Quant Interview Problems

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1. PG 4.1 coin toss game (61)

   $\boxed{\textbf{SOLUTION}}$ Consider it as a sequential problem, then there're 2 cases where A has more tops than B: A and B has the same number of tops in the first n tosses, and then A tosses a top, or A has already more tops than B in the first n tosses. Denote the probability of the condition of case 1 as x and the probability of case 2 as y, then $2y + x = 1$. What we need is merely $y + x/2 = 1/2$.
2. PG 4.1 card game (61)

   $\boxed{\textbf{SOLUTION}}$ Let's calculate it sequentially. In total there're P_{52}^2 plays and 13 kinds of my card with each kind for 4 suits, and for each order $i \in \{1, 2, \ldots, 13\}$ we must have the dealer to take a card from $(i-1)\cdot 4$ kinds. Hence the probability is $16\sum_{i=1}^{13}(i-1)/P_{52}^2 = (16\cdot 12\cdot 13)/(2\cdot 52\cdot 51) = 8/17$.
3. PG 4.1 drunk passenger (62)

   $\boxed{\textbf{SOLUTION}}$ The probability of my seat being untaken is equal to the probability that no one's ever taken it. This can be decomposed into three parts: #1 takes his own seat, he takes my seat, or he takes any seat in between. While the first two cases are equivalent, the last case automatically lead to another drunk guy whose designated seat is #1, so it's again symmetric. As a result the probability is $1/2$.
4. PG 4.1 n points on a circle (63)

   $\boxed{\textbf{SOLUTION}}$ Take any point $i \in 1, 2, \ldots, n$ for example and consider if all other points are on the semicircle starting from point i. The probability is easy to calculate and it's $1/2^{n-1}$. Since all points are symmetric in consideration, the resulting probability should be $N/2^{n-1}$.
5. PG 4.2 poker hands (65)

   $\boxed{\textbf{SOLUTION}}$ There're in total C_{52}^5 cases. Four-of-a-kind: $13\cdot(52-4)$; Full-house: $C_{13}^2\cdot 2\cdot C_4^3\cdot C_4^2$; Two-pair: $C_{13}^2\cdot(C_4^2)^2\cdot(52-4)$. The probabilities are merely these numbers devided by the total number of cases.
6. PG 4.2 hopping rabbit (66)

   $\boxed{\textbf{SOLUTION}}$ We solve it recursively. Let the number of ways to reach stair k be $f(k)$, then we have recursion $f(k+2) = f(k+1) + f(k)$ with initial conditions $f(0) = 1$ and $f(1) = 1$. It's, therefore, a standard Fibonacci sequence and we know $f(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$.
7. PG 4.2 screwy pirates 2 (67)

   $\boxed{\textbf{SOLUTION}}$ In order to forbid any 5-combination of pirates to open the safe while allow exactly the rest 6 to be able to do so, the number of locks needed is given by the number of suck combinations, either 6 out of 11 or 5 out of 11, and the number is C_{11}^5. Each lock must corresponds to at least 6 keys and by symmetry the number of keys for each pirate should be $C_{11}^5\cdot 6/11$.
8. PG 4.2 chess tournament (68)

   $\boxed{\textbf{SOLUTION}}$ In order to have 1 to meet 2 in the final, they must be in different half-games. This means a total number of $2(2^n-2)!(2^{n-1})^2$ cases, while in total there is $2^n!$ combinations. So the probability should be $2(2^n-2)!(2^{n-1})^2/2^n! = 2^{n-1}/(2^n-1)$.
9. PG 4.2 application letters (69)

   $\boxed{\textbf{SOLUTION}}$ This is called the dearrangement number and we can solve it recursively. Denote $f(k)$ as the number of ways to put all letters in wrong envelopes. Assume that letter 1 is in some envelope $i \in \{2, 3, \ldots, k\}$, then we have two cases: letter k being in envelope 1 or not. This leads to the recursion $f(k) = k[f(k-1) + f(k-2)]$ with initial conditions $f(1) = 0$ and $f(2) = 1$. Let $f(k) := k!\cdot g(k)$ and we can derive $g(k)$.
10. PG 4.2 birthday problem (71)

