DEFINITION 2 A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term* a and the *common ratio* r are real numbers.

Remark: A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

EXAMPLE 2 The sequences $\{b_n\}$ with $b_n = (-1)^n$, $\{c_n\}$ with $c_n = 2 \cdot 5^n$, and $\{d_n\}$ with $d_n = 6 \cdot (1/3)^n$ are geometric progressions with initial term and common ratio equal to 1 and -1; 2 and 5; and 6 and $1/3$, respectively, if we start at $n = 0$. The list of terms $b_0, b_1, b_2, b_3, b_4, \ldots$ begins with

$$1, -1, 1, -1, 1, \ldots;$$

the list of terms $c_0, c_1, c_2, c_3, c_4, \ldots$ begins with

$$2, 10, 50, 250, 1250, \ldots;$$

and the list of terms $d_0, d_1, d_2, d_3, d_4, \ldots$ begins with

$$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots.$$ ◀

DEFINITION 3 An *arithmetic progression* is a sequence of the form

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

where the *initial term* a and the *common difference* d are real numbers.

Remark: An arithmetic progression is a discrete analogue of the linear function $f(x) = dx + a$.

EXAMPLE 3 The sequences $\{s_n\}$ with $s_n = -1 + 4n$ and $\{t_n\}$ with $t_n = 7 - 3n$ are both arithmetic progressions with initial terms and common differences equal to -1 and 4, and 7 and -3, respectively, if we start at $n = 0$. The list of terms $s_0, s_1, s_2, s_3, \ldots$ begins with

$$-1, 3, 7, 11, \ldots,$$

and the list of terms $t_0, t_1, t_2, t_3, \ldots$ begins with

$$7, 4, 1, -2, \ldots.$$ ◀

Sequences of the form $a_1, a_2, \ldots, a_n$ are often used in computer science. These finite sequences are also called **strings**. This string is also denoted by $a_1a_2 \ldots a_n$. (Recall that bit strings, which are finite sequences of bits, were introduced in Section 1.1.) The **length** of a string is the number of terms in this string. The **empty string**, denoted by λ, is the string that has no terms. The empty string has length zero.

EXAMPLE 4 The string *abcd* is a string of length four. ◀

Recurrence Relations

In Examples 1–3 we specified sequences by providing explicit formulas for their terms. There are many other ways to specify a sequence. For example, another way to specify a sequence is to provide one or more initial terms together with a rule for determining subsequent terms from those that precede them.

DEFINITION 4

A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation. (A recurrence relation is said to *recursively define* a sequence. We will explain this alternative terminology in Chapter 5.)

EXAMPLE 5 Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$, and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Solution: We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$. It then follows that $a_2 = 5 + 3 = 8$ and $a_3 = 8 + 3 = 11$. ◀

EXAMPLE 6 Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$, and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Solution: We see from the recurrence relation that $a_2 = a_1 - a_0 = 5 - 3 = 2$ and $a_3 = a_2 - a_1 = 2 - 5 = -3$. We can find a_4, a_5, and each successive term in a similar way. ◀

The **initial conditions** for a recursively defined sequence specify the terms that precede the first term where the recurrence relation takes effect. For instance, the initial condition in Example 5 is $a_0 = 2$, and the initial conditions in Example 6 are $a_0 = 3$ and $a_1 = 5$. Using mathematical induction, a proof technique introduced in Chapter 5, it can be shown that a recurrence relation together with its initial conditions determines a unique solution.

Hop along to Chapter 8 to learn how to find a formula for the Fibonacci numbers.

Next, we define a particularly useful sequence defined by a recurrence relation, known as the **Fibonacci sequence**, after the Italian mathematician Fibonacci who was born in the 12th century (see Chapter 5 for his biography). We will study this sequence in depth in Chapters 5 and 8, where we will see why it is important for many applications, including modeling the population growth of rabbits.

DEFINITION 5

Links

The *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, is defined by the initial conditions $f_0 = 0$, $f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \ldots$.

EXAMPLE 7 Find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6.

Solution: The recurrence relation for the Fibonacci sequence tells us that we find successive terms by adding the previous two terms. Because the initial conditions tell us that $f_0 = 0$ and $f_1 = 1$, using the recurrence relation in the definition we find that

$$\begin{aligned}
f_2 &= f_1 + f_0 = 1 + 0 = 1,\\
f_3 &= f_2 + f_1 = 1 + 1 = 2,\\
f_4 &= f_3 + f_2 = 2 + 1 = 3,\\
f_5 &= f_4 + f_3 = 3 + 2 = 5,\\
f_6 &= f_5 + f_4 = 5 + 3 = 8.
\end{aligned}$$
◀

EXAMPLE 8 Suppose that $\{a_n\}$ is the sequence of integers defined by $a_n = n!$, the value of the factorial function at the integer n, where $n = 1, 2, 3, \ldots$. Because $n! = n((n-1)(n-2)\ldots 2 \cdot 1) = n(n-1)! = na_{n-1}$, we see that the sequence of factorials satisfies the recurrence relation $a_n = na_{n-1}$, together with the initial condition $a_1 = 1$. ◀

We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a **closed formula**, for the terms of the sequence.

