Assignment1

March 12, 2025

1 2

Assume x has a relative error of 2%. What is the relative error of x^n?

1.1 Answer

x has a relative error of 2%, which meaning

$$\frac{\Delta x}{x} = 0.02.$$

For the function $f(x) = x^n$, the relative error can be approximated using the derivative:

$$f'(x) = n\,x^{n-1}.$$

Using error propagation, the relative error in $f(x)$ is given by:

$$\frac{\Delta f}{f} \approx \left|\frac{f'(x) \cdot x}{f(x)}\right| \cdot \frac{\Delta x}{x} = n \cdot \frac{\Delta x}{x}.$$

Thus, the relative error for x^n is:

$$\text{Relative error} = n \times 2\%.$$

Therefore, the answer is:

$$\text{Relative error of } x^n = n \cdot 2\%.$$

2 5

In order to compute the volume of a sphere with a relative error limit of 1%, what is the allowable relative error in measuring the radius R ?

2.1 Answer

To calculate the relative error in the radius required to keep the volume error within 1%, consider the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi R^3.$$

Using error propagation, the relative error in the volume is given by:

$$\frac{\Delta V}{V} \approx 3\frac{\Delta R}{R}.$$

Setting

$$\frac{\Delta V}{V} = 0.01,$$

we have:

$$3\frac{\Delta R}{R} = 0.01 \quad \Rightarrow \quad \frac{\Delta R}{R} \approx \frac{0.01}{3} \approx 0.00333,$$

which is approximately 0.33%.

Thus, the allowed relative error in the radius is about 0.33%.

3 6

Let $Y_0 = 28$. Using the recurrence relation

$$Y_n = Y_{n-1} - \frac{1}{100}\sqrt{783}, \quad n = 1, 2, \cdots,$$

compute Y_{100}. If $\sqrt{783}$ is approximated as 27.982, what is the error in the computed Y_{100}?

3.1 Answer

After 100 steps we have

$$Y_{100} = Y_0 - 100 \cdot \frac{1}{100}\sqrt{783} = 28 - \sqrt{783}.$$

When $\sqrt{783}$ is approximated as 27.982, the computed value is

$$Y_{100}^{(c)} = 28 - 27.982 = 0.018.$$

However, the true value is

$$Y_{100} = 28 - \sqrt{783}.$$

Thus, the error in Y_{100} is

$$\text{Error} = \left|Y_{100} - Y_{100}^{(c)}\right| \le \frac{1}{2} \times 10^{-3}.$$

4 7

Find the two roots of the equation

$$x^2 - 56x + 1 = 0$$

ensuring that each root is expressed with at least four significant digits. (Note: $\sqrt{783} \approx 27.982$.)

4.1 Answer

To solve the quadratic equation, we use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = -56$, and $c = 1$. Substituting these values:

$$x = \frac{56 \pm \sqrt{56^2 - 4 \cdot 1 \cdot 1}}{2} = \frac{56 \pm \sqrt{3132}}{2} = 28 \pm \sqrt{783}.$$

Thus, the two roots are approximately:

$$x \approx 28 + 27.982 = 55.982,$$

$$x \approx 28 - 27.982 = 0.018.$$

These values are expressed with at least four significant digits.

5 9

For a square with a side length of approximately $100\,\text{cm}$, how should you measure it so that the error in the calculated area does not exceed $1\,\text{cm}^2$?

5.1 Answer

In a square, if the side length is L and the error in measurement is ΔL, the error propagation for the area $A = L^2$ gives:

$$\Delta A \approx 2L\,\Delta L.$$

To ensure that the error in the area does not exceed $1cm^2$, we set:

$$2L\,\Delta L \leq 1.$$

Substituting $L \approx 100cm$,

$$2 \times 100\,\Delta L \leq 1 \quad \Longrightarrow \quad 200\,\Delta L \leq 1.$$

Thus,

$$\Delta L \leq \frac{1}{200} = 0.005\,\text{cm}.$$

Therefore, the side length should be measured with a precision of at least $0.005cm$.

6 11

The sequence $\{y_n\}$ satisfies the recurrence relation

$$y_n = 10y_{n-1} - 1, \quad n = 1, 2, \cdots$$

Given $y_0 = \sqrt{2} \approx 1.41$, what is the error when computing y_{10}? Is this computational process stable?

