MULTIPLICATIVE STRUCTURE IN THE STABLE SPLITTING OF $\Omega SL_n(\mathbb{C})$

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ABSTRACT. The space of based loops in $SL_n(\mathbb{C})$, also known as the affine Grassmannian of $SL_n(\mathbb{C})$, admits an $\mathbb{E}_2$ or fusion product. Work of Mitchell and Richter proves that this based loop space stably splits as an infinite wedge sum. We prove that the Mitchell–Richter splitting is coherently multiplicative, but not $\mathbb{E}_2$. Nonetheless, we show that the splitting becomes $\mathbb{E}_2$ after base-change to complex cobordism. Our proof of the $\mathbb{A}_\infty$ splitting involves on the one hand an analysis of the multiplicative properties of Weiss calculus, and on the other a use of Beilinson–Drinfeld Grassmannians to verify a conjecture of Mahowald and Richter. Other results are obtained by explicit, obstruction-theoretic computations.

CONTENTS

1. INTRODUCTION

We study the homotopy type of the affine Grassmannian of $SL_n(\mathbb{C})$, which is equivalent to the space $\Omega SU(n)$ of based loops in $SU(n)$. There are essentially two multiplications on this homotopy type, one arising from the composition of loops and the other from the group multiplication on $SL_n(\mathbb{C})$. Together, these two multiplications interact to give $\Omega SU(n)$ the structure of an $\mathbb{E}_2$ or chiral algebra. In geometric representation theory, this structure is witnessed by the existence of the Beilinson–Drinfeld Grassmannian.

Either of the above (homotopy equivalent) products make $H_*(\Omega SU(n); \mathbb{Z})$ into a graded ring. To describe this ring, let us first name some of its elements. For each one-dimensional subspace $V \subset \mathbb{C}^n$, there is a loop $\lambda_V : S^1 \to U(n)$ given by the formula

$$\lambda_V(z) = \begin{pmatrix} z & 0 \\ 0 & I \end{pmatrix},$$

with the matrix presented in terms of the decomposition $\mathbb{C}^n \cong V \oplus V^\perp$. Fixing a particular line $W \subset \mathbb{C}^n$, the construction $V \mapsto \lambda_W^{-1} \cdot \lambda_V$ defines a well-known map

$$\mathbb{CP}^{n-1} \to \Omega SU(n).$$

For $1 \leq i \leq n-1$, let $b_i \in H_{2i}(\Omega SU(n); \mathbb{Z})$ denote the image of the generator of $H_{2i}(\mathbb{CP}^{n-1}; \mathbb{Z})$. It is a result of Bott [Bot58] that

$$H_*(\Omega SU(n); \mathbb{Z}) \cong \mathbb{Z}[b_1, b_2, \cdots, b_{n-1}],$$

with the latter denoting the polynomial algebra on the classes b_i.

Notice that $H_*(\Omega SU(n); \mathbb{Z})$ is a *bigraded* ring: there is, in addition to the usual homological grading $*$, a word length grading that assigns each b_i degree 1. Mahowald observed that the action of the Steenrod algebra on $H_*(\Omega SU(n); \mathbb{F}_2)$ preserves word length, and he conjectured a geometric splitting to be responsible.

Motivated by Mahowald's conjecture, Mitchell [Mit86] (and, independently, Segal [Seg89]) constructed a filtration

$$* = F_{n,0} \longrightarrow F_{n,1} \longrightarrow F_{n,2} \longrightarrow \cdots \longrightarrow \Omega SU(n).$$

Following Mitchell, we name this the *Bott filtration* of $\Omega SU(n)$. The homology of $F_{n,k}$ consists of words of length at most k, and the inclusion $F_{n,1} \to \Omega SU(n)$ is given by the above map $\mathbb{CP}^{n-1} \to \Omega SU(n)$. For $k > 1$, the homotopy type $F_{n,k}$ may be modeled as a singular algebraic variety, and there is a surjective resolution of singularities $(\mathbb{CP}^{n-1})^{\times k} \longrightarrow F_{n,k}$. The exact construction of the Bott filtration is somewhat involved, and we review it in Section 6–it is a subfiltration of the Bruhat ordering on (closures of) Iwahori orbits.

Confirming Mahowald's intuition, Mitchell and Richter [CM88, Theorem 2.1] proved that the Bott filtration splits after taking suspension spectra. In short, there is a wedge sum decomposition

$$\Sigma_+^\infty \Omega SU(n) \simeq \bigvee_k \Sigma^\infty F_{n,k}/F_{n,k-1} \simeq \mathbb{S} \vee \Sigma^\infty \mathbb{CP}^{n-1} \vee \cdots.$$

Remark 1.1. Readers unfamiliar with stable homotopy theory may prefer to think of the suspension spectrum functor Σ_+^∞ as analogous to taking the underlying motive of a variety. The statement is then that this motive splits; in particular, its homology will split in any homology theory.

Example 1.2. In the case $n = 2$, the Bott filtration of $\Omega SU(2) \simeq \Omega S^3$ is the classical James filtration of $\Omega \Sigma S^2$. The Mitchell–Richter splitting recovers the stable James splitting.

In this paper, we will be interested in *multiplicative* aspects of the Bott filtration and its splitting. In Section 2, we review the symmetric monoidal structures on the (∞)-categories of filtered and graded spectra; they are given by Day convolution. This allows us to talk about $\mathbb{E}_n$-algebras in filtered and graded spectra, providing the language necessary to state our first main theorem (proven in Section 6):

Theorem 1.3. *The suspension of the Bott filtration*

$$\mathbb{S} \longrightarrow \Sigma_+^\infty \mathbb{CP}^{n-1} \simeq \Sigma_+^\infty F_{n,1} \longrightarrow \Sigma_+^\infty F_{n,2} \longrightarrow \cdots \longrightarrow \Sigma_+^\infty \Omega SU(n).$$

is an $\mathbb{A}_\infty$-algebra object in filtered spectra.

Remark 1.4. The Bott filtration is multiplicative before suspension, but for technical reasons we prefer to phrase our results in terms of filtered spectra instead of filtered spaces.

Question 1. Is the Bott filtration an $\mathbb{E}_2$ filtration? We do not know the answer–for some thoughts about the problem, see Remark 6.10.

The proof of Theorem 1.3 is fairly straightforward, once given access to the sophisticated machinery behind the Beilinson–Drinfeld Grassmannian. For example, we will explain in Section 6 that this machinery immediately dispenses with a conjecture of Mahowald and Richter [MR93]. Nonetheless, there are some subtleties involved, and it is these subtleties that prevent us from determining if the Bott filtration is $\mathbb{E}_2$. The problem is readily visible in the case $n = \infty$:

Example 1.5. The limiting case of the Bott filtration of $\Omega SU(n)$ as n tends to ∞ is the filtration

$$* \longrightarrow BU(1) \longrightarrow BU(2) \longrightarrow BU(3) \longrightarrow \cdots \longrightarrow BU \simeq \Omega SU.$$

It is easy to see that $\coprod BU(n)$ is a graded $\mathbb{E}_2$-algebra in spaces (in fact, it is a graded $\mathbb{E}_\infty$-algebra, being the nerve of the category of vector spaces). However, the filtered object is much more subtle. For example, the squares

$$
\begin{array}{ccc}
BU(i) \times BU(j) & \longrightarrow & BU(i) \times BU(j+1) \\
\downarrow & & \downarrow \\
BU(i+1) \times BU(j) & \longrightarrow & BU(i+1) \times BU(j+1)
\end{array}
$$

do not commute on the nose, but only up to non-canonical homotopy.

In Section 2, we discuss an *associated graded* construction that transforms filtered $\mathbb{E}_n$-algebras into graded $\mathbb{E}_n$-algebras. The central result of our paper may be thought of as a multiplicatively structured version of the Mitchell–Richter splitting:

Theorem 1.6. *As an $\mathbb{A}_\infty$-algebra object in filtered spectra, the Bott filtration of $\Sigma^\infty_+ \Omega SU(n)$ is equivalent to its associated graded.*

Corollary 1.6.1. *For any multiplicative homology theory E, $E_*(\Omega SU(n))$ is a bigraded ring. One grading is given by $*$, and the other by the associated graded of the Bott filtration.*

Example 1.7. In the case $n = \infty$ of Example 1.5, the Mitchell–Richter splitting

$$\Sigma^\infty_+ BU \simeq \bigvee_n \Sigma^\infty BU(n)/BU(n-1). \tag{1}$$

recovers an older result of Snaith [Sna79]. Snaith further showed that (1) is an equivalence of homotopy commutative ring spectra, and our Theorem 1.6 gives an equivalence of $\mathbb{A}_\infty$-ring spectra.

The left-hand side of (1) is an $\mathbb{A}_\infty$-algebra by virtue of the fact that BU is a loop space, while the right-hand side acquires its $\mathbb{A}_\infty$-algebra structure from an associated graded construction. To understand this latter $\mathbb{A}_\infty$ structure, it may help to know that $BU(n)/BU(n-1) \simeq MU(n)$, the Thom space of the canonical bundle over $BU(n)$. We therefore recognize the right-hand side of (1) as the Thom spectrum of the J-homomorphism

$$\coprod BU(n) \xrightarrow{J} Pic(\mathbb{S}).$$

Since J is a loop map, its Thom spectrum acquires an $\mathbb{A}_\infty$-algebra structure, and this turns out to agree with the $\mathbb{A}_\infty$ associated graded of the Bott filtration.

Of course, BU is not just a loop space, but in fact an infinite loop space. Similarly, J is not just a loop map, but furthermore an infinite loop map. Thus, both sides of (1) are naturally $\mathbb{E}_\infty$-ring spectra. Perhaps surprisingly, these $\mathbb{E}_\infty$-rings are **not** equivalent.

Remark 1.8. Snaith used his splitting (1) to give an equivalence of homotopy commutative ring spectra

$$\Sigma^\infty_+ BU[\beta^{-1}] \simeq MUP, \tag{2}$$

where MUP denotes periodic complex bordism. We will have more to say about the coherence of (2) in forthcoming work.

In fact, the $\mathbb{A}_\infty$-splitting provided by Theorem 1.6 is the best result possible:

Theorem 1.9. *Suppose $n \geq 4$. If the Bott filtration of $\Sigma_+^\infty \Omega SU(n)$ may be made into an $\mathbb{E}_2$-algebra object in filtered spectra, then it is **not** equivalent to its $\mathbb{E}_2$ associated graded. More generally, any extension of the graded $\mathbb{A}_\infty$-algebra of Theorem 1.6 to a graded $\mathbb{E}_2$-algebra must fail to have the usual $\mathbb{E}_2$-algebra structure on its underlying ungraded $\mathbb{E}_2$-ring.*

However, if one is willing to work in a complex-oriented theory, such as ordinary homology, the situation improves:

Theorem 1.10. *Let MU denote the $\mathbb{E}_\infty$-ring spectrum of complex bordism. Suppose that R_1 and R_2 are any two (ungraded) $\mathbb{E}_2$-algebras with the same underlying $\mathbb{A}_\infty$-ring $\Sigma_+^\infty \Omega SU(n)$. Then there there is an equivalence of $\mathbb{E}_2$-MU-algebras*

$$MU \wedge R_1 \simeq MU \wedge R_2.$$

For Theorem 1.10 to be of interest, it is necessary to exhibit exotic $\mathbb{E}_2$-algebra structures on $\Sigma_+^\infty \Omega SU(n)$. Example 1.7 gives some idea of how this may be accomplished via Thom spectra, and we end Section 6 with a sketch of the following:

Construction 1.11. There exists a graded $\mathbb{E}_2$-algebra structure on $\mathrm{gr}(\Sigma_+^\infty\{F_{n,k}\})$ extending the canonical graded $\mathbb{A}_\infty$-algebra structure.

Remark 1.12. Theorem 1.10, combined with Construction 1.11, may be seen as a once-looped analogue of work of Kitchloo [Kit01]. Kitchloo studied a splitting, due to Miller [Mil85], of $\Sigma_+^\infty SU(n)$. His theorem is that, *for complex-oriented E, the corresponding direct sum decomposition of $E_*(SU(n))$ is multiplicative.*

Our proof of Theorem 1.10 is by obstruction theory. We show in Section 9 that all obstructions to an $\mathbb{E}_2$ equivalence vanish. On the other hand, we prove Theorem 1.9 by explicitly calculating a non-zero obstruction in Section 10. It remains to discuss Theorem 1.6, the $\mathbb{A}_\infty$ splitting of $\Sigma_+^\infty \Omega SU(n)$, which is the central result of our paper.

To prove that a filtered spectrum

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \cdots$$

splits, it suffices to provide splitting maps in the form of a "cofiltered spectrum"

$$A_0 \longleftarrow A_1 \longleftarrow A_2 \longleftarrow A_3 \longleftarrow \cdots$$

In Section 2 we make this statement precise (with the proof in Appendix B) by explaining the following theorem:

Theorem 2.6. *Let $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil}(\mathrm{Sp}))$ be an $\mathbb{E}_n$ filtered spectrum. Suppose there exists an $\mathbb{E}_n$ cofiltered spectrum $Y \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Cofil}(\mathrm{Sp}))$ with the following two properties:*
  (1) *There is an equivalence $\mathrm{colim} X \simeq \lim Y$ of $\mathbb{E}_n$-algebras in spectra.*
  (2) *The resulting natural maps $X_i \to Y_i$ are equivalences.*
Then, the filtered spectrum X is $\mathbb{E}_n$-split.

We wish to apply this theorem in the case where X is the $\mathbb{A}_\infty$ filtered spectrum in Theorem 1.3. In Section 7, we produce the corresponding $\mathbb{A}_\infty$ cofiltered spectrum; the proof is then finished in Section 8. To do this, we extend the methods of [Aro01], who used Weiss calculus to give an elegant second proof of the Mitchell–Richter splitting (without multiplicative structure).

Arone's idea is to use additional functoriality present in the filtration of $\Omega SU(n)$. Let $\mathcal{J}$ denote the topological category of complex vector spaces and embeddings, fix a complex vector space V, and consider the functor

$$G : \mathcal{J} \longrightarrow \mathcal{S}\text{paces}$$

given by $G(W) = \mathcal{J}(V, V \oplus W)$. Observe that the special unitary group $SU(V \oplus \mathbb{C})$ arises as the value $G(\mathbb{C}) = \mathcal{J}(V, V \oplus \mathbb{C})$. Weiss calculus provides a toolbox with which to study functors similar to G – a brief review of the theory is provided at the beginning of Section 7.

The Bott filtration in fact arises from a sequence of functors

$$F_0(W) \to F_1(W) \to F_2(W) \to \cdots \to F(W) := \Sigma_+^\infty \Omega\, \mathcal{J}(V, V \oplus W)$$

from $\mathcal{J}$ to spectra. This filtration has the key property that the successive quotients F_n/F_{n-1} are *homogeneous functors of degree n*. Arone then applies an argument of Goodwillie [Goo03, Example 1.20] to see that in this situation, the Weiss polynomial approximation $P_n F(W)$ is precisely the functor $F_n(W)$. The approximations

$$P_n F(W) \to P_{n-1} P_n F(W) \simeq P_{n-1} F(W)$$

provide Arone with splitting maps.

To obtain an $\mathbb{A}_\infty$ splitting, it is no longer sufficient to merely provide splitting maps. As explained by Theorem 2.6, we instead require an $\mathbb{A}_\infty$ structure on the whole system of splitting maps, considered as a cofiltered spectrum. This will arise from combining two statements: the first is that the functor $F(W)$ takes values in $\mathbb{A}_\infty$ ring spectra, and thus gives an $\mathbb{A}_\infty$ object in the category $\mathrm{Sp}^{\mathcal{J}}$ of functors from $\mathcal{J}$ to spectra. The second is that the Weiss tower can be viewed as a symmetric monoidal functor from $\mathrm{Sp}^{\mathcal{J}}$ to cofiltered objects in $\mathrm{Sp}^{\mathcal{J}}$. This observation, which may be of independent interest, is likely known to experts and was suggested to us by Jacob Lurie, but we could not locate in the literature. We have proven it in the following form, where $\mathrm{Sp}^{\mathcal{J}}_{\mathrm{conv}}$ denotes restriction to certain conveniently convergent functors:

Theorem 7.10. *The Weiss tower defines a symmetric monoidal functor*

$$Tow : \mathrm{Sp}^{\mathcal{J}}_{conv} \to \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}}_{conv}).$$

Remark. In fact, the convergence hypotheses were made for convenience and are not necessary. Furthermore, the theorem works just as well in the context of Goodwillie calculus – see Remark 7.11.

