18 积分变换的应用

本章介绍求解偏微分方程定解问题的另一种方法 — 积分变换. 常用的积分变换有 Laplace 变换和 Fourier 变换.

18.1 Laplace 变换

Laplace 变换常用于求解含时间的偏微分方程定解问题. 对于系数与 t 无关的偏微分方程, 变换后微分变量的个数比原来减少一. 一般说来, 后者比较容易求解. 当然, 这样求得的是定解的像函数, 还必须反演, 才能得到原始问题的解.

Example 18.1 *用 Laplace 变换法求解定解问题*

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le l \tag{1a}$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \tag{1b}$$

$$u\Big|_{x=0} = 0 \qquad \frac{\partial u}{\partial x}\Big|_{x=l} = E \sin \omega t \tag{1c}$$

Solution 1. 作 Laplace 变换

$$\mathscr{L}\left\{u(x,t)\right\} = U(x,p) \tag{2}$$

于是

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = p^2 U(x,p) \tag{3}$$

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{\mathrm{d}^2 U(x,p)}{\mathrm{d}x^2} \tag{4}$$

这里, 在写出 (3) 时, 已经利用了初始条件. 定解问题变为

$$a^2 \frac{\mathrm{d}^2 U(x,p)}{\mathrm{d}x^2} - p^2 U(x,p) = 0 \tag{5a}$$

$$U(x,p)\Big|_{x=0} = 0 \qquad \frac{\mathrm{d}U(x,p)}{\mathrm{d}x}\Big|_{x=0} = E\frac{\omega}{p^2+\omega^2} \tag{5b}$$

原来两变量的偏微分方程变为单变量 x 的常微分方程 (另一变量 p 视为参数). 微分方程的通解为

$$U(x,p) = C_1 \cosh \frac{px}{a} + C_2 \sinh \frac{px}{a} \tag{6}$$

由边界条件得

$$C_1 = 0 \qquad C_2 = \frac{E\omega a}{p(p^2+\omega^2)\cosh \frac{pl}{a}}$$

所以

$$U(x,p) = \frac{E\omega a}{p(p^2+\omega^2)} \frac{\sinh \dfrac{px}{a}}{\cosh \dfrac{pl}{a}} \tag{7}$$

2. 求反演

由 Laplace 变换普遍反演公式得

$$u(x,t) = \frac{1}{2\pi \mathrm{i}} \int_{s-\mathrm{i}\infty}^{s+\mathrm{i}\infty} \frac{E\omega a \mathrm{e}^{pt}}{p(p^2+\omega^2)} \frac{\sinh \dfrac{px}{a}}{\cosh \dfrac{pl}{a}} \mathrm{d}p \tag{8}$$

假设 $F(p)$ 满足条件:

(a) $F(p)$ 除 $\mu < \mathrm{Re}p < \gamma$ 间有极点外, 处处解析.

(b) 存在 $k > 1$, $|p^k F(p)|$ 在 $\mathrm{Re}p \geq \gamma$ 有界.

(c) $|F(p)|$ 在 $\mathrm{Re}p \leq \mu$ 有界.

(d) 有一组水平直线 $\mathrm{Im}F(p) = \pm\sigma_n$, $\sigma_1 < \sigma_2 < ...$, $\lim_{n\to\infty} \sigma_n = \infty$, 使得 $n \to \infty$ 时, $|F(s \pm \sigma_n \mathrm{i})|$ 对所有 $s \leq \gamma$ 一致趋向于 0.

则 $f(t) = \mathscr{L}^{-1}\{F(p)\} = \sum\limits_n \mathrm{res}\{\mathrm{e}^{pt}F(p)\}$,

这里求和的每项为 $\sigma_n < \mathrm{Re}p < \sigma_{n+1}$ 和 $-\sigma_{n+1} < \mathrm{Re}p < \sigma_n$ 间的极点留数和.

被积函数有可数个简单奇点

$$\begin{aligned} p_j =& 0, \quad \pm\mathrm{i}\omega, \\ & \pm\,\mathrm{i}\frac{\pi a}{l}\left(k - \frac{1}{2}\right) \equiv \pm\mathrm{i}\omega_k \quad (k = 1, 2, 3, ...) \end{aligned}$$

若 $\mathrm{Re}p \geq \gamma > 0$,

$$\left|\frac{\sinh\dfrac{px}{a}}{\cosh\dfrac{pl}{a}}\right| = \left|\frac{\mathrm{e}^{p(x-l)/a} - \mathrm{e}^{-p(x+l)/a}}{1 + \mathrm{e}^{-2pl/a}}\right| < \frac{2}{1 - \mathrm{e}^{-2\gamma l/a}}.$$

所以

$$|p^2 F(p)|\text{有界}$$

若 $\mathrm{Re}p \leq \mu < 0$, 由对称性(奇偶性), 仍有

$$|p^2 F(p)|\text{有界}$$

所以

$$|F(p)|\text{有界}$$

取 $\sigma_n = n\pi a/l$, 则 $p = s \pm \mathrm{i}\sigma_n$ 时, $s \geq 0$

$$\left|\frac{\sinh\dfrac{px}{a}}{\cosh\dfrac{pl}{a}}\right| = \left|\frac{\mathrm{e}^{p(x-l)/a} - \mathrm{e}^{-p(x+l)/a}}{1 + \mathrm{e}^{-2sl/a}}\right| < 2.$$

$s < 0$, 由对称性也同样一致有界. 于是

$$u(x,t) = \sum_j \mathrm{res}\left(\frac{E\omega a e^{pt}}{p(p^2+\omega^2)}\frac{\sinh\dfrac{px}{a}}{\cosh\dfrac{pl}{a}}\right) \tag{9}$$

