Assignment11

April 8, 2025

1 5.21(e)

计算下列信号的离散傅立叶变换

$$x[n] = \left(\frac{1}{2}\right)^{|n|} \cos\left(\frac{\pi}{8}(n-1)\right)$$

1.1 Answer

令 $x[n] = x_1[n] \cdot x_2[n-1], \quad x_1[n] = \left(\frac{1}{2}\right)^{|n|}, \quad x_2[n-1] = \cos\left(\frac{\pi}{8}(n-1)\right)$

则,

$$\begin{aligned}
X_1(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\omega}\right)^n + \sum_{m=1}^{\infty} \left(\frac{1}{2}e^{j\omega}\right)^m \\
&= \frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{\frac{1}{2}e^{j\omega}}{1-\frac{1}{2}e^{j\omega}} \\
&= \frac{3}{5-4\cos\omega}
\end{aligned}$$

我们有

$$x_2[n] = \cos\Big(\frac{\pi}{8}n\Big) = \frac{1}{2}\Big(e^{j\frac{\pi}{8}n} + e^{-j\frac{\pi}{8}n}\Big).$$

因此(利用 DTFT 中冲激函数对),

$$X_2(e^{j\omega}) = \pi\Big[\delta\Big(\omega - \frac{\pi}{8}\Big) + \delta\Big(\omega + \frac{\pi}{8}\Big)\Big].$$

利用时移性质:

$$\mathcal{DTFT}\{x_2[n-1]\} = e^{-j\omega\cdot 1} X_2(e^{j\omega}) = e^{-j\omega}\,\pi\Big[\delta\Big(\omega - \frac{\pi}{8}\Big) + \delta\Big(\omega + \frac{\pi}{8}\Big)\Big].$$

因为

$$x[n] = x_1[n] \cdot x_2[n-1],$$

所以它的 DTFT 为

$$X(e^{j\omega}) = \frac{1}{2\pi}\Big[X_1(e^{j\omega}) * \Big\{e^{-j\omega}\,\pi\Big[\delta(\omega - \frac{\pi}{8}) + \delta(\omega + \frac{\pi}{8})\Big]\Big\}\Big].$$

也就是说,

$$X(e^{j\omega}) = \frac{1}{2}\int_{-\pi}^{\pi} X_1(e^{j\theta})\, e^{-j(\omega-\theta)}\,\Big[\delta((\omega-\theta) - \frac{\pi}{8}) + \delta((\omega-\theta) + \frac{\pi}{8})\Big] d\theta.$$

利用冲激函数的抽样性质, 积分将 “采样” 到两点: - 当 $\omega - \theta = \frac{\pi}{8}$ 时, 即 $\theta = \omega - \frac{\pi}{8}$; - 当 $\omega - \theta = -\frac{\pi}{8}$ 时, 即 $\theta = \omega + \frac{\pi}{8}$。

于是

$$X(e^{j\omega}) = \frac{1}{2}\Big\{e^{-j\frac{\pi}{8}}\, X_1\Big(e^{j(\omega-\frac{\pi}{8})}\Big) + e^{j\frac{\pi}{8}}\, X_1\Big(e^{j(\omega+\frac{\pi}{8})}\Big)\Big\}.$$

将 $X_1(e^{j\theta}) = \frac{3}{5-4\cos\theta}$ 代入, 得

$$\boxed{X(e^{j\omega}) = \frac{1}{2}\Big[e^{-j\frac{\pi}{8}}\frac{3}{5-4\cos\big(\omega-\frac{\pi}{8}\big)} + e^{j\frac{\pi}{8}}\frac{3}{5-4\cos\big(\omega+\frac{\pi}{8}\big)}\Big].}$$

2 5.22(a)

求相应于下列傅立叶变换的信号

$$X(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{4} \le |\omega \le \frac{3\pi}{4} \\ 0, & \frac{3\pi}{4} \le |\omega| \le \pi, 0 \le |\omega| < \frac{\pi}{4} \end{cases}$$

2.1 Answer

观察给定的 $X(e^{j\omega})$ 可看作两个方波函数的差:

- 令
$$X_1(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{3\pi}{4}, \\ 0, & \text{otherwise}, \end{cases}$$
- 令
$$X_2(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{\pi}{4}, \\ 0, & \text{otherwise}. \end{cases}$$

则显然有

$$X(e^{j\omega}) = X_1(e^{j\omega}) - X_2(e^{j\omega}).$$

利用标准结论: 对于

$$\tilde{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n} d\omega,$$

计算可得

$$\tilde{x}[n] = \frac{\sin(Wn)}{\pi n}, \quad n \neq 0, \quad \tilde{x}[0] = \frac{W}{\pi}.$$

因此：

- 对于 $X_1(e^{j\omega})$（这里 $W = \frac{3\pi}{4}$）有

$$x_1[n] = \frac{\sin\left(\frac{3\pi n}{4}\right)}{\pi n}, \quad n \neq 0, \quad x_1[0] = \frac{3\pi/4}{\pi} = \frac{3}{4}\,.$$

- 对于 $X_2(e^{j\omega})$（这里 $W = \frac{\pi}{4}$）有

$$x_2[n] = \frac{\sin\left(\frac{\pi n}{4}\right)}{\pi n}, \quad n \neq 0, \quad x_2[0] = \frac{\pi/4}{\pi} = \frac{1}{4}\,.$$

由频域的线性叠加（差分）关系，其对应的时域信号为

$$x[n] = x_1[n] - x_2[n].$$

即，对于 $n \neq 0$ 有

$$x[n] = \frac{\sin\left(\frac{3\pi n}{4}\right) - \sin\left(\frac{\pi n}{4}\right)}{\pi n},$$

而 $n = 0$ 时，

$$x[0] = x_1[0] - x_2[0] = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}\,.$$

综上，

$$\boxed{x[n] = \begin{cases} \frac{1}{2}, & n = 0, \\ \frac{1}{\pi n}\left[\sin\left(\frac{3\pi n}{4}\right) - \sin\left(\frac{\pi n}{4}\right)\right], & n \neq 0\,. \end{cases}}$$