

# Structured Splittings of $\Omega SU(n)$ and Snaith's Construction of Periodic Complex Bordism

Jeremy Hahn and Allen Yuan

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## Contents

### 1 Introduction

Write this last after we have filled in all the theorems with complete proofs.

### 2 Filtered and Graded Ring Spectra

Here we review a framework from [?] for studying graded and filtered objects. The reader is referred to [?] for a more thorough treatment and all proofs.

Let  $\mathcal{C}$  be a stable  $\infty$ -category. Denote by  $\mathbb{Z}_{\geq 0}$  the poset of non-negative integers, and by  $\mathbb{Z}_{\geq 0}^{ds}$  the corresponding discrete category. The reader is warned that our numbering conventions are opposite the ones in [?].

**Definition 2.1.** Let  $\mathrm{Gr}(\mathcal{C})$  denote the functor category  $\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{ds}, \mathcal{C})$ . We shall refer to  $\mathrm{Gr}(\mathcal{C})$  as the category of graded objects in  $\mathcal{C}$ . Its objects can be thought of as sequences  $X_0, X_1, X_2, \dots \in \mathcal{C}$ .

**Definition 2.2.** Let  $\mathrm{Fil}(\mathcal{C})$  denote the functor category  $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{C})$ . We shall refer to  $\mathrm{Fil}$  as the category of filtered spectra. Its objects can be thought of as sequences  $Y_0 \rightarrow Y_1 \rightarrow Y_2 \rightarrow \dots$  of spectra filtering  $\mathrm{colim}_i Y_i$ .

The obvious map  $\mathbb{Z}_{\geq 0}^{ds} \rightarrow \mathbb{Z}_{\geq 0}$  induces a restriction functor  $\mathrm{res} : \mathrm{Fil} \rightarrow \mathrm{Gr}$ . The restriction is right adjoint to a functor  $I : \mathrm{Gr} \rightarrow \mathrm{Fil}$  given by left Kan extension. The functor  $I$  can be thought of as taking a graded spectrum  $X_0, X_1, X_2, \dots$  to the filtered spectrum  $X_0 \rightarrow X_0 \oplus X_1 \rightarrow X_0 \oplus X_1 \oplus X_2 \rightarrow \dots$ . There is also an associated graded functor  $\mathrm{gr} : \mathrm{Fil} \rightarrow \mathrm{Gr}$  such that the composite  $\mathrm{gr} \circ I : \mathrm{Gr} \rightarrow \mathrm{Gr}$  is an equivalence.

The categories  $\mathrm{Gr}$  and  $\mathrm{Fil}$  are given symmetric monoidal structures via the Day convolution; we denote the operation in both cases by  $\otimes$ . The unit  $\mathbb{S}^{gr}$  of  $\otimes$  in graded spectra is  $S^0$  in degree 0 and  $*$  otherwise; the unit  $\mathbb{S}^{fil}$  in filtered spectra is  $I\mathbb{S}^{gr}$ .

The functors  $I$  and  $\mathrm{gr}$  can be given symmetric monoidal structures such that the composite  $\mathrm{gr} \circ I : \mathrm{Gr} \rightarrow \mathrm{Gr}$  is a symmetric monoidal equivalence. It follows in particular that they extend to functors on between the categories of  $\mathbb{E}_n$  algebras in  $\mathrm{Gr}$  and  $\mathrm{Fil}$ . Thus, given an  $\mathbb{E}_n$  algebra  $Y$  in filtered spectra, we obtain a canonical  $\mathbb{E}_n$  structure on its associated graded  $\mathrm{gr}(Y)$ . Conversely, given  $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Fil})$

**Definition 2.3.** An object  $X \in \text{Alg}_{\mathbb{E}_n}(\text{Fil})$  is called  $\mathbb{E}_n$ -split if it is in the essential image of  $I$ .

Given an  $\mathbb{E}_n$ -split filtered spectrum  $X$ , we can recover the underlying graded spectrum by taking the associated graded.