求出各奇点处的留数后, 得

$$\begin{aligned} u(x,t) = 0 &+ \frac{Ea}{2\mathrm{i}\omega}\frac{\sin\dfrac{\omega x}{a}}{\cos\dfrac{\omega l}{a}}\mathrm{e}^{\mathrm{i}\omega t} - \frac{Ea}{2\mathrm{i}\omega}\frac{\sin\dfrac{\omega x}{a}}{\cos\dfrac{\omega l}{a}}\mathrm{e}^{-\mathrm{i}\omega t} \\ &+ \sum_{k=1}^{\infty}\left(\frac{E\omega a}{\mathrm{i}\omega_k(\omega_k^2 - \omega^2)}\frac{a\sin\dfrac{\omega_k x}{a}}{(-1)^k l}\mathrm{e}^{\mathrm{i}\omega_k t}\right. \\ &\left. - \frac{E\omega a}{\mathrm{i}\omega_k(\omega_k^2 - \omega^2)}\frac{a\sin\dfrac{\omega_k x}{a}}{(-1)^k l}\mathrm{e}^{-\mathrm{i}\omega_k t}\right) \end{aligned}$$

$$u(x,t)=\frac{Ea}{\omega}\frac{\sin\dfrac{\omega x}{a}}{\cos\dfrac{\omega l}{a}}\sin\omega t$$
$$+\frac{2E\omega a^2}{l}\sum_{k=1}^{\infty}(-1)^k\frac{\sin\dfrac{\omega_k x}{a}}{\omega_k(\omega_k^2-\omega^2)}\sin\omega_k t \tag{10}$$

□

以上例题取自教科书的例题14.4, 具有非齐次的边界条件. 可以采用分离变量法求解, 但需先将边界条件齐次化. 采用 Laplace 变换法, 无需先将边界条件齐次化; 也不像分离变量法那样由初始条件定系数, 因为初始条件已通过 Laplace 变换进入 $U(x,p)$ 的方程.

Example 18.2 求解无界杆的热传导问题

$$\frac{\partial u}{\partial t}-\kappa\frac{\partial^2 u}{\partial x^2}=f(x,t), \qquad -\infty<x<\infty, \qquad t>0; \tag{11a}$$
$$u\big|_{t=0}=0, \qquad -\infty<x<\infty. \tag{11b}$$

Solution 在这种无界区间的定解问题中, 往往并不明确列出边界条件. 实际上, 无界区间只是一个物理上的抽象: $f(x,t)$ 为外界在 t 时刻在 x 处单位时间产生的热量 (通过化学反应, 电流, 等等). 物理上, 外界的影响只应发生在有限的空间内, 即 $f(x,t)$ 应满足

$$f(\infty,t)=0 \qquad f(-\infty,t)=0$$

于是, 无穷远处的温度应保持不变. 如果要完整地列出定解问题的话, 则还应有边界条件

$$u(x,t)\big|_{x\to\pm\infty}\to 0 \tag{12}$$

现在做 Laplace 变换. 令

$$\mathscr{L}\{u(x,t)\}=U(x,p) \tag{13}$$
$$\mathscr{L}\{f(x,t)\}=F(x,p) \tag{14}$$

定解问题变成

$$pU(x,p)-\kappa\frac{\mathrm{d}^2U(x,p)}{\mathrm{d}x^2}=F(x,p) \tag{15}$$

边界条件则为

$$U(\pm\infty,p)=0 \tag{16}$$

方程为二阶线性非齐次微分方程, 用*常数变易法*求解. 相应齐次微分方程的通解为

$$C_1\exp(\sqrt{\frac{p}{k}}x)+C_2\exp(-\sqrt{\frac{p}{k}}x)$$

非齐次方程的解则设为

$$U(x,p)=C_1(x)\exp(\sqrt{\frac{p}{k}}x)+C_2(x)\exp(-\sqrt{\frac{p}{k}}x) \tag{17}$$

并设

$$C_1'(x)(\exp(\sqrt{\frac{p}{k}}x)+C_2'(x)\exp(-\sqrt{\frac{p}{k}}x)=0 \tag{18}$$

则

$$
\begin{aligned}
&\frac{\mathrm{d}U(x,p)}{\mathrm{d}x}\\
=&C_1(x)\frac{\mathrm{d}}{\mathrm{d}x}\exp(\sqrt{\frac{p}{k}}x)+C_2(x)\frac{\mathrm{d}}{\mathrm{d}x}\exp(-\sqrt{\frac{p}{k}}x)\\
&\frac{\mathrm{d}^2U(x,p)}{\mathrm{d}x^2}\\
=&C_1'(x)\frac{\mathrm{d}}{\mathrm{d}x}\exp(\sqrt{\frac{p}{k}}x)+C_2'(x)\frac{\mathrm{d}}{\mathrm{d}x}\exp(-\sqrt{\frac{p}{k}}x)\\
+&C_1(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\exp(\sqrt{\frac{p}{k}}x)+C_2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\exp(-\sqrt{\frac{p}{k}}x)
\end{aligned}
$$

代入非齐次方程, 并注意到 $\exp(\pm\sqrt{\frac{p}{k}}x)$ 为相应齐次方程的解, 得

$$
\begin{aligned}
-\kappa\Big[&C_1'(x)\frac{\mathrm{d}}{\mathrm{d}x}\exp(\sqrt{\frac{p}{k}}x)\\
&+C_2'(x)\frac{\mathrm{d}}{\mathrm{d}x}\exp(-\sqrt{\frac{p}{k}}x)\Big]=F(x,p)
\end{aligned}
\tag{19}
$$

