1(3)

$$f(1)=0,\quad f(-1)=-3,\quad f(2)=4.$$

$$x_0=1,\quad x_1=-1,\quad x_2=2$$

牛顿插值法:

$$f[x_0,x_1]=\frac{-3-0}{-1-1}=\frac{3}{2}$$
$$f[x_1,x_2]=\frac{4-(-3)}{2-(-1)}=\frac{7}{3}$$
$$f[x_0,x_1,x_2]=\frac{\frac{7}{3}-\frac{3}{2}}{2-1}=\frac{5}{6}$$

$$P(x)=\frac{3}{2}(x-1)+\frac{5}{6}(x-1)(x+1)$$

3

$$\delta\leq 0.5\times 10^{-5}$$

$h=1'=\pi/10800\text{rad}$

$$E\leq\frac{1}{8}\left(\frac{\pi}{10800}\right)^2$$
$$\text{总误差}\leq E+\delta$$

4(2)

证明：对任意整数 $1\leq k\leq n$,

$$\sum_{j=0}^{n}(x_j-x)^k\ell_j(x)\equiv 0$$

$$
\begin{aligned}
\sum_{j=0}^{n}(x_j-x)^k\ell_j(x) &\equiv \sum_{m=0}^{k}\sum_{j=0}^{n}(-1)^m x^m x_j^{k-m}\ell_j(x)\\
&\equiv \sum_{m=0}^{k}(-1)^m x^m \sum_{j=0}^{n} x_j^{k-m}\ell_j(x)\\
&\equiv \sum_{m=0}^{k}(-1)^m x^m x^{k-m}\\
&\equiv (x-x)^k\\
&\equiv 0
\end{aligned}
$$

6

二次插值误差 $\leq 10^{-6}$，求 h:

$$
R(x)=\frac{f^{(3)}(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2),
$$

令，

$$
x_1=x_0+h,\quad x_2=x_0+2h.
$$

$$
t=x-x_0,\qquad t\in[0,2h].
$$

有

$$
R(t)=\frac{f^{(3)}(\xi)}{6}\,t\,(t-h)\,(t-2h).
$$

$$
|P(t)|=|t\,(t-h)\,(t-2h)|
$$

求导解得级值点

$$
t=h\Big(1-\frac{1}{\sqrt{3}}\Big),
$$

带入得到

$$
\max_{t\in[0,2h]}|t\,(t-h)\,(t-2h)|=\frac{2h^3}{3\sqrt{3}}.
$$

又因为，

$$M_3 = e^4.$$

所以

$$\frac{e^4\,h^3}{9\sqrt{3}} \le 10^{-6}.$$

7

证明：

1.

$$F = cf \Rightarrow F[...] = c\cdot f[...]$$

- 归纳基

$$F[x_0] = F(x_0) = c\,f(x_0) = c\,f[x_0].$$

- 归纳步

$$F[x_0, x_1, \ldots, x_n] = \frac{c\,f[x_1, x_2, \ldots, x_n] - c\,f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$$

$$F[x_0, x_1, \ldots, x_n] = c\,f[x_0, x_1, \ldots, x_n].$$

2.

$$F = f + g \Rightarrow F[...] = f[...] + g[...]$$

- 归纳基

$$F[x_0] = F(x_0) = f(x_0) + g(x_0) = f[x_0] + g[x_0].$$

- 归纳步

$$\begin{aligned} F[x_0, x_1, \ldots, x_n] &= \frac{\Big(f[\ldots, x_n] + g[\ldots, x_n]\Big) - \Big(f[\ldots, x_{n-1}] + g[\ldots, x_{n-1}]\Big)}{x_n - x_0} \\ &= f[\ldots, x_n] + g[\ldots, x_n]. \end{aligned}$$

8

$$f(x) = x^7 + x^4 + 3x + 1$$

$$f[x_0, x_1, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

7次多项式8阶导为0，7阶导数为 $7!$

9

$$\Delta(f_k g_k) = f_k \Delta g_k + g_{k+1} \Delta f_k$$

证明：

$$f_{k+1}g_{k+1} - f_k g_k = \Big[f_{k+1}g_{k+1} - f_k g_{k+1}\Big] + \Big[f_k g_{k+1} - f_k g_k\Big].$$

$$f_{k+1}g_{k+1} - f_k g_{k+1} = g_{k+1}\big(f_{k+1} - f_k\big) = g_{k+1}\,\Delta f_k,$$

$$f_k g_{k+1} - f_k g_k = f_k\big(g_{k+1} - g_k\big) = f_k\,\Delta g_k.$$

10

证明求和公式：

$$\sum f_k \Delta g_k = f_n g_n - f_0 g_0 - \sum g_{k+1} \Delta f_k$$

证明：

$$f_n g_n - f_0 g_0 = \sum_{k=0}^{n-1} \Big(f_{k+1}g_{k+1} - f_k g_k\Big).$$

$$\Delta(f_k g_k) = f_k\,\Delta g_k + g_{k+1}\,\Delta f_k.$$

$$f_{k+1}g_{k+1} - f_k g_k = f_k \Delta g_k + g_{k+1} \Delta f_k.$$

$$f_n g_n - f_0 g_0 = \sum_{k=0}^{n-1} \Big(f_k \Delta g_k + g_{k+1} \Delta f_k\Big)$$

$$\sum_{k=0}^{n-1} f_k \Delta g_k = f_n g_n - f_0 g_0 - \sum_{k=0}^{n-1} g_{k+1} \Delta f_k.$$

11

$$\sum \Delta^2 y_j = \Delta y_n - \Delta y_0$$

证明：

$$\Delta y_n - \Delta y_0 = \sum_{j=0}^{n-1} \Delta y_{j+1} - \Delta y_j = \sum_{j=0}^{n-1} \Delta^2 y_j$$

12

$$\sum \frac{x_j^k}{f'(x_j)} = \begin{cases} 0, & 0 \le k \le n-2 \\ a_n^{-1}, & k = n-1 \end{cases}$$

设

$$\omega_n(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

原式可化为

$$\sum_{j=1}^{n} \frac{x_j^k}{f'(x_j)} = \sum_{j=1}^{n} \frac{x_j^k}{a_n \omega_n'(x_j)}$$

$$\omega_n'(x_j) = \prod_{k \ne j} (x_j - x_k)$$

令 $g(x) = x^k$，则 $g[\dots] = \sum_{j=1}^{n} \frac{x_j^k}{\omega_n'(x_j)}$

又因为

$$g[\cdots] = \frac{g^{(n-1)}(\xi)}{(n-1)!}$$

故

$$
原式 = \frac{1}{a_n} g[\dots] = \begin{cases} 0, & k \le n-2 \\ a_n^{-1}, & \text{otherwise} \end{cases}
$$

13

构造满足条件的三次多项式

$$
\begin{aligned}
P(x_0) &= f(x_0), \\
P'(x_0) &= f'(x_0), \\
P''(x_0) &= f''(x_0), \\
P(x_1) &= f(x_1).
\end{aligned}
$$

设

$$
P(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3.
$$

$$
P'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2, \quad P'(x_0) = a_1,
$$

$$
P''(x) = 2a_2 + 6a_3(x-x_0), \quad P''(x_0) = 2a_2.
$$

故

$$
\begin{aligned}
a_0 &= f(x_0), \\
a_1 &= f'(x_0), \\
2a_2 &= f''(x_0) \quad \Longrightarrow \quad a_2 = \frac{f''(x_0)}{2}.
\end{aligned}
$$