There are a number of natural and presumably approachable open questions suggested by our work here. In addition to Question 1, we highlight the following:

Question 2. Is the Mitchell–Richter filtration of the loop space of a Stiefel manifold always filtered $\mathbb{A}_\infty$? This is the only obstruction to promoting Theorem 1.6 to a result about all such loop spaces.

Question 3. What are the proper motivic analogues of our results, phrased in the category of $\mathbb{A}^1$-invariant Nisnevich sheaves?

Question 4. What are the proper equivariant analogues of our result? See for example [Ull12], [Tyn17].

Notation: We use Sp to denote the category of spectra, and $\mathcal{S}$ the category of spaces. As through the introduction, we will freely use the word category to refer to a not-necessarily truncated ∞-category.

2. Filtered and Graded Ring Spectra

It will be important for us to have a precise language for discussing filtered and graded spectra, what it means to be split, what it means to take associated graded, and the multiplicative aspects of these constructions. Here we review a framework from [Lur15] for studying graded and filtered objects. The reader is referred to [Lur15] for a more thorough treatment and all proofs.

2.1. **First definitions.** Let $\mathcal{D}$ be an ∞-category which we will regard as the diagram category. Our filtered objects will be valued in the functor category $\mathrm{Sp}^{\mathcal{D}}$. This will be no more difficult than just ordinary spectra because limits, colimits, and smash products will be considered pointwise; in any case, we will refer to objects of $\mathrm{Sp}^{\mathcal{D}}$ as functors or simply as spectra.

Denote by $\mathbb{Z}_{\geq 0}$ the poset of non-negative integers thought of as an ordinary category where $\mathrm{Hom}(a,b)$ is a singleton if $a \leq b$, and empty otherwise. Denote by $\mathbb{Z}_{\geq 0}^{ds}$ the corresponding discrete category. We will implicitly take nerves to obtain ∞-categories which will serve as the indexing sets for filtered and graded spectra. The reader is warned that our numbering conventions are opposite the ones in [Lur15].

Definition 2.1. Let $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ denote the functor category $\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{ds}, \mathrm{Sp}^{\mathcal{D}})$. We shall refer to $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ as the category of graded objects in $\mathrm{Sp}^{\mathcal{D}}$. Its objects can be thought of as sequences $X_0, X_1, X_2, \cdots \in \mathrm{Sp}^{\mathcal{D}}$.

Definition 2.2. Let $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ denote the functor category $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathrm{Sp}^{\mathcal{D}})$. We shall refer to $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ as the category of filtered objects in $\mathrm{Sp}^{\mathcal{D}}$. Its objects can be thought of as sequences $Y_0 \to Y_1 \to Y_2 \to \cdots \in \mathrm{Sp}^{\mathcal{D}}$ filtering $\mathrm{colim}_i Y_i$.

The obvious map $\mathbb{Z}_{\geq 0}^{ds} \to \mathbb{Z}_{\geq 0}$ induces a restriction functor $\mathrm{res} : \mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ which can be thought of as forgetting the maps in the filtered object. The restriction fits into an adjunction

$$I : \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) \rightleftarrows \mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}) : \mathrm{res}$$

where the left adjoint $I : \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ is given by left Kan extension. The functor I can be described explicitly as taking a graded object $X_0, X_1, X_2, \cdots$ to the filtered object

$$I(X_0, X_1, \cdots) = (X_0 \to X_0 \vee X_1 \to X_0 \vee X_1 \vee X_2 \to \cdots).$$

Inverse to this, there is an associated graded functor $\mathrm{gr} : \mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ such that the composite $\mathrm{gr} \circ I : \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ is an equivalence. This can be thought of pointwise by the formula

$$\mathrm{gr}(X_0 \to X_1 \to X_2 \to \cdots) = (X_0, X_1/X_0, X_2/X_1, \cdots).$$

As the names suggest, one may recover from a filtered or graded functor the underlying object. For filtered objects, this is a functor

$$\mathrm{colim} : \mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}) \to \mathrm{Sp}^{\mathcal{D}}$$

given by Kan extending along $\mathbb{Z}_{\geq 0} \to *$. It can be thought of as taking the colimit. For graded objects, the underlying object is simply the direct sum of all the graded piece. We will systematically abuse notation by conflating a graded spectrum with its underlying spectrum; however, when there is any ambiguity, we will specify whether we are referring to the graded object or the underlying.

2.2. **Monoidal structures.** We now begin studying the monoidal structures on graded and filtered spectra. We confine ourselves to a basic discussion here, leaving a more technical discussion for Appendix A.

By [Gla13] or [Lur16, Example 2.2.6.17], the categories $\mathbf{Gr}(\mathrm{Sp})$ and $\mathbf{Fil}(\mathrm{Sp})$ may be given symmetric monoidal structures via the Day convolution. Then, via the identifications $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) = \mathbf{Gr}(\mathrm{Sp})^{\mathcal{D}}$ and $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}) = \mathbf{Fil}(\mathrm{Sp})^{\mathcal{D}}$, the categories $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ and $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ may be given symmetric monoidal structures pointwise on $\mathcal{D}$. In both cases, we denote the resulting operation by $\otimes$. Explicitly, the filtered tensor product

$$(X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots) \otimes (Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \cdots)$$

of two filtered spectra is computed as

$$X_0 \wedge Y_0 \longrightarrow \mathrm{colim} \begin{pmatrix} X_0 \wedge Y_1 \\ \uparrow \\ X_0 \wedge Y_0 \longrightarrow X_1 \wedge Y_0 \end{pmatrix} \longrightarrow \mathrm{colim} \begin{pmatrix} X_0 \wedge Y_2 \\ \uparrow \\ X_0 \wedge Y_1 \longrightarrow X_1 \wedge Y_1 \\ \uparrow \qquad \uparrow \\ X_0 \wedge Y_0 \longrightarrow X_1 \wedge Y_0 \longrightarrow X_2 \wedge Y_0 \end{pmatrix} \longrightarrow \cdots.$$

For graded spectra, the analogous formula is:

$$(A_0, A_1, A_2, \cdots) \otimes (B_0, B_1, B_2, \cdots) \simeq \left(A_0 \wedge B_0, (A_1 \wedge B_0) \vee (A_0 \wedge B_1), \cdots, \bigvee_{i+j=n} A_i \wedge B_j, \cdots \right).$$

The unit $\mathbb{S}_{\mathcal{D}}^{gr}$ of $\otimes$ in $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ is the constant diagram at S^0 in degree 0 and $*$ otherwise; the unit $\mathbb{S}_{\mathcal{D}}^{fil}$ in $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ is $I\mathbb{S}_{\mathcal{D}}^{gr}$. We may then talk about $\mathbb{E}_n$-algebras in $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ and $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$.

The functors I and gr can be given symmetric monoidal structures such that the composite $\mathrm{gr} \circ I : \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ is a symmetric monoidal equivalence by [Lur15, Proposition 3.2.1]. It follows in particular that they extend to functors between the categories of $\mathbb{E}_n$-algebras in $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ and $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$. Thus, given an $\mathbb{E}_n$-algebra Y in filtered spectra, we obtain a canonical $\mathbb{E}_n$ structure on its associated graded $\mathrm{gr}(Y)$. Conversely, given $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}))$, we obtain $IX \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}))$.

Definition 2.3. A filtered $\mathbb{E}_n$-algebra $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}))$ is called $\mathbb{E}_n$-*split* if there exists some $Y \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}))$ with an equivalence $X \simeq IY$ in $\mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}))$.

Given an $\mathbb{E}_n$-split filtered spectrum X, we can recover the underlying graded spectrum by taking the associated graded.

Example 2.4. Let $X \in \mathcal{S}$ be a connected space and $n > 0$. The Snaith splitting can be seen as giving $\Sigma_+^\infty \Omega^n \Sigma^n X$ the structure of a $\mathbb{E}_n$-split filtered spectrum. To see this, first note that there is a commutative square of ∞-categories:

$$\begin{CD} \mathbf{Gr}(\mathrm{Sp}) @>{F_{\mathbb{E}_n}}>> \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}(\mathrm{Sp})) \\ @VVV @VVV \\ \mathrm{Sp} @>{F_{\mathbb{E}_n}}>> \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp}) \end{CD}$$

where the downward arrows forget the grading and the rightward arrows take free algebras. This is essentially because the formula for the free graded $\mathbb{E}_n$-algebra on a graded spectrum is the same as in the ungraded case. Let $\Sigma^\infty X[1]$ denote the graded spectrum with $\Sigma^\infty X$ in degree 1. The above square implies that $F_{\mathbb{E}_n}(\Sigma^\infty X[1]) \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr})$ is a lift of the $\mathbb{E}_n$ ring spectrum $F_{\mathbb{E}_n}(\Sigma^\infty X) \simeq \Sigma_+^\infty \Omega^n \Sigma^n X$. As a result, the filtered spectrum $I(F_{\mathbb{E}_n}\Sigma^\infty X[1])$ exhibits the Snaith filtration on $\Sigma_+^\infty \Omega^n \Sigma^n X$ as an $\mathbb{E}_n$-split $\mathbb{E}_n$ filtered spectrum.

In this paper, we will be interested in when a given $\mathbb{E}_n$ filtered spectrum is $\mathbb{E}_n$-split. Disregarding the multiplicative structure, a filtered spectrum

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots,$$

will split if and only if there are maps going the other way:

$$X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots,$$

with the property that the relevant composites are equivalences. To systematically talk about these backwards maps, we need the following definition:

Definition 2.5. Let $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{D}})$ denote the functor category $\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathrm{Sp}^{\mathcal{D}})$. We shall refer to $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{D}})$ as the category of cofiltered objects in $\mathrm{Sp}^{\mathcal{D}}$. Its objects can be thought of as towers of functors $Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots \in \mathrm{Sp}^{\mathcal{D}}$.

In Appendix A, we show how to give $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{D}})$ the structure of a symmetric monoidal ∞-category. One might then correctly surmise that producing a multiplicatively structured splitting involves the multiplicativity of this opposite filtration. In particular, we have the following criterion, which we prove in Appendix B (we now switch from $\mathrm{Sp}^{\mathcal{D}}$ to Sp for ease of notation):

Theorem 2.6. *Let* $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil}(\mathrm{Sp}))$ *be an* $\mathbb{E}_n$ *filtered spectrum. Suppose there exists an* $\mathbb{E}_n$ *cofiltered spectrum* $Y \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Cofil}(\mathrm{Sp}))$ *with the following two properties:*
  (1) *There is an equivalence* $\mathrm{colim} X \simeq \lim Y$ *of* $\mathbb{E}_n$-*algebras in spectra.*
  (2) *The resulting natural maps* $X_i \to Y_i$ *are equivalences.*
Then, the filtered spectrum X *is* $\mathbb{E}_n$-*split.*

As a final remark, we note that the category of filtered spectra may be recovered as a module category inside the category of graded spectra. Specifically, consider $A = \Sigma_+^\infty \mathbb{Z}_{\geq 0}^{ds}$, the suspension of the nerve of the symmetric monoidal category $\mathbb{Z}_{\geq 0}^{ds}$, as an $\mathbb{E}_\infty$-algebra in $\mathbf{Gr}(\mathrm{Sp})$. The spectrum underlying A is an infinite wedge of copies of $\mathbb{S}^0$. Then the following may be shown by the argument in [Lur15, Proposition 3.1.6]:

Lemma 2.7. *The functor* res *lifts to an equivalence*

$$\mathbf{Fil}(\mathrm{Sp}) \xrightarrow{\simeq} \mathbf{Mod}_A(\mathbf{Gr}(\mathrm{Sp}))$$

of symmetric monoidal ∞-categories.

3. Revision Merge

In the next section we will define the Mitchell–Richter Bott filtration on $\Omega SU(n)$ and prove that it makes $\Sigma_+^\infty \Omega SU(n)$ into a filtered $\mathbb{A}_\infty$-ring spectrum. The filtration, especially as a multiplicative object, is most naturally described in the language of algebraic geometry. Our aim in this section will be to recall the complex points of these algebro-geometric objects, and extract from both the algebro-geometric and topological literature all of the basic facts about these complex points that we will later need. None of the ideas in this section are original: the objects we study are due to Beilinson and Drinfeld **CITE**, and their translation into algebraic topology is due to Jacob Lurie [Lur16, §5.5].

Fix for the moment a smooth, reductive, affine algebraic group G over $\mathbb{C}$. Through the rest of the paper, we will be interested only in the cases $G = \mathbb{G}_m, SL_n,$ or GL_n, and so the reader may safely restrict his attention to those cases for concreteness. As we will explain in detail, an algebro-geometric model for $\Omega G(\mathbb{C})$ is the *affine Grassmannian* Gr_G. The $\mathbb{E}_2$-algebra structure present on $\Omega G(\mathbb{C})$ is encoded by a more elaborate object, the *Beilinson–Drinfeld Grassmannian*. A good general reference for both of these objects is [Zhu16], whose presentation we will more or less follow below.

We use D to denote the formal disk $\mathrm{Spec}(\mathbb{C}[[t]])$ and D^* to denote the punctured disk $\mathrm{Spec}(\mathbb{C}((t)))$. For R a $\mathbb{C}$-algebra, we use D_R to denote $\mathrm{Spec}(\mathbb{C}[[t]]\hat{\otimes}R)$ and D_R^* to denote $\mathrm{Spec}(\mathbb{C}((t))\hat{\otimes}R)$. We refer to a point in the space $G(D^*) = G(\mathbb{C}((t)))$ as an algebraic free (i.e., unbased) loop in G. Such points are exactly automorphisms of the trivial G-torsor $\mathcal{E}^0$ over D^*.

Definition 3.1. The *affine Grassmannian* Gr_G of G is the Ind-scheme with functor of points

$$Gr_G(R) = \{(\mathcal{E}, \beta)\}, \text{ where}$$

$\mathcal{E}$ is a G-torsor over D_R and $\beta : \mathcal{E}|_{D_R^*} \cong \mathcal{E}^0_{D_R^*}$ is a trivialization over D_R^*.

The complex points $Gr_G(\mathbb{C})$ are a model for the topological space ΩG CITE. These complex points are given by the homogeneous space CITE

$$G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]),$$

which up to homotopy is the quotient of the free loop space on G by the action of G.

The above functor of points is not representable by a scheme, but it is a filtered colimit of schemes $Gr_{G,\leq\mu}$. We will define below at least the complex points $Gr_{G,\leq\mu}(\mathbb{C})$ of these schemes, which are specific topological subspaces of $Gr_G(\mathbb{C})$. First, it is necessary to introduce a bit more notation.

Let T denote a maximal torus inside G. We use $\mathbb{X}^\bullet$ to denote the lattice of weights $\mathrm{Hom}(T, \mathbb{G}_m)$, and $\mathbb{X}_\bullet$ to denote the dual lattice of coweights. Inside $\mathbb{X}^\bullet$ is the set Φ of roots. We fix a particular Borel subgroup $B \subset G$, determining a choice of positive roots $\Phi^+ \subset \Phi$ and a semi-group of dominant coweights $\mathbb{X}_\bullet^+ \subset \mathbb{X}_\bullet$. There is a natural bijection

$$\mathbb{X}_\bullet^+ \cong G(\mathbb{C}[[t]])\backslash G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$$

of dominant coweights with the above double cosets. Each coweight $\mu \in \mathrm{Hom}(\mathbb{G}_m, T)$ defines via the inclusion of T into G an element t^μ in $G(\mathbb{C}((t)))$. There is a double-coset decomposition of the algebraic free loop space

$$G(\mathbb{C}((t))) \cong \coprod_{\mu \in \mathbb{X}_\bullet^+} G(\mathbb{C}[[t]])t^\mu G(\mathbb{C}[[t]]).$$

Projecting onto the affine Grassmannian, one learns that the $G(\mathbb{C}[[t]])$-orbits of Gr_G are indexed by $\mu \in \mathbb{X}_\bullet^+$. We will use $Gr_{G,\leq\mu}$ to denote the *closure* of the orbit corresponding to μ. The closure $Gr_{G,\leq\mu_1}$ contains $Gr_{G,\leq\mu_2}$ if and only if $\mu_1 - \mu_2$ is a sum of dominant coroots.

