回顾.

(五) 最速下降法

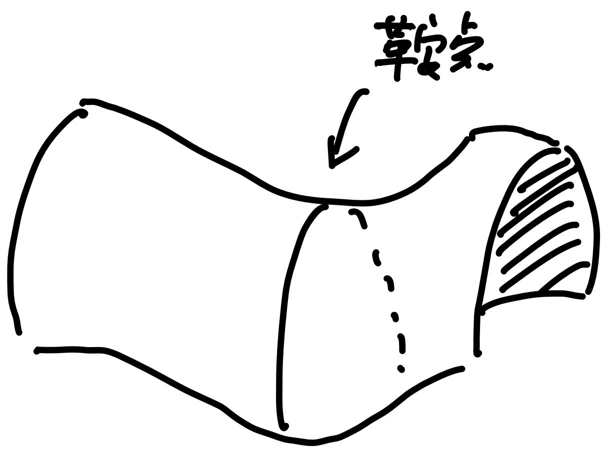
$$I(x)=\int_c g(z)\,e^{xf(z)}\,dz \qquad x\to\infty$$

主要贡献来自鞍点

$$f'(z)=0 \qquad f=u+iv$$

$$\Downarrow$$

$$u_x=0=u_y \qquad \nabla^2 u=0 \Leftarrow \text{调和函数}$$



在鞍点附近.

$$f(z)=f(z_0)+\frac{1}{2}f''(z_0)(z-z_0)^2+O((z-z_0)^3)$$

$$f''(z_0)=\rho e^{i\theta} \qquad z-z_0=se^{i\phi}$$

$$f(z)=f(z_0)+\frac{1}{2}\rho e^{i\theta}s^2e^{2i\phi}+\cdots$$

$$= f(z_0) + \frac{1}{2}\rho s^2 \left(\underset{-1}{\underset{\Uparrow}{\cos(\theta+2\phi)}} + i \underset{0}{\underset{\Uparrow}{\sin(\theta+2\phi)}}\right) + \cdots$$

$$\Rightarrow \quad \theta + 2\phi = \pm\pi, \quad \Rightarrow \quad \phi = \pm\frac{\pi}{2} - \frac{\theta}{2}$$

例：Airy 函数

$$\frac{d^2y}{dx^2} + x y(x) = 0$$

$$\tilde{y}(k) = \int_{-\infty}^{+\infty} dx \, e^{-ikx} y(x)$$

$$(ik)^2 \tilde{y}(k) + i \frac{d\tilde{y}}{dk} = 0$$

$$i \frac{d\tilde{y}}{\tilde{y}} = k^2 dk \quad \Rightarrow \quad i \ln \tilde{y} = \frac{1}{3} k^3 + C_1 .$$

$$\Rightarrow \quad \tilde{y} = \exp\left[-i \frac{1}{3} k^3\right] C_1$$

$$\Rightarrow \quad Ai(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \exp\left[-i \frac{1}{3} k^3\right] e^{ikx}$$

$$= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \exp\left[i\left(kx - \frac{1}{3} k^3\right)\right]$$

令 $k = \sqrt{x}\,\tilde{k}$，

$$Ai(x) = \int_{-\infty}^{+\infty} \sqrt{x}\,\frac{d\tilde{k}}{2\pi} \exp\left[i x^{3/2} \underbrace{\left(\tilde{k} - \frac{1}{3}\tilde{k}^3\right)}_{f(\tilde{k})} \right]$$

