渐近展开

回忆：收敛展开。

$f(x)=\sum_n a_n(x)$ e.g. $a_n(x)=a_n x^n$

收敛：$S_N(x)=\sum_{n=1}^{N} a_n(x)$ $\lim_{N\to\infty} S_N(x)=f(x)$

收敛半径：$R=\lim_{n\to\infty}\frac{a_n}{a_{n+1}}$.

收敛总是在某个区域内有效。

渐近级数：(大部分物理例子)

$$f(x)\sim\sum_n a_n(x) \quad x\to x_0^+ \iff \underline{\lim_{N\to\infty}\lim_{x\to x_0^+}}\frac{|f(x)-S_N(x)|}{|\cancel{a_N(x)}|}\to 0$$

~~$O(a_N(x))$ 渐近级数~~.

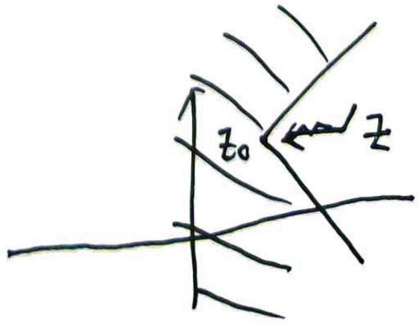
(1) $\lim_{x\to x_0^+}\frac{a_{n+1}(x)}{a_n(x)}=0$

(2) $\lim_{x_0^+}$ 与 $\lim_{x_0^-}$ 通常不同.

推广到复平面：

$f(z)\sim\sum_n a_n(z)$ $z\to z_0$

$$\lim_{\substack{z\to z_0\\ z \text{ in a wedge}}}\left|\frac{f(z)-S_N(z)}{\cancel{a_{N+1}(z)}}\right|\to 0$$



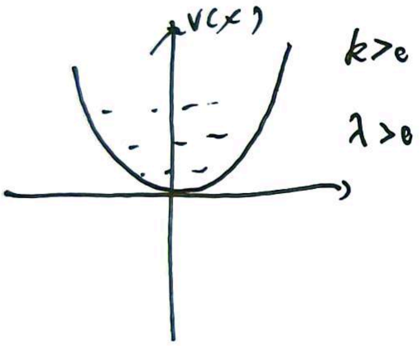
· 渐近级数(为什么)?

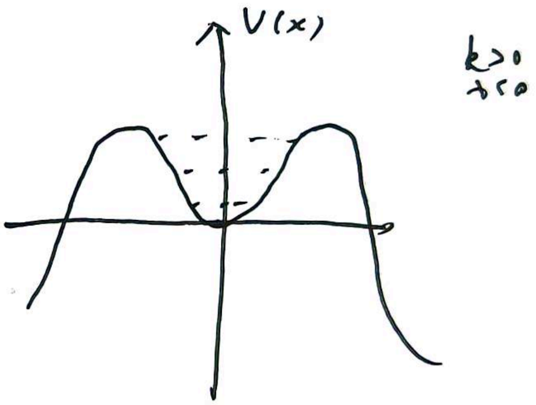
(1) 微扰展开，

给定一个物理理论，有一个小参数 a.

自然界对 a 的展开是非解析的，

例: $V(x) = \frac{k}{2}x^2 + \frac{\lambda}{4}x^4$. 非谐振子 (量子力学)





$a=0$ 点物理行为完全改变.

例: Dyson 关于 α_e 的例子.

(2) 奇异微扰论:

$$\varepsilon^2 \frac{d^2y}{dx^2} + y(x) = 0$$

$\varepsilon \to 0$ 微分方程阶发生改变，故解的个数发生改变.

(3) 积分展开:

$$f(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\,dt \qquad x \to 0_+$$

$x<0$ 时积分无定义.

$$f(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\,dt \qquad x\to 0^+$$

$$\overset{?}{=} \int_0^\infty e^{-t} \sum_{n=0}^{\infty} (-xt)^n \, dt$$

$$\text{IBP} \overset{?}{=} \sum_{n=0}^{\infty} \int_0^\infty e^{-t} (-xt)^n \, dt$$

$$= \sum_{n=0}^{\infty} (-x)^n \int_0^\infty e^{-t} t^n \, dt$$

$$= \sum_{n=0}^{\infty} (-x)^n \Gamma(n+1)$$

$$= \sum_{n=0}^{\infty} (-x)^n n!$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = n+1 = \infty = \frac{1}{R} \Rightarrow R=0$$

收敛半径为 0!

