ECE 457- Microwave Devices and Circuits

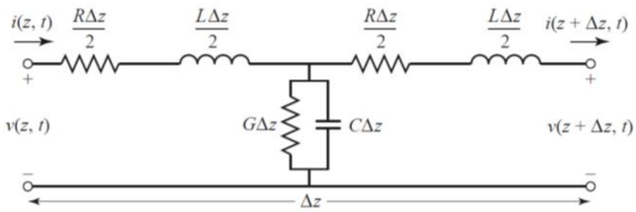
Spring 2024

Homework #2 - Transmission lines

Due: Friday Feb 16th 5pm

1. Pozar, 2.7.

Show that the T-model of a transmission line shown in the accompanying figure also yields the telegrapher equations derived in Section 2.1.



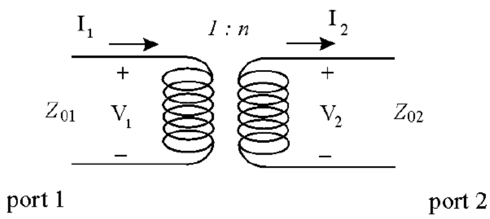
2. A lossless transmission line of characteristic impedance $Z_0 = 100\,\Omega$ is terminated with a load impedance of $Z_L = 110 + j70\,\Omega$. The transmission line is 0.692λ long.

(a) Determine the reactance which, if connected across the input terminals to the line, makes the total impedance purely resistive.

(b) Under this condition, what is the input impedance of the line?

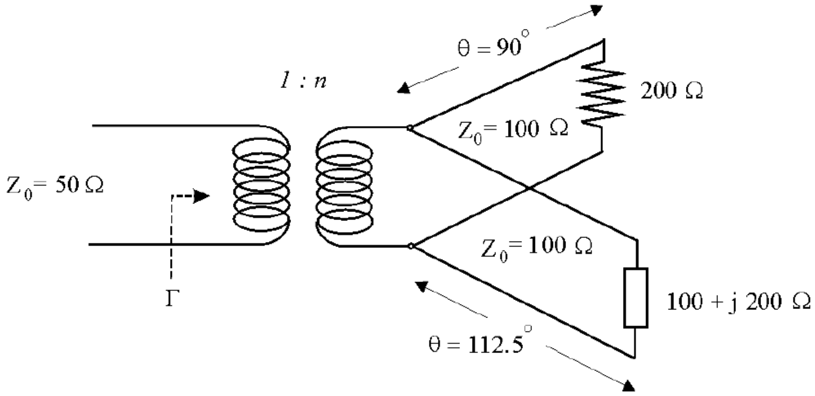
(c) Suggest a method for realizing this required reactance using only transmission lines.

3. Consider an ideal transformer with a turns ratio of $1 : n$ connecting two transmission lines, Z_{01} and Z_{02}, as shown below:

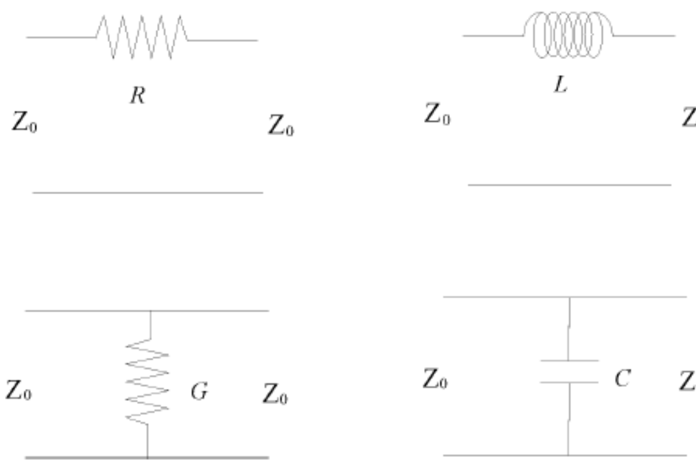


(a) Find the reflection and transmission coefficient of the transformer for a wave incident from the left (on transmission line Z_{01}). Express your answer in terms of the normalized impedance $z = Z_{02}/Z_{01}$ and the turns ratio, n.

