STRUCTURED SPLITTINGS OF $\Omega SU(n)$ AND SNAITH'S CONSTRUCTION OF PERIODIC COMPLEX BORDISM

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Contents

1. Introduction

The complex cobordism spectrum MU has played a central role in homotopy theory ever since Quillen's 19XX arguments **CITE** connected its homotopy with the theory of one-dimensional formal group laws. As the Thom spectrum of the canonical bundle over BU, MU naturally acquires an immense amount of structure: it is an $\mathbb{E}_\infty$-ring spectrum. To this day, however, much remains unknown about the full nature of this $\mathbb{E}_\infty$ structure. For example, almost nothing is known about the k-invariants of the spectrum of units $gl_1(MU)$ **CITE**.

As a spectrum, MU is classically constructed via the sequence of natural maps

$$MU(n) \longrightarrow \cdots$$

This suggests that perhaps the more elemental object is $\bigvee MU(n)$, the Thom spectrum of the J-homomorphism

$$\mathbf{Vect} \xrightarrow{J} \mathrm{Pic}(S)$$

that takes a vector space V to its one-point compactification $J(V) = S^V$. Since

$$J(V \oplus W) \simeq S^V \wedge S^W,$$

the Thom spectrum $\bigvee MU(n)$ is naturally an $\mathbb{E}_\infty$-ring spectrum. Inverting the Bott element $\beta \in \pi_2(MU(1)) \cong \pi_2(MU(1))$, one obtains the *periodic* complex cobordism spectrum

$$MUP \simeq \left(\bigvee MU(n)\right)[\beta^{-1}].$$

This periodic spectrum MUP is a minor variation on MU itself: there is a wedge decomposition $MUP \simeq \bigvee_{a\in\mathbb{Z}} \Sigma^{2a} MU$ and the inclusion $MU \to MUP$ onto the $a = 0$ factor is an an $\mathbb{E}_\infty$-ring homomorphism.

In 19XX, Victor Snaith **CITE** gave another presentation of periodic complex cobordism

Theorem 1.1 (Snaith)**.** *As homotopy commutative ring spectra, $MUP \simeq \Sigma^\infty_+ BU[\beta^{-1}]$. More generally, there is an equivalence of homotopy commutative rings*

$$\bigvee MU(n) \simeq \Sigma^\infty_+ BU.$$

In the above, $\Sigma^\infty_+ BU$ acquires an $\mathbb{E}_\infty$-structure (and hence a homotopy commutative ring structure) from the fact that BU is an infinite loop space. One may think of $\Sigma^\infty_+ BU$ as the group ring of the topological group BU.

The genesis of this paper was an attempt to use Snaith's theorem to study the $\mathbb{E}_\infty$-ring structure on MUP. This seemed like an especially reasonable idea in light of two facts.

(1) By another theorem of Snaith, periodic K-theory KU may be constructed as

$$KU \simeq \Sigma^\infty_+ \mathbb{CP}^\infty[\beta^{-1}].$$

This is an equivalence of $\mathbb{E}_\infty$-ring spectra **CITE**.

(2) In **CITE**, Gepner and Snaith prove a motivic analogue of Theorem 1.1. In particular, **BLAH**. They then use the $\mathbb{E}_\infty$-ring structure on **BLAH** to **define** an $\mathbb{E}_\infty$-ring structure on periodic MGL.

However, as it turns out (though it is not at all obvious from the modern literature and in particular not mentioned in **GepnerSnaith**) another old theorem of Snaith **CITE** shows that our idea was entirely unreasonable.

Theorem 1.2 (Snaith)**.** *As $\mathbb{E}_\infty$-rings,*

$$MUP \not\simeq \Sigma^\infty_+ BU[\beta^{-1}].$$

In Section 7 we refine Snaith's results in to what we consider their definitive form:

Theorem 1.3. *There is an equivalence of $\mathbb{E}_2$-ring spectra*

$$MUP \simeq \Sigma^\infty_+ BU[\beta^{-1}],$$

but $MUP \not\simeq \Sigma^\infty_+ BU[\beta^{-1}]$ as $\mathbb{E}_3$-ring spectra. There is an equivalence of $\mathbb{A}_\infty$-ring spectra

$$\bigvee MU(n) \simeq \Sigma^\infty_+ BU,$$

but $\bigvee MU(n) \not\simeq \Sigma^\infty_+ BU$ as $\mathbb{E}_2$-ring spectra.

Remark 1.4. We freely use the language of ∞-categories throughout this paper, refering to an ∞-category simply as a category.

