数学物理方法（上）第一次作业

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题 1. 计算 $(1+i)^n+(1-i)^n$，其中 $n\in\mathbb{Z}$。

解.

$$
\begin{aligned}
I &= (1+i)^n+(1-i)^n \\
&= 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)+2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right) \\
&= 2^{\frac{n}{2}}\left(2\cos\frac{n\pi}{4}\right) \\
&= 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}
\end{aligned}
$$

□

题 2. 画出 $\mathrm{Re}\left(\frac{z-z_1}{z-z_2}\right)=0$ 所描述的图形。

解.

$$\mathrm{Re}\left((z-z_1)(z-z_2)^*\right)=0$$

$$\vec{r_1}=z-z_1 \qquad \vec{r_2}=z-z_2$$

$$\vec{r_1}\cdot\vec{r_2}=0$$

故描述的是以 z_1z_2 为直径的圆，不包括 z_2 点。

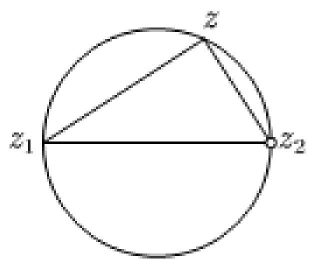

图 1: 第二题区域

□

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题 3. 将下列和式表示成有限形式：

$$\cos\phi + \cos 2\phi + \cos 3\phi + \cdots + \cos n\phi.$$

解.

$$\begin{aligned}
\sum_{k=1}^{n} \cos k\phi &= \frac{1}{2}\sum_{k=1}^{n}\left(e^{ik\phi}+e^{-ik\phi}\right) \\
&= \frac{1}{2}\sum_{k=1}^{n} e^{ik\phi} + \frac{1}{2}\sum_{k=1}^{n} e^{-ik\phi} \\
&= \frac{1}{2}e^{i\phi}\frac{1-e^{in\phi}}{1-e^{i\phi}} + \frac{1}{2}e^{-i\phi}\frac{1-e^{-in\phi}}{1-e^{-i\phi}} \\
&= \frac{1}{2}e^{i\frac{\phi}{2}}\frac{1-e^{in\phi}}{e^{-i\frac{\phi}{2}}-e^{i\frac{\phi}{2}}} + \frac{1}{2}e^{-i\frac{\phi}{2}}\frac{1-e^{-in\phi}}{e^{i\frac{\phi}{2}}-e^{-i\frac{\phi}{2}}} \\
&= \frac{e^{-i\frac{\phi}{2}} - e^{-i\left(n+\frac{1}{2}\right)\phi} - e^{-i\frac{\phi}{2}} + e^{i\left(n+\frac{1}{2}\right)\phi}}{4i\sin\frac{\phi}{2}} \\
&= \frac{sin\left[\left(n+\frac{1}{2}\right)\phi\right] - \sin\frac{\phi}{2}}{2\sin\frac{\phi}{2}} \\
&= \cos\left(\frac{n+1}{2}\phi\right)\frac{\sin\frac{n}{2}\phi}{\sin\frac{\phi}{2}}
\end{aligned}$$

□

题 4. 求出极坐标下的柯西-黎曼方程。

解. 直角坐标系中，Cauchy-Riemann 方程为：

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

在极坐标下，$z = x + iy = re^{i\theta}$，$r = \sqrt{x^2+y^2}$,$\theta = \arctan\frac{y}{x}$。则

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \frac{x}{r}\frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \frac{y}{r}\frac{\partial}{\partial r} + \frac{x}{r^2}\frac{\partial}{\partial \theta}$$

代入 Cauchy-Riemann 方程，得到：

$$\frac{x}{r}\frac{\partial u}{\partial r} - \frac{y}{r^2}\frac{\partial u}{\partial \theta} = \frac{y}{r}\frac{\partial v}{\partial r} + \frac{x}{r^2}\frac{\partial v}{\partial \theta}$$

$$\frac{y}{r}\frac{\partial u}{\partial r} + \frac{x}{r^2}\frac{\partial u}{\partial \theta} = -\frac{x}{r}\frac{\partial v}{\partial r} + \frac{y}{r^2}\frac{\partial v}{\partial \theta}$$

为满足上述方程，有

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -\frac{1}{r}\frac{\partial v}{\partial r}$$

此即为极坐标下的 Cauchy-Riemann 方程。 □

题 5. 证明复变函数的导数满足链式法则：

$$\Big(f[g(z)]\Big)' = f'[g(z)]g'(z)$$

解.

$$\begin{aligned}\Big(f[g(z)]\Big)' &= \lim_{\Delta z\to 0}\frac{f[g(z+\Delta z)]-f[g(z)]}{\Delta z}\\ &= \lim_{\Delta z\to 0}\frac{f[g(z+\Delta z)]-f[g(z)]}{g(z+\Delta z)-g(z)}\frac{g(z+\Delta z)-g(z)}{\Delta z}\\ &= f'[g(z)]g'(z)\end{aligned}$$

若 $g'(z)=0$，则

$$\Big(f[g(z)]\Big)' = 0$$
$$f[g(z)]' = 0$$

仍有

$$\Big(f[g(z)]\Big)' = f'[g(z)]g'(z)$$

□

题 6. 已知解析函数 $f(z)$ 的实部

$$u=\frac{2\sin(2x)}{e^{2y}+e^{-2y}-2\cos(2x)},$$

且 $f(\pi/2)=0$。求 $f(z)$ 。

解. 做变量替换，$x=\frac{z+z^*}{2}$,$y=\frac{z-z^*}{2i}$，则

$$\begin{aligned}u(x,y) &= u(z,z^*)\\ &= \frac{2\sin(z+z^*)}{\cos(z-z^*)-cos(z+z^*)}\\ &= \frac{\sin z\cos z^*+\cos z\sin z^*}{2\sin z\sin z^*}\\ &= \frac{1}{2}\cot z+\frac{1}{2}\cot z^*\end{aligned}$$

因为 $f=u+iv$ 是解析函数，故 $f=f(z)$ 且 $u=u(x,y)$ 为实函数，故

$$v=-\frac{i}{2}\cot z+\frac{i}{2}\cot z^*+const$$

由 $f(\frac{\pi}{2})=0$，得到

$$f(\frac{\pi}{2})=0\Rightarrow const=0$$
$$f(z)=\cot z$$

□