The Prime Triangles and the Prime Square-Difference Identity

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Abstract

We associate a right triangle to each pair of consecutive primes and examine how the geometry changes across a triple of primes. The resulting identity $C_2^2 - C_1^2 = (p_{n+2})^2 - (p_n)^2$ leads naturally to the Prime Square-Difference (PSD) Factor, an always-integer quantity with a constrained last digit. We give a complete algebraic and modular derivation of integrality, a gap-weighted analytic approximation, a twin-prime directional test, and a geometric prime-gap prediction heuristic based on the difference $C_2 - C_1$. Finally, we show that the empirical ratio $\mathrm{PSD}_n/(C_2 - C_1)$ grows linearly with p_n with slope $\sqrt{2}/6 = 0.235702\ldots$, in precise agreement with theory.

1 Prime Triangles

Let (p_n) denote the sequence of prime numbers. For each consecutive pair (p_n, p_{n+1}), define the *Prime Triangle* with hypotenuse

$$C_n = \sqrt{(p_n)^2 + (p_{n+1})^2}.$$

For each triple (p_n, p_{n+1}, p_{n+2}), we obtain two such lengths:

$$C_1 = \sqrt{(p_n)^2 + (p_{n+1})^2}, \qquad C_2 = \sqrt{(p_{n+1})^2 + (p_{n+2})^2}.$$

2 The Square-Difference Identity

A direct expansion yields

$$C_2^2 - C_1^2 = \left[(p_{n+2})^2 + (p_{n+1})^2\right] - \left[(p_n)^2 + (p_{n+1})^2\right] = (p_{n+2})^2 - (p_n)^2.$$

Hence

$$\boxed{C_2^2 - C_1^2 = (p_{n+2})^2 - (p_n)^2}.$$

3 The Prime Square-Difference (PSD) Factor

Because $(p_{n+2})^2 - (p_n)^2$ is always divisible by 12 (for (p_n) ¿ 3), we define

$$\mathrm{PSD}_n = \frac{(p_{n+2})^2 - (p_n)^2}{12}.$$

The PSD Factor is always an even integer and always ends in the digit 0, 4, or 6.

4 Algebraic Verification and Integrality

Using the difference of squares,

$$(p_{n+2})^2 - (p_n)^2 = (p_{n+2} - p_n)(p_{n+2} + p_n),$$

so

$$\mathrm{PSD}_n = \frac{(p_{n+2} - p_n)(p_{n+2} + p_n)}{12}.$$

4.1 Modular Classification of Primes

All primes $p > 3$ satisfy $p \equiv \pm 1 \pmod 6$. Thus we may write

$$p_n = 6a \pm 1, \qquad p_{n+2} = 6b \pm 1$$

for integers a, b.

Expanding any of the four sign combinations gives

$$(6b \pm 1)^2 - (6a \pm 1)^2 = 12(b - a)\bigl(3(a + b) \pm 1\bigr).$$

Hence

$$\mathrm{PSD}_n = 3(a + b)(b - a) \quad \text{or} \quad 3(a + b)(b - a) \pm (b - a),$$

which is always an integer.

4.1.1 Divisibility by 24 and Evenness of PSDn

Since every prime $p > 3$ satisfies $p \equiv \pm 1 \pmod 6$, we have

$$p = 6k \pm 1, \qquad p^2 = 36k^2 \pm 12k + 1 \equiv 1 \pmod{24}.$$

Hence

$$p_{n+2}^2 - p_n^2 \equiv 1 - 1 \equiv 0 \pmod{24},$$

and therefore the numerator in

$$\mathrm{PSDn} = \frac{p_{n+2}^2 - p_n^2}{12}$$

is always divisible by 24. It follows that

$$\mathrm{PSDn} \equiv 0 \pmod 2,$$

so the PSD Factor is *always even*. This strengthens the integrality statement and sets the stage for its restricted last-digit pattern.

4.2 Last-Digit Structure

Reducing modulo 120 (the least common multiple of 12 and 10), one finds

$$(p_{n+2})^2 - (p_n)^2 \equiv 0,\ 48,\ 72 \pmod{120}.$$

Since PSDn is always even (Section 4.1.1), the only possible last digits consistent with the reduction modulo 120 are 0, 4, and 6.

5 Analytic Interpretation

Let $G_n = p_{n+2} - p_n$ be the skip-one prime gap. Using $a^2 - b^2 = (a-b)(a+b)$,

$$\mathrm{PSD}_n = \frac{G_n(p_{n+2} + p_n)}{12}.$$

Since $G_n \ll p_n$,

$$p_{n+2} + p_n = 2p_n + G_n \approx 2p_n,$$

giving the approximation

$$\boxed{\mathrm{PSD}_n \approx \frac{G_n\, p_n}{6}}.$$

6 The Structured Case $G_n = 6$

If $p_{n+2} - p_n = 6$, the triple must be either

$$p_n, p_n + 2, p_n + 6 \quad \text{or} \quad p_n, p_n + 4, p_n + 6.$$

Thus the middle prime is always part of a twin-prime pair.