(b) Consider the circuit shown below. Find the turns ratio for the transformer that *minimizes* the magnitude of the reflection coefficient, Γ.

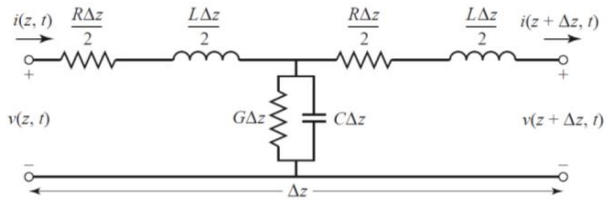


4. Consider the sections of transmission line with four different lumped elements shown below.
   (a) Find the reflection coefficient at each lumped element for a wave incident from the left (assume the line is infinitely long in both directions).
   (b) Find simplified expressions for Γ that apply when R, G, L, and C are small (but non-zero).
   (c) If the incident waveform is a pulse, sketch what the reflected pulse would look like in each case.



1. Pozar, 2.7.

Show that the T-model of a transmission line shown in the accompanying figure also yields the telegrapher equations derived in Section 2.1.



KVL: $v(z,t) - i(z,t)\frac{R\Delta z}{2} - \frac{L\Delta z}{2}\frac{di(z,t)}{dt} - i(z+\Delta z,t)\frac{R\Delta z}{2} - \frac{L\Delta z}{2}\frac{di(z+\Delta z,t)}{dt}$
$- v(z+\Delta z,t) = 0$

voltage through inductor: $V_{ind} = L\frac{di}{dt}$

$$\underbrace{\lim_{\Delta z\to 0}\frac{v(z+\Delta z,t)-v(z,t)}{\Delta z}} \overset{\lim_{\Delta z\to 0}}{=} \frac{-i(z,t)\frac{R\cancel{\Delta z}}{2} - \frac{L\cancel{\Delta z}}{2}\frac{di(z,t)}{dt} - i(z+\Delta z,t)\frac{R\cancel{\Delta z}}{2} - \frac{L\cancel{\Delta z}}{2}\frac{di(z+\Delta z,t)}{dt}}{\cancel{\Delta z}}$$

$$= \frac{\partial v(z,t)}{\partial t} = -i(z,t)\frac{R}{2} - \frac{L}{2}\frac{di(z,t)}{dt} - i(z,t)\frac{R}{2} - \frac{L}{2}\frac{di(z,t)}{dt}$$

$$\boxed{\frac{\partial v(z,t)}{\partial t} = -R\,i(z,t) - L\frac{di(z,t)}{dt}}$$

KCL:

$i_1 = i(z,t)$ $\quad i_2 = i(z+\Delta z,t)$

i_3

Current across capacitor: $i = C\frac{dv}{dt}$

$$i(z,t) = i_3 + i_2(z+\Delta z,t) \quad \text{where } i_3 = V_3Y_3 = \left(v(z,t) - i(z,t)\frac{R\Delta z}{2} - \frac{L\Delta z}{2}\frac{di(z,t)}{dt}\right)(\Delta z\, G)$$
$$+ C\Delta z\frac{\partial}{\partial t}\left(v(z,t) - i(z,t)\frac{R\Delta z}{2} - \frac{L\Delta z}{2}\frac{di(z,t)}{dt}\right)$$

$$i_2(z+\Delta z,t) - i(z,t) = -i_3$$

$$\underbrace{\lim_{\Delta z\to 0}\frac{i_2(z+\Delta z,t)-i(z,t)}{\Delta z}} = \lim_{\Delta z\to 0}\frac{-i_3}{\Delta z}$$

