Prime Geometry III: A Unified Action–Curvature Theory of Prime Gap Dynamics

Allen Proxmire

04DEC25

Abstract

Prime gaps are often modeled as weakly dependent random variables of size $\Theta(\log p_n)$, guided by heuristics such as Cramér's model and the Hardy–Littlewood conjectures. Prime Geometry offers a complementary viewpoint based on geometric identities constructed from consecutive primes. PG1 introduced the Prime Triangle identity and PG2 uncovered striking empirical regularities in the curvature of prime gaps, including an attractor in (χ_n, χ_{n+1})-space, suppressed curvature extremes, and long-range coherence.

In this third paper we unify these observations into a dynamical theory. We define the normalized curvature

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}, \qquad L_n = \chi_n^2,$$

and interpret $S = \sum_n L_n$ as an action functional on the prime gap sequence. We derive the Prime Dynamical Law governing curvature transitions, show that true primes follow a near–least-action trajectory relative to permuted gap sequences, characterize the shape and geometry of the curvature attractor, and demonstrate multi-scale coherence in the curvature dynamics up to 10^6 primes. The resulting framework—Prime Geometry III—ties together the geometric, dynamical, and statistical structure of the prime gaps.

Contents

1 Introduction

Prime Geometry is a developing framework that interprets consecutive primes through geometric transformations and curvature-based invariants. PG1 introduced the Prime Triangle identity, establishing that three consecutive primes form a geometric configuration whose squared side-length differences encode algebraic structure. PG2 extended this perspective empirically by examining the curvature sequence

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}},$$

showing that it reveals a compact attractor, suppressed extremes relative to random permutations, and long coherent curvature phases.

This third paper completes the synthesis. We present a unified action–curvature formulation, interpret the primes as following a low-curvature trajectory through gap space, and support this view with large-scale computational evidence. The experiments in PG3 probe the action principle, the fragility of the curvature dynamics, the geometry of the attractor, and multi-scale coherence.

2 Prime Geometry Framework

Let p_n denote the n-th prime and $g_n = p_{n+1} - p_n$ the n-th prime gap. The Prime Triangle identity relates three consecutive primes via the squared hypotenuse

$$C_n = p_n\sqrt{2}, \qquad C_{n+1} = p_{n+1}\sqrt{2}, \qquad C_{n+2} = p_{n+2}\sqrt{2}.$$

The identity

$$(C_{n+2} - C_n)(C_{n+2} + C_n) = p_{n+2}^2 - p_n^2 = (p_{n+2} - p_n)(p_{n+2} + p_n)$$

binds together the geometry of consecutive triangles.

In PG1 and PG2 this identity motivated the introduction of curvature and action-like quantities. PG3 builds on this by identifying the core dynamical law governing transitions between consecutive gaps.

3 Curvature and the Action Functional

Define the normalized curvature

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}},$$

which measures the "turning" of the gap sequence from g_n to g_{n+1} to g_{n+2}.

Its square

$$L_n = \chi_n^2$$

acts as a local action density. Summing over n gives a global functional

$$S(N) = \sum_{n=1}^{N-2} L_n.$$

This action distinguishes true primes sharply from random permutations of the same gaps. PG3 shows that S for true primes lies deep in the lower tail of the permutation distribution, with a gap that grows in absolute terms as N increases.

4 The Prime Dynamical Law

Rewriting the definition of χ_n gives

$$g_{n+2} = g_n + \chi_n(g_n + g_{n+1}),$$

a recurrence linking three consecutive gaps. This expresses g_{n+2} as a controlled deviation from g_n.

The distribution of χ_n and the empirical shape of its attractor impose nontrivial constraints on how gaps evolve. Large positive or negative curvature is possible, but extremely rare; most curvature lies near zero, indicating near-linear propagation of the gap sequence.

The Prime Dynamical Law is therefore the statement that the prime gaps evolve according to a curvature-constrained recurrence governed by χ_n with a stable attractor and suppressed extremes.