Definition 3.2. An $\mathbb{X}_\bullet^+$-filtered topological space K is a poset of topological subspaces of K, indexed by $\mathbb{X}_\bullet^+$.

An example is of course given by $Gr_G(\mathbb{C})$, filtered by the $Gr_{G,\leq\mu}(\mathbb{C})$. We will refer to this filtration on $Gr_G(\mathbb{C})$ as the *Schubert filtration*.

Remark 3.3. Suppose that γ is a principal G-bundle on $\mathbb{A}^1$, and that p is the origin in $\mathbb{A}^1$. Then restriction of γ to an infinitesimal neighborhood of p gives a principal G-bundle on the formal disk D. If one is then further given a trivialization of γ away from p, there is then an associated

point $x \in Gr_G(\mathbb{C})$. This point x lives in a certain $G(\mathbb{C}[[t]])$-orbit, and thus has well-defined *monodromy* $\mu \in \mathbb{X}_\bullet^+$.

Now, if p is *any* point in $\mathbb{A}^1$, then a formal neighborhood of p is isomorphic but not equal to the formal disk D. If given a trivialization of γ away from p, this means that one does not get a well-defined point $x \in Gr_G(\mathbb{C})$, but one *does* obtain a well-defined $G(\mathbb{C}[[t]])$-orbit inside of $Gr_G(\mathbb{C})$. In other words, the trivialization of γ away from p has well-defined monodromy $\mu \in \mathbb{X}_\bullet^+$. Notice, in fact, that to define μ one needs not a trivialization on all of $\mathbb{A}^1 \backslash \{p\}$, but only a trivialization defined away from p in a neighborhood of p.

Example 3.4. Suppose $G = SL_2(\mathbb{C})$ with its usual maximal torus. A coweight $\mu \in \mathbb{X}_\bullet$ consists of a pair (a, b) of integers with $a + b = 0$. We choose a Borel so that a coweight is dominant if $a \geq b$. The conjugation action of $SL_2(\mathbb{C})$ on $\Omega SL_2(\mathbb{C})$ has one orbit for each pair $(a, -a)$ with $a \geq 0$. The orbit corresponding to $(a, -a)$ contains the loop $\mathbb{G}_m \to \Omega SL_2(\mathbb{C})$ given by

$$t \mapsto \left(\begin{array}{cc} t^a & 0 \\ 0 & t^{-a} \end{array} \right).$$

The closure of the $(a, -a)$ orbit contains the $(b, -b)$ orbit if and only if $b \leq a$. To topologists, $\Omega SL_2(\mathbb{C}) \simeq \Omega \Sigma S^2$ is recognizable as the free $\mathbb{A}_\infty$-algebra on the pointed space S^2. In particular, $Gr_{SL_2}(\mathbb{C})$ is naturally equipped with the James filtration by word length. The closure of the $(a, -a)$ orbit, denoted $Gr_{SL_2, \leq (a,-a)}(\mathbb{C})$, turns out to be the $(2a)$th component of the James filtration.

Thus, the Schubert filtration is strictly coarser than the James filtration. In particular, the S^2 that appears as the first James filtered piece of $\Omega SL_2(\mathbb{C})$ is not closed under the $SL_2(\mathbb{C})$ conjugation action. Only the collection of words of length 2 or less is closed under the $SL_2(\mathbb{C})$ action.

In this paper, we are interested in a filtration called the *Bott filtration* of $\Omega SL_n(\mathbb{C})$. For $\Omega SL_2(\mathbb{C})$, the Bott filtration corresponds to the James filtration on ΩS^3, and in particular is **not** given by the Schubert filtration on $Gr_{SL_2}(\mathbb{C})$. However, in Section CITE, we will explain how to obtain the Bott filtration from the Schubert filtration, while in the remainder of this section we will explain how the Schubert filtration interacts with the $\mathbb{E}_2$-algebra structure on $\Omega SL_2(\mathbb{C})$.

The $\mathbb{E}_2$-algebra structure on ΩG is elegantly encoded in algebraic geometry through the notion of the Beilinson–Drinfeld Grassmannian. In this section we explain this structure and why it gives $\mathbb{E}_{\not{2}}$ algebras in topology. We start by explaining the picture in topology following [Lur16, §5.5].

Let X be a topological space. We first construct a space $\mathrm{Ran}(X)$ called the *Ran space* of X whose points are non-empty finite subsets of X. Note that X determines a contravariant functor from the category $\mathrm{Fin}_{\mathrm{surj}}$ of finite sets and surjections to the category Top of topological spaces carrying a finite set S to X^S and a surjection $S \twoheadrightarrow T$ to the natural diagonal map $X^T \to X^S$.

Definition 3.5. Let $\mathrm{Fin}_{\mathrm{surj}}^{\leq n}$ denote the category of finite sets of cardinality at most n. Define the topological spaces

$$\mathrm{Ran}^{\leq n}(X) := \mathop{\mathrm{colim}}_{S \in \mathrm{Fin}_{\mathrm{surj}}^{\leq n}} X^S$$

and

$$\mathrm{Ran}(X) := \mathop{\mathrm{colim}}_{n \to \infty} \mathrm{Ran}^{\leq n}(X)$$

where the colimits are taken in the 1-category of topological spaces.

Remark 3.6. There is another topology on the Ran space, given for instance in [Lur16]. The results that we cite from [Lur16] work equally well for either topology, so we will use the colimit topology throughout.

Classically, given a symmetric monoidal category $\mathcal{C}$, one thinks of an $\mathbb{E}_2$-algebra A as an object with multiplications parametrized by embeddings of disjoint unions of disks. We can think of this as giving for each point in the plane $x \in \mathbb{C}$ a copy A_x of A. On an infinitesimal scale, the embeddings of disks can be thought of as the collision of points in the plane. The Ran space precisely tracks this collision data, and it is through this that the Ran space gives another way of thinking about an $\mathbb{E}_2$-algebra. More precisely, given an $\mathbb{E}_2$-algebra A and a finite set of points in the plane $\{x_i\}_{i \in I} \in \mathrm{Ran}(\mathbb{C})$, one can form the tensor product $\bigotimes_{x \in X} A_x$. These tensor products turn out to be stalks of a cosheaf on $\mathrm{Ran}(\mathbb{C})$; the functoriality in the Ran space is expresses the idea of points colliding determining the multiplicative structure.

This story can be made precise in the language of ∞-operads. The disk philosophy is captured by an ∞-operad $\mathrm{Disk}(\mathbb{C})^{\otimes}$ whose algebras A can be thought of as giving an object $A(D) \in \mathcal{C}$ for each disk $D \subset \mathbb{C}$ and coherent maps $A(D_1) \otimes A(D_2) \otimes \cdots \otimes A(D_k) \to A(D)$ for inclusions $D_1 \coprod \cdots \coprod D_n \to D$ of disks. The following theorem relates these to $\mathbb{E}_2$-algebras[1]:

Theorem 3.7 (Proposition 5.4.5.15, [Lur16])**.** *There is a fully faithful functor* $\mathrm{Alg}_{\mathbb{E}_2}^{\mathrm{nu}}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Disk}(\mathbb{C})}^{\mathrm{nu}}(\mathcal{C})$ *with essential image the full subcategory of* $\mathrm{Disk}(\mathbb{C})_{\mathrm{nu}}$*-algebras A which are* locally constant *in the sense that for every embedding $D \hookrightarrow D'$ of disks, the natural map $A(D) \to A(D')$ is an equivalence.*

The Ran viewpoint is captured by an ∞-operad $\mathrm{Fact}(\mathbb{C})^{\otimes}$ whose algebras A are the data of an object $A(U) \in \mathcal{C}$ functorially for every open set $U \subset \mathrm{Ran}(\mathbb{C})$ together with structure of *factorization* maps $A(U) \otimes A(V) \to A(U \times V)$ whenever U and V are independent subsets of $\mathrm{Ran}(\mathbb{C})$ (and similar data for more opens).[2]

As alluded to earlier, the notions of $\mathrm{Fact}(\mathbb{C})$ algebra and $\mathrm{Disk}(\mathbb{C})_{\mathrm{nu}}$ algebra are closely related. There is a map of ∞-operads $\mathrm{Disk}(\mathbb{C})_{\mathrm{nu}}^{\otimes} \to \mathrm{Fact}(\mathbb{C})^{\otimes}$. Concretely, this means that one can extract from any $\mathrm{Fact}(\mathbb{C})$ algebra A an underlying $\mathrm{Disk}(\mathbb{C})_{\mathrm{nu}}$ algebra $\overline{A}$ by the formula $\overline{A}(D) = A(\mathrm{Ran}(D))$. The following theorem of Lurie describes the relationship more precisely:

Theorem 3.8 (Theorem 5.5.4.10, [Lur16])**.** *Let $\mathcal{C}$ be a symmetric monoidal ∞-category which admits all colimits and such that the tensor product preserves colimits separately in each variable. Then operadic left Kan extension along the functor $\mathrm{Disk}(\mathbb{C})_{\mathrm{nu}}^{\otimes} \to \mathrm{Fact}(\mathbb{C})^{\otimes}$ defines a fully faithful functor $F : \mathrm{Alg}_{\mathrm{Disk}(\mathbb{C})_{\mathrm{nu}}}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Fact}(\mathbb{C})}(\mathcal{C})$. Moreover, F restricts to an equivalence between the full subcategory of locally constant $\mathrm{Disk}(\mathbb{C})_{\mathrm{nu}}$ algebras and the full subcategory of $\mathrm{Fact}(\mathbb{C})$ algebras A which are constructible cosheaves and for which the factorization maps $A(U) \otimes A(V) \to A(U \times V)$ are equivalences.*

Remark 3.9. In fact, this theorem will be logically unnecessary for what follows and we include it only for completeness. As such, we omit the definition of the term *constructible cosheaf*.

We summarize the preceding discussion by drawing the following corollary of Theorem 3.7:

Corollary 3.9.1. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and $A \in \mathcal{C}$ be a $\mathrm{Fact}(\mathbb{C})$ algebra. Suppose that A has the property that for any inclusion $D \subset D' \subset \mathbb{C}$ of disks, the natural map $A(\mathrm{Ran}(D)) \to A(\mathrm{Ran}(D'))$ is an equivalence. Then, for any disk $D \subset \mathbb{C}$, $A(\mathrm{Ran}(D))$ inherits the structure of a non-unital $\mathbb{E}_2$-algebra in $\mathcal{C}$.*

The structure described in the previous section arises naturally in algebraic geometry.

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[1]For technical reasons, we will work with the non-unital (abbreviated 'nu') variants of $\mathbb{E}_2$ and $\mathrm{Disk}(\mathbb{C})$-algebras from now on.

[2]The reader unfamiliar with the Ran space can keep in mind the example when $U = \mathrm{Ran}(D_1)$, $V = \mathrm{Ran}(D_2)$ for disjoint disks $D_1, D_2 \subset \mathbb{C}$, in which case $U \times V = \mathrm{Ran}(D_1 \coprod D_2)$.

Definition 3.10. The *Ran space* $\mathrm{Ran}_{\mathbb{A}^1}$ is the presheaf that assigns to every $\mathbb{C}$-algebra R the set of non-empty finite subsets of $\mathrm{Spec}(R) \times \mathbb{A}^1$ over $\mathrm{Spec}(R)$. The Beilinson–Drinfeld Grassmannian is the presheaf $Gr_{G,\mathrm{Ran}}$ that assigns to each $\mathbb{C}$-algebra R the set of triplets $(x, \mathcal{E}, \beta)$, where $x \in \mathrm{Ran}(\mathbb{A}^1)(R)$, $\mathcal{E}$ is a G-torsor on $\mathbb{A}^1 \times \mathrm{Spec}(R)$, and β is a trivialization of $\mathcal{E}$ away from the graph of x in $\mathrm{Spec}(R) \times \mathbb{A}^1$.

The Beilinson–Drinfeld Grassmannian is naturally fibered over the Ran space. Again, we shall be primarily interested in complex points. As explained above, a point x in $\mathrm{Ran}_{\mathbb{A}^1}(\mathbb{C})$ consists of a non-empty finite subset $I \subset \mathbb{C}$ of points in $\mathbb{C}$. The fiber of the Beilinson–Drinfeld Grassmannian $Gr_{G,\mathrm{Ran}}(\mathbb{C})$ over x is the moduli of G-bundles on $\mathbb{A}^1$ equipped with a trivialization away from the points in I. This fiber is non-canonically isomorphic to the product of $|I|$ copies of affine Grassmannian $Gr_G(\mathbb{C})$.

Let us now explain how to filter $Gr_{G,\mathrm{Ran}}(\mathbb{C})$ by spaces $Gr_{G,\mathrm{Ran},\leq\mu}$, allowing us to view $Gr_{G,\mathrm{Ran}}(\mathbb{C})$ as an $\mathbb{X}_{\bullet}^+$-filtered topological space in the sense of Definition 4.2.

To do so, consider an arbitrary point $p \in \mathrm{Gr}_{G,\mathrm{Ran}}(\mathbb{C})$, consisting of a G-bundle γ on $\mathbb{A}^1$ trivialized away from a collection of points $I = (i_1, i_2, ..., i_\ell) \subset \mathbb{C}$. For each point $i_k \in \mathbb{C}$, the restriction of γ to a formal neighborhood of i_k has well-defined monodromy μ_k, in the sense of Remark 4.3. We say that p lives in $\mathrm{Gr}_{G,\mathrm{Ran},\leq\mu}(\mathbb{C})$ if and only if

$$\sum_{i=1}^{k} \mu_k \leq \mu.$$

The following fact is key to us:

Lemma 3.11. *Suppose that G is reductive. For each positive integer n and each $\mu \in \mathbb{X}_{\bullet}^+$, define P to make the following square into a pullback in the category of topological spaces:*

$$
\begin{array}{ccc}
P & \longrightarrow & Gr_{G,Ran,\leq\mu}(\mathbb{C}) \\
\downarrow{\scriptstyle f} & & \downarrow \\
\mathbb{C}^n & \longrightarrow & Ran_{\mathbb{A}^1}(\mathbb{C}).
\end{array}
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Then the map f is a proper map of topological spaces.

The authors understand the above fact to be folklore in the geometric representation theory literature. A reference is BLAH. A sketch of a proof in the algebro-geometric context can be found in BLAH. **Figure out a better way to sell this**

4. Jeremy's revision

In the next section we will define the Mitchell–Richter Bott filtration on $\Omega SU(n)$ and prove that it makes $\Sigma_+^\infty \Omega SU(n)$ into a filtered $\mathbb{A}_\infty$-ring spectrum. The filtration, especially as a multiplicative object, is most naturally described in the language of algebraic geometry. Our aim in this section will be to recall the complex points of these algebro-geometric objects, and extract from both the algebro-geometric and topological literature all of the basic facts about these complex points that we will later need. None of the ideas in this section are original: the objects we study are due to Beilinson and Drinfeld **CITE**, and their translation into algebraic topology is due to Jacob Lurie **CITE**.

Fix for the moment a smooth, reductive, affine algebraic group G over $\mathbb{C}$. Through the rest of the paper, we will be interested only in the cases $G = \mathbb{G}_m, SL_n$, or GL_n, and so the reader may safely restrict his attention to those cases for concreteness. As we will explain in detail, an algebro-geometric model for $\Omega G(\mathbb{C})$ is the *affine Grassmannian* Gr_G. The $\mathbb{E}_2$-algebra structure present on $\Omega G(\mathbb{C})$ is encoded by a more elaborate object, the *Beilinson–Drinfeld Grassmannian*.

