Assignment7

March 23, 2025

1 3.28(c)

We are given a discrete-time periodic signal $x[n]$ with period $N = 4$ defined over one period by

$$x[n] = 1 - \sin\left(\frac{\pi n}{4}\right), \quad n = 0, 1, 2, 3.$$

Our task is to compute its Fourier series coefficients

$$a[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, 2, 3,$$

and plot for each $a[k]$ its magnitude and phase.

2 Computation of Fourier Series Coefficients

For $N = 4$, note that the complex exponential becomes

$$e^{-j\frac{2\pi}{4}kn} = e^{-j\frac{\pi}{2}kn}.$$

2.1 Step 1. Evaluate $x[n]$ exactly

For $n = 0, 1, 2, 3$:

- $x[0] = 1 - \sin(0) = 1.$
- $x[1] = 1 - \sin\left(\frac{\pi}{4}\right) = 1 - \frac{\sqrt{2}}{2}.$
- $x[2] = 1 - \sin\left(\frac{\pi}{2}\right) = 1 - 1 = 0.$
- $x[3] = 1 - \sin\left(\frac{3\pi}{4}\right) = 1 - \frac{\sqrt{2}}{2}.$

2.2 Step 2. Compute each $a[k]$ exactly

1. **For $k = 0$:**

$$
\begin{aligned}
a[0] &= \frac{1}{4}\Big[x[0] + x[1] + x[2] + x[3]\Big] \\
&= \frac{1}{4}\Big[1 + \Big(1 - \frac{\sqrt{2}}{2}\Big) + 0 + \Big(1 - \frac{\sqrt{2}}{2}\Big)\Big] \\
&= \frac{1}{4}\Big[3 - \sqrt{2}\Big].
\end{aligned}
$$

2. **For $k = 1$:**

   With
   $$e^{-j\frac{\pi}{2}} = -j \quad \text{and} \quad e^{-j\frac{3\pi}{2}} = j,$$

   $$
   \begin{aligned}
   a[1] &= \frac{1}{4}\Big[x[0] + x[1]\, e^{-j\frac{\pi}{2}} + x[2]\, e^{-j\pi} + x[3]\, e^{-j\frac{3\pi}{2}}\Big] \\
   &= \frac{1}{4}\Big[1 + \Big(1 - \frac{\sqrt{2}}{2}\Big)(-j) + 0 + \Big(1 - \frac{\sqrt{2}}{2}\Big)(j)\Big] \\
   &= \frac{1}{4}\Big[1 + 0\Big] \\
   &= \frac{1}{4}.
   \end{aligned}
   $$

3. **For $k = 2$:**

   Recall $e^{-j\pi} = -1$ and $e^{-j3\pi} = -1$:

   $$
   \begin{aligned}
   a[2] &= \frac{1}{4}\Big[x[0] + x[1]\, e^{-j\pi} + x[2]\, e^{-j2\pi} + x[3]\, e^{-j3\pi}\Big] \\
   &= \frac{1}{4}\Big[1 + \Big(1 - \frac{\sqrt{2}}{2}\Big)(-1) + 0 + \Big(1 - \frac{\sqrt{2}}{2}\Big)(-1)\Big] \\
   &= \frac{1}{4}\Big[1 - 2\Big(1 - \frac{\sqrt{2}}{2}\Big)\Big] \\
   &= \frac{1}{4}\Big[1 - 2 + \sqrt{2}\Big] \\
   &= \frac{\sqrt{2} - 1}{4}.
   \end{aligned}
   $$

4. **For $k = 3$:**

   Note that
   $$e^{-j\frac{3\pi}{2}} = j \quad \text{and} \quad e^{-j\frac{9\pi}{2}} = e^{-j(4\pi + \frac{\pi}{2})} = e^{-j\frac{\pi}{2}} = -j.$$

   $$
   \begin{aligned}
   a[3] &= \frac{1}{4}\Big[x[0] + x[1]\, e^{-j\frac{3\pi}{2}} + x[2]\, e^{-j\frac{3\pi}{2}\cdot 2} + x[3]\, e^{-j\frac{9\pi}{2}}\Big] \\
   &= \frac{1}{4}\Big[1 + \Big(1 - \frac{\sqrt{2}}{2}\Big)j + 0 + \Big(1 - \frac{\sqrt{2}}{2}\Big)(-j)\Big] \\
   &= \frac{1}{4}\,[1 + 0] \\
   &= \frac{1}{4}.
   \end{aligned}
   $$

Summary of Exact Coefficients:

$$
\begin{aligned}
a[0] &= \frac{3-\sqrt{2}}{4}, \\
a[1] &= \frac{1}{4}, \\
a[2] &= \frac{\sqrt{2}-1}{4}, \\
a[3] &= \frac{1}{4}.
\end{aligned}
$$

Since all $a[k]$ are positive real numbers, their phases are 0 radians.

