Assignment5

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We consider the system

$$\frac{dy(t)}{dt} + 2\,y(t) = x(t), \quad t \geq 0,$$

with zero initial conditions (i.e. initial relaxation: $y(0) = 0$).

We solve the differential equation by multiplying it by the integrating factor

$$\mu(t) = e^{\int 2\,dt} = e^{2t}.$$

Multiplying both sides by e^{2t} gives

$$e^{2t}\frac{dy(t)}{dt} + 2e^{2t}y(t) = e^{2t}x(t).$$

The left-hand side is the derivative of the product $e^{2t}y(t)$; therefore,

$$\frac{d}{dt}\Big[e^{2t}y(t)\Big] = e^{2t}x(t).$$

Integrate both sides from 0 to t:

$$e^{2t}y(t) = \int_0^t e^{2\tau}x(\tau)\,d\tau.$$

Thus, the general solution is

$$y(t) = e^{-2t}\int_0^t e^{2\tau}x(\tau)\,d\tau.$$

1.1 (a)

1.1.1 (i) When $x_1(t) = e^{3t}u(t)$

Substitute $x_1(\tau) = e^{3\tau}$ into the integral:

$$y_1(t) = e^{-2t}\int_0^t e^{2\tau}e^{3\tau}\,d\tau = e^{-2t}\int_0^t e^{5\tau}\,d\tau.$$

Evaluating the integral,

$$\int_0^t e^{5\tau}\,d\tau = \frac{e^{5t}-1}{5}.$$

Thus,

$$y_1(t) = e^{-2t}\cdot\frac{e^{5t}-1}{5} = \frac{1}{5}e^{3t} - \frac{1}{5}e^{-2t}.$$

1.1.2 (ii) When $x_2(t) = e^{2t}u(t)$

Substitute $x_2(\tau) = e^{2\tau}$ into the general solution:

$$y_2(t) = e^{-2t} \int_0^t e^{2\tau} e^{2\tau}\, d\tau = e^{-2t} \int_0^t e^{4\tau}\, d\tau.$$

Evaluating the integral,

$$\int_0^t e^{4\tau}\, d\tau = \frac{e^{4t} - 1}{4}.$$

Thus,

$$y_2(t) = e^{-2t} \cdot \frac{e^{4t} - 1}{4} = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}.$$

1.1.3 (iii) When

$$x_3(t) = \alpha\, e^{3t}u(t) + \beta\, e^{2t}u(t), \quad \alpha, \beta \in \mathbb{R}$$

By linearity, the response is given by

$$y_3(t) = e^{-2t} \int_0^t e^{2\tau} \Big(\alpha\, e^{3\tau} + \beta\, e^{2\tau}\Big)\, d\tau.$$

Splitting the integral and using the previous results:

$$y_3(t) = \alpha\, e^{-2t} \int_0^t e^{5\tau}\, d\tau + \beta\, e^{-2t} \int_0^t e^{4\tau}\, d\tau,$$

which yields

$$y_3(t) = \alpha \left[\frac{e^{5t} - 1}{5} e^{-2t}\right] + \beta \left[\frac{e^{4t} - 1}{4} e^{-2t}\right].$$

That is,

$$y_3(t) = \alpha \left(\frac{1}{5}e^{3t} - \frac{1}{5}e^{-2t}\right) + \beta \left(\frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}\right).$$

This clearly shows that

$$y_3(t) = \alpha\, y_1(t) + \beta\, y_2(t),$$

which confirms the superposition property and the linearity of the system.

1.2 (b)

1.2.1 (i) When $x_1(t) = K\, e^{2t}\, u(t)$

For $t \geq 0$, substitute $x_1(\tau) = K\, e^{2\tau}$ into the formula:

$$y_1(t) = e^{-2t} \int_0^t e^{2\tau} \Big[K\, e^{2\tau}\Big]\, d\tau = K\, e^{-2t} \int_0^t e^{4\tau}\, d\tau.$$

Evaluate the integral:

$$\int_0^t e^{4\tau}\, d\tau = \frac{e^{4t}-1}{4}.$$

Thus, the output becomes

$$y_1(t) = K\, e^{-2t} \cdot \frac{e^{4t}-1}{4} = \frac{K}{4}\Big(e^{2t} - e^{-2t}\Big).$$

Including the unit step for clarity,

$$y_1(t) = \frac{K}{4}\Big(e^{2t} - e^{-2t}\Big) u(t).$$

1.2.2 (ii) When $x_2(t) = K\, e^{2(t-T)}\, u(t-T)$

Here the input is delayed; note that $x_2(t) = 0$ for $t < T$. In the solution formula, when integrating we have

$$y_2(t) = e^{-2t} \int_0^t e^{2\tau}\, x_2(\tau)\, d\tau.$$

Since $x_2(\tau) = K\, e^{2(\tau-T)}\, u(\tau - T)$, the integrand is nonzero only for $\tau \geq T$. Hence we can write:

$$y_2(t) = e^{-2t} \int_T^t e^{2\tau} \Big[K\, e^{2(\tau-T)}\Big] d\tau, \quad t \geq T.$$

Combine the exponentials:

$$e^{2\tau}\, e^{2(\tau-T)} = e^{4\tau-2T}.$$

Then

$$y_2(t) = K\, e^{-2t} \int_T^t e^{4\tau-2T}\, d\tau = K\, e^{-2t-2T} \int_T^t e^{4\tau}\, d\tau.$$

Evaluate the integral:

$$\int_T^t e^{4\tau}\, d\tau = \frac{e^{4t}-e^{4T}}{4}.$$

Thus,

$$y_2(t) = \frac{K}{4}\, e^{-2t-2T}\Big(e^{4t} - e^{4T}\Big) = \frac{K}{4}\Big(e^{2t-2T} - e^{-2t+2T}\Big).$$

Recognize that the expression depends on $t - T$. In fact, we may write

$$y_2(t) = \frac{K}{4}\Big(e^{2(t-T)} - e^{-2(t-T)}\Big) u(t-T).$$

Verification of Time Invariance From part (i) we found

$$y_1(t) = \frac{K}{4}\Big(e^{2t} - e^{-2t}\Big) u(t).$$

Thus, if we replace t by $t - T$ we obtain

$$y_1(t-T) = \frac{K}{4}\Big(e^{2(t-T)} - e^{-2(t-T)}\Big) u(t-T),$$

which is exactly the expression we obtained for $y_2(t)$. Therefore,

$$y_2(t) = y_1(t-T).$$