Quant Interview Problems

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SOLUTION By dynamci programming we can derive that the minimum total time is given by $T = \text{constant} + \min\{2c, b + d\}$ and thus here evaluated at $T = 10 + 4 + 3 + 2 \times 2 = 17$.

56. PG 4.1 coin toss game (61)

   SOLUTION Consider it as a sequential problem, then there're 2 cases where A has more tops than B: A and B has the same number of tops in the first n tosses, and then A tosses a top, or A has already more tops than B in the first n tosses. Denote the probability of the condition of case 1 as x and the probability of case 2 as y, then $2y + x = 1$. What we need is merely $y + x/2 = 1/2$.

57. PG 4.1 card game (61)

   SOLUTION Let's calculate it sequentially. In total there're P_{52}^2 plays and 13 kinds of my card with each kind for 4 suits, and for each order $i \in \{1, 2, \ldots, 13\}$ we must have the dealer to take a card from $(i - 1) \cdot 4$ kinds. Hence the probability is $16 \sum_{i=1}^{13} (i - 1)/P_{52}^2 = (16 \cdot 12 \cdot 13)/(2 \cdot 52 \cdot 51) = 8/17$.

58. PG 4.1 drunk passenger (62)

   SOLUTION The probability of my seat being untaken is equal to the probability that no one's ever taken it. This can be decomposed into three parts: #1 takes his own seat, he takes my seat, or he takes any seat in between. While the first two cases are equivalent, the last case automatically lead to another drunk guy whose designated seat is #1, so it's again symmetric. As a result the probability is $1/2$.

59. PG 4.1 n points on a circle (63)

   SOLUTION Take any point $i \in 1, 2, \ldots, n$ for example and consider if all other points are on the semicircle starting from point i. The probability is easy to calculate and it's $1/2^{n-1}$. Since all points are symmetric in consideration, the resulting probability should be $N/2^{n-1}$.

60. PG 4.2 poker hands (65)

   SOLUTION There're in total C_{52}^5 cases. Four-of-a-kind: $13 \cdot (52 - 4)$; Full-house: $C_{13}^2 \cdot 2 \cdot C_4^3 \cdot C_4^2$; Two-pair: $C_{13}^2 \cdot (C_4^2)^2 \cdot (52 - 4)$. The probabilities are merely these numbers devided by the total number of cases.

61. PG 4.2 hopping rabbit (66)

   SOLUTION We solve it recursively. Let the number of ways to reach stair k be $f(k)$, then we have recursion $f(k+2) = f(k+1) + f(k)$ with initial conditions $f(0) = 1$ and $f(1) = 1$. It's, therefore, a standard Fibonacci sequence and we know $f(n) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right]$.

62. PG 4.2 screwy pirates 2 (67)

   SOLUTION In order to forbid any 5-combination of pirates to open the safe while allow exactly the rest 6 to be able to do so, the number of locks needed is given by the number of suck combinations, either 6 out of 11 or 5 out of 11, and the number is C_{11}^5. Each lock must corresponds to at least 6 keys and by symmetry the number of keys for each pirate should be $C_{11}^5 \cdot 6/11$.

63. PG 4.2 chess tournament (68)

   SOLUTION In order to have 1 to meet 2 in the final, they must be in different half-games. This means a total number of $2(2^n - 2)!(2^{n-1})^2$ cases, while in total there is $2^n!$ combinations. So the probability should be $2(2^n - 2)!(2^{n-1})^2/2^n! = 2^{n-1}/(2^n - 1)$.

64. PG 4.2 application letters (69)

   SOLUTION This is called the dearrangement number and we can solve it recursively. Denote $f(k)$ as the number of ways to put all letters in wrong envelopes. Assume that letter 1 is in some envelope $i \in \{2, 3, \ldots, k\}$, then we have two cases: letter k being in envelope 1 or not. This leads to the recursion $f(k) = k[f(k-1) + f(k-2)]$ with initial conditions $f(1) = 0$ and $f(2) = 1$. Let $f(k) := k! \cdot g(k)$ and we can derive $g(k)$.

65. PG 4.2 birthday problem (71)

   SOLUTION Let $\Pi_{i=0}^k (365 - i)/365^k < 1/2$. Enumeration tells that the smallest k is 22 and thus there

should be at least 23 people.

66. PG 4.2 100th digit (71)

$\boxed{\textbf{SOLUTION}}$ Notice that the binomial expansions of $(1+\sqrt{2})^{3000}$ and $(1-\sqrt{2})^{3000}$ have irrational, $\sqrt{2}$ to singular powers, yet with the opposite signs. This indicates that $(1+\sqrt{2})^{3000} + (1-\sqrt{2})^{3000}$ is an integer while since $|1-\sqrt{2}| < 1$, we know $0 < (1-\sqrt{2})^{3000} \ll 10^{-100}$ and the 100th digit of $(1+\sqrt{2})^{3000}$ is 9.

67.

$\boxed{\textbf{SOLUTION}}$ We only consider the last two digits. There're 100 possible cases, namely from 00 up until 99, which can be written as $10a + b$. The binomial expansion of $(10a + b)^3 = 1000a^3 + 300a^2b + 30ab^2 + b^3$ indicates the only solution, which is $b = 1$ and $a = 7$. So the probability is 1%.

68.

$\boxed{\textbf{SOLUTION}}$ Denote A as having at least one boy out of the two. Denote B as having two boys, C as having a boy and a girl, and D as having two girls. Also, we know $\text{P}[A \mid B] = 1$, $\text{P}[A \mid C] = 1$ and $\text{P}[A \mid D] = 0$. By Bayes' formula we have $\text{P}[B \mid A] = \frac{\text{P}[A|B]\cdot\text{P}[B]}{\text{P}[A|B]\cdot\text{P}[B]+\text{P}[A|C]\cdot\text{P}[C]+\text{P}[A|D]\cdot\text{P}[D]}$.

· We have $\text{P}[B] = 1/4$, $\text{P}[C] = 1/2$ and $\text{P}[D] = 1/4$. Therefore, By Bayes' formula we have $\text{P}[B \mid A] = 1/3$.