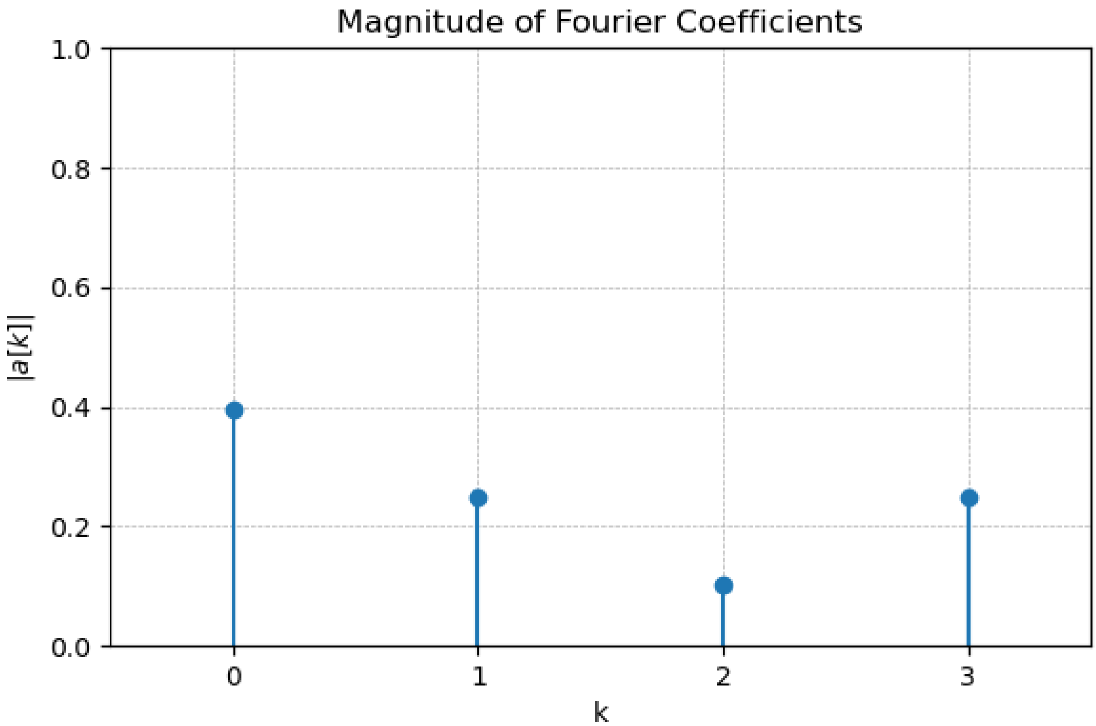
note

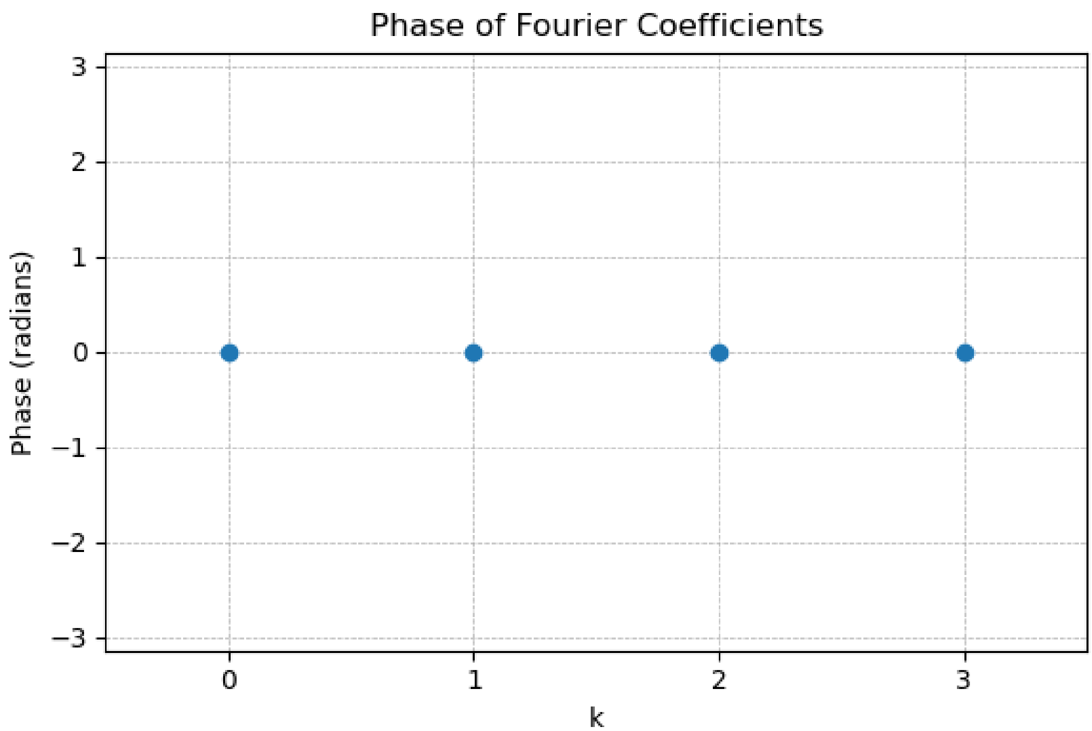
[5]:

```python
import numpy as np
import matplotlib.pyplot as plt

# Period
N = 4
n = np.arange(N)

# Define the signal x[n] exactly:
# x[0] = 1,
# x[1] = 1 - sin( /4) = 1 - (√2/2),
# x[2] = 0,
# x[3] = 1 - sin(3 /4) = 1 - (√2/2)
x = np.array([1, 1 - np.sqrt(2)/2, 0, 1 - np.sqrt(2)/2])

# Compute Fourier series coefficients a[k]
a = np.zeros(N, dtype=complex)
for k in range(N):
    a[k] = (1 / N) * np.sum(x * np.exp(-1j * (2 * np.pi / N) * k * n))

# Extract magnitudes and phases
magnitude = np.abs(a)
phase = np.angle(a)

# Print the Fourier coefficients (numerical values)
print("Fourier Coefficients a[k]:")
for k in range(N):
    print(f"a[{k}] = {a[k]}")
    # Expected:
    # a[0] = (3 - √2) / 4, a[1] = 1/4, a[2] = (√2 - 1) / 4, a[3] = 1/4

# Plot the magnitude and phase
fig, (ax1, ax2) = plt.subplots(2, 1, figsize=(6, 8))
```

```python
# Plot magnitude using stem plot (without use_line_collection)
ax1.stem(np.arange(N), magnitude, basefmt=" ")