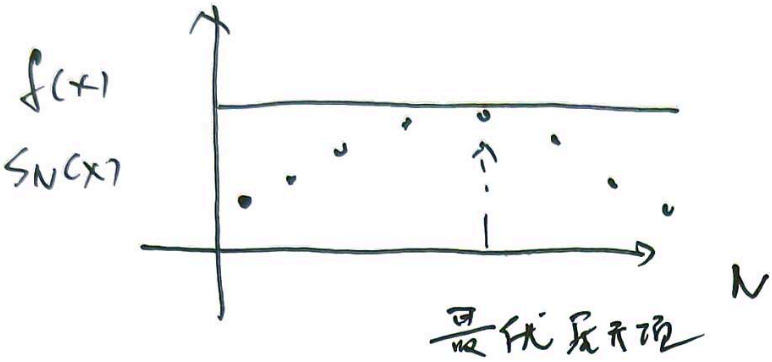
但: ~~$f(x) - S_N(x) = \int_0^\infty e^{-t} \frac{(-xt)^{N+1}}{1+xt}\,dt$~~

$$f(x) - S_N(x) = (-1)^{N+1}(N+1)\,x^{N+1}\int_0^\infty \frac{e^{-t}\,dt}{(1+xt)^{N+2}}$$

$$|f(x) - S_N(x)| \le (N+1)\,x^{N+1}$$

对于固定的 x, $N\to\infty$ 发散.

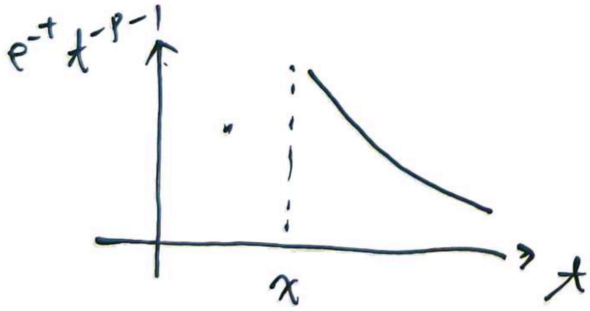
但对于固定的 N, $x\to 0^+$ 趋0.



§一、分部积分法.（最大贡献来自积分端点）

例1 不完备伽马函数

$$\Gamma(-p,x)=\int_x^{\infty}e^{-t}t^{-p-1}dt \qquad x\to+\infty$$



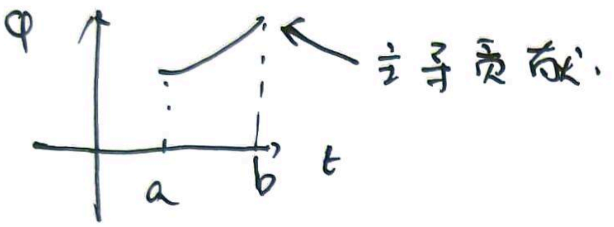
$$\Gamma(-p,x)=\int_x^{\infty}e^{-t}t^{-p-1}dt=\int_x^{\infty}t^{-p-1}d(-e^{-t})$$

$$=-e^{-t}t^{-p-1}\Big|_x^{\infty}+\int_x^{\infty}-(p+1)\,t^{-p-2}e^{-t}\,dt$$

$$=\frac{e^{-x}}{x^{p+1}}-\frac{e^{-x}(p+1)}{x^{p+2}}+\frac{e^{-x}(p+1)(p+2)}{x^{p+3}}+\cdots$$

练习：证明这是渐近级数.

$$I(x)=\int_a^b f(t)\, e^{x\varphi(t)}\, dt \qquad x\to+\infty$$



$\mathrm{Re}\,\varphi(a) < \mathrm{Re}\,\varphi(b)$

$$I(x)=\int_a^b \frac{f(t)}{x\varphi'(t)}\, d\, e^{x\varphi(t)}$$

$$=\frac{1}{x}\left(\frac{f}{\varphi'}e^{x\varphi}\right)\Bigg|_a^b-\frac{1}{x}\int_a^b \frac{d}{dt}\left(\frac{f}{\varphi'}\right)e^{x\varphi(t)}dt+\cdots$$

$$=\frac{1}{x}\left[\frac{f(b)}{\varphi'(b)}e^{x\varphi(b)}-\frac{f(a)}{\varphi'(a)}e^{x\varphi(a)}\right]+\cdots$$