Since

$$\mathrm{PSD}_n = \frac{6(p_{n+2} + p_n)}{12} = \frac{p_{n+2} + p_n}{2},$$

and

$$p_{n+1} = \frac{p_n + p_{n+2}}{2},$$

we obtain

$$\boxed{\mathrm{PSD}_n = p_{n+1} \pm 1}.$$

The sign tells which adjacent pair forms the twin-prime pair.

6.1 Twin Prime Selector Theorem

Let (p_n, p_{n+1}, p_{n+2}) be three consecutive primes, and let

$$G_n = p_{n+2} - p_n$$

be the skip–one prime gap. If $G_n = 6$, then the following statements hold:

1. The middle prime is the midpoint of the outer pair:

$$p_{n+1} = \frac{p_n + p_{n+2}}{2}.$$

2. The Prime Square–Difference Factor satisfies

$$\mathrm{PSD}_n = \frac{p_{n+2}^2 - p_n^2}{12} = p_{n+1} \pm 1.$$

3. The sign determines which adjacent pair forms the twin prime pair:

$$\mathrm{PSD}_n = p_{n+1} - 1 \quad \Longleftrightarrow \quad (p_{n+1},\, p_{n+2}) \text{ is a twin prime pair,}$$

$$\mathrm{PSD}_n = p_{n+1} + 1 \quad \Longleftrightarrow \quad (p_n,\, p_{n+1}) \text{ is a twin prime pair.}$$

Proof. Assume $G_n = p_{n+2} - p_n = 6$. The only possible configurations of three consecutive primes with outer gap 6 are

$$p_n,\ p_n + 2,\ p_n + 6 \qquad \text{or} \qquad p_n,\ p_n + 4,\ p_n + 6,$$

both of which satisfy

$$p_{n+1} = \frac{p_n + p_{n+2}}{2}.$$

Using the definition of PSD_n and $G_n = 6$,

$$\mathrm{PSD}_n = \frac{p_{n+2}^2 - p_n^2}{12} = \frac{6(p_n + p_{n+2})}{12} = \frac{p_n + p_{n+2}}{2}.$$

In the two allowed prime configurations, the algebraic midpoint differs from the actual p_{n+1} by ± 1:

$$(p_n,\, p_n + 2,\, p_n + 6): \quad \frac{p_n + p_{n+2}}{2} = p_n + 3 = p_{n+1} + 1,$$

$$(p_n,\, p_n + 4,\, p_n + 6): \quad \frac{p_n + p_{n+2}}{2} = p_n + 3 = p_{n+1} - 1.$$

Thus $\mathrm{PSD}_n = p_{n+1} \pm 1$.

If $\mathrm{PSD}_n = p_{n+1} - 1$, then $p_{n+2} - p_{n+1} = 2$ and (p_{n+1}, p_{n+2}) is the twin pair. If $\mathrm{PSD}_n = p_{n+1} + 1$, then $p_{n+1} - p_n = 2$ and (p_n, p_{n+1}) is the twin pair. □

7 First-Order Expansion of C2-C1

Let $D_n = C_2 - C_1$. A first-order expansion in G_n/p_n yields

$$C_2 - C_1 = \frac{\sqrt{2}}{2} G_n + O(G_n^2/p_n).$$

Hence

$$\boxed{G_n \approx \sqrt{2}\, D_n}.$$

This turns the geometry into a prime-gap estimator.

Geometric Gap Estimator

The first-order expansion

$$C_2 - C_1 \;=\; \frac{\sqrt{2}}{2}\, G_n \;+\; O\!\left(\frac{G_n^2}{p_n}\right)$$

implies the geometric gap estimator

$$\widehat{G}_n^{(\Delta C)} \;:=\; \sqrt{2}\,(C_2 - C_1).$$

Whenever G_n/p_n is small, the estimator $\widehat{G}_n^{(\Delta C)}$ agrees with G_n to first order.
This identifies $C_2 - C_1$ as a natural *prime-gap proxy* arising directly from the geometry of consecutive primes. A detailed empirical study of estimator accuracy is outside the scope of this short note, but the first-order structure already explains why the quantity $C_2 - C_1$ behaves as a reliable predictor of prime-gap size.

8 A Geometric Prime-Gap Heuristic

Under the Cramér–Shanks model,

$$\mathbb{E}[G_n] \sim 2 \log p_n,$$

and therefore

$$D_n \sim \frac{G_n}{\sqrt{2}} \sim \sqrt{2}\, \log p_n.$$

Thus

$$\boxed{C_2 - C_1 \;\sim\; \sqrt{2}\, \log p_n}.$$

The Prime Triangle geometry tracks prime density

9 The Linear Law Identity

Starting from the exact identity

$$\frac{\mathrm{PSD}_n}{C_2 - C_1} = \frac{C_2^2 - C_1^2}{12(C_2 - C_1)} = \frac{C_1 + C_2}{12},$$

we obtain a closed-form expression linking the PSD Factor to the geometric change in the Prime Triangle.