与 (18) 联立求解, 得

$$
C_1'(x)=-\frac{1}{2\sqrt{\kappa p}}F(x,p)\exp(-\sqrt{\frac{p}{k}}x) \tag{20a}
$$

$$
C_2'(x)=\frac{1}{2\sqrt{\kappa p}}F(x,p)\exp(\sqrt{\frac{p}{k}}x) \tag{20b}
$$

所以

$$
C_1(x)=-\frac{1}{2\sqrt{\kappa p}}\int^x F(x',p)\exp(-\sqrt{\frac{p}{k}}x')\mathrm{d}x' \tag{21a}
$$

$$
C_2(x)=\frac{1}{2\sqrt{\kappa p}}\int^x F(x',p)\exp(\sqrt{\frac{p}{k}}x')\mathrm{d}x' \tag{21b}
$$

考虑到边界条件 (16), 有

$$
C_1(x)=\frac{1}{2\sqrt{\kappa p}}\int_x^\infty F(x',p)\exp(-\sqrt{\frac{p}{k}}x')\mathrm{d}x' \tag{22a}
$$

$$
C_2(x)=\frac{1}{2\sqrt{\kappa p}}\int_{-\infty}^x F(x',p)\exp(\sqrt{\frac{p}{k}}x')\mathrm{d}x' \tag{22b}
$$

最后得到

$$
\begin{aligned}
U(x,p)=&\frac{1}{2\sqrt{\kappa p}}\int_x^\infty F(x',p)\exp[\sqrt{\frac{p}{k}}(x-x')]\mathrm{d}x'\\
&+\frac{1}{2\sqrt{\kappa p}}\int_{-\infty}^x F(x',p)\exp[\sqrt{\frac{p}{k}}(x'-x)]\mathrm{d}x'\\
=&\frac{1}{2\sqrt{\kappa p}}\int_{-\infty}^\infty F(x',p)\exp[-\sqrt{\frac{p}{k}}|x-x'|]\mathrm{d}x'
\end{aligned}
\tag{23}
$$

根据 Laplace 变换的反演公式

$$\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{p}}\exp(-\alpha\sqrt{p})\right\} = \frac{1}{\sqrt{\pi t}}\exp\left(-\frac{\alpha^2}{4t}\right) \tag{24}$$

再由卷积定理, 最后得到

$$\begin{aligned} & u(x,t) \\ =& \frac{1}{2\sqrt{\kappa\pi}}\int_{-\infty}^{\infty}\mathrm{d}x'\int_0^t \exp\left[-\frac{(x-x')^2}{4\kappa(t-\tau)}\right]\frac{f(x',\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau \end{aligned} \tag{25}$$

□

Example 18.3 用 *Laplace* 变换求解无界弦的波动问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad -\infty < x < \infty,\ t > 0 \tag{26a}$$

$$u\Big|_{t=0} = \phi(x), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x) \qquad -\infty < x < \infty \tag{26b}$$

Solution 作为无界空间的定解问题, 其边界条件为

$$u(\pm\infty, t) = 0 \tag{27}$$

设在 Laplace 变换下

$$\mathscr{L}\left\{u(x,t)\right\} = U(x,p) \tag{28}$$

于是, 原来的定解问题化为

$$p^2U(x,p) - p\phi(x) - \psi(x) - a^2\frac{\mathrm{d}^2U(x,p)}{\mathrm{d}x^2} = 0 \tag{29}$$

边界条件则为

$$U(\pm\infty, p) = 0 \tag{30}$$

方程为非齐次方程, 仍用常数变易法. 设

$$U(x,p) = C_1(x)\exp(\frac{p}{a}x) + C_2(x)\exp(-\frac{p}{a}x) \tag{31}$$

并且

$$C_1'(x)\exp(\frac{p}{a}x) + C_2'(x)\exp(-\frac{p}{a}x) = 0 \tag{32}$$

可得

$$\begin{aligned} & -a^2\left[C_1'(x)\frac{\mathrm{d}}{\mathrm{d}x}\exp(\frac{p}{a}x) + C_2'(x)\frac{\mathrm{d}}{\mathrm{d}x}\exp(-\frac{p}{a}x)\right] \\ & = p\phi(x) + \psi(x) \end{aligned} \tag{33}$$

于是

$$\exp(\frac{p}{a}x)C_1'(x) = -\frac{1}{2a}\phi(x) - \frac{1}{2ap}\psi(x) \tag{34a}$$

$$\exp(-\frac{p}{a}x)C_2'(x) = \frac{1}{2a}\phi(x) + \frac{1}{2ap}\psi(x) \tag{34b}$$

得

$$
\begin{aligned}
C_1(x) = & -\frac{1}{2a}\int^x \phi(x')\exp(-\frac{p}{a}x')\mathrm{d}x' \\
& -\frac{1}{2ap}\int^x \psi(x')\exp(-\frac{p}{a}x')\mathrm{d}x' \qquad (35a) \\
C_2(x) = & \frac{1}{2a}\int^x \phi(x')\exp(\frac{p}{a}x')\mathrm{d}x' \\
& +\frac{1}{2ap}\int^x \psi(x')\exp(\frac{p}{a}x')\mathrm{d}x' \qquad (35b)
\end{aligned}
$$

再考虑到边界条件, 应有

$$
\begin{aligned}
C_1(x) = & \frac{1}{2a}\int_x^\infty \phi(x')\exp(-\frac{p}{a}x')\mathrm{d}x' \\
& +\frac{1}{2ap}\int_x^\infty \psi(x')\exp(-\frac{p}{a}x')\mathrm{d}x' \qquad (36a) \\
C_2(x) = & \frac{1}{2a}\int_{-\infty}^x \phi(x')\exp(\frac{p}{a}x')\mathrm{d}x' \\
& +\frac{1}{2ap}\int_{-\infty}^x \psi(x')\exp(\frac{p}{a}x')\mathrm{d}x' \qquad (36b)
\end{aligned}
$$