$$\lim_{\Delta z\to 0}\frac{-i_3}{\Delta z} = -\Big[\Big(v(z,t) - \cancel{i(z,t)\frac{R\Delta z}{2}}^{0} - \cancel{\frac{L\Delta z}{2}\frac{di(z,t)}{dt}}^{0}\Big)G$$
$$+ C\frac{\partial}{\partial t}\Big(v(z,t) - \cancel{i(z,t)\frac{R\Delta z}{2}}^{0} - \cancel{\frac{L\Delta z}{2}\frac{di(z,t)}{dt}}^{0}\Big)\Big]$$

$$= -G\,v(z,t) - C\frac{\partial}{\partial t}v(z,t)$$

$$\frac{\partial i(z,t)}{\partial z} = \lim_{\Delta z\to 0} -\frac{i_3}{\Delta z}$$

$$\boxed{\frac{\partial i(z,t)}{dt} = -G\,v(z,t) - C\frac{\partial v(z,t)}{dt}}$$

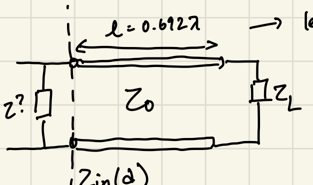
2. A lossless transmission line of characteristic impedance $Z_0 = 100\,\Omega$ is terminated with a load impedance of $Z_L = 110 + j70\,\Omega$. The transmission line is 0.692λ long.

(a) Determine the reactance which, if connected across the input terminals to the line, makes the total impedance purely resistive. $z_L = 1.1 + 0.7j$

(b) Under this condition, what is the input impedance of the line?

(c) Suggest a method for realizing this required reactance using only transmission lines.

2a)



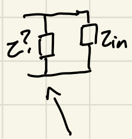
→ length of 0.692λ is equivalent to length of $0.692\lambda - 0.5\lambda = 0.192\lambda$.

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{110 + j70 - 100}{110 + j70 + 100} = \frac{10 + j70}{210 + j70} = \frac{1 + j7}{21 + j7} \Rightarrow 0.33e^{j1.234}$$

$$Z_{in}(d) = Z_0 \frac{1 + \Gamma_0 e^{-j2\beta d}}{1 - \Gamma_0 e^{-j2\beta d}} \quad \text{where } \beta d = \frac{2\pi}{\lambda} \cdot 0.192\lambda = 0.384\pi$$

$$Z_{in}(d) \approx 96.08 - 65.969j \qquad Y_{in} = 0.7 + 0.48j$$

⇒ the line can be simplified to this



what is z? such that z? ∥ Zin is purely resistive

parallel impedances can be viewed as inverse of the sum of admittances

$$\frac{1}{Z_T} = \frac{1}{Z_?} + \frac{1}{Z_{in}}$$

$$\frac{1}{Z_T} = \frac{1}{jX} + \frac{1}{Z_{in}}$$

$$\therefore Z_T = R = \frac{Z_{in} jX}{jX + Z_{in}} \quad \begin{matrix} \rightarrow \text{complex} \\ \text{complex} \end{matrix}$$

$$Z_{in} jX = X(96.08j + 65.969)$$

$$jX + Z_{in} = 96.08 - 65.969j + jX$$
$$96.08 + (x - 65.969)j$$

$$\therefore Z_T = \frac{(96.08Xj + 65.969X)(96.08 - (X - 65.969)j)}{(96.08)^2 + (x - 65.969)^2}$$

$$= 96.08^2 Xj + 96.08X(X - 65.969) + 65.969X(96.08) - 65.969X(X - 65.969)j$$

in order for Z_T to be real, then imaginary components have to sum to 0.