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$$Gr_G(R) = \{(\mathcal{E}, \beta)\}, \text{ where}$$
$\mathcal{E}$ is a G-torsor over D_R and $\beta : \mathcal{E}|_{D_R^*} \cong \mathcal{E}_{D_R^*}^0$ is a trivialization over D_R^*.

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$$G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]),$$
which up to homotopy is the quotient of the free loop space on G by the action of G.

The above functor of points is not representable by a scheme, but it is a filtered colimit of schemes $Gr_{G,\leq\mu}$. We will define below at least the complex points $Gr_{G,\leq\mu}(\mathbb{C})$ of these schemes, which are specific topological subspaces of $Gr_G(\mathbb{C})$. First, it is necessary to introduce a bit more notation.

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of dominant coweights with the above double cosets. Each coweight $\mu \in \mathrm{Hom}(\mathbb{G}_m, T)$ defines via the inclusion of T into G an element t^μ in $G(\mathbb{C}((t)))$. There is a double-coset decomposition of the algebraic free loop space
$$G(\mathbb{C}((t))) \cong \coprod_{\mu \in \mathbb{X}_\bullet^+} G(\mathbb{C}[[t]])t^\mu G(\mathbb{C}[[t]]).$$

Projecting onto the affine Grassmannian, one learns that the $G(\mathbb{C}[[t]])$-orbits of Gr_G are indexed by $\mu \in \mathbb{X}_\bullet^+$. We will use $Gr_{G,\leq\mu}$ to denote the *closure* of the orbit corresponding to μ. The closure $Gr_{G,\leq\mu_1}$ contains $Gr_{G,\leq\mu_2}$ if and only if $\mu_1 - \mu_2$ is a sum of dominant coroots.

Definition 4.2. An $\mathbb{X}_\bullet^+$-filtered topological space K is a poset of topological subspaces of K, indexed by $\mathbb{X}_\bullet^+$.

An example is of course given by $Gr_G(\mathbb{C})$, filtered by the $Gr_{G,\leq\mu}(\mathbb{C})$. We will refer to this filtration on $Gr_G(\mathbb{C})$ as the *Schubert filtration*.

Remark 4.3. Suppose that γ is a principal G-bundle on $\mathbb{A}^1$, and that p is the origin in $\mathbb{A}^1$. Then restriction of γ to an infinitesimal neighborhood of p gives a principal G-bundle on the formal disk D. If one is then further given a trivialization of γ away from p, there is then an associated point $x \in Gr_G(\mathbb{C})$. This point x lives in a certain $G(\mathbb{C}[[t]])$-orbit, and thus has well-defined *monodromy* $\mu \in \mathbb{X}_\bullet^+$.

Now, if p is *any* point in $\mathbb{A}^1$, then a formal neighborhood of p is isomorphic but not equal to the formal disk D. If given a trivialization of γ away from p, this means that one does not get a well-defined point $x \in Gr_G(\mathbb{C})$, but one *does* obtain a well-defined $G(\mathbb{C}[[t]])$-orbit inside of $Gr_G(\mathbb{C})$. In other words, the trivialization of γ away from p has well-defined monodromy

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$$t \mapsto \begin{pmatrix} t^a & 0 \\ 0 & t^{-a} \end{pmatrix}.$$

The closure of the $(a, -a)$ orbit contains the $(b, -b)$ orbit if and only if $b \leq a$. To topologists, $\Omega SL_2(\mathbb{C}) \simeq \Omega\Sigma S^2$ is recognizable as the free $\mathbb{A}_\infty$-algebra on the pointed space S^2. In particular, $Gr_{SL_2}(\mathbb{C})$ is naturally equipped with the James filtration by word length. The closure of the $(a, -a)$ orbit, denoted $Gr_{SL_2, \leq (a, -a)}(\mathbb{C})$, turns out to be the $(2a)$th component of the James filtration.

Thus, the Schubert filtration is strictly coarser than the James filtration. In particular, the S^2 that appears as the first James filtered piece of $\Omega SL_2(\mathbb{C})$ is not closed under the $SL_2(\mathbb{C})$ conjugation action. Only the collection of words of length 2 or less is closed under the $SL_2(\mathbb{C})$ action.

The Bott filtration on $\Omega SL_2(\mathbb{C})$ corresponds to the James filtration on ΩS^3, and in particular is **not** given by the Schubert filtration on $Gr_{SL_2}(\mathbb{C})$. In Section CITE, we will explain how to obtain the Bott filtration from the Schubert filtration, while in the remainder of this section we will explain how the Schubert filtration interacts with the $\mathbb{E}_2$-algebra structure on $\Omega SL_2(\mathbb{C})$.

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Definition 4.5. The *Ran space* $\mathrm{Ran}_{\mathbb{A}^1}$ is the presheaf that assigns to every $\mathbb{C}$-algebra R the set of non-empty finite subsets of $\mathrm{Spec}(R) \times \mathbb{A}^1$ over $\mathrm{Spec}(R)$. The Beilinson–Drinfeld Grassmannian is the presheaf $Gr_{G,\mathrm{Ran}}$ that assigns to each $\mathbb{C}$-algebra R the set of triplets $(x, \mathcal{E}, \beta)$, where $x \in \mathrm{Ran}(\mathbb{A}^1)(R)$, $\mathcal{E}$ is a G-torsor on $\mathbb{A}^1 \times \mathrm{Spec}(R)$, and β is a trivialization of $\mathcal{E}$ away from the graph of x in $\mathrm{Spec}(R) \times \mathbb{A}^1$.

The Beilinson–Drinfeld Grassmannian is naturally fibered over the Ran space. Again, we shall be primarily interested in complex points. A point x in $\mathrm{Ran}_{\mathbb{A}^1}(\mathbb{C})$ consists of a non-empty finite subset $I \subset \mathbb{C}$ of points in $\mathbb{C}$. **INTRODUCE TOPOLOGY ON RAN SPACE** The fiber of the Beilinson–Drinfeld Grassmannian $Gr_{G,\mathrm{Ran}}(\mathbb{C})$ over x is the moduli of G-bundles on $\mathbb{A}^1$ equipped with a trivialization away from the points in I. This fiber is non-canonically isomorphic to the product of $|I|$ copies of $Gr_G(\mathbb{C})$.

Let us now explain how to filter $Gr_{G,\mathrm{Ran}}(\mathbb{C})$ by spaces $Gr_{G,\mathrm{Ran}, \leq \mu}$, allowing us to view $Gr_{G,\mathrm{Ran}}(\mathbb{C})$ as an $\mathbb{X}_\bullet^+$-filtered topological space in the sense of Definition 4.2.

To do so, consider an arbitrary point $p \in \mathrm{Gr}_{G,\mathrm{Ran}}(\mathbb{C})$, consisting of a G-bundle γ on $\mathbb{A}^1$ trivialized away from a collection of points $I = (i_1, i_2, ..., i_\ell) \subset \mathbb{C}$. For each point $i_k \in \mathbb{C}$, the restriction of γ to a formal neighborhood of i_k has well-defined monodromy μ_k, in the sense of Remark 4.3. We say that p lives in $\mathrm{Gr}_{G,\mathrm{Ran}, \leq \mu}(\mathbb{C})$ if and only if

$$\sum_{i=1}^k \mu_k \leq \mu.$$

The following fact is key to us:

Lemma 4.6. *Suppose that G is reductive. For each positive integer n and each $\mu \in \mathbb{X}_\bullet^+$, define P to make the following square into a pullback in the category of topological spaces:*

$$
\begin{array}{ccc}
P & \longrightarrow & Gr_{G,Ran,\leq\mu}(\mathbb{C}) \\
\downarrow{\scriptstyle f} & & \downarrow \\
\mathbb{C}^n & \longrightarrow & Ran_{\mathbb{A}^1}(\mathbb{C}).
\end{array}
$$

Then the map f is a proper map of topological spaces.

The authors understand the above fact to be folklore in the geometric representation theory literature. A reference is BLAH. A sketch of a proof in the algebro-geometric context can be found in BLAH. **Figure out a better way to sell this**

5. Allen's revision

6. The Bott filtration on $\Omega SU(n)$

In this section we recall the Mitchell–Segal Bott filtration [Mit87] on $\Omega SU(n)$. We prove that the Bott filtration is at least $\mathbb{A}_\infty$, meaning in particular that its suspension is an $\mathbb{A}_\infty$-filtered spectrum in the sense of Section 2.

It is most efficient to describe the filtration in the language of algebraic geometry, and in particular we will need to recall the theory of affine and Beilinson–Drinfeld Grassmannians. A good general reference is [Zhu16]. We use D to denote the formal disk $\mathrm{Spec}(\mathbb{C}[[t]])$ and D^* to denote the punctured disk $\mathrm{Spec}(\mathbb{C}((t)))$. For R a $\mathbb{C}$-algebra, we use D_R to denote $\mathrm{Spec}(\mathbb{C}[[t]]\hat{\otimes}R)$ and D_R^* to denote $\mathrm{Spec}(\mathbb{C}((t))\hat{\otimes}R)$.

Definition 6.1. Let G denote a smooth affine algebraic group over $\mathbb{C}$ (we will be interested only in the cases $G = SL_n, GL_n$). The *affine Grassmannian Gr_G* of G is the Ind-scheme with functor of points

$$Gr_G(R) = \{(\mathcal{E}, \beta)\}, \text{ where}$$

$\mathcal{E}$ is a G-torsor over D_R and $\beta : \mathcal{E}|_{D_R^*} \cong \mathcal{E}_{D_R^*}^0$ is a trivialization over D_R^*.

The complex points $Gr_G(\mathbb{C})$ are a model for the topological space ΩG. The idea is that $\mathrm{Hom}(D^*, G) \cong G(\mathbb{C}((t)))$ is the space of algebraic free (i.e., unbased) loops in G. A point in this algebraic free loop space is a trivialization of a G-bundle over the punctured disk D^*, while $G(\mathbb{C}[[t]])$ parametrizes those trivializations that extend over the entire disk. One may therefore think of the complex points of Gr_G as the homogeneous space

$$G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]),$$

which up to homotopy is the quotient of the free loop space on G by the action of G.

We use $\mathbb{X}^\bullet$ to denote the lattice of weights $\mathrm{Hom}(G, \mathbb{G}_m)$, and $\mathbb{X}_\bullet$ to denote the dual lattice of coweights. Inside $\mathbb{X}^\bullet$ is the set Φ of roots. We fix a particular Borel subgroup $B \subset G$, determining a choice of positive roots $\Phi^+ \subset \Phi$ and a semi-group of dominant coweights $\mathbb{X}_\bullet^+ \subset \mathbb{X}_\bullet$. There is a natural bijection

$$\mathbb{X}_\bullet^+ \cong G(\mathbb{C}[[t]])\backslash G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$$

of dominant coweights with the above double cosets. Each coweight $\mu \in \mathrm{Hom}(\mathbb{G}_m, G)$ may be thought of as a point t^μ in the free loop space of G, and hence under projection as a point in ΩG.

There is a double-coset decomposition of the algebraic free loop space

$$G(\mathbb{C}((t))) \cong \coprod_{\mu \in \mathbb{X}_\bullet^+} G(\mathbb{C}[[t]]) t^\mu G(\mathbb{C}[[t]]).$$

Projecting onto the affine Grassmannian, one learns that the $G(\mathbb{C}[[t]])$-orbits of Gr_G are indexed by $\mu \in \mathbb{X}_\bullet^+$. We will use $Gr_{G,\leq\mu}$ to denote the *closure* of the orbit corresponding to μ. The closure $Gr_{G,\leq\mu_1}$ contains $Gr_{G,\leq\mu_2}$ if and only if $\mu_1 - \mu_2$ is a sum of dominant coroots. We call $\{Gr_{G,\leq\mu}|\mu \in \mathbb{X}_\bullet^+\}$ the *Schubert filtration* of Gr_G.

Example 6.2. Suppose $G = SL_2(\mathbb{C})$. Then a coweight $\mu \in \mathbb{X}_\bullet$ consists of a pair (a,b) of integers with $a+b = 0$. We choose a Borel so that a coweight is dominant if $a \geq b$. The conjugation action of $SL_2(\mathbb{C})$ on $\Omega SL_2(\mathbb{C})$ has one orbit for each pair $(a,-a)$ with $a \geq 0$. The orbit corresponding to $(a,-a)$ contains the loop $\mathbb{G}_m \to \Omega SL_2(\mathbb{C})$ given by

$$t \mapsto \left(\begin{array}{cc} t^a & 0 \\ 0 & t^{-a} \end{array} \right).$$

The closure of the $(a,-a)$ orbit contains the $(b,-b)$ orbit if and only if $b \leq a$. To topologists, $\Omega SL_2(\mathbb{C}) \simeq \Omega\Sigma S^2$ is recognizable as the free $\mathbb{A}_\infty$-algebra on the pointed space S^2. In particular, $Gr_{SL_2}(\mathbb{C})$ is naturally equipped with the James filtration by word length. The closure of the $(a,-a)$ orbit turns out to be the $(2a)$th component of the James filtration, so that the Schubert filtration is strictly coarser than the James filtration. In other words, the S^2 that appears as the first James filtered piece of $\Omega SL_2(\mathbb{C})$ is not closed under the $SL_2(\mathbb{C})$ conjugation action. Only the collection of words of length 2 or less is closed under the $SL_2(\mathbb{C})$ action.

The $\mathbb{E}_2$-algebra structure on ΩG is elegantly encoded in algebraic geometry through the notion of the Beilinson–Drinfeld Grassmannian:

Definition 6.3. The *Ran space* $\mathrm{Ran}(\mathbb{A}^1)$ is the presheaf that assigns to every $\mathbb{C}$-algebra R the set of non-empty finite subsets of $\mathrm{Spec}(R) \times \mathbb{A}^1$. The Beilinson–Drinfeld Grassmannian is the presheaf $Gr_{G,Ran}$ that assigns to each $\mathbb{C}$-algebra R the set of triplets $(x, \mathcal{E}, \beta)$, where $x \in \mathrm{Ran}(\mathbb{A}^1)(R)$, $\mathcal{E}$ is a G-torsor on $\mathbb{A}^1 \times \mathrm{Spec}(R)$, and β is a trivialization of $\mathcal{E}$ away from the graph of x in $\mathrm{Spec}(R) \times \mathbb{A}^1$.

One thinks of the Beilinson–Drinfeld Grassmannian as fibered over the Ran space. In other words, for every finite collection of points $I \subset \mathbb{A}^1$, there is a corresponding point x in the Ran space. The fiber of the Beilinson–Drinfeld Grassmannian over x is the moduli of G-bundles on $\mathbb{A}^1$ equipped with a trivialization away from the points in I. This fiber is naturally isomorphic to the product of $|I|$ copies of Gr_G. The multiplication on Gr_G is encoded by degeneration of fibers as points collide in $\mathbb{A}^1$. For more details, see [Zhu16, §3].

The connection of the above structure with the notion of $\mathbb{E}_2$-algebra in homotopy theory was spelled out explicitly by Jacob Lurie in [Lur16, §5.5]. In the language of Lurie's work, the complex points of the Beilinson–Drinfeld Grassmannian form a factorizable cosheaf, valued in spaces, on $\mathrm{Ran}(\mathbb{C})$. Lurie proves [Lur16, Theorem 5.5.4.10] that this is enough to equip the complex points of Gr_G (namely ΩG) with the structure of a non-unital $\mathbb{E}_2$ algebra. This in turn makes $\Sigma_+^\infty \Omega G$ into a unital (in fact augmented) $\mathbb{E}_2$-ring spectrum.

It is through the Beilinson–Drinfeld perspective that we can most easily see the interaction of the Schubert filtration on Gr_G with its $\mathbb{E}_2$-algebra structure. The key point is the fact (see, e.g., [Zhu16, 3.1.14]) that, as points collide in the Beilinson–Drinfeld Grassmannian, the fiber $Gr_{G,\leq\mu_1} \times Gr_{G,\leq\mu_2}$ degenerates to $Gr_{G,\leq\mu_1+\mu_2}$.