所以

$$
\begin{aligned}
U(x,p) = & \frac{1}{2a}\int_x^\infty \phi(x')\exp[\frac{p}{a}(x-x')]\mathrm{d}x' \\
& +\frac{1}{2ap}\int_x^\infty \psi(x')\exp[\frac{p}{a}(x-x')]\mathrm{d}x' \\
& +\frac{1}{2a}\int_{-\infty}^x \phi(x')\exp[\frac{p}{a}(x'-x)]\mathrm{d}x' \\
& +\frac{1}{2ap}\int_{-\infty}^x \psi(x')\exp[\frac{p}{a}(x'-x)]\mathrm{d}x' \\
& =\frac{1}{2a}\int_{-\infty}^\infty \left[\phi(x')+\frac{1}{p}\psi(x')\right]\exp[-\frac{p}{a}|x-x'|]\mathrm{d}x' \qquad (37)
\end{aligned}
$$

再进行反演

$$\mathscr{L}^{-1}\left\{\frac{1}{p}\right\}=\eta(t)$$

利用延迟定理

$$\mathscr{L}^{-1}\left\{\frac{1}{p}\exp[-\frac{p}{a}|x-x'|]\right\}=\eta(t-\frac{|x-x'|}{a})$$

于是

$$
\begin{aligned}
& \mathscr{L}^{-1}\left\{\frac{1}{2a}\int_{-\infty}^\infty \frac{1}{p}\psi(x')\exp[-\frac{p}{a}|x-x'|]\mathrm{d}x'\right\} \\
= & \frac{1}{2a}\int_{-\infty}^\infty \eta(t-\frac{|x-x'|}{a})\psi(x')\mathrm{d}x' \\
= & \frac{1}{2a}\int_{x-at}^{x+at} \psi(x')\mathrm{d}x' \qquad (38)
\end{aligned}
$$

另外, 利用公式

$$\mathscr{L}\{f'(t)\} = pF(p) - f(0)$$

得

$$\begin{aligned}
&\mathscr{L}\left\{\frac{\partial}{\partial t}\frac{1}{2a}\int_{x-at}^{x+at}\phi(x')\mathrm{d}x'\right\} \\
=&\mathscr{L}\left\{\frac{1}{2}[\phi(x+at)+\phi(x-at)]\right\} \\
=&p\frac{1}{2a}\int_{-\infty}^{\infty}\frac{1}{p}\phi(x')\exp[-\frac{p}{a}|x-x'|]\mathrm{d}x'
\end{aligned} \tag{39}$$

最后得

$$u(x,t) = \frac{1}{2}[\phi(x+at)+\phi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(x')\mathrm{d}x' \tag{40}$$

□

18.2 Fourier 变换

Fourier 变换可对空间变量进行变换.

对于一维无界区间 $(-\infty,\infty)$ 上的函数 $f(x)$, 如果在任意有限区间上只有有限个极大极小和有限个第一类间断点, 且积分 $\displaystyle\int_{-\infty}^{\infty}|f(x)|\mathrm{d}x$ 收敛, 则函数 $f(x)$ 的 Fourier 变换存在,

$$\mathscr{F}\{f(x)\} \equiv F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x \tag{41}$$

而逆变换 (反演) 是

$$\mathscr{F}^{-1}\{F(k)\} \equiv f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k \tag{42}$$

这里的 Fourier 变换和逆变换的形式与高等数学中的形式略有不同, 上述形式更加对称, 常在物理学中采用.

下面列出 Fourier 变换的性质:

1. **有界性** $F(k)$ 在 $(-\infty,\infty)$ 上连续且
$$\lim_{k\to\pm\infty}F(k) = 0 \tag{43}$$

2. **线性性质** 设 c_1, c_2 为常数, 则
$$\mathscr{F}\{c_1f(x)+c_2g(x)\} = c_1F(k)+c_2G(k) \tag{44}$$

3. **延迟性质**
$$\mathscr{F}\{f(x-x_0)\} = \mathrm{e}^{-\mathrm{i}kx_0}F(k) \tag{45}$$

4. **频移性质**
$$\mathscr{F}\left\{\mathrm{e}^{-\mathrm{i}k_0x}f(x)\right\} = F(k+k_0) \tag{46}$$

5. **微商性质** 若 $\displaystyle\int_{-\infty}^{\infty}\left|f^{(n)}(x)\right|\mathrm{d}x$ 收敛, 则
$$\mathscr{F}\left\{f^{(n)}(x)\right\} = (\mathrm{i}k)^nF(k) \tag{47}$$

6. **积分性质**

$$\mathscr{F}\left\{\int_{-\infty}^{x} f(x')\mathrm{d}x'\right\} = \frac{1}{\mathrm{i}k}F(k) \tag{48}$$

7. **卷积定理**

$$\mathscr{L}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')g(x-x')\mathrm{d}x'\right\} = F(k)G(k) \tag{49}$$

下面用 Fourier 变换重新求解例 18.2 和例 18.3.

