MULTIPLICATIVE STRUCTURE IN THE STABLE SPLITTING OF $\Omega SU(n)$

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Abstract. The space of based loops in $SU(n)$, also known as the affine Grassmannian of $SL_n(\mathbb{C})$, admits an $\mathbb{E}_2$ or fusion product. Work of Mitchell and Richter proves that this based loop space stably splits as an infinite wedge sum. We prove that the Mitchell–Richter splitting is $\mathbb{A}_\infty$, but not $\mathbb{E}_2$. Furthermore, we show that the splitting becomes $\mathbb{E}_2$ after base-change to complex cobordism. Our proofs involve on the one hand an analysis of the multiplicative properties of Weiss calculus, and on the other a use of Beilinson–Drinfeld Grassmannians to verify a conjecture of Mahowald and Richter.

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1. Introduction

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The space $\Omega SU(n)$ of based loops in $SU(n)$ is well-studied by both algebraic topologists and geometric representation theorists.

In algebraic topology, it was a long-standing conjecture of Mahowald, eventually proven by Mitchell and Richter [CM88, Theorem 2.1], that the suspension spectrum

$$\Sigma^\infty_+ \Omega SU(n) \simeq \mathbb{S} \vee \Sigma^\infty \mathbb{CP}^{n-1} \vee \cdots$$

splits as an infinite wedge sum. On the other hand, since $\Omega SU(n) \simeq \Omega^2 BSU(n)$ is a double loop space, its suspension spectrum is naturally an $\mathbb{E}_2$-ring spectrum (or, in other words, the affine Grasmannian of $SL_n(\mathbb{C})$ admits a factorization structure). It is the main purpose of our work here to study the interaction of the $\mathbb{E}_2$-ring structure with the splitting of Mitchell and Richter. Roughly speaking, we will prove that the splitting respects the underlying $\mathbb{A}_\infty$-ring structure, but does not respect the $\mathbb{E}_2$-ring structure before base change to complex cobordism.

As we will recall in much more detail in Sections 2 and 3 below, the theory of the Belinson-Drinfeld Grassmannian provides a natural $\mathbb{E}_2$ filtration of the space $\Omega SU(n)$. This *Schubert filtration* is indexed by the coweights of $SU(n)$, but restriction along the diagonal produces a coarser, integer-graded $\mathbb{E}_2$ filtration that Mitchell names the *Bott filtration* of $\Omega SU(n)$. Suspending, we obtain a Bott filtration of $\Sigma^\infty_+ \Omega SU(n)$ that forms an $\mathbb{E}_2$-algebra object in the symmetric monoidal category of filtered spectra. What we prove is as follows:

Theorem 1.1. *As an $\mathbb{A}_\infty$-algebra object in filtered spectra, the Bott filtration of $\Sigma^\infty_+ \Omega SU(n)$ is equivalent to its associated graded.*

Theorem 1.2. *As an $\mathbb{E}_2$-algebra object in filtered spectra, the Bott filtration of $\Sigma^\infty_+ \Omega SU(n)$ is not equivalent to its associated graded.*

Theorem 1.3. *As an $\mathbb{E}_2$-algebra object in filtered MU-module spectra, the Bott filtration of $MU \wedge \Sigma^\infty_+ \Omega SU(n)$ is equivalent to its associated graded.*

The final result above, regarding MU-module spectra, can be seen as a once-looped analogue of work of Kitchloo [Kit01]. Kitchloo studied a splitting, due to Miller [Mil85], of $\Sigma^\infty_+ SU(n)$. His theorem is that, *for complex-oriented E*, the corresponding direct sum decomposition of $E_*(SU(n))$ is multiplicative.

Our proof of Theorem 1.3 is by obstruction theory. We show in Section 7 that all obstructions to an $\mathbb{E}_2$-equivalence vanish. On the other hand, we prove Theorem 1.2 by explicitly calculating a non-zero obstruction in Section 9.

It remains to discuss Theorem 1.1, the $\mathbb{A}_\infty$ splitting. **SOMETHING ABOUT STIEFEL MANIFOLDS**

Our original interest in the subject arose from the limiting case $n \to \infty$. There, one considers a stable splitting of $\Omega SU \simeq BU$ due to Snaith [Sna79]. In particular, Snaith shows that

$$\Sigma^\infty_+ BU \simeq \bigvee MU(n),$$

where $MU(n)$ is the Thom spectrum of the canonical bundle over $BU(n)$. Inverting the Bott element, Snaith uses his splitting to note that $\Sigma^\infty_+ BU[\beta^{-1}]$ is equivalent to the periodic complex bordism spectrum MUP as a homotopy commutative ring spectrum.

It was our hope that Snaith's equivalence of homotopy commutative ring spectra could be promoted to an $\mathbb{E}_\infty$-equivalence, and so used to profitably study the $\mathbb{E}_\infty$-ring structure on MUP. Indeed, in the motivic setting Gepner and Snaith [GS09] use $\Sigma^\infty_+ BGL[\beta^{-1}]$ to *define* an $\mathbb{E}_\infty$-ring structure on $PMGL$. Though it is not made obvious in the literature, it was pointed out to us by Jacob Lurie that Snaith [Sna77] made a power operations computation precluding his equivalence from being $\mathbb{E}_\infty$. In the final Section **??** of this paper, we refine Snaith's results:

Theorem 1.4. *There is an equivalence of $\mathbb{E}_2$-ring spectra*

$$MUP \simeq \Sigma^\infty_+ BU[\beta^{-1}],$$

but $MUP \not\simeq \Sigma^\infty_+ BU[\beta^{-1}]$ as $\mathbb{E}_3$-ring spectra.

We end with two open questions regarding natural extensions of our work:

What is the structure of the equivariant splitting?

What is the proper motivic analogue of our result?

2. Filtered and Graded Ring Spectra

It will be important for us to have a precise language for discussing filtered and graded spectra, what it means to be split, what it means to take associated graded, and the multiplicative aspects of these constructions. Here we review a framework from [Lur15] for studying graded and filtered objects. The reader is referred to [Lur15] for a more thorough treatment and all proofs.

2.1. **First definitions.** Let $\mathcal{D}$ be an ∞-category which we will regard as the diagram category. Our filtered objects will be valued in the functor category $\mathrm{Sp}^{\mathcal{D}}$. This will be no more difficult than just ordinary spectra because limits, colimits, and smash products will be considered pointwise.

Denote by $\mathbb{Z}_{\geq 0}$ the poset of non-negative integers, denoted $[n]$, thought of as an ordinary category where $\mathrm{Hom}([a],[b])$ is a singleton if $a \leq b$, and empty otherwise. Denote by $\mathbb{Z}^{ds}_{\geq 0}$ the corresponding discrete category. We may then take nerves to obtain ∞-categories $N(\mathbb{Z}_{\geq 0})$ and $N(\mathbb{Z}^{ds}_{\geq 0})$, which will serve as the indexing sets for filtered and graded spectra. The reader is warned that our numbering conventions are opposite the ones in [Lur15].

Definition 2.1. Let $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ denote the functor category $\mathrm{Fun}(\mathbb{Z}^{ds}_{\geq 0}, \mathrm{Sp}^{\mathcal{D}})$. We shall refer to $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ as the category of graded objects in $\mathrm{Sp}^{\mathcal{D}}$. Its objects can be thought of as sequences $X_0, X_1, X_2, \cdots \in \mathrm{Sp}^{\mathcal{D}}$.

Definition 2.2. Let $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ denote the functor category $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathrm{Sp}^{\mathcal{D}})$. We shall refer to $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ as the category of filtered objects in $\mathrm{Sp}^{\mathcal{D}}$. Its objects can be thought of as sequences $Y_0 \to Y_1 \to Y_2 \to \cdots \in \mathrm{Sp}^{\mathcal{D}}$ filtering $\mathrm{colim}_i\, Y_i$.

