Assignment2

March 26, 2025

1 1(3)

We are given the following function values:

$$f(1) = 0, \quad f(-1) = -3, \quad f(2) = 4.$$

We wish to find the quadratic interpolation polynomial in Newton form using the nodes

$$x_0 = 1, \quad x_1 = -1, \quad x_2 = 2.$$

The Newton interpolation polynomial is expressed as

$$P(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1).$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{-3 - 0}{-1 - 1} = \frac{-3}{-2} = \frac{3}{2}.$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{4 - (-3)}{2 - (-1)} = \frac{7}{3}.$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{7}{3} - \frac{3}{2}}{2 - 1}.$$

Find a common denominator:

$$\frac{7}{3} - \frac{3}{2} = \frac{14}{6} - \frac{9}{6} = \frac{5}{6}.$$

Thus,

$$f[x_0, x_1, x_2] = \frac{5/6}{1} = \frac{5}{6}.$$

Plug the values into the Newton form:

$$P(x) = 0 + \frac{3}{2}(x - 1) + \frac{5}{6}(x - 1)(x - (-1)).$$

Since $x - (-1) = x + 1$, we have:

$$\boxed{P(x) = \frac{3}{2}(x - 1) + \frac{5}{6}(x - 1)(x + 1).}$$

Optionally, expanding the product:

$$(x-1)(x+1) = x^2 - 1,$$

so

$$P(x) = \frac{3}{2}(x-1) + \frac{5}{6}(x^2-1).$$

To express $P(x)$ in standard form, combine the terms:

$$\begin{aligned}
P(x) &= \frac{5}{6}x^2 + \frac{3}{2}x - \left(\frac{3}{2} + \frac{5}{6}\right) \\
&= \frac{5}{6}x^2 + \frac{3}{2}x - \frac{9+5}{6} \\
&= \frac{5}{6}x^2 + \frac{3}{2}x - \frac{14}{6} \\
&= \frac{5x^2 + 9x - 14}{6}.
\end{aligned}$$

Thus, the quadratic interpolation polynomial in Newton form is:

$$\boxed{P(x) = \frac{3}{2}(x-1) + \frac{5}{6}(x-1)(x+1), \quad \text{or equivalently, } P(x) = \frac{5x^2+9x-14}{6}.}$$

2 3

1. **Rounding Error in the Table Entries**

   Since the table has 5 significant digits, the rounding error is bounded by

   $$\delta \leq 0.5 \times 10^{-5}$$

   Thus, we have an error of roughly

   $$\delta \lesssim 5 \times 10^{-6}.$$

2. **Linear Interpolation Error**

   When using linear interpolation between two adjacent table points z_i and z_{i+1} (with spacing $h = 1'$), the interpolation error for a C^2 function $f(z)$ is bounded by

   $$E_{\text{interp}} \leq \frac{M}{8}(z_{i+1} - z_i)^2,$$

   where

   $$M = \max_{z \in [z_i,\, z_{i+1}]} |f''(z)|.$$

   Here $f(z) = \cos z$. Taking z in radians, the second derivative is

   $$f^{(2)}(z) = -\cos z$$

   and thus $|f''(z)| \leq 1$.

we have

$$E_{\text{interp}} \leq \frac{1}{8}\left(\frac{\pi}{10800}\right)^2.$$

Numerically,

$$\left(\frac{\pi}{10800}\right)^2 = \frac{\pi^2}{10800^2} \approx \frac{9.87}{116640000} \approx 8.46 \times 10^{-8},$$

so

$$E_{\text{interp}} \leq \frac{8.46 \times 10^{-8}}{8} \approx 1.06 \times 10^{-8}.$$

This interpolation error is negligible compared to the rounding error.

2.1 Total Error Bound

The overall error when using linear interpolation is the sum of the table rounding error and the interpolation error. Since

$$E_{\text{total}} \lesssim \delta + E_{\text{interp}} \approx 5 \times 10^{-6} + 1.06 \times 10^{-8},$$

the total error bound is dominated by the rounding error.