Example 18.4 用 *Fourier* 变换求解无界杆的热传导问题

$$\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} = f(x,t), \qquad -\infty < x < \infty, \qquad t > 0; \tag{50a}$$

$$u\big|_{t=0} = 0, \qquad -\infty < x < \infty. \tag{50b}$$

Solution 假设 $u(x,t)$ 的 Fourier 变换存在

$$U(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x \tag{51}$$

并设

$$F(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,t)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x \tag{52}$$

在作 Fourier 变换后, 定解问题变为

$$\frac{\mathrm{d}U(k,t)}{\mathrm{d}t} + \kappa k^2 U(k,t) = F(k,t) \tag{53a}$$

$$U(k,t)\big|_{t=0} = 0 \tag{53b}$$

用常数变易法求解这个一阶常微分方程的初值问题, 就得到

$$U(k,t) = \mathrm{e}^{-\kappa k^2 t}\int_0^t F(k,\tau)\mathrm{e}^{\kappa k^2\tau}\mathrm{d}\tau \tag{54}$$

首先算出

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\kappa k^2 t}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k\\
=&\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{x^2}{4\kappa t}\right]\int_{-\infty}^{\infty}\exp\left[-\kappa\left(k-\frac{\mathrm{i}x}{2\kappa t}\right)^2 t\right]\mathrm{d}k\\
=&\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{x^2}{4\kappa t}\right]\int_{-\infty}^{\infty}\exp\left[-\kappa k^2 t\right]\mathrm{d}k\\
=&\frac{1}{\sqrt{2\kappa t}}\exp\left[-\frac{x^2}{4\kappa t}\right]
\end{aligned}$$

利用 Fourier 变换的卷积公式, 最后就能得到

$$\begin{aligned}
&u(x,t)\\
=&\int_0^t\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{f(\xi,\tau)}{\sqrt{2\kappa(t-\tau)}}\exp\left[-\frac{(x-\xi)^2}{4\kappa(t-\tau)}\right]\mathrm{d}\xi\right\}\mathrm{d}\tau\\
=&\frac{1}{2\sqrt{\kappa\pi}}\int_0^t\left\{\int_{-\infty}^{\infty}f(\xi,\tau)\exp\left[-\frac{(x-\xi)^2}{4\kappa(t-\tau)}\right]\mathrm{d}\xi\right\}\frac{\mathrm{d}\tau}{\sqrt{t-\tau}}
\end{aligned} \tag{55}$$

与上节用 Laplace 变换得到的解的形式完全一样. □

Example 18.5 用 *Fourier* 变换求解无界弦的波动问题

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad -\infty < x < \infty,\ t > 0 \tag{56a}$$

$$u\Big|_{t=0} = \phi(x), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x) \qquad -\infty < x < \infty \tag{56b}$$

Solution 仍设 $u(x,t)$ 的 Fourier 变换存在

$$U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x \tag{57}$$

并设

$$\Phi(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x,t)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x \tag{58}$$

$$\Psi(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,t)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x \tag{59}$$

在作 Fourier 变换后, 定解问题变为

$$\frac{\mathrm{d}^2 U(k,t)}{\mathrm{d}t^2} + a^2k^2U(k,t) = 0 \tag{60a}$$

$$U(k,t)\Big|_{t=0} = \Phi(k), \quad \frac{\mathrm{d}U(k,t)}{\mathrm{d}t}\Big|_{t=0} = \Psi(k) \tag{60b}$$

这是一个二阶齐次常微分方程的初值问题, 解之即得

$$U(k,t) = \Phi(k)\cos kat + \Psi(k)\frac{\sin kat}{ka} \tag{61}$$

根据 Fourier 变换的反演公式, 就可以得到

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\Phi(k)\cos kat + \Psi(k)\frac{\sin kat}{ka}\right] \times \mathrm{e}^{\mathrm{i}kx}\mathrm{d}k \tag{62}$$

注意到

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(k)\cos kat\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k \\
=&\frac{1}{\sqrt{2\pi}}\frac{1}{2} \int_{-\infty}^{\infty} \Phi(k)\left[\mathrm{e}^{\mathrm{i}k(x+at)} + \mathrm{e}^{\mathrm{i}k(x-at)}\right]\mathrm{d}k \\
=&\frac{1}{2}\left[\phi(x+at) + \phi(x-at)\right]
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(k)\frac{\sin kat}{ka}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k \\
=&\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(k)\left[\int_0^t \cos ka\tau\mathrm{d}\tau\right]\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k \\
=&\int_0^t \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(k)\cos ka\tau\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k\right]\mathrm{d}\tau \\
=&\frac{1}{2}\int_0^t \left[\psi(x+a\tau) + \psi(x-a\tau)\right]\mathrm{d}\tau \\
=&\frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\mathrm{d}\xi
\end{aligned}$$

最后得到的解式与上节结果完全相同

$$u(x,t)=\frac{1}{2}\left[\phi(x+at)+\phi(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi \tag{63}$$

□

18.3 F-L 混和变换法

对于无界空间中含时间的偏微分方程的定解问题, 例如: 无界区间 x-t 两变量的定解问题, 若施行两次变换 — 对 x 作 Fourier 变换和对 t 作 Laplace 变换, 往往更为简便; 我们称这种求解方法为 F-L 混和变换法. 我们以例 18.2 为例.