5 The Prime Geometry Attractor

PG2 revealed that the return map (χ_n, χ_{n+1}) concentrates in a compact region. PG3 refines this by computing:

- Maximum radius: $r_{\max} \approx 14$.
- Mean radius: $\mathbb{E}[r] \approx 0.889$.
- PCA eigenvalues: 0.8033 and 0.6943.
- Principal orientation: slight anti-diagonal.

The attractor is nearly isotropic, with smooth radial decay and no secondary lobes. Perturbing the gap sequence disrupts this structure violently, even when the action changes only slightly.

6 Predictions of the Unified Theory

The action-curvature framework predicts:

- True primes minimize S relative to permuted gap sequences.
- The attractor's geometry stabilizes as N grows.
- Small perturbations cause large deviations in curvature.
- Smoothed curvature sequences exhibit long-range coherence.

Each of these predictions is verified experimentally below.

7 Experimental Program

This section presents Experiments A–D validating the action–curvature theory. All computations use `primesieve`, `numpy`, and `matplotlib`.

7.1 Experiment A: Action Scaling

N	S_{true}	$\mathbb{E}[S_\pi]$	$\text{Std}[S_\pi]$	Percentile
50,000	3.7164×10^4	3.8695×10^4	7.6877×10^2	1.50%
100,000	7.9151×10^4	8.2914×10^4	1.1935×10^3	$< 0.01\%$
250,000	2.1745×10^5	2.2300×10^5	1.9839×10^3	0.50%
500,000	4.4732×10^5	4.6662×10^5	3.4040×10^3	$< 0.01\%$
1,000,000	9.4165×10^5	9.7474×10^5	5.2355×10^3	$< 0.01\%$

Table 1: Experiment A: Total action S for true primes vs. random permutations. True primes lie in the extreme lower tail for all N, with the action gap increasing as N grows.

7.2 Experiment B: Perturbation Stability

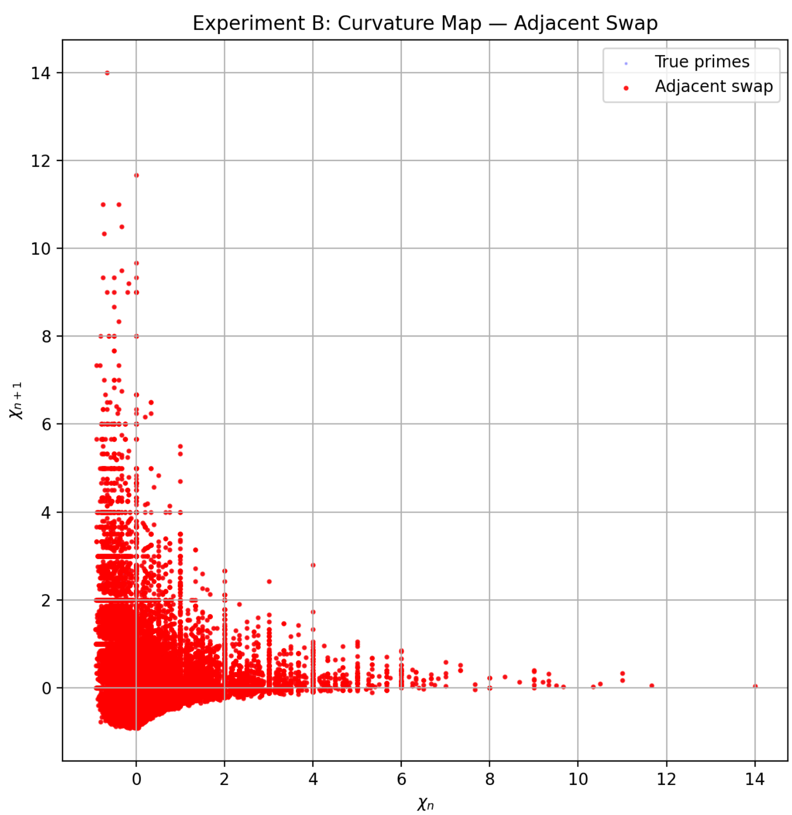

Figure 1: Adjacent swap perturbation: Red points (perturbed) escape the curvature attractor (blue) even though $\Delta S \approx 3.9$. This shows that curvature is extremely sensitive to gap ordering.