Thus, we have

$$\boxed{E_{\text{total}} \lesssim 5 \times 10^{-6}.}$$

3 4(2)

We wish to prove that if $x_0, x_1, \dots, x_n$ are distinct nodes and the Lagrange basis polynomials are given by

$$\ell_j(x) = \prod_{\substack{0 \leq m \leq n \\ m \neq j}} \frac{x - x_m}{x_j - x_m}, \quad j = 0, 1, \dots, n,$$

then for any integer k with $1 \leq k \leq n$ the following holds identically:

$$\sum_{j=0}^{n} (x_j - x)^k \ell_j(x) \equiv 0.$$

3.1 Proof

$$\sum_{j=0}^{n} (x_j - x)^k \ell_j(x) \equiv \sum_{j=0}^{n} \ell_j(x) \sum_{m=0}^{k} (-1)^m x^m x_j^{k-m}$$

Since we know,

$$\sum_{i=0}^{n} x_i^k \ell_i(x) \equiv x^k$$

Therefore,

$$
\begin{aligned}
\sum_{j=0}^{n}(x_j - x)^k \ell_j(x) &\equiv \sum_{m=0}^{k}\sum_{j=0}^{n}(-1)^m x^m x_j^{k-m}\ell_j(x) \\
&\equiv \sum_{m=0}^{k}(-1)^m x^m \sum_{j=0}^{n} x_j^{k-m}\ell_j(x) \\
&\equiv \sum_{m=0}^{k}(-1)^m x^m x^{k-m} \\
&\equiv (x-x)^k \\
&\equiv 0
\end{aligned}
$$

4 6

We wish to approximate $f(x) = e^x$ on the interval

$$-4 \le x \le 4,$$

by quadratic interpolation using an equidistant function table. We require that the truncation (interpolation) error be no more than

$$10^{-6}.$$

4.1 Step 1. Error Formula for Quadratic Interpolation

For quadratic interpolation (i.e. interpolation by a polynomial of degree 2) on three nodes x_0, x_1, x_2, the interpolation error at any point x in $[x_0, x_2]$ is given by

$$R(x) = \frac{f^{(3)}(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2),$$

for some ξ in the interval. In our case, the nodes are equally spaced with step size h.

4.2 Step 2. Maximum Error over a Subinterval

Let the three successive nodes be

$$x_0, \quad x_1 = x_0 + h, \quad x_2 = x_0 + 2h.$$

Define a shifted variable

$$t = x - x_0, \quad \text{with } t \in [0, 2h].$$

Then the error becomes

$$R(t) = \frac{f^{(3)}(\xi)}{6}\, t\,(t-h)\,(t-2h).$$

We wish to bound the maximum of

$$|P(t)| = |t\,(t-h)\,(t-2h)|$$

for $t \in [0, 2h]$. It can be shown (by finding the critical points) that the maximum absolute value is attained at

$$t = h\Big(1 - \frac{1}{\sqrt{3}}\Big),$$

so

$$\max_{t\in[0,2h]} |t\,(t-h)\,(t-2h)| = \frac{2h^3}{3\sqrt{3}}.$$

Thus the maximum interpolation error over any subinterval is bounded by

$$|R(x)| \le \frac{M_3}{6}\cdot\frac{2h^3}{3\sqrt{3}} = \frac{M_3\,h^3}{9\sqrt{3}},$$

where

$$M_3 = \max_{x\in[-4,4]} |f^{(3)}(x)|.$$

4.3 Step 3. Estimating M_3 for $f(x) = e^x$

Since

$$f(x) = e^x, \quad f^{(3)}(x) = e^x,$$

we have

$$|f^{(3)}(x)| = e^x.$$

On the interval $[-4, 4]$ the maximum occurs at $x = 4$ so that

$$M_3 = e^4.$$

4.4 Step 4. Imposing the Error Tolerance

We require

$$\frac{e^4\,h^3}{9\sqrt{3}} \le 10^{-6}.$$

Solving for h, we have

$$h^3 \le \frac{9\sqrt{3}\,10^{-6}}{e^4}.$$

Taking cube roots:

$$h \le \left(\frac{9\sqrt{3}\,10^{-6}}{e^4}\right)^{\frac{1}{3}}.$$

4.5 Step 5. Evaluating the Expression

Using the approximate value $e^4 \approx 54.598$ and $\sqrt{3} \approx 1.732$, we have:

$$9\sqrt{3} \approx 9 \times 1.732 \approx 15.588,$$

so that

$$\frac{9\sqrt{3}\,10^{-6}}{e^4} \approx \frac{15.588 \times 10^{-6}}{54.598} \approx 2.853 \times 10^{-7}.$$

Taking the cube root,

$$h \le (2.853 \times 10^{-7})^{1/3} \approx 6.56 \times 10^{-3}.$$

4.6 Final Answer

To keep the quadratic interpolation truncation error below 10^{-6}, the step size h for the equidistant function table should be chosen approximately as

$$\boxed{h \approx 6.6 \times 10^{-3}.}$$

5 7

We wish to prove the following two properties of the divided differences.

5.1 Property (1)

Statement:
If

$$F(x) = c\, f(x),$$

then

$$F[x_0, x_1, \dots, x_n] = c\, f[x_0, x_1, \dots, x_n].$$

Proof:
We proceed by induction on n.