Example 18.6 *用 F-L 混和变换法求解无界杆的热传导问题*

$$\frac{\partial u}{\partial t}-\kappa\frac{\partial^2 u}{\partial x^2}=f(x,t), \qquad -\infty<x<\infty, \qquad t>0; \tag{64a}$$

$$u\big|_{t=0}=0, \qquad -\infty<x<\infty. \tag{64b}$$

Solution 1. 作 Fourier 变换和 Laplace 变换

$$\mathscr{F}\left\{u(x,t)\right\}=U_F(k,t) \tag{65}$$

$$\mathscr{F}\left\{f(x,t)\right\}=F_F(k,t) \tag{66}$$

$$\mathscr{L}\left\{U_F(k,t)\right\}=U_{FL}(k,p) \tag{67}$$

$$\mathscr{L}\left\{F_F(k,t)\right\}=F_{FL}(k,p) \tag{68}$$

在作 Fourier 变换后, 定解问题变为

$$\frac{\mathrm{d}U_F(k,t)}{\mathrm{d}t}+\kappa k^2 U_F(k,t)=F_F(k,t) \tag{69a}$$

$$U_F(k,t)\big|_{t=0}=0 \tag{69b}$$

于是, 再作 Laplace 变换后, 偏微分方程变成代数方程

$$pU_{FL}(k,p)+\kappa k^2 U_{FL}(k,p)=F_{FL}(k,p) \tag{70}$$

其解为

$$U_{FL}(k,p)=\frac{1}{p+\kappa k^2}F_{FL}(k,p) \tag{71}$$

2. 求 Laplace 反演和 Fourier 反演

$$\mathscr{L}^{-1}\left\{U_{FL}(k,p)\right\}=U_F(k,t)$$

$$\mathscr{L}^{-1}\left\{F_{FL}(k,p)\right\}=F_F(k,t)$$

$$\mathscr{L}^{-1}\left\{\frac{1}{p+\kappa k^2}\right\}=\mathrm{e}^{-\kappa k^2 t}$$

利用 Laplace 变换的卷积定理, 得到

$$U_F(k,t)=\int_0^t F_F(k,\tau)\mathrm{e}^{-\kappa k^2(t-\tau)}\mathrm{d}\tau \tag{72}$$

因为

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\kappa k^2 t}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k=\frac{1}{\sqrt{2\kappa t}}\exp\left[-\frac{x^2}{4\kappa t}\right]$$

利用 Fourier 变换的卷积公式, 最后就能得到

$$\begin{aligned}
&u(x,t)\\
=&\int_0^t\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{f(\xi,\tau)}{\sqrt{2\kappa(t-\tau)}}\right.\\
&\left.\times\ \exp\left[-\frac{(x-\xi)^2}{4\kappa(t-\tau)}\right]\mathrm{d}\xi\right\}\mathrm{d}\tau\\
=&\frac{1}{2\sqrt{\kappa\pi}}\int_0^t\left\{\int_{-\infty}^{\infty}f(\xi,\tau)\right.\\
&\left.\times\ \exp\left[-\frac{(x-\xi)^2}{4\kappa(t-\tau)}\right]\mathrm{d}\xi\right\}\frac{\mathrm{d}\tau}{\sqrt{t-\tau}}
\end{aligned}\tag{73}$$

解的形式仍然不变. □

18.4 关于积分变换的一般讨论

可以把积分变换统一写成

$$F(k)=\int_a^b K(k,x)f(x)\mathrm{d}x\tag{74}$$

其中 $K(k,x)$ 是积分变换的核.

- Laplace 变换: $K(k,x)=\mathrm{e}^{-kx}$
- Fourier 变换: $K(k,x)=\mathrm{e}^{-\mathrm{i}kx}$
- 正弦变换: $K(k,x)=\sin kx$
- 余弦变换: $K(k,x)=\cos kx$
- Hankel 变换: $K(k,x)=xJ_n(kx)$
- Mellin 变换: $K(k,x)=x^{k-1}$

就一个具体的偏微分方程定解问题而言, 应当选用一种适合的积分变换.

例如, 对于偏微分方程

$$\big[\boldsymbol{L}_1(x)+\boldsymbol{L}_2(y)\big]u(x,y)=f(x,y),$$

其中 $\boldsymbol{L}_1(x)$ 和 $\boldsymbol{L}_2(y)$ 分别是关于 x 和 y的微分算符. 假定算符 $\boldsymbol{L}_1(x)$ 中的系数都是 x 的实函数, 再设 $u(x,y)$ 也是实函数, 则在积分变换

$$\int_a^b K(k,x)u(x,y)\mathrm{d}x=U(k,y)\tag{75}$$

之下，方程变为

$$\begin{aligned}\int_a^b K(k,x)f(x,y)\mathrm{d}x=&\int_a^b K(k,x)\big[\boldsymbol{L}_1(x)u(x,y)\big]\mathrm{d}x\\
&+\boldsymbol{L}_2(y)\int_a^b K(k,x)u(x,y)\mathrm{d}x\end{aligned}$$

分部积分后, 可改写为

$$\int_a^b \big[\boldsymbol{M}_1(x)K(k,x)\big]u(x,y)\mathrm{d}x + \boldsymbol{L}_2(y)U(k,y) = F(k,y), \tag{76}$$

其中 $\boldsymbol{M}_1(x)$ 称为算符 $\boldsymbol{L}_1(x)$ 的伴算符. 为了保证方程(76)是关于 $U(k,y)$ 的微分方程, 可以要求

$$\boldsymbol{M}_1(x)K(k,x) = \lambda K(k,x), \tag{77}$$

即 $\boldsymbol{M}_1(x)K(k,x)$ 仍然回到 $K(k,x)$. 这就限定了所能选择的变换核 $K(k,x)$. 例如, 在柱坐标系中求解 Laplace 方程或 Poisson 方程时, 对于变量 r, 就应该选择 Hankel 变换.