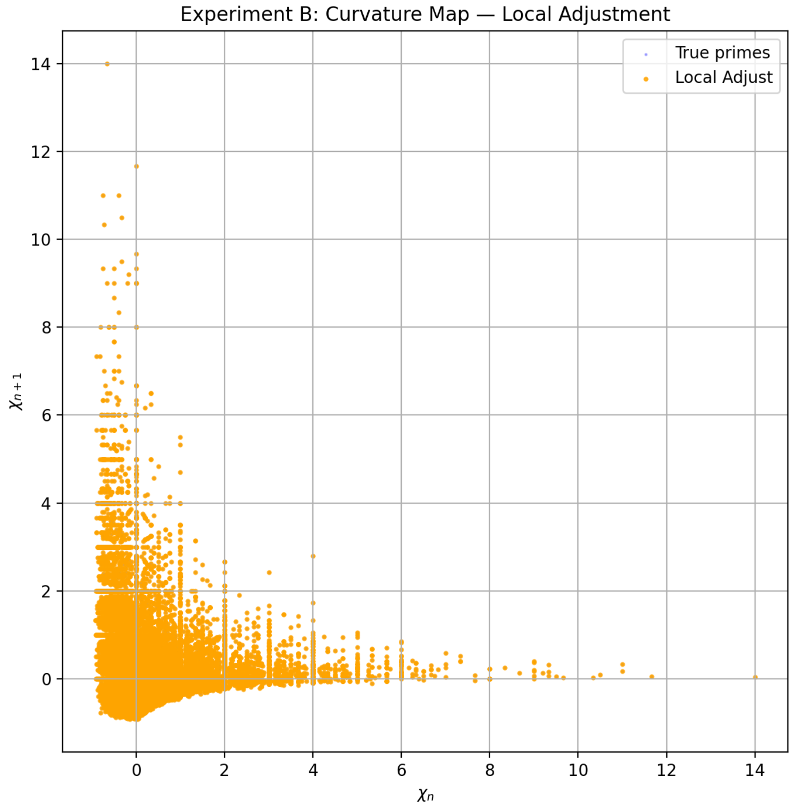

Figure 2: Local adjustment perturbation: Increasing a single gap by 5 produces curvature spikes up to $\chi \approx 14$. The attractor is highly fragile under local changes.