- **Base Case ($n = 0$):**
  The zeroth divided difference is just the function value evaluated at the node:

  $$F[x_0] = F(x_0) = c\, f(x_0) = c\, f[x_0].$$

  Thus, the property holds for $n = 0$.

- **Inductive Step:**
  Assume the property is true for all orders less than n. For $n \geq 1$, by the definition of divided differences (for distinct nodes),

  $$F[x_0, x_1, \dots, x_n] = \frac{F[x_1, x_2, \dots, x_n] - F[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}.$$

  By the induction hypothesis, we have:

  $$F[x_1, x_2, \dots, x_n] = c\, f[x_1, x_2, \dots, x_n] \quad \text{and} \quad F[x_0, x_1, \dots, x_{n-1}] = c\, f[x_0, x_1, \dots, x_{n-1}].$$

  Thus,

  $$F[x_0, x_1, \dots, x_n] = \frac{c\, f[x_1, x_2, \dots, x_n] - c\, f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0} = c\, \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}.$$

  By definition, the last factor is $f[x_0, x_1, \dots, x_n]$. Hence,

  $$F[x_0, x_1, \dots, x_n] = c\, f[x_0, x_1, \dots, x_n].$$

This completes the proof of Property (1).

5.2 Property (2)

Statement:
If

$$F(x) = f(x) + g(x),$$

then

$$F[x_0, x_1, \dots, x_n] = f[x_0, x_1, \dots, x_n] + g[x_0, x_1, \dots, x_n].$$

Proof:
Again, we use induction on n.

- **Base Case ($n = 0$):**

$$F[x_0] = F(x_0) = f(x_0) + g(x_0) = f[x_0] + g[x_0].$$

  Therefore, the property holds for $n = 0$.

- **Inductive Step:**
  Assume the result is true for orders less than n. For $n \geq 1$, by the definition of divided differences,

$$F[x_0, x_1, \dots, x_n] = \frac{F[x_1, x_2, \dots, x_n] - F[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}.$$

  Using the induction hypothesis,

$$F[x_1, x_2, \dots, x_n] = f[x_1, x_2, \dots, x_n] + g[x_1, x_2, \dots, x_n],$$

  and

$$F[x_0, x_1, \dots, x_{n-1}] = f[x_0, x_1, \dots, x_{n-1}] + g[x_0, x_1, \dots, x_{n-1}].$$

  Substituting these into the formula,

$$\begin{aligned} F[x_0, x_1, \dots, x_n] &= \frac{\Big(f[x_1, x_2, \dots, x_n] + g[x_1, x_2, \dots, x_n]\Big) - \Big(f[x_0, x_1, \dots, x_{n-1}] + g[x_0, x_1, \dots, x_{n-1}]\Big)}{x_n - x_0} \\ &= \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0} + \frac{g[x_1, x_2, \dots, x_n] - g[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0} \\ &= f[x_0, x_1, \dots, x_n] + g[x_0, x_1, \dots, x_n]. \end{aligned}$$

This completes the proof of Property (2).

6 8

We are given

$$f(x) = x^7 + x^4 + 3x + 1,$$

and we wish to compute the divided differences

$$f[2^0, 2^1, \dots, 2^7] \quad \text{and} \quad f[2^0, 2^1, \dots, 2^8].$$

Since we know

$$f[x_0, x_1, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

6.1 Application to $f(x) = x^7 + x^4 + 3x + 1$

1. **For the nodes $2^0, 2^1, \ldots, 2^7$:**

   There are 8 nodes, so the highest (7th) divided difference is

   $$f[2^0, 2^1, \ldots, 2^7].$$

   Since $f(x)$ is a polynomial of degree 7 with leading coefficient 1 (from the x^7 term), it follows that

   $$f[2^0, 2^1, \ldots, 2^7] = 1.$$

2. **For the nodes $^0, 2^1, \ldots, 2^8$:**

   Here we have 9 nodes. Since $f(x)$s of degree 7, any divided difference of order 8 must be zero. That is,

   $$f[2^0, 2^1, \ldots, 2^8] = 0.$$

6.2 Final Answer

$$\boxed{f[2^0, 2^1, \ldots, 2^7] = 1 \quad \text{and} \quad f[2^0, 2^1, \ldots, 2^8] = 0.}$$

7 9

We need to prove that for the forward difference operator defined by

$$\Delta a_k = a_{k+1} - a_k,$$

the following product rule holds:

$$\Delta(f_k g_k) = f_k \, \Delta g_k + g_{k+1} \, \Delta f_k.$$

Proof:

1. By the definition of the forward difference,

   $$\Delta(f_k g_k) = f_{k+1} g_{k+1} - f_k g_k.$$

2. Now, add and subtract $f_k g_{k+1}$to the right-hand side:

   $$f_{k+1} g_{k+1} - f_k g_k = \Big[f_{k+1} g_{k+1} - f_k g_{k+1}\Big] + \Big[f_k g_{k+1} - f_k g_k\Big].$$

3. Factor out common factors in each bracket:

   $$f_{k+1} g_{k+1} - f_k g_{k+1} = g_{k+1}(f_{k+1} - f_k) = g_{k+1} \, \Delta f_k,$$

   and

   $$f_k g_{k+1} - f_k g_k = f_k(g_{k+1} - g_k) = f_k \, \Delta g_k.$$

4. Thus, combining these, we obtain:

$$\Delta(f_k g_k) = g_{k+1}\,\Delta f_k + f_k\,\Delta g_k,$$

which is exactly the desired result:

$$\boxed{\Delta(f_k g_k) = f_k\,\Delta g_k + g_{k+1}\,\Delta f_k.}$$

This completes the proof.

8 10

We want to prove the following identity:

$$\sum_{k=0}^{n-1} f_k \Delta g_k = f_n g_n - f_0 g_0 - \sum_{k=0}^{n-1} g_{k+1} \Delta f_k,$$

where the forward difference operator is defined by

$$\Delta f_k = f_{k+1} - f_k \quad \text{and} \quad \Delta g_k = g_{k+1} - g_k.$$

Proof:

1. Write the telescoping sum:

$$f_n g_n - f_0 g_0 = \sum_{k=0}^{n-1} \Big(f_{k+1} g_{k+1} - f_k g_k \Big).$$

2. Use the product rule for finite differences. Recall that (as proved in a previous exercise)

$$\Delta(f_k g_k) = f_k\,\Delta g_k + g_{k+1}\,\Delta f_k.$$

   Hence, for each k,

$$f_{k+1} g_{k+1} - f_k g_k = f_k \Delta g_k + g_{k+1} \Delta f_k.$$

3. Substitute the above into the telescoping sum:

$$f_n g_n - f_0 g_0 = \sum_{k=0}^{n-1} \Big(f_k \Delta g_k + g_{k+1} \Delta f_k \Big) = \sum_{k=0}^{n-1} f_k \Delta g_k + \sum_{k=0}^{n-1} g_{k+1} \Delta f_k.$$

4. Rearranging the result gives:

$$\sum_{k=0}^{n-1} f_k \Delta g_k = f_n g_n - f_0 g_0 - \sum_{k=0}^{n-1} g_{k+1} \Delta f_k.$$

This completes the proof.

9 11

Prove that

$$\sum_{j=0}^{n-1} \Delta^2 y_j = \Delta y_n - \Delta y_0$$

Proof:

Write the telescoping sum

$$\Delta y_n - \Delta y_0 = \sum_{j=0}^{n-1} \Delta y_{j+1} - \Delta y_j = \sum_{j=0}^{n-1} \Delta^2 y_j$$

This completes the proof.

10 12

We are given a polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i = a_n\,(x - x_1)(x - x_2)\cdots(x - x_n)$$

with n distinct real roots $x_1, x_2, \dots, x_n$. We wish to prove that

$$\sum_{j=1}^{n} \frac{x_j^k}{f'(x_j)} = \begin{cases} 0, & 0 \le k \le n-2, \\ a_n^{-1}, & k = n-1. \end{cases}$$

A common strategy is to relate the fractions $\frac{x_j^k}{f'(x_j)}$ to the partial fractions expansion of a rational function. In particular, note that for any integer (with $\le k \le n-1$) the rational function

$$\frac{x^k}{f(x)}$$

can be written in the partial-fractions form (its poles are simple at $x_1, \dots, x_n$):

$$\frac{x^k}{f(x)} = \sum_{j=1}^{n} \frac{A_j}{x - x_j}, \quad \text{with } A_j = \lim_{x \to x_j} (x - x_j)\frac{x^k}{f(x)} = \frac{x_j^k}{f'(x_j)}.$$

Thus,

$$\frac{x^k}{f(x)} = \sum_{j=1}^{n} \frac{x_j^k}{f'(x_j)} \cdot \frac{1}{x - x_j}.$$

Now, let us compare the asymptotic behavior as $x \to \infty$.