$$f'(\tilde{k}) = 0 \Rightarrow i(1 - \tilde{k}^2) = 0$$

$$\Rightarrow \tilde{k}_0 = \pm 1 \quad \Leftarrow \text{鞍点}$$

$$f''(k_0) = -2i\tilde{k}_0 = \pm 2i$$

① $\tilde{k} = 1 + se^{i\phi}$

$$f(\tilde{k}) \sim f(1) + \frac{1}{2}(-2i)s^2 e^{2i\phi}$$

$$= \frac{2}{3}i + s^2 e^{2i\phi - \frac{i}{2}\pi}$$

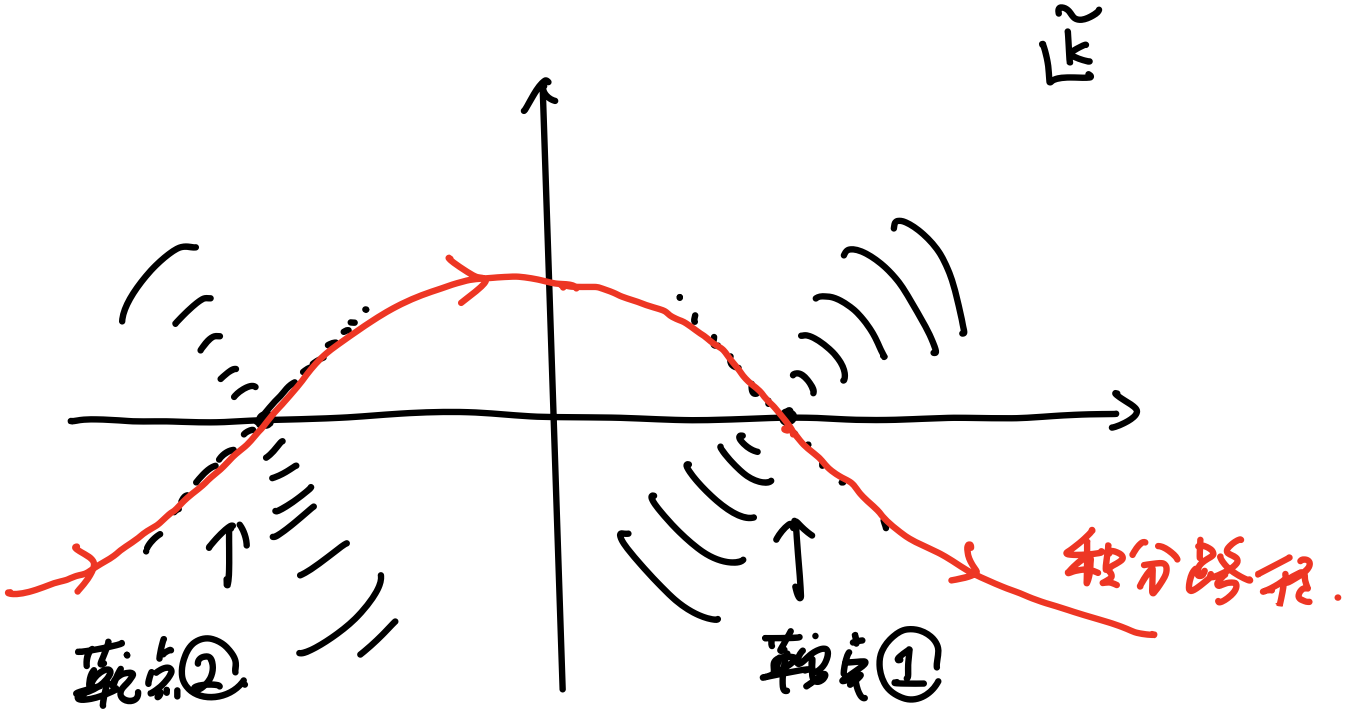
$$2\phi - \frac{\pi}{2} = \pm\pi, \Rightarrow \phi = \frac{3}{4}\pi \text{ 或 } \phi = -\frac{\pi}{4}$$

② $\tilde{k} = -1 + se^{i\phi}$

$$f(\tilde{k}) \sim f(-1) + \frac{1}{2}(2i)s^2 e^{2i\phi}$$

$$= -\frac{2}{3}i + s^2 e^{2i\phi + \frac{i}{2}\pi}$$

$$2\phi + \frac{\pi}{2} = \pm\pi, \quad \phi = \frac{\pi}{4} \text{ 或 } \phi = -\frac{3}{4}\pi$$