In Section 2 we will review the symmetric monoidal categories **Fil** and **Gr** of filtered and graded spectra, respectively. A filtered spectrum is an infinite sequence

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \cdots$$

of spectra. The tensor product

$$(X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots) \otimes (Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \cdots)$$

of two filtered spectra is computed as a Day convolution

$$X_0 \otimes Y_0 \longrightarrow \operatorname{colim} \left(\begin{tikzcd} X_0 \wedge Y_1 & \\ X_0 \wedge Y_0 \arrow[u] \arrow[r] & X_1 \wedge Y_0 \end{tikzcd} \right) \longrightarrow \operatorname{colim} \left(\begin{tikzcd} X_0 \wedge Y_2 & & \\ X_0 \wedge Y_1 \arrow[u] \arrow[r] & X_1 \wedge Y_1 & \\ X_0 \wedge Y_0 \arrow[u] \arrow[r] & X_1 \wedge Y_0 \arrow[u] \arrow[r] & X_2 \wedge Y_0 \end{tikzcd} \right) \longrightarrow \cdots.$$

A graded spectrum, on the other hand, is simply an ordered sequence $(A_0, A_1, A_2, \cdots)$ of spectra. The tensor product is computed as

$$(A_0, A_1, A_2, \cdots) \otimes (B_0, B_1, B_2, \cdots) \simeq \left(A_0 \wedge B_0, (A_1 \wedge B_0) \vee (A_0 \wedge B_1), \cdots, \bigvee_{i+j=n} A_i \wedge B_j, \cdots \right).$$

There is a sequence of symmetric monoidal functors

$$\mathbf{Gr} \xrightarrow{\quad I \quad} \mathbf{Fil} \xrightarrow{\quad \mathrm{colim} \quad} \mathrm{Sp},$$

where I sends the graded spectrum $(A_0, A_1, A_2, \cdots)$ to the filtered spectrum

$$I(A_0, A_1, A_2, \cdots) = (A_0 \longrightarrow A_0 \vee A_1 \longrightarrow A_0 \vee A_1 \vee A_2 \longrightarrow \cdots).$$

Definition 1.5. We say that an $\mathbb{E}_n$-ring spectrum is **$\mathbb{E}_n$-split** if it is equivalent to the image of an $\mathbb{E}_n$-algebra in **Gr** under the composite $\mathrm{colim} \circ I$. Similarly, a filtered $\mathbb{E}_n$-algebra is $\mathbb{E}_n$-split if it is equivalent to the image under I of an $\mathbb{E}_n$-algebra in **Gr**.

In section 5 we will show:

Theorem 1.6. *The ??? filtration on $\Sigma^\infty_+ \Omega SU(n)$ is $\mathbb{E}_2$-split after smashing with MU.*

2. Filtered and Graded Ring Spectra

Here we review a framework from [Lur15] for studying graded and filtered objects. The reader is referred to [Lur15] for a more thorough treatment and all proofs.

Let $\mathcal{C}$ be a stable ∞-category. Denote by $\mathbb{Z}_{\geq 0}$ the poset of non-negative integers, and by $\mathbb{Z}^{ds}_{\geq 0}$ the corresponding discrete category. The reader is warned that our numbering conventions are opposite the ones in [Lur15].

Definition 2.1. Let $\mathbf{Gr}(\mathcal{C})$ denote the functor category $\mathrm{Fun}(\mathbb{Z}^{ds}_{\geq 0}, \mathcal{C})$. We shall refer to $\mathbf{Gr}(\mathcal{C})$ as the category of graded objects in $\mathcal{C}$. Its objects can be thought of as sequences $X_0, X_1, X_2, \cdots \in \mathcal{C}$.

Definition 2.2. -Let $\mathbf{Fil}(\mathcal{C})$ denote the functor category $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{C})$. We shall refer to **Fil** as the category of filtered spectra. Its objects can be thought of as sequences $Y_0 \to Y_1 \to Y_2 \to \cdots$ of spectra filtering $\mathrm{colim}_i\, Y_i$.

The obvious map $\mathbb{Z}^{ds}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ induces a restriction functor $\mathrm{res} : \mathbf{Fil} \to \mathbf{Gr}$. The restriction is right adjoint to a functor $I : \mathbf{Gr} \to \mathbf{Fil}$ given by left Kan extension. The functor I can be thought of as taking a graded spectrum $X_0, X_1, X_2, \cdots$ to the filtered spectrum $X_0 \to X_0 \oplus X_1 \to X_0 \oplus X_1 \oplus X_2 \to \cdots$. There is also an associated graded functor $\mathrm{gr} : \mathbf{Fil} \to \mathbf{Gr}$ such that the composite $\mathrm{gr} \circ I : \mathbf{Gr} \to \mathbf{Gr}$ is an equivalence.

The categories **Gr** and **Fil** are given symmetric monoidal structures via the Day convolution; we denote the operation in both cases by $\otimes$. The unit $\mathbb{S}^{gr}$ of $\otimes$ in graded spectra is S^0 in degree 0 and $*$ otherwise; the unit $\mathbb{S}^{fil}$ in filtered spectra is $I\,\mathbb{S}^{gr}$.

