Prime Geometry I: The Prime Triangle Identity and Its Consequences

Core Identity, Energy, Curvature, and Dimensionless Structure in Prime Triangles

Allen Proxmire

02DEC25

1 Introduction

Prime Geometry begins with a simple geometric observation: consecutive primes naturally define right triangles. Given primes p_n, p_{n+1}, p_{n+2}, form the hypotenuse lengths

$$C_1 = \sqrt{p_n^2 + p_{n+1}^2}, \qquad C_2 = \sqrt{p_{n+1}^2 + p_{n+2}^2}.$$

These "Prime Triangles" satisfy an exact algebraic–geometric identity. From that identity arise the notions of geometric energy, its curvature, and a dimensionless Lagrangian describing local variation in the prime-gap landscape. This document summarizes the core structure developed in Prime Geometry.

2 The Prime Triangle Identity (One Equation Core)

For any three consecutive primes $p_n < p_{n+1} < p_{n+2}$, the hypotenuse lengths satisfy the exact identity

$$\boxed{(C_2 - C_1)(C_1 + C_2) = p_{n+2}^2 - p_n^2.}$$

Equivalently, using the Prime Square-Difference (PSD) factor,

$$\boxed{\mathrm{PSD}_n = \frac{(C_2 - C_1)(C_1 + C_2)}{12}.}$$

Interpretation: Square-jump = (local geometric energy) × (local geometric scale).

3 Prime Triangle Energy

Define the *energy* of the transition between consecutive Prime Triangles:

$$E_n := C_2 - C_1.$$

A first-order expansion in the size of the prime gaps gives

$$E_n \approx \frac{\sqrt{2}}{2}\,(p_{n+2} - p_n) = \frac{\sqrt{2}}{2}\,G_n,$$

where $G_n = g_n + g_{n+1}$ is the skip-one gap.

Gap–Energy Constant

$$\boxed{\frac{E_n}{G_n} \to \frac{\sqrt{2}}{2} \qquad (n \to \infty).}$$

Bias term (gap asymmetry)

A more refined expansion yields

$$\frac{E_n}{G_n} \approx \frac{\sqrt{2}}{2} - \frac{g_n - g_{n+1}}{2\sqrt{2}(2p_n + G_n)}.$$

Thus the energy is modulated by the asymmetry of consecutive gaps.

4 PSD and the Prime Scale Law

The skip-one square jump is

$$p_{n+2}^2 - p_n^2 = 12\,\mathrm{PSD}_n.$$

Direct algebra gives

$$\mathrm{PSD}_n = \frac{G_n(2p_n + G_n)}{12}.$$

First-order expansion

$$\mathrm{PSD}_n \approx \frac{p_n G_n}{6}.$$

PSD Scale Constant

$$\boxed{\frac{\mathrm{PSD}_n}{p_n G_n} \to \frac{1}{6}.}$$

Exact bias decomposition

$$\boxed{\frac{\mathrm{PSD}_n}{p_n G_n} = \frac{1}{6} + \frac{G_n}{12p_n}.}$$

The deviation from $1/6$ is explicit, positive, and vanishes like $(\log p_n)/p_n$.

5 Energy Curvature (Second Difference)

Energy describes the local geometric cost of moving between Prime Triangles. Its *curvature* is defined by the second difference

$$K_n := E_{n+1} - E_n.$$

Using first-order expansions,

$$E_n \approx \frac{\sqrt{2}}{2}(g_n + g_{n+1}), \qquad E_{n+1} \approx \frac{\sqrt{2}}{2}(g_{n+1} + g_{n+2}),$$

hence

$$\boxed{K_n \approx \frac{\sqrt{2}}{2}(g_{n+2} - g_n).}$$

K_n detects whether gaps are opening ($g_{n+2} > g_n$) or closing ($g_{n+2} < g_n$).

6 Dimensionless Shape Curvature

Normalize curvature by energy to obtain a pure, scale-free invariant:

$$\chi_n := \frac{K_n}{E_n}.$$

Using previous expressions,

$$\boxed{\chi_n \approx \frac{g_{n+2} - g_n}{g_n + g_{n+1}}.}$$

This measures the local *shape* of three consecutive gaps:

- $\chi_n = 0 \quad \Rightarrow$ gaps are locally symmetric;
- $\chi_n > 0 \quad \Rightarrow$ forward gap exceeds backward gap;
- $\chi_n < 0 \quad \Rightarrow$ forward gap is smaller.

7 The Prime Geometry Lagrangian

Define the *local action density*

$$\boxed{\mathcal{L}_n := \chi_n^2 = \left(\frac{g_{n+2} - g_n}{g_n + g_{n+1}}\right)^2.}$$

This is a dimensionless measure of local curvature in the prime-gap landscape.

The corresponding *Prime Action* over an interval $[N_1, N_2]$ is

$$\boxed{S = \sum_{n=N_1}^{N_2} \mathcal{L}_n.}$$

Low S indicates globally smoother variation in prime gaps; high S indicates sharp local fluctuations.

8 Structural Invariants of Prime Geometry

Prime Geometry contains the following foundational invariants:

1. Gap–Energy Constant:
$$\frac{E_n}{G_n} \to \frac{\sqrt{2}}{2};$$
2. PSD Scale Constant:
$$\frac{\mathrm{PSD}_n}{p_n G_n} \to \frac{1}{6};$$
3. PSD Bias:
$$\frac{G_n}{12 p_n};$$
4. Shape Curvature:
$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}.$$

These arise naturally from the Prime Triangle Identity and govern the geometry of primes at first and second order.

9 Unified Picture

All components of Prime Geometry descend from the One Equation Core:

$$(C_2 - C_1)(C_1 + C_2) = p_{n+2}^2 - p_n^2.$$

This identity ties together:

- local geometry of Prime Triangles,
- the skip-one square jump between primes,
- geometric energy E_n,
- energy curvature K_n,
- scale laws and constants,
- dimensionless shape curvature χ_n.

Prime Geometry therefore describes the growth and bending of prime gaps through geometric energies and the curvature of their variation.

10 Outlook

Natural directions beyond this Roadmap include:

- incorporating $\Delta\alpha$ into the energy–curvature hierarchy;
- connecting the scale term $C_1 + C_2$ to zeta zeros via the Prime–Zero Ratio;
- analyzing the distribution of χ_n and the global action S;
- comparing S for real primes to randomized gap sequences;
- developing higher-order geometric invariants.

These extend the core structure presented here; the foundational framework is complete.