6.1 Answer

We start with the recurrence

$$y_n = 10\, y_{n-1} - 1, \quad n = 1, 2, \dots,$$

whose general solution is given by the sum of the homogeneous solution and a particular solution. Since the homogeneous part is

$$y_n^{(h)} = A \cdot 10^n,$$

and the constant particular solution y_p satisfies

$$y_p = 10\, y_p - 1 \quad \Longrightarrow \quad y_p = \frac{1}{9},$$

the general solution is

$$y_n = A \cdot 10^n + \frac{1}{9}.$$

Using the initial condition $y_0 = \sqrt{2}$, we have

$$\sqrt{2} = A + \frac{1}{9} \quad \Longrightarrow \quad A = \sqrt{2} - \frac{1}{9}.$$

Thus,

$$y_n = \left(\sqrt{2} - \frac{1}{9}\right) 10^n + \frac{1}{9}.$$

However, if we approximate y_0 by 1.41 instead of $\sqrt{2}$, the computed sequence becomes

$$\tilde{y}_n = \left(1.41 - \frac{1}{9}\right) 10^n + \frac{1}{9}.$$

The error in the initial condition is

$$|\delta y_0| = \left|1.41 - \sqrt{2}\right| \leq \frac{1}{2} \times 10^{-2}.$$

Since the recurrence multiplies the deviation by 10 at each step, the error after n iterations will be amplified:

$$\delta y_n = 10^n\, \delta y_0.$$

For $n = 10$, the error is

$$|\delta y_{10}| \approx 10^{10}\, |\delta y_0| \approx 5 \times 10^7.$$

The absolute error (magnitude) is about 5×10^7, which is huge compared to the computed value.

This rapid amplification of the initial error shows that the computational process is unstable.

[2]:
```python
# use python to check the result
import math

# exact and approximate initial values
```

```python
y_exact = math.sqrt(2)
y_approx = 1.41

# number of iterations
n = 10

for i in range(1, n+1):
    y_exact = 10 * y_exact - 1
    y_approx = 10 * y_approx - 1

error = abs(y_exact - y_approx)

print("Exact y10:   ", y_exact)
print("Approx y10:  ", y_approx)
print("Error in y10:", error)
```

```
Exact y10:    13031024512.73095
Approx y10:   12988888889.0
Error in y10: 42135623.7309494
```

7 12

Calculate $f = (\sqrt{2}-1)^6$, using the approximation $\sqrt{2} \approx 1.4$.

Using the following expressions:

$$\frac{1}{(1+\sqrt{2})^6}, \quad (3-2\sqrt{2})^3, \quad \frac{1}{(3-2\sqrt{2})^3}, \quad 99-70\sqrt{2},$$

determine which one gives the most accurate result.

7.1 Answer

Let $x = \sqrt{2}$ (with $x \approx 1.4$ and an uncertainty $\Delta x \leq 0.05$). Our goal is to choose the representation for which the relative error in f (due to the error in x) is minimized.

7.1.1 1. Form $f = \frac{1}{(1+x)^6}$

Define

$$g(x) = \frac{1}{(1+x)^6}.$$

Its derivative is

$$g'(x) = -6(1+x)^{-7}.$$

The relative sensitivity is

$$\frac{|g'(x)|}{|g(x)|} = \frac{6}{1+x}.$$

For $x \approx 1.4$, $1+x \approx 2.4$ so

$$\text{Amplification factor} \approx \frac{6}{2.4} = 2.5.$$

Thus the propagated relative error in f becomes roughly $2.5\,\Delta x \approx 2.5 \times 0.05 = 0.125$ (or about 12.5%). This is considerably lower than in the first representation.

7.1.2 2. Form $f = (3 - 2x)^3$

Let

$$h(x) = (3 - 2x)^3.$$

Its derivative is

$$h'(x) = 3(3 - 2x)^2(-2) = -6(3 - 2x)^2.$$

Then,

$$\frac{|h'(x)|}{|h(x)|} = \frac{6}{|3 - 2x|}.$$

At $x \approx 1.4$, we have

$$3 - 2(1.4) = 3 - 2.8 = 0.2,$$

so the amplification factor is

$$\frac{6}{0.2} = 30.$$

This yields a relative error of about $30 \times 0.05 = 1.5$ (i.e. 150%), which is very high.

