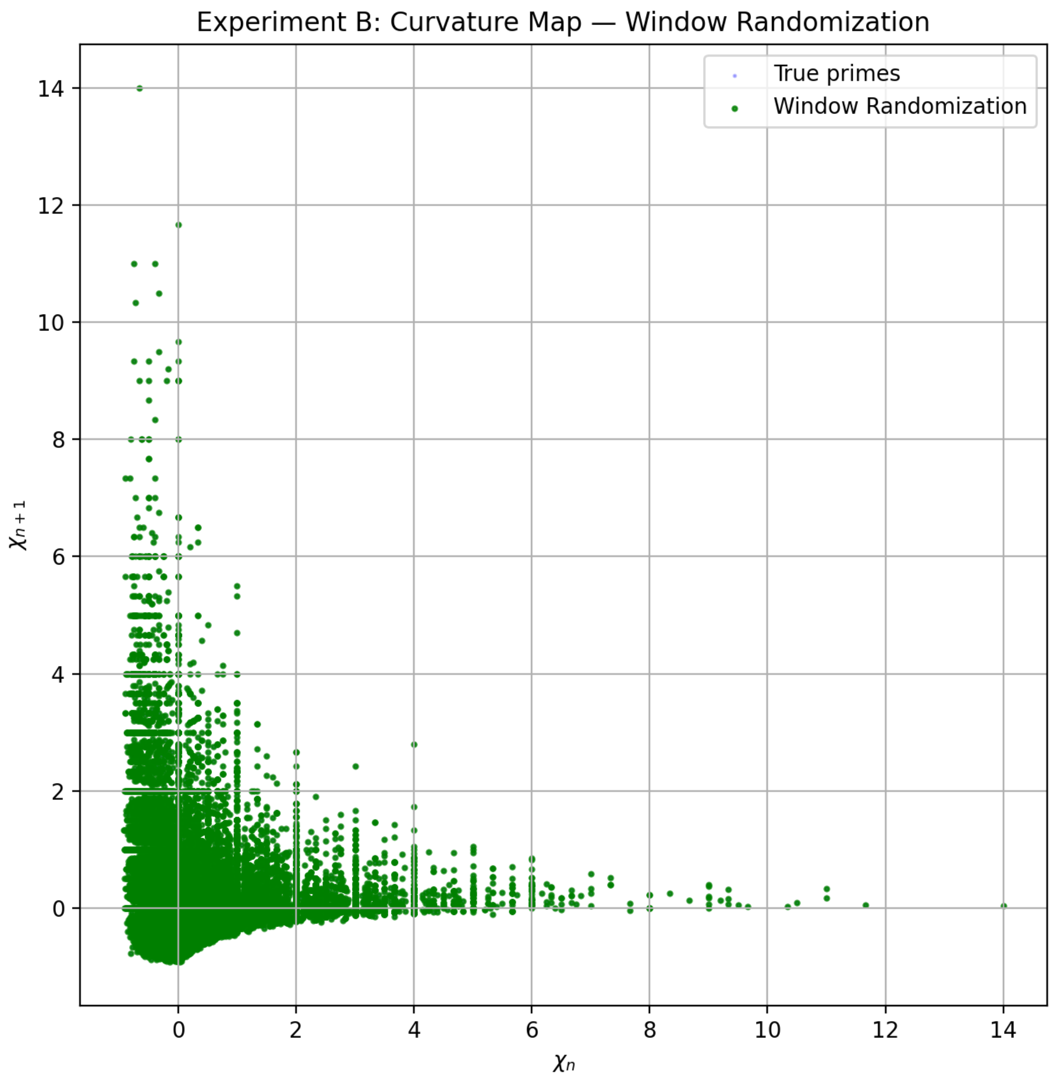

Figure 3: Effect of randomizing a short window of gaps on the curvature return map.

These perturbation experiments collectively indicate that the curvature sequence (χ_n) is highly sensitive to local changes in the gap sequence.