### 3 The (Segal-Mitchell-Richter?) Filtration on $\Omega SU(n)$

I believe Mitchell shows in [?] that the filtration is filtered  $\mathbb{A}_\infty$ . We need to check this.

**Conjecture 3.1.** *The filtration is  $\mathbb{E}_2$ .*

I guess now we know this is just true. We should probably thank Jacob for bringing to our attention that this conjecture of Mahowald is actually well-known by geometric representation theorists.

### 4 An $\mathbb{A}_\infty$ -splitting by Weiss Calculus

In this section, we extend the methods of [?] to produce  $\mathbb{A}_\infty$  stable splittings of Stiefel manifolds.

#### 4.1 Weiss Calculus

Let  $\mathcal{J}$  be the  $\infty$ -category which is the nerve of the topological category whose objects are finite dimensional complex vector spaces equipped with a Hermitian inner product and whose morphisms are spaces of linear isometries.

#### 4.2 General splitting machinery

Let  $[n]$  denote the linearly ordered set of integers  $0 \leq i \leq n$ . Define  $\text{Fil}_n = \text{Fun}([n], \text{Sp}^{\mathcal{J}})$  and  $\text{Cofil}_n = \text{Fun}([n]^{\text{op}}, \text{Sp}^{\mathcal{J}})$ . These categories admit functors to  $\text{Sp}^{\mathcal{J}}$  by taking colimit and limit, respectively. Let  $\mathcal{C}_n = \text{Fil}_n \times_{\text{Sp}^{\mathcal{J}}} \text{Cofil}_n$ . Finally, let  $\text{Gr}_n = \text{Fun}([n]^{\text{ds}}, \text{Sp}^{\mathcal{J}})$  where  $[n]^{\text{ds}}$  denotes the underlying discrete category. We have the following lemma:

**Lemma 4.1.** *For all integers  $n > 0$ , there is a fully faithful functor  $i_n : \text{Gr}_{n+1} \rightarrow \mathcal{C}_n$ .*

*Proof.* The proof is by induction. □

We may then take inverse limits to get a category  $\mathcal{C}_\infty = \text{Fil}(\text{Sp}^{\mathcal{J}}) \times_{\text{Sp}^{\mathcal{J}}} \text{Cofil}(\text{Sp}^{\mathcal{J}})$  and a functor  $i : \text{Gr} \rightarrow \mathcal{C}_\infty$ .

**Corollary 4.1.1.** *The functor  $i$  is fully faithful.*

*Proof.* □

at the end, restrict connectivity so that it's monoidal

## 5 An $\mathbb{E}_2$ -splitting in Complex Cobordism

Let  $R$  denote any  $\mathbb{E}_2$ -ring spectrum with even-concentrated homotopy groups. I claim that any  $\mathbb{A}_\infty$ -morphism  $\Omega SU(n) \rightarrow R$  lifts to an  $\mathbb{E}_2$ -morphism.

The argument is inspired by [?]

*Proof.* The obstructions live in  $H_*(\Omega SU(n); \pi_{*+1}(R))$ , and no matter the value of  $\pi_{*+1}(R)$  this is 0.  $\square$

**Corollary 5.0.1.** *If  $R$  is the relevant split  $MU$ - $\mathbb{E}_2$ -algebra, this yields an  $MU$ - $\mathbb{E}_2$ -algebra equivalence  $\Omega SU(n) \wedge MU \rightarrow R$  by adjunction from the  $\mathbb{E}_2$ -map  $\Omega SU(n) \rightarrow R$ .*

## 6 Obstructions to a general $\mathbb{E}_2$ -splitting

Suppose there is a map  $\Sigma_+^\infty \Omega SU(n) \rightarrow \Sigma_+^\infty \mathbb{C}P^{n-1}$ . This is adjoint to a double loop map  $\Omega SU(n) \rightarrow GL_1(\Sigma_+^\infty \mathbb{C}P^{n-1})$  which lands in the component  $SL_1(\Sigma_+^\infty \mathbb{C}P^{n-1}) \simeq \Omega^2 \Sigma^2 \mathbb{C}P^{n-1}$ . BLAH BLAH