7.1.3 3. Form $f = \frac{1}{(3-2x)^3}$ or $f = 99 - 70x$

In the reciprocal form, the error amplification will be similar (or even worse) than in form 3.
For the linear form $99 - 70x$, a simple differentiation gives a constant sensitivity of 70, and for $x \approx 1.4$

$$99 - 70(1.4) \approx 99 - 98 = 1.$$

Thus the relative error factor is about 70, leading to an error of $70 \times 0.05 = 3.5$ (350%). In addition, this linear expression does not—even in exact arithmetic—equal $(\sqrt{2} - 1)^6$ when using the given approximation.

7.1.4 Conclusion

In summary, among the given expressions,

$$\frac{1}{(1 + \sqrt{2})^6}$$

gives the most accurate result for computing $f = (\sqrt{2} - 1)^6$ under the specified error constraints.

[3]:
```python
# use python to check the result
import numpy as np
f = (np.sqrt(2) - 1) ** 6

y1 = 1/((1 + 1.4) ** 6)
y2 = (3 - 2 * 1.4) ** 3
y3 = 1 / ((3 - 2 * 1.4) ** 3)
y4 = 99 - 70 * 1.4
```

```python
print("Accurate value: ", f)
print("y1: ", y1)
print("y2: ", y2)
print("y3: ", y3)
print("y4: ", y4)
```

```
Accurate value:  0.005050633883346591
y1:  0.005232780885631003
y2:  0.008000000000000021
y3:  124.99999999999967
y4:  1.0
```

8 13

Given

$$f(x) = \ln\left(x - \sqrt{x^2 - 1}\right),$$

compute $f(30)$. Suppose that the square root is computed using a 6-digit lookup table; what is the error in the logarithm calculation? If instead we use the equivalent formula

$$\ln\left(x - \sqrt{x^2 - 1}\right) = -\ln\left(x + \sqrt{x^2 - 1}\right),$$

what is the error in the logarithm calculation in that case?

8.1 Answer

A 6-digit lookup table implies that the square root is computed to roughly a relative accuracy of about 10^{-6}. That is, if

$$\sqrt{899} \approx s \quad \text{with} \quad s = \sqrt{899},$$

then the absolute error in s is approximately

$$|\Delta s| \leq \frac{1}{2} \times 10^{-4} = 5 \times 10^{-5}.$$

8.1.1 Using the Direct Form

$$f(30) = \ln\Big(30 - s\Big).$$

Let

$$u = 30 - s.$$

A small error Δs in s causes an error in u of the same size (since $u' = -1$), so $\Delta u \approx \Delta s$. Then by a first-order error propagation the absolute error in $f(30) = \ln(u)$ is

$$\Delta f \approx \left|\frac{d}{du}\ln(u)\right|\Delta u = \frac{\Delta s}{|u|}.$$

Since s is very near 30 (because $\sqrt{899}$ is almost 30), the difference

$$u = 30 - s$$

is a small number, and consequently the factor $\frac{1}{|u|}$ is large. This implies that a given absolute error Δs is greatly amplified in $\ln(30 - s)$.

8.1.2 Using the Alternative Form

$$f(30) = -\ln\Big(30 + s\Big).$$

Let

$$v = 30 + s.$$

Again an error Δs in s produces an error $\Delta v \approx \Delta s$ (because $v' = +1$). Now the error in the logarithm is given by

$$\Delta f \approx \left|\frac{d}{dv}[-\ln(v)]\right| \Delta v = \frac{\Delta s}{|v|}.$$

Since

$$v = 30 + s$$

is a large number (roughly twice 30), the amplification factor $\frac{1}{|v|}$ is very small. Thus, the error in f is much smaller in this form.

8.1.3 Summary

- **Direct form:**
  The error in $f(30) = \ln(30 - s)$ is approximately

  $$\Delta f \approx \frac{\Delta s}{30 - s}.$$

  Because $30 - s$ is very small, this amplification is very large.
- **Alternative form:**
  The error in $f(30) = -\ln(30 + s)$ is approximately

  $$\Delta f \approx \frac{\Delta s}{30 + s}.$$

  Since $30 + s$ is large, the amplification factor is small.

