手写VIO第三章作业讲解

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作业

1 样例代码给出了使用 LM 算法来估计曲线 $y = \exp(ax^2 + bx + c)$ 参数 a, b, c 的完整过程。

1. 请绘制样例代码中 LM 阻尼因子 μ 随着迭代变化的曲线图
2. 将曲线函数改成 $y = ax^2 + bx + c$，请修改样例代码中残差计算，雅克比计算等函数，完成曲线参数估计。
3. 实现其他更优秀的阻尼因子策略，并给出实验对比（选做，评优秀），策略可参考论文[a] 4.1.1 节。

2 公式推导，根据课程知识，完成 $\mathbf{F}, \mathbf{G}$ 中如下两项的推导过程：

$$\mathbf{f}_{15} = \frac{\partial \boldsymbol{\alpha}_{b_i b_{k+1}}}{\partial \delta \mathbf{b}_k^g} = -\frac{1}{4}(\mathbf{R}_{b_i b_{k+1}}[(\mathbf{a}^{b_{k+1}} - \mathbf{b}_k^a)]_\times \delta t^2)(-\delta t)$$

$$\mathbf{g}_{12} = \frac{\partial \boldsymbol{\alpha}_{b_i b_{k+1}}}{\partial \mathbf{n}_k^g} = -\frac{1}{4}(\mathbf{R}_{b_i b_{k+1}}[(\mathbf{a}^{b_{k+1}} - \mathbf{b}_k^a)]_\times \delta t^2)(\frac{1}{2}\delta t)$$

3 证明式(9)。

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[a] Henri Gavin. "The Levenberg-Marquardt method for nonlinear least squares curve-fitting problems". In: *Department of Civil and Environmental Engineering, Duke University* (2011), pp. 1–15.

第一题

- 1.1 可以输出两种lambda，分别是：
  - 确实在迭代过程中让误差下降的lambda，在代码里对应让IsGoodStepInLM=true的lambda

```
while (!stop && (iter < iterations)) {
    std::cout << "iter: " << iter << " , chi= " << currentChi_ << " , Lambda= " << currentLambda_
            << std::endl;
    outfile<< "iter: " << iter << " , chi= " << currentChi_ << " , Lambda= " << currentLambda_
            << std::endl;
```

另一种是在while（！oneStepSuccess）中不断被尝试和更新的lambda，可以选择在IsGoodStepInLM（）中打印

第一题

残差和雅克比（优化变量是a,b,c）：

$$e_i = a x_i^2 + b x_i + c - y_i$$

$$J_i = [x_i^2, x_i, 1]$$

- 1.2
  - 修改残差和雅克比，对应函数ComputeResidual(), ComputeJacobians()
  - 原始数据拟合可能效果较差，大家可以尝试调整数据，比如数据点个数，噪声大小（方差），增大数据范围，直到得到一个满意的拟合结果

1.3 实现其他更新阻尼因子策略

策略1：

1. $\lambda_0 = \lambda_o$; λ_o is user-specified [8].
   use eq'n (13) for $\boxed{\boldsymbol{h}_{\mathsf{lm}}}$ and eq'n (16) for ρ
   if $\rho_i(\boldsymbol{h}) > \epsilon_4$: $\boldsymbol{p} \leftarrow \boldsymbol{p} + \boldsymbol{h}$; $\boxed{\lambda_{i+1}} = \max[\lambda_i / L_\downarrow, 10^{-7}]$;
   otherwise: $\lambda_{i+1} = \min[\lambda_i L_\uparrow, 10^7]$;