① : $\tilde{k} - 1 = s e^{-i\frac{\pi}{4}}$

$$\mathrm{Ai}(x)\Big|_{①} \sim \frac{\sqrt{x}}{2\pi}\int_{-\infty}^{+\infty} ds\, e^{\frac{2}{3}i x^{3/2}} e^{-i\frac{\pi}{4}} e^{-x^{3/2}s^2}$$

$$= \frac{\sqrt{x}}{2\pi} e^{\frac{2}{3}i x^{3/2} - \frac{i}{4}\pi} \sqrt{\frac{\pi}{x^{3/2}}}$$

$$\mathrm{Ai}(x)\Big|_{②} \sim \frac{\sqrt{x}}{2\pi}\int_{-\infty}^{+\infty} ds\, e^{-\frac{2}{3}i x^{3/2}} e^{\frac{i}{4}\pi} e^{-x^{3/2}s^2}$$

$$= \frac{\sqrt{x}}{2\pi} e^{-\frac{2}{3}i x^{3/2}} e^{\frac{i}{4}\pi} \sqrt{\frac{\pi}{x^{3/2}}}$$

$$\mathrm{Ai}(x) \underset{x\to+\infty}{\sim} \sqrt{\frac{1}{\pi}}\, x^{-\frac{1}{4}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right)$$

$x \to -\infty$

令$_1$ $x = -|x| \qquad |x| \to \infty$

$$Ai(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{i\left(kx - \frac{k^3}{3}\right)}$$

$$= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{-i\left(k|x| + \frac{k^3}{3}\right)}$$

令$_2$ $k = \sqrt{|x|}\,\tilde{k}$

$$Ai(x) = \int_{-\infty}^{+\infty} \sqrt{|x|}\,\frac{d\tilde{k}}{2\pi} e^{-i|x|^{3/2}\left(\tilde{k} + \frac{\tilde{k}^3}{3}\right)}$$

鞍点： $1 + \tilde{k}^2 = 0 \Rightarrow \tilde{k} = \pm i$

① $\tilde{k} = +i + s e^{i\phi} \qquad f''(\tilde{k}) = -2i\tilde{k}$

$$f(\tilde{k}) = -i \cdot \left(\frac{2}{3} i\right) + \frac{1}{2} \cdot 2 \cdot s^2 e^{2i\phi}$$

$$= \frac{2}{3} + s^2 e^{2i\phi}$$

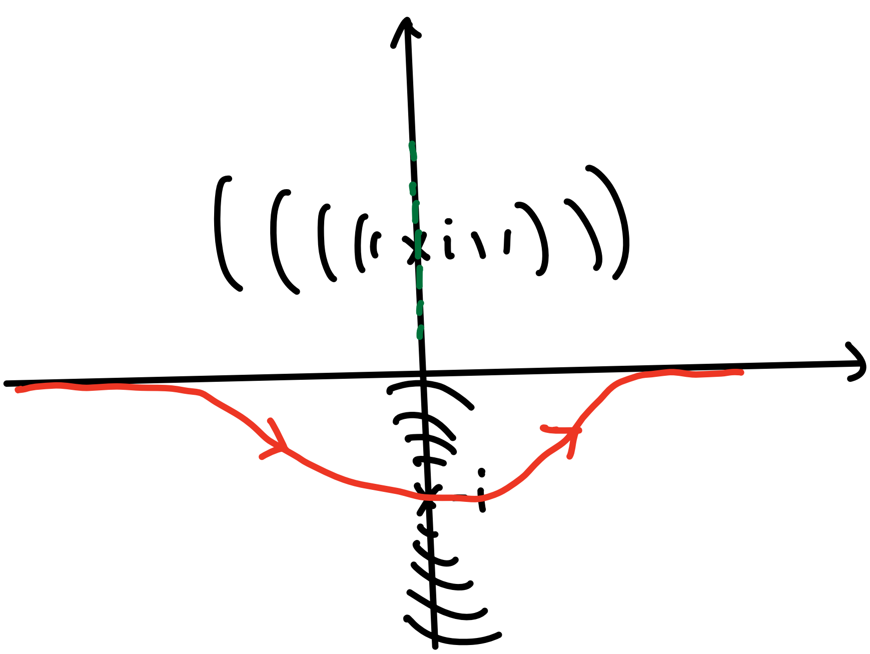
$\Rightarrow 2\phi = \pm\pi, \qquad \Rightarrow \phi = \pm\frac{\pi}{2}$