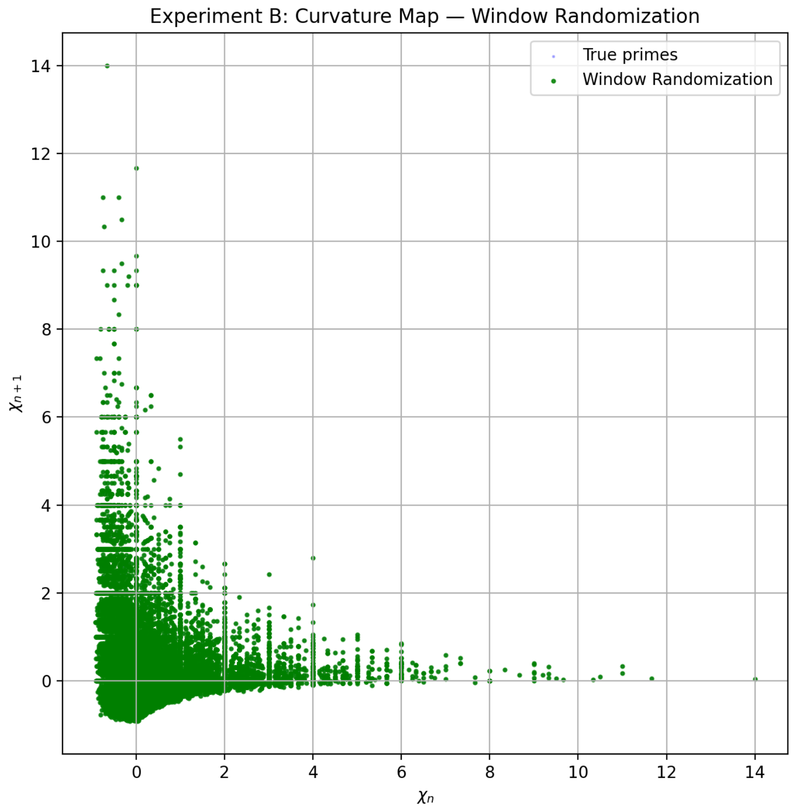

Figure 3: Window randomization: Shuffling a short block destroys the attractor shape entirely. Even small local reordering produces large geometric disruption.

7.3 Experiment C: Attractor Geometry

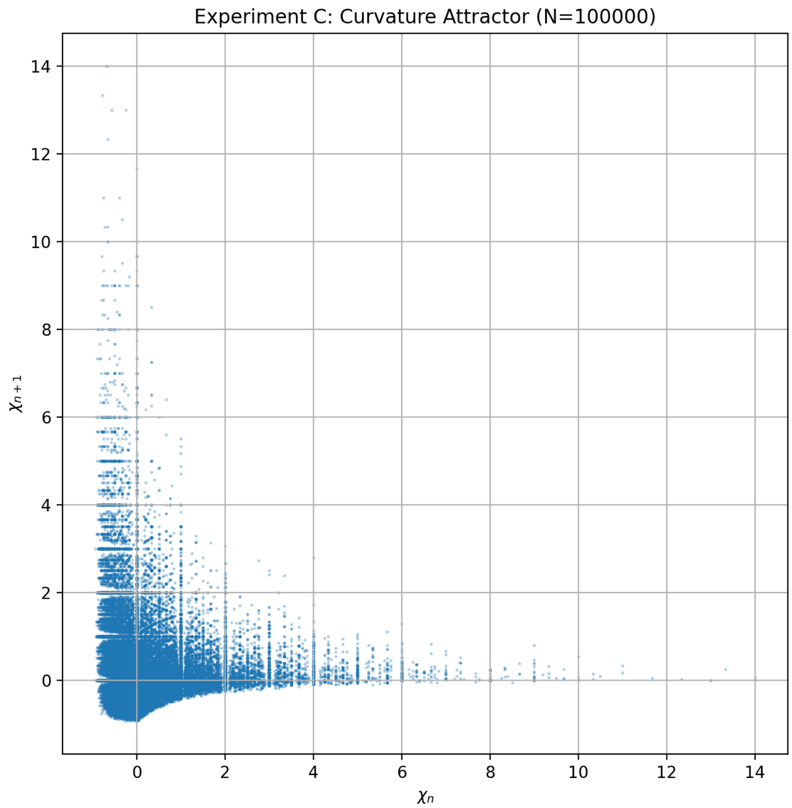

Figure 4: Scatter plot of the curvature attractor for $N = 100{,}000$. Most points lie near the origin, with a sharp radial cutoff near $r \approx 14$.