ax1.set_title("Magnitude of Fourier Coefficients")
ax1.set_xlabel("k")
ax1.set_ylabel(r"$|a[k]|$")
ax1.set_xticks(np.arange(N))
ax1.set_ylim(0, 1)
ax1.set_xlim(-0.5, N - 0.5)
ax1.grid(True, which="both", linestyle="--", linewidth=0.5)

# Plot phase using stem plot
ax2.stem(np.arange(N), phase, basefmt=" ")
ax2.set_title("Phase of Fourier Coefficients")
ax2.set_xlabel("k")
ax2.set_ylabel("Phase (radians)")
ax2.set_xticks(np.arange(N))
ax2.set_ylim(-np.pi, np.pi)
ax2.set_xlim(-0.5, N - 0.5)
ax2.grid(True, which="both", linestyle="--", linewidth=0.5)

# Ensure the axes (spines) are clearly visible
for ax in [ax1, ax2]:
    for spine in ax.spines.values():
        spine.set_visible(True)

plt.tight_layout()
plt.show()
```

```
Fourier Coefficients a[k]:
a[0] = (0.39644660940672627+0j)
a[1] = (0.25+0j)
a[2] = (0.10355339059327379-3.5869074291185986e-17j)
a[3] = (0.25000000000000006+0j)
```


Magnitude of Fourier Coefficients
|a[k]|
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
k


