Beta-Binomial conjugation

pf: Note that X here is the case of a binomial likelihood

$X = \sum_{i=1}^{N} x_i$ bernoulli trials

$$P(x|\theta) = \frac{P(x|\theta)\cdot P(\theta)}{P(x)} = \frac{P(x,\theta)}{\int_0^1 P(x,\theta)\,d\theta} \rightarrow \text{joint}$$

$$P(x|\theta) = \binom{N}{x}\cdot\theta^{x}\cdot(1-\theta)^{N-x}, \text{ where } P(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\,\theta^{a-1}\cdot(1-\theta)^{b-1}$$

$$P(x,\theta) = \frac{\Gamma(a+b)\Gamma(N+1)}{\Gamma(a)\Gamma(b)\Gamma(N-x+1)\Gamma(x+1)}\,\theta^{a+x-1}\cdot(1-\theta)^{N+b-x-1}$$

~~Assume~~ that $r = \frac{\Gamma(a+b)\Gamma(N+1)}{\Gamma(a)\Gamma(b)\Gamma(N-x+1)\Gamma(x+1)}$

$$\therefore P(x,\theta) = r\cdot\theta^{x+a-1}\cdot(1-\theta)^{N+b-x-1}$$

$$= \frac{\Gamma(a+b+N)}{\Gamma(x+a)\Gamma(N+b-x)}\cdot\theta^{x+a-1}\cdot(1-\theta)^{N+b-x-1}\cdot r \times \frac{\Gamma(x+a)\Gamma(N+b-x)}{\Gamma(a+b+N)}$$

$$\text{where } \int_0^1 P(x,\theta)\,d\theta = \underbrace{\int_0^1 \frac{\Gamma(a+b+N)}{\Gamma(a+x)\Gamma(N+b-x)}\,\theta^{x+a-1}\cdot(1-\theta)^{N+b-x-1}\,d\theta}\cdot r\cdot\frac{\Gamma(x+a)\Gamma(N+b-x)}{\Gamma(a+b+N)}$$

Within this integral is a beta distribution. It's just a beta distribution of a new value of a' and b'. $\therefore$ This must integrate to one.

$$\therefore P(x,\theta) = r\cdot\theta^{x+a-1}\cdot(1-\theta)^{N+b-x-1}. \quad P(x) = r\cdot\frac{\Gamma(x+a)\Gamma(N+b-x-1)}{\Gamma(N+a+b)}$$

$$P(\theta|x) = \frac{r\cdot\theta^{x+a-1}\cdot(1-\theta)^{N+b-x-1}}{r\cdot\Gamma(x+a)\Gamma(N+b-x)}\cdot\Gamma(N+a+b)$$

$$= \frac{\Gamma(N+a+b)}{\Gamma(x+a)\Gamma(N+b-x)}\cdot\theta^{x+a-1}\cdot(1-\theta)^{N+b-x-1}$$

$$= \text{Beta}(x+a,\ N+b-x) \quad \#$$