Local Correlation Structure in Renormalized Prime Gaps

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Abstract

In a companion preprint, we established scale stability and local stationarity of the renormalized prime gap sequence $g_n/\log p_n$. In this paper we investigate whether consecutive renormalized gaps are statistically independent, or whether they exhibit measurable local correlation structure. Using geometric observables constructed from triples of consecutive gaps, we show that prime gap triples occupy a significantly lower-dimensional region of space than randomized null models with identical marginal statistics. This structure persists locally along the prime sequence, decays over a finite correlation length of only a few gaps, and exceeds what can be explained by trivial local dependence alone. Complementary variance-based diagnostics confirm the absence of long-range accumulation in gap fluctuations. Together, these results demonstrate the presence of short-range, scale-stable correlation structure in the prime gap sequence.

1 Introduction

Let p_n denote the nth prime and $g_n = p_{n+1} - p_n$ the corresponding prime gaps. In [1], we showed that the renormalized gaps

$$\tilde{g}_n = \frac{g_n}{\log p_n}$$

exhibit scale-stable marginal statistics and local stationarity across wide ranges of primes.

Scale stability alone, however, does not determine whether consecutive gaps are independent. If the prime gap sequence were generated by an independent random process, then collections of consecutive renormalized gaps would fill space isotropically, apart from trivial constraints.

The purpose of this paper is to test this independence hypothesis empirically. Rather than assuming a dynamical model, we construct simple geometric observables from consecutive renormalized gaps and compare their behavior to randomized null models that preserve marginal statistics while destroying correlations.

2 Data and Null Models

All computations use primes up to 10^7, corresponding to approximately 6.6×10^5 consecutive gaps.

We employ three null models:

(i) **Permutation null:** the sequence $\{g_n\}$ is randomly permuted, preserving the marginal gap distribution while destroying all ordering.

(ii) **Block-permutation null:** gaps are permuted in contiguous blocks of fixed size, preserving short-range dependence while destroying longer-range structure.

(iii) **Lagged constructions:** renormalized gaps separated by a fixed lag k are grouped to probe correlation length.

3 Geometric Representation of Gap Triples

To probe local structure, we consider triples of consecutive renormalized gaps,

$$\mathbf{v}_n = (\tilde{g}_n,\ \tilde{g}_{n+1},\ \tilde{g}_{n+2}).$$

If the $\tilde{g}_n$ were independent samples from a fixed distribution, the resulting cloud of points would be approximately isotropic in $\mathbb{R}^3$. Deviations from isotropy indicate correlation or constraint.

We quantify this geometry using principal component analysis (PCA). Let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ denote the eigenvalues of the covariance matrix of the triple cloud. We summarize dimensionality by the thickness ratio

$$R = \frac{\lambda_3}{\lambda_1}.$$

4 Global PCA of Renormalized Gap Triples

Figure 1 shows the PCA eigenvalues of renormalized gap triples for primes and for the permutation null.

For primes we find $R \approx 0.87$, compared to $R \approx 0.97$ for the null, demonstrating substantial geometric compression.

5 Local Persistence of Structure

To test whether this effect is global or local, we compute the thickness ratio in sliding windows along the prime sequence.

As shown in Figure 2, the geometric constraint persists across the entire range of primes examined.

6 Lag Dependence and Correlation Length

We generalize the triple construction to include a lag k,

$$(\tilde{g}_n,\ \tilde{g}_{n+k},\ \tilde{g}_{n+2k}),$$

and compute the thickness ratio $R(k)$.

Figure 3 shows that the structure decays rapidly and becomes indistinguishable from null behavior by approximately $k \approx 5$.

7 Block-Permutation Robustness

Figure 4 compares lagged PCA thickness for primes and a block-permutation null, showing that nearest-neighbor structure exceeds trivial local dependence.

Figure 5 shows that increasing block size gradually eliminates differences between primes and nulls, indicating a fully local structure.

8 Complementary Variance-Based Diagnostics

Figures 6 and 7 examine first differences of gaps and confirm the absence of long-range accumulation, consistent with the PCA results.

9 Discussion

The results demonstrate that consecutive renormalized prime gaps are not statistically independent. Instead, they obey short-range correlation constraints that manifest as geometric compression in gap triples. These constraints are local, decay rapidly, and do not produce long-range accumulation.