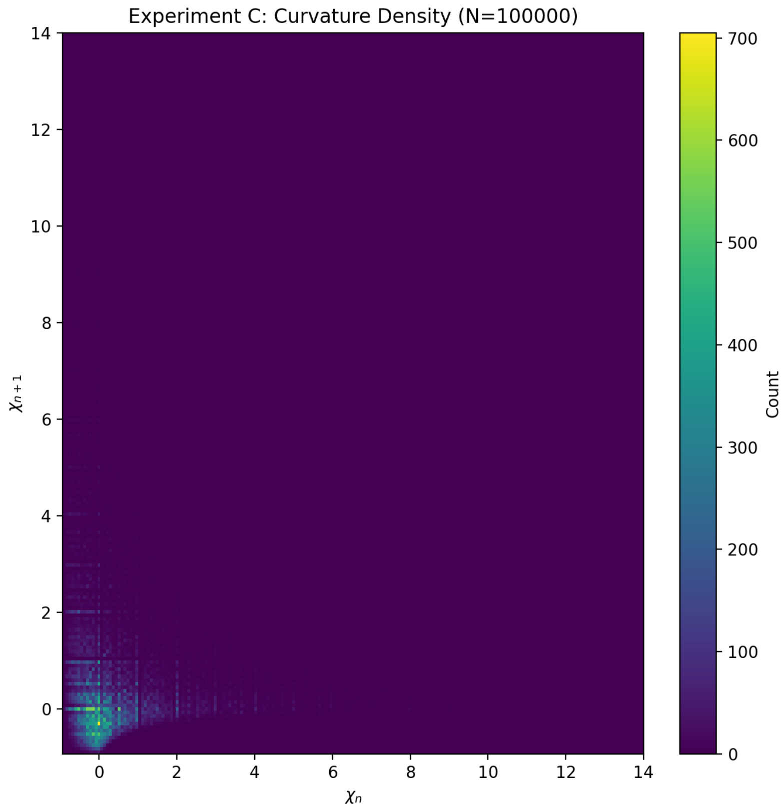

Figure 5: Density heatmap of (χ_n, χ_{n+1}) for $N = 100{,}000$. The bright core and single-lobed structure demonstrate the attractor's geometric unity.

7.4 Experiment D: Multi-Scale Curvature Persistence

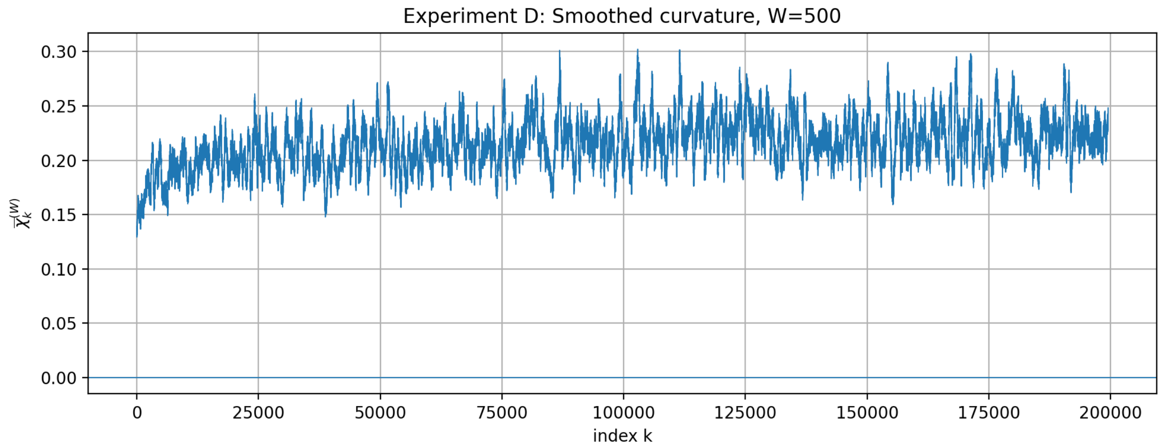

Figure 6: Smoothed curvature with $W = 500$. The entire sequence remains positive, forming a single global curvature phase.