The functors I and gr can be given symmetric monoidal structures such that the composite $\mathrm{gr} \circ I : \mathbf{Gr} \to \mathbf{Gr}$ is a symmetric monoidal equivalence. It follows in particular that they extend to functors on between the categories of $\mathbb{E}_n$ algebras in **Gr** and **Fil**. Thus, given an $\mathbb{E}_n$ algebra Y in filtered spectra, we obtain a canonical $\mathbb{E}_n$ structure on its associated graded $\mathrm{gr}(Y)$. Conversely, given $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil})$

Definition 2.3. An object $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil})$ is called *$\mathbb{E}_n$-split* if it is in the essential image of I.

Given an $\mathbb{E}_n$-split filtered spectrum X, we can recover the underlying graded spectrum by taking the associated graded.

3. The (Segal-Mitchell-Richter?) Filtration on $\Omega SU(n)$

I believe Mitchell shows in [Mit86] that the filtration is filtered $\mathbb{A}_\infty$. We need to check this.

Conjecture 3.1. *The filtration is $\mathbb{E}_2$.*

I guess now we know this is just true. We should probably thank Jacob for bringing to our attention that this conjecture of Mahowald is actually well-known by geometric representation theorists.

4. An $\mathbb{A}_\infty$-splitting by Weiss Calculus

In this section, we extend the methods of [Aro01] to produce $\mathbb{A}_\infty$ stable splittings of Stiefel manifolds.

4.1. **Weiss Calculus.** Let $\mathcal{J}$ be the ∞-category which is the nerve of the topological category whose objects are finite dimensional complex vector spaces equipped with a Hermitian inner product and whose morphisms are spaces of linear isometries.

4.2. **General splitting machinery.** Let $[n]$ denote the linearly ordered set of integers $0 \leq i \leq n$. Define $\mathbf{Fil}_n = \mathrm{Fun}([n], \mathrm{Sp}^{\mathcal{J}})$ and $\mathbf{Cofil}_n = \mathrm{Fun}([n]^{\mathrm{op}}, \mathrm{Sp}^{\mathcal{J}})$. These categories admit functors to $\mathrm{Sp}^{\mathcal{J}}$ by taking colimit and limit, respectively. Let $\mathcal{C}_n = \mathbf{Fil}_n \times_{\mathrm{Sp}^{\mathcal{J}}} \mathbf{Cofil}_n$. Finally, let $\mathbf{Gr}_n = \mathrm{Fun}([n]^{\mathrm{ds}}, \mathrm{Sp}^{\mathcal{J}})$ where $[n]^{\mathrm{ds}}$ denotes the underlying discrete category. We have the following lemma:

Lemma 4.1. *For all integers $n > 0$, there is a fully faithful functor $i_n : \boldsymbol{Gr}_{n+1} \to \mathcal{C}_n$.*

Proof. An element of $\mathcal{C}_n$ is given by a sequence of maps
The proof is by induction. □

We may then take inverse limits to get a category $\mathcal{C}_\infty = \mathbf{Fil}(\mathrm{Sp}^{\mathcal{J}}) \times_{\mathrm{Sp}^{\mathcal{J}}} \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$ and a functor $i : \mathbf{Gr} \to \mathcal{C}_\infty$.

Corollary 4.1.1. *The functor i is fully faithful.*

Proof. □

at the end, restrict connectivity so that it's monoidal

5. An $\mathbb{E}_2$-splitting in Complex Cobordism

Let R denote any $\mathbb{E}_2$-ring spectrum with even-concentrated homotopy groups. I claim that any $\mathbb{A}_\infty$-morphism $\Omega SU(n) \to R$ lifts to an $\mathbb{E}_2$-morphism.

The argument is inspired by [CM15]

Proof. The obstructions live in $H_*(\Omega SU(n); \pi_{*+1}(R))$, and no matter the value of $\pi_{*+1}(R)$ this is 0. □

Corollary 5.0.2. *If R is the relevant split MU-$\mathbb{E}_2$-algebra, this yields an MU-$\mathbb{E}_2$-algebra equivalence $\Omega SU(n) \wedge MU \to R$ by adjunction from the $\mathbb{E}_2$-map $\Omega SU(n) \to R$.*

6. Obstructions to a general $\mathbb{E}_2$-splitting

Suppose there is a map $\Sigma^\infty_+ \Omega SU(n) \to \Sigma^\infty_+ \mathbb{CP}^{n-1}$. This is adjoint to a double loop map $\Omega SU(n) \to GL_1(\Sigma^\infty_+ \mathbb{CP}^{n-1})$ which lands in the component $SL_1(\Sigma^\infty_+ \mathbb{CP}^{n-1}) \simeq \Omega^2 \Sigma^2 \mathbb{CP}^{n-1}$. BLAH BLAH