Importantly, no dynamical law is assumed or required. The results place strong empirical constraints on any prospective explanation of prime gap behavior.

10 Conclusion

We have shown that the renormalized prime gap sequence exhibits short-range, scale-stable correlation structure absent in randomized models with identical marginal statistics. The structure is local, finite in extent, and robust under multiple null comparisons. These findings complement the baseline statistical results of [1] and provide a firm empirical foundation for further theoretical investigation.

References

[1] A. Proxmire, *Scale–Stable Statistics of Renormalized Prime Gaps*, companion preprint, 2025.

Figures

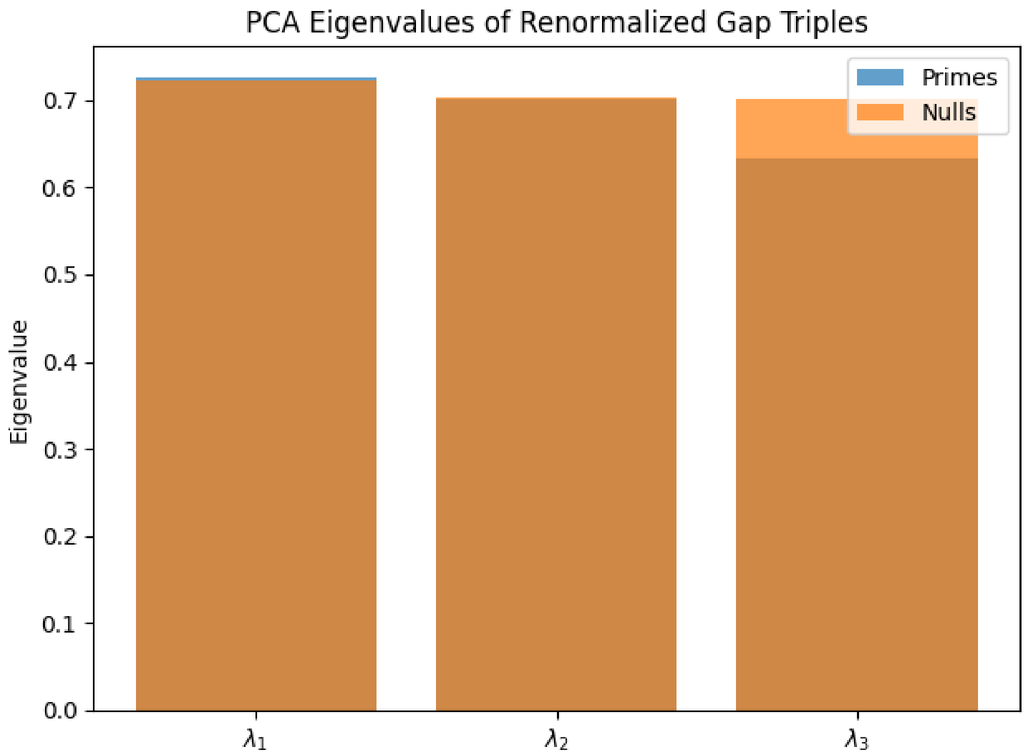

Figure 1: PCA eigenvalues of renormalized gap triples for primes and a permutation null. Prime triples occupy a significantly thinner region of space, as indicated by a reduced λ_3/λ_1 ratio.