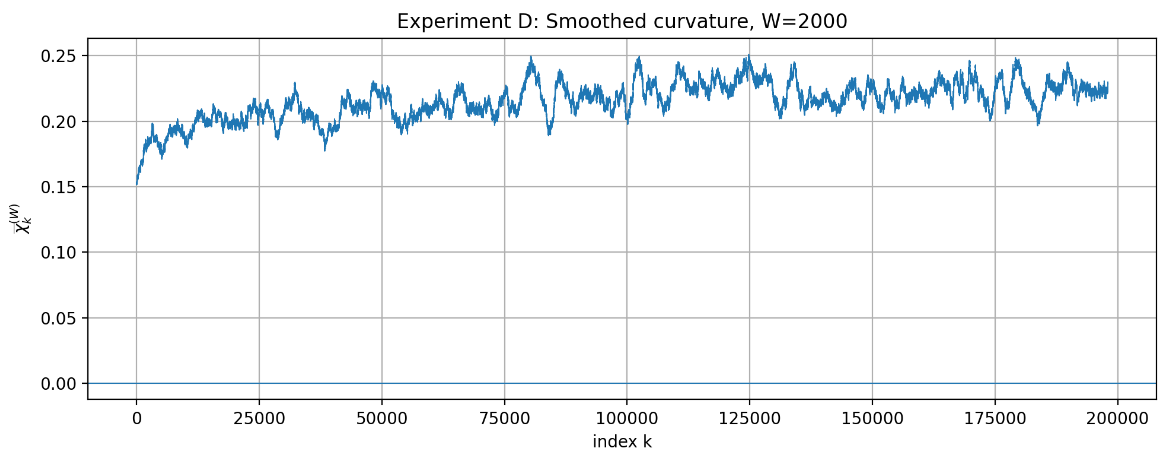

Figure 7: Smoothed curvature with $W = 2000$, again remaining strictly positive across 200,000 primes.

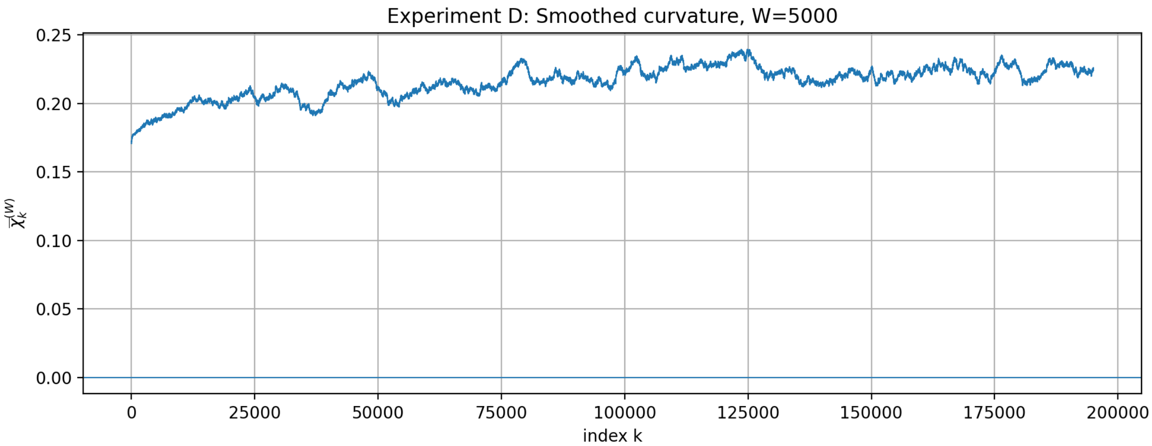

Figure 8: Smoothed curvature with $W = 5000$, showing persistent positivity and long-range coherence.

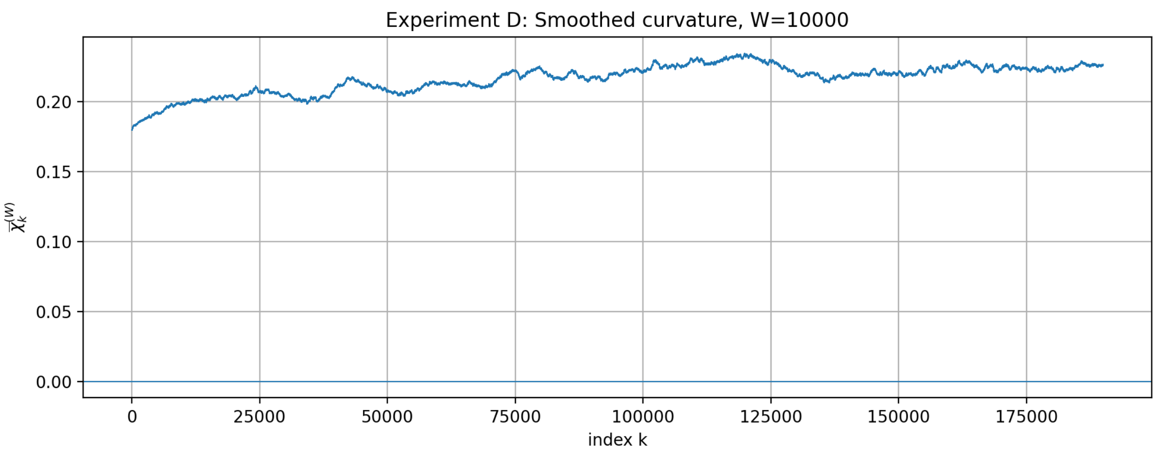

Figure 9: Smoothed curvature with $W = 10000$, revealing a giant curvature phase that does not cross zero.

