Assignment4

March 11, 2025

1 2.21(a)

$$\begin{cases} y[n] &= x[n] * h[n] \\ x[n] &= \alpha^n u[n] \\ h[n] &= \beta^n u[n] \\ \alpha &\neq \beta \end{cases}$$

1.1 Answer

Based on the definition of convolution, we have

$$y[n] = \sum_{k=0}^{n} x[k]\, h[n-k].$$

Substituting $x[k] = \alpha^k\, u[k]$ and $h[n-k] = \beta^{n-k}\, u[n-k]$ gives

$$y[n] = \sum_{k=0}^{n} \alpha^k\, \beta^{n-k}.$$

Factor out β^n to obtain

$$y[n] = \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k.$$

Using the finite geometric series formula (with $\alpha \neq \beta$)

$$\sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k = \frac{1-\left(\frac{\alpha}{\beta}\right)^{n+1}}{1-\frac{\alpha}{\beta}}.$$

Substituting the above into the equation, we get

$$y[n] = \beta^n\, \frac{1-\left(\frac{\alpha}{\beta}\right)^{n+1}}{1-\frac{\alpha}{\beta}} = \frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}.$$

Including the unit step function $u[n]$, the final result is

$$y[n] = \frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}\, u[n].$$

Thus, the final answer is:

$$y[n] = \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} u[n].$$

2 2.22(c)

2.1 Answer

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{\max(0,t-3)}^{\min(2,t-1)} 2\sin(\pi\tau)d\tau. \end{aligned}$$

For different values of t, the integration limits change:

- For $t < 1$, the interval is empty, so $y(t) = 0$.
- For $1 \leq t \leq 3$, the limits are $\tau = 0$ to $\tau = t - 1$, and

$$y(t) = 2\int_{0}^{t-1} \sin(\pi\tau)d\tau \qquad = \frac{2}{\pi}\Big[1 - \cos(\pi(t-1))\Big].$$

- For $3 \leq t \leq 5$, the limits are $\tau = t - 3$ to $\tau = 2$, and

$$y(t) = 2\int_{t-3}^{2} \sin(\pi\tau)d\tau \qquad = \frac{2}{\pi}\Big[\cos(\pi(t-3)) - 1\Big].$$

- For $t > 5$, the interval is again empty, so $y(t) = 0$.

Thus, the convolution result is:

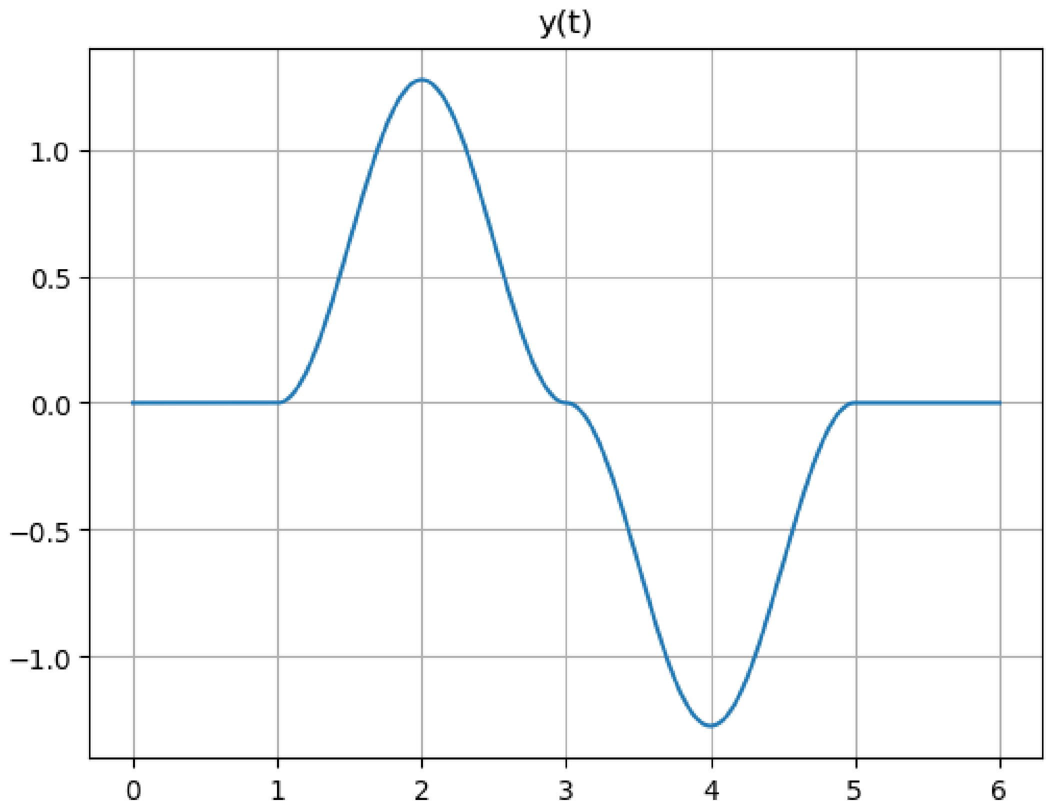
$$y(t) = \begin{cases} 0, & t < 1, \\ \frac{2}{\pi}\Big[1 - \cos(\pi(t-1))\Big], & 1 \leq t \leq 3, \\ \frac{2}{\pi}\Big[\cos(\pi(t-3)) - 1\Big], & 3 \leq t \leq 5, \\ 0, & t > 5. \end{cases}$$