Hankel 变换

变换式

$$U(p) = \int_0^\infty u(r)J_\nu(pr)r\mathrm{d}r$$

反演式

$$u(r) = \int_0^\infty U(p)J_\nu(pr)p\mathrm{d}p$$

Hankel 变换与 Fourier 变换

考虑二维 Fourier 变换

$$F(\boldsymbol{p}) = \frac{1}{2\pi}\int f(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\mathrm{d}^2\boldsymbol{r},$$
$$f(\boldsymbol{r}) = \frac{1}{2\pi}\int F(\boldsymbol{p})\mathrm{e}^{+\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\mathrm{d}^2\boldsymbol{p}.$$

变换到极坐标系

$$F(p,\phi) = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} f(r,\theta)\mathrm{e}^{-\mathrm{i}pr\cos(\theta-\phi)}r\mathrm{d}r\mathrm{d}\theta,$$
$$f(r,\theta) = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} F(p,\phi)\mathrm{e}^{+\mathrm{i}pr\cos(\theta-\phi)}p\mathrm{d}p\mathrm{d}\phi.$$

对角度展开为 Fourier 级数

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r)\mathrm{e}^{\mathrm{i}n\theta},$$
$$F(p,\phi) = \sum_{n=-\infty}^{\infty} F_n(p)\mathrm{e}^{\mathrm{i}n\phi},$$

其中

$$f_n(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta,$$
$$F_n(p) = \frac{1}{2\pi}\int_0^{2\pi} F(p,\phi)\mathrm{e}^{-\mathrm{i}n\phi}\mathrm{d}\phi.$$

而

$$\mathrm{e}^{\mathrm{i}pr\cos(\theta-\phi)} = \sum_{n=-\infty}^{\infty} J_n(pr)\mathrm{i}^n\mathrm{e}^{\mathrm{i}n(\theta-\phi)}$$

得

$$F_n(p) = (-\mathrm{i})^n \int_0^\infty f_n(r) J_n(pr) r \mathrm{d}r$$

和

$$f_n(r) = \mathrm{i}^n \int_0^\infty F_n(p) J_n(pr) p \mathrm{d}p$$

下面就举一个应用 Hankel 变换的例子.

Example 18.7 (应用积分变换求解带电导体圆盘的静电势) 采用柱坐标系, 定解问题是

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = 0, \qquad 0 < r < \infty,\ z > 0; \tag{78a}$$

$$u\big|_{r=0}\text{有界}, \qquad u\big|_{r\to\infty} \to 0; \tag{78b}$$

$$u\big|_{z=0} = u_0, \qquad r < a; \tag{78c}$$

$$\frac{\partial u}{\partial z}\bigg|_{z=0} = 0, \qquad r > a; \tag{78d}$$

$$u\big|_{z\to\infty} \to 0. \tag{78e}$$

Solution 作 Hankel 变换, 令

$$U(p,\, z) = \int_0^\infty u(r,\, z) J_0(pr) r \mathrm{d}r. \tag{79}$$

容易证明, 在边界条件(78b)之下,

$$\int_0^\infty \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) J_0(pr)\, r\, \mathrm{d}r = -p^2 U(p,\, z). \tag{80}$$

所以, 方程(78a)和边界条件(78b)就变换为

$$\frac{\mathrm{d}^2 U(p,\, z)}{\mathrm{d}z^2} - p^2 U(p,\, z) = 0, \tag{81a}$$

同样, 边界条件(78e)就变换为

$$U(p,\, z)\Big|_{z\to\infty} \to 0. \tag{81b}$$

解之即得

$$U(p,\, z) = A(p)\mathrm{e}^{-pz}.$$

现在的问题是, 难以将平面 $z = 0$ 上的边界条件(78c)和(78d)也代入 Hankel 变换, 因为这一组边界条件给出的是 $0 \le r < a$ 时的 $u(r,\, z)\big|_{z=0}$ 值和 $r > a$ 时的 $\partial u/\partial z\big|_{z=0}$ 值. 我们先求反演, 而得到了定解问题的积分形式的解

$$u(r,\, z) = \int_0^\infty A(p)\mathrm{e}^{-pz} J_0(pr) p \mathrm{d}p. \tag{82}$$

然后再设法定出函数 $A(p)$. 为此, 将上式代入边界条件(78c)和(78d), 可以得到一对方程

$$\int_0^\infty A(p)\, J_0(pr)\, p\, \mathrm{d}p = u_0, \quad 0 < r < a;$$

$$\int_0^\infty A(p)\, J_0(pr)\, p^2\, \mathrm{d}p = 0, \quad r > a.$$

这是包含 Bessel 函数的**对偶积分方程** — 未知函数在区间 $(0, \infty)$ 的一部分上满足一个积分方程, 而在区间的其他部分上满足另一积分方程.

□

一个有用的积分公式

设

$$f(x)=x^{\nu}(a^2-x^2)^{\mu}\eta(a-x),\quad \mu>-1.$$

做 Hankel 变换, 用 $J_\nu(kx)$ 展开

$$\begin{aligned}F_\nu(k)&=\int_0^a x^{\nu+1}(a^2-x^2)^{\mu}J_\nu(kx)\mathrm{d}x\\&=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(\nu+m+1)}\left(\frac{k}{2}\right)^{\nu+2m}\\&\qquad\times\int_0^a x^{2\nu+2m+1}(a^2-x^2)^{\mu}\mathrm{d}x\end{aligned}$$

上式中的积分可用 B 函数计算, 得

$$\begin{aligned}F_\nu(k)=&\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(\mu+1)a^{2\mu+2\nu+2m+2}}{2\,m!\Gamma(\mu+\nu+m+2)}\left(\frac{k}{2}\right)^{\nu+2m}\\=&2^{\mu}a^{\mu+\nu+1}k^{-\mu-1}\Gamma(\mu+1)J_{\nu+\mu+1}(ak)\end{aligned}$$

求反演! 得

$$\begin{aligned}f(x)=&\int_0^{\infty}F_\nu(k)J_\nu(xk)k\mathrm{d}k\\=&2^{\mu}a^{\mu+\nu+1}\Gamma(\mu+1)\int_0^{\infty}J_\nu(xk)J_{\nu+\mu+1}(ak)k^{-\mu}\mathrm{d}k.\end{aligned}$$

