Assignment6

March 19, 2025

1 2.39(b)

The given differential equation is

$$\frac{d\,y(t)}{dt} + 3y(t) = x(t),$$

which can be rewritten as

$$\frac{d\,y(t)}{dt} = x(t) - 3y(t).$$

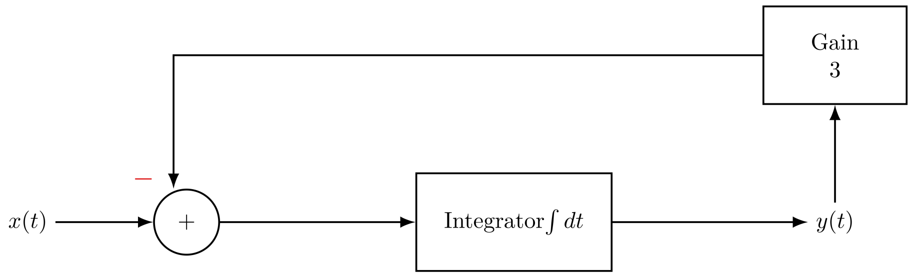
A common block diagram realization for a causal LTI system described by this equation is as follows:

1. **Summer/Adder:**
   Subtract the feedback term $3y(t)$ from the input $x(t)$ to form the derivative dy/dt.
2. **Integrator:**
   Integrate the derivative dy/dt to obtain the output $y(t)$.
3. **Feedback Path:**
   Multiply the output $y(t)$ by 3 and feed it back (with a negative sign) into the summer.

The block diagram can be drawn as:

[20]:
```
import matplotlib.image as mpimg
import matplotlib.pyplot as plt
img = mpimg.imread('./fig/test2.png')
plt.rcParams['figure.dpi'] = 1500
plt.axis('off')
plt.imshow(img)
```

[20]: <matplotlib.image.AxesImage at 0x15d5c2360>



2 2.20

We wish to evaluate the following integrals:

1.
$$I_1 = \int_{-\infty}^{\infty} u_0(t)\cos(t)\,dt,$$

2.
$$I_2 = \int_0^5 \sin(2\pi t)\,\delta(t+3)\,dt,$$

3.
$$I_3 = \int_{-5}^{5} u_1(1-\tau)\cos(2\pi\tau)\,d\tau.$$

(i) Evaluation of I_1 Since
$$u_0(t) = \delta(t),$$
by the sifting property we have:
$$I_1 = \int_{-\infty}^{\infty} \delta(t)\cos(t)\,dt = \cos(0) = 1.$$

(ii) Evaluation of I_2 We have
$$I_2 = \int_0^5 \sin(2\pi t)\,\delta(t+3)\,dt.$$
The delta function $\delta(t+3)$ "picks out" the value at $t=-3$. Since $-3 \notin [0,5]$, the integrand is zero over the integration interval. Thus,
$$I_2 = 0.$$

(iii) Evaluation of I_3 Now, the integrand involves the unit doublet. Since

$$u_1(1-\tau) = \delta'(1-\tau),$$

it is more convenient to rewrite the derivative of the delta function as follows. Using the property

$$\delta'(1-\tau) = -\delta'(\tau-1),$$

we have

$$I_3 = \int_{-5}^{5} \delta'(1-\tau)\cos(2\pi\tau)\,d\tau = -\int_{-5}^{5} \delta'(\tau-1)\cos(2\pi\tau)\,d\tau.$$

Apply the sifting property for the derivative of the delta function with $f(\tau) = \cos(2\pi\tau)$:

$$\int_{-\infty}^{\infty} \delta'(\tau-1)\cos(2\pi\tau)\,d\tau = -\frac{d}{d\tau}\cos(2\pi\tau)\Big|_{\tau=1}.$$

(The integration limits $[-5,5]$ cover $\tau = 1$, so the identity holds.)

Now, compute the derivative:

$$\frac{d}{d\tau}\cos(2\pi\tau) = -2\pi\sin(2\pi\tau).$$

Evaluating at $\tau = 1$ gives:

$$\frac{d}{d\tau}\cos(2\pi\tau)\Big|_{\tau=1} = -2\pi\sin(2\pi) = -2\pi\cdot 0 = 0.$$

Hence,

$$\int_{-\infty}^{\infty} \delta'(\tau-1)\cos(2\pi\tau)\,d\tau = -\big[-2\pi\sin(2\pi)\big] = 0.$$

Returning to our expression for I_3:

$$I_3 = -\times 0 = 0.$$

3 3.21

We are given a real and periodic signal with period

$$T = 8,$$

so that the fundamental angular frequency is

$$\omega_0 = \frac{2\pi}{8} = \frac{\pi}{4}.$$

Its (complex) Fourier series coefficients are given by

$$a_1 = j, \quad a_{-1} = a_1^* = -j, \quad a_5 = 2, \quad a_{-5} = 2,$$

with all other coefficients zero.