· Different from the previous problem, now define A^- as the case where the **first** child is a boy. Now, $\text{P}[A^- \mid B] = 1$, $\text{P}[A^- \mid C] = 1/2$ and $\text{P}[A^- \mid D] = 0$. So $\text{P}[B \mid A^-] = \frac{\text{P}[A^-|B]\cdot\text{P}[B]}{\text{P}[A^-|B]\cdot\text{P}[B]+\text{P}[A^-|C]\cdot\text{P}[C]+\text{P}[A^-|D]\cdot\text{P}[D]} = 1/2$.

69.

$\boxed{\textbf{SOLUTION}}$ The fraction of girls is solely given by nature, and thus is 50%.

70.

$\boxed{\textbf{SOLUTION}}$ Denote event A as the coin tosses being the unfair one, B as a consecutive sequence of 10 tosses gives all heads. We know that $\text{P}[A] = 1/1000$ and $\text{P}[\bar{A}] = 999/1000$. We also know that $\text{P}[B \mid A] = 1$ and $\text{P}[B \mid \bar{A}] = 2^{-10}$. Then by Bayes' theorem, we have $\text{P}[A \mid B] = \text{P}[B \mid A] \cdot \text{P}[A] \cdot (\text{P}[B \mid A] \cdot \text{P}[A] + \text{P}[B \mid \bar{A}] \cdot \text{P}[\bar{A}])^{-1} = 0.506$.

71.

$\boxed{\textbf{SOLUTION}}$ Toss twice and let HT be win, TH be loss, otherwise again.

72.

$\boxed{\textbf{SOLUTION}}$ This is about the probability that the last throw does not lead to the best score. Any of the three throws can be in that case so the probability is 1/3.

73.

$\boxed{\textbf{SOLUTION}}$ Say that you choose to be in position $n \geq 1$, then the probability that you get that ticket is $\text{P}[n] = \frac{365!}{365^{n-1}(365-n+1)!}\frac{n-1}{365}$. In order to have $P[n] > P[n-1]$ and $P[n] > P[n+1]$ we have $n = 20$.

74.

$\boxed{\textbf{SOLUTION}}$ The total number of events that're required is $n = C_6^3$ while the number of all possible events is $N = 6^3$. So the probability is $\text{P} = C_6^3/6^3 = 5/54$.

75.

$\boxed{\textbf{SOLUTION}}$ There're two cases.

· If I chose the door with car, which has a probability of 1/3, then switching leads to P[switch and car | car] = 0.

· If I chose the door with goat, which has a probability of 2/3, then switching leads to P[switch and car | goat] = 1.

Therefore, P[switch and win] = 2/3.

76. PG 4.3 amoeba population (79)

$\boxed{\textbf{SOLUTION}}$ Solve the fixed point of the generating function $f(x) = (1 + x + x^2 + x^3)/4$. The only reasonable solution is $x = \sqrt{2} - 1$.

77.

$\boxed{\textbf{SOLUTION}}$ If blue is the last, then the conditional probability that green is the last among all red and green candies is $\text{P}_1 = 3/4$. If green is the last, then the conditional probability that blue is the last among all red and blue candies is $\text{P}_2 = 2/3$. As a result, the probability that all reds end before blue and green is $\text{P} = \text{P}_1/3 + \text{P}_2/2 = 1/4 + 1/3 = 7/12$.

78.

$\boxed{\textbf{SOLUTION}}$ Let's say the probability is P[A]. Then $\text{P}[B] = 1 - \text{P}[A]$. We condition on A's first toss. If it's

a tail, then it's become the opposite case where B is the first one to toss, and thus $P[A \mid T] = P[B] = 1 - P[A]$. If it's a head, then we have two sub-cases: (a) if B tosses a tail, then B's won; (b) is B tosses a head, then the game starts again. So we have $P[A \mid H] = 0 + (1 - P[A \mid H])/2$. Together with these two equations we have $P[A] = P[A \mid H]/2 + P[A \mid T]/2 = 4/9$. This should be smaller than $1/2$ because A can never win in the first toss, while B can.

79. PG 4.3 russian roulette series (81)

SOLUTION We solve it one by one.

- No it does not matter. The probability is always $1/2$.
- Let the probability be p if we choose to play first, then to loss means $p = 1/6 + 5/6 \cdot (1 - p)$, so $p = 6/11$. Therefore, we should choose to play second which gives a loss probability of $5/11$.
- It's just $2/6$ and $2/5$. You definitely should spin.
- Say that the 2 bullets are in bezels 1 and 2. Then the opponent can have chosen anywhere between 3 and 6, so the probability that I can live is $3/4$ if I don't spin. In contrast, if I spin, the probability should be indifferent to what he's chosen and the living probability is $4/6 = 2/3$, which is less. So I should not spin.

80. PG 4.3 aces (82)

SOLUTION Total number of events is $N = \frac{52!}{(13!)^4}$. The total number of events where each player has an ace is $n = 4! \cdot \frac{48!}{(12!)^4}$. So the probability is $n/N = \frac{4! \cdot 48! \cdot (13!)^4}{52! \cdot (12!)^4} = \frac{13^4 \cdot 24}{52 \cdot 51 \cdot 50 \cdot 49} = 0.105$.

81. PG 4.3 gambler's ruin problem (83)

SOLUTION Let P_i be the winning probability starting from fortune i, then $P_i = P_{i-1}p + P_{i+1}q$, which together with boundary conditions $P_0 = 0$ and $P_N = 1$ gives

$$P_i = \begin{cases} \frac{1 - (q/p)^i}{1 - (q/p)^N} & \text{if } p \neq 1/2, \\ i/N & \text{if } p = 1/2. \end{cases}$$

82. PG 4.3 basketball scores (84)

SOLUTION Denote (n, k) as n throws with k scores. Then $P[(3,1)] = P[(3,2)] = 1/2$. Further, we have

- $P[(4,1)] = P[(4,1) \mid (3,1)]P[(3,1)] = 2/3 \cdot 1/2 = 1/3$;
- $P[(4,2)] = P[(4,2) \mid (3,1)]P[(3,1)] + P[(4,2) \mid (3,2)]P[(3,2)] = 1/3 \cdot 1/2 + 1/3 \cdot 1/2 = 1/3$;
- $P[(4,3)] = P[(4,3) \mid (3,2)]P[(3,2)] = 2/3 \cdot 1/2 = 1/3$.