$$\sim e^{x\varphi(b)}\sum_n A_n x^{-n}, \quad x\to\infty$$

如果 $\varphi'(t)=0$ for $t\in(a,b)$

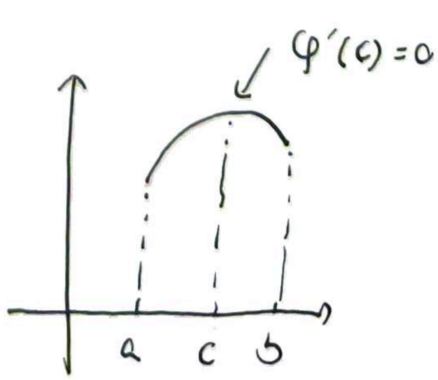
则此方法无效！

(二) 拉普拉斯方法.

$$I(x)=\int_a^b dt\, f(t)\, e^{x\varphi(t)}dt$$

$x\to+\infty$

$\varphi'(c)=0,\quad \varphi''(c)<0$



在 c 附近展开：

$$\varphi(t) \simeq \varphi(c) + \underset{\to 0}{\varphi'(c)}(t-c) + \frac{(t-c)^2}{2}\varphi''(c) + \cdots$$

$$I(x) \underset{x\to+\infty}{\sim} \int_{c-\varepsilon}^{c+\varepsilon} \left(f(c)+f'(c)(t-c)+\cdots\right) e^{x\left(\varphi(c)+0+\frac{1}{2}\varphi''(c)(t-c)^2+\cdots\right)} dt$$

$$\underset{\substack{t\sim c\\ \text{最重要}}}{\sim} \int_{-\infty}^{+\infty} f(c)\, e^{x\varphi(c)} \cdot e^{x\varphi''(c)\frac{1}{2}(t-c)^2} dt$$

$$\sim f(c)\, e^{x\varphi(c)} \sqrt{\frac{2\pi}{-x\varphi''(c)}} \quad , x\to+\infty$$

注意到渐近展开仅以在包含 $\frac{1}{x}$ 项。

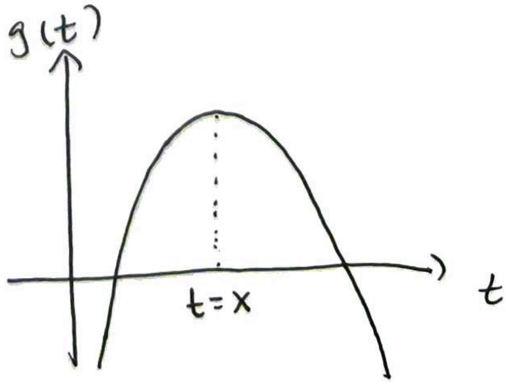
例：斯特林公式：

$\Gamma(n+1) = n!$ $\quad n\to\infty$?

$\Gamma(x) = \int_0^\infty e^{-t} t^{x} dt$ 并不是 Laplace 型积分.

$$= \int_0^\infty e^{-t+x\ln t} dt$$

$g(t) = -t + x\ln t$



可移动极大值.

变量替换： $t = sx$， $dt = x\,ds$

$$\Gamma(x+1) = \int_0^\infty e^{-sx + x\ln(sx)}\, x\,ds$$ 极大值 $s=1$.

$$= x^{x+1}\int_0^\infty e^{-sx + x\ln s}\, ds$$

$\tilde{\varphi}(1) = -1$

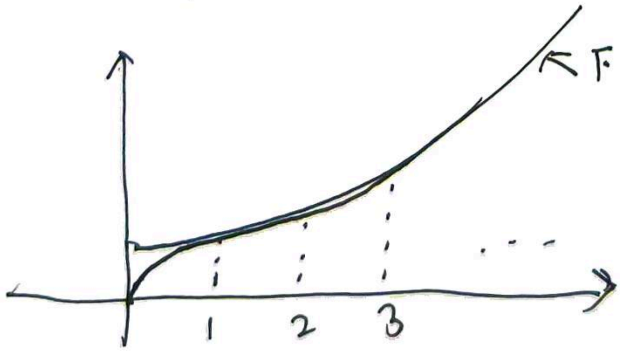
$\tilde{\varphi}(s) = -sx + x\ln s$， $\tilde{\varphi}''(s=1) = -\frac{1}{s^2} = 1$.