[29]:
```
import matplotlib.pyplot as plt
import numpy as np

def draw(x, y, title):
    plt.plot(x, y)
    plt.grid()
    plt.title(title)
    plt.show()

x = np.linspace(0, 6, 1000)
```

```python
y = []
for t in x:
    if t < 1:
        y.append(0)
    elif t <= 3:
        y.append(2/np.pi * (1 - np.cos(np.pi * (t - 1))))
    elif t <= 5:
        y.append(2/np.pi * (np.cos(np.pi * (t - 3)) - 1))
    else:
        y.append(0)
draw(x, y, "y(t)")
```



3 2.47

Given a linear time-invariant system with impulse response $h_0(t)$, its output $y_0(t)$ for the input $x_0(t)$ is given by

$$y_0(t) = \begin{cases} \frac{1}{2}t, & 0 \le t \le 2, \\ 0, & \text{otherwise} \end{cases}$$

In each case, determine whether there is sufficient information to uniquely determine the output $y(t)$ when the input is $x(t)$ and the system's impulse response is $h_0(t)$.

3.1 (b)

$$x(t) = x_0(t) - x_0(t-2), \quad h(t) = h_0(t)$$

3.1.1 Answer

$$\begin{aligned} y(t) &= y_0(t) - y_0(t-2) \\ &= \begin{cases} \frac{1}{2}t, & 0 \le t < 2, \\ -\frac{1}{2}(t-2), & 2 \le t < 4, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Using linearity and time invariance, we can uniquely determine $y(t)$ from the known $y_0(t)$.

3.2 (d)

$$x(t) = x_0(-t), \quad h(t) = h_0(t)$$

3.2.1 Answer

$$y(t) = \int_{-\infty}^{\infty} x_0(-\tau)\, h_0(t-\tau)\, d\tau.$$

Even though we know the convolution result

$$y_0(t) = \int_{-\infty}^{\infty} x_0(\tau)\, h_0(t-\tau)\, d\tau,$$

the transformation of the input (time reversal) does not yield a simple relation between $y(t)$ and $y_0(t)$ unless additional properties (such as symmetry) of $h_0(t)$ or more details about $x_0(t)$ are assumed. In general, with only the knowledge of $y_0(t)$ and without further information about $x_0(t)$ or $h_0(t)$, we do not have sufficient information to uniquely determine $y(t)$.

3.3 (e)

$$x(t) = x_0(t), \quad h(t) = h_0(-t)$$

3.3.1 Answer

Using time-reversal properties, we can show that

$$y(t) = \int_{-\infty}^{+\infty} x_0(-\tau) h_0(-(t-\tau)) d\tau = \int_{-\infty}^{+\infty} x_0(u) h_0(-t-u) du = y_0(-t).$$

Since

$$y_0(t) = \begin{cases} \frac{1}{2}t, & 0 \le t \le 2, \\ 0, & \text{otherwise,} \end{cases}$$

it follows that

$$y(t) = y_0(-t) = \begin{cases} \frac{1}{2}(-t), & -2 \le t \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the output is uniquely determined by time reversing the known response.

3.4 (f)

$$x(t) = x_0'(t), \quad h(t) = h_0'(t)$$

3.4.1 Answer

Using the differentiation property of convolutions, we have

$$\mathcal{F}\{x_0'(t)\} = j\omega X_0(\omega), \quad \mathcal{F}\{h_0'(t)\} = j\omega H_0(\omega).$$

Thus,

$$\mathcal{F}\{y(t)\} = (j\omega)^2 X_0(\omega) H_0(\omega) = -\omega^2 Y_0(\omega),$$

which implies

$$y(t) = y_0''(t).$$

Since

$$y_0(t) = \begin{cases} \frac{1}{2}t, & 0 \le t \le 2, \\ 0, & \text{otherwise,} \end{cases}$$