So we guess that $P[(n,k)] \equiv (n-1)^{-1}$. By induction this is proved, and thus the answer is $1/99$.

83. PG 4.3 cars on road (85)

SOLUTION Solve $(1 - p)^4 = 1 - 609/625$ and we have $p = 3/5$.

84. PG 4.4 meeting probability (88)

SOLUTION The probability is the area of $|X - Y| \leq 5$, and thus is $P = (60^2 - 55^2)/60^2 = 23/144$.

85. PG 4.4 probability of triangle (89)

SOLUTION Assume the first cut is x and the second y. To make it possible to form a triangle, we need:

- $\max\{x, y\} > 1/2$;
- $\min\{x, y\} < 1/2$;
- $|y - x| < 1/2$.

Therefore, we can calculate the area and get the probability as $P = 1/4$.

86. PG 4.4 property of Poisson process (90)

SOLUTION Exponential waiting time is memoriless both forward and backward time: $f(t) = \lambda e^{-\lambda t}$. The expected waiting time is 10 minutes for both.

87. PG 4.4 moments of normal distribution (91)

SOLUTION We have the central moments of standard normal distribution as $\mu_1 = 0$, $\mu_2 = 1$, $\mu_3 = 0$ and $\mu_4 = 3$. More generally, we have $\mu_{2k} = (2k-1)!!$ and $\mu_{2k+1} = 0$ for all $k \in \mathbb{N}$. The k-th moment is calculated by taking the k-th derivative of $M(t) = E[e^{tX}]$.

88. PG 4.5 connecting noodles (93)

SOLUTION Assume we have n noodles, i.e. $2n$ ends, then there're in total $C_{2n}^2 = n(2n-1)$ combinations in which n makes 1 circle. So we have $E[f(n)] = E[f(n-1)] + (2n-1)^{-1}$.

89. PG 4.5 optimal hedge ratio (94)

> **SOLUTION** Assume we short Δ shares of B, then $\mathrm{Var}[A - \Delta B] = \sigma_A^2 + \Delta^2 \sigma_B^2 - 2\Delta \sigma_A \sigma_B \rho$. Take the first derivative of Δ to this equation and evaluate it to zero, then we have $\Delta \sigma_B^2 = \sigma_A \sigma_B \rho \Rightarrow \Delta = \rho \sigma_A / \sigma_B$.

90. <mark>PG 4.5 dice game (94)</mark>

> **SOLUTION** With $1/2$ probability we have expected payoff $E_1 = 2$; with the other $1/2$ probability we have $E_2 = 5 + E$. So we have $E = E_1/2 + E_2/2 = 7/2 + E/2 \Rightarrow E = 7$.

91. <mark>PG 4.5 card game (95)</mark>

> **SOLUTION** Define 5 regions devided by the 4 aces, then the probability of a card being in any region is $1/5$, and thus the expected position of the first ace is $E = 1 + 48/5 = 10.6$ (which is really smart).

92. PG 4.5 sum of random variables (96)

> **SOLUTION** This is the area in a unit hyper-cube, that is, $P = 1 \times 1 \times \frac{1}{2} \times \frac{1}{3} \times \cdots \times \frac{1}{n} = 1/n!$ (Also, we can use $P[\sum_{i=1}^{n} x_i < 1] = \int_0^1 P[\sum_{i=1}^{n-1} x_i < 1 - x_n] dx_n = P[\sum_{i=1}^{n-1} x_i < 1]/n = \cdots = 1/n!$).

93. <mark>PG 4.5 coupon collection (97)</mark>

> **SOLUTION** We solve part A and B separately.
>
> · Part A. Assume the expected number of tries is $E[X_n]$ to get the n-th kind of coupon in a sequence, then, since $E[X_n] = \frac{N}{N-n+1}$ and the total number of tries is merely the sum, we have $E[X] = \sum_{n=1}^{N} E[X_i] = \sum_{n=1}^{N} \frac{N}{N-n+1} = N(1^{-1} + 2^{-1} + \cdot + N^{-1})$.
>
> · Part B. Define inticator variable I_i as 1 if at least 1 coupon is of type 1 in the n-coupon set and 0 if there's not. Then we know $P[I_i = 0] = (\frac{N-1}{N})^n$ for any i. This gives $E[I_i] = 1 \cdot P[I_i = 1] = 1 - P[I_i = 0] = 1 - (\frac{N-1}{N})^n$. Also we have the total number of types is $Y = \sum_{i=1}^{n} I_i$ and thus $E[Y] = \sum_{i=1}^{n} E[I_i] = 1 - N(\frac{N-1}{N})^n$.

94. <mark>PG 4.5 joint default probability (98)</mark>

> **SOLUTION** The largest probability to have at least one bond to default is $50\% + 30\% = 80\%$ which requires $\rho = -1$.

95. PG 4.6 expected value of max and min (99)

> **SOLUTION** For any $n > 0$ we have $F(z) = P[Z_n \leq z] = z^n \Rightarrow f(z) = nz^{n-1}$ and thus $E[Z_n] = n \int_0^1 z^n dz = \frac{n}{n+1}$. Symmetrically, we have $E[Y_n] = \frac{1}{n+1}$.