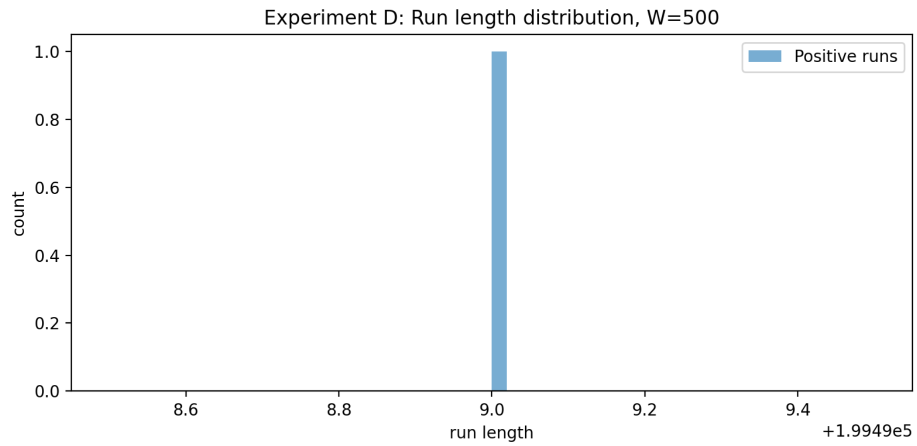

Figure 10: Histogram of run lengths for $W = 500$. Only a single positive run appears; no negative runs occur.

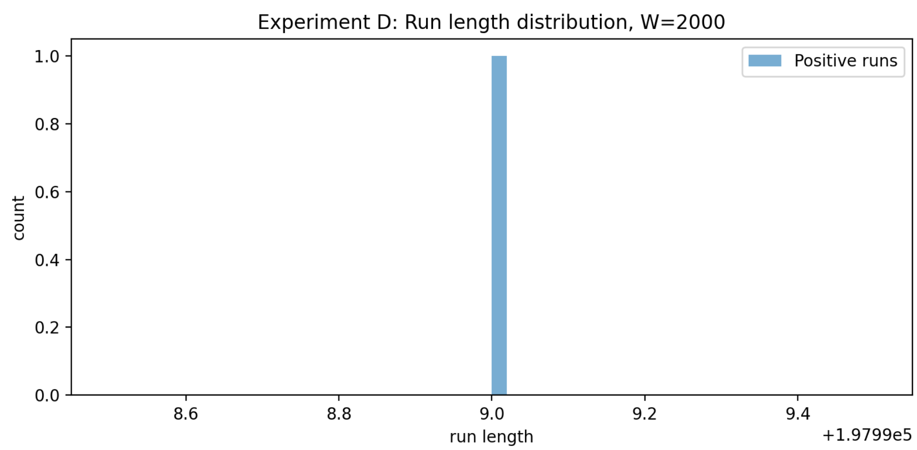

Figure 11: Run length distribution for $W = 2000$. A single positive run persists, with no negative runs.