$$96.08^2 Xj - 65.969 X^2 j + 65.969^2 X j = 0$$

$$X(96.08^2 - 65.969X + 65.969^2) = 0$$

$$X = 0 \qquad 96.08^2 + 65.969^2 = 65.969X$$

trivial case, since $Z_T = 0$

$$X = \frac{96.08^2 + 65.969^2}{65.969}$$

$$X \approx 205.906 \Omega$$

$\boxed{\text{Reactance of around } 205.906 j\Omega \text{ needed}}$

b) Using a parallel calculation:

$$Z_{in,total} = \frac{Z_{in,line} \cdot jX}{jX + Z_{in,line}} \approx \boxed{141.375\Omega}$$

c) This reactance can be realized with an open stub transmission line in parallel.

$$Z_{in,open} = -jZ_0 \cot \beta d$$
$$= -j(100)\cot\left(\frac{2\pi}{\cancel{\lambda}}(\ell' \cancel{\lambda})\right)$$

$$Z_{in,open} = -\cancel{j}100 \cot(2\pi \ell') = 205.906\cancel{j}$$

$$\cot(2\pi\ell') = \frac{-205.906}{100}$$

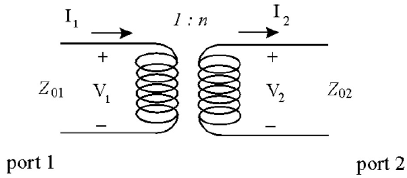
$$2\pi\ell' = \arctan\left(\frac{-100}{205.906}\right)$$

$$\ell' = \frac{1}{2\pi}\left(\arctan\left(\frac{-100}{205.906}\right) + \pi\right)$$

$$\ell' \approx 0.428\lambda$$

$\therefore$ an open stub t-line with a length of 0.428λ can be used to realize the reactance found in part a.

3. Consider an ideal transformer with a turns ratio of $1 : n$ connecting two transmission lines, Z_{01} and Z_{02}, as shown below:



(a) Find the reflection and transmission coefficient of the transformer for a wave incident from the left (on transmission line Z_{01}). Express your answer in terms of the normalized impedance $z = Z_{02}/Z_{01}$ and the turns ratio, n.

a) Ideal transformer:

$$\frac{V_1}{n_1} = \frac{V_2}{n_2} \qquad \text{where } \frac{n_1}{n_2} = \frac{1}{n} \qquad \therefore n_2 = n \cdot n_1$$

$$\therefore V_2 = V_1 \frac{n_2}{n_1} \rightarrow V_2 = V_1 \frac{n \cdot \cancel{n_1}}{\cancel{n_1}} \rightarrow V_2 = n \cdot V_1$$

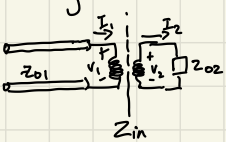
$$P_{left} = P_{right}$$

$$V_1 i_1 = V_2 i_2$$

$$\cancel{V_1} i_1 = n \cancel{V_1} i_2$$

$$i_1 = n i_2$$

Assuming that line 2 is infinite, we can draw the line as



$$i_1 = n i_2$$

$$Z_{in} = \frac{V_1}{I_1} = \frac{V_2}{n} \cdot \frac{1}{n i_2} = \frac{V_2}{i_2} \cdot \frac{1}{n^2}$$

$$Z_{in} = Z_{02} \cdot \frac{1}{n^2}$$

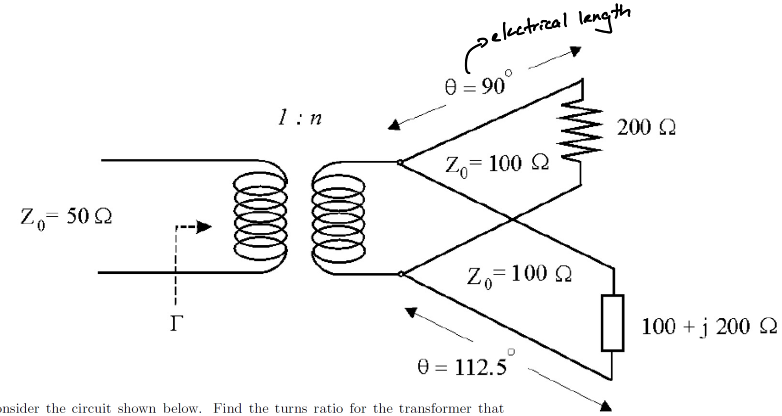
$$Z_{in} = \frac{Z_{02}}{n^2}$$

$$\rightarrow \Gamma = \frac{Z_{in} - Z_{01}}{Z_{in} + Z_{01}} \qquad \text{where } Z_{in} = \frac{Z_{02}}{n^2}$$