$$\Rightarrow \Gamma(x+1) \sim x^{x+1} e^{-x}\sqrt{\frac{2\pi}{x}} = \left(\frac{x}{e}\right)^x \sqrt{2\pi x}$$

斯特林公式：

$$n! = \Gamma(n+1) \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

$$\ln n! = n\ln n$$



〈三〉傅立叶型积分.

$$I(x) = \int_a^b dt\, f(t)\, e^{ix\psi(t)}\, dt \qquad x \to +\infty$$

$\psi'(t) \neq 0$ 在 $t \in (a, b)$. IBP.

$$I(x) = \int_a^b f(t)\frac{1}{xi\psi'(t)}\, d\, e^{ix\psi(t)}$$

是否足够小？

$$= \frac{f(t)}{xi\psi'(t)} e^{ix\psi(t)}\Big|_a^b - \frac{1}{ix}\int_a^b \frac{d}{dt}\left(\frac{f(t)}{\psi'(t)}\right) e^{ix\psi(t)}\, dt$$

黎曼－勒贝格引理.

如果 $\int_a^b |f(t)|dt < \infty$

$$\lim_{x\to+\infty}\int_a^b e^{ixt} f(t)\,dt = 0$$

① 由于 $\psi'(t)\neq 0$，令 $\tau = \psi(t)$

$$d\tau = \psi'(t)\,dt$$

$$\int_a^b e^{ix\psi(t)} f(t)\,dt \to \int e^{ix\tau}\frac{f}{\psi'}d\tau \to 0$$

$$\Rightarrow I(x) \sim \left[\frac{f(t)}{ix} e^{i\psi(t)x}\right]_a^b$$

例： $I(x) = \int_0^1 \frac{e^{ixt}}{1+t}dt \qquad x\to\infty$

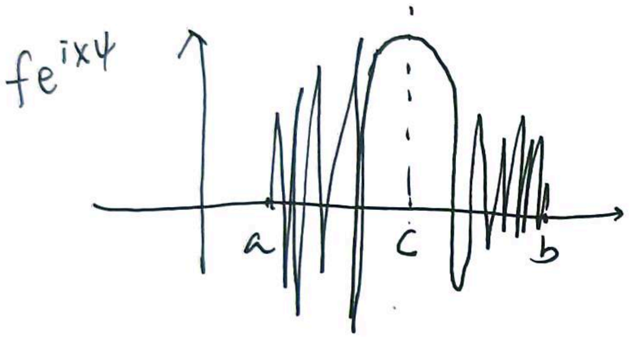
$$= \frac{1}{ix}\int_0^1 \frac{de^{ixt}}{1+t}$$

$$= \underbrace{\frac{1}{ix}\left.\frac{e^{ixt}}{1+t}\right|_0^1}_{\text{领头项}} + \frac{1}{ix}\int_0^1 \frac{e^{ixt}}{(1+t)^2}dt$$

(四) 稳定相位法.

$\psi(t)=0$ 在 $t\in(a,b)$ 时如何办?

$$I(x)=\int_a^b f(t)e^{ix\psi(t)}dt. \qquad \psi'(c)=0 \quad a<c<b$$



由 R-L 引理, 假设积分为 $t\sim c$ 附近主导.

$$I(x)=\left[\int_a^{c-\varepsilon}+\int_{c-\varepsilon}^{c+\varepsilon}+\int_{c+\varepsilon}^{b}\right]f(t)e^{ix\left(\psi(c)+\frac{(t-c)^2}{2}\psi''(c)+\cdots\right)}dt$$

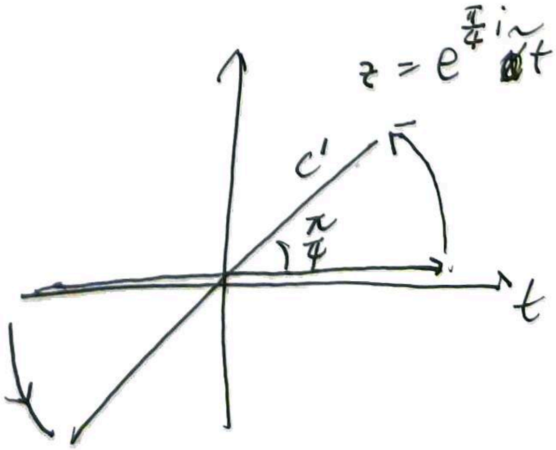
$$\overset{x\to+\infty}{\sim} f(c)e^{ix\psi(c)}\int_{-\infty}^{+\infty}e^{ix\psi''(c)\frac{1}{2}(t-c)^2}dt$$