Δx

μ

注意：算Δx和ρ与之前不一致

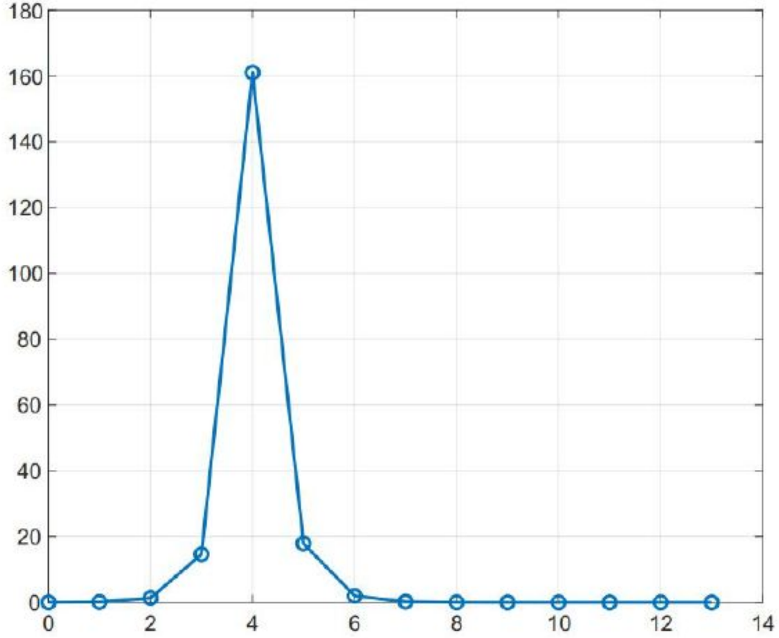
$$\left[\boldsymbol{J}^{\mathsf{T}}\boldsymbol{W}\boldsymbol{J} + \lambda\, \boxed{\mathsf{diag}(\boldsymbol{J}^{\mathsf{T}}\boldsymbol{W}\boldsymbol{J})}\right]\boldsymbol{h}_{\mathsf{lm}} = \boldsymbol{J}^{\mathsf{T}}\boldsymbol{W}(\boldsymbol{y} - \hat{\boldsymbol{y}})$$

$$\rho_i(\boldsymbol{h}_{\mathsf{lm}}) = \frac{\chi^2(\boldsymbol{p}) - \chi^2(\boldsymbol{p} + \boldsymbol{h}_{\mathsf{lm}})}{\boldsymbol{h}_{\mathsf{lm}}^{\mathsf{T}}\left(\lambda_i \boxed{\mathsf{diag}(\boldsymbol{J}^{\mathsf{T}}\boldsymbol{W}\boldsymbol{J})}\boldsymbol{h}_{\mathsf{lm}} + \boldsymbol{J}^{\mathsf{T}}\boldsymbol{W}(\boldsymbol{y} - \hat{\boldsymbol{y}}(\boldsymbol{p}))\right)}$$

需要修改AddLambdatoHessianLM、RemoveLambdaHessianLM和IsGoodStepInLM。

1.3 实现其他更新阻尼因子策略

策略1阻尼因子变化（更少的迭代次数）：



1.3 实现其他更新阻尼因子策略

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策略2：

2. $\lambda_0 = \lambda_o \max\left[\mathrm{diag}[\boldsymbol{J}^\top \boldsymbol{W} \boldsymbol{J}]\right]$; λ_o is user-specified.
use eq'n (12) for $\boldsymbol{h}_{\mathrm{lm}}$ and eq'n (15) for ρ
$\alpha = \left(\left(\boldsymbol{J}^\top \boldsymbol{W}(\boldsymbol{y} - \hat{\boldsymbol{y}}(\boldsymbol{p}))\right)^\top \boldsymbol{h}\right) / \left(\left(\chi^2(\boldsymbol{p}+\boldsymbol{h}) - \boxed{\chi^2(\boldsymbol{p})}\right)/2 + 2\left(\boldsymbol{J}^\top \boldsymbol{W}(\boldsymbol{y} - \hat{\boldsymbol{y}}(\boldsymbol{p}))\right)^\top \boldsymbol{h}\right)$;
if $\rho_i(\alpha \boldsymbol{h}) > \epsilon_4$: $\boldsymbol{p} \leftarrow p + \alpha \boldsymbol{h}$; $\lambda_{i+1} = \max\left[\lambda_i/(1+\alpha), 10^{-7}\right]$;
otherwise: $\lambda_{i+1} = \lambda_i + |\chi^2(\boldsymbol{p}+\alpha\boldsymbol{h}) - \chi^2(\boldsymbol{p})|/(2\alpha)$;

