10. 解：

(1)

令 $\varsigma^2 = c\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2$

$$
\begin{aligned}
E(\varsigma^2) &= cE[\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2] = c\sum_{i=1}^{n-1}E(X_{i+1}^2-2X_{i+1}X_i+X_i^2) \\
&= c\sum_{i=1}^{n-1}(D(X_{i+1})+[E(X_{i+1})]^2-2E(X_{i+1})E(X_i)+D(X_i)+[E(X_{i+1})]^2) \\
&= c(n-1)(\sigma^2+\mu^2-2\mu^2+\sigma^2+\mu^2) = \sigma^2
\end{aligned}
$$

故有，

$$c = \frac{1}{2(n-1)}$$

(2)

$$E(\overline{X}^2 - cS^2) = E(\overline{X}^2) - cE(S^2) = (\frac{\sigma^2}{n}+\mu^2) - c\sigma^2 = \mu^2$$

故，

$$c = \frac{1}{n}$$

13. 解：

(1)

由题知，$E(\hat{\theta}) = \theta, D(\hat{\theta}) > 0, \hat{\theta}^2 = (\hat{\theta})^2$

故，

$$E(\hat{\theta}^2) = E((\hat{\theta})^2) = D(\hat{\theta}) + E(\hat{\theta})^2 = D(\hat{\theta}) + \theta^2 > \theta^2$$

(2)

似然函数为,

$$L(\theta)=\begin{cases}\frac{1}{\theta^n} & 0<x_1,x_2,...,x_n\le\theta\\ 0 & \text{otherwise}\end{cases}$$

故最大似然估计量 $\hat{\theta}$ 为,

$$\hat{\theta}=\max\{x_1,x_2,...,x_n\}$$

总体 X 分布函数为,

$$F(X)=\begin{cases}0 & x<0\\ \frac{x}{\theta} & 0\le x<\theta\\ 1 & x\ge\theta\end{cases}$$

故,

$$F_{\hat{\theta}}(x)=F_{\max}(x)=(F(x))^n=\begin{cases}0 & x<0\\ (\frac{x}{\theta})^n & 0\le x<\theta\\ 1 & x\ge\theta\end{cases}$$

故,

$$f_{\hat{\theta}}(x)=\begin{cases}\frac{n}{\theta}(\frac{x}{\theta})^{n-1} & 0\le z\le\theta\\ 0 & \text{otherwise}\end{cases}$$

故,

$$E(\hat{\theta})=\int_{-\infty}^{+\infty}xf_{\hat{\theta}}(x)\mathrm{d}x=\int_0^{\theta}n(\frac{x}{\theta})^n\mathrm{d}x=\frac{n\theta}{n+1}\neq\theta$$

15. 解:

$$E(\hat{\theta})=E(\sum_{i=1}^{n}a_iX_i)=\sum_{i=1}^{n}a_iE(X_i)=\theta\sum_{i=1}^{n}a_i$$

故有,

$$\sum_{i=1}^{n} a_i = 1$$

$$D(\hat{\theta}) = \sum_{i=1}^{n} a_i^2 D(X_i) = \sum_{i=1}^{n} a_i^2 \sigma_i^2$$

作函数

$$g(a_1, a_2, ..., a_n, \lambda) = \sum_{i=1}^{n} a_i^2 \sigma_i^2 + \lambda(\sum_{i=1}^{n} a_i - 1)$$

令

$$\frac{\partial g}{\partial a_i} = 2\sigma_i^2 a_i + \lambda = 0$$
$$\frac{\partial g}{\partial \lambda} = \sum_{i=1}^{n} a_i - 1 = 0$$

故有 $a_i = -\frac{\lambda}{2\sigma_i^2}$，代入得，

$$-\frac{\lambda}{2} \sum_{i=1}^{n} \frac{1}{\sigma_i^2} = 1$$

故，

$$\lambda = -\frac{2}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}}$$
$$a_i = \frac{1}{\sigma_i^2 \sum_{j=1}^{n} \frac{1}{\sigma_j^2}}$$