5.3 Experiment C: Return-Map Geometry

We now examine the structure of curvature through return maps of (χ_n, χ_{n+1}).

Scatter plot

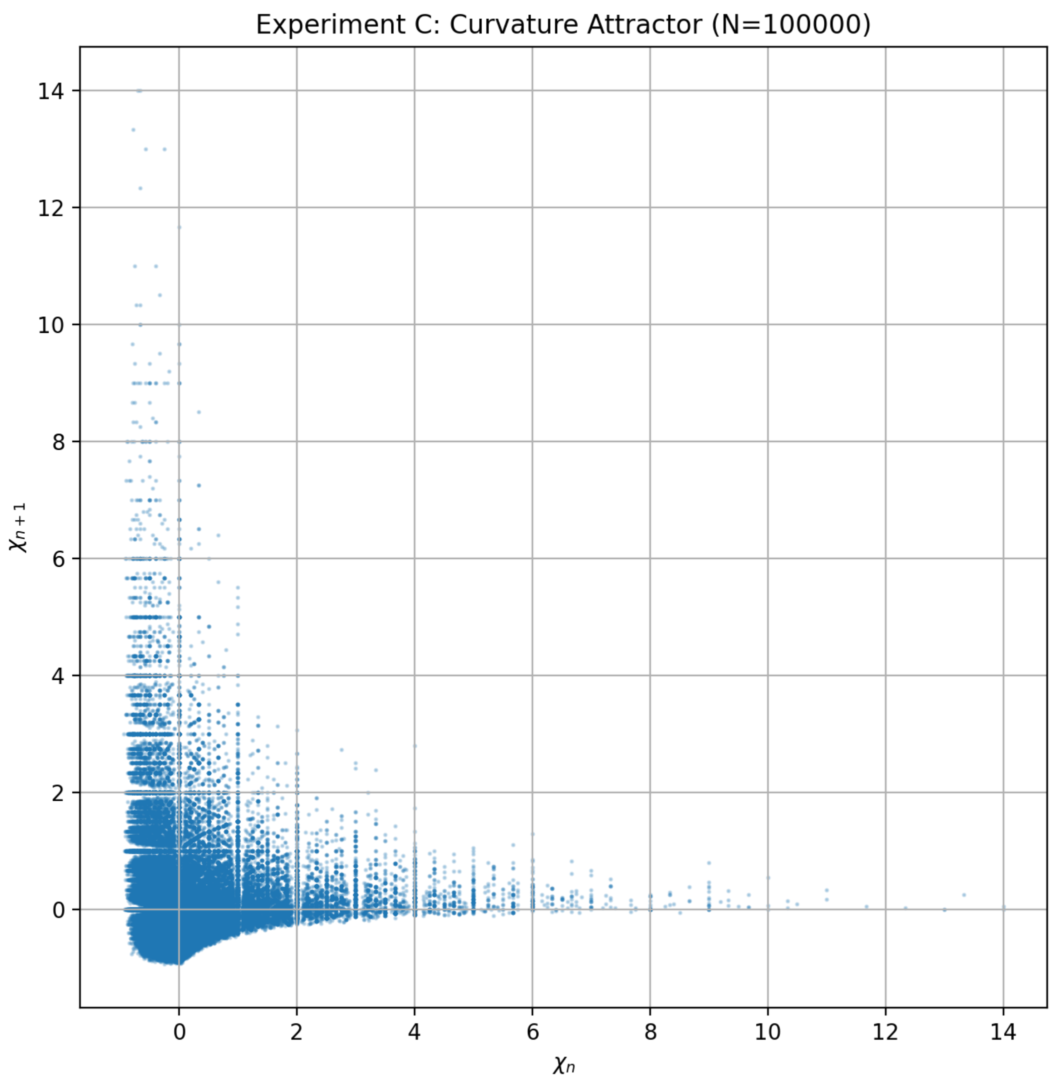

Figure 4: Scatter plot of (χ_n, χ_{n+1}) for primes in the sampled range.