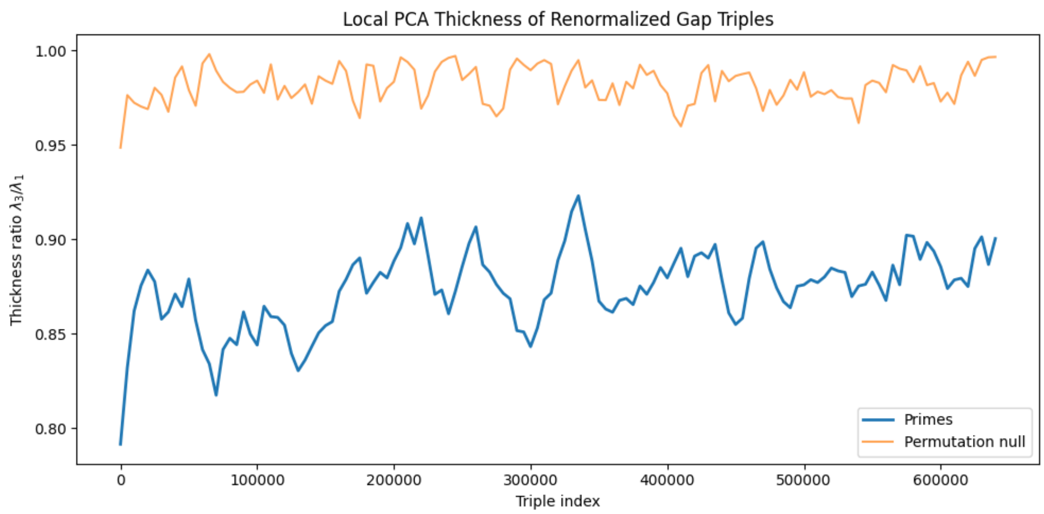

Figure 2: Local PCA thickness ratio computed in sliding windows along the prime sequence. The reduced thickness persists throughout, indicating local rather than global structure.

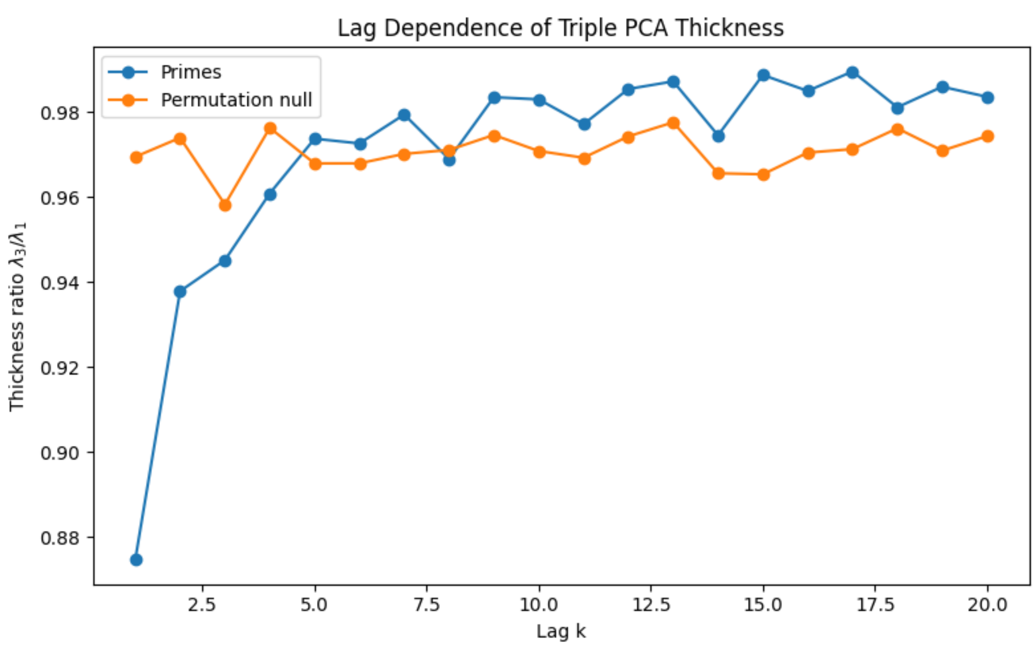

Figure 3: Thickness ratio $R(k)$ as a function of lag k. The structure decays rapidly and becomes indistinguishable from null behavior by approximately $k \approx 5$.