10.1 Asymptotic Expansion

Since

$$f(x) = a_n x^n + (\text{lower order terms}),$$

we have for large x

$$\frac{x^k}{f(x)} = \frac{x^k}{a_n x^n}\Big(1 + O(1/x)\Big) = \frac{1}{a_n}\, x^{k-n}\Big(1 + O(1/x)\Big).$$

On the other hand, using the expansion of each term in the partial-fractions expansion, for large x

$$\frac{1}{x - x_j} = \frac{1}{x}\left(1 + \frac{x_j}{x} + \frac{x_j^2}{x^2} + \cdots\right).$$

Thus,

$$\sum_{j=1}^{n} \frac{x_j^k}{f'(x_j)} \frac{1}{x - x_j} = \frac{1}{x} \sum_{j=1}^{n} \frac{x_j^k}{f'(x_j)} \left[1 + \frac{x_j}{x} + \frac{x_j^2}{x^2} + \cdots\right].$$

Collecting the first-order term (i.e. the term proportional to $1/x$), we have

$$\frac{1}{x} \sum_{j=1}^{n} \frac{x_j^k}{f'(x_j)} + \cdots.$$

10.2 Case 1: $\le k \le n - 2$

In this case, the exponent in the asymptotic expansion of the left-hand side is

$$x^{k-n}, \quad \text{with } k - n \le -2,$$

so the left-hand side decays like $1/x^2$(or faster). Thus, there is no \$/x\$term in its expansion. Equating the $1/x$ coefficients on both sides yields

$$\sum_{j=1}^{n} \frac{x_j^k}{f'(x_j)} = 0.$$

10.3 Case 2: $= n - 1$

For $= n - 1$, the left-hand side behaves as

$$\frac{x^{n-1}}{f(x)} \sim \frac{1}{a_n} \frac{1}{x},$$

so the coefficient of $1/x$ is $1/a_n$ On the other hand, the contribution in the partial-fractions expansion is exactly

$$\frac{1}{x} \sum_{j=1}^{n} \frac{x_j^{n-1}}{f'(x_j)} + \cdots.$$

Equating the two $1/x$ coefficients gives

$$\sum_{j=1}^{n} \frac{x_j^{n-1}}{f'(x_j)} = \frac{1}{a_n}.$$

10.4 Conclusion

We have shown that

$$\sum_{j=1}^{n} \frac{x_j^k}{f'(x_j)} = \begin{cases} 0, & 0 \le k \le n-2, \\ \dfrac{1}{a_n}, & k = n-1, \end{cases}$$

which is the desired result.

$$\boxed{\sum_{j=1}^{n} \frac{x_j^k}{f'(x_j)} = \begin{cases} 0, & 0 \le k \le n-2, \\ a_n^{-1}, & k = n-1. \end{cases}}$$

11 13

We wish to find a polynomial $P(x)$ of degree at most 3 such that

$$\begin{aligned} P(x_0) &= f(x_0), \\ P'(x_0) &= f'(x_0), \\ P''(x_0) &= f''(x_0), \\ P(x_1) &= f(x_1). \end{aligned}$$

Write

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3.$$

Then

$$\begin{aligned} P(x_0) &= a_0, \\ P'(x) &= a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2, \quad \text{so} \quad P'(x_0) = a_1, \\ P''(x) &= 2a_2 + 6a_3(x - x_0), \quad \text{so} \quad P''(x_0) = 2a_2. \end{aligned}$$

From the conditions,

$$\begin{aligned} a_0 &= f(x_0), \\ a_1 &= f'(x_0), \\ 2a_2 &= f''(x_0) \quad \Longrightarrow \quad a_2 = \frac{f''(x_0)}{2}. \end{aligned}$$

We require

$$P(x_1) = f(x_1).$$

Since

$$P(x_1) = a_0 + a_1(x_1 - x_0) + a_2(x_1 - x_0)^2 + a_3(x_1 - x_0)^3,$$

substitute the expressions for a_0, a_1, a_2:

$$f(x_0) + f'(x_0)(x_1 - x_0) + \frac{f''(x_0)}{2}(x_1 - x_0)^2 + a_3(x_1 - x_0)^3 = f(x_1).$$

Solve for a_3:

$$a_3 = \frac{f(x_1) - f(x_0) - f'(x_0)(x_1 - x_0) - \frac{f''(x_0)}{2}(x_1 - x_0)^2}{(x_1 - x_0)^3}.$$

11.1 Final Form

Thus, the desired polynomial is

$$\boxed{f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f(x_1) - f(x_0) - f'(x_0)(x_1 - x_0) - \frac{f''(x_0)}{2}(x_1 - x_0)^2}{(x_1 - x_0)^3}(x - x_0)^3.}$$

This is the unique polynomial of degree at most 3 that satisfies the given interpolation and derivative conditions.