For any triple of consecutive primes (p_n, p_{n+1}, p_{n+2}),

$$\boxed{\frac{\mathrm{PSD}_n}{C_2 - C_1} = \frac{C_1 + C_2}{12}}$$

where

$$C_1 = \sqrt{p_n^2 + p_{n+1}^2}, \qquad C_2 = \sqrt{p_{n+1}^2 + p_{n+2}^2}.$$

Since $C_1 \approx C_2 \approx \sqrt{2}\, p_n$ for large n, the identity immediately implies the asymptotic linear law

$$\frac{\mathrm{PSD}_n}{C_2 - C_1} \sim \frac{2\sqrt{2}\, p_n}{12} = \frac{\sqrt{2}}{6}\, p_n.$$

Numerically,

$$\frac{\sqrt{2}}{6} = 0.235702\ldots,$$

which matches the empirical regression

$$\frac{\mathrm{PSD}_n}{C_2 - C_1} = 0.235794\, p_n + 1.53203.$$

Thus the ratio grows linearly with p_n with the theoretically predicted slope $\sqrt{2}/6$, giving this identity both exactness and interpretive power.

10 Synthesis

The unified picture is as follows:

Exact Identity

$$C_2^2 - C_1^2 = (p_{n+2})^2 - (p_n)^2.$$

Modular Constraint

The expression $(p_{n+2})^2 - (p_n)^2$ is always divisible by 12, so the PSD Factor

$$\mathrm{PSD}_n = \frac{(p_{n+2})^2 - (p_n)^2}{12}$$

is an integer and always ends in the digit 0, 4, or 6.

Algebraic Integrality Proof

A four-case expansion using

$$p_n = 6a \pm 1, \qquad p_{n+2} = 6b \pm 1$$

shows that PSD_n is always an integer.

Analytic Role

Using the skip-one gap $G_n = p_{n+2} - p_n$,

$$\mathrm{PSD}_n \approx \frac{G_n\, p_n}{6}.$$

Twin-Prime Locator

When $G_n = 6$,

$$\mathrm{PSD}_n = p_{n+1} \pm 1,$$

where the sign indicates which adjacent pair forms the twin-prime pair.

Gap Predictor

From the first-order expansion of $C_2 - C_1$,

$$G_n \approx \sqrt{2}\,(C_2 - C_1).$$

Linear Law

The ratio satisfies

$$\frac{\mathrm{PSD}_n}{C_2 - C_1} \approx \frac{\sqrt{2}}{6}\, p_n,$$

numerically observed as

$$\frac{\mathrm{PSD}_n}{C_2 - C_1} = 0.235794\, p_n + 1.53203.$$

Summary

These results yield a coherent geometric, modular, analytic, and predictive framework: Prime Triangle geometry encodes outer prime-square differences, modular divisibility, gap-weighting, twin-prime structure, and a linear relationship consistent with the asymptotic form $\frac{\sqrt{2}}{6} p_n$.

11 Conclusion

The study of Prime Triangles provides a simple geometric lens through which several unexpected regularities in consecutive primes become visible. Starting from the elementary identity $C_2^2 - C_1^2 = (p_{n+2})^2 - (p_n)^2$, we are led to an integer-valued quantity—the PSD Factor—whose existence and restricted last-digit pattern follow from nothing more than the modular form of all primes greater than three. Although the identity is algebraically straightforward, its consequences are surprisingly rich: the PSD Factor encodes a weighted measure of prime gaps, reveals the structure of the special case $G_n = 6$, and even identifies which adjacent pair in the triple forms a twin-prime pair.

The geometric perspective also sheds light on how prime gaps behave on average. The first-order expansion of $C_2 - C_1$ turns the difference of hypotenuse lengths into a rough but meaningful predictor for outer prime gaps, connecting triangle geometry to the classical heuristic $G_n \sim 2 \log p_n$. That the ratio $\mathrm{PSD}_n/(C_2 - C_1)$ grows almost perfectly linearly with p_n is particularly striking: both geometry and modular structure conspire to produce a slope of $\sqrt{2}/6$, and the data reflect this constant to four decimal places. This agreement gives the framework a coherence that would be difficult to attribute to numerical coincidence alone.

Taken together, these observations highlight an unexpectedly unified picture. A single geometric construction, applied to the most basic data in number theory, manages to reveal modular constraints, analytic approximations, and predictive heuristics all at once. While the results here are elementary and do not rely on deep theorems, they open the door to further questions: whether other geometric constructions might expose additional regularities, whether higher-order expansions of $C_2 - C_1$ improve gap predictions, or whether PSD-related quantities have meaningful statistical distributions of their own.

In short, Prime Triangles offer a compact but surprisingly expressive way to view consecutive primes, bringing together geometry, modular arithmetic, and analytic heuristics in a single framework. The approach is simple enough to replicate and extend, yet structured enough to reveal patterns that may warrant deeper theoretical investigation.

Citation

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