That is already enough to prove that, for example, the Schubert filtration on $\Sigma_+^\infty \Omega SU(2)$ described in Example 6.2 is an $\mathbb{E}_2$-filtered spectrum in the sense of Section 2. What we will actually want to be $\mathbb{E}_2$, or at least $\mathbb{A}_\infty$, is the James filtration on $\Sigma_+^\infty \Omega SU(2)$. In general, it turns out that the Schubert filtration on the Beilinson–Drinfeld Grassmannian for $SL_n(\mathbb{C})$ provides only direct access to every nth piece of the Bott filtration on $\Sigma_+^\infty \Omega SL_n(\mathbb{C})$. We will follow Segal [Seg89] and access the Bott filtration on Gr_{SL_n} in a somewhat indirect manner, by considering not Gr_{SL_n} but Gr_{GL_n}:

Definition 6.4. Consider the affine Grassmannian Gr_{GL_n}. We denote by $F_{n,k}$ the subset of Gr_{GL_n} that is the closure of the $GL_n(\mathbb{C}[[t]])$ orbit containing:

$$t \mapsto \begin{pmatrix} t^k & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In other words, $F_{n,k} = Gr_{GL_n, \leq (k,0,\cdots,0)}$.

We draw the following lemma, affirming a conjecture of Mahowald and Richter, as an immediate corollary of the abstract machinery of Beilinson–Drinfeld [Zhu16, 3.1.14] and [Lur16, 5.5.4.10]:

Lemma 6.5 (Conjecture of Mahowald–Richter [MR93])**.** *The inclusion*

$$\coprod_k F_{n,k} \subset \Omega GL_n(\mathbb{C})$$

may be made into a map of non-unital $\mathbb{E}_2$-algebras. The suspension

$$\Sigma_+^\infty \coprod_k F_{n,k}$$

is a graded $\mathbb{E}_2$-algebra in the sense of Section 2.

As explained by Segal [Seg89], the coproduct $\coprod_k F_{n,k}$ may be viewed as the subspace of loops in $U(n)$ 'of positive winding number.' The kth piece $F_{n,k}$ consists of loops of winding number exactly k, and the group completion of $\coprod_k F_{n,k}$ is $\Omega U(n)$.

Example 6.6. For any n, $F_{n,1}$ is equivalent to $\mathbb{CP}^{n-1}$. The space $F_{2,k}$ is the kth stage of the James filtration of ΩS^3, consisting of all words of length $\leq k$.

It is not at all obvious from the above construction that there should exist maps $F_{n,k} \to F_{n,k+1}$. To make such maps requires some way of identifying the various connected components of $\Omega U(n)$, each of which is individually equivalent to $\Omega SU(n)$. Following Segal [Seg89, pg. 3–4], one makes this identification by multiplying by powers of

$$\lambda = \begin{pmatrix} t & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In other words, there is a map from the space of loops of winding number k to loops of winding number 0 given by multiplication by λ^{-k}.

Definition 6.7. The Bott filtration on $\Omega SL_n(\mathbb{C})$ is the filtration with kth piece given by $\lambda^{-k} F_{n,k}$. We will refer to the associated filtered spectrum

$$\mathbb{S} \to \Sigma^\infty \lambda^{-1} F_{n,1} \simeq \Sigma^\infty \mathbb{CP}^{n-1} \to \Sigma^\infty \lambda^{-2} F_{n,2} \to \cdots$$

by $\Sigma_+^\infty \{F_{n,k}\}$.

The above constructions make $\Sigma_+^\infty \{F_{n,k}\}$ into a filtered spectrum whose underlying graded spectrum is $\mathbb{E}_2$. We will now discuss the problem of making the filtered spectrum itself $\mathbb{E}_2$, or at least $\mathbb{A}_\infty$. For this, recall from Lemma 2.7 that there is a graded $\mathbb{E}_\infty$ ring $A = \Sigma_+^\infty \mathbb{Z}_{\geq 0}^{ds}$ so that filtered spectra may be described as A-modules in graded spectra. We now discuss the following theorem, which implies Theorem 1.3 from the Introduction, and which again follows easily from the machinery of Beilinson–Drinfeld Grassmannians:

Theorem 6.8. *There is a map of $\mathbb{E}_2$-algebra objects in graded spectra*

$$A \simeq \Sigma_+^\infty \mathbb{Z}_{\geq 0}^{ds} \longrightarrow \Sigma_+^\infty \coprod_k F_{n,k}.$$

In particular, $\Sigma_+^\infty \coprod_k F_{n,k}$ is an $\mathbb{A}_\infty$-algebra in A-modules, and so $\Sigma_+^\infty \{F_{n,k}\}$ is a filtered $\mathbb{A}_\infty$-algebra.

Remark 6.9. The $\mathbb{E}_2$-algebra map $A \to \Sigma_+^\infty \coprod_k F_{n,k}$ sits in a commutative diagram of $\mathbb{E}_2$-algebras

$$\begin{array}{ccc}
A & \longrightarrow & \Sigma_+^\infty \coprod_k F_{n,k} \\
\downarrow & & \downarrow \\
\Sigma_+^\infty \mathbb{Z} & \longrightarrow & \Sigma_+^\infty \Omega U(n).
\end{array}$$

The map $\Sigma_+^\infty \mathbb{Z} \to \Sigma_+^\infty \Omega U(n)$ may be described as the suspension of the natural map

$$\Omega^2(BU(1) \to BU(n)).$$

Remark 6.10. The fact that there is an $\mathbb{E}_2$-algebra map $A \to \Sigma_+^\infty \coprod_k F_{n,k}$ is stronger than the fact that $\Sigma_+^\infty \{F_{n,k}\}$ is $\mathbb{A}_\infty$ filtered, but it is weaker than the claim that $\Sigma_+^\infty \{F_{n,k}\}$ is $\mathbb{E}_2$ filtered. We do not know if the Bott filtration is $\mathbb{E}_2$ or not, but would be very interested to learn the answer.

The machinery of Beilinson–Drinfeld Grassmannians proves that the coarsened filtration consisting of every nth piece of the Bott filtration (i.e. $\Sigma_+^\infty \{F_{n,nk}\}$) is an $\mathbb{E}_2$-filtration. The question is equivalent to the production of an $\mathbb{E}_3$-algebra map from A to the $\mathbb{E}_3$-center of the $\mathbb{E}_2$-algebra $\Sigma_+^\infty \coprod \{F_{n,k}\}$. After group completion, this would in particular imply the existence of an $\mathbb{E}_3$-algebra map

$$\mathbb{Z} \longrightarrow (\Omega U(n))^{hU(n)}.$$

We do not know whether even this last map exists.

Proof of Theorem 6.8. Consider $Gr_{\mathbb{G}_m}$, the affine Grassmannian for the multiplicative group. This is a model for ΩS^1 and so has $\mathbb{Z}$ many contractible connected components. Choosing a dominant coweight corresponding to a loop of winding number 1 identifies a copy of $\mathbb{Z}_{\geq 0}^{ds}$ inside of $Gr_{\mathbb{G}_m}$. The Beilinson–Drinfeld Grassmannian for the group $G = \mathbb{G}_m$ then describes $\Sigma_+^\infty \mathbb{Z}_{\geq 0}^{ds}$ as a sub-$\mathbb{E}_2$-algebra of $\Sigma_+^\infty Gr_{\mathbb{G}_m}$.

Now, the map of groups $\mathbb{G}_m \to GL_n(\mathbb{C})$ given by the dominant coweight $(1, 0, \cdots, 0)$ induces a map of Beilinson–Drinfeld Grassmannians. Applying Lurie's Theorem [Lur16, Theorem 5.5.4.10], we obtain the desired map of graded $\mathbb{E}_2$-algebras. $\qquad\square$

We end this section by sketching what is named Construction 1.11 in the Introduction:

Construction 6.11. The graded $\mathbb{A}_\infty$-algebra $gr(\Sigma_+^\infty \{F_{n,k}\})$ may be equipped with the structure of a graded $\mathbb{E}_2$-algebra.

Proof sketch. As explained above, the $\mathbb{E}_2$-algebra in spaces $\coprod F_{n,k}$ receives a natural $\mathbb{E}_2$-map from $\mathbb{Z}_{\geq 0}^{ds}$. We may thus view $\coprod F_{n,k}$ as an $\mathbb{E}_2$-algebra over $\mathbb{Z}_{\geq 0}^{ds}$ (by, e.g., the straightening and unstraightening correspondence). There is a diagram of $\mathbb{E}_2$-algebras:

$$\begin{array}{ccccccc}
\coprod_k F_{n,k} & \longrightarrow & \Omega U(n) & \longrightarrow & \Omega U \simeq BU \times \mathbb{Z} & \xrightarrow{J} & Pic(\mathbb{S}) \subset \mathrm{Sp} \\
\downarrow & & & & & & \\
\mathbb{Z}_{\geq 0}^{ds}. & & & & & &
\end{array}$$

In [Seg89, 1.7], it is proven that the colimit of the functor $\coprod F_{n,k} \longrightarrow \mathrm{Sp}$ is equivalent (as a spectrum) to $gr(\Sigma_+^\infty\{F_{n,k}\})$. Note that this colimit is more classically described as the Thom spectrum of the map $\coprod F_{n,k} \longrightarrow BU \times \mathbb{Z}$.

One may also compute this colimit by first making a left Kan extension along the map

$$\coprod F_{n,k} \to \mathbb{Z}_{\geq 0}^{ds},$$

and then taking the coproduct of the images of the resulting map

$$\mathbb{Z}_{\geq 0}^{ds} \to \mathrm{Sp}.$$

Taking an operadic left Kan extension as in [Lur16, 3.1.2], one learns that the left Kan extension $\mathbb{Z}_{\geq 0}^{ds} \longrightarrow \mathrm{Sp}$ is lax $\mathbb{E}_2$-monoidal. The properties of Day convolution (explained in, e.g., Appendix A) then imply that the Thom spectrum is naturally an $\mathbb{E}_2$-graded spectrum.

To see that the underlying $\mathbb{A}_\infty$-graded spectrum agrees with the associated graded of the Bott filtration, notice that the zero-section of the Thom construction is a map of graded $\mathbb{E}_2$-algebras. This zero-section is, on the kth graded piece, a model for the map

$$\Sigma^\infty F_{n,k} \longrightarrow \Sigma^\infty F_{n,k}/F_{n,k-1}.$$

The sequence of graded $\mathbb{E}_2$-algebra maps

$$\Sigma_+^\infty \mathbb{Z}_{\geq 0}^{ds} \longrightarrow \Sigma_+^\infty \coprod_k F_{n,k} \longrightarrow \mathrm{Thom}\left(\coprod_k F_{n,k}\right)$$

then implies the result. $\qquad\square$

7. Multiplicative Aspects of Weiss Calculus

In this section, we determine the multiplicative properties of the Weiss calculus polynomial approximation functors [Wei95]. More precisely, for a functor F, we aim to understand the Taylor tower of $F \wedge F$ in terms of the tower for F. The results in this section are likely known to experts, but the authors were not able to locate it in the literature. They thank Jacob Lurie for suggesting that Theorem 7.10 is true.

7.1. **Review of Weiss calculus.** We briefly review notions of Weiss calculus to set notation. The reader is referred to [Wei95] for proofs and additional details. We note that the discussion there is in the case of real vector spaces, but the results work just the same in the complex case. We shall also work in the language of ∞-categories rather than topological categories, and Remark 7.2 justifies this passage.

Let $\mathcal{J}$ be the ∞-category which is the nerve of the topological category whose objects are finite dimensional complex vector spaces equipped with a Hermitian inner product and whose morphisms are spaces of linear isometries.

Weiss calculus studies functors out of $\mathcal{J}$ in a way analogous to Goodwillie calculus, by understanding successive "polynomial approximations" to these functors. Here, we will discuss only the stable setting where we apply the theory to the functor category $\mathrm{Sp}^{\mathcal{J}}$. The central definition is:

Definition 7.1. A functor $F \in \mathrm{Sp}^{\mathcal{J}}$ is polynomial of degree n if the natural map

$$F(V) \to \lim_U F(U \oplus V)$$

is an equivalence, where the limit is indexed over the ∞-category of nonzero subspaces $U \subset \mathbb{C}^{n+1}$.

As in Goodwillie calculus, the inclusion of the full subcategory $\mathrm{Poly}^{\leq n}(\mathrm{Sp}^{\mathcal{J}}) \subset \mathrm{Sp}^{\mathcal{J}}$ of functors which are polynomial of degree n admits a left adjoint

$$P_n : \mathrm{Sp}^{\mathcal{J}} \rightleftarrows \mathrm{Poly}^{\leq n}(\mathrm{Sp}^{\mathcal{J}}) : j_n.$$

The unit η_n of this adjunction provides for each $F \in \mathrm{Sp}^{\mathcal{J}}$ a natural transformation $F \to P_n F$ which we will refer to as the *degree n polynomial approximation* of F.

Remark 7.2. This universal property was not explicitly stated in [Wei95], but it follows formally from Weiss's results as follows: the functor P_n and the transformation η_n can be defined explicitly as in [Wei95] by iteratively applying the functor $\tau_n : \mathrm{Sp}^{\mathcal{J}} \to \mathrm{Sp}^{\mathcal{J}}$ defined by the formula

$$\tau_n F(V) = \lim_U F(U \oplus V)$$

with the limit indexed as in Definition 7.1. The facts required of the functors P_n in the proof of Theorem 6.1.1.10 in [Lur16] are precisely the content of Theorem 6.3 of [Wei95].

Given this universal property, Proposition 5.4 of [Wei95] ensures the existence of a natural Taylor tower

$$F \longrightarrow \cdots \longrightarrow P_n F \xrightarrow{p_{n-1}} P_{n-1} F \longrightarrow \cdots \longrightarrow P_0 F$$

living under any functor $F \in \mathrm{Sp}^{\mathcal{J}}$. The fiber $D_n F$ of p_{n-1} has the special property that it is polynomial of degree n and $P_{n-1} D_n F \simeq 0$. Such a functor is called *n-homogeneous*; such functors are completely classified by the following theorem:

Theorem 7.3 ([Wei95, Theorem 7.3]). *Let $F \in \mathrm{Sp}^{\mathcal{J}}$. Then F is an n-homogeneous functor if and only if there exists a spectrum Θ with an action of the unitary group $U(n)$ such that*

$$F(V) = (\Theta \wedge S^{nV})_{hU(n)}.$$

7.2. **The Taylor tower.** It is helpful, for our study of multiplicative properties, to package all polynomial approximations into a single object–the following construction makes this precise:

Construction 7.4. We now construct a functor

$$\mathrm{Tow} : \mathrm{Sp}^{\mathcal{J}} \to \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$$

with the property that it sends a functor $F \in \mathrm{Sp}^{\mathcal{J}}$ to its Taylor tower

$$\mathrm{Tow}(F) = P_0 F \longleftarrow P_1 F \longleftarrow P_2 F \longleftarrow \cdots.$$

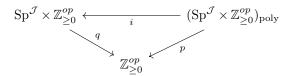Recall that the P_n functors are given as left adjoints of the fully faithful inclusions

$$\mathrm{Poly}^{\leq n}(\mathrm{Sp}^{\mathcal{J}}) \subset \mathrm{Sp}^{\mathcal{J}}.$$

We proceed by telling a parametrized version of this story that includes all n simultaneously. The proper framework for such a story is the formalism of *relative adjunctions*; these are developed in the ∞-categorical context in [Lur16, Section 7.3.2].

Consider the category $\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op}$ together with the full subcategory

$$(\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op})_{\mathrm{poly}} \subset \mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op}$$

on the pairs $(F, [n])$ such that $F \in \mathrm{Poly}^{\leq n}(\mathrm{Sp}^{\mathcal{J}})$. Via projection, these fit into a diagram

$$\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op} \xleftarrow{\quad i \quad} (\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op})_{\mathrm{poly}}$$
$$\searrow{q} \qquad \swarrow{p}$$
$$\mathbb{Z}_{\geq 0}^{op}$$

This will be relevant to us because the category of sections of q are precisely $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$. The sections of p can be thought of as those cofiltered functors such t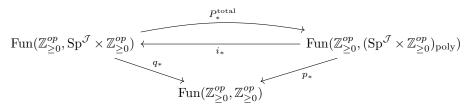hat the nth piece is polynomial of degree n. We will denote this category of sections of p by $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})_{\mathrm{poly}}$.

