HA.1 Parameter estimation

(a) $\underset{\hat{\mu},\hat{\sigma}^2}{\operatorname{argmax}}\ P(x_1,\dots x_N \mid \hat{\mu},\hat{\sigma}^2)$

$= \underset{\hat{\mu},\hat{\sigma}^2}{\operatorname{argmax}} \prod_{i=1}^{N} P(x_i \mid \hat{\mu},\hat{\sigma}^2) = \underset{\hat{\mu},\hat{\sigma}^2}{\operatorname{argmax}} \left(\frac{1}{(2\pi\hat{\sigma}^2)^{N/2}} \exp\left(-\frac{\sum_{i=1}^{N}(x_i-\hat{\mu})^2}{2\hat{\sigma}^2} \right) \right)$

map to $\Rightarrow l = \log\left(\frac{1}{(2\pi\hat{\sigma}^2)^{N/2}} \exp\left(-\frac{\sum_{i=1}^{N}(x_i-\hat{\mu})^2}{2\hat{\sigma}^2} \right) \right)$

$= -\frac{N}{2}\left(\log(2\pi\hat{\sigma}^2)\right) - \frac{\sum_{i=1}^{N}(x_i-\hat{\mu})^2}{2\hat{\sigma}^2}$

$\frac{\partial l}{\partial \hat{\mu}} = \frac{\partial}{\partial \hat{\mu}} \frac{-\sum_{i=1}^{N}(x_i-\hat{\mu})^2}{2\hat{\sigma}^2} = \frac{\partial}{\partial \hat{\mu}} \frac{-\sum_{i=0}^{N}(x_i-\hat{\mu})^T(x_i-\hat{\mu})}{2\hat{\sigma}^2}$

$= \frac{\partial}{\partial \hat{\mu}} \frac{-1}{2\hat{\sigma}^2} \sum_{i=0}^{N} (\cancel{x_i^T x_i} - 2x_i^T\hat{\mu} + \hat{\mu}^T\hat{\mu}) \overset{\text{set to}}{=} 0$

$\Rightarrow \sum(-2x_i + 2\hat{\mu}) = 0$

$\Rightarrow \boxed{\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i}$

$\frac{\partial l}{\partial \hat{\sigma}^2} = \frac{1}{\partial \hat{\sigma}^2}\left(-\frac{N}{2}\log\hat{\sigma}^2 - \frac{\sum_{i=0}^{N}(x_i-\hat{\mu})^2}{2\hat{\sigma}^2} \right)$

$= -\frac{N}{2\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4}\sum_{i=0}^{N}(x_i-\hat{\mu})^2 \overset{\text{set to}}{=} 0,$

$\Rightarrow \boxed{\hat{\sigma}^2 = \frac{1}{N-1}\sum_{i=0}^{N}(x_i-\hat{\mu})^2}$

(b).

$$l_0 = \frac{-N_0}{2}\log|2\pi\Sigma_0| - \frac{1}{2}\sum_{i=1}^{N_0}\left((x_i-\mu_0)^T\Sigma_0^{-1}(x_i-\mu_0)\right)$$

$$\frac{\partial l_0}{\partial \mu_0} = \frac{\partial}{\partial \mu_0} -\frac{1}{2}\sum_{i=1}^{N_0} \cancel{x_i^T\Sigma_0^{-1}x_i} - x_i^T\Sigma_0^{-1}\mu_0 - \mu_0^T\Sigma_0^{-1}x_i + \mu_0^T\Sigma_0^{-1}\mu_0$$

$$= -\frac{1}{2}\sum_{i=1}^{N_0} -2x_i\Sigma_0^{-1} + 2\mu_0\Sigma_0^{-1} \overset{\text{set to}}{=} 0.$$