$$= \frac{\frac{Z_{02}}{n^2} - Z_{01}}{\frac{Z_{02}}{n^2} + Z_{01}} \cdot \frac{\frac{1}{Z_{01}}}{\frac{1}{Z_{01}}}$$

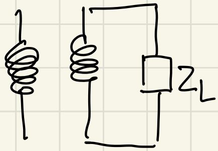
$$= \frac{\frac{z}{n^2} - 1}{\frac{z}{n^2} + 1}$$

$$\boxed{\Gamma = \frac{z - n^2}{z + n^2}} \qquad \tau = 1 + \Gamma \quad \therefore \boxed{\tau = \frac{2z}{z + n^2}}$$



(b) Consider the circuit shown below. Find the turns ratio for the transformer that *minimizes* the magnitude of the reflection coefficient, Γ.

b) first we must determine the load impedance



the line terminated w/ $200\,\Omega$

has a $Z_{in,1} = Z_0 \dfrac{1+\Gamma_1 e^{-j2\beta d}}{1-\Gamma_1 e^{-j2\beta d}}$ where $\Gamma_1 = \dfrac{Z_{L,1}-100}{Z_{L,1}+100} = \dfrac{100}{300} = \dfrac{1}{3}$

$$Z_{in,1} = 100\left(\frac{1+\frac{1}{3}e^{-j2(90^\circ)}}{1-\frac{1}{3}e^{-j2(90^\circ)}}\right)$$

$$Z_{in,1} = 100\left(\frac{1+\frac{1}{3}e^{-j(180^\circ)}}{1-\frac{1}{3}e^{-j180^\circ}}\right)$$

$$= 100\left(\frac{\frac{2}{3}}{\frac{4}{3}}\right) = 100\left(\frac{1}{2}\right) = 50\,\Omega$$

$Z_{in,2} = Z_0 \dfrac{1+\Gamma_2 e^{-j2\beta d}}{1-\Gamma_2 e^{-j2\beta d}}$ where $\Gamma_2 = \dfrac{100+200j-100}{100+200j+100}$

where $2\cdot\beta d = 2\cdot 112.5^\circ = 225^\circ$ $= \dfrac{200j}{200+200j}$

$$\Gamma_2 = \frac{j(1-j)}{2}$$

$$\Gamma_2 = \frac{j+1}{2}$$

$$Z_{in,2} = Z_0 \frac{1+\Gamma_2 e^{-j225^\circ}}{1-\Gamma_2 e^{-j225^\circ}}$$

$$Z_{in,2} = 100\,\frac{1+\sqrt{2}e^{j45^\circ}e^{-j225^\circ}}{1-\sqrt{2}e^{j45^\circ}e^{-j225^\circ}}$$

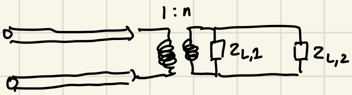
$$= 100\,\frac{1+\frac{\sqrt{2}}{2}e^{-j180^\circ}}{1-\frac{\sqrt{2}}{2}e^{-j180^\circ}}$$

$$= 100\,\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}$$

$$= 100\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)$$

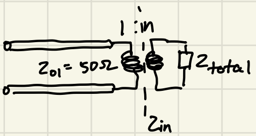
$$\approx 17.157\,\Omega$$

now the lines can be drawn as:



$$\therefore Z_L \text{ total} = \frac{Z_{L,1} \cdot Z_{L,2}}{Z_{L,1} + Z_{L,2}} = \frac{50 \cdot 17.157}{50 + 17.157} \approx 12.774\,\Omega$$

now the line is drawn as:



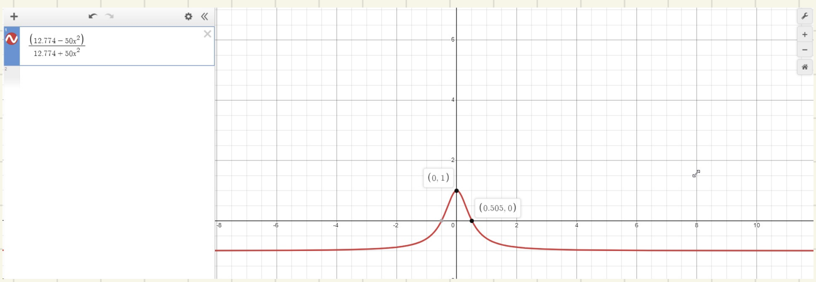
we know from part a) $Z_{in} = \frac{Z_{02}}{n^2}$ where $Z_{02} = Z_{total}$

$$\therefore Z_{in} = \frac{Z_{total}}{n^2}$$

$$\therefore Z_{in} = \frac{12.774}{n^2}$$

$$\Gamma = \frac{Z_{in} - Z_{01}}{Z_{in} + Z_{01}} = \frac{\frac{12.774}{n^2} - 50}{\frac{12.774}{n^2} + 50}$$

$$\Gamma = \frac{12.774 - 50n^2}{12.774 + 50n^2}$$ find n that minimizes this



we can see that Γ is minimized (to 0) when $n \approx \frac{1}{2}$ $\therefore$ coil 1 should have twice as many turns as coil 2.

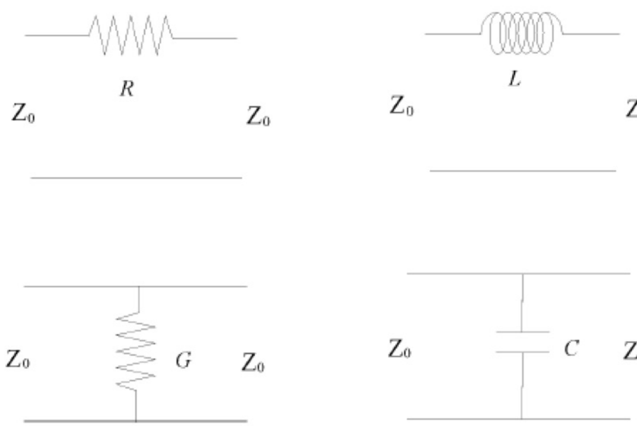
i.e. a $\boxed{2:1 \text{ ratio}}$

4. Consider the sections of transmission line with four different lumped elements shown below.

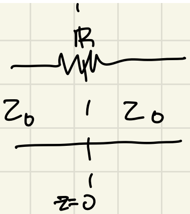
(a) Find the reflection coefficient at each lumped element for a wave incident from the left (assume the line is infinitely long in both directions).

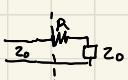
(b) Find simplified expressions for Γ that apply when R, G, L, and C are small (but non-zero).