The obvious map $N(\mathbb{Z}^{ds}_{\geq 0}) \to N(\mathbb{Z}_{\geq 0})$ induces a restriction functor $\mathrm{res} : \mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ which can be thought of as forgetting the maps in the filtered object. The restriction fits into an adjunction

$$I : \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) \rightleftharpoons \mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}) : \mathrm{res}$$

where the left adjoint $I : \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ is given by left Kan extension. The functor I can be described explicitly as taking a graded object $X_0, X_1, X_2, \cdots$ to the filtered object

$$I(X_0, X_1, \cdots) = (X_0 \to X_0 \oplus X_1 \to X_0 \oplus X_1 \oplus X_2 \to \cdots).$$

2.2. **Monoidal structures, I.** We now begin studying the monoidal structures on graded and filtered spectra. We confine ourselves to a basic discussion in this section, saving the finer details and definitions for the next subsection. By [Lur16, Example 2.2.6.17], the categories $\mathbf{Gr}(\mathrm{Sp})$ and $\mathbf{Fil}(\mathrm{Sp})$ may be given symmetric monoidal structures via the Day convolution. Then, via the identifications $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) = \mathbf{Gr}(\mathrm{Sp})^{\mathcal{D}}$ and $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}) = \mathbf{Fil}(\mathrm{Sp})^{\mathcal{D}}$, the categories $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ and $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ may be given symmetric monoidal structures pointwise on $\mathcal{D}$. In both cases, we denote the resulting operation by $\otimes$. Explicitly, the filtered tensor product

$$(X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots) \otimes (Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \cdots)$$

of two filtered spectra is computed as

$$X_0 \otimes Y_0 \longrightarrow \mathrm{colim} \left(\begin{tikzcd} X_0 \wedge Y_1 & \\ X_0 \wedge Y_0 \arrow[u] \arrow[r] & X_1 \wedge Y_0 \end{tikzcd} \right) \longrightarrow \mathrm{colim} \left(\begin{tikzcd} X_0 \wedge Y_2 & & \\ X_0 \wedge Y_1 \arrow[u] \arrow[r] & X_1 \wedge Y_1 & \\ X_0 \wedge Y_0 \arrow[u] \arrow[r] & X_1 \wedge Y_0 \arrow[u] \arrow[r] & X_2 \wedge Y_0 \end{tikzcd} \right) \longrightarrow \cdots .$$

For graded spectra, the analogous formula is:

$$(A_0, A_1, A_2, \cdots) \otimes (B_0, B_1, B_2, \cdots) \simeq \left(A_0 \wedge B_0, (A_1 \wedge B_0) \vee (A_0 \wedge B_1), \cdots, \bigvee_{i+j=n} A_i \wedge B_j, \cdots \right).$$

The unit $\S 0^{gr}_{\mathcal{D}}$ of $\otimes$ in $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ is the constant diagram at S^0 in degree 0 and $*$ otherwise; the unit $\S 0^{fil}_{\mathcal{D}}$ in $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ is $I\S 0^{gr}_{\mathcal{D}}$. We may then talk about $\mathbb{E}_n$-algebras in $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ and $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$.

There is also an associated graded functor $\mathrm{gr}\ : \mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ such that the composite $\mathrm{gr}\ \circ I : \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ is an equivalence. This can be thought of pointwise by the formula

$$\mathrm{gr}(X_0 \to X_1 \to X_2 \to \cdots) = X_0, X_1/X_0, X_2/X_1, \cdots .$$

The functors I and gr can be given symmetric monoidal structures such that the composite $\mathrm{gr}\ \circ I : \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ is a symmetric monoidal equivalence. It follows in particular that they extend to functors between the categories of $\mathbb{E}_n$-algebras in $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ and $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$. Thus, given an $\mathbb{E}_n$-algebra Y in filtered spectra, we obtain a canonical $\mathbb{E}_n$ structure on its associated graded $\mathrm{gr}(Y)$. Conversely, given $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}))$, we obtain $IX \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}))$.

Definition 2.3. An object $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}))$ is called $\mathbb{E}_n$*-split* if there exists $Y \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}))$ and an equivalence $X \simeq IY$ in $\mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}))$.

Given an $\mathbb{E}_n$-split filtered spectrum X, we can recover the underlying graded spectrum by taking the associated graded.

***Maybe it's worth making a remark about what these mean in terms of power operations, or say, the spectral sequence associated to the filtered object; maybe there's some sort of power operation on the E_2 page and you can say something about it...

2.3. **Monoidal structures, II.** Here we discuss some additional constructions and results that we will need for the more technical parts of this paper.

The monoidal structures on our categories will arise from Day convolution. This was studied for ∞-categories by Glasman [Gla13] and Lurie [Lur15, Lur16] at varying levels of generality. We will find it convenient to use the formulation from Section 2.2.6 of [Lur16].

Theorem 2.4 ([Lur16], Example 2.2.6.9)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories. Then there is an ∞-operad $Fun(\mathcal{C}, \mathcal{D})^{\otimes}$ with the following properties:*

1. *The underlying ∞-category of $Fun(\mathcal{C}, \mathcal{D})^{\otimes}$ is the functor category $Fun(\mathcal{C}, \mathcal{D})$.*
2. *The ∞-category $Alg_{\mathbb{E}_\infty}(Fun(\mathcal{C}, \mathcal{D})^{\otimes})$ of $\mathbb{E}_\infty$ algebras in $Fun(\mathcal{C}, \mathcal{D})^{\otimes}$ is equivalent to the category of lax symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.*

In order for the ∞-operad $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}$ to actually be a symmetric monoidal ∞-category, one needs to make additional assumptions.

Proposition 2.5 ([Lur16], Proposition 2.2.6.16)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories. Suppose that κ is an uncountable regular cardinal such that:*

1. *$\mathcal{C}$ is essentially κ-small.*
2. *$\mathcal{D}$ admits κ-small colimits.*
3. *The tensor product on $\mathcal{D}$ preserves κ-small colimits separately in each variable.*

Then $Fun(\mathcal{C}, \mathcal{D})^{\otimes}$ is a symmetric monoidal ∞-category.

Recall that the Day convolution is defined classically via left Kan extension. Assumptions (1) and (2) ensure that the relevant Kan extensions exist. Assumption (3) then ensures that the multiplication is associative by allowing the colimits taken in the formula for left Kan extension to commute with the tensor product.

As stated before, Proposition 2.5 is sufficient to construct symmetric monoidal ∞-categories $\mathbf{Fil}(\mathrm{Sp})$ and $\mathbf{Gr}(\mathrm{Sp})$. However, we wish to understand the interaction of the Weiss calculus with multiplicative structure; there, the filtrations go the other way.

Definition 2.6. Let $\mathcal{D}$ be an ∞-category. Let $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{D}})$ denote the functor category $\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathrm{Sp}^{\mathcal{D}})$. We shall refer to $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{D}})$ as the category of cofiltered objects in $\mathrm{Sp}^{\mathcal{D}}$. Its objects can be thought of as towers of functors $Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots \in \mathrm{Sp}^{\mathcal{D}}$.

We would like to make **Cofil**(Sp) a symmetric monoidal ∞-category by putting the Day convolution on its opposite, $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathrm{Sp}^{op})$. However, the smash product of spectra does not preserve small colimits separately in each variable. Nevertheless, it does preserve *finite* colimits separately in each variable. In fact, these are the only colimits that are needed in the case at hand and so we have the following variant of Proposition 2.5:

Variant 2.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories. Suppose that:*

(1) *For every $C \in \mathcal{C}$, the slice category $\mathcal{C}_{/C}$ is finite.*
(2) *$\mathcal{D}$ admits finite colimits.*
(3) *The tensor product on $\mathcal{D}$ preserves finite colimits separately in each variable.*

Then $Fun(\mathcal{C}, \mathcal{D})^{\otimes}$ is a symmetric monoidal ∞-category.

Proof. This follows directly from the same arguments as Proposition 2.5. In [Lur16, Corollary 2.2.6.14], the assumptions are used to guarantee the existence of a left Kan extension; this again exists by assumptions (1) and (2) and [Lur17, Lemma 4.3.2.13]. Similarly, the proof of [Lur16, Proposition 2.2.6.16] only makes reference to commuting tensor products in $\mathcal{D}$ with finite colimits, which is ensured by assumption (3). $\square$

In Section 4, we will need to consider not only the Day convolution monoidal structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ but its functoriality as $\mathcal{C}$ varies. For instance, we would for symmetric monoidal functors $\mathcal{C}_1 \to \mathcal{C}_2$ to induce symmetric monoidal functors $\mathrm{Fun}(\mathcal{C}_1, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_2, \mathcal{D})$ via left Kan extension.

We give a very close variant of [Nik16, Corollary 3.8] in our current framework:

Proposition 2.8. *Let $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{D}$ be symmetric monoidal ∞-categories such that the pairs $(\mathcal{C}_1, \mathcal{D})$ and $(\mathcal{C}_2, \mathcal{D})$ satisfy the hypotheses of Proposition 2.5 or of Variant 2.7. Let $f : \mathcal{C}_1 \to \mathcal{C}_2$ be a symmetric monoidal functor. Then there is an adjunction*

$$f_! : Fun(\mathcal{C}_1, \mathcal{D}) \rightleftharpoons Fun(\mathcal{C}_2, \mathcal{D}) : f_*$$

where f_* denotes restriction and $f_!$ denotes left Kan extension. Moreover, the functor f_* is lax symmetric monoidal and $f_!$ is symmetric monoidal.

Proof. The universal property of $\mathrm{Fun}(\mathcal{C}_1, \mathcal{D})^{\otimes}$ immediately implies the existence of a map of ∞-operads $\mathrm{Fun}(\mathcal{C}_2, \mathcal{D})^{\otimes} \to \mathrm{Fun}(\mathcal{C}_1, \mathcal{D})^{\otimes}$, which makes f_* a lax symmetric monoidal functor.

Assumptions (1) and (2) of Proposition 2.5 guarantee that the adjunction exists at the level of ∞-categories. The rest of the proof from [Nik16, Corollary 3.8] carries over verbatim. $\square$

3. The Bott filtration on $\Omega SU(n)$

In this section we recall the Mitchell–Segal Bott filtration [Mit87] on $\Omega SU(n)$. We prove that the Bott filtration is at least $\mathbb{A}_\infty$, meaning in particular that its suspension is an $\mathbb{A}_\infty$-filtered spectrum in the sense of Section 2.