    $\boxed{\textbf{SOLUTION}}$ Let $\Pi_{i=0}^k(365-i)/365^k < 1/2$. Enumeration tells that the smallest k is 22 and thus there should be at least 23 people.
11. PG 4.2 100th digit (71)

    $\boxed{\textbf{SOLUTION}}$ Notice that the binomial expansions of $(1+\sqrt{2})^{3000}$ and $(1-\sqrt{2})^{3000}$ have irrational, $\sqrt{2}$ to singular powers, yet with the opposite signs. This indicates that $(1+\sqrt{2})^{3000}+(1-\sqrt{2})^{3000}$ is an integer while since $|1-\sqrt{2}| < 1$, we know $0 < (1-\sqrt{2})^{3000} \ll 10^{-100}$ and the 100th digit of $(1+\sqrt{2})^{3000}$ is 9.
12. PG 4.2 cubic of integer (72)

    $\boxed{\textbf{SOLUTION}}$ We only consider the last two digits. There're 100 possible cases, namely from 00 up until 99, which can be written as $10a + b$. The binomial expansion of $(10a + b)^3 = 1000a^3 + 300a^2b + 30ab^2 + b^3$ indicates the only solution, which is $b = 1$ and $a = 7$. So the probability is 1%.
13. PG 4.3 boys and girls (73)

    $\boxed{\textbf{SOLUTION}}$ Denote A as having at least one boy out of the two. Denote B as having two boys, C as having a boy and a girl, and D as having two girls. Also, we know $\mathrm{P}[A \mid B] = 1$, $\mathrm{P}[A \mid C] = 1$ and $\mathrm{P}[A \mid D] = 0$. By Bayes' formula we have $\mathrm{P}[B \mid A] = \frac{\mathrm{P}[A|B]\cdot\mathrm{P}[B]}{\mathrm{P}[A|B]\cdot\mathrm{P}[B]+\mathrm{P}[A|C]\cdot\mathrm{P}[C]+\mathrm{P}[A|D]\cdot\mathrm{P}[D]}$.

- We have $\mathrm{P}[B] = 1/4$, $\mathrm{P}[C] = 1/2$ and $\mathrm{P}[D] = 1/4$. Therefore, By Bayes' formula we have $\mathrm{P}[B \mid A] = 1/3$.
- Different from the previous problem, now define A^- as the case where the **first** child is a boy. Now, $\mathrm{P}[A^- \mid B] = 1$, $\mathrm{P}[A^- \mid C] = 1/2$ and $\mathrm{P}[A^- \mid D] = 0$. So $\mathrm{P}[B \mid A^-] = \frac{\mathrm{P}[A^- | B] \cdot \mathrm{P}[B]}{\mathrm{P}[A^- | B] \cdot \mathrm{P}[B] + \mathrm{P}[A^- | C] \cdot \mathrm{P}[C] + \mathrm{P}[A^- | D] \cdot \mathrm{P}[D]} = 1/2$.

14. PG 4.3 all-girl world (74)

    SOLUTION The fraction of girls is solely given by nature, and thus is 50%.

15. PG 4.3 unfair coin (74)

    SOLUTION Denote event A as the coin tosses being the unfair one, B as a consecutive sequence of 10 tosses gives all heads. We know that $\mathrm{P}[A] = 1/1000$ and $\mathrm{P}[\bar{A}] = 999/1000$. We also know that $\mathrm{P}[B \mid A] = 1$ and $\mathrm{P}[B \mid \bar{A}] = 2^{-10}$. Then by Bayes' theorem, we have $\mathrm{P}[A \mid B] = \mathrm{P}[B \mid A] \cdot \mathrm{P}[A] \cdot (\mathrm{P}[B \mid A] \cdot \mathrm{P}[A] + \mathrm{P}[B \mid \bar{A}] \cdot \mathrm{P}[\bar{A}])^{-1} = 0.506$.

16. PG 4.3 fair probability from an unfair coin (75)

    SOLUTION Toss twice and let HT be win, TH be loss, otherwise again.

17. PG 4.3 dart game (75)

    SOLUTION This is about the probability that the last throw does not lead to the best score. Any of the three throws can be in that case so the probability is $1/3$.