② $\tilde{k} = -i + s e^{i\phi}$

$$f(\tilde{k}) = -i \cdot \left(-\frac{2}{3} i\right) + \frac{1}{2} \cdot (-2)\, s^2 e^{2i\phi}$$

$$= -\frac{2}{3} + s^2 e^{2i\phi + i\pi}$$

$\Rightarrow \quad \phi + \pi = \pm \pi, \quad \phi = 0$ 或 $\phi = -\pi$



$$Ai(x) \underset{x \to -\infty}{\sim} \frac{\sqrt{|x|}}{2\pi} \int_{-\infty}^{+\infty} d\tilde{k}\, e^{-\frac{2}{3}|x|^{3/2} - |x|^{3/2} s^2}$$

$$= \frac{\sqrt{|x|}}{2\pi} e^{-\frac{2}{3}|x|^{3/2}} \sqrt{\frac{\pi}{|x|^{3/2}}}$$

Schordinger Equation

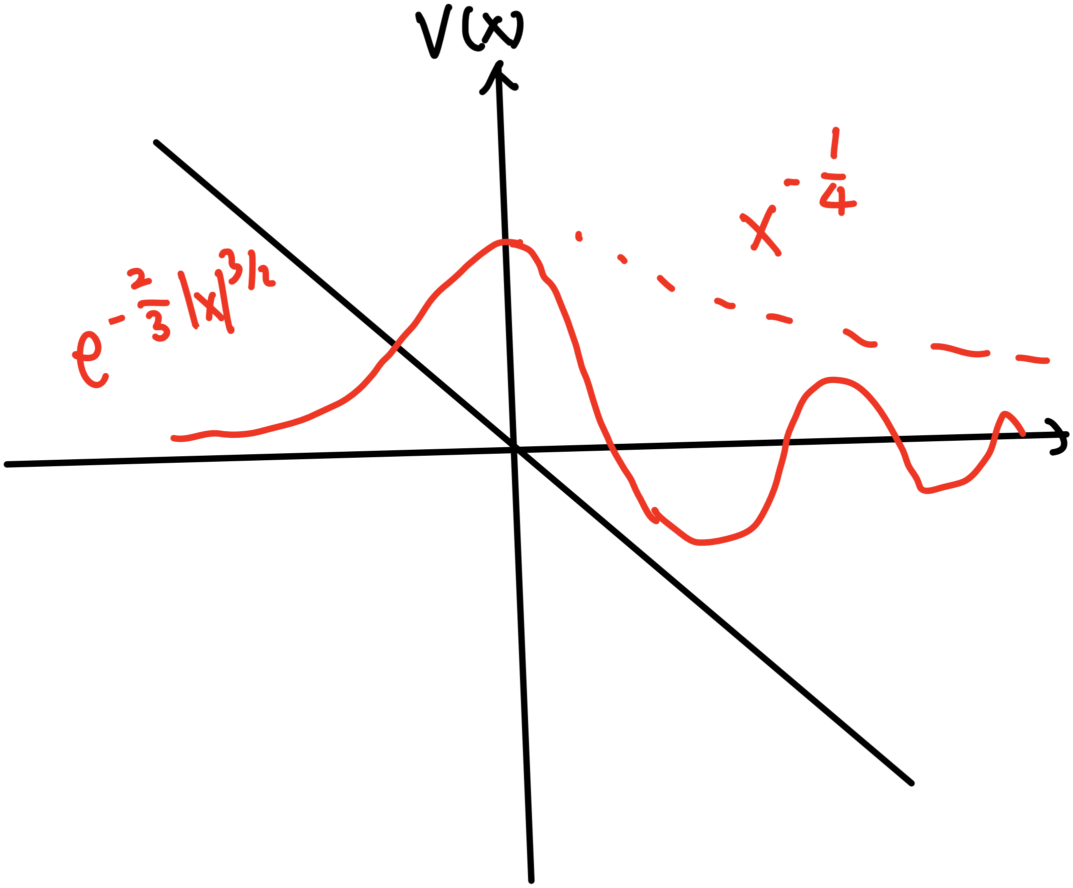
$$\frac{d^2y}{dx^2} = -xy$$

$$H\psi = E\psi \qquad H = -\frac{d^2}{dx^2} + V$$

$$\Rightarrow \frac{d^2}{dx^2}\psi = (V-E)\psi$$

$E=0$

$$\Rightarrow \frac{d^2}{dx^2}\psi = V(x)\psi \qquad V(x) = -x$$



常微分方程的级数解

线性二阶齐次

$$y''(x) + P(x)y' + Q(x)y = 0$$

两个线性无关解 y_1，y_2，则

$c_1y_1 + c_2y_2$ 也是齐次解

对于非齐次方程

$$y''(x) + P(x)y' + Q(x)y(x) = F(x)$$

如果有特解 $y_p(x)$，则最一般解为

$$y(x) = \underbrace{c_1y_1 + c_2y_2}_{\text{由边界条件决定}} + y_p$$

* 奇点分类：

(1) x_0 是齐次方程的常点

$\Leftrightarrow$ $P(x_0)$，$Q(x_0)$ 有限.

(2) 正则奇点 x_0，

$p(x_0)$ 或 $Q(x_0)$ 发散，但 $(x-x_0)p(x)$ 和 $(x-x_0)^2Q(x)$ 在 x_0 处有泰勒展开。 [富克斯型方程]

(3) 非正则奇点

$p(x)$ 比 $\frac{1}{x-x_0}$ 发散更快，或 $Q(x)$ 比 $\frac{1}{(x-x_0)^2}$ 发散更快。

* 无穷远处奇点 通过代换 $x=\frac{1}{z}$ 分析。

例：贝塞尔方程：

$$x^2y''+xy'+(x^2-n^2)y=0$$

