The Prime Square-Difference Identity: A Modular Constraint and the PSD Factor

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Abstract

We examine the identity

$$(p_{n+2})^2 = (p_n)^2 + 12x,$$

which holds for all primes p_n and p_{n+2} (not necessarily consecutive). The integer

$$x = \frac{(p_{n+2})^2 - (p_n)^2}{12}$$

defines what we term the Prime Square-Difference Factor (PSD Factor). We give a complete modular classification showing that x always ends in digit 0, 4, or 6, and we develop both an analytic approximation and a precise structural interpretation of x in the special case when $p_{n+2} - p_n = 6$. In that case the PSD Factor encodes the location of a twin-prime pair and determines which of the outer primes is twinned with the middle prime.

1 Statement of the Identity

For any primes p_n and p_{n+2} (not necessarily consecutive), define

$$(p_{n+2})^2 = (p_n)^2 + 12x. \tag{1}$$

Solving for x gives the Prime Square-Difference Factor

$$x = \frac{(p_{n+2})^2 - (p_n)^2}{12}. \tag{2}$$

We now establish that x is always an integer and determine its last digit.

2 Algebraic Derivation

Using the difference-of-squares identity,

$$(p_{n+2})^2 - (p_n)^2 = (p_{n+2} - p_n)(p_{n+2} + p_n),$$

we obtain

$$x = \frac{(p_{n+2} - p_n)(p_{n+2} + p_n)}{12}. \tag{3}$$

Integrality is not immediately obvious from this form, but follows from the mod-6 structure of all primes greater than 3.

2.1 Mod-6 Representation of Primes

Every prime $p > 3$ satisfies

$$p \equiv \pm 1 \pmod 6,$$

so we may write

$$p_n = 6a \pm 1, \qquad p_{n+2} = 6b \pm 1,$$

for integers a, b. There are four sign choices. A direct expansion shows that

$$(6b \pm 1)^2 - (6a \pm 1)^2 = 12(b - a)\big(3(a + b) \pm 1\big),$$

which always contains a factor of 12. Thus x is always an integer.

2.2 Last-Digit Classification

To determine the last digit of x, reduce modulo 120, the least common multiple of 10 and 12. In each sign case we obtain

$$(p_{n+2})^2 - (p_n)^2 \equiv 0, 48, 72 \pmod{120}.$$

Dividing by 12 yields

$$x \equiv 0, 4, 6 \pmod{10}.$$

Hence the PSD Factor always ends in digit 0, 4, or 6.

3 Analytic Interpretation: The PSD Factor

Let $g = p_{n+2} - p_n$ be the skip-one prime gap. Then

$$x = \frac{g\,(p_{n+2} + p_n)}{12}.$$

Since

$$p_{n+2} = p_n + g \quad \Longrightarrow \quad p_{n+2} + p_n = 2p_n + g,$$

and $g \ll p_n$ for all sufficiently large primes, we obtain the approximation

$$p_{n+2} + p_n \approx 2p_n.$$

Thus the PSD Factor satisfies the analytic approximation

$$x \approx \frac{g\,p_n}{6}. \tag{4}$$

In this sense, the PSD Factor behaves like a gap-weighted prime value: it amplifies the skip-one prime gap g by the surrounding prime magnitude.

4 Twin-Prime Structure When the Skip-One Gap is 6

A particularly structured case occurs when

$$p_{n+2} - p_n = 6.$$

Because primes are ± 1 modulo 6, the three primes must take one of the two forms:

$$p_n,\ p_n + 2,\ p_n + 6 \qquad \text{or} \qquad p_n,\ p_n + 4,\ p_n + 6.$$

In the first pattern, the pair (p_n, p_{n+1}) is a twin-prime pair, and in the second pattern the pair (p_{n+1}, p_{n+2}) is a twin-prime pair. Thus the middle prime p_{n+1} always belongs to a twin-prime pair whenever the skip-one gap is 6.

4.1 Exact Formula for the PSD Factor in the Gap-6 Case

From the identity,

$$x = \frac{(p_{n+2})^2 - (p_n)^2}{12} = \frac{6\,(p_{n+2} + p_n)}{12} = \frac{p_{n+2} + p_n}{2},$$

we see that x is the exact midpoint of the two outer primes. Since

$$p_{n+1} = p_n + 3 = \frac{p_n + p_{n+2}}{2},$$

we obtain the exact relation

$$x = p_{n+1} \pm 1,$$

where the sign depends on the mod-6 configuration.

4.2 Determining the Twin-Prime Direction

The deviation of the PSD Factor from the middle prime p_{n+1} identifies which twin-prime pair contains p_{n+1}:

- If $x = p_{n+1} + 1$, then p_{n+1} is twinned with p_n.
- If $x = p_{n+1} - 1$, then p_{n+1} is twinned with p_{n+2}.

Thus, when the skip-one gap is 6, the PSD Factor not only reproduces the midpoint structure but also encodes the direction of the twin-prime attachment for the prime p_{n+1}.

5 Conclusion

The Prime Square-Difference Identity

$$(p_{n+2})^2 = (p_n)^2 + 12x$$

gives rise to an integer PSD Factor x with a rigid last-digit pattern $\{0, 4, 6\}$. The factor admits both a modular explanation and a natural analytic approximation $x \approx g\,p_n/6$ tied to the magnitude of the skip-one prime gap. In the highly structured case where the skip-one gap is 6, the PSD Factor recovers (up to ± 1) the middle prime of a twin-prime pair and determines which adjacent prime forms the pair. This combination of algebraic rigidity, modular restriction, and prime-gap structure gives the PSD Factor a rich internal behavior and suggests several directions for further exploration.