18. PG 4.3 birthday line (76)

    SOLUTION Say that you choose to be in position $n \geq 1$, then the probability that you get that ticket is $\mathrm{P}[n] = \frac{365!}{365^{n-1}(365-n+1)!} \frac{n-1}{365}$. In order to have $P[n] > P[n-1]$ and $P[n] > P[n+1]$ we have $n = 20$.

19. PG 4.3 dice order (78)

    SOLUTION The total number of events that're required is $n = C_6^3$ while the number of all possible events is $N = 6^3$. So the probability is $\mathrm{P} = C_6^3/6^3 = 5/54$.

20. PG 4.3 monty hall (78)

    SOLUTION There're two cases.

    - If I chose the door with car, which has a probability of $1/3$, then switching leads to $\mathrm{P}[\text{switch and car} \mid \text{car}] = 0$.
    - If I chose the door with goat, which has a probability of $2/3$, then switching leads to $\mathrm{P}[\text{switch and car} \mid \text{goat}] = 1$.

    Therefore, $\mathrm{P}[\text{switch and win}] = 2/3$.

21. PG 4.3 amoeba population (79)

    SOLUTION Solve the fixed point of the generating function $f(x) = (1+x+x^2+x^3)/4$. The only reasonable solution is $x = \sqrt{2} - 1$.

22. PG 4.3 candies in a jar (79)

    SOLUTION If blue is the last, then the conditional probability that green is the last among all red and green candies is $\mathrm{P}_1 = 3/4$. If green is the last, then the conditional probability that blue is the last among all red and blue candies is $\mathrm{P}_2 = 2/3$. As a result, the probability that all reds end before blue and green is $\mathrm{P} = \mathrm{P}_1/3 + \mathrm{P}_2/2 = 1/4 + 1/3 = 7/12$.

23. PG 4.3 coin toss game (80)

    SOLUTION Let's say the probability is $\mathrm{P}[A]$. Then $\mathrm{P}[B] = 1 - \mathrm{P}[A]$. We condition on A's first toss. If it's a tail, then it's become the opposite case where B is the first one to toss, and thus $\mathrm{P}[A \mid T] = \mathrm{P}[B] = 1 - \mathrm{P}[A]$. If it's a head, then we have two sub-cases: (a) if B tosses a tail, then B's won; (b) is B tosses a head, then the game starts again. So we have $\mathrm{P}[A \mid H] = 0 + (1 - \mathrm{P}[A \mid H])/2$. Together with these two equations we have $\mathrm{P}[A] = \mathrm{P}[A \mid H]/2 + \mathrm{P}[A \mid T]/2 = 4/9$. This should be smaller than $1/2$ because A can never win in the first toss, while B can.

24. PG 4.3 russian roulette series (81)

    SOLUTION We solve it one by one.

    - No it does not matter. The probability is always $1/2$.
    - Let the probability be p if we choose to play first, then to loss means $p = 1/6 + 5/6 \cdot (1 - p)$, so $p = 6/11$. Therefore, we should choose to play second which gives a loss probability of $5/11$.
    - It's just $2/6$ and $2/5$. You definitely should spin.
    - Say that the 2 bullets are in bezels 1 and 2. Then the opponent can have chosen anywhere between 3 and 6, so the probability that I can live is $3/4$ if I don't spin. In contrast, if I spin, the probability should be indifferent to what he's chosen and the living probability is $4/6 = 2/3$, which is less. So I should not spin.

25. PG 4.3 aces (82)

    SOLUTION Total number of events is $N = \frac{52!}{(13!)^4}$. The total number of events where each player has an ace is $n = 4! \cdot \frac{48!}{(12!)^4}$. So the probability is $n/N = \frac{4! \cdot 48! \cdot (13!)^4}{52! \cdot (12!)^4} = \frac{13^4 \cdot 24}{52 \cdot 51 \cdot 50 \cdot 49} = 0.105$.