(c) If the incident waveform is a pulse, sketch what the reflected pulse would look like in each case.



a)

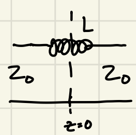


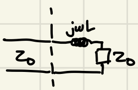
∴ line is 

where $Z_{total} = R + Z_0$

$$\therefore \Gamma = \frac{R+Z_0-Z_0}{R+Z_0+Z_0} = \frac{R}{R+2Z_0}$$

$$\boxed{\Gamma = \frac{R}{R+2Z_0}}$$

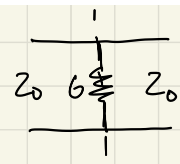


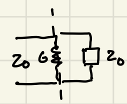
∴ line is: 

where $Z_{total} = j\omega L + Z_0$

$$\therefore \Gamma = \frac{j\omega L + Z_0 - Z_0}{j\omega L + Z_0 + Z_0}$$

$$\boxed{\Gamma = \frac{j\omega L}{j\omega L + 2Z_0}}$$



∴ line is: 

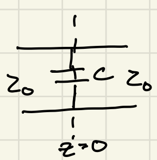
$$\therefore Z_{tot} = \frac{\frac{Z_0}{G}}{\frac{1}{G} + Z_0} = \frac{Z_0}{1+GZ_0}$$

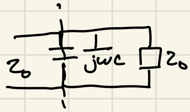
$$\Gamma = \frac{\frac{Z_0}{1+GZ_0} - Z_0}{\frac{Z_0}{1+GZ_0} + Z_0} \cdot \frac{1+GZ_0}{1+GZ_0}$$

$$\Gamma = \frac{Z_0 - Z_0 - GZ_0^2}{Z_0 + Z_0 + GZ_0^2}$$

$$\Gamma = \frac{1-1-GZ_0}{1+1+GZ_0}$$

$$\boxed{\Gamma = \frac{-GZ_0}{2+GZ_0}}$$



→ line is: 

$$\therefore Z_{total} = \frac{\frac{Z_0}{j\omega C}}{\frac{1}{j\omega C} + Z_0} = \frac{Z_0}{1+j\omega CZ_0}$$

$$\Gamma = \frac{\frac{Z_0}{1+j\omega CZ_0} - Z_0}{\frac{Z_0}{1+j\omega CZ_0} + Z_0} = \frac{1-1-j\omega CZ_0}{1+1+j\omega CZ_0} = \boxed{\frac{-j\omega CZ_0}{2+j\omega CZ_0}}$$

b) for $\Gamma = \frac{R}{R+2Z_0}$ as $R \to 0$, Γ will also approach 0 since it will be a very small # divided by $2Z_0$

$\therefore \boxed{\Gamma_R \approx 0^+}$

for $\Gamma_L = \frac{j\omega L}{j\omega L + 2Z_0}$ as $L \to 0$, Γ will also approach 0 since it will be a small number divided by $2Z_0$.

it will also advance in phase by $90°$

$\therefore \boxed{\Gamma_L \approx 0^+}$

for $\Gamma_G = \frac{-GZ_0}{2+GZ_0}$ as $G \to 0$, Γ will approach 0 since a very small # divided by 2 will also be small, however it will be a negative number since it will approach 0 from negative side.

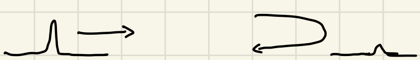
$\boxed{\Gamma_G \approx 0^-}$

for $\Gamma_C = \frac{-j\omega CZ_0}{2+j\omega CZ_0}$ as $C \to 0$, Γ will approach 0 since a very small # divided by 2 will also be small, just like Γ_G, it will be negative.

return wave also lagged in phase $90°$.

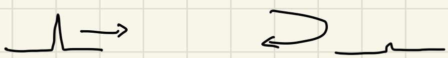
$\boxed{\Gamma_C \approx 0^-}$

c) The pulse for Γ_R will look like:



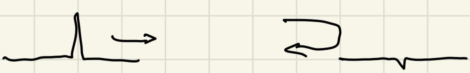
1) amplitude smaller

The pulse for Γ_L will look like



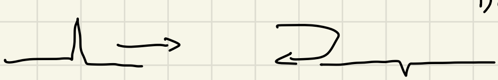
1) amplitude smaller

The pulse for Γ_G will look like



1) amplitude smaller
2) sign flip

The pulse for Γ_C will look like



1) amplitude smaller
2) Sign flip.