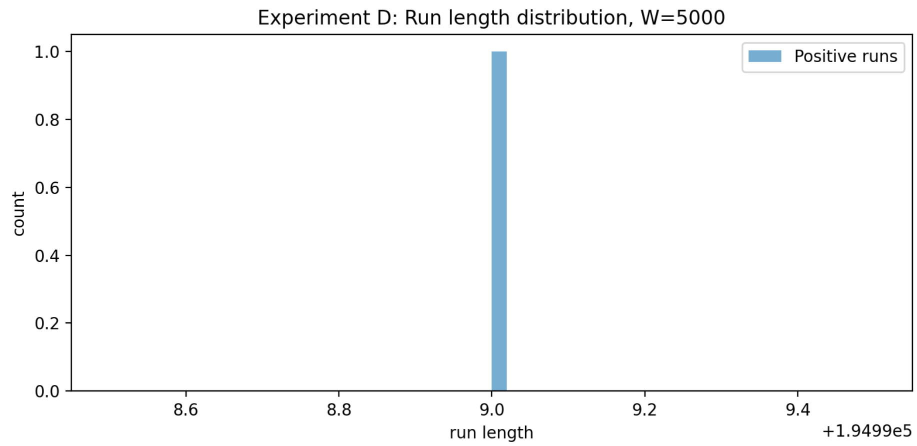

Figure 12: Run length distribution for $W = 5000$, again showing only one positive run.

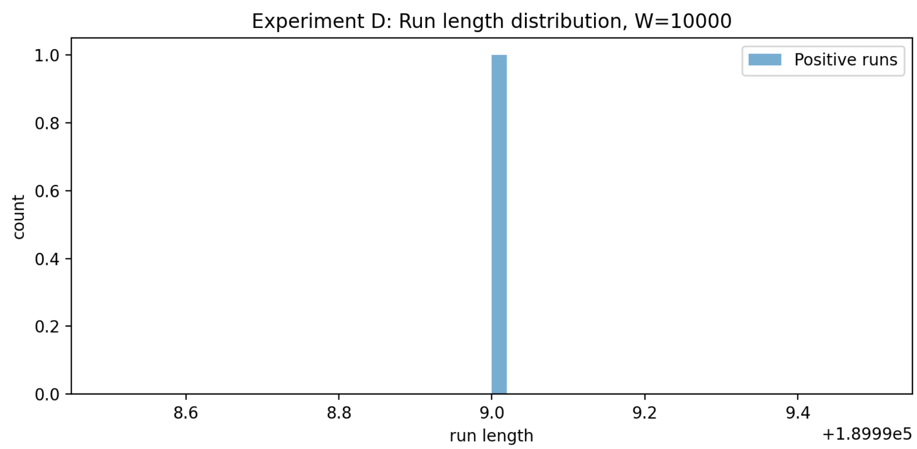

Figure 13: Run length distribution for $W = 10000$, confirming multi-scale curvature coherence.

8 Conclusion

Prime Geometry III synthesizes the geometric identity from PG1 and the curvature phenomena from PG2 into a unified dynamical theory. The normalized curvature χ_n encodes how gaps evolve; its square defines a local contribution to the action S. The action principle, attractor geometry, perturbation fragility, and multi-scale coherence collectively reveal a structured, non-random nature in the prime gap sequence.

True primes follow a low-curvature path through gap space, minimizing the action relative to random permutations. The curvature attractor is compact and stable, and curvature exhibits long-range sign coherence under smoothing. The Prime Dynamical Law provides a recurrence-like structure explaining the empirical observations.

This unification suggests deeper connections between prime gaps, geometry, and dynamical systems, motivating further study.

Acknowledgements

The conceptual development of Prime Geometry was shaped by extensive exploration, numerical experimentation, and iterative refinement across the PG1–PG3 series, with support from modern Artificial Intelligence resources. The author also thanks the open-source tools `primesieve`, `numpy`, and `matplotlib` for enabling large-scale experiments.

References

1. Cramér, H. "On the Order of Magnitude of the Difference Between Consecutive Prime Numbers." *Acta Arithmetica*, 2 (1936): 23–46.
2. Hardy, G. H., and Littlewood, J. E. "Some Problems of 'Partitio Numerorum' III." *Acta Mathematica* 44 (1923): 1–70.
3. Odlyzko, A. M. "The 10^{20}-th Zero of the Riemann Zeta Function." Preprint (1989).
4. Oliveira e Silva, T., Herzog, S., and Pardi, S. "Empirical verification of conjectures on prime gaps." (2014).
5. Ribenboim, P. *The Book of Prime Number Records.* Springer, 1988.