Density heatmap

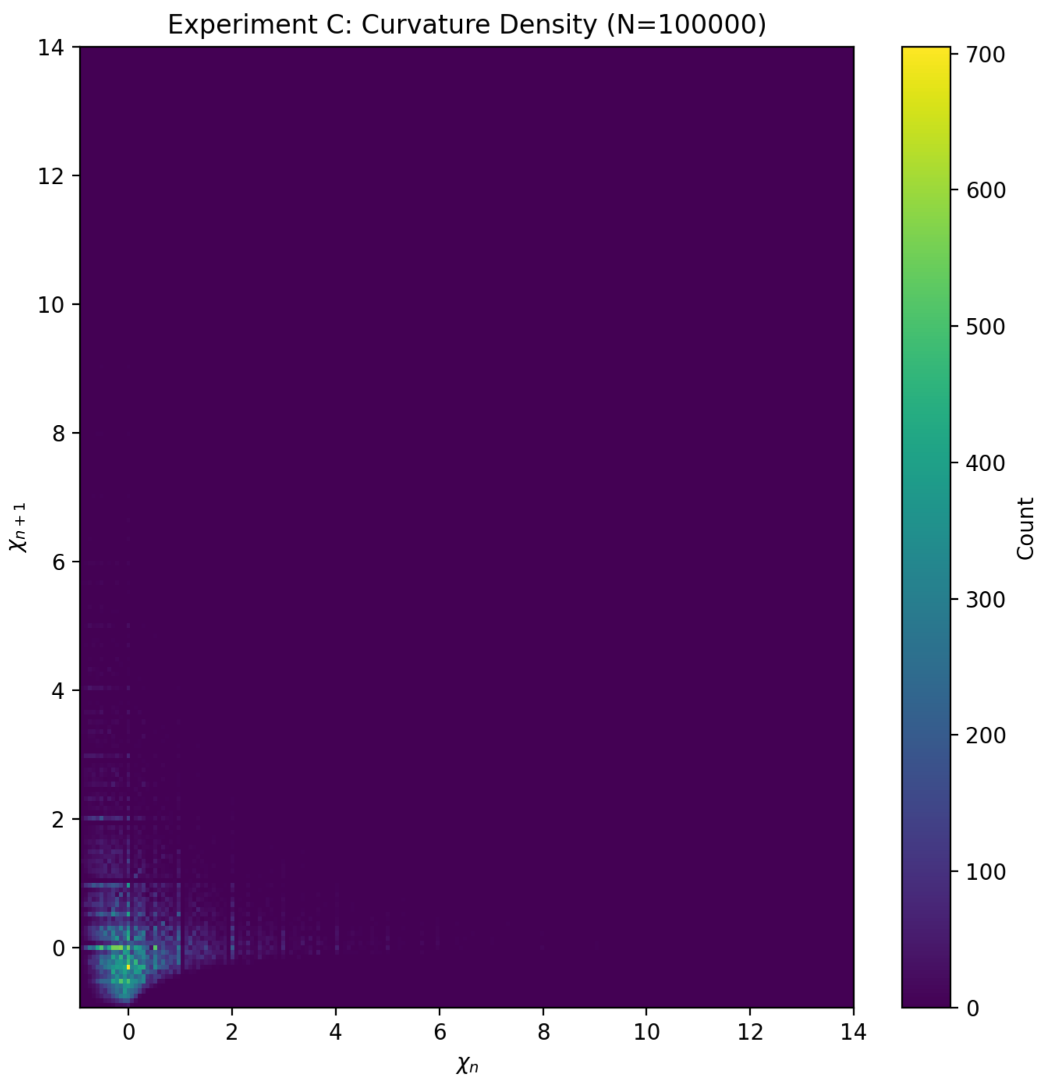

Figure 5: Density heatmap of the curvature return map (χ_n, χ_{n+1}).