96. PG 4.6 correlation of max and min (100)

> **SOLUTION** We need $E[Z]$, $E[Y]$, $E[YZ]$, $\mathrm{std}[Z]$ and $\mathrm{std}[Y]$. We already have $E[Z] = 2/3$ and $E[Y] = 1/3$ from the previous question. We also have $E[Z^2] = n \int_0^1 z^{n+1} dz = \frac{n}{n+2} = 1/2$ and thus $\mathrm{std}[Z] = \sqrt{E[Z^2] - E[Z]^2} = \sqrt{1/2 - 4/9} = \sqrt{1/18}$. Symmetrically we have $\mathrm{std}[Y] = \mathrm{std}[Z] = \sqrt{1/18}$. Lastly, we have $E[YZ] = E[X_1 X_2] = E[X_1]E[X_2] = 1/4$ because of independence. As a conclusion, $\mathrm{Cov}[Z,Y] = E[YZ] - E[Y]E[Z] = 1/4 - 2/9 = 1/36$ and $\mathrm{Corr}[Z,Y] = \mathrm{Cov}[Z,Y]/18 = 1/2$.

97. PG 4.6 random ants (102)

> **SOLUTION** This is merely the expected value of the maximum of $n = 500$ i.i.d. uniform r.v.s. which we already calculated in problem on pg. 99, i.e. $E[\max X] = \frac{n}{n+1} = \frac{500}{501}$.

98. PG 5.1 gambler's ruin problem (107)

> **SOLUTION** Assume the winning probability starting from \$1 is a_1, and similarly define a_2, a_3 and a_0. Then $a_0 = 0$, $a_3 = 1$, $a_1 = 2/3 \cdot a_2 + 1/3 \cdot a_0$ and $a_2 = 1/3 \cdot a_1 + 2/3 \cdot a_3$. Finally we have $a_1 = 4/7$.

99. PG 5.1 dice question (108)

> **SOLUTION** We use conditional probability. Denote F as the sum of first throw, and A being A wins, then $P[A] = P[A \mid F = 12] \cdot P[F = 12] + P[A \mid F = 7] \cdot P[F = 7] + P[A \mid F \notin \{7, 12\}] \cdot P[F \notin \{7, 12\}] = 7/13$.

100. <mark>PG 5.1 coin triplets (109)</mark>

101. <mark>PG 5.2 coin sequence (119)</mark>

102. PG 5.3 dice game (123)

> **SOLUTION** In the third throw we have $1/6$ probability for each result x, which leads to the expected value $E[V_3] = 3.5$, while we're not necessarily optimal to choose to throw the third time. Actually, our

possible values for the second throw are

$$V_2 = \begin{cases} 3.5 & \text{if we get } 1, 2, 3, \\ 4 & \text{if we get } 4, \\ 5 & \text{if we get } 5, \\ 6 & \text{if we get } 6 \end{cases}$$

and each has probability $1/6$, so $\mathrm{E}[V_2] = 4.25$. Further, conditional on the first throw, we can actually have better choices, namely

$$V_1 = \begin{cases} 4.25 & \text{if we get } 1, 2, 3, 4, \\ 5 & \text{if we get } 5, \\ 6 & \text{if we get } 6 \end{cases}$$

and thus $\mathrm{E}[V_1] = 14/3$.

103. <mark>PG 5.3 world series (123)</mark>
104. PG 5.3 dynamic dice game (126)

$\boxed{\textbf{SOLUTION}}$ First we calculate the expected game return n. Assume we continue playing whenever we have no more than n money, then $\sum_{i=1}^{5}(n+i)/6 = 5n/6 + 2.5 > n$, which gives $n < 15$. Therefore, we need to calculate all $\mathrm{E}[f(n)]$ where $n = 0, 1, \ldots, 19$ and $f(\cdot)$ is the return. For $n = 15, 16, \ldots, 19$, we have $\mathrm{E}[f(n)] = f(n) = n$; for $n = 0, 1, \ldots, 14$, we have $\mathrm{E}[f(n)] = \sum_{i=1}^{5} \mathrm{E}[f(n+i)]/6$. By backward-induction we have $\mathrm{E}[f(0)] = 6.15$ which is exactly the expected amount of money when our current value is zero.

105. <mark>PG 5.3 dynamic card game (127)</mark>
106. <mark>PG 5.4 Brownian motion (129)</mark>

$\boxed{\textbf{SOLUTION}}$ We solve these problems one by one.

A. The Brownian motion is defined as the stochastic process that starts from $B(0) = 0$ and has normal i.i.d. increments with zero mean and variance equal to its length t. Some important properties are:

- continuous
- $\mathrm{E}[B(t)] = 0$, $\mathrm{E}[B(t)^2] = t$, $B(t) \sim \mathcal{N}(0, t)$
- martingale: $\mathrm{E}[B(t+s) \mid B(t)] = B(t)$
- Markov: $\mathrm{E}[B(t+0) \mid B(s) \forall s \in [0, t)] = \mathrm{E}[B(t+0) \mid B(t)]$
- $Y(t) = B(t)^2 - t$ is a martingale
- $Z(t) = \exp(\lambda B(t) - \lambda^2 t/2)$ is a martingale (called **exponential martingale**)

B. The correlation of a Brownian motion and its square is zero, because $\mathrm{Cov}[B, B^2] = \mathrm{E}[B^3] - \mathrm{E}[B]\mathrm{E}[B^2] = 0 - 0 = 0$.

C. To calculate the probability of $B_1 > 0$ and $B_2 < 0$, we need, by independence of increments, $B_1 > 0$ and $B_2 - B_1 < -B_1$ where B_1 and $B_2 - B_1$ are independent. Therefore, we equivalently have two i.i.d. normal random variables X and Y, and need to calculate $\mathrm{P}[Y > X > 0] = \mathrm{P}[X > 0] \cdot \mathrm{P}[Y > 0] \cdot \mathrm{P}[Y > X] = 1/8$.

107. PG 5.4 stopping time / first passage time (131)

$\boxed{\textbf{SOLUTION}}$ We solve them separately.

A. As we said above, $B_t^2 - t$ is a martingale, so is $B_T^2 - T$. Therefore, we have $\mathrm{E}[B_T^2 - T] = 0$. Because we assume T the time of first passage, we have $|B_t| = 1$ and thus $\mathrm{E}[T] = \mathrm{E}[B_T^2] = 1$.