$$\Rightarrow \boxed{\mu_0 = \frac{1}{N_0}\sum_{i=1}^{N_0} x_i}$$

$$\frac{\partial l_0}{\partial \Sigma_0^{-1}} = \frac{\partial}{\partial \Sigma_0^{-1}} \cancel{-\frac{N_0}{2}\log|2\pi|} + \frac{N_0}{2}\log|\Sigma_0| - \frac{1}{2}\sum_{i=1}^{N_0}\left((x_i-\mu_0)^T\Sigma_0^{-1}(x_i-\mu_0)\right).$$

$$= \frac{-N_0\Sigma_0}{2} - \frac{1}{2}\sum_{i=1}^{N_0}\left((x_i-\mu_0)^T(x_i-\mu_0)\right) \overset{\text{set to}}{=} 0.$$

$$\Rightarrow \boxed{\Sigma_0 = \frac{1}{N_0}\sum_{i=1}^{N_0}\left((x_i-\mu_0)^T(x_i-\mu_0)\right)}$$

similarly,

$$\mu_1 = \frac{1}{N_1}\sum_{i=1}^{N_1} x_i.$$

$$\Sigma_1 = \frac{1}{N_1}\sum_{i=1}^{N_1}\left((x_i-\mu_1)^T(x_i-\mu_1)\right). \#$$

H4.2

(a)

$$\log p(y=1|\bar{x}) \underset{0}{\overset{1}{\gtrless}} \log p(y=0|\bar{x})$$

$$\Rightarrow \cancel{\log}\left(\cancel{\frac{1}{\sqrt{2\pi\Sigma_1}}}\,\cancel{\exp}\left(\frac{-1}{2}(\bar{x}-\mu_1)^T\Sigma_1^{-1}(\bar{x}-\mu_1)\right)\right) \underset{0}{\overset{1}{\gtrless}}$$

$$\cancel{\log}\left(\cancel{\frac{1}{\sqrt{2\pi\Sigma_0}}}\,\cancel{\exp}\left(\frac{-1}{2}(\bar{x}-\mu_0)^T\Sigma_0^{-1}(\bar{x}-\mu_0)\right)\right)$$

$$\Rightarrow -(\bar{x}-\mu_1)^T\Sigma_1^{-1}(\bar{x}-\mu_1) + (\bar{x}-\mu_0)^T\Sigma_0^{-1}(\bar{x}-\mu_0) \underset{0}{\overset{1}{\gtrless}} 0.$$

$$\Rightarrow \cancel{-\bar{x}^T\Sigma_1^{-1}\bar{x}} + 2\mu_1^T\Sigma_1^{-1}\bar{x} - \mu_1^T\Sigma_1^{-1}\mu_1 + \cancel{\bar{x}^T\Sigma_0^{-1}\bar{x}} - 2\mu_0^T\Sigma_0^{-1}\bar{x} + \mu_0^T\Sigma_0^{-1}\mu_0 \underset{0}{\overset{1}{\gtrless}} 0$$

