Structured Splittings of $\Omega SU(n)$ and Snaith's Construction of Periodic Complex Bordism

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1 Introduction

Write this last after we have filled in all the theorems with complete proofs.

2 Filtered and Graded Ring Spectra

Here we review a framework from [**?**] for studying graded and filtered objects. The reader is referred to [**?**] for a more thorough treatment and all proofs.

Let $\mathcal{C}$ be a stable ∞-category. Denote by $\mathbb{Z}_{\geq 0}$ the poset of non-negative integers, and by $\mathbb{Z}^{ds}_{\geq 0}$ the corresponding discrete category. The reader is warned that our numbering conventions are opposite the ones in [**?**].

Definition 2.1. Let $\mathbf{Gr}(\mathcal{C})$ denote the functor category $\mathrm{Fun}(\mathbb{Z}^{ds}_{\geq 0}, \mathcal{C})$. We shall refer to $\mathbf{Gr}(\mathcal{C})$ as the category of graded objects in $\mathcal{C}$. Its objects can be thought of as sequences $X_0, X_1, X_2, \dots \in \mathcal{C}$.

Definition 2.2. Let $\mathbf{Fil}(\mathcal{C})$ denote the functor category $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{C})$. We shall refer to $\mathbf{Fil}$ as the category of filtered spectra. Its objects can be thought of as sequences $Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \cdots$ of spectra filtering $\mathrm{colim}_i\, Y_i$.

The obvious map $\mathbb{Z}_{\geq 0}^{ds} \longrightarrow \mathbb{Z}_{\geq 0}$ induces a restriction functor $\mathrm{res} : \mathbf{Fil} \longrightarrow \mathbf{Gr}$. The restriction is right adjoint to a functor $I : \mathbf{Gr} \longrightarrow \mathbf{Fil}$ given by left Kan extension. The functor I can be thought of as taking a graded spectrum $X_0, X_1, X_2, \cdots$ to the filtered spectrum $X_0 \longrightarrow X_0 \oplus X_1 \longrightarrow X_0 \oplus X_1 \oplus X_2 \longrightarrow \cdots$. There is also an associated graded functor $\mathrm{gr} : \mathbf{Fil} \longrightarrow \mathbf{Gr}$ such that the composite $\mathrm{gr} \circ I : \mathbf{Gr} \longrightarrow \mathbf{Gr}$ is an equivalence.

The categories $\mathbf{Gr}$ and $\mathbf{Fil}$ are given symmetric monoidal structures via the Day convolution; we denote the operation in both cases by $\otimes$. The unit $\mathbb{S}^{gr}$ of $\otimes$ in graded spectra is S^0 in degree 0 and $*$ otherwise; the unit $\mathbb{S}^{fil}$ in filtered spectra is $I\,\mathbb{S}^{gr}$.

The functors I and gr can be given symmetric monoidal structures such that the composite $\mathrm{gr} \circ I : \mathbf{Gr} \longrightarrow \mathbf{Gr}$ is a symmetric monoidal equivalence. It follows in particular that they extend to functors on between the categories of $\mathbb{E}_n$ algebras in $\mathbf{Gr}$ and $\mathbf{Fil}$. Thus, given an $\mathbb{E}_n$ algebra Y in filtered spectra, we obtain a canonical $\mathbb{E}_n$ structure on its associated graded $\mathrm{gr}(Y)$. Conversely, given $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil})$

Definition 2.3. An object $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil})$ is called *$\mathbb{E}_n$-split* if it is in the essential image of I.

Given an $\mathbb{E}_n$-split filtered spectrum X, we can recover the underlying graded spectrum by taking the associated graded.

3 The (Segal-Mitchell-Richter?) Filtration on $\Omega SU(n)$

I believe Mitchell shows in [**?**] that the filtration is filtered $\mathbb{A}_\infty$. We need to check this.

Conjecture 3.1. *The filtration is* $\mathbb{E}_2$.

I guess now we know this is just true. We should probably thank Jacob for bringing to our attention that this conjecture of Mahowald is actually well-known by geometric representation theorists.

4 An $\mathbb{A}_\infty$-splitting by Weiss Calculus

In this section, we extend the methods of [**?**] to produce $\mathbb{A}_\infty$ stable splittings of Stiefel manifolds.

4.1 Weiss Calculus

Let $\mathcal{J}$ be the ∞-category which is the nerve of the topological category whose objects are finite dimensional complex vector spaces equipped with a Hermitian inner product and whose morphisms are spaces of linear isometries.

4.2 General splitting machinery

Let $[n]$ denote the linearly ordered set of integers $0 \leq i \leq n$. Define $\mathbf{Fil}_n = \mathrm{Fun}([n], \mathrm{Sp}^{\mathcal{J}})$ and $\mathbf{Cofil}_n = \mathrm{Fun}([n]^{\mathrm{op}}, \mathrm{Sp}^{\mathcal{J}})$. These categories admit functors to $\mathrm{Sp}^{\mathcal{J}}$ by taking colimit and limit, respectively. Let $\mathcal{C}_n = \mathbf{Fil}_n \times_{\mathrm{Sp}^{\mathcal{J}}} \mathbf{Cofil}_n$. Finally, let $\mathbf{Gr}_n = \mathrm{Fun}([n]^{\mathrm{ds}}, \mathrm{Sp}^{\mathcal{J}})$ where $[n]^{\mathrm{ds}}$ denotes the underlying discrete category. We have the following lemma:

Lemma 4.1. *For all integers $n > 0$, there is a fully faithful functor $i_n : \boldsymbol{Gr}_{n+1} \longrightarrow \mathcal{C}_n$.*

Proof. An element of $\mathcal{C}_n$ is given by a sequence of maps
The proof is by induction. □

We may then take inverse limits to get a category $\mathcal{C}_\infty = \mathbf{Fil}(\mathrm{Sp}^{\mathcal{J}}) \times_{\mathrm{Sp}^{\mathcal{J}}} \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$ and a functor $i : \mathbf{Gr} \longrightarrow \mathcal{C}_\infty$.

