

The Prime Square-Difference Identity: A Modular Constraint Between Squared Primes

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1 Statement of the Identity

Let p_n and p_{n+2} be primes (not necessarily consecutive). Consider the relation

$$(p_{n+2})^2 = 12x + (p_n)^2. \quad (1)$$

Solving for x gives

$$x = \frac{(p_{n+2})^2 - (p_n)^2}{12}. \quad (2)$$

Empirically, for all tested values, x ends in digit 0, 4, or 6. We prove this below.

2 Algebraic Verification

Using the difference of squares,

$$(p_{n+2})^2 - (p_n)^2 = (p_{n+2} - p_n)(p_{n+2} + p_n), \quad (3)$$

so

$$x = \frac{(p_{n+2} - p_n)(p_{n+2} + p_n)}{12}. \quad (4)$$

Integrality is not immediately obvious from this expression, but follows from modular analysis.

3 Modular Classification of Primes

All primes greater than 3 satisfy

$$p \equiv \pm 1 \pmod{6}. \quad (5)$$

Hence we may write

$$p_n = 6a \pm 1, \quad p_{n+2} = 6b \pm 1, \quad (6)$$

for integers a, b .

There are four sign cases to consider. Expanding yields

$$(6b \pm 1)^2 - (6a \pm 1)^2 = 12(b - a)(3(a + b) \pm 1). \quad (7)$$

Thus

$$x = 3(a + b)(b - a) \quad \text{or} \quad x = 3(a + b)(b - a) \pm (b - a), \quad (8)$$

which is always an integer.¹

4 Last-Digit Classification

To determine the final digit of x , reduce modulo 120 (the least common multiple of 12 and 10).

For each of the four sign cases, one obtains

$$(p_{n+2})^2 - (p_n)^2 \equiv 0, 48, 72 \pmod{120}. \quad (9)$$

Dividing through by 12 yields

$$x \equiv 0, 4, 6 \pmod{10}. \quad (10)$$

Therefore, the last digit of x is always in $\{0, 4, 6\}$.

5 Interpretation and Significance

This identity is not commonly cited in prime literature and does not appear to be a standard result in analytic number theory. Its significance lies in:

- revealing a hidden divisibility-by-12 structure relating prime squares,
- producing a restricted last-digit pattern $\{0, 4, 6\}$,
- offering a new prime-derived integer sequence,
- enabling statistical study of modular fingerprints of primes.

Possible further directions include studying correlations between x and the prime gap $g = p_{n+2} - p_n$, the density of each residue class, or potential links to known conjectures on prime distribution.

6 Conclusion

The identity

$$(p_{n+2})^2 = 12x + (p_n)^2 \quad (11)$$

exhibits a structured modular constraint based on the arithmetic form of primes. The resulting integer x has only three possible last digits, forming a simple but nontrivial signature within the prime sequence.

This suggests additional structure worthy of further study and offers an approachable, verifiable, and novel observation suitable for sequence submission or exploratory research.

¹Because $(p_{n+2})^2 - (p_n)^2$ always contains a factor of 12 under the mod-6 structure.