On the fibers over an integer $[n] \in \mathbb{Z}_{\geq 0}^{op}$, we see the inclusion $\mathrm{Sp}^{\mathcal{J}} \leftarrow \mathrm{Poly}^{\leq n}(\mathrm{Sp}^{\mathcal{J}})$. It is in this sense that the current picture is a parametrized version of the ordinary polynomial approximations. We now claim that i admits a left adjoint $P^{\mathrm{total}} : \mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op} \to (\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op})_{\mathrm{poly}}$ *relative* to $\mathbb{Z}_{\geq 0}^{op}$. The strategy is to use Proposition 7.3.2.6 of [Lur16], which tells us that we need to check the following three statements:

(1) The functors p and q are locally Cartesian categorical fibrations.
(2) For each $[n] \in \mathbb{Z}_{\geq 0}^{op}$, the functor on fibers $i|_{p^{-1}[n]} : p^{-1}[n] \to q^{-1}[n]$ admits a right adjoint.
(3) The functor i carries locally p-Cartesian morphisms of $(\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op})_{\mathrm{poly}}$ to locally q-Cartesian morphisms of $\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op}$.

Condition (2) is clear from the existence of polynomial approximations in Weiss calculus. To see conditions (1) and (3), we first note that q is in fact a Cartesian fibration because it is a projection from a product. Moreover, the q-Cartesian morphisms are precisely those morphisms which are equivalences on the $\mathrm{Sp}^{\mathcal{J}}$ coordinate. Now suppose we are given a pair $(F, [m]) \in \mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op}$ such that $F \in \mathrm{Poly}^{\leq m}(\mathrm{Sp}^{\mathcal{J}})$ and morphism $\sigma : [n] \to [m]$. Any q-Cartesian edge lying over σ with target $(F, [m])$ has source equivalent to $(F, [n])$ and thus is also in the full subcategory $(\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op})_{\mathrm{poly}}$ because $m \leq n$. Since p is certainly an inner fibration (by construction as a full subcategory), this implies that p is also a Cartesian fibration and that the inclusion i carries p-Cartesian edges to q-Cartesian edges. Since any Cartesian fibration is a categorical fibration ([Lur17, Proposition 3.3.1.7]), conditions (1) and (3) are verified.

We now wish to look at the adjunction at the level of sections of q and p. Considering functors from $\mathbb{Z}_{\geq 0}^{op}$ into Diagram 7.4, we obtain a new diagram

$$\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op}) \xleftarrow[\quad i_* \quad]{\quad\quad\quad} \overset{P_*^{\mathrm{total}}}{\xrightarrow{\quad\quad}} \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, (\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op})_{\mathrm{poly}})$$

$$\xrightarrow{q_*} \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathbb{Z}_{\geq 0}^{op}) \xleftarrow{p_*}$$

which exhibits P_*^{total} as a left adjoint of i_* relative to $\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathbb{Z}_{\geq 0}^{op})$. Proposition 7.3.2.5 of [Lur16] ensures that there is an adjunction at the level of fibers above $\mathrm{id} \in \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathbb{Z}_{\geq 0}^{op})$:

$$\mathcal{P} : \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}}) \rightleftharpoons \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})_{\mathrm{poly}} : j.$$

Finally, observe that the unique functor $r : \mathbb{Z}_{\geq 0}^{op} \to *$ induces an adjunction

$$r^* : \mathrm{Sp}^{\mathcal{J}} \rightleftharpoons \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}}) : \lim$$

where r^* is the constant functor and $\lim$ is the same as right Kan extension along r. We now compose these adjunctions, denoting $\mathrm{Tow} = \mathcal{P} \circ r^*$ to obtain:

$$\mathrm{Tow} : \mathrm{Sp}^{\mathcal{J}} \rightleftharpoons \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})_{\mathrm{poly}} : \lim.$$

By construction, $\mathrm{Tow}(F)$ is the cofiltered spectrum

$$P_0 F \longleftarrow P_1 F \longleftarrow P_2 F \longleftarrow \cdots.$$

This concludes the construction of Tow.

7.3. **Multiplicativity of Tow.** The next task is to understand the multiplicative structure of Tow. The idea is that we would like to express $\mathrm{Tow}(F \wedge F)$ in terms of $\mathrm{Tow}(F)$ and the Day convolution monoidal structure on $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$. We start with the following lemma:

Lemma 7.5. *The Weiss tower functor* Tow *defines an oplax symmetric monoidal functor*

$$\mathrm{Tow} : \mathrm{Sp}^{\mathcal{J}} \to \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}}).$$

Proof. Recall that the Weiss tower functor was defined as a composite $\mathrm{Tow} = \mathcal{P} \circ r^*$. The functor r^* is just the constant functor, so it is symmetric monoidal. On the other hand, $\mathcal{P}$ is adjoint to the inclusion $j : \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})_{\mathrm{poly}} \to \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$. Since the class of functors which are polynomial of degree n is closed under finite limits, the subcategory $(\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}}))_{\mathrm{poly}}$ is closed under the convolution tensor product. We may therefore give it the structure of a symmetric monoidal ∞-category such that j is symmetric monoidal. This makes the left adjoint $\mathcal{P}$ an oplax symmetric monoidal functor, which induces an oplax symmetric monoidal structure on Tow. $\qquad\square$

Concretely, this oplax structure can be described on the nth filtered piece as follows: suppose $F, G \in \mathrm{Sp}^{\mathcal{J}}$; since $\mathrm{Tow}(F) \otimes \mathrm{Tow}(G) \in \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})_{\mathrm{poly}}$, the filtered piece $(\mathrm{Tow}(F) \otimes \mathrm{Tow}(G))_n$ is polynomial of degree n. It follows that the natural map from $F \wedge G$ factors through a map $\varphi_n : P_n(F \wedge G) \to (\mathrm{Tow}(F) \otimes \mathrm{Tow}(G))_n$.

In order to show that Tow is a symmetric monoidal functor, one would need to show that each φ_n is an equivalence for all n. We will do this after restricting to a smaller subcategory of functors with nice convergence properties:

Definition 7.6. Let $F \in \mathrm{Sp}^{\mathcal{J}}$ be a functor. Call F *rapidly convergent* if F takes values in connective spectra and there exist real numbers $c, \alpha > 0$ such that the natural map $F(W) \to P_nF(W)$ is $(\alpha n)\dim(W) - c$ connected. We denote by $\mathrm{Sp}^{\mathcal{J}}_{\mathrm{conv}}$ the category of rapidly convergent functors.

Example 7.7. Let $V \in \mathcal{J}$ be a complex vector space. The functor $F_V \in \mathrm{Sp}^{\mathcal{J}}$ defined by

$$F_V(W) = \Sigma^{\infty}_+ \Omega\, \mathcal{J}(V, V \oplus W)$$

is rapidly convergent. Indeed, [Aro01] shows that its homogeneous layers are given by

$$D_nF_V(W) = \Omega^{\infty}(s_n(V) \wedge S^{nW})_{hU(n)}$$

where $s_n(V)$ is the suspension spectrum of a space. Since colimits do not lower connectivity, this implies $D_nF_V(W)$ is at least $(2n)\dim(W) - 1$-connected. It follows from the Milnor sequence that F_V is rapidly convergent, where α can be taken to be 2.

Observe that rapidly convergent functors in particular have convergent Weiss towers. However, the following lemma of Weiss about functors "agreeing up to order n" allows us to say more:

Lemma 7.8 ([Wei98]). *Let $F, G \in \mathrm{Sp}^{\mathcal{J}}$ be functors, $\eta : F \to G$ a natural transformation, and $n \geq 0$ an integer. Suppose that there exists $c > 0$ such that for all $W \in \mathcal{J}$, the map of spectra $\eta_W : F(W) \to G(W)$ is $(n + 1)\dim(W) - c$ connected. Then the natural transformation $P_n\eta : P_nF \to P_nG$ is an equivalence.*

The following corollary is immediate:

Corollary 7.8.1. *Let $F, G \in \mathrm{Sp}^{\mathcal{J}}_{conv}$ be rapidly convergent and $n > 0$ an integer. Then there exists an integer M such that for $m > M$, the natural transformation $F \wedge G \to P_mF \wedge P_mG$ is an equivalence after applying P_k for all integers $0 \leq k \leq n$.*

We will show that this implies the following further corollary:

Corollary 7.8.2. *The map φ_n constructed above is an equivalence for all n when $F, G \in \mathrm{Sp}_{conv}^{\mathcal{J}}$ are rapidly convergent functors.*

The proof will require the following basic lemma whose proof we will record at the end of this section:

Lemma 7.9. *Let $X, Y \in \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$ and $n > 0$ an integer. Then we have the following formula for the successive fibers:*

$$\mathrm{fib}((X \otimes Y)_n \to (X \otimes Y)_{n-1}) \simeq \coprod_{p+q=n} \mathrm{fib}(X_p \to X_{p-1}) \wedge \mathrm{fib}(Y_q \to Y_{q-1}).$$

We now prove the corollary:

Proof of Corollary 7.8.2. Corollary 7.8.2 implies that by replacing F and G by appropriate polynomial approximations, it suffices to consider the case where F and G are polynomial of degree m for some m (and thus, have finite Weiss towers). Note further that Lemma 7.9 applied to the case $X = \mathrm{Tow}(F)$, $Y = \mathrm{Tow}(G)$ implies that

$$\mathrm{fib}((\mathrm{Tow}(F) \otimes \mathrm{Tow}(G))_{n+1} \to (\mathrm{Tow}(F) \otimes \mathrm{Tow}(G))_n)$$

is homogeneous of degree $n + 1$. It follows that the fiber of the natural map

$$F \wedge G \to (\mathrm{Tow}(F) \otimes \mathrm{Tow}(G))_n$$

is a finite limit of functors killed by P_n. Since P_n commutes with finite limits, we conclude that the natural map $\varphi_n : P_n(F \wedge G) \to (\mathrm{Tow}(F) \otimes \mathrm{Tow}(G))_n$ is an equivalence. $\qquad\square$

The proof shows further that rapidly convergent functors are closed under the tensor product. In total, we have now proven the following theorem:

Theorem 7.10. *The Weiss tower defines a symmetric monoidal functor*

$$\mathrm{Tow} : \mathrm{Sp}_{conv}^{\mathcal{J}} \to \mathbf{Cofil}(\mathrm{Sp}_{conv}^{\mathcal{J}}).$$

Remark 7.11. Theorem 7.10 and its proof work equally well in Goodwillie calculus. There, the hypothesis on convergence can be removed, and the adjoining discussion can be replaced with the observation [Goo03, Lemma 6.10] that the smash product of an n-reduced functor and an m-reduced functor is $(n + m)$-reduced. In Weiss calculus, this fact is also true but not in the literature, so we have opted to give the above proof which is sufficient for our application.

Combining this with Example 7.7, we obtain:

Corollary 7.11.1. *Let $V \in \mathcal{J}$ be a complex vector space. The functor $F_V \in \mathrm{Sp}^{\mathcal{J}}$ defined by*

$$F_V(W) = \Sigma_+^\infty \Omega \, \mathcal{J}(V, V \oplus W)$$

determines a cofiltered associative algebra $\mathrm{Tow}(F_V) \in \mathrm{Alg}_{\mathbb{A}_\infty}(\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}}))$.

We finish with a proof omitted earlier:

Proof of Lemma 7.9. Let $A_n \subset \mathbb{Z}_{\geq 0}^{op} \times \mathbb{Z}_{\geq 0}^{op}$ be the full subcategory spanned by pairs (p, q) with $p + q \leq n$.

Define a functor $T : \mathbb{Z}_{\geq 0}^{op} \times \mathbb{Z}_{\geq 0}^{op} \to \mathrm{Sp}^{\mathcal{J}}$ by the formula $T(p, q) = X_p \wedge Y_q$. We have by definition that $\lim T|_{A_n} \simeq (X \otimes Y)_n$.

Define the functor $\overline{T_n} : A_n \to \mathrm{Sp}^{\mathcal{J}}$ as the right Kan extension of $T|_{A_{n-1}}$ along the inclusion $A_{n-1} \to A_n$. Then, $\overline{T_n}$ has the following properties:

(1) $\lim \overline{T_n} = \lim T|_{A_{n-1}}$.
(2) $\overline{T_n}|_{A_{n-1}} = T|_{A_{n-1}}$.

(3) $\overline{T_n}(n,0) = X_{n-1} \wedge Y_0$ and $\overline{T_n}(0,n) = X_0 \wedge Y_{n-1}$.
(4) $\overline{T_n}(p,q) = X_{p-1} \wedge Y_q \times_{X_{p-1} \wedge Y_{q-1}} X_p \wedge Y_{q-1}$ for $p+q = n$, $p,q \geq 1$.

We may therefore compute

$$\mathrm{fib}\big(\lim_{A_n} T|_{A_n} \to \lim_{A_{n-1}} T|_{A_{n-1}}\big) = \mathrm{fib}\big(\lim_{A_n} T|_{A_n} \to \lim_{A_n} \overline{T_n}\big) = \lim_{A_n} \mathrm{fib}\big(T|_{A_n} \to \overline{T_n}\big).$$

The conclusion now follows immediately from the usual fact that

$$\mathrm{fib}(X_p \wedge Y_q \to X_{p-1} \wedge Y_q \times_{X_{p-1} \wedge Y_{q-1}} X_p \wedge Y_{q-1}) = \mathrm{fib}(X_p \to X_{p-1}) \wedge \mathrm{fib}(Y_q \to Y_{q-1}).$$

$\square$

8. Stable $\mathbb{A}_\infty$ Splittings

In this brief section, we assemble results proved above in order to obtain what is labeled Theorem 1.6 in the Introduction:

Theorem 8.1. *As an $\mathbb{A}_\infty$-algebra object in filtered spectra, the Bott filtration of $\Sigma_+^\infty \Omega SU(n)$ is equivalent to its associated graded.*

Proof. The construction of the stable Bott filtration as an $\mathbb{A}_\infty$-filtered spectrum is our Theorem 6.8. According to our Theorem 2.6, to complete the proof of Theorem 8.1 it suffices to produce an $\mathbb{A}_\infty$-cofiltered spectrum with limit $\Sigma_+^\infty \Omega SU(n)$ and with the property that certain composites are equivalences.

As explained in the Introduction, we follow Arone [Aro01] in producing the desired cofiltered spectrum by means of Weiss calculus. Consider, in the notation of Section 7 and particularly Example 7.7, the functor

$$F_V : \mathcal{J} \longrightarrow \mathrm{Sp}$$

given by $F_V(W) = \Sigma_+^\infty \Omega \mathcal{J}(V, V \oplus W)$. Corollary 7.11.1 implies that the Taylor tower of F_V, applied to W, is an $\mathbb{A}_\infty$-cofiltered spectrum with limit $\Sigma_+^\infty \Omega \mathcal{J}(V, V \oplus W)$.

Specializing to the case $V = \mathbb{C}^{n-1}$, $W = \mathbb{C}$, it is straightforward to see that $\mathcal{J}(\mathbb{C}^{n-1}, \mathbb{C}^{n-1} \oplus \mathbb{C})$ is equivalent to $SU(n)$. Roughly speaking, this is because any embedding of $\mathbb{C}^{n-1}$ into $\mathbb{C}^n$ may be extended in a unique way to an automorphism of $\mathbb{C}^n$ that is unitary of determinant one. Thus, applying Corollary 7.11.1 in the case $V = \mathbb{C}^{n-1}$, $W = \mathbb{C}$ gives an $\mathbb{A}_\infty$-cofiltered spectrum with the desired limit. To complete the proof of Theorem 8.1 it suffices then to check that certain composites are equivalences. In fact, one of the main results of [Aro01] is that those composites are equivalences (see the proof of [Aro01, Theorem 1.2]). $\square$

Remark 8.2. Mitchell and Richter constructed [Cra87] a filtration not just of

$$\Omega SU(n) \simeq \Omega \mathcal{J}(\mathbb{C}^{n-1}, \mathbb{C}^n),$$

but also of $\Omega \mathcal{J}(V, V \oplus W)$ for a general V and W. Arone showed in [Aro01, Theorem 1.2] that this Mitchell–Richter filtration always stably splits, and Corollary 7.11.1 provides an $\mathbb{A}_\infty$-cofiltered spectrum inducing this splitting. We do not know, however, whether the Mitchell–Richter filtration is always $\mathbb{A}_\infty$-split because it is not known whether the filtration is $\mathbb{A}_\infty$.