$F(x)$

算法流程：

1、计算 Δx

2、计算 $F(x+\Delta x)$（累加edge的残差）

3、计算 $\alpha = \dfrac{-b^T \Delta x}{\dfrac{F(x+\Delta x) - F(x)}{2} - 2b^T \Delta x}$

4、计算 $\Delta x \leftarrow \alpha \Delta x$（需要先RollbackStates）

5、计算 $\rho = \dfrac{F(x) - F(x+\Delta x)}{\frac{1}{2}\Delta x^T(\mu \Delta x + b)}$

6、根据5选择更新策略

1.3 实现其他更新阻尼因子策略

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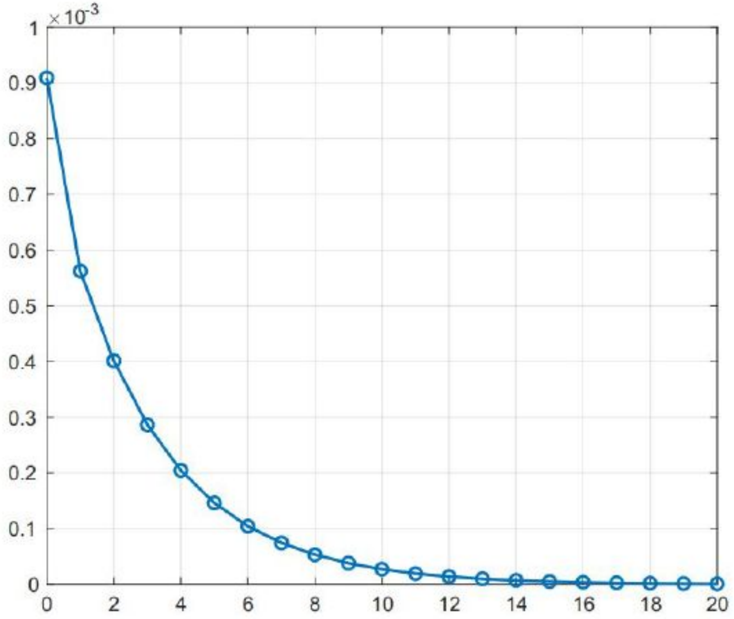
策略2阻尼因子变化：



三个策略（成功迭代次数/总迭代次数）

策略1：9/13

策略2：20/20

策略3：11/18

可以发现策略2虽然增加了迭代次数，但没有失败的迭代（每次迭代误差都在下降）。

2 公式推导 f_{15}

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$$\boldsymbol{\alpha}_{b_ib_{k+1}} = \boldsymbol{\alpha}_{b_ib_k} + \boldsymbol{\beta}_{b_ib_k}\delta t + \frac{1}{2}\boldsymbol{a}\delta t^2$$

$$\begin{aligned}\boldsymbol{a}\delta t^2 &= \frac{1}{2}\left(\boldsymbol{q}_{b_ib_k}\left(\boldsymbol{a}^{b_k} - \boldsymbol{b}_k^a\right) + \boldsymbol{q}_{b_ib_{k+1}}\left(\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right)\right)\delta t^2 \\ &= \frac{1}{2}\left(\boldsymbol{q}_{b_ib_k}\left(\boldsymbol{a}^{b_k} - \boldsymbol{b}_k^a\right) + \left(\boldsymbol{q}_{b_ib_k} \otimes \begin{pmatrix} 1 \\ \frac{1}{2}\boldsymbol{\omega}\delta t\end{pmatrix}\right)\left(\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right)\delta t^2\right)\end{aligned}$$

