STRUCTURED SPLITTINGS OF $\Omega SU(n)$ AND SNAITH'S CONSTRUCTION OF PERIODIC COMPLEX BORDISM

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Abstract. We show that the ??? filtration on the affine Grassmannian of $SL_n(\mathbb{C})$, known to topologists as the Bott filtration on $\Omega SU(n)$, stably splits as an $\mathbb{A}_\infty$ but not as an $\mathbb{E}_2$ filtration. We further prove that the splitting becomes $\mathbb{E}_2$ after smashing with any complex-oriented homology theory. As a limiting case, we study the coherence of Snaith's construction of periodic complex cobordism. We determine that it is a construction of MUP as an $\mathbb{E}_2$-ring spectrum, but not as an $\mathbb{E}_3$-ring spectrum.

Contents

1. Introduction

The complex cobordism spectrum MU has played a central role in homotopy theory ever since Quillen [?] connected its homotopy with the theory of one-dimensional formal group laws. As the Thom spectrum of the canonical bundle over BU, MU naturally acquires an immense amount of structure: it is an $\mathbb{E}_\infty$-ring spectrum. To this day, however, much remains unknown about the full nature of this $\mathbb{E}_\infty$ structure. For example, almost nothing is known about the k-invariants of the spectrum of units $gl_1(MU)$ [?].

As a spectrum, MU is classically constructed as the direct limit of the sequence

$$\mathbb{S} = MU(0) \longrightarrow \Sigma^{-2}MU(1) \longrightarrow \cdots \longrightarrow \Sigma^{-2n}MU(n) \longrightarrow \cdots,$$

where $MU(n)$ is the Thom spectrum of the canonical bundle over $BU(n)$.

This suggests that perhaps the more elemental object is

$$\bigvee MU(n),$$

the Thom spectrum of the J-homomorphism

$$\mathbf{Vect} \xrightarrow{J} \mathrm{Pic}(\mathbb{S})$$

that takes a vector space V to its one-point compactification $J(V) = S^V$. Since

$$J(V \oplus W) \simeq S^V \wedge S^W,$$

the Thom spectrum $\bigvee MU(n)$ is naturally an $\mathbb{E}_\infty$-ring spectrum. Inverting the Bott element $\beta \in \pi_2(MU(1)) \cong \pi_2(\mathbb{CP}^\infty)$, one obtains the *periodic* complex cobordism spectrum

$$MUP \simeq \left(\bigvee MU(n)\right)[\beta^{-1}].$$

This periodic spectrum MUP is a minor variation on MU itself–there is a wedge decomposition

$$MUP \simeq \bigvee_{a \in \mathbb{Z}} \Sigma^{2a} MU,$$

and the inclusion $MU \to MUP$ onto the $a = 0$ factor is an an $\mathbb{E}_\infty$-ring homomorphism.

In 1979, Victor Snaith [**?**] gave a second, and fundamentally different, presentation of periodic complex cobordism:

Theorem 1.1 (Snaith)**.** *As homotopy commutative ring spectra,*

$$MUP \simeq \Sigma^\infty_+ BU[\beta^{-1}].$$

More generally, there is an equivalence of homotopy commutative rings

$$\bigvee MU(n) \simeq \Sigma^\infty_+ BU.$$

Remark 1.2. In the above, $\Sigma^\infty_+ BU$ acquires an $\mathbb{E}_\infty$-structure (and hence a homotopy commutative ring structure) from the fact that BU is an infinite loop space. One may think of $\Sigma^\infty_+ BU$ as the group ring of the topological group BU.

The genesis of this paper was an attempt to use Snaith's theorem to study the $\mathbb{E}_\infty$-ring structure on MUP. This seemed like an especially reasonable idea in light of two facts:

(1) By another theorem of Snaith, periodic K-theory KU may be constructed as

$$KU \simeq \Sigma^\infty_+ \mathbb{CP}^\infty[\beta^{-1}].$$

This is an equivalence of $\mathbb{E}_\infty$-ring spectra **CITE**.

(2) In [GS09], Gepner and Snaith prove a motivic analogue of Theorem 1.1. In particular, they prove an equivalence

$$\Sigma^\infty_+ BGL[\beta^{-1}] \simeq PMGL.$$

They then use the natural $\mathbb{E}_\infty$-ring structure on $\Sigma^\infty_+ BGL[\beta^{-1}]$ to **define** an $\mathbb{E}_\infty$-ring structure on $PMGL$.