9. $\mathbb{E}_2$ Splittings in Complex Cobordism

We remark in this section that the $\mathbb{A}_\infty$ splitting

$$\Sigma_+^\infty \Omega SU(n) \simeq gr(\Sigma_+^\infty \{F_{n,k}\})$$

becomes $\mathbb{E}_2$ after base-change to complex bordism. More precisely, suppose that $gr(\Sigma_+^\infty \{F_{n,k}\})$ is equipped with some graded $\mathbb{E}_2$-ring structure extending the natural graded $\mathbb{A}_\infty$-ring structure. Construction 6.11 provides one possible way to do this. We will by abuse of notation use

$gr(\Sigma^\infty_+\{F_{n,k}\})$ also to denote the underlying *ungraded* $\mathbb{E}_2$-algebra, and our main theorem is that there is an equivalence of (ungraded) $\mathbb{E}_2$-MU-algebras:

$$MU \wedge \Sigma^\infty \Omega SU(n) \simeq MU \wedge gr(\Sigma^\infty_+\{F_{n,k}\}).$$

Notice that the results of Section 8 give an $\mathbb{A}_\infty$-equivalence between these two MU-algebras. This $\mathbb{A}_\infty$-MU-algebra equivalence is adjoint to a map of $\mathbb{A}_\infty$-$\mathbb{S}$-algebras

$$\Sigma^\infty_+ \Omega SU(n) \longrightarrow MU \wedge gr(\Sigma^\infty_+\{F_{n,k}\}). \tag{3}$$

Our task in this section will be to show that (3) may be refined to a morphism of $\mathbb{E}_2$-ring spectra. We do so by obstruction theory–the key fact powering our proof is that

$$MU_{2*+1}(\Omega SU(n)) \cong 0.$$

This classical vanishing result may be proven via Atiyah–Hirzerburch spectral sequence, using the even cell-decomposition of $Gr_{SL_n}(\mathbb{C})$.

Inspired by [CM15], we prove the following general result (implying in particular Theorem 1.10):

Theorem 9.1. *Suppose that R is an $\mathbb{E}_2$-ring spectrum with no homotopy groups in odd degrees. Then any homotopy commutative ring homomorphism*

$$\Sigma^\infty_+ \Omega SU(n) \to R$$

lifts to a morphism of $\mathbb{E}_2$-ring spectra. Moreover, any chosen $\mathbb{A}_\infty$ lift may be extended to an $\mathbb{E}_2$ lift.

Proof. By taking connective covers, one learns that any ring homomorphism

$$\Sigma^\infty_+ \Omega SU(n) \to R$$

must factor through the natural $\mathbb{E}_2$-algebra map $\tau_{\geq 0} R \to R$. Thus, without loss of generality we will assume that R is (-1)-connected.

It is clear that the composite ring homomorphism

$$\Sigma^\infty_+ \Omega SU(n) \longrightarrow R \longrightarrow \tau_{\leq 0} R \simeq H\pi_0(R)$$

may be lifted to an $\mathbb{E}_2$-ring homomorphism factoring through $\tau_{\leq 0}\Sigma^\infty_+\Omega SU(n) \simeq H\mathbb{Z}$. Suppose now for $q > 0$ that we have chosen an $\mathbb{E}_2$-ring homomorphism

$$\Sigma^\infty_+ \Omega SU(n) \longrightarrow \tau_{\leq q-1} R$$

We will show that there is no obstruction to the existence of a further $\mathbb{E}_2$-lift

$$\Sigma^\infty_+ \Omega SU(n) \longrightarrow \tau_{\leq q} R,$$

and that one may be chosen lifting any specified $\mathbb{A}_\infty$ map $\Sigma^\infty_+\Omega SU(n) \to \tau_{\leq q} R$.

According to [CM15, Theorem 4.1], there is a diagram of principal fibrations

$$
\begin{array}{ccc}
\mathbb{E}_2\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\leq q} R) & \longrightarrow & \mathbb{A}_\infty\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\leq q} R) \\
\downarrow & & \downarrow \\
\mathbb{E}_2\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\leq q-1} R) & \longrightarrow & \mathbb{A}_\infty\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\leq q-1} R) \\
\downarrow & & \downarrow \\
\mathcal{S}_*(BSU(n), K(\pi_q R, q+3)) & \longrightarrow & \mathcal{S}_*(SU(n), K(\pi_q R, q+2))
\end{array}
$$

For q odd, $\tau_{\leq q-1} R \simeq \tau_{\leq q} R$, so there is no obstruction. Let us therefore assume that q is even.

Since the cohomology of $BSU(n)$ is even-concentrated with coefficients in any abelian group, we have that $\pi_0 \, \mathcal{S}_*(BSU(n), K(\pi_q R, q+3)) \cong H^{q+3}(BSU(n); \pi_q R)$ is zero. It follows then that the given class

$$x \in \pi_0 \mathbb{E}_2\text{-Ring}(\Sigma^\infty_+ \Omega SU(n), \tau_{\leq q-1} R)$$

admits some lift

$$\widetilde{x} \in \mathbb{E}_2\text{-Ring}(\Sigma^\infty_+ \Omega SU(n), \tau_{\leq q} R).$$

We may need to modify $\widetilde{x}$ to match our chosen $\mathbb{A}_\infty$-ring homomorphism. This is always possible so long as the map

$$\pi_1(\mathcal{S}_*(BSU(n), K(\pi_q R, q+3))) \longrightarrow \pi_1(\mathcal{S}_*(SU(n), K(\pi_q R, q+2)))$$

is surjective. Said in other terms, this is just the map

$$H^{2q+2}(BSU(n); \pi_q R) \longrightarrow H^{2q+1}(SU(n); \pi_q R) \cong H^{2q+2}(\Sigma SU(n); \pi_q R)$$

induced by the natural map $\Sigma SU(n) \to BSU(n)$. It is a classical fact that this map is surjective (it follows from a calculation with the bar spectral sequence, using the fact that the cohomology of $SU(n)$ is exterior). $\qquad \square$

10. Obstructions to a General $\mathbb{E}_2$ Splitting

Let $3 < n \leq \infty$ be an integer. The $\mathbb{A}_\infty$ filtered equivalence of Theorem 1.6 gives an equivalence of $\mathbb{A}_\infty$ ring spectra

$$\Sigma^\infty_+ \Omega SU(n) \simeq gr(\Sigma^\infty_+ \{F_{n,k}\}).$$

The right-hand side is the associated graded of the stable Bott filtration $\Sigma^\infty_+ \{F_{n,k}\}$, which we showed is $\mathbb{A}_\infty$ in Theorem 1.3, but which is not known to be $\mathbb{E}_2$ (see Question 1).

In this section, we show that the graded spectrum on the right-hand side cannot be given a graded $\mathbb{E}_2$ structure which makes the above equivalence $\mathbb{E}_2$ on underlying ring spectra. This proves Theorem 1.9, and in particular says that even if the Bott filtration is $\mathbb{E}_2$, it will not be $\mathbb{E}_2$-split before smashing with MU.

The proof is via a power operation computation. In particular, the $\mathbb{A}_\infty$ splitting map takes the stabilization of the bottom cell $\beta_l : S^2 \to \mathbb{CP}^{n-1} \to \Omega SU(n)$ on the left-hand side to the stabilization of the bottom cell $\beta_r : S^2 \to F_{n,1} \simeq \mathbb{CP}^{n-1}$ on the right-hand side. We construct a power operation ν^s and show that $\nu^s(\Sigma^\infty \beta_l) \neq \nu^s(\Sigma^\infty \beta_r)$. The obstruction will be 2-primary, so we will implicitly complete at 2 for the remainder of the section.

Observation 10.1. Let $Y \in \text{Alg}_{\mathbb{E}_2}(\mathcal{S})$, and suppose we are given a map $S^2 \to Y$. This extends to an $\mathbb{E}_2$ map $\Omega^2 S^4 \to Y$. We may precompose with the map $h : S^5 \to \Omega^2 S^4$ adjoint to the Hopf map $S^7 \to S^4$. This procedure determines a natural operation

$$\nu^u : \pi_2(Y) \to \pi_5(Y)$$

in the homotopy of any $\mathbb{E}_2$-algebra in spaces.

Correspondingly, for any $X \in \text{Alg}_{\mathbb{E}_2}(\text{Sp})$, a class in $\pi_2(X)$ determines an $\mathbb{E}_2$ map $\Sigma^\infty_+ \Omega^2 S^4 \to X$. The above map h then determines an operation $\nu^s : \pi_2(X) \to \pi_5(X)$ via precomposition. This has the property that for $Y \in \text{Alg}_{\mathbb{E}_2}(\mathcal{S})$ and $\beta \in \pi_2(Y)$, we have $\nu^s(\Sigma^\infty \beta) = \Sigma^\infty \nu^u(\beta)$.

Remark 10.2. The notation is meant to hint at the fact that if $Y = \Omega^\infty X$ comes from a spectrum, then the operation ν^u is given by multiplication by the element $\nu \in \pi_3(\mathbb{S})^\wedge_2$ from the 2-primary homotopy groups of the sphere spectrum. Thus, ν^u is an unstable version of ν that is already seen in any $\mathbb{E}_2$ algebra in spaces.

We now compute the operation ν^s on $\Sigma^\infty \beta_l$ and $\Sigma^\infty \beta_r$.

(1) For $n > 3$, observe that the natural map $\Omega SU(n) \to BU$ is an isomorphism in homology up to degree 7. This implies that $\pi_5(\Omega SU(n)) \simeq \pi_5(BU) \simeq 0$ because BU is even. Consequently, $\nu^u(\beta_l) = 0$ and so $\nu^s(\Sigma^\infty \beta_l) = 0$.

(2) For β_r, we use the assumption that $gr(\Sigma_+^\infty \{F_{n,k}\})$ is an $\mathbb{E}_2$ graded spectrum. The map $\beta_r : S^2 \to F_{n,1}$ extends to an $\mathbb{E}_2$ map of underlying $\mathbb{E}_2$-algebras

$$\Sigma_+^\infty \Omega^2 S^4 \to gr(\Sigma_+^\infty \{F_{n,k}\}).$$

Since $\Sigma^\infty \beta_r$ hits the degree 1 piece, we may lift this to an $\mathbb{E}_2$ map of graded spectra

$$F_{\mathbb{E}_2}(\Sigma^\infty S^2[1]) \to gr(\Sigma_+^\infty \{F_{n,k}\})$$

from the free graded $\mathbb{E}_2$ algebra on $\Sigma^\infty S^2$ in degree 1 (see Example 2.4).

We aim to show that $\nu^s(\Sigma^\infty \beta_r)$ is nonzero. From the graded statement, it suffices to see that its component in grading 1 is nonzero; it is given by the composite

$$\Sigma^\infty S^5 \to \Sigma_+^\infty \Omega^2 S^4 \to \Sigma^\infty S^2 \xrightarrow{\Sigma^\infty \beta_r} \Sigma^\infty F_{n,1} = \Sigma^\infty \mathbb{CP}^{n-1}$$

where the middle map is given by projection onto the first graded piece (it's the map from the Snaith splitting). It is easy to see that the first composite $\Sigma^\infty S^5 \to \Sigma^\infty S^2$ is simply $\nu \in \pi_3(\mathbb{S})_2^\wedge$. Therefore, the whole composite is given by the product $\nu \cdot (\Sigma^\infty \beta_r)$. However, it was computed in [Liu63, Theorem II.8] that $\pi_5(\Sigma^\infty \mathbb{CP}^\infty) = \mathbb{Z}/2$ generated by $\nu \cdot (\Sigma^\infty \beta_r)$. Moreover, the natural map $\Sigma^\infty \mathbb{CP}^{n-1} \to \Sigma^\infty \mathbb{CP}^\infty$ is an isomorphism on π_5 for $n > 3$. We conclude that $\nu \cdot (\Sigma^\infty \beta_r) \neq 0$ and thus $\nu^s(\Sigma^\infty \beta_r) \neq 0$. This contradicts the existence of an $\mathbb{E}_2$ splitting.

Remark 10.3. Taking the limit as $n \to \infty$, we see from the above computations that the $\mathbb{E}_2$ power operations on the bottom cells of BU and $Q\mathbb{CP}^\infty$ do not agree. The bottom of the Weiss tower for the functor $V \mapsto BU(V)$ gives a well-known loop map $s : BU \to Q\mathbb{CP}^\infty$, implementing the splitting principle. The obstruction of this section recovers the classical fact that s is not a double loop map.

Appendix A. Further Properties of Day Convolution

Here we discuss some additional constructions and results that we will need for the more technical parts of this paper.

The monoidal structures on our categories will arise from Day convolution. This was studied for ∞-categories by Glasman [Gla13] and Lurie [Lur15, Lur16] at varying levels of generality. We will find it convenient to use the formulation from Section 2.2.6 of [Lur16].

Theorem A.1 ([Lur16], Example 2.2.6.9). *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories. Then there is an ∞-operad $Fun(\mathcal{C}, \mathcal{D})^\otimes$ with the following properties:*

(1) *The underlying ∞-category of $Fun(\mathcal{C}, \mathcal{D})^\otimes$ is the functor category $Fun(\mathcal{C}, \mathcal{D})$.*
(2) *The ∞-category $\mathrm{Alg}_{\mathbb{E}_\infty}(Fun(\mathcal{C}, \mathcal{D})^\otimes)$ of $\mathbb{E}_\infty$ algebras in $Fun(\mathcal{C}, \mathcal{D})^\otimes$ is equivalent to the category of lax symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.*

In order for the ∞-operad $Fun(\mathcal{C}, \mathcal{D})^\otimes$ to actually be a symmetric monoidal ∞-category, one needs to make additional assumptions.

Proposition A.2 ([Lur16], Proposition 2.2.6.16). *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories. Suppose that κ is an uncountable regular cardinal such that:*

(1) *$\mathcal{C}$ is essentially κ-small.*
(2) *$\mathcal{D}$ admits κ-small colimits.*
(3) *The tensor product on $\mathcal{D}$ preserves κ-small colimits separately in each variable.*

Then $Fun(\mathcal{C}, \mathcal{D})^\otimes$ is a symmetric monoidal ∞-category.

Recall that the Day convolution is defined classically via left Kan extension. Assumptions (1) and (2) ensure that the relevant Kan extensions exist. Assumption (3) then ensures that the multiplication is associative by allowing the colimits taken in the formula for left Kan extension to commute with the tensor product.

As stated before, Proposition A.2 is sufficient to construct symmetric monoidal ∞-categories **Fil**(Sp) and **Gr**(Sp). However, we wish to understand the interaction of the Weiss calculus with multiplicative structure; there, the filtrations go the other way.

We would like to make **Cofil**(Sp) a symmetric monoidal ∞-category by putting the Day convolution on its opposite, $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathrm{Sp}^{op})$. However, the smash product of spectra does not preserve small colimits separately in each variable. Nevertheless, it does preserve *finite* colimits separately in each variable. In fact, these are the only colimits that are needed in the case at hand and so we have the following variant of Proposition A.2:

Variant A.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories. Suppose that:*

(1) *Let I be a nonempty finite set and consider the multiplication map $\Pi_{i\in I}\,\mathcal{C} \to \mathcal{C}$. For every $C \in \mathcal{C}$, the slice category $\Pi_{i\in I}\,\mathcal{C} \times_{\mathcal{C}} \mathcal{C}_{/C}$ has a finite cofinal subcategory.*
(2) *$\mathcal{D}$ admits finite colimits.*
(3) *The tensor product on $\mathcal{D}$ preserves finite colimits separately in each variable.*

Then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}$ is a symmetric monoidal ∞-category.