Corollary 4.1.1. *The functor i is fully faithful.*

Proof. □

at the end, restrict connectivity so that it's monoidal

5 An $\mathbb{E}_2$-splitting in Complex Cobordism

Let R denote any $\mathbb{E}_2$-ring spectrum with even-concentrated homotopy groups. I claim that any $\mathbb{A}_\infty$-morphism $\Omega SU(n) \to R$ lifts to an $\mathbb{E}_2$-morphism.

The argument is inspired by [?]

Proof. The obstructions live in $H_*(\Omega SU(n); \pi_{*+1}(R))$, and no matter the value of $\pi_{*+1}(R)$ this is 0. □

Corollary 5.0.1. *If R is the relevant split MU-$\mathbb{E}_2$-algebra, this yields an MU-$\mathbb{E}_2$-algebra equivalence $\Omega SU(n) \wedge MU \to R$ by adjunction from the $\mathbb{E}_2$-map $\Omega SU(n) \to R$.*

6 Obstructions to a general $\mathbb{E}_2$-splitting

Suppose there is a map $\Sigma^\infty_+ \Omega SU(n) \to \Sigma^\infty_+ \mathbb{CP}^{n-1}$. This is adjoint to a double loop map $\Omega SU(n) \to GL_1(\Sigma^\infty_+ \mathbb{CP}^{n-1})$ which lands in the component $SL_1(\Sigma^\infty_+ \mathbb{CP}^{n-1}) \simeq \Omega^2 \Sigma^2 \mathbb{CP}^{n-1}$. BLAH BLAH

7 Snaith's Construction of Periodic Complex Bordism

A classical theorem of Snaith [?] gives an equivalence of homotopy commutative ring spectra

$$\Sigma^\infty_+ BU[\beta^{-1}] \simeq MUP.$$

The equivalence arises from considering the total MU-Chern class map $BU \longrightarrow GL_1(MUP)$. It is known from [?] that the total Chern class in integral homology is not an infinite loop map. It follows from the existence of an $\mathbb{E}_\infty$ map $MUP \longrightarrow H\,\mathbb{Z}\,P$ from periodic complex bordism to periodic integral homology that Snaith's equivalence is not an equivalence of $\mathbb{E}_\infty$ ring spectra. The following theorem refines this observation:

Theorem 7.1. *The equivalence $\Sigma^\infty_+ BU[\beta^{-1}] \simeq MUP$ is $\mathbb{E}_2$ but not $\mathbb{E}_3$.*

Proof. Proof goes here □

Comment now about GepnerSnaith. We should cite at some point here or the introduction all of [?], [?], and the Snaith book with the original splitting.

8 Miscellaneous stuff here

It would be nice to at some point deal with showing the associated graded E_2 structure of BU is the thom spectrum VMU(n). I've directly pasted in some writing from a previous argument I claimed, but it definitely uses that $\coprod BU(n)$ is an E_2 algebra over $\mathbb{Z}_{\geq 0}$ which I never got straight an actual proof of.

Proposition 8.1. *The associated graded of $\Sigma^\infty_+ BU$ is E_2 equivalent to the Thom spectrum $\bigvee MU(n)$.*

Proof. Let $R = BU$ with its natural filtration, and let $R^\oplus = \coprod BU(n)$ with its natural filtration. Let M be the (E_∞) filtered spectrum which is $MU(n)$ in degree n, and all maps are 0. In other words, $\bigvee MU(n)$ with its natural filtration is $I(res(M))$.

We begin with a filtered E_∞ map $z : R^\oplus \longrightarrow I(res(M))$ coming from the zero section. Then, $R^\oplus$ comes with the structure of an E_2 algebra over $\mathbb{Z}^{fil}_{\geq 0}$. In fact, $I(res(M))$ has a trivial structure as an E_∞-algebra over $\mathbb{Z}^{fil}_{\geq 0}$ via the augmentation $\mathbb{Z}^{fil}_{\geq 0} \longrightarrow S^{0,fil} \longrightarrow I(res(M))$. We may then tensor z along the augmentation to get a map of E_2 filtered spectra $z' : R \longrightarrow I(res(M))$.

There is a canonical equivalence $I(res(M)) \otimes \mathbb{A} \simeq M$ because M is in the image of $\mathbb{A} \otimes I(-)$ (that is, all the maps in the filtration of M were zero). As such, M acquires a canonical structure as an $\mathbb{A}$ algebra such that the map $M \otimes \mathbb{A} \longrightarrow M$ is a map of E_2 rings (in fact I think it's E_∞?).

Finally, we observe that we may tensor z' with $\mathbb{A}$ and compose with the multiplication map to get an E_2 map $R \otimes A \longrightarrow M \otimes A \longrightarrow M$ which is the right thing up to homotopy, so it's an equivalence. $\square$

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What is $\Sigma^\infty_+ \Omega SU(n)[\beta^{-1}]$, by the way? Is it related to a periodic version of the $X(n)$-filtration of MU??