The PSD Gap-Bias Identity: A Closed-Form Accuracy Theorem for a Prime-Gap Estimator

Allen Proxmire

29 November 2025

Abstract

In earlier work introducing the *Prime Square-Difference* (PSD) factor,

$$\mathrm{PSDn} = \frac{(p_{n+2})^2 - (p_n)^2}{12},$$

we observed that the quantity PSDn/p_n serves as a natural estimator for the "skip-one" prime gap $G_n = p_{n+2} - p_n$. In this short note we prove an exact closed-form identity describing the bias of this estimator. The result is elementary, unconditional, and valid for every triple of consecutive primes.

1 Definitions

Let $p_n < p_{n+1} < p_{n+2}$ be consecutive primes. Define the *skip-one prime gap*

$$G_n := p_{n+2} - p_n,$$

and the *Prime Square-Difference* (PSD) factor

$$\mathrm{PSDn} := \frac{(p_{n+2})^2 - (p_n)^2}{12}.$$

The PSD-based estimator for the outer gap is

$$\widehat{G}_n := \frac{6\,\mathrm{PSDn}}{p_n}.$$

2 Main Result

[Proximire PSD Gap-Bias Identity] For any triple of consecutive primes p_n, p_{n+1}, p_{n+2}, the PSD-based estimator

$$\widehat{G}_n = \frac{6\,\mathrm{PSDn}}{p_n}$$

satisfies the exact identity

$$\widehat{G}_n = G_n \left(1 + \frac{G_n}{2p_n}\right).$$

In particular,

$$\widehat{G}_n - G_n = \frac{G_n^2}{2p_n} > 0,$$

so the estimator always overestimates the true outer gap by a precise amount.

Proof. Using the difference of squares,

$$(p_{n+2})^2 - (p_n)^2 = (p_{n+2} - p_n)(p_{n+2} + p_n) = G_n(2p_n + G_n).$$

Thus

$$\text{PSDn} = \frac{G_n(2p_n + G_n)}{12}.$$

By definition,

$$\widehat{G}_n = \frac{6}{p_n}\,\text{PSDn} = \frac{6}{p_n} \cdot \frac{G_n(2p_n + G_n)}{12} = G_n \left(\frac{2p_n + G_n}{2p_n}\right) = G_n \left(1 + \frac{G_n}{2p_n}\right).$$

□

3 Corollary: Asymptotic Exactness

Since $G_n/p_n \to 0$ as $n \to \infty$, the relative error satisfies

$$\frac{\widehat{G}_n}{G_n} \to 1, \qquad \widehat{G}_n - G_n \to 0.$$

Proof. Divide the identity from the theorem by G_n:

$$\frac{\widehat{G}_n}{G_n} = 1 + \frac{G_n}{2p_n}.$$

Since $G_n = o(p_n)$, the term $G_n/(2p_n)$ vanishes. □

4 Remarks

- Under standard heuristics such as $G_n \sim 2 \log p_n$, the relative bias becomes

  $$\frac{\widehat{G}_n - G_n}{G_n} = \frac{G_n}{2p_n} \sim \frac{\log p_n}{p_n},$$

  which decays exceptionally quickly.

- The identity is purely algebraic and holds for *every* triple of consecutive primes; no analytic number theory is required.

- This result gives an explicit law for the deviation of a natural prime-gap estimator arising from the "Prime Triangle" geometric framework.

Acknowledgements

This note builds on the geometric and modular framework introduced in *Prime Triangles and the Prime Square-Difference Identity* (Proxmire, 2025).