周道高斯积分:

$$\int_{-\infty}^{+\infty}e^{i\alpha t^2}dt\to\int_{c}e^{i\alpha z^2}dz$$

$\alpha>0$

$z=e^{\frac{\pi}{4}i}\tilde{t}$



$$=\int_{-\infty}^{+\infty}e^{\frac{\pi}{4}i}e^{-\alpha\tilde{t}^2}d\tilde{t}$$

$$=e^{i\frac{\pi}{4}}\sqrt{\frac{\pi}{\alpha}} \qquad \alpha>0 \qquad \left(\alpha<0 \text{ 时 } e^{-i\frac{\pi}{4}}\sqrt{\frac{\pi}{-\alpha}}\right)$$

$$\Rightarrow I(x) \underset{x\to\infty}{\sim} f(c)e^{ix\psi(c)}\, e^{\pm\frac{\pi}{4}i}\sqrt{\frac{2\pi}{x(\pm\psi''(c))}}$$

例：零阶贝塞尔函数：

$$J_0(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\cos\theta)\, d\theta$$

$$= \frac{2}{\pi}\,\mathrm{Re}\int_0^{\pi/2} e^{ix\cos\theta}\, d\theta \qquad x\to\infty$$

稳定相：$\cos'\theta = -\sin\theta = 0 \Rightarrow \theta = 0$.

临界点位于端点.

$$J_0(x) \sim \frac{2}{\pi}\,\mathrm{Re}\int_0^{\pi/2} e^{ix(1-\frac{1}{2}\theta^2)}\, d\theta$$

$$\sim \frac{2}{\pi}\,\mathrm{Re}\; e^{ix}\cdot\int_0^{\infty} e^{-ix\frac{\theta^2}{2}}\, d\theta$$

$$\sim \frac{2}{\pi}\,\mathrm{Re}\left[e^{ix}e^{-i\pi/4}\,\frac{1}{2}\sqrt{\frac{2\pi}{x}}\right]$$

$$\sim \sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi}{4}\right) \qquad x\to\infty$$

(五) 最速下降法：

$$I(x) = \int_c g(z)\, e^{xf(z)}\, dz$$

最重要贡献来自复平面上的稳定点，

$f'(z)=0$ 通过围道形变跳过临界点并用其计算积分.

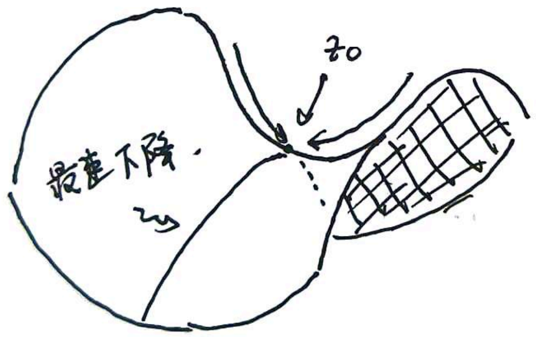
令 $f(z) = u(x,y) + iv(x,y)$

$$|e^{x f(z)}| = e^{x u(x,y)}$$

在临界点，$z_0 = x_0 + iy_0$，

$$f'(z_0) = 0 \Rightarrow \cancel{u_x \to 0}\ u_x(x_0,y_0) = 0 \quad u_y(x_0,y_0) = 0$$

事实上，z_0 必须是鞍点！ $\nabla^2 u \equiv 0$ 调和函数.



在 z_0 附近，

$$f(z) \simeq f(z_0) + \frac{1}{2} f''(z_0)(z-z_0)^2 + \cdots$$

令 $f''(z_0) = \rho e^{i\theta}$ ， $(z-z_0) = s e^{i\varphi}$

$$f(z) \approx f(z_0) + \frac{1}{2}\rho e^{i\theta} \frac{1}{2} s^2 e^{2i\varphi} + \cdots$$

$$\simeq f(z_0) + \frac{1}{2}\rho s^2 \left(\cos(\theta+2\varphi) + i\sin(\theta+2\varphi)\right) + \cdots$$

最速下降路径： $\cos(\theta+2\varphi) = -1$

$\sin(\theta+2\varphi) = 0$

两个解： $\theta + 2\varphi = \pm\pi$.

例：Airy 函数.