Allen, there is something I am confused about. In Gepner-Snaith, they claim that there is an $\mathbb{E}_\infty$ ring homomorphism from $PMGL$ to KU given by inverting β in the suspension of the determinant map

$$BU \to \mathbb{CP}^\infty.$$

I thought though that we decided there is no $\mathbb{E}_\infty$-ring map to $H\mathbb{Z}P$. What's up with that?

However, as it turns out (though it is not at all obvious from the modern literature and in particular not mentioned in [GS09]), another old theorem of Snaith [Sna77] shows that our idea was entirely unreasonable:

Theorem 1.3 (Snaith)**.** *As $\mathbb{E}_\infty$-rings,*

$$MUP \not\simeq \Sigma^\infty_+ BU[\beta^{-1}].$$

In the final Section 7 of this paper, we refine Snaith's results in to what we consider their definitive form:

Theorem 1.4. *There is an equivalence of* $\mathbb{E}_2$*-ring spectra*

$$MUP \simeq \Sigma^\infty_+ BU[\beta^{-1}],$$

but $MUP \not\simeq \Sigma^\infty_+ BU[\beta^{-1}]$ *as* $\mathbb{E}_3$*-ring spectra. There is an equivalence of* $\mathbb{A}_\infty$*-ring spectra*

$$\bigvee MU(n) \simeq \Sigma^\infty_+ BU,$$

but $\bigvee MU(n) \not\simeq \Sigma^\infty_+ BU$ *as* $\mathbb{E}_2$*-ring spectra.*

With the focus now on $\mathbb{E}_2$-algebras, it is natural to view Snaith's splitting result as the limiting case of a sequence of other stable splittings. Consider the filtration

$$* \simeq \Omega SU(1) \longrightarrow \Omega SU(2) \longrightarrow \cdots \longrightarrow \Omega SU(n) \longrightarrow \cdots \longrightarrow \Omega SU \simeq BU,$$

where the last equivalence is by Bott periodicity. Taking suspension spectra, we obtain a filtration of $\mathbb{E}_2$-ring spectra

$$\mathbb{S} \longrightarrow \Sigma^\infty_+ \Omega SU(2) \longrightarrow \cdots \longrightarrow \Sigma^\infty_+ \Omega SU(n) \longrightarrow \cdots \longrightarrow \Sigma^\infty_+ BU.$$

It is a theorem of Crabb and Mitchell [**?**] that, for $n > 1$, $\Sigma^\infty_+ \Omega SU(n)$ splits as an infinite wedge sum. We study the coherence of their splitting in Sections 4 and **BLAH**:

Theorem 1.5. *The Crabb–Mitchell stable splitting of* $\Sigma^\infty_+ \Omega SU(n)$ *is a splitting of* $\mathbb{A}_\infty$*-ring spectra, but not of* $\mathbb{E}_2$*-ring spectra.*

To be precise about what we mean by a splitting of $\mathbb{E}_n$-ring spectra, we need to introduce a bit of abstract terminology.

Remark 1.6. We freely use the language of ∞-categories throughout this paper, refering to an ∞-category simply as a category.

In Section 2 we will review the symmetric monoidal categories **Fil** and **Gr** of filtered and graded spectra, respectively. A filtered spectrum is an infinite sequence

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \cdots$$

of spectra. The tensor product

$$(X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots) \otimes (Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \cdots)$$

of two filtered spectra is computed as a Day convolution

$$X_0 \otimes Y_0 \longrightarrow \operatorname{colim} \left(\begin{array}{ccc} X_0 \wedge Y_1 & & \\ \uparrow & & \\ X_0 \wedge Y_0 & \longrightarrow & X_1 \wedge Y_0 \end{array} \right) \longrightarrow \operatorname{colim} \left(\begin{array}{ccccc} X_0 \wedge Y_2 & & & & \\ \uparrow & & & & \\ X_0 \wedge Y_1 & \longrightarrow & X_1 \wedge Y_1 & & \\ \uparrow & & \uparrow & & \\ X_0 \wedge Y_0 & \longrightarrow & X_1 \wedge Y_0 & \longrightarrow & X_2 \wedge Y_0 \end{array} \right) \longrightarrow \cdots.$$