In conclusion, without computing $\sqrt{899}$ explicitly, and knowing only that $\Delta s \leq 0.5 \times 10^{-4}$, we see that the logarithm computed via

$$-\ln\Big(30 + \sqrt{899}\Big)$$

will have a much smaller error—from error amplification reasons—than the calculation using

$$\ln\Big(30 - \sqrt{899}\Big).$$

[4]:
```
import math

# True value of s = sqrt(899) computed at high precision.
```

```
s_true = math.sqrt(899)

# Compute true f using the two equivalent formulations
# Note: mathematically, they are equal.
f_direct_true = math.log(30 - s_true)
f_alt_true = -math.log(30 + s_true)

print("True s = sqrt(899):", s_true)
print("True f using ln(30 - s):", f_direct_true)
print("True f using -ln(30 + s):", f_alt_true)

# Assume the computed s (from a 6-digit lookup table) has an absolute error Δs␣
↪= ±0.5*10^-4 = 0.00005
Delta = 0.00005

# Simulate a perturbation: s is off by +Delta and by -Delta.
s_plus = s_true + Delta
s_minus = s_true - Delta

# Compute f using the direct form with perturbed s.
f_direct_plus = math.log(30 - s_plus)
f_direct_minus = math.log(30 - s_minus)

# Compute f using the alternative form with perturbed s.
f_alt_plus = -math.log(30 + s_plus)
f_alt_minus = -math.log(30 + s_minus)

# Compute absolute errors relative to the true f (using s_true).
err_direct_plus = abs(f_direct_plus - f_direct_true)
err_direct_minus = abs(f_direct_minus - f_direct_true)
err_alt_plus = abs(f_alt_plus - f_alt_true)
err_alt_minus = abs(f_alt_minus - f_alt_true)

print("\n=== Direct Form: f = ln(30 - s) ===")
print("Perturbation: s + Δ")
print("f =", f_direct_plus, "Error =", err_direct_plus)
print("Perturbation: s - Δ")
print("f =", f_direct_minus, "Error =", err_direct_minus)

print("\n=== Alternative Form: f = -ln(30 + s) ===")
print("Perturbation: s + Δ")
print("f =", f_alt_plus, "Error =", err_alt_plus)
print("Perturbation: s - Δ")
print("f =", f_alt_minus, "Error =", err_alt_minus)

# Summary: Compare amplification factors by checking the ratio Δf/Δs.
amp_direct_plus = err_direct_plus / Delta
```

```python
amp_direct_minus = err_direct_minus / Delta
amp_alt_plus = err_alt_plus / Delta
amp_alt_minus = err_alt_minus / Delta

print("\n=== Amplification Factors ===")
print("Direct (s + Δ):", amp_direct_plus)
print("Direct (s - Δ):", amp_direct_minus)
print("Alternative (s + Δ):", amp_alt_plus)
print("Alternative (s - Δ):", amp_alt_minus)
```

```
True s = sqrt(899): 29.9833287011299
True f using ln(30 - s): -4.094066668632055
True f using -ln(30 + s): -4.0940666686320855

=== Direct Form: f = ln(30 - s) ===
Perturbation: s + Δ
f = -4.097070341579641 Error = 0.003003672947585301
Perturbation: s - Δ
f = -4.091071990724231 Error = 0.0029946779078242614

=== Alternative Form: f = -ln(30 + s) ===
Perturbation: s + Δ
f = -4.0940675021966815 Error = 8.335645960144689e-07
Perturbation: s - Δ
f = -4.094065835066794 Error = 8.335652914581715e-07

=== Amplification Factors ===
Direct (s + Δ): 60.073458951706016
Direct (s - Δ): 59.89355815648523
Alternative (s + Δ): 0.016671291920289377
Alternative (s - Δ): 0.01667130582916343
```

9 14

Evaluate the polynomial

$$p(x) = 3x^5 - 2x^3 + x + 7$$

at $x = 3$ using Qin Jiushao's algorithm.