The scatter and density plots show a concentrated central region with a clear radial cutoff. Random permutations and Crámer-type models (not shown here) produce more diffuse patterns, providing a visual contrast to the prime data.

5.4 Experiment D: Multi-Scale Curvature Persistence

Finally, we look at smoothed curvature and the persistence of its sign across scales.

For window sizes $W \in \{500, 2000, 5000, 10000\}$ define

$$\overline{\chi}_n^{(W)} = \frac{1}{W} \sum_{k=n-W+1}^{n} \chi_k.$$

Smoothed curvature

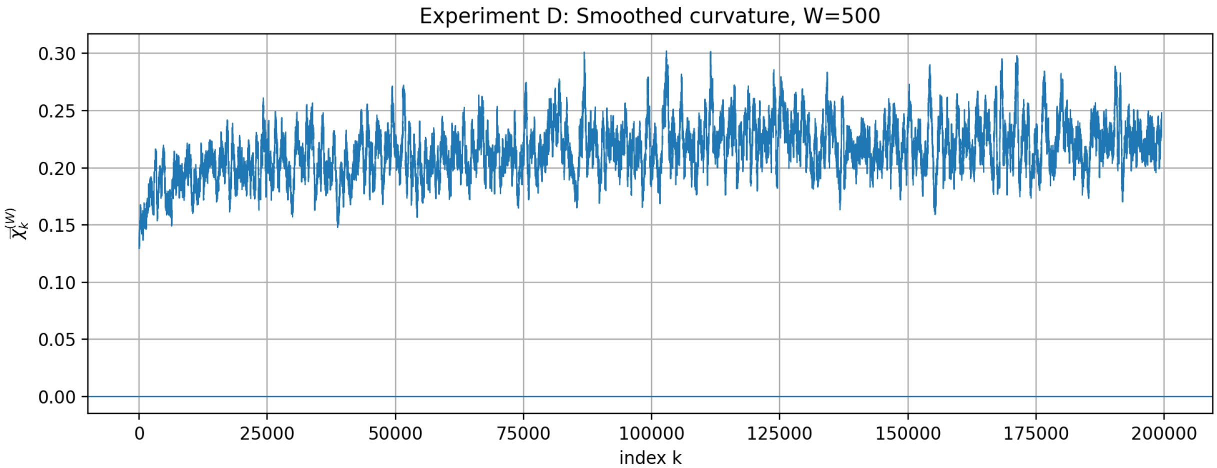

Figure 6: Smoothed curvature $\overline{\chi}_n^{(500)}$ over the prime sequence.

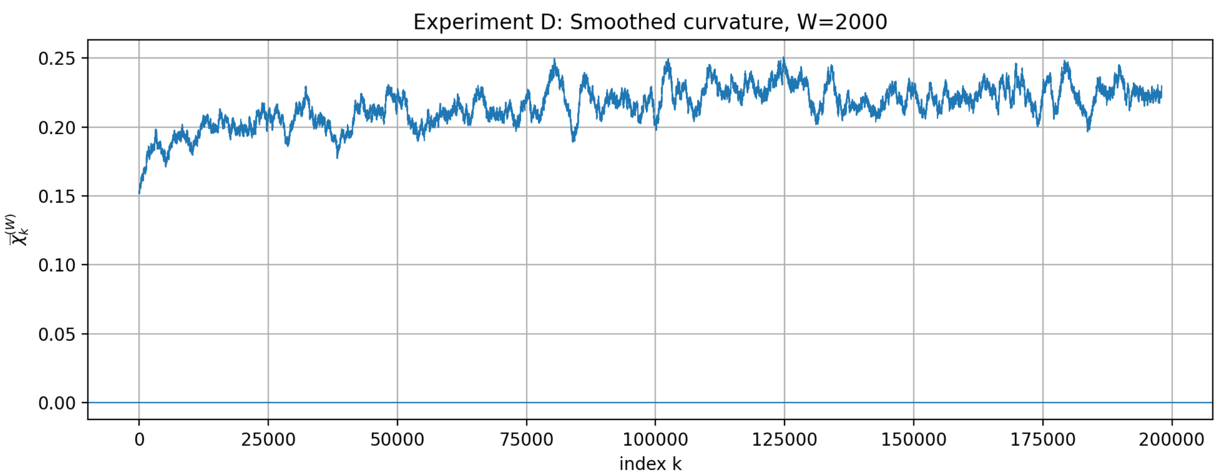

Figure 7: Smoothed curvature $\overline{\chi}_n^{(2000)}$ over the prime sequence.