EXAMPLE 9 Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$. Answer the same question where $a_n = 2^n$ and where $a_n = 5$.

Solution: Suppose that $a_n = 3n$ for every nonnegative integer n. Then, for $n \geq 2$, we see that $2a_{n-1} - a_{n-2} = 2(3(n-1)) - 3(n-2) = 3n = a_n$. Therefore, $\{a_n\}$, where $a_n = 3n$, is a solution of the recurrence relation.

Suppose that $a_n = 2^n$ for every nonnegative integer n. Note that $a_0 = 1, a_1 = 2$, and $a_2 = 4$. Because $2a_1 - a_0 = 2 \cdot 2 - 1 = 3 \neq a_2$, we see that $\{a_n\}$, where $a_n = 2^n$, is not a solution of the recurrence relation.

Suppose that $a_n = 5$ for every nonnegative integer n. Then for $n \geq 2$, we see that $a_n = 2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$. Therefore, $\{a_n\}$, where $a_n = 5$, is a solution of the recurrence relation. ◀

Many methods have been developed for solving recurrence relations. Here, we will introduce a straightforward method known as iteration via several examples. In Chapter 8 we will study recurrence relations in depth. In that chapter we will show how recurrence relations can be used to solve counting problems and we will introduce several powerful methods that can be used to solve many different recurrence relations.

EXAMPLE 10 Solve the recurrence relation and initial condition in Example 5.

Solution: We can successively apply the recurrence relation in Example 5, starting with the initial condition $a_1 = 2$, and working upward until we reach a_n to deduce a closed formula for the sequence. We see that

$$\begin{aligned}
a_2 &= 2 + 3\\
a_3 &= (2 + 3) + 3 = 2 + 3 \cdot 2\\
a_4 &= (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3\\
&\vdots\\
a_n &= a_{n-1} + 3 = (2 + 3 \cdot (n-2)) + 3 = 2 + 3(n-1).
\end{aligned}$$

We can also successively apply the recurrence relation in Example 5, starting with the term a_n and working downward until we reach the initial condition $a_1 = 2$ to deduce this same formula. The steps are

$$\begin{aligned}
a_n &= a_{n-1} + 3\\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2\\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3\\
&\vdots\\
&= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1).
\end{aligned}$$

At each iteration of the recurrence relation, we obtain the next term in the sequence by adding 3 to the previous term. We obtain the nth term after $n-1$ iterations of the recurrence relation. Hence, we have added $3(n-1)$ to the initial term $a_0 = 2$ to obtain a_n. This gives us the closed formula $a_n = 2 + 3(n-1)$. Note that this sequence is an arithmetic progression. ◀

The technique used in Example 10 is called **iteration**. We have iterated, or repeatedly used, the recurrence relation. The first approach is called **forward substitution** – we found successive terms beginning with the initial condition and ending with a_n. The second approach is called **backward substitution**, because we began with a_n and iterated to express it in terms of falling terms of the sequence until we found it in terms of a_1. Note that when we use iteration, we essential guess a formula for the terms of the sequence. To prove that our guess is correct, we need to use mathematical induction, a technique we discuss in Chapter 5.

In Chapter 8 we will show that recurrence relations can be used to model a wide variety of problems. We provide one such example here, showing how to use a recurrence relation to find compound interest.

EXAMPLE 11 **Compound Interest** Suppose that a person deposits \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Extra Examples

Solution: To solve this problem, let P_n denote the amount in the account after n years. Because the amount in the account after n years equals the amount in the account after $n-1$ years plus interest for the nth year, we see that the sequence $\{P_n\}$ satisfies the recurrence relation

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}.$$

The initial condition is $P_0 = 10{,}000$.

We can use an iterative approach to find a formula for P_n. Note that

$$\begin{aligned}
P_1 &= (1.11)P_0\\
P_2 &= (1.11)P_1 = (1.11)^2P_0\\
P_3 &= (1.11)P_2 = (1.11)^3P_0\\
&\vdots\\
P_n &= (1.11)P_{n-1} = (1.11)^nP_0.
\end{aligned}$$

When we insert the initial condition $P_0 = 10{,}000$, the formula $P_n = (1.11)^n 10{,}000$ is obtained.

Inserting $n = 30$ into the formula $P_n = (1.11)^n 10{,}000$ shows that after 30 years the account contains

$$P_{30} = (1.11)^{30}10{,}000 = \$228{,}922.97.$$ ◀

Special Integer Sequences

A common problem in discrete mathematics is finding a closed formula, a recurrence relation, or some other type of general rule for constructing the terms of a sequence. Sometimes only a few terms of a sequence solving a problem are known; the goal is to identify the sequence. Even though the initial terms of a sequence do not determine the entire sequence (after all, there are infinitely many different sequences that start with any finite set of initial terms), knowing the first few terms may help you make an educated conjecture about the identity of your sequence. Once you have made this conjecture, you can try to verify that you have the correct sequence.