CheatSheet

Γ函数

$$\Gamma(z)=\int_0^\infty \mathrm{e}^{-t}t^{z-1}\mathrm{d}t=\int_1^\infty \mathrm{e}^{-t}t^{z-1}\mathrm{d}t+\sum_{n=0}^{\infty}\frac{(-)^n}{n!}\frac{1}{n+z}$$

$$\Gamma(1)=1\qquad \Gamma(\frac{1}{2})=\sqrt{\pi}\qquad \Gamma(z+1)=z\Gamma(z)=z!$$

$$\mathrm{res}\Gamma(-n)=\frac{(-1)^n}{n!}\qquad \Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$$

$$\Gamma(2z)=2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2})$$

$$\Gamma(z)\sim z^{z-1/2}\mathrm{e}^{-z}\sqrt{2\pi}\left(1+\frac{1}{12z}+\frac{1}{288z^2}+\ldots\right)$$

$$\Gamma(z)=\frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^{z}\right]$$

$$\frac{1}{\Gamma(z)}=z\mathrm{e}^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)\mathrm{e}^{-z/n}\right]$$

$$\text{n维球体积}\quad V_n(a)=\frac{\pi^{n/2}}{\Gamma(1+n/2)}a^n$$

$$\gamma=-\int_0^\infty \mathrm{e}^{-\mathrm{u}}\ln u\mathrm{d}u=-\frac{\partial}{\partial\alpha}\int_0^\infty u^{\alpha-1}e^{-u}\mathrm{d}u|_{alpha=1}=-\Gamma'(1)$$

ψ函数

$$\psi(z)=\frac{\mathrm{d}\ln\Gamma(z)}{\mathrm{d}z}=-\gamma+\sum_{n=1}^{\infty}\frac{1}{n}-\sum_{n=0}^{\infty}\frac{1}{n+z}=-\gamma+1+\frac{1}{2}+\ldots+\frac{1}{z-1}$$

$$\psi(z+1)=\psi(z)+\frac{1}{z}\quad \psi(z+n)=\psi(z)+\frac{1}{z}+\ldots+\frac{1}{z+n-1}$$

$$\psi(1-z)=\psi(z)+\pi\cot(\pi z)\quad \psi(z)-\psi(-z)=-\frac{1}{z}-\pi\cot(\pi z)$$

$$\psi(2z)=\frac{1}{2}\psi(z)+\frac{1}{2}\psi(z+\frac{1}{2})+\ln 2\qquad \lim_{n\to\infty}[\psi(z+n)-\ln n]=0$$

$$\psi'(z)=\sum_{n=0}^{\infty}\frac{1}{(n+z)^2}$$

B函数

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t=2\int_0^{\frac{\pi}{2}}\sin^{2p-1}\theta\cos^{2q-1}\theta\mathrm{d}\theta=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

ζ函数

$$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

Fourier 变换

$$F(k)=\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x \qquad f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k$$

$$\mathscr{F}\{f(ax)\}=\frac{1}{a}F\left(\frac{k}{a}\right) \qquad \mathscr{F}\{f(x-x_0)\}=\mathrm{e}^{-\mathrm{i}kx}F(k)$$

$$\mathscr{F}\{\mathrm{e}^{\mathrm{i}k_0x}f(x)\}=F(k-k_0) \qquad \mathscr{F}\{f'(x)\}=\mathrm{i}kF(k) \qquad \mathscr{F}\{\int^{x} f(\xi)\mathrm{d}\xi\}=\frac{1}{\mathrm{i}k}F(k)$$

$$f\circ g(t)=\int_{-\infty}^{\infty} f(t-\tau)g(\tau)\mathrm{d}\tau \qquad \mathscr{F}\{f\circ g(t)\}=F(\omega)G(\omega)$$

$$\int_{-\infty}^{\infty} f(t)g(-t)\mathrm{d}t=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G(\omega)\mathrm{d}\omega \qquad \int_{-\infty}^{\infty} f(t)g(t)\mathrm{d}t=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G(-\omega)\mathrm{d}\omega$$

$$\int_{-\infty}^{\infty} f(t)g(t)^*\mathrm{d}t=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G(\omega)^*\mathrm{d}\omega$$

$$\int_{-\infty}^{\infty} \delta(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k=1 \qquad \delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\mathrm{d}k \qquad \delta(x-a)=\sum_{n=0}^{\infty}\frac{(-)^n}{n!}\delta^{(n)}(x)a^n$$

$$D_0(|f|^2)D_0(F^2)\geq\frac{\pi}{2}\left(\int_{-\infty}^{\infty}|f(t)|^2dt\right)^2$$

$$\int_a^b f(x)\delta(g(x))\mathrm{d}x=\frac{f(x_0)}{g'(x_0)} \qquad \mathscr{L}\{H(x)\}=-\mathcal{P}\frac{\mathrm{i}}{k}+\pi\delta(k)$$