For a real signal the cosine (or magnitude–phase) form is

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2|a_k|\cos(k\omega_0 t + \phi_k),$$

where $\phi_k = \angle a_k$ and

$$|a_k| = \sqrt{(\Re\{a_k\})^2 + (\Im\{a_k\})^2}.$$

Since no a_0 is given we take

$$a_0 = 0.$$

For the nonzero coefficients:

1. For $k = 1$:
   - $a_1 = j$ so that
     $$|a_1| = 1, \quad \phi_1 = \angle(j) = \frac{\pi}{2}.$$
   - Thus the cosine term for $k = 1$ is
     $$2|a_1|\cos\Big(\frac{\pi}{4}t + \frac{\pi}{2}\Big) = 2\cos\Big(\frac{\pi}{4}t + \frac{\pi}{2}\Big).$$
2. For $k = 5$:
   - $a_5 = 2$ (a real number) so that
     $$|a_5| = 2, \quad \phi_5 = 0.$$
   - Then the cosine term for $k = 5$ is
     $$2|a_5|\cos\Big(5\frac{\pi}{4}t + 0\Big) = 4\cos\Big(\frac{5\pi}{4}t\Big).$$

Thus the Fourier series representation in cosine form becomes

$$\boxed{x(t) = 2\cos\Big(\frac{\pi}{4}t + \frac{\pi}{2}\Big) + 4\cos\Big(\frac{5\pi}{4}t\Big).}$$

This is the desired expression for $x(t)$ in the form

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \phi_k).$$

Here, $A_1 = 2$ with $\phi_1 = \frac{\pi}{2}$ and $A_5 = 4$ with $\phi_5 = 0$; all other $A_k = 0$.

4 3.23(c)

We are given that the periodic signal $x(t)$ has period

$$T = 4,$$

so its fundamental angular frequency is

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}.$$

The Fourier series coefficients are given by

$$a_k = \begin{cases} jk, & |k| < 3, \\ 0, & \text{otherwise}, \end{cases}$$

so the only nonzero coefficients occur for

$$k = -2, -1, 0, 1, 2.$$

Since $a_0 = j \cdot 0 = 0$, the nonzero coefficients are:

$$a_{-2} = j\,(-2) = -2j, \quad a_{-1} = -j, \quad a_1 = j, \quad a_2 = 2j.$$

The synthesis equation for $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

Because only $k = -2, -1, 1, 2$ are nonzero, we have

$$x(t) = a_{-2} e^{-j2\frac{\pi}{2}t} + a_{-1} e^{-j\frac{\pi}{2}t} + a_1 e^{j\frac{\pi}{2}t} + a_2 e^{j2\frac{\pi}{2}t}.$$

Notice that

$$2\frac{\pi}{2}t = \pi t.$$

Thus,

$$x(t) = -2j\, e^{-j\pi t} - j\, e^{-j\frac{\pi}{2}t} + j\, e^{j\frac{\pi}{2}t} + 2j\, e^{j\pi t}.$$

Group the terms into two pairs (for $k = \pm 2$ and $k = \pm 1$):

1. For $k = \pm 2$:

$$-2j\, e^{-j\pi t} + 2j\, e^{j\pi t} = 2j\Big(e^{j\pi t} - e^{-j\pi t}\Big) = 2j\Big(2j\sin(\pi t)\Big) = -4\sin(\pi t),$$

   since $j \cdot j = -1$.

2. For $k = \pm 1$:

$$-j\, e^{-j\frac{\pi}{2}t} + j\, e^{j\frac{\pi}{2}t} = j\Big(e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}\Big) = j\Big(2j\sin\Big(\frac{\pi}{2}t\Big)\Big) = -2\sin\Big(\frac{\pi}{2}t\Big).$$

Thus, the reconstructed signal is

$$\boxed{x(t) = -4\sin(\pi t) - 2\sin\Big(\frac{\pi}{2}t\Big).}$$

5 3.22(b)

We are given a periodic signal with period

$$T = 2,$$

and fundamental angular frequency

$$\omega_0 = \frac{2\pi}{T} = \pi.$$

The signal is defined over one period by

$$x(t) = e^{-t}, \quad -1 < t < 1,$$

and then extended periodically with period 2.

The Fourier series representation of a periodic signal is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t},$$

with Fourier coefficients

$$a_k = \frac{1}{T} \int_{T_0}^{T_0+T} x(t) \, e^{-jk\omega_0 t} \, dt.$$

A convenient integration interval is $[-1, 1]$. With $T = 2$ and $\omega_0 = \pi$ we have

$$a_k = \frac{1}{2} \int_{-1}^{1} e^{-t} \, e^{-jk\pi t} \, dt = \frac{1}{2} \int_{-1}^{1} e^{-t(1+jk\pi)} \, dt.$$

Let

$$\mu = 1 + jk\pi.$$

Then

$$a_k = \frac{1}{2} \int_{-1}^{1} e^{-\mu t} \, dt = \frac{1}{2} \left[\frac{e^{-\mu t}}{-\mu} \right]_{t=-1}^{t=1} = \frac{1}{2(-\mu)} \left(e^{-\mu \cdot 1} - e^{-\mu \cdot (-1)} \right).$$

That is,

$$a_k = \frac{-1}{2\mu} \left(e^{-\mu} - e^{\mu} \right) = \frac{1}{2\mu} \left(e^{\mu} - e^{-\mu} \right).$$

Recognizing the hyperbolic sine,

$$e^{\mu} - e^{-\mu} = 2 \sinh(\mu),$$

we obtain

$$a_k = \frac{1}{2\mu} \cdot 2 \sinh(\mu) = \frac{\sinh(1 + jk\pi)}{1 + jk\pi}.$$

Thus, the Fourier series representation of $x(t)$ is

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} \frac{\sinh(1 + jk\pi)}{1 + jk\pi} \, e^{jk\pi t}.}$$