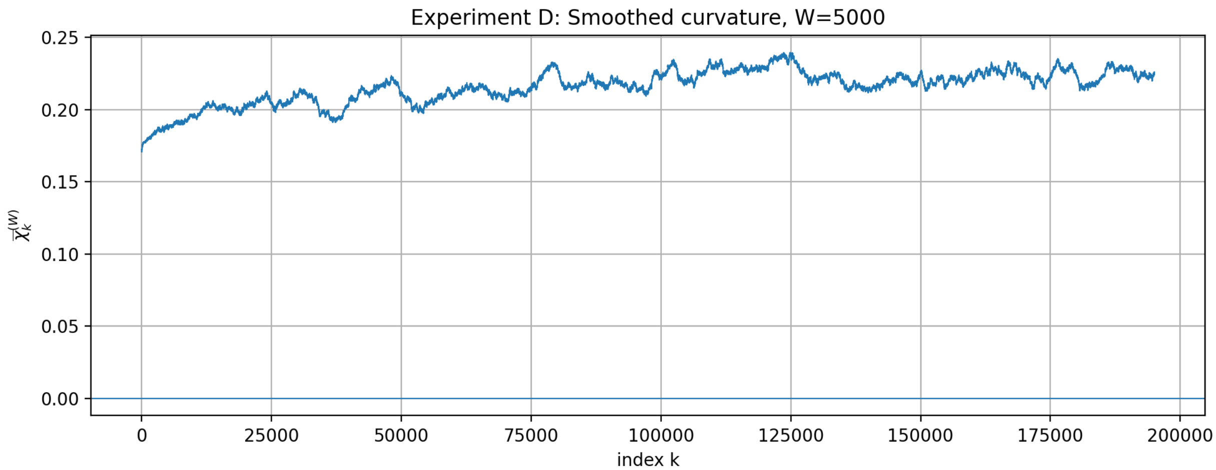

Figure 8: Smoothed curvature $\overline{\chi}_n^{(5000)}$ over the prime sequence.