Laplace 变换

$$\mathscr{L}\{f(t)\}=F(p)=\int_0^{\infty} \mathrm{e}^{-pt}f(t)\mathrm{d}t \qquad f(t)=0 \quad for \quad t<0$$

$$f(t)=\frac{1}{2\pi\mathrm{i}}\int_{s-\mathrm{i}\infty}^{s+\mathrm{i}\infty} F(p)\mathrm{e}^{pt}\mathrm{d}p \qquad for \quad s>s_0$$

$$\mathscr{L}\{f(at)\}=\frac{1}{a}F\left(\frac{p}{a}\right) \qquad \mathscr{L}\{f(t-\tau)\}=\mathrm{e}^{-p\tau}F(p)$$

$$\mathscr{L}\{\mathrm{e}^{p_0t}f(t)\}=F(p-p_0) \qquad F^{(n)}(p)=\int_0^{\infty}(-t)^nf(t)\mathrm{e}^{-pt}\mathrm{d}t$$

$$\mathscr{L}\{\int_0^t f(\tau)\mathrm{d}\tau\}=\frac{F(p)}{p} \qquad \mathscr{L}\{f'(t)\}=pF(p)-f(0)$$

$$\mathscr{L}\{\frac{f(t)}{t}\}=\int_p^{\infty} F(q)\mathrm{d}q \qquad \int_0^{\infty}\frac{f(t)}{t}\mathrm{d}t=\int_0^{\infty}F(p)\mathrm{d}p$$

$$\mathscr{L}\{\int_0^{\infty} f(t,\tau)\mathrm{d}\tau\}=\int_0^{\infty} F(p,\tau)\mathrm{d}\tau \qquad \mathscr{L}\{\frac{\partial f(x,t)}{\partial x}\}=\frac{\mathrm{d}F(x,p)}{\mathrm{d}x} \qquad \mathscr{L}\{\int_t^{\infty}\frac{f(\tau)}{\tau}\mathrm{d}\tau\}=\frac{1}{p}\int_0^p F(q)\mathrm{d}q$$

$$f_1\circ f_2(t)=\int_0^t f_1(\tau)f_2(t-\tau)\mathrm{d}\tau \qquad \mathscr{L}\{f_1\circ f_2(t)\}=F_1(p)F_2(p)$$

$$\mathscr{L}\{\sin\omega t\}=\frac{\omega}{p^2+\omega^2} \qquad \mathscr{L}\{\cos\omega t\}=\frac{p}{p^2+\omega^2}$$

$$\mathscr{L}\{\sinh\omega t\}=\frac{\omega}{p^2-\omega^2} \qquad \mathcal{L}\{\cosh\omega t\}=\frac{p}{p^2-\omega^2}$$

$$\mathscr{L}\left\{t^n\right\}=\int_0^{\infty} e^{-st}t^ndt=\frac{\Gamma(n+1)}{s^{n+1}}, \quad s>0,n>-1$$

$$\mathscr{L}\{-\gamma-\ln t\}=\frac{\ln p}{p}$$

二阶线性ODE

$$w_2(z)=Aw_1(z)\int^z\left[\frac{1}{w_1(z)^2}\exp\left(-\int^z p(\zeta)\mathrm{d}\zeta\right)\right]$$

$$z(1-z)y''(z)+[c-(a+b+1)z]y'(z)-aby(z)=0$$

$${}_2F_1(a,b,c;z)=\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{n!\Gamma(a)\Gamma(b)\Gamma(c+n)}z^n=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\mathrm{d}t$$

· 非正则奇点展开与WKB近似

$$y''+P(x)y'+Q(x)y=0\qquad y=fu$$

$$fu''+(2f'+fP)u'+(f''+Pf'+Qf)u=0\qquad 2f'+Pf=0$$

$$f=\exp\left(-\frac{1}{2}\int^x P(x')\mathrm{d}x'\right)$$

$$u''+\hat{Q}u=0\qquad \hat{Q}=Q-\frac{1}{2}P'-\frac{1}{4}P^2\qquad \text{let}\quad u=\mathrm{e}^{\mathrm{i}\phi(x)}$$

$$\text{if}\quad \phi''(x)\text{是小量}\quad u=c_1\hat{Q}^{-\frac{1}{4}}\mathrm{e}^{\mathrm{i}\int^x\sqrt{\hat{Q}}\mathrm{d}x'}+c_2\hat{Q}^{-\frac{1}{4}}\mathrm{e}^{-\mathrm{i}\int^x\sqrt{\hat{Q}}\mathrm{d}x'}$$