$$p(x)=\frac{1}{x},\quad Q(x)=1-\frac{n^2}{x^2}$$

$x=0$ 是正则奇点。

令 $z=\frac{1}{x}$，
$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=-z^2\frac{dy}{dz}$$
$$\frac{d^2y}{dx^2}=\frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx}=-z^2\cdot\left(-2z\frac{dy}{dz}-z^2\frac{d^2y}{dz^2}\right)$$

$$\Rightarrow \left(+z^4\frac{d^2y}{dz^2}+2z^3\frac{dy}{dz}\right)\frac{1}{z^2}+\frac{1}{z}\cdot\left(-z^2\frac{dy}{dz}\right)+\left(\frac{1}{z^2}-n^2\right)y(z)=0$$

$$\Rightarrow +z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}+\left(\frac{1}{z^2}-n^2\right)y=0$$

$$P(z)=\frac{z}{z^2}=\frac{1}{z}\qquad Q(z)=\frac{1}{z^4}-\frac{n^2}{z^2}$$

$z=\infty$ 是非正则奇点。

* 常点附近的级数解.

例： $y''-y=0$

瞪眼法： $y_1=\cosh x$

$y_2=\sinh x$

级数法求 $x=0$ 处的级数解

设 $y=\sum_{k=0}^{\infty}a_kx^k$

代入原方程

$$\sum_{k=2}^{\infty}k\cdot(k-1)a_kx^{k-2}-\sum_{k=0}^{\infty}a_kx^k=0$$

$$\Rightarrow \sum_{k=0}^{\infty}\left[(k+2)(k+1)a_{k+2} - a_k\right]x^k = 0$$

$$\Rightarrow \quad a_{k+2} = \frac{a_k}{(k+2)(k+1)}$$

两个初始系数 a_0 和 a_1

$$a_2 = \frac{a_0}{2\cdot 1}$$

$$a_4 = \frac{a_2}{4\cdot 3} = \frac{a_0}{4\cdot 3\cdot 2\cdot 1}$$

$$a_6 = \frac{a_4}{6\cdot 5} = \frac{a_0}{6!}$$

$$\vdots$$

$$a_{2n} = \frac{a_0}{(2n)!}$$

$$a_3 = \frac{a_1}{3\cdot 2}$$

$$a_5 = \frac{a_3}{5\cdot 4} = \frac{a_1}{5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$\vdots$$