26. PG 4.3 gambler's ruin problem (83)

    SOLUTION Let P_i be the winning probability starting from fortune i, then $\mathrm{P}_i = \mathrm{P}_{i-1}p + \mathrm{P}_{i+1}q$, which together with boundary conditions $\mathrm{P}_0 = 0$ and $\mathrm{P}_N = 1$ gives

$$\mathrm{P}_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 1/2, \\ i/N & \text{if } p = 1/2. \end{cases}$$

27. PG 4.3 basketball scores (84)
 SOLUTION Denote (n,k) as n throws with k scores. Then $\mathrm{P}[(3,1)] = \mathrm{P}[(3,2)] = 1/2$. Further, we have
   - $\mathrm{P}[(4,1)] = \mathrm{P}[(4,1) \mid (3,1)]\mathrm{P}[(3,1)] = 2/3 \cdot 1/2 = 1/3$;
   - $\mathrm{P}[(4,2)] = \mathrm{P}[(4,2) \mid (3,1)]\mathrm{P}[(3,1)] + \mathrm{P}[(4,2) \mid (3,2)]\mathrm{P}[(3,2)] = 1/3 \cdot 1/2 + 1/3 \cdot 1/2 = 1/3$;
   - $\mathrm{P}[(4,3)] = \mathrm{P}[(4,3) \mid (3,2)]\mathrm{P}[(3,2)] = 2/3 \cdot 1/2 = 1/3$.

   So we guess that $\mathrm{P}[(n,k)] \equiv (n-1)^{-1}$. By induction this is proved, and thus the answer is $1/99$.
28. PG 4.3 cars on road (85)
 SOLUTION Solve $(1-p)^4 = 1 - 609/625$ and we have $p = 3/5$.
29. PG 4.4 meeting probability (88)
 SOLUTION The probability is the area of $|X - Y| \le 5$, and thus is $\mathrm{P} = (60^2 - 55^2)/60^2 = 23/144$.
30. PG 4.4 probability of triangle (89)
 SOLUTION Assume the first cut is x and the second y. To make it possible to form a triangle, we need:
   - $\max\{x, y\} > 1/2$;
   - $\min\{x, y\} < 1/2$;
   - $|y - x| < 1/2$.

