二阶线性 ODE (2)

$$y'' + P(x)y' + Q(x)y = 0$$

如 P、Q 在 x_0 解析，则称 x_0 为常点、

存在级数解（泰勒）

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

我们看到通常级数解依赖于两个未知变量：两个独立解 y_1，y_2，

则任意解

$$y = c_1 y_1 + c_2 y_2$$

设有边界条件：

$y(0) = Y_0$ $y'(0) = Y_1$，则有方程

$$\begin{cases} c_1 y_1(0) + c_2 y_2(0) = Y_0 \\ c_1 y_1'(0) + c_2 y_2'(0) = Y_1 \end{cases} \quad ①$$

何时有解？

定义：朗斯其行列式

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

则方程①有解 $\Leftrightarrow$ $W(0) \neq 0$

如果 $W(x) \neq 0$，则称两个解是"线性无关"的。

例：$y'' - 3y' + 2y = 0$

$$y_1 = e^x, \quad y_2 = e^{2x}$$

$$W(x) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x} \neq 0$$

- 类比：在线性代数中，两个 2D 向量 $\vec{v}, \vec{u}$ 线性无关的条件：

$$W = \begin{vmatrix} \vec{v} & \vec{u} \end{vmatrix} = \begin{vmatrix} V_1 & u_1 \\ V_2 & u_2 \end{vmatrix} = V_1u_2 - u_1V_2 \neq 0$$

定理：如果 $W(x_0)\neq 0$，则 $W(x)$ 处处不为 0.

证明： $W(x) = y_1 y_2' - y_2 y_1'$

$$\frac{dW}{dx} = y_1 y_2'' - y_2 y_1''$$

而 $$\begin{cases} y_1'' + P y_1' + Q y_1 = 0 \\ y_2'' + P y_2' + Q y_2 = 0 \end{cases}$$

$$\Rightarrow \quad y_1 y_2'' + P y_1 y_2' + \cancel{Q y_1 y_2} = y_2 y_1'' + P y_2 y_1' + \cancel{Q y_2 y_1}$$

$$\Rightarrow \quad y_1 y_2'' - y_2 y_1'' = P(y_2 y_1' - y_1 y_2')$$

$$= -PW$$

$$\therefore \quad \frac{dW}{W} = -P\,dx$$

式 $$\ln W = -\int^x P(x')\,dx' + C_1$$

$$W = C_1 \exp\left[-\int^x P(x')\,dx'\right]$$

已知一个解，Wronskian 可用来求解另一解：

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{y_2'}{y_1} - \frac{y_2}{y_1^2} y_1'$$

$$\therefore \quad y_1^2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right) = W$$

$$\therefore \quad \frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{W}{y_1^2}$$

$$\Rightarrow \quad \boxed{y_2 = y_1 \exp\left[\int^x \frac{W(x')}{y_1^2(x')} dx'\right]}$$

正则奇点.

$$P(x) = \frac{p(x)}{x}, \quad p(x) = p_0 + p_1 x + \cdots$$

$$Q(x) = \frac{q(x)}{x^2}, \quad q(x) = q_0 + q_1 x + \cdots$$

Indicial Equation:

$$r(r-1) + p_0 r + q_0 = 0 \qquad \text{roots}: r_1, r_2$$

(1) $r_1 \neq r_2$, $r_1 - r_2 \notin \mathbb{Z}$.

Solutions given by two Frobenius series:

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad , \quad a_0 \neq 0$$

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n$$

(2) $r_1 = r_2$,

only one Frobenius Solution

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \qquad a_0 \neq 0$$

$$y_2 = y_1 \ln x + x^{r_1} \sum_{n=0}^{\infty} b_n x^n$$

(3) $r_1 \neq r_2 \quad r_1 - r_2 = \mathbb{N}_+$

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \qquad a_0 \neq 0$$

$$y_2 = c y_1 \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n , \quad b_0 \neq 0$$

Q: What's special about regular singularities?

How to understand the different cases?

A: Consider the extreme case,

$p_1 = p_2 = \cdots = 0 , \quad q_1 = q_2 = \cdots = 0$

$$\Rightarrow y'' + \frac{p_0}{x} y' + \frac{q_0}{x^2} y = 0$$

or: $x^2 y'' + p_0 x y' + q_0 y = 0$

let $z = \ln x$, then $x = e^z$

$$y' = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \frac{1}{x}$$

$$y'' = \frac{d}{dz}\left(\frac{dy}{dz} \frac{1}{x}\right) \frac{1}{x} = \frac{d^2y}{dz^2} \frac{1}{x^2} + \frac{dy}{dz} \frac{d}{dz}\left(\frac{1}{x}\right) \frac{1}{x}$$

$$= \frac{d^2y}{dz^2} \frac{1}{x^2} + \frac{dy}{dz}\left(-\frac{1}{x^3}\right) \cdot x$$

$\therefore$ our equation become (Cauchy-Euler Eq)

$$\frac{d^2y}{dz^2} + (p_0 - 1) \frac{dy}{dz} + q_0 y = 0$$

Let's make an ansatz:

$y = e^{rz} = x^r$, substitute into C-E Eq,

$r^2 + (p_0 - 1) r + q_0 = 0$ [Indicial Eq. !]

