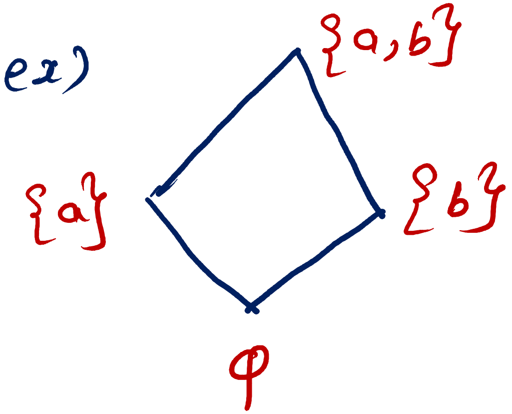
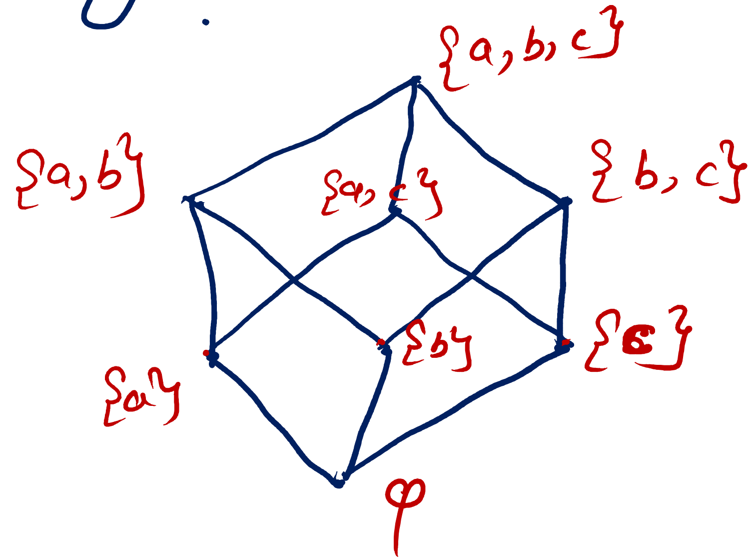
Boolean Algebra:

Lattice: A Lattice is a partially ordered set $(L, \leq)$ in which every pair of elements $a, b \in L$ has a greatest lower bound and a least upper bound (GLB and LUB)

(ex) Let S be any set and $P(S)$ be its power set. The partially ordered set $(P(S), \subseteq)$ is a lattice in which the meet and join are same as operations $\cap$ and $\cup$.

ex)





Defn: <u>Boolean Algebra</u>: A Boolean Algebra is a complemented, distributive Lattice.

* Electronic circuits are working with rules of Boolean Algebra.

Note: A Boolean algebra will generally be denoted by $(B, *, \oplus, ', 0, 1)$ in $(B, *, \oplus)$ is a lattice with two binary operations $*$ and $\oplus$ called meet and join. complementation is denoted by $'$

(1)

$*$	0	1
0	0	0
1	0	1

$\oplus$	0	1
0	0	1
1	1	1

x	x'
0	$0' = 1$
1	$1' = 0$

$(B, *, \oplus, ', 0, 1)$ is a Boolean Algebra.

A Boolean Algebra satisfies the following properties:

$(B, *, \oplus)$ is a Lattice

$\{ * : \wedge \ , \ \oplus : \vee \}$

L1. $a * a = a$ L1'. $a \oplus a = a$

L2. $a * b = b * a$ L2'. $a \oplus b = b \oplus a$ (Commutative Law

L3. $(a * b) * c = a * (b * c)$ L3': $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ $\{$ associative Law

L4. $a * (a \oplus b) = a$ L4': $a \oplus (a * b) = a$

$(B, *, \oplus)$ is a distributive Lattice

D1. $a * (b \oplus c) = (a * b) \oplus (a * c)$

D2. $a \oplus (b * c) = (a \oplus b) * (a \oplus c)$

$\}$ Distributive Laws.

D3. $(a*b)\oplus(b*c)\oplus(c*a) = (a\oplus b)*(b\oplus c)*(c\oplus a)$

D4. $a*b = a*c$ and $a\oplus b = a\oplus c \Rightarrow b=c$

$(B, *, \oplus)$ is a bounded lattice:

$\left\{ \begin{array}{l} * : \wedge \\ \oplus : \vee \end{array} \right.$

B1. $0 \le a \le 1$

B2. $a*0=0$ — B_2': $a\oplus 1 = 1$ [Dominance Laws]

B3. $a*1=a$ — B_3': $a\oplus 0 = a$ [Identity Laws].

$(B, *, \oplus)$ is a complemented lattice:

C1. $a*a'=0$ — C1': $a\oplus a' = 1$ {complement laws

$C_2 : 0' = 1$ $\quad$ $C_2' : 1' = 0$ $\quad$ { Complement laws

$C_3 : (a * b)' = a' \oplus b'$ $\quad$ $C_3' : (a \oplus b)' = a' * b'$ $\quad$ { De-Morgan's laws

There exists a partial ordering relation $\leq$ on B such that:

$P_1 : a * b = GLB\{a, b\}$ $\quad$ $P_1' : a \oplus b = LUB\{a, b\}$

$P_2 : \boxed{a \leq b \iff a * b = a \iff a \oplus b = b}$

$P_3 : a \leq b \iff a * b' = 0 \iff b' \leq a' \iff a' \oplus b = 1$

(ex) $(P(A), \cup, \cap, \subseteq)$ is a Boolean Algebra.

pbm: 1 Show that in any Boolean Algebra,

$$(a+b)(a'+c) = ac + a'b' + bc$$

Soln Let $(B, +, \cdot, ')$ be Boolean Algebra.

let $a, b, c \in B$

$$LHS = (a+b)(a'+c) = (a+b)a' + (a+b)c$$

$$= aa' + ba' + ac + bc$$

$$= 0 + a'b + ac + bc$$

$$= ac + a'b + bc = RHS.$$

2) In any Boolean Algebra, show that $a = b$ if and only if $a\bar{b} + \bar{a}b = 0$

<u>soln:</u> Let $(B, \cdot, +, -)$ be any Boolean algebra

let $a, b \in B$ and $a = b$.

<u>T.P</u> $a\bar{b} + \bar{a}b = 0$

$\{ a = b \therefore \bar{a} = \bar{b}$

$$a\bar{b} + \bar{a}b = a\bar{a} + \bar{a}a$$

$$= 0 + 0 = 0$$

$$\therefore a\bar{b} + \bar{a}b = 0$$

let $a\bar{b} + \bar{a}b = 0$

then, $a + a\bar{b} + \bar{a}b = a$ [cancellation law, left

$a + \bar{a}b = a$ {Absorption, $a + a\bar{b} = a$

$(a+\bar{a})\cdot(a+b) = a$ {distributive, $+$ over $\cdot$

$1\cdot(a+b) = a$ {$a+\bar{a} = 1$

$a+b = a$ —— (1)

Consider, $a\bar{b} + \bar{a}b = 0$

$a\bar{b} + \bar{a}b + b = b$ {cancellation, right

$a\bar{b} + b = b$ {$\bar{a}b + b = b$

$(a+b) \cdot (b+\bar{b}) = b$ { distributive '+' over '·'

$(a+b) = b$ { $b+\bar{b}=1$

$\quad$ ②

from ① and ②

$$\boxed{a=b}$$