傅立叶变换.

$$\frac{d^2y}{dx^2}+xy=0 \Rightarrow Ai(x)=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{i(\omega x-\frac{\omega^3}{3})}$$

ω 鞍点：$\left[x\left(\omega-\frac{\omega^3}{3x}\right)\right]'=0 \Rightarrow 1=\frac{\omega^2}{x} \Rightarrow \omega^2=x$

$\omega=\pm\sqrt{x}$. 活动鞍点.

变量替换：$\omega=\sqrt{x}\,\tilde\omega \Rightarrow \tilde\omega=\pm1$.

在 $\tilde\omega$ 复平面.

$f(\tilde\omega)=i(\tilde\omega-\tilde\omega^3/3)$

而 $Ai(x)=\sqrt{x}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{x^{3/2}f(\tilde\omega)}$

$f(\tilde\omega)$ $f'(\tilde\omega)=0 \Rightarrow \tilde\omega=\pm1$. $f''(\tilde\omega)=-2i\tilde\omega$

① $\tilde\omega=1+se^{i\varphi}$.

$$f'(1+se^{i\varphi})=\frac{2}{3}i+\frac{1}{2}s^2e^{2i\varphi}\underbrace{f''(1)}_{(-2i)}+\cdots$$

$$=\frac{2}{3}i+s^2e^{2i\varphi-\frac{i}{2}\pi}+\cdots$$

最速下降：$2\varphi-\frac{\pi}{2}=\pm\pi$.

$\Rightarrow \varphi=\frac{3}{4}\pi$ 或 $\varphi=-\frac{\pi}{4}$.

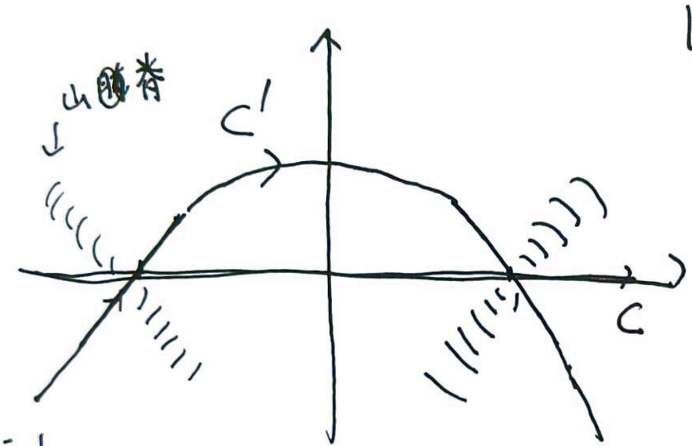
② $\tilde{w} = -1 + se^{i\varphi}$

$$f'(-1+se^{i\varphi}) = -\frac{2}{3}i + \frac{1}{2}s^2 e^{2i\varphi}(2i) + \cdots$$

$$= -\frac{2}{3}i + s^2 e^{2i\varphi + \frac{\pi}{2}i}$$

$2\varphi + \frac{\pi}{2} = \pm\pi$, $\Rightarrow \varphi = \frac{\pi}{4}$ 或 $\varphi = -\frac{3}{4}\pi$.

$\tilde{w}$



$\tilde{w} = 1$ $\qquad \tilde{w} - 1 = se^{-\frac{\pi}{4}i}$

$$I(x) \simeq \frac{\sqrt{x}}{2\pi}\int_{-\infty}^{+\infty} ds\, e^{x^{3/2}\left(\frac{2}{3}i - s^2\right)} e^{\frac{\pi}{4}i}$$

$$\sim \frac{\sqrt{x}}{2\pi} e^{\frac{2}{3}ix^{3/2}} e^{-\frac{\pi}{4}i} \sqrt{\frac{\pi}{x^{3/2}}}$$

$\tilde{w} \simeq -1$ $\qquad \tilde{w} + 1 = se^{+\frac{\pi}{4}i}$

$$I \cong \frac{\sqrt{x}}{2\pi}\int_{-\infty}^{+\infty} ds\, e^{x^{3/2}\left(-\frac{2}{3}i - s^2\right)} e^{+\frac{\pi}{4}i}$$

$$= \frac{\sqrt{x}}{2\pi} e^{-\frac{2}{3}ix^{\frac{3}{2}}} e^{+\frac{\pi}{4}i} \sqrt{\frac{\pi}{x^{3/2}}}$$

相加后得到:

$$Ai(x) \sim \frac{1}{2}\sqrt{\frac{1}{\pi}}\, x^{-\frac{1}{4}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right)$$