It is most efficient to describe the filtration in the language of algebraic geometry, and in particular we will need to recall the theory of affine and Beilinson–Drinfeld Grassmannians. A good general reference is [Zhu16]. We use D to denote the formal disk $\mathrm{Spec}(\mathbb{C}[[t]])$ and D^* to denote the punctured disk $\mathrm{Spec}(\mathbb{C}((t)))$. For R a $\mathbb{C}$-algebra, we use D_R to denote $\mathrm{Spec}(\mathbb{C}[[t]]\hat{\otimes} R)$ and D_R^* to denote $\mathrm{Spec}(\mathbb{C}((t))\hat{\otimes} R)$.

Definition 3.1. Let G denote a smooth affine algebraic group over $\mathbb{C}$ (we will be interested only in the cases $G = SL_n, GL_n$). The *affine Grassmannian Gr_G* of G is the Ind-scheme with functor of points

$$Gr_G(R) = \{(\mathcal{E}, \beta)\}, \text{ where}$$

$\mathcal{E}$ is a G-torsor over D_R and $\beta : \mathcal{E}|_{D_R^*} \cong \mathcal{E}^0_{D_R^*}$ is a trivialization over D_R^*.

The complex points $Gr_G(\mathbb{C})$ are a model for the topological space ΩG. The idea is that $\mathrm{Hom}(D^*, G)$ is the space of algebraic free (i.e., unbased) loops in G. One thinks of the complex points of Gr_G as the homogeneous space

$$G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]),$$

which up to homotopy is the quotient of the free loop space on G by the action of G.

We use $\mathbb{X}^\bullet$ to denote the lattice of weights $\mathrm{Hom}(G, \mathbb{G}_m)$, and $\mathbb{X}_\bullet$ to denote the dual lattice of coweights. Inside $\mathbb{X}^\bullet$ is the set Φ of roots. We fix a particular Borel subgroup $B \subset G$, determining a choice of positive roots $\Phi^+ \subset \Phi$ and a semi-group of dominant coweights $\mathbb{X}_\bullet^+ \subset \mathbb{X}_\bullet$. There is a natural bijection

$$\mathbb{X}_\bullet^+ \cong G(\mathbb{C}[[t]])\backslash G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$$

of dominant coweights with the above double cosets. Each coweight $\mu \in \mathrm{Hom}(\mathbb{G}_m, G)$ may be thought of as a specific loop t^μ in the free loop space of G, and hence under projection as a point in ΩG.

There is a double-coset decomposition of the free loop space

$$\coprod_{\mu \in \mathbb{X}_\bullet^+} G(\mathbb{C}[[t]])t^\mu G(\mathbb{C}[[t]]).$$

Projecting onto the affine Grassmannian, one learns that the $G(\mathbb{C}[[t]])$-orbits of Gr_G are indexed by $\mu \in \mathbb{X}_\bullet^+$. We will use $Gr_{G,\leq \mu}$ to denote the *closure* of the orbit corresponding to μ. The closure $Gr_{G,\leq \mu_1}$ contains $Gr_{G,\leq \mu_2}$ if and only if $\mu_1 - \mu_2$ is a sum of dominant coroots. We call $\{Gr_{G,\leq \mu} | \mu \in \mathbb{X}_\bullet^+\}$ the *Schubert filtration* of Gr_G.

Example 3.2. Suppose $G = SL_2(\mathbb{C})$. Then a coweight $\mu \in \mathbb{X}_\bullet$ consists of a pair (a, b) of integers with $a+b = 0$. We choose a Borel so that a coweight is dominant if $a \geq b$. The conjugation action of $SL_2(\mathbb{C})$ on $\Omega SL_2(\mathbb{C})$ has one orbit for each pair $(a, -a)$ with $a \geq 0$. The orbit corresponding to $(a, -a)$ contains the loop $\mathbb{G}_m \to \Omega SL_2(\mathbb{C})$ given by

$$t \mapsto \begin{pmatrix} t^a & 0 \\ 0 & t^{-a} \end{pmatrix}.$$

The closure of the $(a, -a)$ orbit contains the $(b, -b)$ orbit if and only if $b \leq a$. To topologists, $\Omega SL_2(\mathbb{C}) \simeq \Omega\Sigma S^2$ is recognizable as the free $\mathbb{A}_\infty$-algebra on the pointed space S^2. In particular, $Gr_{SL_2}(\mathbb{C})$ is naturally equipped with the James filtration by word length. The closure of the $(a, -a)$ orbit turns out to be the $(2a)$th component of the James filtration, so that the Schubert filtration is strictly coarser than the James filtration. In other words, the S^2 that appears as the first James filtered piece of $\Omega SL_2(\mathbb{C})$ is not closed under the $SL_2(\mathbb{C})$ conjugation action. Only the collection of words of length 2 or less is closed under the $SL_2(\mathbb{C})$ action.

The $\mathbb{E}_2$-algebra structure on ΩG is elegantly encoded in algebraic geometry through the notion of Beilinson–Drinfeld Grassmannian:

Definition 3.3. The *Ran space* $\mathrm{Ran}(\mathbb{A}^1)$ is the presheaf that assigns to every $\mathbb{C}$-algebra R the set of non-empty finite subsets of $\mathrm{Spec}(R) \times \mathbb{A}^1$. The Beilinson–Drinfeld Grassmannian is the presheaf $Gr_{G,Ran}$ that assigns to each $\mathbb{C}$-algebra R the set of triplets $(x, \mathcal{E}, \beta)$, where $x \in \mathrm{Ran}(\mathbb{A}^1)(R)$, $\mathcal{E}$ is a G-torsor on $\mathbb{A}^1 \times \mathrm{Spec}(R)$, and β is a trivialization of $\mathcal{E}$ away from the graph of x in $\mathbb{A}^1 \times \mathrm{Spec}(R)$.

One thinks of the Beilinson–Drinfeld Grassmannian as fibered over the Ran space. In other words, for every collection of points $I \subset \mathbb{A}^1$, there is a corresponding point x in the Ran space. The fiber of the Beilinson–Drinfeld Grassmannian over x is the moduli of G-bundles on $\mathbb{A}^1$ equipped with a trivialization away from the points in I. This fiber is naturally isomorphic to the product of $|I|$ copies of Gr_G. The multiplication on Gr_G is encoded by degeneration of fibers as points collide in the Ran space. For more details, see [Zhu16, §3].

The connection of the above structure with the notion of $\mathbb{E}_2$-algebra in homotopy theory was spelled out explicitly by Jacob Lurie in [Lur16, §5.5]. In the language of Lurie's work, the complex points of the Beilinson–Drinfeld Grassmannian form a factorizable cosheaf, valued in spaces, on $Ran(\mathbb{C})$. Lurie proves [Lur16, Theorem 5.5.4.10] that this is enough to equip the complex points of Gr_G (namely ΩG) with the structure of a non-unital $\mathbb{E}_2$ algebra. This in turn makes $\Sigma^\infty_+\Omega G$ into a unital (in fact augmented) $\mathbb{E}_2$-ring spectrum.

It is through the Beilinson–Drinfeld perspective that we can most easily see the interaction of the Schubert filtration on Gr_G with its $\mathbb{E}_2$-algebra structure. The key point is the fact (see, e.g., [Zhu16, 3.1.14]) that, as points collide in the Beilinson–Drinfeld Grassmannian, the fiber $Gr_{G,\leq\mu_1} \times Gr_{G,\leq\mu_2}$ degenerates to $Gr_{G,\leq\mu_1+\mu_2}$.

That is already enough to prove that, for example, the Schubert filtration on $\Sigma^\infty_+\Omega SU(2)$ described in Example 3.2 is an $\mathbb{E}_2$-filtered spectrum in the sense of Section 2. What we will actually want to be $\mathbb{E}_2$, or at least $\mathbb{A}_\infty$, is the James filtration on $\Sigma^\infty_+\Omega SU(2)$. In general, it turns out that the Schubert filtration on the Beilinson–Drinfeld Grassmannian for $SL_n(\mathbb{C})$ provides only direct access to every nth piece of the Bott filtration on $\Sigma^\infty_+\Omega SL_n(\mathbb{C})$. We will follow Segal [Seg89] and access the Bott filtration on Gr_{SL_n} in a somewhat indirect manner, by considering not Gr_{SL_n} but Gr_{GL_n}:

Definition 3.4. Consider the affine Grassmannian Gr_{GL_n}. We denote by $F_{n,k}$ the subset of Gr_{GL_n} that is the closure of the $GL_n(\mathbb{C}[[t]])$ orbit containing:

$$t \mapsto \begin{pmatrix} t^k & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In other words, $F_{n,k} = Gr_{GL_n,\leq(k,0,\cdots,0)}$.

We draw the following lemma, affirming a conjecture of Mahowald and Richter, as an immediate corollary of the abstract machinary of Beilinson–Drinfeld [Zhu16, 3.1.14] and [Lur16, 5.5.4.10]:

Lemma 3.5 (Conjecture of Mahowald–Richter [MR93])**.** *The inclusion*

$$\coprod_k F_{n,k} \subset \Omega GL_n(\mathbb{C})$$

may be made into a map of non-unital $\mathbb{E}_2$-algebras.

