

Assignment 5

March 18, 2025

1 2.33

We consider the system

$$\frac{dy(t)}{dt} + 2y(t) = x(t), \quad t \geq 0,$$

with zero initial conditions (i.e. initial relaxation: $y(0) = 0$).

We solve the differential equation by multiplying it by the integrating factor

$$\mu(t) = e^{\int 2 dt} = e^{2t}.$$

Multiplying both sides by e^{2t} gives

$$e^{2t} \frac{dy(t)}{dt} + 2e^{2t}y(t) = e^{2t}x(t).$$

The left-hand side is the derivative of the product $e^{2t}y(t)$; therefore,

$$\frac{d}{dt} [e^{2t}y(t)] = e^{2t}x(t).$$

Integrate both sides from 0 to t :

$$e^{2t}y(t) = \int_0^t e^{2\tau}x(\tau) d\tau.$$

Thus, the general solution is

$$y(t) = e^{-2t} \int_0^t e^{2\tau}x(\tau) d\tau.$$

1.1 (a)

1.1.1 (i) When $x_1(t) = e^{3t}u(t)$

Substitute $x_1(\tau) = e^{3\tau}$ into the integral:

$$y_1(t) = e^{-2t} \int_0^t e^{2\tau}e^{3\tau} d\tau = e^{-2t} \int_0^t e^{5\tau} d\tau.$$

Evaluating the integral,

$$\int_0^t e^{5\tau} d\tau = \frac{e^{5t} - 1}{5}.$$

Thus,

$$y_1(t) = e^{-2t} \cdot \frac{e^{5t} - 1}{5} = \frac{1}{5}e^{3t} - \frac{1}{5}e^{-2t}.$$

1.1.2 (ii) When $x_2(t) = e^{2t}u(t)$

Substitute $x_2(\tau) = e^{2\tau}$ into the general solution:

$$y_2(t) = e^{-2t} \int_0^t e^{2\tau} e^{2\tau} d\tau = e^{-2t} \int_0^t e^{4\tau} d\tau.$$

Evaluating the integral,

$$\int_0^t e^{4\tau} d\tau = \frac{e^{4t} - 1}{4}.$$

Thus,

$$y_2(t) = e^{-2t} \cdot \frac{e^{4t} - 1}{4} = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}.$$

1.1.3 (iii) When

$$x_3(t) = \alpha e^{3t}u(t) + \beta e^{2t}u(t), \quad \alpha, \beta \in \mathbb{R}$$

By linearity, the response is given by

$$y_3(t) = e^{-2t} \int_0^t e^{2\tau} (\alpha e^{3\tau} + \beta e^{2\tau}) d\tau.$$

Splitting the integral and using the previous results:

$$y_3(t) = \alpha e^{-2t} \int_0^t e^{5\tau} d\tau + \beta e^{-2t} \int_0^t e^{4\tau} d\tau,$$

which yields

$$y_3(t) = \alpha \left[\frac{e^{5t} - 1}{5} e^{-2t} \right] + \beta \left[\frac{e^{4t} - 1}{4} e^{-2t} \right].$$

That is,

$$y_3(t) = \alpha \left(\frac{1}{5}e^{3t} - \frac{1}{5}e^{-2t} \right) + \beta \left(\frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t} \right).$$

This clearly shows that

$$y_3(t) = \alpha y_1(t) + \beta y_2(t),$$

which confirms the superposition property and the linearity of the system.

1.2 (b)

1.2.1 (i) When $x_1(t) = K e^{2t} u(t)$

For $t \geq 0$, substitute $x_1(\tau) = K e^{2\tau}$ into the formula:

$$y_1(t) = e^{-2t} \int_0^t e^{2\tau} [K e^{2\tau}] d\tau = K e^{-2t} \int_0^t e^{4\tau} d\tau.$$

Evaluate the integral:

$$\int_0^t e^{4\tau} d\tau = \frac{e^{4t} - 1}{4}.$$

Thus, the output becomes

$$y_1(t) = K e^{-2t} \cdot \frac{e^{4t} - 1}{4} = \frac{K}{4} (e^{2t} - e^{-2t}).$$

Including the unit step for clarity,

$$y_1(t) = \frac{K}{4} (e^{2t} - e^{-2t}) u(t).$$

1.2.2 (ii) When $x_2(t) = K e^{2(t-T)} u(t-T)$

Here the input is delayed; note that $x_2(t) = 0$ for $t < T$. In the solution formula, when integrating we have

$$y_2(t) = e^{-2t} \int_0^t e^{2\tau} x_2(\tau) d\tau.$$

Since $x_2(\tau) = K e^{2(\tau-T)} u(\tau-T)$, the integrand is nonzero only for $\tau \geq T$. Hence we can write:

$$y_2(t) = e^{-2t} \int_T^t e^{2\tau} [K e^{2(\tau-T)}] d\tau, \quad t \geq T.$$

Combine the exponentials:

$$e^{2\tau} e^{2(\tau-T)} = e^{4\tau-2T}.$$

Then

$$y_2(t) = K e^{-2t} \int_T^t e^{4\tau-2T} d\tau = K e^{-2t-2T} \int_T^t e^{4\tau} d\tau.$$

Evaluate the integral:

$$\int_T^t e^{4\tau} d\tau = \frac{e^{4t} - e^{4T}}{4}.$$

Thus,

$$y_2(t) = \frac{K}{4} e^{-2t-2T} (e^{4t} - e^{4T}) = \frac{K}{4} (e^{2t-2T} - e^{-2t+2T}).$$

Recognize that the expression depends on $(t-T)$. In fact, we may write

$$y_2(t) = \frac{K}{4} (e^{2(t-T)} - e^{-2(t-T)}) u(t-T).$$

Verification of Time Invariance From part (i) we found

$$y_1(t) = \frac{K}{4} (e^{2t} - e^{-2t}) u(t).$$

Thus, if we replace (t) by $(t-T)$ we obtain

$$y_1(t-T) = \frac{K}{4} (e^{2(t-T)} - e^{-2(t-T)}) u(t-T),$$

which is exactly the expression we obtained for $(y_2(t))$. Therefore,

$$y_2(t) = y_1(t-T).$$