Assignment1

March 11, 2025

1 2

Assume x has a relative error of 2%. What is the relative error of x^n?

1.1 Answer

x has a relative error of 2%, which meaning

$$\frac{\Delta x}{x} = 0.02.$$

For the function $f(x) = x^n$, the relative error can be approximated using the derivative:

$$f'(x) = n\,x^{n-1}.$$

Using error propagation, the relative error in $f(x)$ is given by:

$$\frac{\Delta f}{f} \approx \left| \frac{f'(x) \cdot x}{f(x)} \right| \cdot \frac{\Delta x}{x} = n \cdot \frac{\Delta x}{x}.$$

Thus, the relative error for x^n is:

$$\text{Relative error} = n \times 2\%.$$

Therefore, the answer is:

$$\text{Relative error of } x^n = n \cdot 2\%.$$

2 5

In order to compute the volume of a sphere with a relative error limit of 1%, what is the allowable relative error in measuring the radius R ?

2.1 Answer

To calculate the relative error in the radius required to keep the volume error within 1%, consider the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi R^3.$$

Using error propagation, the relative error in the volume is given by:

$$\frac{\Delta V}{V} \approx 3\frac{\Delta R}{R}.$$

Setting

$$\frac{\Delta V}{V} = 0.01,$$

we have:

$$3\frac{\Delta R}{R} = 0.01 \quad \Rightarrow \quad \frac{\Delta R}{R} \approx \frac{0.01}{3} \approx 0.00333,$$

which is approximately 0.33%.

Thus, the allowed relative error in the radius is about 0.33%.

3 6

Let $Y_0 = 28$. Using the recurrence relation

$$Y_n = Y_{n-1} - \frac{1}{100}\sqrt{783}, \quad n = 1, 2, \cdots,$$

compute Y_{100}. If $\sqrt{783}$ is approximated as 27.982, what is the error in the computed Y_{100}?

3.1 Answer

After 100 steps we have

$$Y_{100} = Y_0 - 100 \cdot \frac{1}{100}\sqrt{783} = 28 - \sqrt{783}.$$

When $\sqrt{783}$ is approximated as 27.982, the computed value is

$$Y_{100}^{(c)} = 28 - 27.982 = 0.018.$$

However, the true value is

$$Y_{100} = 28 - \sqrt{783}.$$

Thus, the error in Y_{100} is

$$\text{Error} = \left|28 - \sqrt{783} - 0.018\right| = \left|\sqrt{783} - 27.982\right|.$$

Using a linear approximation for $\sqrt{783}$ (since $783 = 784 - 1$ and $\sqrt{784} = 28$), we get

$$\sqrt{783} \approx 28 - \frac{1}{2 \cdot 28} = 28 - \frac{1}{56} \approx 28 - 0.0178571 = 27.9821429.$$

Therefore, the error is approximately

$$|27.9821429 - 27.982| \approx 0.0001429.$$

So, the computed Y_{100} has an error of about 1.43×10^{-4}.

4 7

Find the two roots of the equation

$$x^2 - 56x + 1 = 0$$

ensuring that each root is expressed with at least four significant digits. (Note: $\sqrt{783} \approx 27.982$.)

4.1 Answer

To solve the quadratic equation, we use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = -56$, and $c = 1$. Substituting these values:

$$x = \frac{56 \pm \sqrt{56^2 - 4 \cdot 1 \cdot 1}}{2} = \frac{56 \pm \sqrt{3132}}{2} = 28 \pm \sqrt{783}.$$

Thus, the two roots are approximately:

$$x \approx 28 + 27.982 = 55.982,$$

$$x \approx 28 - 27.982 = 0.018.$$

These values are expressed with at least four significant digits.

5 9

For a square with a side length of approximately 100 cm, how should you measure it so that the error in the calculated area does not exceed $1\,\mathrm{cm}^2$?

5.1 Answer

In a square, if the side length is L and the error in measurement is ΔL, the error propagation for the area $A = L^2$ gives:

$$\Delta A \approx 2L\,\Delta L.$$

To ensure that the error in the area does not exceed $1cm^2$, we set:

$$2L\,\Delta L \leq 1.$$

Substituting $L \approx 100cm$,

$$2 \times 100\,\Delta L \leq 1 \quad \Longrightarrow \quad 200\,\Delta L \leq 1.$$

Thus,

$$\Delta L \leq \frac{1}{200} = 0.005\,\mathrm{cm}.$$

Therefore, the side length should be measured with a precision of at least $0.005cm$.