B. By reflection principle, we know

$$\mathrm{P}[\tau_x \le t] = \mathrm{P}[\tau_x \le t, W(t) > x] + \mathrm{P}[\tau_x \le t, W(t) \le x] = 2\mathrm{P}[\tau_x \le t, W(t) > x] = 2(1 - \mathcal{N}(x/\sqrt{t})).$$

Therefore we have, by taking derivative w.r.t. t,

$$f_{\tau_x}(t) = \frac{x \exp(-x^2/2t)}{t\sqrt{2\pi t}}, \quad \forall x > 0.$$

To answer the question what the expected time to reach any $x > 0$, we're not usign the same process. Instead, we can show from **A** that the expected first passage time to reach $\alpha > 0$ or $-\beta < 0$ is $\mathrm{E}[\tau_{\alpha, -\beta}] = \alpha\beta$.

Therefore, we have $\mathrm{E}[\tau_x] \equiv \mathrm{E}[\tau_{x,-\infty}] = x \cdot \infty = \infty$.

C. Assume the probability of hitting 3 first is P_3, then since B_T is a martingale for T being the first passage time, we know $3\mathrm{P}_3 + (-5)(1-\mathrm{P}_3) = 0$ which gives $\mathrm{P}_3 = 5/8$. Generally, we have $\mathrm{P}_\alpha = \alpha/(\alpha+\beta)$. For $B_t = W_t + mt$, since $\exp(\lambda W_t + \lambda^2 t/2)$ is a martingale (see exponential martingale), we know $X_t = \exp(\lambda B_t + \lambda^2 t/2)$ is also a martingale and thus $\mathrm{E}[X_t] = 0$. Notice this holds for any λ and thus we can let $\lambda = -2m$, which gives $\mathrm{E}[X_t] = \mathrm{E}[\exp(-2mB_t)] = 1$. We know a martingale stopped at the stopping time is also a martingale, i.e. X_T is still a martingale for T being the time first B_t hits 3 or -5, so $\mathrm{E}[X_T] = \mathrm{E}[\exp(-2mB_T)] = \mathrm{P}_3 \exp(-2m \cdot 3) + (1-\mathrm{P}_3)\exp(2m \cdot 5) = \mathrm{P}_3 \exp(-6m) + (1-\mathrm{P}_3)\exp(10m) = 1$. Therefore, we have $\mathrm{P}_3 = \frac{\exp(10m)-1}{\exp(10m)-\exp(-6m)}$.

D. Again we have $\mathrm{E}[\exp(-2mB_t)] = 1$ while here we have $m = 1$, and thus $\mathrm{E}[\exp(-2B_t)] = \mathrm{P}_{-1}\exp(2) + (1-\mathrm{P}_{-1})\exp(-\infty) = \exp(2)\mathrm{P}_{-2} = 1$. So $\mathrm{P}_{-1} = \exp(-2)$.

108. PG 5.4 Ito's lemma (135)

$\boxed{\textbf{SOLUTION}}$ First let's recite the Ito's lemma:

$$\mathrm{d}f = \left(\frac{\partial f}{\partial t} + \alpha\frac{\partial f}{\partial x} + \frac{\beta^2}{2}\frac{\partial^2 f}{\partial x^2}\right)\mathrm{d}t + \beta\frac{\partial f}{\partial x}\mathrm{d}w$$

which directly gives the drift rate $\mathrm{d}f/\mathrm{d}t$.

A. Since $Z_t = \sqrt{t}B_t$, we have for constant t, Z_t is symmetric around 0 and thus has $\mathrm{E}[Z_t] = 0$ and $\mathrm{Var}[Z_t] = t^2$. However, to determine whether Z_t is a martingale or not we need either to use its definition, which says conditional expectation is equal to the current expectation, or to prove that it's a Wiener process with no drift term. Use Ito's lemma and we can prove Z_t has non-zero drift term, so it's not a martingale.

B. Similar to **A**, we need to use Ito's lemma and can actually show $f = W^3$ has a non-zero drift term, so it's not a martingale either.

109.

$\boxed{\textbf{SOLUTION}}$ The co-movements are summarized below.

variable	Eu. call	Eu. put	Am. call	Am. put
$S\uparrow$	↑	↓	↑	↓
$K\uparrow$	↓	↑	↓	↑
$\tau\uparrow$	?	?	↑	↑
$\sigma\uparrow$	↑	↑	↑	↑
$r\uparrow$	↑	↓	↑	↓
$D\uparrow$	↓	↑	↓	↑

110.

$\boxed{\textbf{SOLUTION}}$ First let's cite the put-call parity below. For European options:

$$c - p = S - K\exp(-r\tau)$$

and for American options:

$$C - P \in [S - K\exp(-r\tau) - D, S - K\exp(-r\tau)].$$

This can be proved easily if we consider the LHS as a forward with payoff $\max(S_T-K,0)-\max(K-S_T,0) = S_T - K$. By discounting using the risk-free rate we have the equation.

111.

$\boxed{\textbf{SOLUTION}}$ We've got two ways to answer this question.

· First is that exercising a call is giving its intrinsic value $S-K$ while neglecting its time value $K-K\exp(-r\tau)$ and the protection put's value p. Instead of exercising it early, it's better off to sell it before maturity.
· Second, we try to use the Jensen's inequality. It's clear that the $C(S) = (S-K)^+$ is a convex function, and thus we have $\mathrm{E}[C(S_T)] \geq C(\mathrm{E}[S_T])$.

112.

113. PG 6.1 BS formula (143)

$\boxed{\textbf{SOLUTION}}$ First let's cite the original BS formula:

$$c = S\exp(-y\tau)\mathcal{N}(d_1) - K\exp(-r\tau)\mathcal{N}(d_2) \quad \text{and} \quad p = K\exp(-r\tau)\mathcal{N}(-d_2) - S\exp(-y\tau)\mathcal{N}(-d_1)$$

where

$$d_1 = \frac{\ln(S/K) + (r - y + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

and

$$d_2 = d_1 - \sigma\sqrt{\tau}.$$

Note that if the underlying asset is an option, $y = 0$; if the underlying asset is a future then $y = r$; if the underlying asset is a foreign currency, then $y = r_f$ where r_f is the foreign risk-free rate.

