Prime Triangles and the Prime Square-Difference Identity

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Abstract

We associate a right triangle to each consecutive prime pair (p_n, p_{n+1}) and examine how the geometry changes across a triple of consecutive primes. Comparing the hypotenuse lengths of the resulting Prime Triangles yields a universal identity for the square difference $(p_{n+2})^2 - (p_n)^2$, which is always divisible by 12. This motivates the Prime Square-Difference (PSD) Factor

$$\mathrm{PSD}_n = \frac{(p_{n+2})^2 - (p_n)^2}{12},$$

an integer with a constrained last digit and a natural analytic approximation. We also show that when the skip-one gap $p_{n+2} - p_n$ equals 6, the PSD Factor identifies the direction of the twin-prime pair contained in the triple.

1 Prime Triangles

Let (p_n) be the sequence of prime numbers. For each consecutive pair (p_n, p_{n+1}), we define the *Prime Triangle* as the right triangle with legs p_n and p_{n+1} and hypotenuse

$$C_n = \sqrt{p_n^2 + p_{n+1}^2}.$$

For a triple of consecutive primes (p_n, p_{n+1}, p_{n+2}), we obtain two such hypotenuse lengths:

$$C_1 = \sqrt{p_n^2 + p_{n+1}^2}, \qquad C_2 = \sqrt{p_{n+1}^2 + p_{n+2}^2}.$$

Although defined geometrically, the behavior of C_1 and C_2 encodes several rigid arithmetic relationships among the primes.

2 The Square-Difference Identity

A direct expansion shows:

$$C_2^2 - C_1^2 = (p_{n+2}^2 + p_{n+1}^2) - (p_n^2 + p_{n+1}^2) = p_{n+2}^2 - p_n^2.$$

Thus the difference of the hypotenuse squares depends only on the first and third primes in the triple:

$$C_2^2 - C_1^2 = (p_{n+2})^2 - (p_n)^2.$$

Divisibility by 12

For all primes $p > 3$,

$$p \equiv \pm 1 \pmod 6 \quad \Longrightarrow \quad p^2 \equiv 1 \pmod{12}.$$

Hence

$$(p_{n+2})^2 - (p_n)^2 \equiv 0 \pmod{12}.$$

This motivates the following definition.

Definition (PSD Factor).

$$\mathrm{PSD}_n = \frac{(p_{n+2})^2 - (p_n)^2}{12}.$$

Since the numerator is always divisible by 12, the PSD Factor is an integer.

3 Modular Structure of the PSD Factor

All primes greater than 3 can be written

$$p_n = 6a \pm 1, \qquad p_{n+2} = 6b \pm 1.$$

Expanding each of the four sign combinations yields

$$(p_{n+2})^2 - (p_n)^2 \equiv 0,\ 48,\ 72 \pmod{120},$$

and therefore

$$\mathrm{PSD}_n \equiv 0,\ 4,\ 6 \pmod{10}.$$

Thus the PSD Factor always ends in the digit 0, 4, or 6.

4 Analytic Interpretation

Let

$$G = p_{n+2} - p_n$$

be the skip-one prime gap. Using the factorization $a^2 - b^2 = (a-b)(a+b)$ we obtain

$$\mathrm{PSD}_n = \frac{G(p_{n+2} + p_n)}{12}.$$

Since $G \ll p_n$ for large primes,

$$p_{n+2} + p_n = 2p_n + G \approx 2p_n.$$

Hence the PSD Factor satisfies the approximation

$$\mathrm{PSD}_n \approx \frac{G \cdot p_n}{6}.$$

Thus the PSD Factor behaves like a *gap-weighted prime magnitude*.

5 Twin-Prime Structure When $G = 6$

A particularly structured case occurs when the skip-one gap is 6:

$$p_{n+2} - p_n = 6.$$

Because primes are $\pm 1 \pmod 6$, the triple must be either

$$p_n,\ p_n + 2,\ p_n + 6, \qquad \text{or} \qquad p_n,\ p_n + 4,\ p_n + 6.$$

Thus the middle prime p_{n+1} always participates in a twin-prime pair.

Exact Formula

From the identity,

$$\mathrm{PSD}_n = \frac{(p_{n+2})^2 - p_n^2}{12} = \frac{6(p_{n+2} + p_n)}{12} = \frac{p_{n+2} + p_n}{2},$$

we see that PSD_n is the exact midpoint of the outer primes:

$$\mathrm{PSD}_n = \frac{p_n + p_{n+2}}{2}.$$

Since the middle prime in this case satisfies

$$p_{n+1} = \frac{p_n + p_{n+2}}{2},$$

we obtain the remarkably simple relation

$$\mathrm{PSD}_n = p_{n+1} \pm 1,$$

where the sign corresponds to the mod-6 configuration.

Identifying the Twin-Prime Direction

$$\begin{cases} \mathrm{PSD}_n = p_{n+1} + 1 \quad \Rightarrow \quad (p_n, p_{n+1}) \text{ is the twin-prime pair,} \\ \mathrm{PSD}_n = p_{n+1} - 1 \quad \Rightarrow \quad (p_{n+1}, p_{n+2}) \text{ is the twin-prime pair.} \end{cases}$$

Thus the PSD Factor encodes not only the midpoint structure but also the *direction* of the twin-prime attachment.

Conclusion

The geometry of Prime Triangles leads directly to the rigid identity

$$(p_{n+2})^2 - (p_n)^2 \equiv 0 \pmod{12},$$

and to the integer PSD Factor with a constrained last digit. Its analytic approximation $\mathrm{PSD}_n \approx (Gp_n)/6$ reflects the structure of prime gaps, and in the special case $G = 6$, the PSD Factor encodes the position of the twin-prime pair within the triple. This blend of geometric simplicity and modular rigidity highlights an unexpectedly rich structure underlying consecutive primes.

Appendix: Sample Values

p_n	p_{n+1}	p_{n+2}	G	C_1	C_2	$C_2 - C_1$	PSD_n
5	7	11	6	8.6023	13.0384	4.4361	8
7	11	13	6	13.0384	17.0294	3.9910	10
11	13	17	6	17.0294	21.4009	4.3715	14
13	17	19	6	21.4009	25.4951	4.0942	16
17	19	23	6	25.4951	29.8329	4.3378	20