Phase of Fourier Coefficients
Phase (radians)
3
2
1
0
−1
−2
−3
0
1
2
3
k

3 3.34(c)

3.1 1. Problem Statement

Consider a continuous-time LTI system whose impulse response is

$$h(t) = e^{-4|t|}.$$

The input $x(t)$ is a square-wave function with period $T = 1$. In one period (i.e. for $-\frac{1}{4} \leq t \leq \frac{3}{4}$) the waveform is defined as

$$x(t) = \begin{cases} 1, & -\frac{1}{4} \leq t \leq \frac{1}{4}, \\ 0, & \frac{1}{4} < t \leq \frac{3}{4}, \end{cases}$$

with the function then extended periodically to all t.

The goal is to find the Fourier series representation of the output

$$y(t) = x(t) * h(t)$$

where "(*)" denotes convolution.

3.2 2. Step-by-Step Derivation

3.2.1 (a) Fourier Series of the Input $x(t)$

Since $x(t)$ is periodic with period $T = 1$, its Fourier series coefficients are defined by

$$X_k = \int_{T_0}^{T_0+1} x(t)\, e^{-j2\pi kt}\, dt.$$

A convenient choice of integration interval is

$$\left[-\frac{1}{4}, \frac{3}{4}\right].$$

However, note that $x(t)$ is nonzero only on the interval

$$\left[-\frac{1}{4}, \frac{1}{4}\right].$$

Thus, for any integer k we have

$$X_k = \int_{-1/4}^{1/4} e^{-j2\pi kt}\, dt.$$

For $k \neq 0$, integrating gives

$$X_k = \left[\frac{e^{-j2\pi kt}}{-j2\pi k}\right]_{t=-1/4}^{1/4} = \frac{1}{-j2\pi k}\Big(e^{-j2\pi k(1/4)} - e^{j2\pi k(1/4)}\Big)$$

$$= \frac{1}{-j2\pi k}\Big(-2j\,\sin\Big(\frac{2\pi k}{4}\Big)\Big) = \frac{2\,\sin\Big(\frac{\pi k}{2}\Big)}{2\pi k} = \frac{\sin\left(\frac{\pi k}{2}\right)}{\pi k}.$$

For $k = 0$,

$$X_0 = \int_{-1/4}^{1/4} dt = \frac{1}{4} - \Big(-\frac{1}{4}\Big) = \frac{1}{2}.$$

3.2.2 (b) Frequency Response of the System

The Fourier transform of $h(t) = e^{-4|t|}$ is a standard result:

$$H(j\omega) = \frac{8}{16 + \omega^2}.$$

For a periodic input, we evaluate the frequency response at the harmonic frequencies (= 2 k). Thus,

$$H(j2\pi k) = \frac{8}{16 + (2\pi k)^2} = \frac{8}{16 + 4\pi^2 k^2} = \frac{2}{4 + \pi^2 k^2}.$$

3.2.3 (c) Output Spectrum and Fourier Series of $y(t)$

For an LTI system, when the input has Fourier series coefficients X_k and the frequency response is $H(j2\pi k)$, the output Fourier series coefficients are given by

$$Y_k = X_k \, H(j2\pi k).$$

Thus, for $k = 0$,

$$Y_0 = X_0 \, H(0) = \frac{1}{2} \times \frac{8}{16} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

And for $k \neq 0$,

$$Y_k = \frac{\sin\left(\frac{\pi k}{2}\right)}{\pi k} \cdot \frac{2}{4 + \pi^2 k^2}.$$

Therefore, the Fourier series representation of the output $y(t)$ is

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k \, e^{j2\pi k \, t} = \frac{1}{4} + \sum_{k \neq 0} \frac{2 \, \sin\left(\frac{\pi k}{2}\right)}{\pi k \, \left(4 + \pi^2 k^2\right)} \, e^{j2\pi k \, t}.$$

3.3 3. Final Answer

The Fourier series representation of the output $y(t)$ is

$$\boxed{y(t) = \frac{1}{4} + \sum_{k \neq 0} \frac{2 \, \sin\left(\frac{\pi k}{2}\right)}{\pi k \, \left(4 + \pi^2 k^2\right)} \, e^{j2\pi k \, t}.}$$

This expression shows that the output is composed of a DC term $\frac{1}{4}$ plus the harmonics determined by the product of the input Fourier coefficients and the system's frequency response.