its second derivative (in the distributional sense) is

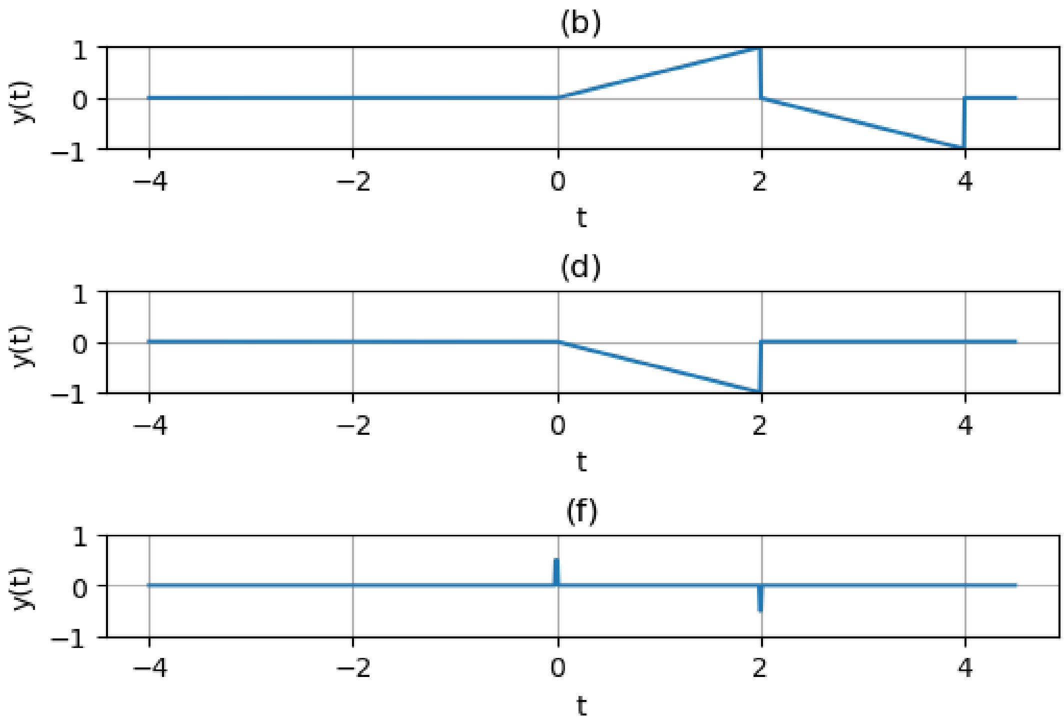
$$y_0''(t) = \frac{1}{2}\delta(t) - \frac{1}{2}\delta(t-2).$$

Hence, the uniquely determined output is

$$y(t) = \frac{1}{2}\delta(t) - \frac{1}{2}\delta(t-2).$$

[30]:
```
def draw_sub(x, y, title, sub):
    plt.subplot(6, 1, sub)
    plt.plot(x, y)
    plt.grid()
    plt.ylabel("y(t)")
    plt.xlabel("t")
    plt.ylim(-1, 1)
    plt.title(title)
x = np.linspace(-4, 4.5, 1000)
y = []
for t in x:
    if 0 <= t < 2:
        y.append(t / 2)
    elif 2 <= t < 4:
        y.append(-1/2*(t - 2))
    else:
        y.append(0)
draw_sub(x, y, "(b)", 1)
y = []
for t in x:
    if 0 <= t < 2:
        y.append(- 1/2 * t)
    else:
        y.append(0)
draw_sub(x, y, "(d)", 3)
eps = 1e-2
def delta(x):
    if abs(x) < eps:
        return 1
    else:
        return 0

y = np.array([1/2 * delta(t) - 1/2 * delta(t - 2) for t in x])
draw_sub(x, y, "(f)", 5)
```



4 2.28(c)

$h[n]$ is the unit impulse response of a discrete-time LTI system. The task is to determine if the system is causal and/or BIBO stable.

$h[n]$ is defined as

$$h[n] = \left(\frac{1}{2}\right)^n u[-n],$$

where $u[-n]$ equals 1 for $n \leq 0$ and 0 for $n > 0$.

4.1 Answer

- A causal system requires $h[n] = 0$ for $n < 0$, but here $h[n]$ is nonzero for some negative n. Therefore, the system is non-causal.

- For BIBO stability, the sum

$$\sum_{n=-\infty}^{\infty} |h[n]|$$

must be finite. For n 0, we have

$$|h[n]| = \left(\frac{1}{2}\right)^n = 2^{-n},$$

so the relevant sum becomes

$$\sum_{n=-\infty}^{0} 2^{-n} = \sum_{m=0}^{\infty} 2^m,$$

which diverges. Hence, the system is unstable.

In summary, the system is neither causal nor stable.

5 2.29(g)

We are given the impulse response of a continuous-time LTI system as

$$h(t) = \left(2e^{-t} - e^{\frac{t-100}{100}}\right)u(t).$$

5.1 Answer

Because of the factor $u(t)$, we have $h(t) = 0$ for $t < 0$. This means the system does not respond before $t = 0$, so it is causal.

To check BIBO stability, we need to evaluate whether

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} \left|2e^{-t} - e^{\frac{t-100}{100}}\right| dt$$

is finite. Notice that the second term can be rewritten as

$$e^{\frac{t-100}{100}} = e^{\frac{t}{100}-1}.$$

For large t, the term $2e^{-t}$ decays to zero, while $e^{\frac{t}{100}-1}$ grows exponentially (albeit slowly). As a result, $|h(t)|$ for large t behaves like

$$|h(t)| \sim e^{\frac{t}{100}-1},$$

and the integral

$$\int_{0}^{\infty} e^{\frac{t}{100}-1} dt = e^{-1} \int_{0}^{\infty} e^{\frac{t}{100}}\, dt$$

diverges. Therefore, the impulse response is not absolutely integrable, implying that the system is not BIBO stable.

In summary:

- The system is causal.
- The system is not BIBO stable.