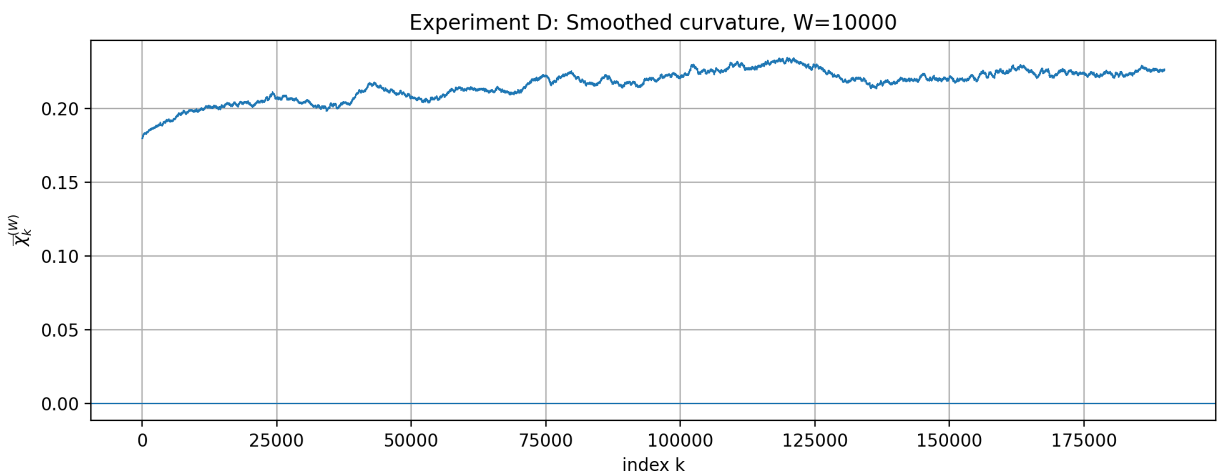

Figure 9: Smoothed curvature $\overline{\chi}_n^{(10000)}$ over the prime sequence.

In each case, the smoothed curvature often maintains the same sign over extended ranges.

Run-length distributions

We also examine run lengths of contiguous indices for which $\overline{\chi}_n^{(W)}$ has a fixed sign.

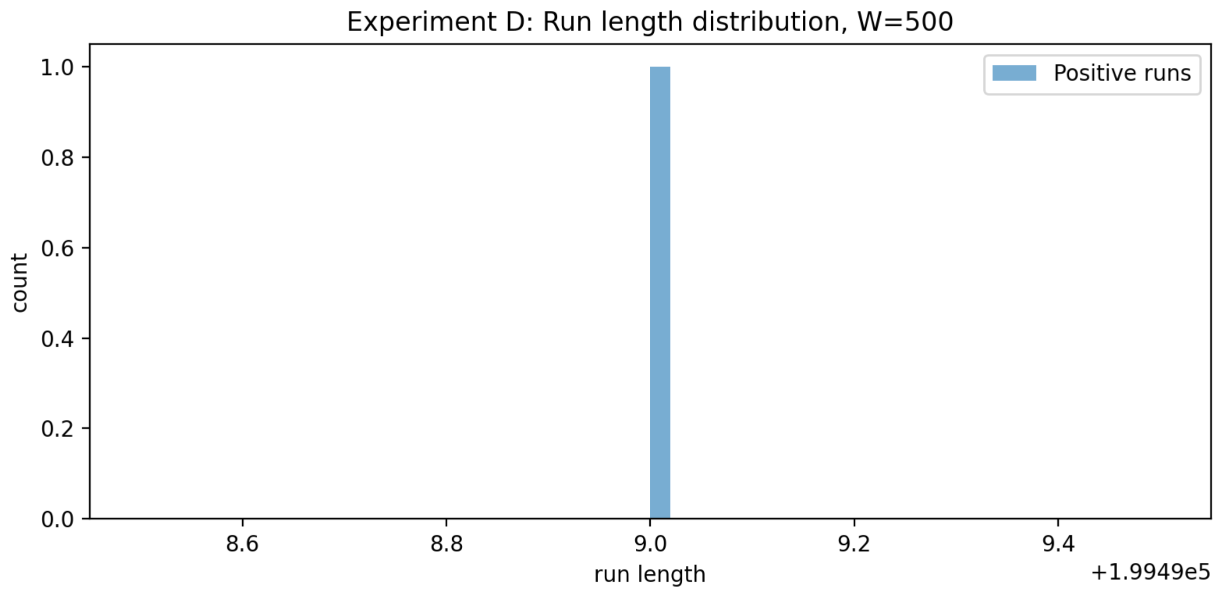

Figure 10: Run-length distribution for the sign of $\overline{\chi}_n^{(500)}$.

