Chapter 1

Set Theory

All mathematical objects can be defined in terms of sets, and the language of set theory is used in every mathematical subject.

1.1 Definitions and the Element Method of Proof

Sets, as defined earlier, are collections of objects, called elements. Using our new knowledge, we can redefine some definitions.

1.1.1 Subsets

We can redefine some definitions of subsets:

$$A \subseteq B \Leftrightarrow \forall x, x \in A \rightarrow x \in B.$$

$$A \nsubseteq B \Leftrightarrow \exists x \mid x \in A \wedge x \notin B.$$

Recall that a **proper subset** is a subset that is not equal to its containing set.

We can prove for two sets X and Y that $X \subseteq Y$ by **supposing** that x is a particular but arbitrarily chosen element of X, and **showing** that x is also an element of Y.

Definition 1: Set Equality

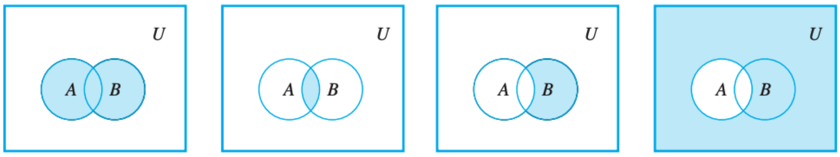
Set A equals set B if, and only if, $A \subseteq B$ and $B \subseteq A$.

1.1.2 Operations on Sets

Definition 2: Operations

Let A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $A \cup B$, is the set of all elements that are in at least one of A or B.
2. The **intersection** of A and B, denoted $A \cap B$, is the set of all elements that are common to both A and B.
3. The **difference** of B minus A (or **relative complement** of A in B), denoted $B - A$, is the set of all elements that are in B and not A.
4. The **complement** of A, denoted A^c, is the set of all elements in U that are not in A.



Shaded region represents $A \cup B$. Shaded region represents $A \cap B$. Shaded region represents $B - A$. Shaded region represents A^c.

1.1.3 Disjoint Sets and Partitions

A group of sets are **mutually disjoint** if the intersection of all pairs of sets is equal to the empty set $\emptyset$.

Definition 3: Partition

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3 \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i
2. The sets $A_1, A_2, A_3 \ldots$ are mutually disjoint.

Definition 4: Power Sets

Given a set A, the **power set** of A, denoted $\wp(A)$, is the set of all subsets of A.

Definition 5: Cartesian Product

In general,

$$A_1 \times A_2 \times \dots A_n = \{(a_1, a_2, \dots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n\}.$$

Note that $A_1 \times A_2 \times A_3$ is not quite the same thing as $(A_1 \times A_2) \times A_3$ because of tuple ordering.

1.2 Properties of Sets

Theorem 1: Some Subset Relations

1. Inclusion of Intersection: $A \cap B \subseteq A$ and vice versa
2. Inclusion in Union: $A \subseteq A \cup B$ and vice versa
3. Transitive Property of Subsets: $A \subseteq B \wedge B \subseteq C \to A \subseteq C$

To prove these theorems, *suppose* that there is some arbitrary element of A and show that it is also in B.

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,
$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$
2. *Associative Laws:* For all sets A, B, and C,
$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$
3. *Distributive Laws:* For all sets, A, B, and C,
$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$
4. *Identity Laws:* For all sets A,
$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$
5. *Complement Laws:*
$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$
6. *Double Complement Law:* For all sets A,
$$(A^c)^c = A.$$
7. *Idempotent Laws:* For all sets A,
$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$
8. *Universal Bound Laws:* For all sets A,
$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$
9. *De Morgan's Laws:* For all sets A and B,
$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$
10. *Absorption Laws:* For all sets A and B,
$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$
11. *Complements of U and $\emptyset$:*
$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$
12. *Set Difference Law:* For all sets A and B,
$$A - B = A \cap B^c.$$

In general, to prove set equality, you prove that set A is a subset of set B, and that set B is a subset of set A. Additionally, to prove that a set X is equal to the empty set $\emptyset$, suppose X has an element and derive a contradiction.

Additionally, casework is helpful when dealing with unions. It may be helpful to split a union into 2 cases.

Chapter 2

Functions

In this chapter we go more in depth into properties of functions and their composition.

2.1 Functions Defined on General Sets

Definition 6: Function

A **function** from a set X to a set Y, denoted $f : X \to Y$, is a relation from X, the **domain**, to Y, the **co-domain**, that satisfies two properties:

1. Every element in X is related to some element in Y
2. No element in X is related to more than one element in Y.

The set of all values of f is called the *range of f* or the *image of X under f*. If there exists some x such that $f(x) = y$, then x is called a **preimage (or inverse image) of y**.
Two functions $F : X \to Y$ and $G : X \to Y$ are considered equal if, for all $x \in X, F(x) = G(x)$

Definition 7: Identity Function

The identity function I_X is a function from $X \to X$ by which $I_X(x) = x \forall x \in X$.

Definition 8: Logarithmic Function

The log function $\log_b x = y$ (from $\mathbb{R}^+$ to $\mathbb{R}$) maps a number to the y in the solution of the equation $b^y = x$.

2.1.1 Well Defined Functions

We say that a function is **not well defined** if it fails to satisfy at least one of the requirements for being a function. A function being well defined really means that it qualifies to be called a function.

2.2 One-to-One and Onto, Inverse Functions

2.2.1 One-to-one

A function is **one-to-one** (or **injective**) if, and only if, every input has a unique output. Symbolically, $\forall x_1, x_2 \in X, f(x_1) = f(x_2) \to x_1 = x_2$.

To prove that f is one-to-one, you **suppose** x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$, and **show** that $x_1 = x_2$.

2.2.2 Onto

A function is **onto** (or **surjective**) if, and only if, the co-domain of the function is equal to its image. Symbolically, $\forall y \in Y, \exists x \in X \mid f(x) = y$.

To prove that f is onto, you **suppose** y is in Y, and **show** that there exists an element in x such that $y = f(x)$.

2.2.3 One-to-one correspondences and Inverse Functions

Definition 9: Bijection

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function that is both one-to-one and onto.

Theorem 2: Inverse Functions

Suppose $F : X \to Y$ is a one-to-one correspondence. Then there is a function $F^{-1} : Y \to X$ where $F^{-1}(y) = x$. Note that F^{-1} is also a one-to-one correspondence.
Additionally, note that

$$f^{-1}(b) = a \Leftrightarrow f(a) = b.$$

Finding an inverse function is done while proving that some function F is onto.

2.3 Composition of Functions

The composition of functions (defined as $(g \circ f)(x) = g(f(x)) \forall x \in X$) for two functions $f : X \to Y$ and $g : Y \to Z$ is $(g \circ f) : X \to Z$. Two compositions are equivalent if they have the same output for every input.

Theorem 3: Composition of a Function with Its Inverse

$f^{-1} \circ f = I_X$ and $f \circ f^{-1} = I_Y$. This can be proved directly using the