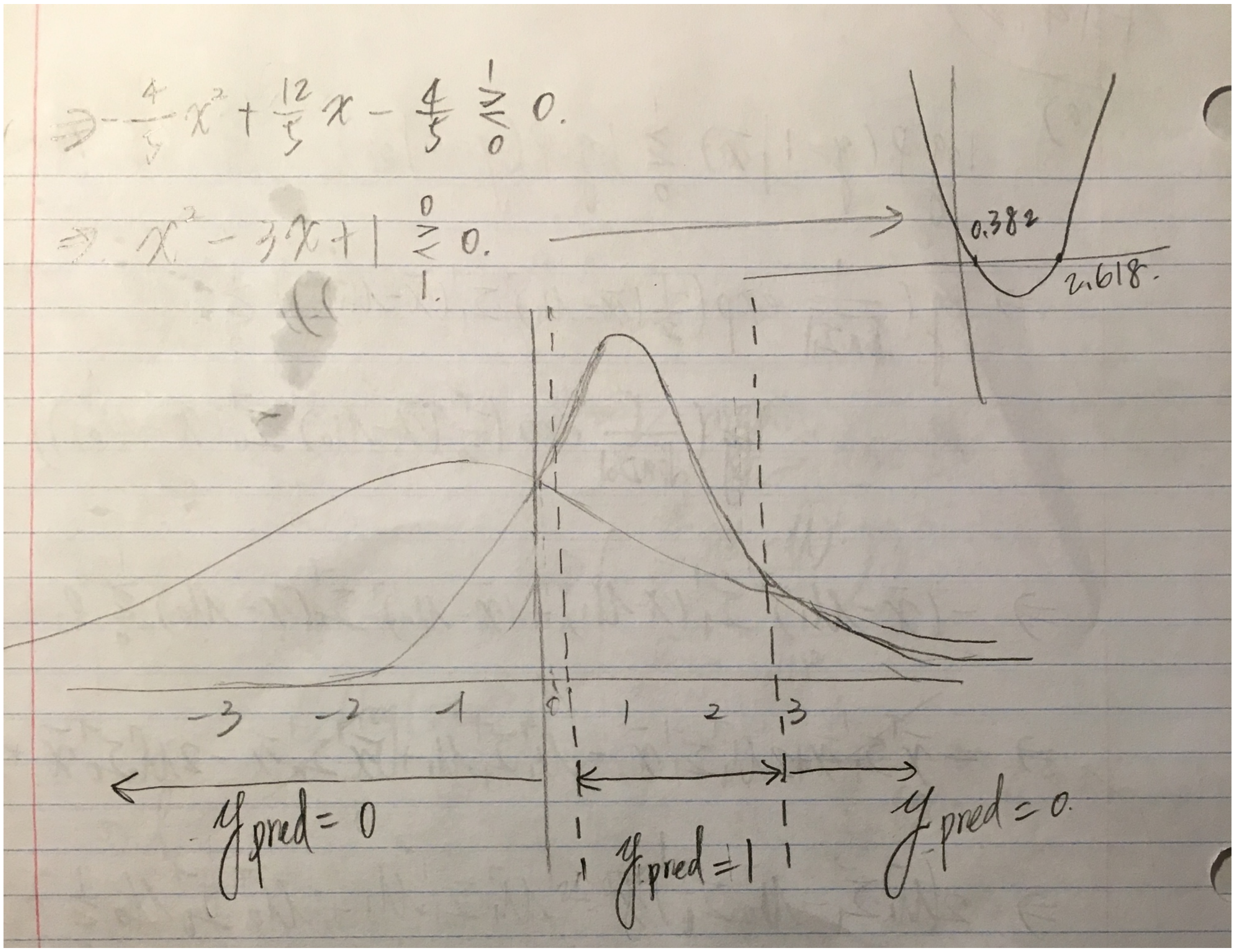
$$\Rightarrow \underbrace{2(\mu_1^T\Sigma_1^{-1} - \mu_0^T\Sigma_0^{-1})\bar{x}}_{w.} \underbrace{- \mu_1^T\Sigma_1^{-1}\mu_1 + \mu_0^T\Sigma_0^{-1}\mu_0}_{b.} \underset{0}{\overset{1}{\gtrless}} 0$$

(b)

$$\Rightarrow \bar{x}^T\underbrace{(-\Sigma_1^{-1} + \Sigma_0^{-1})}_{A}\bar{x} + \underbrace{2(\mu_1^T\Sigma_1^{-1} - \mu_0^T\Sigma_0^{-1})}_{w}\bar{x} + \underbrace{(-\mu_1^T\Sigma_1^{-1}\mu_1 + \mu_0^T\Sigma_0^{-1}\mu_0)}_{b.} \underset{0}{\overset{1}{\gtrless}} 0$$

(c) $\mu_1 = 1,\ \sigma_1^2 = 1,\ \mu_0 = -1,\ \sigma_0^2 = 5.$

$$\Rightarrow A = (-1+\tfrac{1}{5}) = \tfrac{-4}{5},\quad w = 2(1+\tfrac{1}{5}) = \tfrac{12}{5},\quad b = (-1+\tfrac{1}{5}) = \tfrac{-4}{5}$$



H4.3 and H4.4 are present in h4_3.ipynb, and h4_4.ipynb