6 11

The sequence $\{y_n\}$ satisfies the recurrence relation

$$y_n = 10y_{n-1} - 1, \quad n = 1, 2, \cdots$$

Given $y_0 = \sqrt{2} \approx 1.41$, what is the error when computing y_{10}? Is this computational process stable?

6.1 Answer

We start with the recurrence

$$y_n = 10\, y_{n-1} - 1, \quad n = 1, 2, \dots,$$

whose general solution is given by the sum of the homogeneous solution and a particular solution. Since the homogeneous part is

$$y_n^{(h)} = A \cdot 10^n,$$

and the constant particular solution y_p satisfies

$$y_p = 10\, y_p - 1 \quad \Longrightarrow \quad y_p = \frac{1}{9},$$

the general solution is

$$y_n = A \cdot 10^n + \frac{1}{9}.$$

Using the initial condition $y_0 = \sqrt{2}$, we have

$$\sqrt{2} = A + \frac{1}{9} \quad \Longrightarrow \quad A = \sqrt{2} - \frac{1}{9}.$$

Thus,

$$y_n = \left(\sqrt{2} - \frac{1}{9}\right) 10^n + \frac{1}{9}.$$

However, if we approximate y_0 by 1.41 instead of $\sqrt{2}$, the computed sequence becomes

$$\tilde{y}_n = \left(1.41 - \frac{1}{9}\right) 10^n + \frac{1}{9}.$$

The error in the initial condition is

$$\delta y_0 = 1.41 - \sqrt{2} \approx 1.41 - 1.41421356 \approx -0.00421356.$$

Since the recurrence multiplies the deviation by 10 at each step, the error after n iterations will be amplified:

$$\delta y_n = 10^n\, \delta y_0.$$

For $n = 10$, the error is

$$\delta y_{10} \approx 10^{10}\, \delta y_0 \approx 10^{10} \times (-0.00421356) \approx -4.21356 \times 10^7.$$

The absolute error (magnitude) is about 4.21×10^7, which is huge compared to the computed value.

This rapid amplification of the initial error shows that the computational process is unstable.

[3]:
```python
# use python to check the result
import math

# exact and approximate initial values
y_exact = math.sqrt(2)
y_approx = 1.41

# number of iterations
n = 10

for i in range(1, n+1):
    y_exact = 10 * y_exact - 1
    y_approx = 10 * y_approx - 1

error = abs(y_exact - y_approx)

print("Exact y10:   ", y_exact)
print("Approx y10:  ", y_approx)
print("Error in y10:", error)
```

```
Exact y10:     13031024512.73095
Approx y10:    12988888889.0
Error in y10: 42135623.7309494
```

7 12

Calculate $f = (\sqrt{2} - 1)^6$, using the approximation $\sqrt{2} \approx 1.4$.

Using the following expressions:

$$\frac{1}{(1+\sqrt{2})^6}, \quad (3-2\sqrt{2})^3, \quad \frac{1}{(3-2\sqrt{2})^3}, \quad 99-70\sqrt{2},$$

determine which one gives the most accurate result.

7.1 Answer

1. Directly computing
$$f \approx (1.4-1)^6 = (0.4)^6 = 0.004096.$$
2. Using the identity
$$\sqrt{2} - 1 = \frac{1}{1+\sqrt{2}},$$
we get
$$f = \left(\frac{1}{1+\sqrt{2}}\right)^6 \quad \Longrightarrow \quad f \approx \frac{1}{(1+1.4)^6} = \frac{1}{(2.4)^6} \approx \frac{1}{191.102976} \approx 0.005233.$$
3. Writing
$$(\sqrt{2}-1)^2 = 3 - 2\sqrt{2},$$

then

$$f = (\sqrt{2}{-}1)^6 = \left((\sqrt{2}{-}1)^2\right)^3 = (3{-}2\sqrt{2})^3 \quad \Longrightarrow \quad f \approx (3{-}2(1.4))^3 = (3{-}2.8)^3 = (0.2)^3 = 0.008.$$

4. The reciprocal

$$\frac{1}{(3-2\sqrt{2})^3}$$

yields

$$\frac{1}{(0.2)^3} = \frac{1}{0.008} = 125,$$

which is clearly far off.