4. A General Splitting Machine

Given a filtered spectrum

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots,$$

it will split if and only if there are maps going the other way:

$$X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots,$$

with the property that the relevant composites are equivalences. Motivated by this, one could ask: given an $\mathbb{E}_n$ filtered spectrum X, when is it $\mathbb{E}_n$-split? In this section, we answer this question by proving the following:

Theorem 4.1. *Let $X \in Alg_{\mathbb{E}_n}(\textbf{Fil})$ be an $\mathbb{E}_n$ filtered spectrum. Suppose there exists an $\mathbb{E}_n$ cofiltered spectrum $Y \in Alg_{\mathbb{E}_n}(\textbf{Cofil})$ with the following two properties:*

(1) *There is an equivalence* $\operatorname{colim} X \simeq \lim Y$ *of* $\mathbb{E}_n$*-algebras in spectra.*
(2) *The resulting natural maps* $X_i \to Y_i$ *are equivalences.*

Then, the filtered spectrum X *is* $\mathbb{E}_n$*-split.*

We will need a few preliminary definitions. We start by fixing a positive integer n. Let $[n]$ denote the linearly ordered set of integers $0 \leq i \leq n$. For any indexing 1-category $\mathcal{D}$, denote by $\mathcal{D}^{ds}$ the underlying discrete category, and denote by $\mathcal{D}^+$ the the category formed by formally adding a final object, which we will refer to as "+". Define $\mathbf{Fil}_n^+ = \mathrm{Fun}([n]^+, \mathrm{Sp})$ and $\mathbf{Cofil}_n^+ = \mathrm{Fun}(([n]^+)^{op}, \mathrm{Sp})$. These categories admit functors to Sp by restriction to the distinguished point. We define $\mathcal{C}_n$ by the following pullback:

$$\begin{tikzcd} \mathcal{C}_n \arrow[r] \arrow[d] & \mathbf{Cofil}_n^+ \arrow[d] \\ \mathbf{Fil}_n^+ \arrow[r] & \mathrm{Sp} \end{tikzcd}$$

An element of $\mathcal{C}_n$ can be thought of as a sequence of spectra connected by maps:

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \longrightarrow X \simeq Y \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0$$

where the middle arrow is an equivalence, as indicated. This is equivalent to just considering sequences

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \longrightarrow Z \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0,$$

and so we shall refer to general elements by these names below.

Define the subcategory $\mathcal{G}_n \subset \mathcal{C}_n$ as the full subcategory such that for each integer $0 \leq i \leq n$, the composite $X_i \longrightarrow Y_i$ is an equivalence.

Lemma 4.2. *There is an equivalence*

$$\mathcal{G}_n \simeq \boldsymbol{Gr}_n^+ := \mathit{Fun}(([n]^+)^{ds}, \mathit{Sp}),$$

where $([n]^+)^{ds}$ *is, as usual, the underlying set of* $[n]^+$.

Proof. We proceed by induction on n.

For $n = 0$, we are considering the full subcategory of diagrams $X_0 \to Z \to Y_0$ of spectra with the property that the composite is an equivalence. By taking the fiber of the second map, this is equivalent to the category of triples (X_0, Y_0', Z) of spectra together with an equivalence $X_0 \vee Y_0' \xrightarrow{\sim} Z$. This is certainly equivalent to the category of pairs (X_0, Y_0') of spectra, which is $\mathbf{Gr}_0^+$.

Next, assume the statement for $n \leq k$ and consider $\mathcal{G}_{k+1}$. We consider the auxiliary category $\overline{\mathcal{G}}_{k+1}$ which is the full subcategory of $\mathcal{C}_{k+1}$ where only $X_{k+1} \to Y_{k+1}$ is stipulated to be an equivalence. The argument for the base case shows that

$$\overline{\mathcal{G}}_{k+1} = \mathbf{Fil}_{k+1} \times_{\mathrm{Sp}} \mathbf{Gr}_0^+ \times_{\mathrm{Sp}} \mathbf{Cofil}_{k+1}$$

where the fiber products are over the restriction to $0 \in [0]^+$ for $\mathbf{Gr}_0^+$, and over X_{k+1} and Y_{k+1} in the filtered and cofiltered spectra. By commuting the fiber products, we find that

$$\overline{\mathcal{G}}_{k+1} = (\mathbf{Fil}_{k+1} \times_{\mathrm{Sp}} \mathbf{Cofil}_{k+1}) \times_{\mathrm{Sp}} \mathbf{Gr}_0^+ \simeq \mathcal{C}_k \times_{\mathrm{Sp}} \mathbf{Gr}_0^+$$

where we have implicitly used the identifications $\mathbf{Fil}_{k+1} \simeq \mathbf{Fil}_k^+$ and $\mathbf{Cofil}_{k+1} \simeq \mathbf{Cofil}_k^+$. Under this equivalence, the full subcategory $\mathcal{G}_{k+1} \subset \overline{\mathcal{G}}_{k+1}$ corresponds to $\mathcal{G}_k \times_{\mathrm{Sp}} \mathbf{Gr}_0^+ \simeq \mathbf{Gr}_{k+1}^+$ as desired.

□

In fact, the functor $\mathbf{Gr}_n^+ \to \mathcal{C}_n$ can be seen very explicitly as follows: theres a functor

$$I_n^+ : \mathbf{Gr}_n^+ \to \mathbf{Fil}_n^+$$

given by left Kan extension along the inclusion $([n]^+)^{ds} \to [n]^+$ which is completely analogous to the functor I described in Section 2. Dually, there's a functor

$$I_n^{op,+} : \mathbf{Gr}_n^+ \to \mathbf{Cofil}_n^+$$

given by right Kan extension along the inclusion $([n]^+)^{ds} \to ([n]^+)^{op}$ which sends an element $(X_0, X_1, \cdots, X_n, X) \in \mathbf{Gr}_n^+$ to

$$X_0 \longleftarrow X_0 \vee X_1 \longleftarrow \cdots \longleftarrow \bigvee_i X_i \longleftarrow X \vee \bigvee_i X_i.$$

These functors agree on restriction to the distinguished object, and so they define the desired functor $\mathbf{Gr}_n^+ \to \mathcal{C}_n$.

Until this point, we have been working with a fixed n and without regard to the monoidal structure. The key to moving further is the following theorem of Lurie:

Theorem 4.3 ([Lur15], Corollary 2.3.8)**.** *The construction which sends an ∞-category $\mathcal{D}$ to the functor category Fun($\mathcal{D}^{op}$, Sp) determines a symmetric monoidal functor from the ∞-category of ∞-categories to the ∞-category of presentable stable ∞-categories.*

In particular, this implies that if $\mathcal{D}$ is a symmetric monoidal category, then $\mathrm{Fun}(\mathcal{D}, \mathrm{Sp})$ has a natural symmetric monoidal structure, and for any symmetric monoidal functor $\mathcal{D}_1 \to \mathcal{D}_2$, left Kan extension yields a symmetric monoidal functor $\mathrm{Fun}(\mathcal{D}_1, \mathrm{Sp}) \to \mathrm{Fun}(\mathcal{D}_2, \mathrm{Sp})$.

To apply this to our situation, we give $[n]^+$ the structure of a symmetric monoidal category by taking $\mathbb{Z}_{\geq 0}$ under addition and identifying all the integers $m > n$ with the point $+$. There are natural symmetric monoidal functors $[n+1]^+ \to [n]^+$ by successive quotient. By Theorem 4.3 combined with the commutative diagram of symmetric monoidal functors

$$\begin{tikzcd} ([n+1]^+)^{ds} \arrow[r] \arrow[d] & {[n+1]^+} \arrow[d] \\ ([n]^+)^{ds} \arrow[r] & {[n]^+}, \end{tikzcd}$$

we obtain a commutative diagram of symmetric monoidal ∞-categories:

$$\begin{tikzcd} \mathbf{Gr}_{n+1}^+ \arrow[r, "I_{n+1}^+"] \arrow[d] & \mathbf{Fil}_{n+1}^+ \arrow[d] \\ \mathbf{Gr}_n^+ \arrow[r, "I_n^+"] & \mathbf{Fil}_n^+ . \end{tikzcd}$$

We remark that the right vertical functor coincides with the restriction induced by the natural inclusion $[n] \to [n+1]$. As a consequence, we find that $\lim_n \mathbf{Fil}_n^+ = \mathrm{Fun}(\mathrm{colim}_n [n]^+, \mathrm{Sp}) = \mathrm{Fun}(\mathbb{Z}_{\geq 0}^+, \mathrm{Sp}) =: \mathbf{Fil}^+$.

There is a similar diagram for the cofiltered side. (TODO, once we figure out how to say the cofiltered convolution)

Thus, taking the limit in n in the original picture yields a diagram of symmetric monoidal functors:

$$\begin{tikzcd}
\mathcal{G}_\infty \arrow[dr] \arrow[drr, bend left, "I^{+,op}"] \arrow[ddr, bend right, "I^+"'] & & \\
& \mathcal{C}_\infty \arrow[r] \arrow[d] & \mathbf{Cofil}^+ \arrow[d] \\
& \mathbf{Fil}^+ \arrow[r] & \mathrm{Sp}
\end{tikzcd}$$

where the square is Cartesian.