If two different root r_1, r_2,

two independent solutions

$$y_1 = x^{r_1}, \qquad y_2 = x^{r_2}$$

If $r_1 = r_2$, two solution degenerate.

- Connection with Linear Algebra, again.

* Converting 2nd ODE into system of 1st order ODE:

let $Y_1 = y$

$$Y_2 = \frac{dy}{dz}$$

then: $\vec{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$

$$\frac{d}{dz}\vec{Y} = \underbrace{\begin{pmatrix} 0 & 1 \\ -q_0 & 1-p_0 \end{pmatrix}}_{A}\vec{Y}$$

All information about the original equations are contained in A.

Character equation:

$$0 = |A - rI| = \begin{vmatrix} -r & 1 \\ -q_0 & 1-p_0-r \end{vmatrix}$$

Indicial Eq: $\quad 0 = r^2 - r + p_0 r + q_0$

If we can diagonalize the matrix equation, then solution is obvious. When A has 2 different real eigenvalue r_1, r_2, then $\exists\, T$,

$$T^{-1} A T = \begin{pmatrix} r_1 & \\ & r_2 \end{pmatrix}$$

$$\frac{d}{dz} T^{-1} \vec{Y} = T^{-1} A T T^{-1} \vec{Y}$$

$$\frac{d}{dz} \tilde{Y}' = \begin{pmatrix} r_1 & \\ & r_2 \end{pmatrix} \tilde{Y}'$$

Solution: $\tilde{Y}' = \begin{pmatrix} e^{r_1 z} & \\ & e^{r_2 z} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$= \begin{pmatrix} x^{r_1} & \\ & x^{r_2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

T is found by solving the eigenvector equation

$$(A - rI)\vec{x} = 0$$

or $\quad A\vec{x} = r\vec{x} \quad$ for $r = r_1$ or r_2

$$T = (\vec{X}_1 \ \vec{X}_2)$$

However, when $r_1 = r_2$

$(A - rI)\vec{x} = 0$ only give one eigenvector $\vec{x}$,

generalize eigenvector $\vec{x}'$

$$(A - rI)^2 \vec{x}' = 0 \Rightarrow (A - rI)\vec{x}' = \vec{x}$$

In the basis of $\vec{x}, \vec{x}'$,

$$A\vec{x} = r\vec{x}$$

$$A\vec{x}' = r\vec{x}' + \vec{x}$$

$T = (\vec{x} \ \ \vec{x}')$ put A in Jordan form

$$T^{-1}AT = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix}$$

$$\Rightarrow \quad \frac{d}{dz} T^{-1}\vec{Y} = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix} T^{-1}\vec{Y}$$

Solution: $\vec{Y}' = \begin{pmatrix} e^{rz} & z e^{rz} \\ 0 & e^{rz} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$

$$= \begin{pmatrix} e^{rz} C_1 + C_2 z e^{rz} \\ C_2 e^{rz} \end{pmatrix}$$

$$= \begin{pmatrix} x^r C_1 + C_2 \ln x \, x^r \\ C_2 x^r \end{pmatrix}$$

We have therefore seen that $\log x$ comes from degeneracy.

例 : $4x^2y'' - 4x^2y' + (1-2x)y = 0$

$$P(x) = -1 , \quad Q(x) = \frac{1-2x}{4x^2}$$

$$p(x) = xP(x) = 0 - x$$

$$q(x) = x^2Q(x) = \underset{\substack{\Uparrow \\ q_0}}{\frac{1}{4}} - \frac{1}{2}x$$

indicial Eq : $r^2 - r + q_0 = r^2 - r + \frac{1}{4} = 0$

or $(r-\frac{1}{2})^2 = 0$ $\quad r = \frac{1}{2}$ is double root

① Frobenius solution

$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r}$ substitute into the ODE.

$$0 = 4x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$
$$- 4x^2 \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$
$$+ (1-2x) \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= \sum_{n=0}^{\infty} 4(n+r)(n+r-1)\, a_n x^{n+r}$$

$$- \sum_{n=0}^{\infty} 4(n+r)\, a_n x^{n+r+1}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} 2 a_n x^{n+r+1}$$

$$= x^r \cdot [4r(r-1)+1]\, a_0$$

$$+ \sum_{n=1}^{\infty} [4(n+r)(n+r-1)\, a_n - 4(n+r-1)\, a_{n-1}$$
$$+ a_n - 2a_{n-1}]\, x^{n+r}$$