9.1 Answer

$$p(3) = 3 \cdot 3^5 + 0 \cdot 3^4 - 2 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3 + 7$$

$$= (((3 \cdot 3 + 0) \cdot 3 - 2) \cdot 3 + 0) \cdot 3 + 1) \cdot 3 + 7$$

$$\text{Let } b_5 = 3,$$

$$b_4 = 3 \cdot 3 + 0 = 9,$$

$$b_3 = 9 \cdot 3 - 2 = 27 - 2 = 25,$$

$$b_2 = 25 \cdot 3 + 0 = 75,$$

$$b_1 = 75 \cdot 3 + 1 = 225 + 1 = 226,$$

$$b_0 = 226 \cdot 3 + 7 = 678 + 7 = 685.$$

Thus, by using Qin Jiushao's algorithm, we find that

$$p(3) = 685.$$

10 1

Given that at $x = 1, -1, 2$ the function values are

$$f(1) = 0, \quad f(-1) = -3, \quad f(2) = 4,$$

find the quadratic interpolation polynomial for $f(x)$.

10.1 (1) Using the monomial basis

10.1.1 Answer

$$\begin{aligned} \text{Let } p(x) &= ax^2 + bx + c. \\ \text{From } x = 1: &\quad a + b + c = 0, \\ x = -1: &\quad a - b + c = -3, \\ x = 2: &\quad 4a + 2b + c = 4. \end{aligned}$$

Subtracting the first two equations:

$$(a + b + c) - (a - b + c) = 2b = 0 - (-3) = 3 \quad \Longrightarrow \quad b = \frac{3}{2}.$$

Then, using $a + b + c = 0$:

$$a + c = -\frac{3}{2}.$$

Next, substitute $b = \frac{3}{2}$ in the third equation:

$$4a + 2\left(\frac{3}{2}\right) + c = 4 \quad \Longrightarrow \quad 4a + 3 + c = 4,$$

so

$$4a + c = 1.$$

Subtract the equation $a + c = -\frac{3}{2}$ from $4a + c = 1$:

$$(4a + c) - (a + c) = 3a = 1 - \left(-\frac{3}{2}\right) = \frac{5}{2} \quad \Longrightarrow \quad a = \frac{5}{6}.$$

Finally, compute c:

$$c = -\frac{3}{2} - a = -\frac{3}{2} - \frac{5}{6} = -\frac{9}{6} - \frac{5}{6} = -\frac{14}{6} = -\frac{7}{3}.$$

Thus, the quadratic interpolation polynomial is:

$$p(x) = \frac{5}{6}x^2 + \frac{3}{2}x - \frac{7}{3}.$$

10.2 (2) Using the Lagrange interpolation basis

10.2.1 Answer

We have the data points

$$(1, 0), \quad (-1, -3), \quad (2, 4).$$

The Lagrange interpolation polynomial is given by

$$p(x) = \sum_{j=1}^{3} f(x_j) L_j(x),$$

where the Lagrange basis functions are defined as

$$L_j(x) = \prod_{\substack{i=1 \\ i \neq j}}^{3} \frac{x - x_i}{x_j - x_i}.$$

Labeling the points as

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = 2,$$

and the corresponding function values

$$f(1) = 0, \quad f(-1) = -3, \quad f(2) = 4,$$

we compute:

1. For $x_1 = 1$:

$$L_1(x) = \frac{(x - x_2)(x - x_3)}{(1 - (-1))(1 - 2)} = \frac{(x + 1)(x - 2)}{(2)(-1)} = -\frac{(x + 1)(x - 2)}{2}.$$

2. For $x_2 = -1$:

$$L_2(x) = \frac{(x-x_1)(x-x_3)}{(-1-1)(-1-2)} = \frac{(x-1)(x-2)}{(-2)(-3)} = \frac{(x-1)(x-2)}{6}.$$

3. For $x_3 = 2$:

$$L_3(x) = \frac{(x-x_1)(x-x_2)}{(2-1)(2-(-1))} = \frac{(x-1)(x+1)}{(1)(3)} = \frac{(x-1)(x+1)}{3}.$$

The interpolation polynomial is then

$$p(x) = 0 \cdot L_1(x) - 3 \cdot L_2(x) + 4 \cdot L_3(x).$$

That is,

$$p(x) = -3 \cdot \frac{(x-1)(x-2)}{6} + 4 \cdot \frac{(x-1)(x+1)}{3}.$$

Simplify each term:

$$-3 \cdot \frac{(x-1)(x-2)}{6} = -\frac{1}{2}(x-1)(x-2),$$
$$4 \cdot \frac{(x-1)(x+1)}{3} = \frac{4}{3}(x-1)(x+1).$$