A. Now that the asset is an European option, we have $y = 0$, which requires the following assumptions:
· There is no arbitrage, boundary, cost, dividend or tax.
· Stock price is GBM.
· All securities are perfectly divisible.
· The risk-free rate is constant and known.

B. According to the risk-neutral probability measure, the price of an option should be equal to the discounted value of the payoff. This means $V(t) = \exp(-rt)\mathrm{E}[V(T) \mid S(t)]$ when r is a constant. Since S follows a GBM, i.e.

$$\mathrm{d}S = rS\mathrm{d}t + \sigma S\mathrm{d}w,$$

by applying Ito's lemma, we have

$$\mathrm{d}(\ln S) = (r - \sigma^2/2)\mathrm{d}t + \sigma\mathrm{d}w \Rightarrow \ln S \sim \mathcal{N}(\ln S_T + (r - \sigma^2/2)\tau, \sigma^2\tau)$$

and thus $S_T = S\exp[(r - \sigma^2/2)\tau + \sigma\sqrt{T}\varepsilon]$ where $\varepsilon \sim \mathcal{N}(0,1)$. This shall be enough to derive the BS formula.

C. Skipped.

D. The option should value $\$1/H$ because whenever a \$1-stock hits $\$H$ it pays \$1. By no arbitrage assumption we know it must worth $1/H$ of the stock value.

E. Assume $S \sim \mathrm{GBM}(r, \sigma^2)$, then the derivative has payoff $f(S) = 1/S$ and thus by Ito's lemma we have

$$\mathrm{d}f = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}rS + \frac{\partial^2 f}{\partial S^2}\frac{\sigma^2 S^2}{2}\right)\mathrm{d}t + \frac{\partial f}{\partial S}\sigma S\mathrm{d}w = (\sigma^2 - r)f\mathrm{d}t - \sigma f\mathrm{d}w$$

and thus

$$\ln f = \left(\frac{\sigma^2}{2} - r\right)\mathrm{d}t - \sigma\mathrm{d}w.$$

Therefore, we have

$$\ln f(T) = \ln f(t) + \mathcal{N}((\sigma^2/2 - r)\tau, \sigma^2\tau)$$

and thus we know $\mathrm{E}[f(T)] = \mathrm{E}[\exp(\ln f(T))] = \exp(\sigma^2\tau - r\tau)/S_T$. Discount this value using risk-free interest rate r and we have $V = \exp(\sigma^2\tau - 2r\tau)/S$.

114. PG 6.2 delta (149)

$\boxed{\textbf{SOLUTION}}$ Before we solve the problem, let's review the definitions of the greeks:
· Delta: $\Delta = \partial f/\partial S$
· Gamma: $\Gamma = \partial^2 f\partial S^2$
· Theta: $\Theta = \partial f/\partial t$
· Vega: $\nu = \partial f/\partial\sigma$
· Rho: $\rho = \partial f/\partial r$

A. For an European call option with value $c = S\exp(-yt)\mathcal{N}(d_1) - K\exp(-rt)\mathcal{N}(d_2)$ we have

$$\Delta = \partial c/\partial S = \exp(-yt)\mathcal{N}(d_1) = \mathcal{N}(d_1)$$

when there is no dividend payment. However, here cannot directly take out the second part because we also have S in d_1 and d_2. In fact, since $d_2 = d_1 - \sigma\sqrt{\tau}$ we know that

$$\partial d_1/\partial S = \partial d_2/\partial S$$

and thus it can be proved that $\mathcal{N}'(d_2) = \mathcal{N}'(d_1)\exp((r-y)\tau)S/K$, which eventually means the last two parts with $\mathcal{N}'$ in Δ cancels out, so the result is still $\Delta = \mathcal{N}(d_1)$.

B. When there is no dividend and the option is at-the-money, we know that the Delta is given by

$$\Delta = \mathcal{N}(d_1) = \mathcal{N}\left(\left(\frac{r}{\sigma} + \frac{\sigma}{2}\right)\sqrt{\tau}\right)$$

which is always above $1/2$ and gets larger as τ gets larger, i.e. with the option coming to maturity the Delta will get smaller.

C. We need Delta hedging. This means with every 1 option bought we also short Δ stocks.

D. Since it's at-the-money, we have $S = K$. Also, for small r a typical volatility $\sigma < 40\%$ and maturity $\tau < 1/4$ year both tells that the value of the option is approximated by

$$c = S\exp(-yt)\mathcal{N}(d_1) - K\exp(-rt)\mathcal{N}(d_2) \approx S(\mathcal{N}(d_1) - \mathcal{N}(d_2)) \approx \frac{\sigma\sqrt{\tau}S}{\sqrt{2\pi}} \approx 0.4\sigma S\sqrt{\tau}.$$

   SOLUTION The value is bounded by $K_2 - K_1$ and $(K_2 - K_1)S/K_2$, the latter of which is given by cutting the payoff lines vertically at S.
   SOLUTION A vanilla European option ends in the money with probability $\mathcal{N}(d_2)$, which here is exactly the expected payoff of the option. Therefore, the price should be the discounted payoff, i.e. $c = \exp(-rt)\mathcal{N}(d_2)$. To hedge this option, we need $\Delta = \partial c/\partial S = \exp(-rt)\mathcal{N}'(d_2))(S\sigma\sqrt{\tau})^{-1}$. However, when the difference between K and S are not big, or when τ is small, the Delta is extremely volatile and thus cannot be used in hedging well.
   SOLUTION Minimize variance and solve for weights. The result is $6/7$ in A and $1/7$ in B.
   SOLUTION Before solving this problem, let's review the definition of duration and convexity of bonds. The duration is defined as $D = -\frac{1}{p}\frac{dp}{dy}$ where p is the price of the bond and y is the yield to maturity. The convexity of a bond is defined as $C = \frac{1}{p^2}\frac{d^2p}{dy^2}$. Using Taylor's expansion and we have approximation

$$\frac{\Delta p}{p} \approx -D\Delta y$$

Another important concept is the dollar duration, which is defined as

$$\$D = -\frac{dp}{dy} \approx DP$$

and thereby defined $DV01 = \$D/10000$. When a series of bonds form a portfolio, the portfolio duration is the weighted duration by prices, and the dollar duration is mere the sum of all dollar durations.