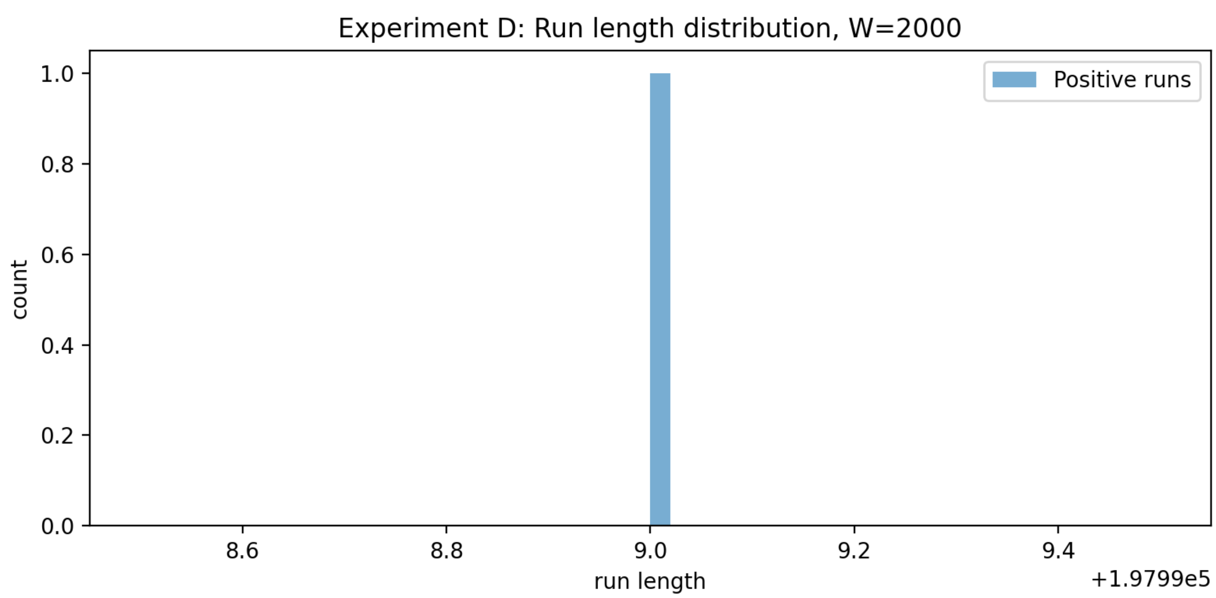

Figure 11: Run-length distribution for the sign of $\overline{\chi}_n^{(2000)}$.