Remark 4.4. While $\mathcal{G}_\infty \to \mathcal{C}_\infty$ remains a fully faithful functor, we warn the reader that $\mathcal{G}_\infty$ is not simply $\mathrm{Fun}((\mathbb{Z}^+_{\geq 0})^{ds}, \mathrm{Sp})$ because the maps in the inverse system for $\mathcal{G}$ are not just the ones induced by the inclusions $([n]^+)^{ds} \to ([n+1]^+)^{ds}$.

Recall that we were interested in understanding when an $\mathbb{E}_n$ filtered spectrum $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil})$ is split - that is, when there exists $Z \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr})$ such that $X \simeq IZ$. The following proposition is key in relating that to our current situation; informally, it allows us to get rid of the $+$'s.

Proposition 4.5. *There exists a diagram of symmetric monoidal ∞-categories and symmetric monoidal functors*

$$\begin{tikzcd}
& \mathcal{G}_\infty \arrow[r, "\pi"] \arrow[d, "I^+"] & \boldsymbol{Gr} \arrow[d, "I"] \\
\boldsymbol{Fil} \arrow[r, "\iota"] & \boldsymbol{Fil}^+ \arrow[r, "\varpi"] & \boldsymbol{Fil}
\end{tikzcd}$$

where the bottom row is a retract and I^+ is induced by the I^+_n at each finite level.

Proof. We have already seen that I^+ is symmetric monoidal. For the bottom row, simply apply Theorem 4.3 to the sequence

$$\mathbb{Z}_{\geq 0} \to \mathbb{Z}^+_{\geq 0} \to \mathbb{Z}_{\geq 0}\,.$$

Finally, to see that π is symmetric monoidal, we claim that at each finite level n, the functor

$$\pi_n : \mathbf{Gr}^+_n \to \mathbf{Gr}_n$$

is symmetric monoidal. This is because it is localization to the full subcategory of $\mathbf{Gr}^+_n$ which restricts to 0 at the element $+ \in ([n]^+)^{ds}$, and the localization is clearly compatible with the monoidal structure (for example, see [Lur16, Proposition 2.2.1.9]). $\square$

We are now ready to prove the main result of this section.

Proof of Theorem 4.1. We use the notations of Proposition 4.5. Since ι is lax monoidal, we obtain an $\mathbb{E}_n$ algebra $\iota X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil}^+)$. Similarly, since the functor $\iota^{op} : \mathbf{Cofil} \to \mathbf{Cofil}^+$ is lax monoidal, we get $\iota^{op} Y \in \mathbf{Cofil}^+$. Condition (1) in the statement of the theorem guarantees that ιX and $\iota^{op} Y$ determine an element $\mathcal{X} \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}_\infty)$. Condition (2) combined with the fact that $\mathcal{G}_\infty \to \mathcal{C}_\infty$ is fully faithful implies that $\mathcal{X} \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbb{G}_\infty)$. Finally, we chase through the diagram of Proposition 4.5 to see that $I\pi\mathcal{X} \simeq \varpi I^+ \mathcal{X} \simeq \varpi\iota X \simeq X$ as $\mathbb{E}_n$ algebras in **Fil**. $\square$

5. Multiplicative Aspects of Weiss Calculus

In this section, we briefly review notions of Weiss calculus to set notation and then prove a statement about its multiplicative properties. The reader is referred to [Wei95] for proofs and additional details. We note that the discussion there is in the case of real vector spaces, but the results work just the same in the complex case. We shall also work in the language of ∞-categories rather than topological categories, and Remark 5.2 justifies this passage.

Let $\mathcal{J}$ be the ∞-category which is the nerve of the topological category whose objects are finite dimensional complex vector spaces equipped with a Hermitian inner product and whose morphisms are spaces of linear isometries.

Weiss calculus studies functors out of $\mathcal{J}$ in a way analogous to Goodwillie calculus, by understanding successive "polynomial approximations" to these functors. Here, we will discuss only the stable setting where we apply the theory to the functor category $\mathrm{Sp}^{\mathcal{J}}$. The central definition is:

Definition 5.1. A functor $F \in \mathrm{Sp}^{\mathcal{J}}$ is polynomial of degree n if the natural map

$$F(V) \to \lim_{U} F(U \oplus V)$$

is an equivalence, where the limit is indexed over the ∞-category of nonzero subspaces $U \subset \mathbb{C}^{n+1}$.

As in Goodwillie calculus, the inclusion of the full subcategory $\mathrm{Poly}^{\leq n}(\mathrm{Sp}^{\mathcal{J}}) \subset \mathrm{Sp}^{\mathcal{J}}$ of functors which are polynomial of degree n admits a left adjoint

$$P_n : \mathrm{Sp}^{\mathcal{J}} \rightleftharpoons \mathrm{Poly}^{\leq n}(\mathrm{Sp}^{\mathcal{J}}) : j_n.$$

The unit η_n of this adjunction provides for each $F \in \mathrm{Sp}^{\mathcal{J}}$ a natural transformation $F \to P_n F$ which we will refer to as the *degree n polynomial approximation* of F.

Remark 5.2. This universal property was not explicitly stated in [Wei95], but it follows formally from Weiss's results as follows: the functor P_n and the transformation η_n can be defined explicitly as in [Wei95] by iteratively applying the functor $\tau_n : \mathrm{Sp}^{\mathcal{J}} \to \mathrm{Sp}^{\mathcal{J}}$ defined by the formula

$$\tau_n F(V) = \lim_{U} F(U \oplus V)$$

with the limit indexed as in Definition 5.1. The facts required of the functors P_n in the proof of Theorem 6.1.1.10 in [Lur16] are precisely the content of Theorem 6.3 of [Wei95].

Given this universal property, Proposition 5.4 of [Wei95] ensures the existence of a natural Taylor tower

$$F \longrightarrow \cdots \longrightarrow P_n F \xrightarrow{p_{n-1}} P_{n-1} F \longrightarrow \cdots \longrightarrow P_0 F$$

living under any functor $F \in \mathrm{Sp}^{\mathcal{J}}$. The fiber $D_n F$ of p_{n-1} has the special property that it is polynomial of degree n and $P_{n-1} D_n F \simeq 0$. Such a functor is called *n-homogeneous*; such functors are completely classified by the following theorem:

Theorem 5.3 ([Wei95, Theorem 7.3])**.** *Let $F \in Sp^{\mathcal{J}}$. Then F is an n-homogeneous functor if and only if there exists a spectrum Θ with an action of the unitary group $U(n)$ such that*

$$F(V) = (\Theta \wedge S^{nV})_{hU(n)}.$$

The observation of Goodwillie, as exploited by [Aro01], is that this provides a canonical way to split certain functorial filtrations whose successive quotients are homogeneous. More precisely, we have the following theorem:

Theorem 5.4 ([Aro01])**.** *Suppose $F \in Sp^{\mathcal{J}}$ is a functor together with an increasing filtration*

$$0 = F^{(0)} \longrightarrow F^{(1)} \longrightarrow F^{(2)} \longrightarrow \cdots F$$

by functors $F^{(i)} \in Sp^{\mathcal{J}}$ with the property that the successive quotients $F^{(n)}/F^{(n-1)}$ are n-homogeneous for all integers $n > 0$. Then, each functor $F^{(n)}$ is polynomial of degree n and each composite $F^{(n-1)} \longrightarrow F^{(n)} \xrightarrow{\eta_{n-1}} P_{n-1}F^{(n)}$ is an equivalence.

In [Aro01], this result is applied to the functor $F \in \mathrm{Sp}^{\mathcal{J}}$ defined by the formula

$$F_V(W) = \Sigma^\infty_+ \Omega\, \mathcal{J}(V, V \oplus W)$$

where $V \in \mathcal{J}$ is a fixed finite dimensional complex vector space. Arone provides a filtration $F_V^{(0)}(W) \subset F_V^{(1)}(W) \subset \cdots F_V(W)$ which is functorial in both V and W, and which satisfies the constraints of Theorem 5.4 for fixed V. This provides a stable splitting of the space $\Omega\, \mathcal{J}(V, V \oplus W)$. Letting $W = \mathbb{C}$ and $V = \mathbb{C}^{n-1}$, he obtains splittings of the loop groups $\Omega SU(n)$, and for higher dimension W, this provides splittings of the loop spaces of Stiefel manifolds.

In order to upgrade the results of [Aro01] to structured multiplicative splittings, we must understand the multiplicative properties of the polynomial approximation functors. More precisely, for a functor $F \in \mathrm{Sp}^{\mathcal{J}}$, we aim to understand the Taylor tower of $F \wedge F$ in terms of the tower for F. The results in this section are likely known to experts, but the authors were not able to locate it in the literature. They thank Jacob Lurie for suggesting that Proposition 5.6 is true.