将 $\mu+\nu+1$ 替换为 μ, x 换为 b, k 换为 t, 得

$$\begin{aligned}&\int_0^{\infty}t^{1-\mu+\nu}J_\mu(at)J_\nu(bt)\mathrm{d}t\\&\qquad=\frac{b^{\nu}(a^2-b^2)^{\mu-\nu-1}}{2^{\mu-\nu-1}a^{\mu}\Gamma(\mu-\nu)}\eta(a-b),\\&\qquad\qquad\mu>\nu.\end{aligned}\tag{83}$$

利用以上 Bessel 函数的积分公式, 求解对偶积分方程

$$\int_0^{\infty}f(t)\,J_\nu(xt)\,\mathrm{d}t=x^{-\nu}M(x),\quad 0<x<1,\tag{84}$$

$$\int_0^{\infty}f(t)\,J_\nu(xt)\,t\,\mathrm{d}t=0,\qquad x>1\tag{85}$$

设

$$f(t)=\sqrt{\frac{2t}{\pi}}\int_0^1 x^{\nu+1/2}\xi(x)\,J_{\nu-1/2}(xt)\,\mathrm{d}x$$

通过分布积分, $f(t)$ 又可表示为

$$\begin{aligned}f(t)=&\sqrt{\frac{2}{\pi t}}\Big[\xi(1)J_{\nu+1/2}(t)\\&\qquad-\int_0^1 x^{\nu+1/2}\xi'(x)\,J_{\nu+1/2}(xt)\,\mathrm{d}x\Big]\end{aligned}$$

当 $x>1$ 而 $y\le 1$ 时

$$\int_0^\infty t^{1/2}J_\nu(xt)J_{\nu+1/2}(yt)\mathrm{d}t=0,$$

所以, 方程(85)自然满足.

代入方程(84), 得

$$\frac{2}{\pi}\int_0^x y^{2\nu}(x^2-y^2)^{-1/2}\xi(y)\mathrm{d}y=M(x)$$

利用 Laplace 变换及其卷积定理解之. 作变换

$$x^2=t,\qquad y^2=u,$$

上式变换为

$$\frac{1}{\pi}\int_0^t u^{\nu-1/2}(t-u)^{-1/2}\xi(u)\mathrm{d}u=M(t).$$

设

$$\begin{aligned}Q(u)=&u^{\nu-1/2}\xi(u),\\ \mathscr{L}\{Q\}=&q,\\ \mathscr{L}\{M\}=&m.\end{aligned}$$

计算, 得

$$\begin{aligned}\mathscr{L}\left\{t^{-1/2}\right\}=&\int_0^\infty t^{-1/2}\mathrm{e}^{-pt}\mathrm{d}t\\ =&\frac{1}{\sqrt{p}}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{\sqrt{p}}.\end{aligned}$$

作 Laplace 变换, 得

$$\frac{1}{\pi}\times\frac{\sqrt{\pi}}{\sqrt{p}}\times q(p)=m(p).$$

于是

$$q(p)=\sqrt{\pi}\sqrt{p}\times m(p).$$

改写成

$$\begin{aligned}q(p)=&p\mathscr{L}\left\{t^{-1/2}\right\}\mathscr{L}\{M(t)\}\\ =&\mathscr{L}\left\{t^{-1/2}\right\}\left[\mathscr{L}\{M'(t)\}+M(0)\right].\end{aligned}$$

再次利用卷积定理求反演, 得

$$Q(t)=M(0)t^{-1/2}+\int_0^t(t-u)^{-1/2}M'(u)du.$$

最后, 得

$$\xi(x)=x^{-2\nu}M(0)+x^{1-2\nu}\int_0^x\left(x^2-y^2\right)^{-1/2}M'(y)\,\mathrm{d}y.$$

当 $\nu = 0$ 时, 有

$$
\begin{aligned}
f(t) &= \frac{2}{\pi}\int_0^1 \xi(x)\,\cos xt\,\mathrm{d}x,\\
\xi(x) &= M(0) + x\int_0^x \left(x^2-y^2\right)^{-1/2} M'(y)\,\mathrm{d}y.
\end{aligned}
$$

再取 $f(p) = pA(p)$, $M(p) = u_0$

$$
\begin{aligned}
\xi(p) =& u_0,\\
pA(p) =& \frac{2u_0}{\pi}\int_0^a \cos xp\mathrm{d}x = \frac{2u_0 \sin ap}{\pi p}.
\end{aligned}
$$

即

$$
A(p) = \frac{2u_0 \sin ap}{\pi p^2}.
$$

代入解式, 并算出积分, 就可以得到带电导体圆盘的静电势

$$
\begin{aligned}
u(r,\,z) &= \frac{2u_0}{\pi}\int_0^\infty \mathrm{e}^{-pz} J_0(pr)\frac{\sin ap}{p}\,\mathrm{d}p\\
&= \frac{2u_0}{\pi}\arcsin\frac{2a}{\sqrt{z^2+(r+a)^2}+\sqrt{z^2+(r-a)^2}}.
\end{aligned}
\tag{86}
$$

以上讨论的都是无界或半无界区间上的积分变换, 它们的共同特点是变量 k 的取值也是连续的. 对于有界区间上的积分变换, 则 k 只能取离散值. 可以把变换核记为 $K_n(x)$. 例如, 常见的有界区间上的积分变换有

- 有限正弦变换: $K_n(x) = \sin nx$
- 有限余弦变换: $K_n(x) = \cos nx$
- Legendre 变换: $K_n(x) = P_n(x)$

可以看出, 这些积分变换的核恰好就是定义在各自区间上的本征函数.