5. Finally,

$$99 - 70\sqrt{2} \quad \Longrightarrow \quad 99 - 70(1.4) = 99 - 98 = 1.$$

Comparing these results to the true value (which is approximately 0.00506 when computed with the exact value of $\sqrt{2}$), the expression

$$\frac{1}{(1+\sqrt{2})^6}$$

gives

$$f \approx 0.005233,$$

which is closest to the true value.

Thus, the most accurate result is obtained using

$$\frac{1}{(1+\sqrt{2})^6}.$$

[4]:
```python
# use python to check the result
import numpy as np
f = (np.sqrt(2) - 1) ** 6

y1 = 1/((1 + 1.4) ** 6)
y2 = (3 - 2 * 1.4) ** 3
y3 = 1 / ((3 - 2 * 1.4) ** 3)
y4 = 99 - 70 * 1.4

print("Accurate value: ", f)
print("y1: ", y1)
print("y2: ", y2)
print("y3: ", y3)
print("y4: ", y4)
```

```
Accurate value:  0.005050633883346591
y1:  0.005232780885631003
y2:  0.008000000000000021
y3:  124.99999999999967
y4:  1.0
```

8 13

Given

$$f(x) = \ln\left(x - \sqrt{x^2 - 1}\right),$$

compute $f(30)$. Suppose that the square root is computed using a 6-digit lookup table; what is the error in the logarithm calculation? If instead we use the equivalent formula

$$\ln\left(x - \sqrt{x^2 - 1}\right) = -\ln\left(x + \sqrt{x^2 - 1}\right),$$

what is the error in the logarithm calculation in that case?

8.1 Answer

A 6-digit lookup table implies that the square root is computed to roughly a relative accuracy of about 10^{-6}. That is, if

$$\sqrt{899} \approx s \quad \text{with} \quad s = \sqrt{899} \quad \text{and} \quad \Delta s \approx 10^{-6}\, s,$$

then the absolute error in s is approximately

$$\Delta s \approx 30 \times 10^{-6} = 3 \times 10^{-5}.$$

Using the direct formula calculate the argument

$$u = 30 - \sqrt{899} \approx 30 - 29.98333 \approx 0.01667.$$

Since $f(30) = \ln(u)$, an error Δs in $\sqrt{899}$ produces an error

$$\Delta f \approx \left|\frac{\partial}{\partial s}\ln(30 - s)\right| \Delta s = \frac{\Delta s}{30 - \sqrt{899}} = \frac{3 \times 10^{-5}}{0.01667} \approx 1.8 \times 10^{-3}.$$

Alternatively, using the equivalent form

$$f(30) = -\ln\Big(30 + \sqrt{899}\Big)$$

the argument becomes

$$v = 30 + \sqrt{899} \approx 30 + 29.98333 \approx 59.98333.$$

Here the error in f is

$$\Delta f \approx \left|\frac{\partial}{\partial s}\Big(-\ln(v)\Big)\right| \Delta s = \frac{\Delta s}{30 + \sqrt{899}} = \frac{3 \times 10^{-5}}{59.98333} \approx 5.0 \times 10^{-7},$$

which is much smaller.

Thus, when computed via

$$\ln\Big(30 - \sqrt{899}\Big),$$

the logarithm has an error of approximately 1.8×10^{-3}. Using the equivalent formula

$$-\ln\Big(30 + \sqrt{899}\Big),$$

the error reduces to roughly 5.0×10^{-7}.

9 14

Evaluate the polynomial

$$p(x) = 3x^5 - 2x^3 + x + 7$$

at $x = 3$ using Qin Jiushao's algorithm.