Proof. This follows directly from the same arguments as Proposition A.2. In [Lur16, Corollary 2.2.6.14], the assumptions are used to guarantee the existence of a left Kan extension; this again exists by assumptions (1) and (2) and [Lur17, Lemma 4.3.2.13]. Similarly, the proof of [Lur16, Proposition 2.2.6.16] only makes reference to commuting tensor products in $\mathcal{D}$ with finite colimits, which is ensured by assumption (3). $\qquad\square$

In Section B, we will need to consider not only the Day convolution monoidal structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ but its functoriality as $\mathcal{C}$ varies. For instance, we would for symmetric monoidal functors $\mathcal{C}_1 \to \mathcal{C}_2$ to induce symmetric monoidal functors $\mathrm{Fun}(\mathcal{C}_1, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_2, \mathcal{D})$ via left Kan extension.

We give a very close variant of [Nik16, Corollary 3.8] in our current framework:

Proposition A.4. *Let $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{D}$ be symmetric monoidal ∞-categories and let $f : \mathcal{C}_1 \to \mathcal{C}_2$ be a symmetric monoidal functor. Suppose that one of the following conditions hold:*

(1) *The pairs $(\mathcal{C}_1, \mathcal{D})$ and $(\mathcal{C}_2, \mathcal{D})$ satisfy the hypotheses of Proposition A.2*
(2) *The pairs $(\mathcal{C}_1, \mathcal{D})$ and $(\mathcal{C}_2, \mathcal{D})$ satisfy the hypotheses of Variant A.3 and for any object $c \in \mathcal{C}_2$, the slice category $\mathcal{C}_1 \times_{\mathcal{C}_2} \mathcal{C}_{2/c}$ has a finite cofinal subset.*

Then there is an adjunction

$$f_! : \mathrm{Fun}(\mathcal{C}_1, \mathcal{D}) \rightleftarrows \mathrm{Fun}(\mathcal{C}_2, \mathcal{D}) : f_*$$

where f_* denotes restriction and $f_!$ denotes left Kan extension. Moreover, the functor f_* is lax symmetric monoidal and $f_!$ is symmetric monoidal.

Proof. The universal property of $\mathrm{Fun}(\mathcal{C}_1, \mathcal{D})^{\otimes}$ immediately implies the existence of a map of ∞-operads $\mathrm{Fun}(\mathcal{C}_2, \mathcal{D})^{\otimes} \to \mathrm{Fun}(\mathcal{C}_1, \mathcal{D})^{\otimes}$, which makes f_* a lax symmetric monoidal functor.

Assumptions (1) and (2) of Proposition A.2 guarantee that the adjunction exists at the level of ∞-categories. The rest of the proof from [Nik16, Corollary 3.8] carries over verbatim. $\qquad\square$

APPENDIX B. PROOF OF THEOREM 2.6

We will need a few preliminary definitions. We start by fixing a positive integer n. Let $[n]$ denote the linearly ordered set of integers $0 \leq i \leq n$. For any indexing 1-category $\mathcal{D}$,

denote by $\mathcal{D}^{ds}$ the underlying discrete category, and denote by $\mathcal{D}^+$ the the category formed by formally adding a final object, which we will refer to as "+". Define $\mathbf{Fil}_n^+ = \mathrm{Fun}([n]^+, \mathrm{Sp})$ and $\mathbf{Cofil}_n^+ = \mathrm{Fun}(([n]^+)^{op}, \mathrm{Sp})$. These categories admit functors to Sp by restriction to the distinguished point. We define $\mathcal{C}_n$ by the following pullback:

$$
\begin{array}{ccc}
\mathcal{C}_n & \longrightarrow & \mathbf{Cofil}_n^+ \\
\downarrow & & \downarrow \\
\mathbf{Fil}_n^+ & \longrightarrow & \mathrm{Sp}
\end{array}
\tag{4}
$$

An element of $\mathcal{C}_n$ can be thought of as a sequence of spectra connected by maps:

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \longrightarrow X \simeq Y \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0$$

where the middle arrow is an equivalence, as indicated. This is equivalent to just considering sequences

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \longrightarrow Z \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0,$$

and so we shall refer to general elements by these names below.

Define the subcategory $\mathcal{G}_n \subset \mathcal{C}_n$ as the full subcategory such that for each integer $0 \le i \le n$, the composite $X_i \longrightarrow Y_i$ is an equivalence.

Lemma B.1. *There is an equivalence*

$$\mathcal{G}_n \simeq \mathbf{Gr}_n^+ := Fun(([n]^+)^{ds}, \mathrm{Sp}).$$

Proof. We proceed by induction on n.

For $n = 0$, we are considering the full subcategory of diagrams $X_0 \to Z \to Y_0$ of spectra with the property that the composite is an equivalence. By taking the fiber of the second map, this is equivalent to the category of triples (X_0, Y_0', Z) of spectra together with an equivalence $X_0 \vee Y_0' \xrightarrow{\sim} Z$. This is certainly equivalent to the category of pairs (X_0, Y_0') of spectra, which is $\mathbf{Gr}_0^+$.

Next, assume the statement for $n \le k$ and consider $\mathcal{G}_{k+1}$. We consider the auxiliary category $\overline{\mathcal{G}}_{k+1}$ which is the full subcategory of $\mathcal{C}_{k+1}$ where only $X_{k+1} \to Y_{k+1}$ is stipulated to be an equivalence. The argument for the base case shows that

$$\overline{\mathcal{G}}_{k+1} = \mathbf{Fil}_{k+1} \times_{\mathrm{Sp}} \mathbf{Gr}_0^+ \times_{\mathrm{Sp}} \mathbf{Cofil}_{k+1}$$

where the fiber products are over the restriction to $0 \in [0]^+$ for $\mathbf{Gr}_0^+$, and over X_{k+1} and Y_{k+1} in the filtered and cofiltered spectra. By commuting the fiber products, we find that

$$\overline{\mathcal{G}}_{k+1} = (\mathbf{Fil}_{k+1} \times_{\mathrm{Sp}} \mathbf{Cofil}_{k+1}) \times_{\mathrm{Sp}} \mathbf{Gr}_0^+ \simeq \mathcal{C}_k \times_{\mathrm{Sp}} \mathbf{Gr}_0^+$$

where we have implicitly used the identifications $\mathbf{Fil}_{k+1} \simeq \mathbf{Fil}_k^+$ and $\mathbf{Cofil}_{k+1} \simeq \mathbf{Cofil}_k^+$. Under this equivalence, the full subcategory $\mathcal{G}_{k+1} \subset \overline{\mathcal{G}}_{k+1}$ corresponds to $\mathcal{G}_k \times_{\mathrm{Sp}} \mathbf{Gr}_0^+ \simeq \mathbf{Gr}_{k+1}^+$ as desired.

$\square$

In fact, the functor $\mathbf{Gr}_n^+ \to \mathcal{C}_n$ can be seen very explicitly as follows: there's a functor

$$I_n^+ : \mathbf{Gr}_n^+ \to \mathbf{Fil}_n^+$$

given by left Kan extension along the inclusion $([n]^+)^{ds} \to [n]^+$ which is completely analogous to the functor I described in Section 2. Dually, there's a functor

$$I_n^{op,+} : \mathbf{Gr}_n^+ \to \mathbf{Cofil}_n^+$$

given by right Kan extension along the inclusion $([n]^+)^{ds} \to ([n]^+)^{op}$ which sends an element $(X_0, X_1, \cdots, X_n, X) \in \mathbf{Gr}_n^+$ to

$$X_0 \longleftarrow X_0 \vee X_1 \longleftarrow \cdots \longleftarrow \bigvee_i X_i \longleftarrow X \vee \bigvee_i X_i.$$

These functors agree on restriction to the distinguished object, and so they define the desired functor $\mathbf{Gr}_n^+ \to \mathcal{C}_n$.

Until this point, we have been working with a fixed n and without regard to the monoidal structure. The results of Appendix A allow us to analyze what happens as n varies.

In particular, give $[n]^+$ the structure of a symmetric monoidal category by taking $\mathbb{Z}_{\geq 0}$ under addition and identifying all the integers $m > n$ with the point $+$. By Proposition A.2, this gives $\mathbf{Fil}_n^+$ the structure of a symmetric monoidal ∞-category. There are natural symmetric monoidal functors $\mathbb{Z}_{\geq 0}^+ \to [n+1]^+ \to [n]^+$ by successive quotient. By Proposition A.4 combined with the commutative diagram of symmetric monoidal functors

$$
\begin{array}{ccc}
([n+1]^+)^{ds} & \longrightarrow & [n+1]^+ \\
\downarrow & & \downarrow \\
([n]^+)^{ds} & \longrightarrow & [n]^+,
\end{array}
$$

we obtain a commutative diagram of symmetric monoidal ∞-categories:

$$
\begin{array}{ccc}
\mathbf{Gr}_{n+1}^+ & \xrightarrow{I_{n+1}^+} & \mathbf{Fil}_{n+1}^+ \\
\downarrow & & \downarrow \\
\mathbf{Gr}_n^+ & \xrightarrow{I_n^+} & \mathbf{Fil}_n^+.
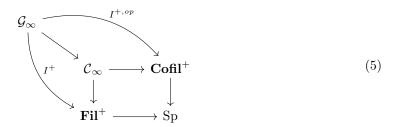\end{array}
$$

We claim that the limit of the right vertical arrows over n is $\mathbf{Fil}^+ := \mathrm{Fun}(\mathbb{Z}_{\geq 0}^+, \mathrm{Sp})$ with its symmetric monoidal structure as given by Proposition A.2. This is because by Proposition A.4, there are symmetric monoidal functors $\mathbf{Fil}^+ \to \mathbf{Fil}_n^+$ induced by the symmetric monoidal functor $\mathbb{Z}_{\geq 0}^+ \to [n]^+$ collapsing $+$ and all integers $m > n$ to a point. This induces a symmetric monoidal functor $\mathbf{Fil}^+ \to \lim_n \mathbf{Fil}_n^+$. It is then easy to check that this is an equivalence.

There is a similar diagram for the cofiltered side by the appropriate analogs of the above statements:

$$
\begin{array}{ccc}
\mathbf{Gr}_{n+1}^+ & \xrightarrow{I_{n+1}^{op,+}} & \mathbf{Cofil}_{n+1}^+ \\
\downarrow & & \downarrow \\
\mathbf{Gr}_n^+ & \xrightarrow{I_n^{op,+}} & \mathbf{Cofil}_n^+
\end{array}
$$

and as before, we have an identification $\mathrm{Fun}((\mathbb{Z}_{\geq 0}^+)^{op}, \mathrm{Sp}) =: \mathbf{Cofil}^+ \simeq \lim_n \mathbf{Cofil}_n^+$ by Variant A.3 and Proposition A.4.

Thus, taking the limit in n in the diagram (4) yields a diagram of symmetric monoidal functors:

$$
\begin{array}{ccc}
\mathcal{G}_\infty & \xrightarrow{\quad I^{+,op} \quad} & \\
\downarrow I^+ & \mathcal{C}_\infty \longrightarrow \mathbf{Cofil}^+ & \\
& \downarrow \qquad\quad \downarrow & \\
& \mathbf{Fil}^+ \longrightarrow \mathrm{Sp} &
\end{array}
\tag{5}
$$

where the square is Cartesian.

Remark B.2. While $\mathcal{G}_\infty \simeq \lim \mathbf{Gr}_n^+$, we warn the reader that $\mathcal{G}_\infty$ is not simply $\mathrm{Fun}((\mathbb{Z}_{\geq 0}^+)^{ds}, \mathrm{Sp})$ because the maps in the inverse system for $\mathcal{G}_\infty$ are not just the ones induced by the inclusions $([n]^+)^{ds} \to ([n+1]^+)^{ds}$.

The functor $\mathcal{G}_\infty \to \mathcal{C}_\infty$ is fully faithful because it is the inverse limit of fully faithful functors. Moreover, the essential image consists of those pairs $(X, Y) \in \mathbf{Fil}^+ \times_{\mathrm{Sp}} \mathbf{Cofil}^+ = \mathcal{C}_\infty$ such that the natural maps $X_i \to Y_i$ are equivalences for all $i \geq 0$. Since all the functors in the diagram (5) were symmetric monoidal, we have for each integer $n \geq 0$ a diagram at the level of algebras:

$$
\begin{array}{ccc}
\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{G}_\infty) & \xrightarrow{\quad I^{+,op} \quad} & \\
\downarrow I^+ & \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}_\infty) \longrightarrow \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Cofil}^+) & \\
& \downarrow \qquad\qquad\qquad \downarrow & \\
& \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil}^+) \longrightarrow \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp}) &
\end{array}
\tag{6}
$$

where the square is Cartesian and the functor $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{G}_\infty) \to \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}_\infty)$ is fully faithful with essential image as described above.

Recall that we were interested in understanding when an $\mathbb{E}_n$ filtered spectrum $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil})$ is split - that is, when there exists $Z \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr})$ such that $X \simeq IZ$. The following proposition relates that to our current situation; informally, it allows us to get rid of the +'s.

Proposition B.3. *There exists a diagram of symmetric monoidal ∞-categories and symmetric monoidal functors*

$$
\begin{array}{ccc}
\mathcal{G}_\infty & \xrightarrow{\;\pi\;} & \mathbf{Gr} \\
\downarrow{\scriptstyle I^+} & & \downarrow{\scriptstyle I} \\
\mathbf{Fil} \xrightarrow{\;\iota\;} \mathbf{Fil}^+ & \xrightarrow{\;\varpi\;} & \mathbf{Fil}
\end{array}
$$

where the bottom row is a retract and I^+ is induced by the I_n^+ at each finite level.

Proof. We have the square at each finite level:

$$
\begin{array}{ccc}
\mathbf{Gr}_n^+ & \longrightarrow & \mathbf{Gr}_n \\
\downarrow{\scriptstyle I_n^+} & & \downarrow{\scriptstyle I_n} \\
\mathbf{Fil}_n^+ & \longrightarrow & \mathbf{Fil}_n \, .
\end{array}
$$

where the right arrow comes from ignoring the +. It is easy to see that all the functors are symmetric monoidal, and so taking the limit gives the square.

For the bottom row, we apply Proposition A.4 to the symmetric monoidal functor $q : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}^+$. The functor ϖ certainly coincides with q_*, and we define $\iota = q_!$, which is symmetric monoidal by Proposition A.4. It is immediate that $\varpi \circ \iota$ is an equivalence, so the bottom row is a retract.

$\square$

We are now ready to prove the main result of this section.

Proof of Theorem 2.6. We use the notations of Proposition B.3. Since ι is a symmetric monoidal functor, we obtain an $\mathbb{E}_n$ algebra $\iota X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil}^+)$. On the cofiltered side, we note that by the universal property of Day convolution, the symmetric monoidal functor $\mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}^+$ induces a map of ∞-operads $\mathrm{Fun}(\mathbb{Z}_{\geq 0}^+, \mathrm{Sp}^{op})^\otimes \to \mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathrm{Sp}^{op})^\otimes$. This amounts to an oplax symmetric monoidal functor $\mathbf{Cofil}^+ \to \mathbf{Cofil}$ which restricts away from the $+$. It is easy to see that this is actually a symmetric monoidal functor, and so its adjoint $\iota^{op} : \mathbf{Cofil} \to \mathbf{Cofil}^+$ by right Kan extension is lax monoidal. Concretely, this is the functor which takes a cofiltered spectrum and adds in its limit. As a result, we obtain an $\mathbb{E}_n$-algebra $\iota^{op} Y \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Cofil}^+)$ with image $\lim Y \in \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})$.

Condition (1) in the statement of the theorem guarantees that ιX and $\iota^{op} Y$ determine an element $\mathcal{X} \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}_\infty)$. Condition (2) then ensures that $\mathcal{X}$ is in the essential image of the fully faithful functor $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{G}_\infty) \to \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}_\infty)$; consequently, we regard $\mathcal{X}$ as an $\mathbb{E}_n$-algebra in $\mathcal{G}_\infty$. Finally, we chase through the diagram of Proposition B.3 to see that $I\pi\mathcal{X} \simeq \varpi I^+ \mathcal{X} \simeq \varpi \iota X \simeq X$ as $\mathbb{E}_n$-algebras in $\mathbf{Fil}$. Hence, X is $\mathbb{E}_n$-split, as desired. $\square$

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