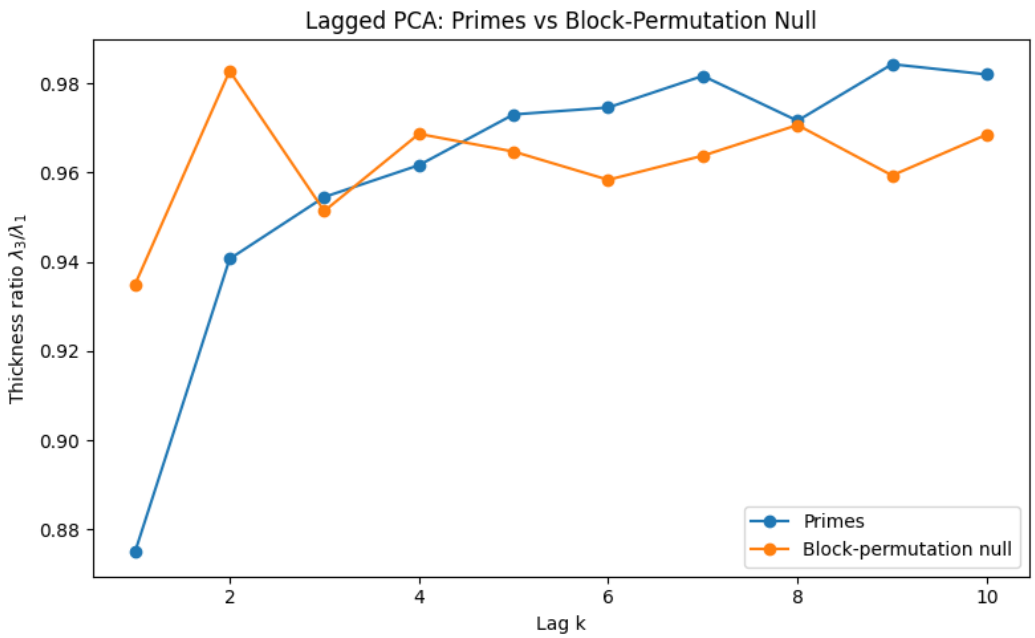

Figure 4: Lagged PCA thickness for primes and a block-permutation null. At nearest neighbor lag ($k = 1$), prime structure exceeds what is explained by trivial local dependence.

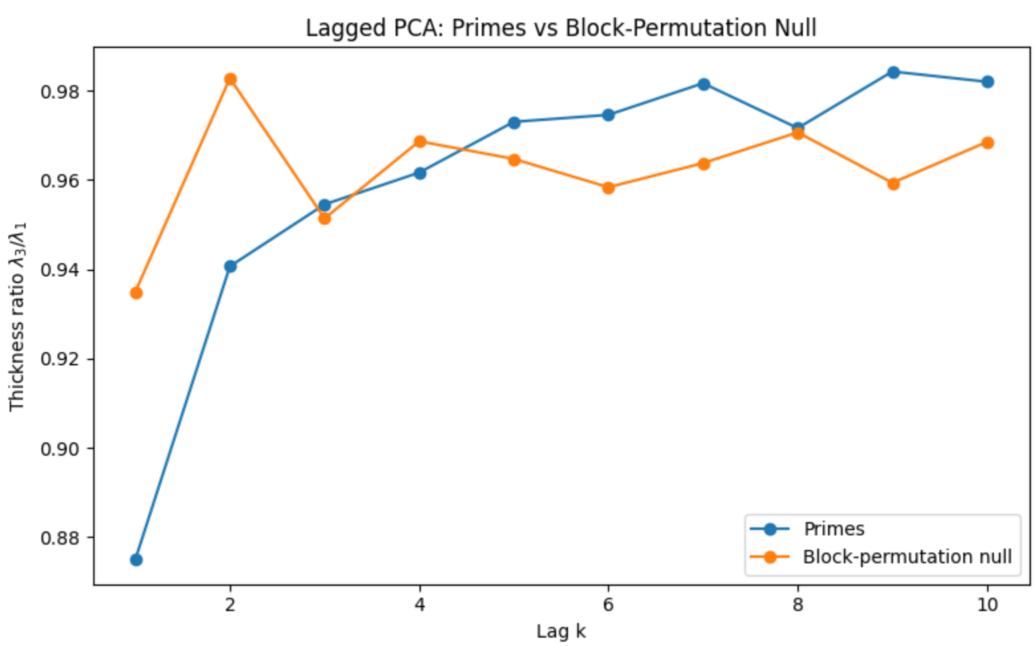

Figure 5: Block-size sweep of lagged PCA thickness. Increasing block size gradually eliminates differences between primes and nulls, indicating a fully local structure.