Weierstrass

(Weierstrass 因子分解定理) 设 $f(z)$ 为整函数，$z=0$ 为 $f(z)$ 的 m 重零点 (m 可以为 0)，其余零点为 $a_1,a_2,\cdots$，满足 $0<|a_n|\leqslant|a_{n+1}|$ 且 $\lim\limits_{n\to\infty}|a_n|=\infty$，则存在非负整数序列 $\{k_n\}_{n=1}^{\infty}$，使得任取 $R>0$ 都有级数 $\sum_{n=1}^{\infty}\left(\frac{R}{|a_n|}\right)^{k_n+1}$ 收敛，此时 $f(z)$ 可以写为

$$f(z)=z^m e^{g(z)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{P_n(z)}$$

其中 $g(z)$ 为整函数，$P_n(z)$ 为多项式，定义为

$$P_n(z)=\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{k_n}\left(\frac{z}{a_n}\right)^{k_n}$$

特别地，若存在非负整数 k 使得 $\sum_{n=1}^{\infty}\frac{1}{|a_n|^{k+1}}$ 收敛，则可以取所有的 $k_n=k$.

$$\sin\pi z=\pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\qquad \cos\pi z=\prod_{n=0}^{\infty}\left(1-\frac{z^2}{(n+1/2)^2}\right)$$

Mittag-Leffler

定理 32 (Mittag-Leffler 定理) 设亚纯函数 $f(z)$ 的单极点位于 $\{a_k\}$，设 $\{a_k\}_{k=1}^{\infty}$ 是 $\mathbb{C}$ 中一列互不相同的复数，满足 $0<|a_1|\leq|a_2|\leq\ldots$ 且 $\lim_{k\to\infty}|a_k|=\infty$。并且在每个极点 a_k 处的留数为 b_k。则该函数可以写为

$$f(z)=f(0)+\sum_{k=1}^{\infty}b_k\left(\frac{1}{z-a_k}+\frac{1}{a_k}\right)$$

辐角原理

设 C 是一条分段光滑的简单闭合围道（按逆时针方向）。设函数 $f(z)$ 在 C 上及其内部是亚纯函数，并且在 C 上没有零点和极点。令 Z 为 $f(z)$ 在 C 内部的零点数目（计入重数），P 为 $f(z)$ 在 C 内部的极点数目（计入阶数）。则

$$\frac{1}{2\pi i}\oint_C\frac{f'(z)}{f(z)}dz=Z-P$$

渐近展开与鞍点近似

对于复变积分(辐角信息归入$g(z)$以保证$\lambda>0$)

$$I(\lambda)=\int_C f(z)\mathrm{e}^{\lambda g(z)}\mathrm{d}z,\quad \text{as }\lambda\to\infty$$

$$I(\lambda) \sim f(z_0)\mathrm{e}^{\lambda g(z_0)}\sqrt{\frac{2\pi}{\lambda|g''(z_0)|}}\mathrm{e}^{\mathrm{i}\phi}, \quad \phi = \frac{\pi - \arg[g''(z_0)]}{2} + k\pi, \quad k \in \mathbb{Z}$$

这种方法对于评估无法通过实变量技术处理的复积分特别有效. 其关键思想是将积分路径变形, 使其沿着积分值减小最快的路径穿过鞍点, 从而使主要贡献来自这些点的邻域.

常见积分

$$\int_0^l x\cos(\frac{n\pi}{l}x)\mathrm{d}x = \frac{l^2}{n^2\pi^2}(-1+\cos(n\pi))$$
$$\int_0^l x^2\cos(\frac{n\pi}{l}x)\mathrm{d}x = \frac{l^3}{n^3\pi^3}(2n\pi\cos(n\pi))$$
$$\int_0^l x^3\cos(\frac{n\pi}{l}x)\mathrm{d}x = \frac{l^4}{n^4\pi^4}(6+3(-2+n^2\pi^2)\cos(n\pi))$$
$$\int_0^l x\sin(\frac{n\pi}{l}x)\mathrm{d}x = -\frac{l^2}{n\pi}\cos(n\pi)$$
$$\int_0^l x^2\sin(\frac{n\pi}{l}x)\mathrm{d}x = \frac{l^3}{n^3\pi^3}(-2+(2-n^2\pi^2)\cos(n\pi))$$
$$\int_0^l x^3\sin(\frac{n\pi}{l}x)\mathrm{d}x = -\frac{l^4}{n^3\pi^3}(-6+n^2\pi^2)\cos(n\pi)$$

解析函数相关公式

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\text{Tayor 展开} \qquad f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n \qquad a_n = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-a)^{n+1}}\mathrm{d}\zeta$$

$$\text{Laurent 展开} \qquad f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n \qquad a_n = \frac{1}{2\pi \mathrm{i}}\oint_C \frac{f(\zeta)}{(\zeta-a)^{n+1}}\mathrm{d}\zeta$$

$$\text{留数} \qquad a_{-1} = \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}(z-b)^m f(z)|_{z=b}$$

$$\mathrm{res} f(\infty) = \frac{1}{2\pi\mathrm{i}}\oint_{C'} f(z)\mathrm{d}z = -f(z)\text{展开中}z^{-1}\text{系数}$$