$\Rightarrow \quad 4r\cdot(r-1)+1=0 \qquad \text{if } a_0 \neq 0$

$\quad r = \frac{1}{2}$

$\Rightarrow \quad [4\cdot(n+r)(n+r-1)+1]\, a_n = [4(n+r-1)+2]\, a_{n-1}$

let $r = \frac{1}{2}$

$$\Rightarrow \quad a_n = \frac{(4\cdot(n-\frac{1}{2})+2)\, a_{n-1}}{4\cdot(n+\frac{1}{2})(n-\frac{1}{2})+1}$$

$$= \frac{4n}{4\cdot(n^2-\frac{1}{4})+1}\, a_{n-1} = \frac{4n}{4n^2}\, a_{n-1} = \frac{1}{n}\, a_{n-1}$$

$$= \frac{1}{n!} a_0$$

$$\therefore \quad y = x^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{x^n}{n!} a_0 = a_0 x^{\frac{1}{2}} e^x$$

② Second solution :

$$Y_2 = \sum_{n=0}^{\infty} b_n x^{n+r} + \ln x \cdot y_1$$

$$= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}} + \ln x \ x^{\frac{1}{2}} e^x$$

substitute into the ODE

$$\Rightarrow \quad b_1 = b_0 - 1, \quad b_2 = \frac{b_1}{2} - \frac{1}{4}, \quad b_3 = \frac{1}{3} b_2 - \frac{1}{18}, \ \dots$$

$$b_1 \cdot (2x - 1) + b_0 - 1 - x = 0$$

例: Bessel Equation :

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

ansatz : $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r}$

substitute :

$$0 = x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$+ x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$+ x^2 \sum_{n=0}^{\infty} a_n x^{n+r} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r}$$

$$+ \sum_{n=0}^{\infty} (n+r) a_n x^{n+r}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r}$$

x^r term:

$$r \cdot (r-1) a_0 + r a_0 - \nu^2 a_0 = 0$$

or $a_0 \cdot (r^2 - \nu^2) = 0$

$r = \pm \nu$. assume $\nu \notin \mathbb{Z}$

Then we will have two Frobenius solution.

x^1 term:

$$(1+r) \cdot r \cdot a_1 + (1+r) a_1 - \nu^2 a_1 = 0$$

substitute $r = \nu$

$$\left[(1+\nu)\nu + (1+\nu) - \nu^2\right] a_1 = 0$$

$\Rightarrow \quad a_1 = 0 \quad$ unless $\quad \nu = -\frac{1}{2}$

Now we let $r = \nu$

$$\sum_{n=2}^{\infty} (n+\nu)(n+\nu-1) a_n x^{n+\nu}$$

$$+ \sum_{n=2}^{\infty} (n+\nu) a_n x^{n+\nu}$$

$$+ \sum_{n=2}^{\infty} a_{n-2} x^{n+\nu} - \nu^2 \sum_{n=2}^{\infty} a_n x^{n+\nu} = 0$$

$$\Rightarrow \left[(n+\nu)(n+\nu-1) + (n+\nu) - \nu^2\right] a_n$$

$$+ a_{n-2} = 0$$

$$\Rightarrow a_n = -\frac{a_{n-2}}{(n+\nu)^2 - \nu^2}$$

$$= -\frac{a_{n-2}}{n^2 + 2n\nu} = -\frac{a_{n-2}}{n \cdot (n+2\nu)}$$

Since we only need even ν.

$$a_{2n} = \frac{-a_{2n-2}}{2n\cdot(2n+2\nu)} = -\frac{a_{2n-2}}{2^2 n\cdot(n+\nu)}$$

$$a_2 = -\frac{a_0}{2^2\cdot(1+\nu)} = -\frac{a_0}{2^2}\frac{\Gamma(1+\nu)}{\Gamma(2+\nu)}$$

$$a_4 = -\frac{a_2}{2^3\cdot(2+\nu)} = \frac{a_0}{2^4\cdot 2}\cdot\frac{\Gamma(1+\nu)}{\Gamma(2+\nu)}\frac{\Gamma(2+\nu)}{\Gamma(3+\nu)}$$

$$= \frac{a_0}{2^4\cdot 2}\cdot\frac{\Gamma(1+\nu)}{\Gamma(3+\nu)}$$

$$a_6 = -\frac{a_4}{2^2\cdot 3\cdot(3+\nu)} = -\frac{a_4}{2^2\cdot 3}\cdot\frac{\Gamma(3+\nu)}{\Gamma(4+\nu)}$$

$$= -\frac{a_0}{2^6\underbrace{3!}_{\Gamma(3+1)}}\frac{\Gamma(1+\nu)}{\Gamma(4+\nu)} \quad \cdots$$

$$a_{2n} = (-1)^n\cdot\frac{\Gamma(1+\nu)}{\Gamma(n+1)\Gamma(n+1+\nu)}\cdot\frac{1}{2^{2n}}\cdot a_0$$

If we let $a_0 = \frac{1}{2^p \Gamma(1+\nu)}$

then we get the Bessel func of first kind:

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+\nu)} \left(\frac{x}{2}\right)^{2n+\nu}$$