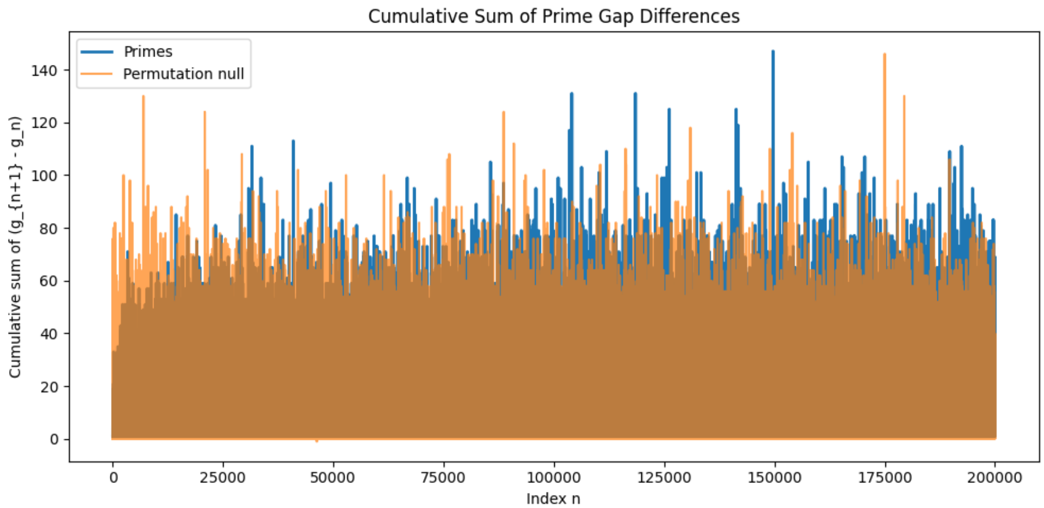

Figure 6: Cumulative sums of gap differences for primes and a permutation null. Both exhibit rapid cancellation, ruling out long-range accumulation.

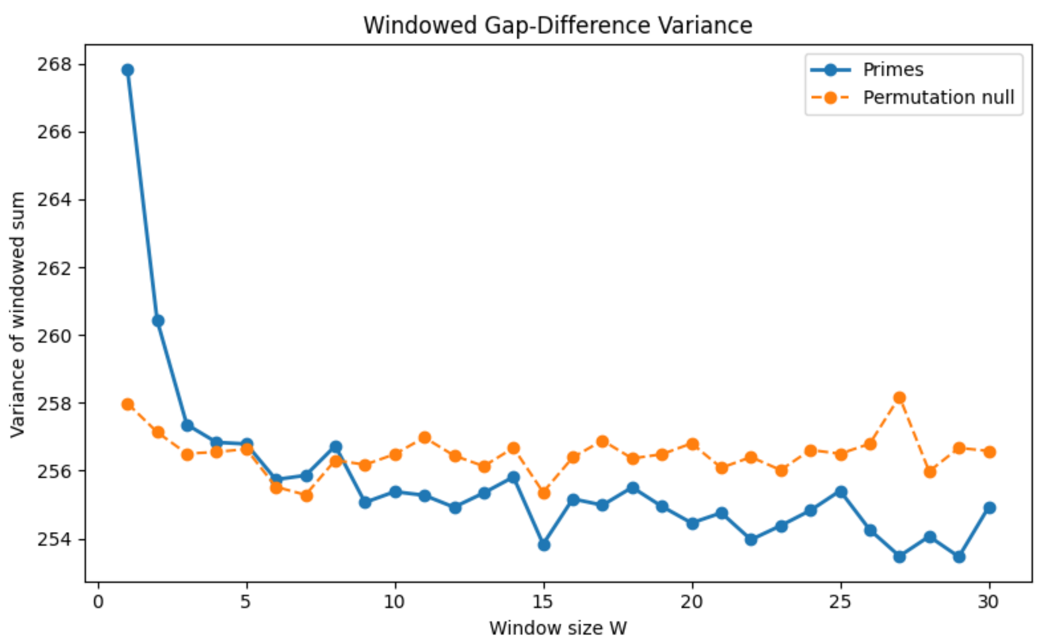

Figure 7: Variance of windowed sums of gap differences as a function of window size. Variance saturates rapidly, consistent with short-range regulation.