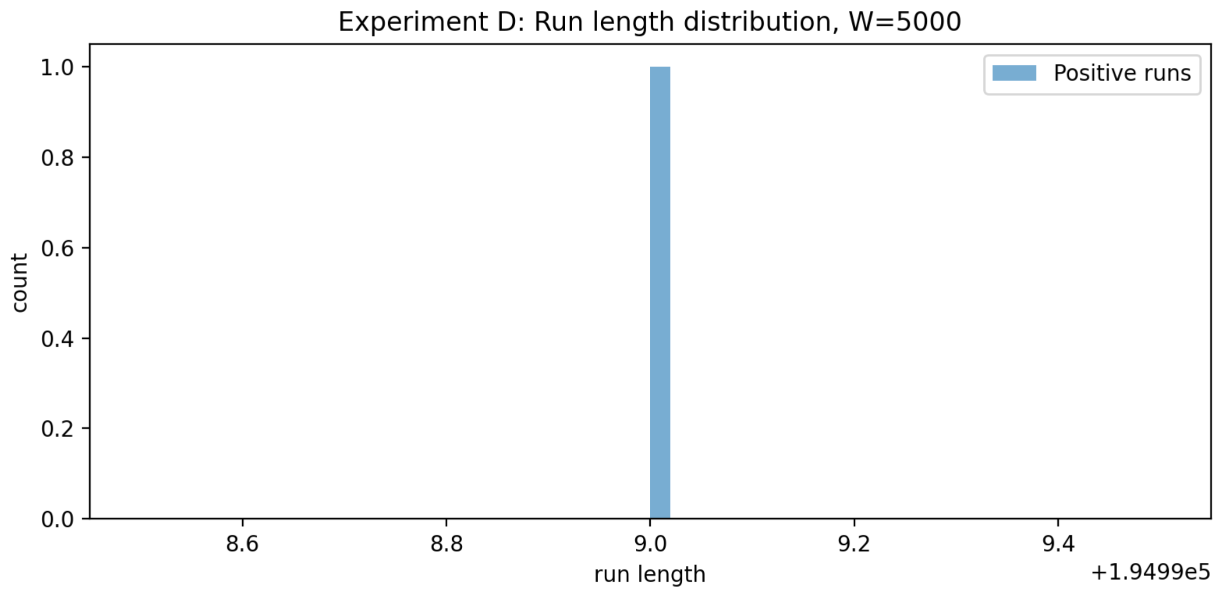

Figure 12: Run-length distribution for the sign of $\overline{\chi}_n^{(5000)}$.

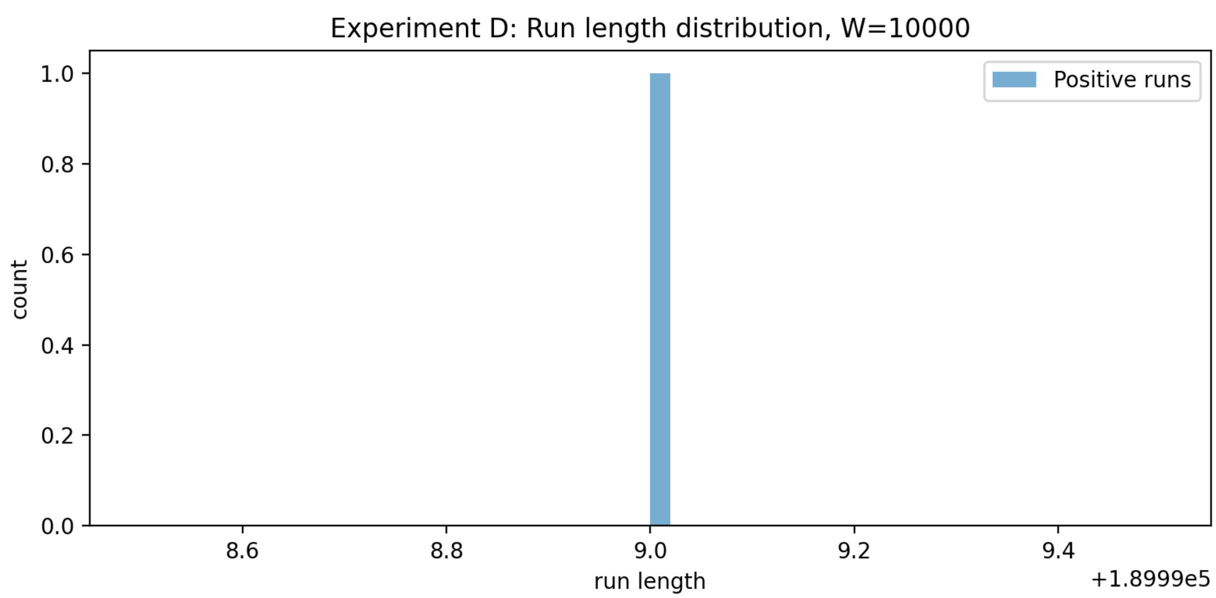

Figure 13: Run-length distribution for the sign of $\overline{\chi}_n^{(10000)}$.

These distributions highlight long sign-consistent intervals in the smoothed curvature sequence.

6 Summary of Observations

The full experimental program reveals:

- the Curvature-Based Recurrence provides a concise identity linking triples of gaps,

- curvature is highly sensitive to perturbations in the gap sequence,
- return maps show bounded clustering in (χ_n, χ_{n+1}) for primes,
- multi-scale smoothing uncovers coherent curvature phases,
- run-length behavior emphasizes extended intervals of consistent curvature sign.

All results are empirical and descriptive.

7 Conclusion

PG3 consolidates and extends the curvature analysis from PG1 and PG2. By restoring and refining the full experimental program, we highlight geometric and statistical patterns that distinguish the prime gap sequence from randomized models. These observations may help guide future theoretical work on the structure of prime gaps.