9.1 Answer

$$p(3) = 3 \cdot 3^5 + 0 \cdot 3^4 - 2 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3 + 7$$

$$= (((3 \cdot 3 + 0) \cdot 3 - 2) \cdot 3 + 0) \cdot 3 + 1) \cdot 3 + 7$$

$$\text{Let } b_5 = 3,$$

$$b_4 = 3 \cdot 3 + 0 = 9,$$

$$b_3 = 9 \cdot 3 - 2 = 27 - 2 = 25,$$

$$b_2 = 25 \cdot 3 + 0 = 75,$$

$$b_1 = 75 \cdot 3 + 1 = 225 + 1 = 226,$$

$$b_0 = 226 \cdot 3 + 7 = 678 + 7 = 685.$$

Thus, by using Qin Jiushao's algorithm, we find that

$$p(3) = 685.$$

10 1

Given that at $x = 1, -1, 2$ the function values are

$$f(1) = 0, \quad f(-1) = -3, \quad f(2) = 4,$$

find the quadratic interpolation polynomial for $f(x)$.

10.1 (1) Using the monomial basis

10.1.1 Answer

$$\begin{aligned} &\text{Let } p(x) = ax^2 + bx + c. \\ \text{From } x = 1: &\quad a + b + c = 0, \\ x = -1: &\quad a - b + c = -3, \\ x = 2: &\quad 4a + 2b + c = 4. \end{aligned}$$

Subtracting the first two equations:

$$(a+b+c)-(a-b+c)=2b=0-(-3)=3 \quad \Longrightarrow \quad b=\frac{3}{2}.$$

Then, using $a+b+c=0$:

$$a+c=-\frac{3}{2}.$$

Next, substitute $b=\frac{3}{2}$ in the third equation:

$$4a+2\left(\frac{3}{2}\right)+c=4 \quad \Longrightarrow \quad 4a+3+c=4,$$

so

$$4a+c=1.$$

Subtract the equation $a+c=-\frac{3}{2}$ from $4a+c=1$:

$$(4a+c)-(a+c)=3a=1-\left(-\frac{3}{2}\right)=\frac{5}{2} \quad \Longrightarrow \quad a=\frac{5}{6}.$$

Finally, compute c:

$$c=-\frac{3}{2}-a=-\frac{3}{2}-\frac{5}{6}=-\frac{9}{6}-\frac{5}{6}=-\frac{14}{6}=-\frac{7}{3}.$$

Thus, the quadratic interpolation polynomial is:

$$p(x)=\frac{5}{6}x^2+\frac{3}{2}x-\frac{7}{3}.$$

10.2 (2) Using the Lagrange interpolation basis

10.2.1 Answer

We have the data points

$$(1,0), \quad (-1,-3), \quad (2,4).$$

The Lagrange interpolation polynomial is given by

$$p(x)=\sum_{j=1}^{3} f(x_j)L_j(x),$$

where the Lagrange basis functions are defined as

$$L_j(x)=\prod_{\substack{i=1\\ i\neq j}}^{3} \frac{x-x_i}{x_j-x_i}.$$

Labeling the points as

$$x_1=1, \quad x_2=-1, \quad x_3=2,$$

and the corresponding function values

$$f(1)=0, \quad f(-1)=-3, \quad f(2)=4,$$

we compute:

1. For $x_1 = 1$:

$$L_1(x) = \frac{(x - x_2)(x - x_3)}{(1 - (-1))(1 - 2)} = \frac{(x + 1)(x - 2)}{(2)(-1)} = -\frac{(x + 1)(x - 2)}{2}.$$

2. For $x_2 = -1$:

$$L_2(x) = \frac{(x - x_1)(x - x_3)}{(-1 - 1)(-1 - 2)} = \frac{(x - 1)(x - 2)}{(-2)(-3)} = \frac{(x - 1)(x - 2)}{6}.$$

3. For $x_3 = 2$:

$$L_3(x) = \frac{(x - x_1)(x - x_2)}{(2 - 1)(2 - (-1))} = \frac{(x - 1)(x + 1)}{(1)(3)} = \frac{(x - 1)(x + 1)}{3}.$$