The idea is to consider all the polynomial approximations at once. The following construction makes this precise:

Construction 5.5. We now construct a functor

$$\mathrm{Tow} : \mathrm{Sp}^{\mathcal{J}} \to \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$$

with the property that it sends a functor $F \in \mathrm{Sp}^{\mathcal{J}}$ to its Taylor tower

$$\mathrm{Tow}(F) = P_0F \longleftarrow P_1F \longleftarrow P_2F \longleftarrow \cdots .$$

Recall that the P_n functors are given as left adjoints of the fully faithful inclusions $\mathrm{Poly}^{\le n}(\mathrm{Sp}^{\mathcal{J}}) \subset \mathrm{Sp}^{\mathcal{J}}$. We proceed by telling a parametrized version of this story that includes all n simultaneously. The proper framework for such a story is the formalism of *relative adjunctions*; these are developed in the ∞-categorical context in [Lur16], Section 7.3.2.

Consider the category $\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\ge 0}$ together with the full subcategory $(\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\ge 0})_{\mathrm{poly}} \subset \mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\ge 0}$ on the pairs $(F, [n])$ such that $F \in \mathrm{Poly}^{\le n}(\mathrm{Sp}^{\mathcal{J}})$. Via projection, these fit into a diagram

$$\begin{tikzcd}
\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\ge 0} \arrow[dr, "q"'] & & (\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\ge 0})_{\mathrm{poly}} \arrow[ll, "i"'] \arrow[dl, "p"] \\
& \mathbb{Z}^{op}_{\ge 0} &
\end{tikzcd}$$

This will be relevant to us because the category of sections of q are precisely $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$. The sections of p can be thought of those cofiltered functors such that the nth piece is polynomial of degree n. We will denote this category of sections of p by $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})_{\mathrm{poly}}$.

On the fibers over an integer $[n] \in \mathbb{Z}^{op}_{\ge 0}$, we see the inclusion $\mathrm{Sp}^{\mathcal{J}} \leftarrow \mathrm{Poly}^{\le n}(\mathrm{Sp}^{\mathcal{J}})$. It is in this sense that the current picture is a parametrized version of the ordinary polynomial approximations. We now claim that i admits a left adjoint $P^{\mathrm{total}} : \mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\ge 0} \to (\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\ge 0})_{\mathrm{poly}}$ *relative* to $\mathbb{Z}^{op}_{\ge 0}$. The strategy is to use Proposition 7.3.2.6 of [Lur16], which tells us that we need to check the following three statements:

(1) The functors p and q are locally Cartesian categorical fibrations.

(2) For each $[n] \in \mathbb{Z}_{\geq 0}^{op}$, the functor on fibers $i|_{p^{-1}[n]} : p^{-1}[n] \to q^{-1}[n]$ admits a right adjoint.
(3) The functor i carries locally p-Cartesian morphisms of $(\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op})_{\mathrm{poly}}$ to locally q-Cartesian morphisms of $\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op}$.

Condition (2) is clear from the existence of polynomial approximations in Weiss calculus. To see conditions (1) and (3), we first note that q is in fact a Cartesian fibration because it is a projection from a product. Moreover, the q-Cartesian morphisms are precisely those morphisms which are equivalences on the $\mathrm{Sp}^{\mathcal{J}}$ coordinate. Now suppose we are given a pair $(F, [m]) \in \mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op}$ such that $F \in \mathrm{Poly}^{\leq m}(\mathrm{Sp}^{\mathcal{J}})$ and morphism $\sigma : [n] \to [m]$. Any q-Cartesian edge lying over σ with target $(F, [m])$ has source equivalent to $(F, [n])$ and thus is also in the full subcategory $(\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op})_{\mathrm{poly}}$ because $m \leq n$. Since p is certainly an inner fibration (by construction as a full subcategory), this implies that p is also a Cartesian fibration and that the inclusion i carries p-Cartesian edges to q-Cartesian edges. Since any Cartesian fibration is a categorical fibration ([Lur17, Proposition 3.3.1.7]), conditions (1) and (3) are verified.

We now wish to look at the adjunction at the level of sections of q and p. Considering functors from $\mathbb{Z}_{\geq 0}^{op}$ into Diagram 5.5, we obtain a new diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op}) \arrow[rr, bend left, "P_*^{\mathrm{total}}"] \arrow[dr, "q_*"'] & & \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, (\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op})_{\mathrm{poly}}) \arrow[ll, "i_*"] \arrow[dl, "p_*"] \\
& \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathbb{Z}_{\geq 0}^{op}) &
\end{tikzcd}$$

which exhibits P_*^{total} as a left adjoint of i_* relative to $\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathbb{Z}_{\geq 0}^{op})$. Proposition 7.3.2.5 of [Lur16] ensures that there is an adjunction at the level of fibers above $\mathrm{id} \in \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathbb{Z}_{\geq 0}^{op})$:

$$\mathcal{P} : \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}}) \rightleftharpoons \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})_{\mathrm{poly}} : j.$$

Finally, observe that the unique functor $r : \mathbb{Z}_{\geq 0}^{op} \to *$ induces an adjunction

$$r^* : \mathrm{Sp}^{\mathcal{J}} \rightleftharpoons \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}}) : \lim$$

where r^* is the constant functor and lim is the same as right Kan extension along r. We now compose these adjunctions, denoting $\mathrm{Tow} = \mathcal{P} \circ r^*$ to obtain:

$$\mathrm{Tow} : \mathrm{Sp}^{\mathcal{J}} \rightleftharpoons \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})_{\mathrm{poly}} : \lim .$$

It remains to check that Tow actually recovers the Weiss tower. **TBD**. Maybe let's just say this: By construction, $\mathrm{Tow}(F)$ is the cofiltered spectrum

$$P_0 F \longleftarrow P_1 F \longleftarrow P_2 F \longleftarrow \cdots .$$

This concludes the construction of Tow.

The next task is to understand the monoidal structure on Tow. The idea is that we would like to express $\mathrm{Tow}(F \wedge F)$ in terms of $\mathrm{Tow}(F)$ and a "Day convolution" monoidal structure on $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$. TODO: actually define it here

We then have the following result about the monoidality of the Weiss tower functor.

Proposition 5.6. *The Weiss tower functor Tow defines a symmetric monoidal functor*

$$Tow : Sp^{\mathcal{J}} \to \boldsymbol{Cofil}(Sp^{\mathcal{J}}).$$

Proof. Recall that the Weiss tower functor was defined as a composite $\mathrm{Tow} = \mathcal{P} \circ r^*$. The functor r^* is just the constant functor, so it is symmetric monoidal. On the other hand, $\mathcal{P}$ is adjoint to the inclusion $j : \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})_{\mathrm{poly}} \to \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$. Since the class of functors which are polynomial of degree n is closed under finite limits, the subcategory $(\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}}))_{\mathrm{poly}}$ is closed under the convolution tensor product. We may therefore give it the structure of a symmetric monoidal ∞-category such that j is symmetric monoidal. This makes the left adjoint $\mathcal{P}$ an oplax monoidal functor, which induces an oplax monoidal structure on Tow.

Concretely, this oplax structure can be described on the nth filtered piece as follows: suppose $F, G \in \mathrm{Sp}^{\mathcal{J}}$; since $\mathrm{Tow}(F) \otimes \mathrm{Tow}(G) \in \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})_{\mathrm{poly}}$, the filtered piece $(\mathrm{Tow}(F) \otimes \mathrm{Tow}(G))_n$ is polynomial of degree n. It follows that the natural map from $F \wedge G$ factors through a map $\varphi_n : P_n(F \wedge G) \to (\mathrm{Tow}(F) \otimes \mathrm{Tow}(G))_n$. To see that Tow is a symmetric monoidal functor, it suffices to show that each φ_n is an equivalence. TODO

□

Remark 5.7. Proposition 5.6 is written in the language of Weiss calculus as that is the present application, but the proof works equally well in Goodwillie calculus.

6. Stable $\mathbb{A}_\infty$ Splittings

The main result of [Aro01] shows that the Mitchell-Richter filtration on $\Omega SU(n)$ (and more generally, for loops on a Stiefel manifold) stably splits. The key insight is that this filtration has extra structure: it is a particular value of a *functor* which has a natural filtration. The tool that allows for the exploitation of this structure is Weiss's theory of orthogonal or unitary calculus.

In this section, we extend the methods of [Aro01] to produce $\mathbb{A}_\infty$ stable splittings of Stiefel manifolds. We will begin this section by reviewing the theory of calculus introduced in [Wei95]. We then make a statement about the multiplicativity of the construction which ...