7. Snaith's Construction of Periodic Complex Bordism

A classical theorem of Snaith [Sna81] gives an equivalence of homotopy commutative ring spectra

$$\Sigma^\infty_+ BU[\beta^{-1}] \simeq MUP.$$

The equivalence arises from considering the total MU-Chern class map $BU \to GL_1(MUP)$. It is known from [Sna77] that the total Chern class in integral homology is not an infinite loop map. It follows from the existence of an $\mathbb{E}_\infty$ map $MUP \to H\,\mathbb{Z}\,P$ from periodic complex bordism to periodic integral homology that Snaith's equivalence is not an equivalence of $\mathbb{E}_\infty$ ring spectra. The following theorem refines this observation:

Theorem 7.1. *The equivalence $\Sigma^\infty_+ BU[\beta^{-1}] \simeq MUP$ is $\mathbb{E}_2$ but not $\mathbb{E}_3$.*

Proof. Proof goes here □

Comment now about GepnerSnaith. We should cite at some point here or the introduction all of [Sna77], [GS09], and the Snaith book with the original splitting.

8. Miscellaneous stuff here

It would be nice to at some point deal with showing the associated graded E_2 structure of BU is the thom spectrum VMU(n). I've directly pasted in some writing from a previous argument I claimed, but it definitely uses that $\coprod BU(n)$ is an E_2 algebra over $\mathbb{Z}_{\geq 0}$ which I never got straight an actual proof of.

Proposition 8.1. *The associated graded of $\Sigma^\infty_+ BU$ is E_2 equivalent to the Thom spectrum $\bigvee MU(n)$.*

Proof. Let $R = BU$ with its natural filtration, and let $R^\oplus = \coprod BU(n)$ with its natural filtration. Let M be the (E_∞) filtered spectrum which is $MU(n)$ in degree n, and all maps are 0. In other words, $\bigvee MU(n)$ with its natural filtration is $I(res(M))$.

We begin with a filtered E_∞ map $z : R^\oplus \to I(res(M))$ coming from the zero section. Then, $R^\oplus$ comes with the structure of an E_2 algebra over $\mathbb{Z}^{fil}_{\geq 0}$. In fact, $I(res(M))$ has a trivial structure as an E_∞-algebra over $\mathbb{Z}^{fil}_{\geq 0}$ via the augmentation $\mathbb{Z}^{fil}_{\geq 0} \to S^{0,fil} \to I(res(M))$. We may then tensor z along the augmentation to get a map of E_2 filtered spectra $z' : R \to I(res(M))$.

There is a canonical equivalence $I(res(M)) \otimes \mathbb{A} \simeq M$ because M is in the image of $\mathbb{A} \otimes I(-)$ (that is, all the maps in the filtration of M were zero). As such, M acquires a canonical structure as an $\mathbb{A}$ algebra such that the map $M \otimes \mathbb{A} \to M$ is a map of E_2 rings (in fact I think it's E_∞?).

Finally, we observe that we may tensor z' with $\mathbb{A}$ and compose with the multiplication map to get an E_2 map $R \otimes A \to M \otimes A \to M$ which is the right thing up to homotopy, so it's an equivalence. □

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What is $\Sigma^\infty_+ \Omega SU(n)[\beta^{-1}]$, by the way? Is it related to a periodic version of the $X(n)$-filtration of MU??

References

[Aro01] Greg Arone, *The Mitchell-Richter filtration of loops on Stiefel manifolds stably splits*, Proc. Amer. Math. Soc. **129** (2001), no. 4, 1207–1211. MR 1814154

[CM15] Steven Greg Chadwick and Michael A. Mandell, *E_n genera*, Geom. Topol. **19** (2015), no. 6, 3193–3232. MR 3447102

[GS09] David Gepner and Victor Snaith, *On the motivic spectra representing algebraic cobordism and algebraic K-theory*, Doc. Math. **14** (2009), 359–396. MR 2540697

[Lur15] Jacob Lurie, *Rotation invariance in algebraic k-theory*, Available at http://www.math.harvard.edu/ lurie/ (2015).

[Mit86] Stephen A. Mitchell, *A filtration of the loops on* $\mathrm{SU}(n)$ *by Schubert varieties*, Math. Z. **193** (1986), no. 3, 347–362. MR 862881

[Sna77] Victor Snaith, *The total Chern and Stiefel-Whitney classes are not infinite loop maps*, Illinois J. Math. **21** (1977), no. 2, 300–304. MR 0433446

[Sna81] ———, *Localized stable homotopy of some classifying spaces*, Math. Proc. Camb. Phil. Soc. **89** (1981), no. 2, 325–330. MR 0600247