The interpolation polynomial is then

$$p(x) = 0 \cdot L_1(x) - 3 \cdot L_2(x) + 4 \cdot L_3(x).$$

That is,

$$p(x) = -3 \cdot \frac{(x - 1)(x - 2)}{6} + 4 \cdot \frac{(x - 1)(x + 1)}{3}.$$

Simplify each term:

$$-3 \cdot \frac{(x - 1)(x - 2)}{6} = -\frac{1}{2}(x - 1)(x - 2),$$

$$4 \cdot \frac{(x - 1)(x + 1)}{3} = \frac{4}{3}(x - 1)(x + 1).$$

Thus,

$$p(x) = -\frac{1}{2}(x^2 - 3x + 2) + \frac{4}{3}(x^2 - 1).$$

Expanding,

$$-\frac{1}{2}x^2 + \frac{3}{2}x - 1 + \frac{4}{3}x^2 - \frac{4}{3}.$$

Combine like terms:

$$x^2 : -\frac{1}{2} + \frac{4}{3} = \frac{-3 + 8}{6} = \frac{5}{6},$$

$$x : \frac{3}{2},$$

$$\text{constant:} \ -1 - \frac{4}{3} = -\frac{3}{3} - \frac{4}{3} = -\frac{7}{3}.$$

Thus, the quadratic interpolation polynomial is

$$p(x) = \frac{5}{6}x^2 + \frac{3}{2}x - \frac{7}{3}.$$

11 2

Given the numerical table for $f(x) = \ln x$:

x	0.4	0.5	0.6	0.7	0.8
$\ln x$	-0.916291	-0.693147	-0.510826	-0.356675	-0.223144

Use linear interpolation and quadratic interpolation to estimate the value of ln(0.54).

11.1 Answer

Linear Interpolation:

Using the tabulated values, choose $x_0 = 0.5$ with $f(x_0) = \ln(0.5) = -0.693147$ and $x_1 = 0.6$ with $f(x_1) = \ln(0.6) = -0.510826$. Then

$$\ln(0.54) \approx \ln(0.5) + \frac{\ln(0.6) - \ln(0.5)}{0.6 - 0.5}(0.54 - 0.5).$$

That is,

$$\ln(0.54) \approx -0.693147 + \frac{-0.510826 + 0.693147}{0.1}(0.04)$$

$$\ln(0.54) \approx -0.693147 + \frac{0.182321}{0.1}(0.04)$$

$$\ln(0.54) \approx -0.693147 + 1.82321 \times 0.04 \approx -0.693147 + 0.072928 \approx -0.620219.$$

Quadratic Interpolation:

Choose the three points $x_0 = 0.5$, $x_1 = 0.6$, and $x_2 = 0.7$ with

$$f(0.5) = -0.693147, \quad f(0.6) = -0.510826, \quad f(0.7) = -0.356675.$$

The Lagrange basis polynomials at $x = 0.54$ are:

$$L_0(0.54) = \frac{(0.54 - 0.6)(0.54 - 0.7)}{(0.5 - 0.6)(0.5 - 0.7)} = \frac{(-0.06)(-0.16)}{(-0.1)(-0.2)} = \frac{0.0096}{0.02} = 0.48,$$

$$L_1(0.54) = \frac{(0.54 - 0.5)(0.54 - 0.7)}{(0.6 - 0.5)(0.6 - 0.7)} = \frac{(0.04)(-0.16)}{(0.1)(-0.1)} = \frac{-0.0064}{-0.01} = 0.64,$$

$$L_2(0.54) = \frac{(0.54 - 0.5)(0.54 - 0.6)}{(0.7 - 0.5)(0.7 - 0.6)} = \frac{(0.04)(-0.06)}{(0.2)(0.1)} = \frac{-0.0024}{0.02} = -0.12.$$

Then the quadratic interpolation estimate is

$$\ln(0.54) \approx f(0.5)L_0(0.54) + f(0.6)L_1(0.54) + f(0.7)L_2(0.54)$$

$$\ln(0.54) \approx (-0.693147)(0.48) + (-0.510826)(0.64) + (-0.356675)(-0.12).$$

Calculating each term:

$$-0.693147 \times 0.48 \approx -0.332711, \quad -0.510826 \times 0.64 \approx -0.326928, \quad -0.356675 \times (-0.12) \approx 0.042801.$$

Thus,

$$\ln(0.54) \approx -0.332711 - 0.326928 + 0.042801 \approx -0.616838.$$

$$\boxed{\begin{array}{ll} \text{Linear interpolation:} & \ln(0.54) \approx -0.6202, \\ \text{Quadratic interpolation:} & \ln(0.54) \approx -0.6168. \end{array}}$$