7. $\mathbb{E}_2$ Splittings in Complex Cobordism

In this brief section, we remark that the $\mathbb{A}_\infty$-splitting

$$\Sigma^\infty_+ \Omega SU(n) \simeq ???$$

becomes $\mathbb{E}_2$ after smashing with MU. More precisely, we show that there is an equivalence of $\mathbb{E}_2$-MU-algebras

$$MU \wedge \Sigma^\infty_+ \Omega SU(n) \simeq ???.$$

The $\mathbb{A}_\infty$-MU-algebra equivalence constructed in Section 6 is realized by a map of $\mathbb{A}_\infty$-$\mathbb{S}$-algebras

$$\Sigma^\infty \Omega SU(n) \longrightarrow ???. \tag{1}$$

Our task is to show that (1) may be refined to a morphism of $\mathbb{E}_2$-ring spectra. We do so by obstruction theory–the key fact powering our proof is that

$$MU_* \left(\Omega SU(n) \right) \cong 0$$

whenever $*$ is odd. **FIND A REFERENCE**. In fact, inspired by [CM15], we prove the following more general result:

Theorem 7.1. *Suppose that R is an $\mathbb{E}_2$-ring spectrum with no homotopy groups in odd degrees. Then any $\mathbb{A}_\infty$-ring homomorphism*

$$\Sigma^\infty_+ \Omega SU(n) \to R$$

lifts to a morphism of $\mathbb{E}_2$-ring spectra.

Proof. By taking connective covers, one learns that any $\mathbb{A}_\infty$-ring homomorphism

$$\Sigma^\infty_+ \Omega SU(n) \to R$$

must factor through the natural $\mathbb{E}_2$-algebra map $\tau_{\geq 0} R \to R$. Thus, without loss of generality we will assume that R is (-1)-connected.

It is clear that the composite $\mathbb{A}_\infty$-ring homomorphism

$$\Sigma^\infty_+ \Omega SU(n) \longrightarrow R \longrightarrow \tau_{\leq 0} R \simeq H\pi_0(R)$$

may be lifted to an $\mathbb{E}_2$-ring homomorphism factoring through $\tau_{\leq 0}\Sigma^\infty_+ \Omega SU(n) \simeq H\mathbb{Z}$. Suppose now for $q > 0$ that we have chosen an $\mathbb{E}_2$-ring homomorphism

$$\Sigma^\infty_+ \Omega SU(n) \longrightarrow \tau_{\leq q-1} R$$

lifting the given $\mathbb{A}_\infty$-algebra map

$$\Sigma^\infty_+ \Omega SU(n) \longrightarrow R \longrightarrow \tau_{\leq q-1} R.$$

We will show that there is no obstruction to the existence of a further $\mathbb{E}_2$-lift

$$\Sigma^\infty_+ \Omega SU(n) \longrightarrow \tau_{\leq q} R.$$

According to [CM15, Theorem 4.1], there is a diagram of principal fibrations

$$\begin{tikzcd}
\mathbb{E}_2\text{-Ring}(\Sigma^\infty_+ \Omega SU(n), \tau_{\leq q} R) \arrow[r] \arrow[d] & \mathbb{A}_\infty\text{-Ring}(\Sigma^\infty_+ \Omega SU(n), \tau_{\leq q} R) \arrow[d] \\
\mathbb{E}_2\text{-Ring}(\Sigma^\infty_+ \Omega SU(n), \tau_{\leq q-1} R) \arrow[r] \arrow[d] & \mathbb{A}_\infty\text{-Ring}(\Sigma^\infty_+ \Omega SU(n), \tau_{\leq q-1} R) \arrow[d] \\
\mathcal{S}_*(BSU(n), K(\pi_q R, q+3)) \arrow[r] & \mathcal{S}_*(SU(n), K(\pi_q R, q+2))
\end{tikzcd}$$

For q odd, $\tau_{\leq q-1} R \simeq \tau_{\leq q} R$, so there is no obstruction. Let us therefore assume that q is even.

Since the cohomology of $BSU(n)$ is even-concentrated with coefficients in any abelian group, we have that $\pi_0\, \mathcal{S}_*(BSU(n), K(\pi_q R, q+3)) \cong H^{q+3}(BSU(n); \pi_q R)$ is zero. It follows then that the given class

$$x \in \pi_0 \mathbb{E}_2\text{-Ring}(\Sigma^\infty_+ \Omega SU(n), \tau_{\leq q-1} R)$$

admits some lift

$$\widetilde{x} \in \mathbb{E}_2\text{-Ring}(\Sigma^\infty_+ \Omega SU(n), \tau_{\leq q} R).$$

We may need to modify $\widetilde{x}$ to match our chosen $\mathbb{A}_\infty$-ring homomorphism. This is always possible so long as the map

$$\pi_1(\mathcal{S}_*(BSU(n), K(\pi_q R, q+3))) \longrightarrow \pi_1(\mathcal{S}_*(SU(n), K(\pi_q R, q+2)))$$

is surjective. Said in other terms, this is just the map

$$H^{2q+2}(BSU(n); \pi_q R) \longrightarrow H^{2q+1}(SU(n); \pi_q R) \cong H^{2q+2}(\Sigma SU(n); \pi_q R)$$

induced by the natural map $\Sigma SU(n) \to BSU(n)$. It is a classical fact that this map is surjective (it follows from a calculation with the bar spectral sequence, using the fact that the cohomology of $SU(n)$ is exterior). **Maybe you can check this Allen**. $\square$

8. Square Zero Algebras

We will now discuss square zero extensions in our framework. For this, it will be convenient to work with the category $\mathbf{Gr}_u$ of *unital* graded spectra in the strong sense that the unit map induces an equivalence in grading 0. Note that there is a fully faithful functor $T : \mathrm{Sp} \to \mathbf{Gr}_u$ which sends a spectrum A to the graded spectrum

$$S^0, A, *, *, \cdots .$$

Its essential image is the full subcategory $i : \mathbf{Gr}_u^{\leq 1} \to \mathbf{Gr}_u$ consisting of unital graded spectra X such that X_k is contractible for $k > 1$. In this section, we analyze graded spectra in this subcategory $\mathbf{Gr}_u^{\leq 1}$. Our goal is to show any such graded spectrum admits an essentially unique $\mathbb{E}_n$-algebra structure for any $0 \leq n \leq \infty$. This goal is realized in Proposition 8.1.

The inclusion i fits into an adjunction

$$L^{\leq 1} : \mathbf{Gr}_u \rightleftharpoons \mathbf{Gr}_u^{\leq 1} : i$$

where the left adjoint $L^{\leq 1}$ can be thought of as truncating above grading 1. The localization $L^{\leq 1}$ is visibly compatible with the monoidal structure in the sense that for any $f : X \to Y$ in $\mathbf{Gr}_u$ such that $L^{\leq 1} f$ is an equivalence and any $Z \in \mathbf{Gr}_u$, the natural map $L^{\leq 1}(X \wedge Z) \to L^{\leq 1}(Y \wedge Z)$ is an equivalence. We are now in the situation of Proposition 2.2.1.9 of [Lur16], and so we may conclude that $\mathbf{Gr}_u^{\leq 1}$ inherits a symmetric monoidal structure such that $L^{\leq 1}$ is symmetric monoidal and the inclusion i is lax monoidal. This monoidal structure can be described explicitly by the formula

$$X \otimes_{\mathbf{Gr}_u^{\leq 1}} Y = L^{\leq 1}(X \otimes_{\mathbf{Gr}_u} Y).$$

We may then apply Remark 7.3.2.13 of [Lur16] to obtain an adjunction at the level of algebras for any integer $0 \leq n \leq \infty$:

$$L^{\leq 1}_{\mathrm{alg}} : \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}_u) \rightleftharpoons \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}_u^{\leq 1}) : i_{\mathrm{alg}}.$$

Since the counit $Li \to \mathrm{id}$ before lifting to algebras is an equivalence, we have that the counit $L^{\leq 1}_{\mathrm{alg}} i_{\mathrm{alg}} \to \mathrm{id}$ is also an equivalence. This implies in particular that i_{alg} is fully faithful. We are now in position to prove the main proposition of this section:

Proposition 8.1. *Let $0 \leq n \leq \infty$ be an integer. Then, there is a sequence of equivalences of categories*

$$Sp \xrightarrow{\bar{T}} \boldsymbol{Gr}_u^{\leq 1} \longrightarrow Alg_{\mathbb{E}_n}(\boldsymbol{Gr}_u^{\leq 1}) \longrightarrow Alg_{\mathbb{E}_n}(\boldsymbol{Gr}_u) \times_{\boldsymbol{Gr}_u} \boldsymbol{Gr}_u^{\leq 1}$$

where the first functor $\bar{T}$ is obtained by restricting the codomain of the functor $T : Sp \to \boldsymbol{Gr}_u$. In particular, for any $X \in \boldsymbol{Gr}_u^{\leq 1}$, the graded spectrum $iX \in \boldsymbol{Gr}_u$ has an essentially unique $\mathbb{E}_n$-algebra structure.

Proof. The third arrow is defined by i_{alg}, and is an equivalence because i_{alg} is fully faithful, so it remains to consider the first two arrows.