$$a_{2n+1} = \frac{a_1}{(2n+1)!}$$

一般级数解：

$$y(x) = \sum_{n=0}^{\infty} \frac{a_0}{(2n)!} x^{n} + \sum_{n=0}^{\infty} \frac{a_1}{(2n+1)!} x^{2n+1}$$

$$= a_0 \cosh x + a_1 \sinh x$$

例： Airy 函数

$$y''(x) - xy = 0 \qquad x=0 \text{ 是常点}$$

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$\Rightarrow \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - \sum_{k=0}^{\infty} a_k x^{k+1} = 0$$

x^0 项： $2 \cdot 1 \cdot a_2 = 0 \Rightarrow a_2 = 0$

$$\sum_{k=0}^{\infty} \Big[(k+3)(k+2) \underset{a_{k+3}}{\underset{\wedge}{}} x^{k+1} - a_k x^{k+1} \Big] = 0$$

$\Rightarrow$

$$a_{k+3} = \frac{a_k}{(k+3)(k+2)} \qquad k = 0, 1, 2, \ldots$$

$$a_3 = \frac{a_0}{3\cdot 2}$$

$$a_6 = \frac{a_3}{6\cdot 5} = \frac{a_0}{6\cdot 5\cdot 3\cdot 2}$$

$$a_9 = \frac{a_6}{9\cdot 8} = \frac{a_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}$$

$$\Rightarrow \quad a_{3n} = \frac{a_0}{3n\,(3n-1)\;3(n-1)\;(3(n-1)-1)\cdots}$$

$$a_4 = \frac{a_1}{4\cdot 3}$$

$$a_7 = \frac{a_4}{7\cdot 6} = \frac{a_1}{7\cdot 6\cdot 4\cdot 3}$$

$$a_{10} = \frac{a_7}{10\cdot 9} = \frac{a_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}$$

$$\vdots$$

$$a_{3n+1} = \frac{a_1}{(3n+1)(3n)\cdot \quad \cdots \quad 4\cdot 3}$$

例：勒让得方程：　　　　l是常数.

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0$$

$x=0$ 是常点.

令 $y(x) = \sum_{k=0}^{\infty} a_k x^k$,

$$\sum_{k=2}^{\infty} k\cdot(k-1)a_k(x^{k-2} - x^k) - \sum_{k=0}^{\infty} 2k a_k x^k$$

$$+ \sum_{k=0}^{\infty} l\cdot(l+1) a_k x^k = 0$$

x^0 项：　$2a_2 + l\cdot(l+1)a_0 = 0$

x^1 项：　$6a_3 - 2a_1 + l\cdot(l+1)a_1 = 0$

x^2 及以上项给出方程

$$\sum_{k=0}^{\infty}\Big[(k+2)(k+1)a_{k+2}x^k - k\cdot(k-1)x^k a_k$$

$$- 2k a_k x^k + l\cdot(l+1)a_k x^k\Big] = 0$$

$$-k^2+k-2k+l^2+l$$

$$=-k^2-k+l^2+l=(l+k)(l-k)+(l-k)$$

$$=(l-k)(l+k+1)$$

$$\therefore\ a_{k+2}=-a_k\frac{(l-k)(l+k+1)}{(k+2)(k+1)}$$

通解:

$$y(x)=a_0\cdot\left[1-\frac{l\cdot(l+1)}{2\cdot 1}x^2+\frac{l\cdot(l+1)(l-2)(l+3)}{4!}x^4+\cdots\right]$$

$$+a_1\cdot\left[x-\frac{(l-1)(l+2)}{3\cdot 2}x^3+\frac{(l-1)(l+2)(l-3)(l+4)}{5\cdot 4\cdot 3\cdot 2\cdot 1}x^5+\cdots\right]$$

对于整数 l

一个截断解，一个无穷级数解.