Thus,

$$p(x) = -\frac{1}{2}(x^2 - 3x + 2) + \frac{4}{3}(x^2 - 1).$$

Expanding,

$$-\frac{1}{2}x^2 + \frac{3}{2}x - 1 + \frac{4}{3}x^2 - \frac{4}{3}.$$

Combine like terms:

$$x^2 : -\frac{1}{2} + \frac{4}{3} = \frac{-3+8}{6} = \frac{5}{6},$$
$$x : \frac{3}{2},$$
$$\text{constant:} \ -1 - \frac{4}{3} = -\frac{3}{3} - \frac{4}{3} = -\frac{7}{3}.$$

Thus, the quadratic interpolation polynomial is

$$p(x) = \frac{5}{6}x^2 + \frac{3}{2}x - \frac{7}{3}.$$

11 2

Given the numerical table for $f(x) = \ln x$:

x	0.4	0.5	0.6	0.7	0.8
$\ln x$	-0.916291	-0.693147	-0.510826	-0.356675	-0.223144

Use linear interpolation and quadratic interpolation to estimate the value of $\ln(0.54)$.

11.1 Answer

Linear Interpolation:

Using the tabulated values, choose $x_0 = 0.5$ with $f(x_0) = \ln(0.5) = -0.693147$ and $x_1 = 0.6$ with $f(x_1) = \ln(0.6) = -0.510826$. Then

$$\ln(0.54) \approx \ln(0.5) + \frac{\ln(0.6) - \ln(0.5)}{0.6 - 0.5}(0.54 - 0.5).$$

That is,

$$\ln(0.54) \approx -0.693147 + \frac{-0.510826 + 0.693147}{0.1}(0.04)$$

$$\ln(0.54) \approx -0.693147 + \frac{0.182321}{0.1}(0.04)$$

$$\ln(0.54) \approx -0.693147 + 1.82321 \times 0.04 \approx -0.693147 + 0.072928 \approx -0.620219.$$

Quadratic Interpolation:

Choose the three points $x_0 = 0.5$, $x_1 = 0.6$, and $x_2 = 0.7$ with

$$f(0.5) = -0.693147, \quad f(0.6) = -0.510826, \quad f(0.7) = -0.356675.$$

The Lagrange basis polynomials at $x = 0.54$ are:

$$L_0(0.54) = \frac{(0.54 - 0.6)(0.54 - 0.7)}{(0.5 - 0.6)(0.5 - 0.7)} = \frac{(-0.06)(-0.16)}{(-0.1)(-0.2)} = \frac{0.0096}{0.02} = 0.48,$$

$$L_1(0.54) = \frac{(0.54 - 0.5)(0.54 - 0.7)}{(0.6 - 0.5)(0.6 - 0.7)} = \frac{(0.04)(-0.16)}{(0.1)(-0.1)} = \frac{-0.0064}{-0.01} = 0.64,$$

$$L_2(0.54) = \frac{(0.54 - 0.5)(0.54 - 0.6)}{(0.7 - 0.5)(0.7 - 0.6)} = \frac{(0.04)(-0.06)}{(0.2)(0.1)} = \frac{-0.0024}{0.02} = -0.12.$$

Then the quadratic interpolation estimate is

$$\ln(0.54) \approx f(0.5)L_0(0.54) + f(0.6)L_1(0.54) + f(0.7)L_2(0.54)$$

$$\ln(0.54) \approx (-0.693147)(0.48) + (-0.510826)(0.64) + (-0.356675)(-0.12).$$

Calculating each term:

$$-0.693147 \times 0.48 \approx -0.332711, \quad -0.510826 \times 0.64 \approx -0.326928, \quad -0.356675 \times (-0.12) \approx 0.042801.$$

Thus,

$$\ln(0.54) \approx -0.332711 - 0.326928 + 0.042801 \approx -0.616838.$$

$$\boxed{\begin{array}{ll} \text{Linear interpolation:} & \ln(0.54) \approx -0.6202, \\ \text{Quadratic interpolation:} & \ln(0.54) \approx -0.6168. \end{array}}$$