$$\boldsymbol{\omega} = \frac{1}{2}\left(\left(\boldsymbol{\omega}^{b_k} - \boldsymbol{b}_k^g\right) + \left(\boldsymbol{\omega}^{b_{k+1}} - \boldsymbol{b}_k^g\right)\right) = \frac{1}{2}\left(\boldsymbol{\omega}^{b_k} + \boldsymbol{\omega}^{b_{k+1}}\right) - \boldsymbol{b}_k^g$$

$$\begin{aligned}\frac{\partial \boldsymbol{\alpha}_{b_ib_{k+1}}}{\partial \delta \boldsymbol{b}_k^g} &= \frac{\partial \frac{1}{2}\cdot\frac{1}{2}\left(\boldsymbol{q}_{b_ib_k} \otimes \begin{pmatrix} 1 \\ \frac{1}{2}\boldsymbol{\omega}\delta t\end{pmatrix}\right)\left(\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right)\delta t^2}{\partial \delta \boldsymbol{b}_k^g} \\
&= \frac{\partial \frac{1}{4}\left(\boldsymbol{q}_{b_ib_k} \otimes \begin{pmatrix} 1 \\ \frac{1}{2}\left(\boldsymbol{\omega} - \delta\boldsymbol{b}_k^g\right)\delta t\end{pmatrix}\right)\left(\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right)\delta t^2}{\partial \delta \boldsymbol{b}_k^g} \\
&= \frac{\partial \frac{1}{4}\left(\boldsymbol{R}_{b_ib_k} e^{\left[\left(\boldsymbol{\omega} - \delta\boldsymbol{b}_k^g\right)\delta t\right]_\times}\right)\left(\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right)\delta t^2}{\partial \delta \boldsymbol{b}_k^g} \\
&\approx \frac{\partial \frac{1}{4}\left(\boldsymbol{R}_{b_ib_k} e^{\left[\boldsymbol{\omega}\delta t\right]_\times} e^{\left[-\boldsymbol{J}_r(\boldsymbol{\omega}\delta t)\delta\boldsymbol{b}_k^g\delta t\right]_\times}\right)\left(\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right)\delta t^2}{\partial \delta \boldsymbol{b}_k^g} \\
&\approx \frac{\partial \frac{1}{4}\left(\boldsymbol{R}_{b_ib_{k+1}}\left(\boldsymbol{I} - \left[\boldsymbol{J}_r(\boldsymbol{\omega}\delta t)\delta\boldsymbol{b}_k^g\delta t\right]_\times\right)\right)\left(\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right)\delta t^2}{\partial \delta \boldsymbol{b}_k^g} \\
&= \frac{\partial \frac{1}{4}\boldsymbol{R}_{b_ib_{k+1}}\left[-\boldsymbol{J}_r(\boldsymbol{\omega}\delta t)\delta\boldsymbol{b}_k^g\delta t\right]_\times\left(\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right)\delta t^2}{\partial \delta \boldsymbol{b}_k^g} \\
&= \frac{\partial -\frac{1}{4}\boldsymbol{R}_{b_ib_{k+1}}\left[\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right]_\times\left(-\boldsymbol{J}_r(\boldsymbol{\omega}\delta t)\delta\boldsymbol{b}_k^g\delta t\right)\delta t^2}{\partial \delta \boldsymbol{b}_k^g} \\
&= -\frac{1}{4}\boldsymbol{R}_{b_ib_{k+1}}\left[\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right]_\times\left(-\boldsymbol{J}_r(\boldsymbol{\omega}\delta t)\delta t\right)\delta t^2 \\
&\approx -\frac{1}{4}\left(\boldsymbol{R}_{b_ib_{k+1}}\left[\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right]_\times\right)\delta t^2(-\delta t))\end{aligned}$$