127. PG 7.1 number swap (172)

SOLUTION We may use ±

```
void swap(int &i, int &j) {
        i = i + j;
        j = i - j;
        i = i - j;
}
```

Or, we can also use bitwise XOR

```
void swap(int &i, int &j) {
        i = i ^ j;
        j = j ^ i;
        i = i ^ j;
}
```

128. PG 7.1 unique elements (173)

129. PG 7.1 Horner's algorithm (174)

SOLUTION Instead of calculating multiplication for individual coefficients, we can iteratively use the results from previous multiplications.

130. PG 7.1 moving average (174)

131. PG 7.1 sorting algorithm (174)

SOLUTION The sorting algorithms are summarized as below.

algorithm	avg. time	space	description	stability
insertion	$O(n^2)$	$O(1)$	insert unsorted into sorted	✓
selection	$O(n^2)$	$O(1)$	select the largest and so on	✗
heap	$O(n \lg n)$	$O(1)$	build a heap and iteratively take away the max i.e. root	✗
bubble	$O(n^2)$	$O(1)$	swap neighbors s.t. they are in order	✓
merge	$O(n \lg n)$	$O(n)$	sort two halves and merge	✓
quick	$O(n \lg n)$	$O(n)$	use pivots for classification	✗

132. PG 7.1 random permutation (176)

133. PG 7.1 search algorithm (177)

134. PG 7.1 Fibonacci numbers (179)

135. PG 7.1 maximum contiguous subarray (180)

136. PG 7.2 power of two (182)

137. PG 7.2 multiplications by 7 (182)

138. PG 7.2 probability simulation (182)

139. PG 7.2 poisonous wine (183)

SOLUTION Since $2^10 = 1024 > 1000$, ten mice is enough to represent all wines and thus allow us to use these ten lab mice simultaneously and retrack the poisonous wine by binary coding.

140. PG 7.3 Monte Carlo simulation (184)

SOLUTION We mainly talk about **C** here.

· antithetic variable: for each ε, calculate the corresponding negative situation to reduce variance
· moment matching: rescale small sample s.t. its moments matches the population moments well
· control variate: if we have another variable Y that has an analytical form, then we can use the same set of random numbers to estimate it first and offset the bias in estimation of X by $\hat{X}' = \hat{X} + Y - \hat{Y}$
· importance sampling: use some fat-tailed distribution to adjust the span of the original distribution
· low-discrepancy sequence: use some random sequence instead of just random sample

141. PG 7.3 finite difference method (189)

142. HS 1.1 (9)

143. HS 1.2 (9)

144. HS 1.3 (9)

145. HS 1.4 (9)

146. HS 1.5 (10)

  SOLUTION First separate the large piles of marbles into three 4-marble piles and denote each pile as a combination of $1 + 3$. Weigh a pair from these three piles and conditioning on the results:

  · If the two piles are of the same weight, the the odd marble is in the rest pile of $1 + 3$. Weigh the 3 with a weighed sub-pile of 3 and if they are equal, then the rest one is odd (2 weighs in total); if they're not equal, then the odd is in the 3-marble sub-pile, weigh two of the three and we can tell (3 weighs).

  · If the two piles are of different weights, then the odd marble is in either of the weighed 4-marble piles. Now swap one 3-marble pile with the rest one's, and weigh again. If they're still different, then the odd one is the one not swapped (2 weighs); if they're now equally heavy, then the odd one is in the sub-pile being swapped. Another weigh can tell the odd one (3 weighs).

147. HS 1.6 (10)
148. HS 1.7 (10)
149. HS 1.8 (11)

  SOLUTION The ratio will always be 1/2. No need of calculation.

150. HS 1.9 (11)
151. HS 1.10 (11)
152. HS 1.11 (11)
153. HS 1.12 (11)
154. HS 1.13 (11)

  SOLUTION 64 has factors 1, 2, 4, 8, 16, 32 and 64. So it's been switched seven times and eventually is illuminated.

155. HS 1.14 (12)
156. HS 1.15 (12)
157. HS 1.16 (12)
158. HS 1.17 (12)
159. HS 1.18 (13)

  SOLUTION There is a conclusion that for a pile of 3^k coins you need at most k weighs to identify the odd one. You can prove it by induction. Therefore, split the 90 coins by $81 = 27 + 27 + 27$ and 9, then identify whether the odd coin is in the 81-coin group by 2 weighs. If so, you need another $\log_3 27 = 3$ trials; if not, you need 2. So in total at most \$500 is paid.

160. HS 1.19 (13)
161. HS 1.20 (13)
162. HS 1.21 (14)
163. HS 1.22 (14)
164. HS 1.23 (14)
165. HS 1.24 (14)
166. HS 1.25 (14)

  SOLUTION Turn on two and turn off one of them after a minute. Then walk into the room and see which is on, which is off and which is warm but off.

167. HS 1.26 (14)
168. HS 1.27 (14)

  SOLUTION Denote this number as X and we have

  $$X = 2a - 1 = 3b - 1 = 4c - 1 = 5d - 1 = 6e - 1 = 7f - 1 = 8g - 1 = 9h - 1 = 10i - 1$$

  for some $a, b, \ldots, i \in \mathbb{Z}_+$. After simple analysis we can shorten this long equation into

  $$X = 6e - 1 = 7f - 1 = 8g - 1 = 9h - 1 = 10i - 1$$

  and thus

  $$X = 2520n - 1$$

  for $n \in \mathbb{Z}_+$ and thus the minimum value is $X = 2519$.

