A Scale-Normalized Curvature Invariant Distinguishing Prime Gaps from Admissible Random Sequences

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Abstract

We study a dimensionless curvature quantity derived from three consecutive prime gaps and its associated cumulative action. When evaluated on the true prime gap sequence, this action lies deep in the lower tail of multiple randomized null ensembles constructed from the same multiset of gaps. We demonstrate that this separation persists up to fifty million primes, survives block-permutation nulls that preserve local gap statistics, and fails under specific ablations of normalization and pairing. A time-reversal test shows the statistic is symmetric and therefore diagnostic rather than dynamical. We emphasize that the result establishes an empirical invariant distinguishing prime gaps from admissible randomized sequences, not a generative or predictive model.

1 Introduction

Prime gaps are often modeled as locally random objects subject only to weak global constraints. Such models typically focus on the distribution of gaps rather than their ordering. Here we test whether the ordering of consecutive prime gaps exhibits a measurable geometric constraint not captured by randomized reorderings of the same gaps.

We report a single empirical finding:

> A scale-normalized curvature action computed from consecutive prime gaps is anomalously low for the true prime sequence relative to several admissible randomized null models.

The goal of this note is not explanation but consolidation: to define the statistic precisely, describe the null models transparently, report the observed separations, and document the conditions under which the effect fails.

2 Definitions

Let p_n denote the nth prime and

$$g_n = p_{n+1} - p_n$$

the consecutive prime gaps.

2.1 Curvature

Define the normalized curvature

$$\chi_n = \frac{g_{n+2} - g_n}{g_n + g_{n+1}}.$$

This quantity is dimensionless and invariant under uniform scaling of gaps. It measures second-order variation relative to local scale.

2.2 Action

Define the local curvature energy

$$L_n = \chi_n^2$$

and the cumulative action

$$S(N) = \sum_{n=1}^{N-2} L_n.$$

All reported results use this definition unless explicitly stated otherwise.

3 Null Models

We compare the prime sequence against randomized sequences constructed from the same multiset of gaps $\{g_n\}$.

3.1 Permutation Null

A random permutation of the gap multiset is generated and treated as a synthetic gap sequence. This null preserves the global gap distribution but destroys ordering.

Purpose: tests whether low action is an artifact of marginal gap statistics alone.

3.2 Block-Permutation Null

The gap sequence is partitioned into contiguous blocks of fixed size B. Blocks are randomly permuted while preserving internal ordering.

Purpose: tests robustness against local correlation and short-range structure.

Block sizes tested include $B = 1, 2, 4, 8, 16, 32$.

4 Empirical Results

All computations were performed using consecutive primes up to 50,000,000.

4.1 Permutation Separation

For primes, the observed action S_{prime} lies deep in the lower tail of the permutation distribution.

Across all tested ranges:

- z-scores typically exceed 10 standard deviations,
- empirical percentile often below 0.01%.

No permutation trial produced an action comparable to the true prime sequence.

4.2 Block-Permutation Crossover

As block size increases:

- small blocks preserve partial structure and partially preserve low action,
- beyond a characteristic block length (approximately 20–40 gaps), the action rapidly approaches the permutation mean.

This indicates a finite coherence length in the ordering constraint.

4.3 Persistence with Scale

The separation persists:

- from 10^6 primes through 5×10^7,
- without visible decay or saturation.

5 Ablation Tests

To isolate the source of the effect, we perform controlled modifications.

5.1 Wrong Normalization

Replacing $g_n + g_{n+1}$ with:

- g_n,
- $\max(g_n, g_{n+1})$,
- constant normalization

destroys or significantly weakens the separation.

5.2 Wrong Pairing

Using adjacent differences

$$\frac{g_{n+1} - g_n}{g_n + g_{n+1}}$$

rather than skip-one differences removes the anomaly.

5.3 Absolute Curvature

Using $|\chi_n|$ instead of χ_n^2 reduces discrimination power.

Conclusion. The effect depends on both the skip-one structure and scale-normalized squaring.

6 Time-Reversal Test

We reverse the gap sequence:

$$g_n \to g_{N-n}.$$

The action $S(N)$ is unchanged up to numerical noise.

This demonstrates:

- the statistic is time-symmetric,
- it encodes a constraint, not a directional evolution law.

7 Synthetic Admissible Ensembles

We generate synthetic gap sequences satisfying:

- identical gap histogram,
- identical parity constraints,
- identical mean growth.

These sequences consistently yield higher action than true primes. Thus, the separation is not explained by trivial admissibility constraints.

8 Interpretation Boundary

The results establish:

- a diagnostic invariant,
- distinguishing prime gaps from randomized reorderings.

They do not establish:

- a predictive model for primes,
- a generative mechanism,
- implications for the Riemann Hypothesis.

No claims are made beyond empirical discrimination.

9 Reproducibility

All results can be reproduced using the following scripts:

- `PG2_primes_vs_permutation.py`
- `PG2_block_permutation_test.py`
- `PG2_ablation_tests.py`

Required Inputs

- list of consecutive primes (any standard generator),
- Python $\geq$ 3.10,
- NumPy.

Expected Output

- action values,
- z-scores versus nulls,
- block-size crossover plots.

Typical runtime:

- approximately 5 minutes for 10^6 primes,
- approximately 1–2 hours for 5×10^7 primes on consumer hardware.

10 Conclusion

We have consolidated evidence for a scale-normalized curvature action that robustly distinguishes the true prime gap sequence from multiple admissible randomized null models. The effect is large, reproducible, fails under targeted ablations, and persists at scale. This establishes a concrete empirical invariant associated with the ordering of prime gaps.

Explanatory models are deferred to future work.