2 公式推导 g_{12}

$$\boldsymbol{\alpha}_{b_ib_{k+1}} = \boldsymbol{\alpha}_{b_ib_k} + \boldsymbol{\beta}_{b_ib_k}\delta t + \frac{1}{2}\boldsymbol{a}\delta t^2$$

$$\begin{aligned}
\boldsymbol{a}\delta t^2 &= \frac{1}{2}\left(\boldsymbol{q}_{b_ib_k}\left(\boldsymbol{a}^{b_k} - \boldsymbol{b}_k^a\right) + \boldsymbol{q}_{b_ib_{k+1}}\left(\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right)\right)\delta t^2 \\
&= \frac{1}{2}\left(\boldsymbol{q}_{b_ib_k}\left(\boldsymbol{a}^{b_k} - \boldsymbol{b}_k^a\right) + \left(\boldsymbol{q}_{b_ib_k} \otimes \begin{pmatrix} 1 \\ \frac{1}{2}\boldsymbol{\omega}\delta t \end{pmatrix}\right)\left(\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right)\delta t^2\right)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\omega} &= \frac{1}{2}\left(\left(\left(\boldsymbol{\omega}^{b_k} + \boldsymbol{n}_k^g\right) - \boldsymbol{b}_k^g\right) + \left(\left(\boldsymbol{\omega}^{b_{k+1}} + \boldsymbol{n}_{k+1}^g\right) - \boldsymbol{b}_k^g\right)\right) \\
&= \frac{1}{2}\left(\boldsymbol{\omega}^{b_k} + \boldsymbol{\omega}^{b_{k+1}}\right) - \boldsymbol{b}_k^g + \frac{1}{2}\boldsymbol{n}_{k+1}^g + \frac{1}{2}\boldsymbol{n}_k^g
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \boldsymbol{\alpha}_{b_ib_{k+1}}}{\partial \delta \boldsymbol{n}_{b_k}^g} &= \frac{\partial \frac{1}{2}\cdot\frac{1}{2}\left(\boldsymbol{q}_{b_ib_k} \otimes \begin{pmatrix} 1 \\ \frac{1}{2}\left(\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{n}_{b_k^g}\right)\delta t \end{pmatrix}\right)\left(\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right)\delta t^2}{\partial \delta \boldsymbol{n}_{b_k}^g} \\
&= \frac{\partial -\frac{1}{4}\boldsymbol{R}_{b_ib_{k+1}}\left[\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right]_\times\left(\boldsymbol{J}_r(\boldsymbol{\omega}\delta t)\frac{1}{2}\boldsymbol{n}_{b_k^g}\delta t\right)\delta t^2}{\partial \delta \boldsymbol{n}_{b_k}^g} \\
&= -\frac{1}{4}\boldsymbol{R}_{b_ib_{k+1}}\left[\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right]_\times\left(\boldsymbol{J}_r(\boldsymbol{\omega}\delta t)\frac{1}{2}\delta t\right)\delta t^2 \\
&\approx -\frac{1}{4}\left(\boldsymbol{R}_{b_ib_{k+1}}\left[\boldsymbol{a}^{b_{k+1}} - \boldsymbol{b}_k^a\right]_\times\right)\delta t^2\left(\frac{1}{2}\delta t\right)
\end{aligned}$$

3 证明式(9)

已知 $(J^T J + \mu I)\Delta x_{lm} = -F'^T$

对 $J^T J$ 做特征值分解：$J^T J = V\Sigma V^T = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} {v_1}^T \\ {v_2}^T \\ \vdots \\ {v_n}^T \end{bmatrix}$

$$\Delta x_{lm} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} (\lambda_1+\mu)^{-1} & & & \\ & (\lambda_2+\mu)^{-1} & & \\ & & \ddots & \\ & & & (\lambda_n+\mu)^{-1} \end{bmatrix} \begin{bmatrix} {v_1}^T \\ {v_2}^T \\ \vdots \\ {v_n}^T \end{bmatrix} \left(-F'^T\right)$$

$$= -\sum_{i=1}^{n} \frac{v_i \cdot {v_i}^T F'^T}{\lambda_i + \mu} \overset{{v_i}^T F'^T \text{是一个数}}{=} -\sum_{i=1}^{n} \frac{{v_i}^T F'^T v_i}{\lambda_i + \mu}$$

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Q&A

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Thanks for Listening