## 7 Snaith's Construction of Periodic Complex Bordism

A classical theorem of Snaith [?] gives an equivalence of homotopy commutative ring spectra

$$\Sigma_+^\infty BU[\beta^{-1}] \simeq MUP.$$

The equivalence arises from considering the total  $MU$ -Chern class map  $BU \rightarrow GL_1(MUP)$ . It is known from [?] that the total Chern class in integral homology is not an infinite loop map. It follows from the existence of an  $\mathbb{E}_\infty$  map  $MUP \rightarrow H\mathbb{Z}P$  from periodic complex bordism to periodic integral homology that Snaith's equivalence is not an equivalence of  $\mathbb{E}_\infty$  ring spectra. The following theorem refines this observation:

**Theorem 7.1.** *The equivalence  $\Sigma_+^\infty BU[\beta^{-1}] \simeq MUP$  is  $\mathbb{E}_2$  but not  $\mathbb{E}_3$ .*

*Proof.* Proof goes here  $\square$

Comment now about GepnerSnaith. We should cite at some point here or the introduction all of [?], [?], and the Snaith book with the original splitting.

## 8 Miscellaneous stuff here

It would be nice to at some point deal with showing the associated graded  $E_2$  structure of  $BU$  is the thom spectrum  $VMU(n)$ . I've directly pasted in some writing from a previous argument I claimed, but it definitely uses that  $\coprod BU(n)$  is an  $E_2$  algebra over  $\mathbb{Z}_{\geq 0}$  which I never got straight an actual proof of.

**Proposition 8.1.** *The associated graded of  $\Sigma_+^\infty BU$  is  $E_2$  equivalent to the Thom spectrum  $\bigvee MU(n)$ .*

*Proof.* Let  $R = BU$  with its natural filtration, and let  $R^\oplus = \coprod BU(n)$  with its natural filtration. Let  $M$  be the  $(E_\infty)$  filtered spectrum which is  $MU(n)$  in degree  $n$ , and all maps are 0. In other words,  $\bigvee MU(n)$  with its natural filtration is  $I(res(M))$ .

We begin with a filtered  $E_\infty$  map  $z : R^\oplus \rightarrow I(res(M))$  coming from the zero section. Then,  $R^\oplus$  comes with the structure of an  $E_2$  algebra over  $\mathbb{Z}_{\geq 0}^{fil}$ . In fact,  $I(res(M))$  has a trivial structure as an  $E_\infty$ -algebra over  $\mathbb{Z}_{\geq 0}^{fil}$  via the augmentation  $\mathbb{Z}_{\geq 0}^{fil} \rightarrow S^{0,fil} \rightarrow I(res(M))$ . We may then tensor  $z$  along the augmentation to get a map of  $E_2$  filtered spectra  $z' : R \rightarrow I(res(M))$ .

There is a canonical equivalence  $I(res(M)) \otimes \mathbb{A} \simeq M$  because  $M$  is in the image of  $\mathbb{A} \otimes I(-)$  (that is, all the maps in the filtration of  $M$  were zero). As such,  $M$  acquires a canonical structure as an  $\mathbb{A}$  algebra such that the map  $M \otimes \mathbb{A} \rightarrow M$  is a map of  $E_2$  rings (in fact I think it's  $E_\infty$ ?).

Finally, we observe that we may tensor  $z'$  with  $\mathbb{A}$  and compose with the multiplication map to get an  $E_2$  map  $R \otimes \mathbb{A} \rightarrow M \otimes \mathbb{A} \rightarrow M$  which is the right thing up to homotopy, so it's an equivalence.  $\square$

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What is  $\Sigma_+^\infty \Omega SU(n)[\beta^{-1}]$ , by the way? Is it related to a periodic version of the  $X(n)$ -filtration of  $MU$ ??