We have already seen that the functor $\bar{T} : \mathrm{Sp} \to \mathbf{Gr}_u^{\leq 1}$ is an equivalence of categories. However, it may be promoted to a symmetric monoidal equivalence when Sp is given the cocartesian monoidal structure - that is, the monoidal structure defined by $\vee$, the coproduct. This monoidal structure has a very special property: by Proposition 2.4.3.9 of [Lur16], there is for each n an equivalence $\mathrm{Sp} \simeq \mathrm{Alg}^{\vee}_{\mathbb{E}_n}(\mathrm{Sp})$, where the superscript $\vee$ indicates that we are considering algebras under the wedge. Informally, this says that any $Y \in \mathrm{Sp}$ admits an essentially unique $\mathbb{E}_n$-algebra structure under the coproduct. It follows that the same holds for any $X \in \mathbf{Gr}_u^{\leq 1}$, and so there is an equivalence $\mathbf{Gr}_u^{\leq 1} \to \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}_u^{\leq 1})$, as desired. $\square$

Terminology 8.2. Let $0 \leq n \leq \infty$ be an integer. By taking composing with the colimit functor, Proposition 8.1 provides a functor

$$\omega_n : \mathrm{Sp} \to \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})$$

which we will refer to as the square zero extension. It sends a spectrum X to a ring with underlying spectrum $S^0 \vee X$. We will call any $\mathbb{E}_n$-algebra structure produced via Proposition 8.1 or ω_n a *square zero* $\mathbb{E}_n$ structure.

Remark 8.3. For any $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}_u)$, we have a map $X \to i_{alg} L^{\leq 1}_{alg} X$ of $\mathbb{E}_n$-algebras. Taking colimits, we obtain a map $\operatorname{colim} X \to \operatorname{colim} i_{alg} L^{\leq 1}_{alg} X$ of $\mathbb{E}_n$ ring spectra. We may summarize this informally by saying that any $\mathbb{E}_n$-split ring spectrum X has an $\mathbb{E}_n$ map to the square zero extension determined by its degree one component X_1.

We will need to understand structured maps into square zero extensions. This amounts to understanding the space of units. In classical algebra, given a commutative ring A and an A-module M, the group of units of the square zero extension are given by the formula

$$(A \oplus M)^\times \simeq A^\times \times M.$$

A similar formula holds in our context for suspension spectra of connected spaces.

Proposition 8.4. *Let $0 \leq n \leq \infty$ be an integer and $X \in \mathcal{S}$ a connected space. There is a canonical equivalence*

$$GL_1(\omega_n(\Sigma^\infty X)) \simeq GL_1(S^0) \times QX$$

of $\mathbb{E}_n$-algebras in spaces, where QX is our notation for $\Omega^\infty \Sigma^\infty X$.

Proof. The functors ω_n are compatible under restriction, so it suffices to prove the statement for $n = \infty$. For this case, we will show that there is a splitting

$$gl_1(\omega_\infty(\Sigma^\infty X)) \simeq gl_1(S^0) \vee \Sigma^\infty X$$

of spectra, where gl_1 denotes the spectrum of units of an $\mathbb{E}_\infty$-ring introduced in [May77]. We first look at what happens on homotopy. Recall that for any $\mathbb{E}_\infty$ ring spectrum Y, we have the formula

$$\pi_*(gl_1(Y)) \simeq (\pi_*(Y))^\times$$

where on the right hand side, we consider $\pi_*(Y)$ as a graded ring. In our case, this yields an identification

$$\pi_*(gl_1(\omega_\infty(\Sigma^\infty X))) \simeq (\pi_*(S^0) \oplus \pi_*(\Sigma^\infty X))^\times \simeq \pi_*(S^0)^\times \times \pi_*(\Sigma^\infty X)$$

where we have used that on homotopy, $\omega_\infty(\Sigma^\infty X)$ is a square zero extension. To conclude the proof, it suffices to show that the two factors on the right hand side can be realized by maps of spectra.

The first factor is realized simply by gl_1 of the unit map $S^0 \to \omega_\infty(\Sigma^\infty X)$. In fact, it is not difficult to see directly that this map is split.

For the second factor, observe that since $\omega_\infty(\Sigma^\infty X)$ is an $\mathbb{E}_\infty$-ring, it receives a canonical $\mathbb{E}_\infty$ map

$$\Sigma^\infty_+ QX \longrightarrow \omega_\infty(\Sigma^\infty X)$$

from the free $\mathbb{E}_\infty$ ring on $\Sigma^\infty X$ which extends the canonical map of spectra $\Sigma^\infty X \to \omega_\infty(\Sigma^\infty X)$. Now, note that there is an adjunction [May77]

$$\Sigma^\infty_+ \Omega^\infty : \mathrm{Sp} \rightleftarrows \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp}) : gl_1$$

under which the above map may be identified with a map

$$b : \Sigma^\infty X \to gl_1(\omega_\infty(\Sigma^\infty X))$$

of spectra. **NEED TO SAY A TINY BIT MORE**

Finally, we may take the map $a \vee b : gl_1(S^0) \vee \Sigma^\infty X \to gl_1(\omega_\infty(\Sigma^\infty X))$ and the above comments show that it is an equivalence, as desired.

□

9. Obstructions to a General $\mathbb{E}_2$ Splitting

Let $3 < n \leq \infty$ be an integer. We will now show that the $\mathbb{A}_\infty$ splitting

$$\Sigma^\infty_+ \Omega SU(n) \simeq ???$$

cannot be promoted to an $\mathbb{E}_2$-splitting before smashing with MU.

Suppose that such a splitting existed. By Remark 8.3, we would obtain an $\mathbb{E}_2$-ring homomorphism $\Sigma^\infty_+ \Omega SU(n) \to \Sigma^\infty_+ \mathbb{CP}^{n-1}$, where $\Sigma^\infty_+ \mathbb{CP}^{n-1}$ is given the square-zero multiplication. Furthermore, the precomposition with the inclusion $\Sigma^\infty_+ \mathbb{CP}^{n-1} \longrightarrow \Sigma^\infty_+ \Omega SU(n)$ must yield the identity map. In particular, the map sends the generator of $\pi_2(\Sigma^\infty \mathbb{CP}^{n-1})$ to the generator of $\pi_2(\Sigma^\infty \Omega SU(n))$.

Recall now that there is an adjunction [May77]

$$\Sigma^\infty_+ : \textbf{Double Loop Spaces} \rightleftharpoons \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp}) : GL_1.$$

Using this, we may form the adjoint $\mathbb{E}_2$ map

$$\Omega SU(n) \to GL_1(\Sigma^\infty_+ \mathbb{CP}^{n-1}).$$

The right hand side is identified as an $\mathbb{E}_2$ algebra by Proposition 8.4. In particular, we obtain an $\mathbb{E}_2$ composite

$$\phi : \Omega SU(n) \to GL_1(\Sigma^\infty_+ \mathbb{CP}^{n-1}) \simeq GL_1(S^0) \times Q\,\mathbb{CP}^{n-1} \to Q\,\mathbb{CP}^{n-1}$$

which has the additional property that it is an isomorphism on π_2.

We now show that such a map ϕ cannot exist due to the operations that exist in the homotopy of an $\mathbb{E}_2$ algebra.

Observation 9.1. Let $Y \in \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})$, and suppose we are given a map $S^2 \to Y$. This extends to an $\mathbb{E}_2$ map $\Omega^2 S^4 \to Y$. We may precompose with the map $S^5 \to \Omega^2 S^4$ adjoint to the Hopf map $S^7 \to S^4$. This procedure determines a natural operation

$$\nu^u : \pi_2(Y) \to \pi_5(Y)$$

in the homotopy of any $\mathbb{E}_2$-algebra in spaces.

Remark 9.2. The notation is meant to hint at the fact that if $Y = \Omega^\infty X$ comes from a spectrum, then the operation ν^u is given by multiplication by the element $\nu \in \pi_3(\mathbb{S})^\wedge_2$ from the 2-primary homotopy groups of the sphere spectrum. Thus, ν^u is an unstable version of ν that is already seen in any $\mathbb{E}_2$ algebra in spaces.

Finally, we show that ϕ cannot be compatible with ν^u on homotopy by directly computing ν^u on either side.

For $n > 3$, observe that the natural map $\Omega SU(n) \to BU$ is an isomorphism in homology up to degree 7. This implies that $\pi_5(\Omega SU(n)) \simeq \pi_5(BU) \simeq 0$ because BU is even. Hence, ν^u is trivial on the generator of $\pi_2(\Omega SU(n))$.

Similarly, the map $Q\,\mathbb{CP}^{n-1} \to Q\,\mathbb{CP}^\infty$ is an isomorphism on π_5 for $n > 3$. However, it was computed in [Liu63, Theorem II.8] that $\pi_5(\mathbb{CP}^\infty) = \mathbb{Z}/2$ generated by ν times the degree 2 generator. Hence, by Remark 9.2, if $\beta \in \pi_2(Q\,\mathbb{CP}^{n-1})$ denotes the generator, then $\nu^u(\beta) \in \pi_5(Q\,\mathbb{CP}^{n-1})$ is nontrivial. This implies that there can be no $\mathbb{E}_2$ map ϕ which induces an isomorphism on π_2 and concludes the proof.

Remark 9.3. Taking the limit as $n \to \infty$, we obtain the statement that the map $BU \to Q\,\mathbb{CP}^\infty$ implementing the splitting principle does not lift to an $\mathbb{E}_2$ map. This map is well-studied: among other places, it appears as the first connecting map in the Weiss tower for the functor $V \mapsto BU(V)$. As such, it can be seen as a "BU-analog" to the Kahn-Priddy map.

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