$\boxed{\textbf{SOLUTION}}$ According to Archemides law, the lifting force, which is equal to the force of gravity of the expelled water, must be equal to the total force of gravity of the boat and the stone. Therefore, throwing the stone away is merely decreasing the water expelled by the same weight as the stone and thus the water level drops. Notice here we're talking about a very small stone so we don't need to consider the volume of it. However, even talking that into account the level will still drop, since the density of a stone must be larger than water and thus water re-expelled will still be less than before, and thus the level drops.

$\boxed{\textbf{SOLUTION}}$ The speed of the light along the coastline, if we only consider $1/4$ revolution that starts from the position the light beam is at right angle toward the coastline, is a function of the angle revolved from the starting position, say θ. The function $v(\theta)$ for $\theta \in [0, \pi/4)$ is given by

$$v(\theta)\mathrm{d}t = L\tan(\theta + \omega\mathrm{d}t)$$

and thus

$$v(\theta) = \lim_{\mathrm{d}t \to 0} L\frac{\tan(\theta + \omega\mathrm{d}t)}{\mathrm{d}t} = L\omega\sec^2\theta.$$

When $\tan\theta = 3 \Rightarrow \sec^2\theta = \tan^2\theta + 1 = 10$ and thus $v(\theta) = L\omega\sec^2\theta = 3 \times \frac{2\pi}{60} \times 10 = \pi$ miles per second.

$\boxed{\textbf{SOLUTION}}$ Denote the expected payoff when there're x red and y black cards as $\mathrm{E}(x, y)$, which is given by

$$\mathrm{E}(x,y) = \begin{cases} 0 & \text{if } x = 0, \\ x & \text{if } y = 0, \\ \max\{0, \frac{x}{x+y}(1 + \mathrm{E}(x-1, y)) + \frac{y}{x+y}(-1 + \mathrm{E}(x, y-1))\} & \text{otherwise} \end{cases}$$

then we can iteratively or recursively caculate $\mathrm{E}(26, 26)$, in Python

```python
def E(x, y):
    if x == 0:
        return 0
    if y == 0:
        return x
    t = x + y
    rx = x / t
    ry = y / t
    return max(0, rx * (1 + E(x - 1, y)) + ry * (-1 + E(x, y - 1)))

print(E(10, 10))
```

or in C++ (where I implemented a cache for faster recursion) as below

```cpp
#include <iostream>
#include <algorithm>
using namespace std;

#define X 26
#define Y 26
#define value -1

double cache[X+1][Y+1];

void printMatrix() {
    for (int i=0; i<X+1; ++i) {
        for (int j=0; j<Y+1; ++j)
            cout << cache[i][j] << " ";
        cout << endl;
    }
}

void fillMatrix() {
    for (int i=0; i<X+1; ++i) {
        for (int j=0; j<Y+1; ++j)
            cache[i][j] = value;
    }
}

double E(int x, int y) {
    if (x == 0) return 0;
    if (y == 0) return x;
    if (cache[x][y] != value) return cache[x][y];
    int t = x + y;
    double rx = (double)x / (double)t;
    double ry = (double)y / (double)t;
    double ans = max(0., rx * (1 + E(x - 1, y)) + ry * (-1 + E(x, y - 1)));
    cache[x][y] = ans;
    return ans;
}

int main() {
    fillMatrix();
    cout << E(X, Y) << endl;
    return 0;
}
```

which gives $E(26, 26) = 2.62448$ eventually.

188. HS 1.47 (18)
189. HS 1.48 (18)
190. HS 1.49 (19)
191. HS 1.50 (19)
192. HS 1.51 (19)
193. HS 1.52 (19)

SOLUTION Light the first one on both ends, then it'll cost 30 seconds to finish. Simultaneously light the second fuse on one end. When the first is finished (so after 30 seconds), start light the second fuse's other end, and it will take another 15 seconds to finish all. In total it's 45 seconds.

194. HS 1.53 (19)
195. HS 1.54 (19)
196. HS 1.55 (19)

197. HS 1.56 (19)

SOLUTION We give a possible sulution as below:

· Day 1: give him 1
· Day 2: give him 2, take back 1
· Day 3: give him 1
· Day 4: give him 4, take back 2 and 1
· Day 5: give him 1
· Day 6: give him 2, take back 1
· Day 7: give him 1

and thus in sum we devide the gold bar into $7 = 1 + 2 + 4$.

206. HS 1.65 (21)

SOLUTION We need to express

$$f(x) = \int_{t=x}^{\infty} \exp\left(-\frac{at^2}{2} + bt\right) \mathrm{d}t = \int_{t=x}^{\infty} \exp\left(-\frac{at^2 - 2bt}{2}\right) \mathrm{d}t = \frac{\exp(b^2/2a)}{\sqrt{a}} \int_{t=x}^{\infty} \exp\left(-\frac{(\sqrt{a}t - b/\sqrt{a})^2}{2}\right) \mathrm{d}\sqrt{a}t$$

which is equivalently

$$\frac{\sqrt{a}}{\exp(b^2/2a)} f(x) = \int_{t=\sqrt{a}x-b/\sqrt{a}}^{\infty} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t$$

in the form of the CDF of normal distribution, i.e.

$$\Phi(x) = \int_{t=-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t.$$

Now, since we know

$$\bar{\Phi}(x) \equiv 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{t=x}^{\infty} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t$$

which gives

$$\sqrt{2\pi}[1 - \Phi(\sqrt{a}x - b/\sqrt{a})] = \int_{t=\sqrt{a}x-b/\sqrt{a}}^{\infty} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t = \frac{\sqrt{a}}{\exp(b^2/2a)} f(x)$$

and thus

$$f(x) = \exp(b^2/2a)\sqrt{2\pi/a}[1 - \Phi(\sqrt{a}x - b/\sqrt{a})].$$

207. HS 1.66 (21)

SOLUTION An easy while not intuitive question.

$$\lim_{x \to \infty} \sqrt{x^2 + x} - x = \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + x} + x} = \lim_{x \to \infty} \frac{1}{\sqrt{1 + 1/x} + 1} = \frac{1}{2}.$$

266. HS 2.57 (32)
267. HS 2.58 (32)
268. HS 2.59 (32)
269. HS 2.60 (32)