   Therefore, we can calculate the area and get the probability as $\mathrm{P} = 1/4$.
31. PG 4.4 property of Poisson process (90)
 SOLUTION Exponential waiting time is memoriless both forward and backward time: $f(t) = \lambda e^{-\lambda t}$. The expected waiting time is 10 minutes for both.
32. PG 4.4 moments of normal distribution (91)
 SOLUTION We have the central moments of standard normal distribution as $\mu_1 = 0$, $\mu_2 = 1$, $\mu_3 = 0$ and $\mu_4 = 3$. More generally, we have $\mu_{2k} = (2k-1)!!$ and $\mu_{2k+1} = 0$ for all $k \in \mathbb{N}$. The k-th moment is calculated by taking the k-th derivative of $M(t) = \mathrm{E}[e^{tX}]$.
33. PG 4.5 connecting noodles (93)
 SOLUTION Assume we have n noodles, i.e. $2n$ ends, then there're in total $C_{2n}^2 = n(2n-1)$ combinations in which n makes 1 circle. So we have $\mathrm{E}[f(n)] = \mathrm{E}[f(n-1)] + (2n-1)^{-1}$.
34. PG 4.5 optimal hedge ratio (94)
 SOLUTION Assume we short Δ shares of B, then $\mathrm{Var}[A - \Delta B] = \sigma_A^2 + \Delta^2\sigma_B^2 - 2\Delta\sigma_A\sigma_B\rho$. Take the first derivative of Δ to this equation and evaluate it to zero, then we have $\Delta\sigma_B^2 = \sigma_A\sigma_B\rho \Rightarrow \Delta = \rho\sigma_A/\sigma_B$.
35. PG 4.5 dice game (94)
 SOLUTION With $1/2$ probability we have expected payoff $\mathrm{E}_1 = 2$; with the other $1/2$ probability we have $\mathrm{E}_2 = 5 + \mathrm{E}$. So we have $\mathrm{E} = \mathrm{E}_1/2 + \mathrm{E}_2/2 = 7/2 + \mathrm{E}/2 \Rightarrow \mathrm{E} = 7$.
36. PG 4.5 card game (95)
 SOLUTION Define 5 regions devided by the 4 aces, then the probability of a card being in any region is $1/5$, and thus the expected position of the first ace is $\mathrm{E} = 1 + 48/5 = 10.6$ (which is really smart).
37. PG 4.5 sum of random variables (96)
 SOLUTION This is the area in a unit hyper-cube, that is, $\mathrm{P} = 1 \times 1 \times \frac{1}{2} \times \frac{1}{3} \times \cdots \times \frac{1}{n} = 1/n!$ (Also, we can use $\mathrm{P}[\sum_{i=1}^n x_i < 1] = \int_0^1 \mathrm{P}[\sum_{i=1}^{n-1} x_i < 1 - x_n]\mathrm{d}x_n = \mathrm{P}[\sum_{i=1}^{n-1} x_i < 1]/n = \cdots = 1/n!$).
38. PG 4.5 coupon collection (97)
 SOLUTION We solve part A and B separately.
   - Part A. Assume the expected number of tries is $\mathrm{E}[X_n]$ to get the n-th kind of coupon in a sequence, then, since $\mathrm{E}[X_n] = \frac{N}{N-n+1}$ and the total number of tries is merely the sum, we have $\mathrm{E}[X] = \sum_{n=1}^N \mathrm{E}[X_i] = \sum_{n=1}^N \frac{N}{N-n+1} = N(1^{-1} + 2^{-1} + \cdot + N^{-1})$.
   - Part B. Define inticator variable I_i as 1 if at least 1 coupon is of type 1 in the n-coupon set and 0 if there's not. Then we know $\mathrm{P}[I_i = 0] = (\frac{N-1}{N})^n$ for any i. This gives $\mathrm{E}[I_i] = 1 \cdot \mathrm{P}[I_i = 1] = 1 - \mathrm{P}[I_i = 0] = 1 - (\frac{N-1}{N})^n$. Also we have the total number of types is $Y = \sum_{i=1}^n I_i$ and thus $\mathrm{E}[Y] = \sum_{i=1}^n \mathrm{E}[I_i] = 1 - N(\frac{N-1}{N})^n$.
39. PG 4.5 joint default probability (98)
 SOLUTION The largest probability to have at least one bond to default is $50\% + 30\% = 80\%$ which requires $\rho = -1$.
40. PG 4.6 expected value of max and min (99)
 SOLUTION For any $n > 0$ we have $F(z) = \mathrm{P}[Z_n \le z] = z^n \Rightarrow f(z) = nz^{n-1}$ and thus $\mathrm{E}[Z_n] = n\int_0^1 z^n \mathrm{d}z = \frac{n}{n+1}$. Symmetrically, we have $\mathrm{E}[Y_n] = \frac{1}{n+1}$.
41. PG 4.6 correlation of max and min (100)
 SOLUTION We need $\mathrm{E}[Z]$, $\mathrm{E}[Y]$, $\mathrm{E}[YZ]$, $\mathrm{std}[Z]$ and $\mathrm{std}[Y]$. We already have $\mathrm{E}[Z] = 2/3$ and $\mathrm{E}[Y] = 1/3$ from the previous question. We also have $\mathrm{E}[Z^2] = n\int_0^1 z^{n+1}\mathrm{d}z = \frac{n}{n+2} = 1/2$ and thus $\mathrm{std}[Z] = \sqrt{\mathrm{E}[Z^2] - \mathrm{E}[Z]^2} = \sqrt{1/2 - 4/9} = \sqrt{1/18}$. Symmetrically we have $\mathrm{std}[Y] = \mathrm{std}[Z] = \sqrt{1/18}$. Lastly, we have $\mathrm{E}[YZ] = \mathrm{E}[X_1X_2] = \mathrm{E}[X_1]\mathrm{E}[X_2] = 1/4$ because of independence. As a conclusion, $\mathrm{Cov}[Z, Y] = \mathrm{E}[YZ] - \mathrm{E}[Y]\mathrm{E}[Z] = 1/4 - 2/9 = 1/36$ and $\mathrm{Corr}[Z, Y] = \mathrm{Cov}[Z, Y]/18 = 1/2$.

42. PG 4.6 random ants (102)
    $\boxed{\textbf{SOLUTION}}$ This is merely the expected value of the maximum of $n = 500$ i.i.d. uniform r.v.s. which we already calculated in problem on pg. 99, i.e. $\mathrm{E}[\max X] = \frac{n}{n+1} = \frac{500}{501}$.
43. PG 5.1 gambler's ruin problem (107)
44. PG 5.1 dice question (108)
45. PG 5.3 dice game (123)
46. PG 5.3 dynamic dice game (126)