A graded spectrum, on the other hand, is simply an ordered sequence $(A_0, A_1, A_2, \cdots)$ of spectra. The tensor product is computed as

$$(A_0, A_1, A_2, \cdots) \otimes (B_0, B_1, B_2, \cdots) \simeq \left(A_0 \wedge B_0, (A_1 \wedge B_0) \vee (A_0 \wedge B_1), \cdots, \bigvee_{i+j=n} A_i \wedge B_j, \cdots \right).$$

There is a sequence of symmetric monoidal functors

$$\mathbf{Gr} \xrightarrow{\quad I \quad} \mathbf{Fil} \xrightarrow{\quad \operatorname{colim} \quad} \mathrm{Sp},$$

where I sends the graded spectrum $(A_0, A_1, A_2, \cdots)$ to the filtered spectrum

$$I(A_0, A_1, A_2, \cdots) = (A_0 \longrightarrow A_0 \vee A_1 \longrightarrow A_0 \vee A_1 \vee A_2 \longrightarrow \cdots).$$

Definition 1.7. We say that an $\mathbb{E}_n$-ring spectrum is **$\mathbb{E}_n$-split** if it is equivalent to the image of an $\mathbb{E}_n$-algebra in **Gr** under the composite $\operatorname{colim} \circ I$. Similarly, a filtered $\mathbb{E}_n$-algebra is $\mathbb{E}_n$-split if it is equivalent to the image under I of an $\mathbb{E}_n$-algebra in **Gr**.

BLAH

In section 5 we will show:

Theorem 1.8. *The ??? filtration on $\Sigma^\infty_+ \Omega SU(n)$ is $\mathbb{E}_2$-split after smashing with MU.*

BLAH

2. Filtered and Graded Ring Spectra

Here we review a framework from [Lur15] for studying graded and filtered objects. The reader is referred to [Lur15] for a more thorough treatment and all proofs.

Let $\mathcal{C}$ be a stable ∞-category. Denote by $\mathbb{Z}_{\geq 0}$ the poset of non-negative integers, and by $\mathbb{Z}^{ds}_{\geq 0}$ the corresponding discrete category. The reader is warned that our numbering conventions are opposite the ones in [Lur15].

Definition 2.1. Let $\mathbf{Gr}(\mathcal{C})$ denote the functor category $\operatorname{Fun}(\mathbb{Z}^{ds}_{\geq 0}, \mathcal{C})$. We shall refer to $\mathbf{Gr}(\mathcal{C})$ as the category of graded objects in $\mathcal{C}$. Its objects can be thought of as sequences $X_0, X_1, X_2, \cdots \in \mathcal{C}$.

Definition 2.2. Let $\mathbf{Fil}(\mathcal{C})$ denote the functor category $\operatorname{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{C})$. We shall refer to **Fil** as the category of filtered objects in $\mathcal{C}$. Its objects can be thought of as sequences $Y_0 \to Y_1 \to Y_2 \to \cdots \in \mathcal{C}$ filtering $\operatorname{colim}_i Y_i$.

For us, the category $\mathcal{C}$ will always be either spectra or the category of functors from an indexing category to spectra. Because limits, colimits, and smash products of functors are taken pointwise, these cases are essentially the same. Thus, we will suppress $\mathcal{C}$ in what follows and refer to its objects as spectra.

The obvious map $\mathbb{Z}^{ds}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ induces a restriction functor $\operatorname{res} : \mathbf{Fil} \to \mathbf{Gr}$. The restriction is right adjoint to a functor $I : \mathbf{Gr} \to \mathbf{Fil}$ given by left Kan extension. The functor I can be thought of as taking a graded spectrum $X_0, X_1, X_2, \cdots$ to the filtered spectrum $X_0 \to X_0 \vee X_1 \to X_0 \vee X_1 \vee X_2 \to \cdots$. There is also an associated graded functor $\operatorname{gr} : \mathbf{Fil} \to \mathbf{Gr}$ such that the composite $\operatorname{gr} \circ I : \mathbf{Gr} \to \mathbf{Gr}$ is an equivalence.

The categories **Gr** and **Fil** are given symmetric monoidal structures via the Day convolution; we denote the operation in both cases by $\otimes$. The unit $\mathbb{S}^{gr}$ of $\otimes$ in graded spectra is S^0 in degree 0 and $*$ otherwise; the unit $\mathbb{S}^{fil}$ in filtered spectra is $I\,\mathbb{S}^{gr}$. We may then talk about $\mathbb{E}_n$-algebras in **Gr** and **Fil**.

The functors I and gr can be given symmetric monoidal structures such that the composite $\operatorname{gr} \circ I : \mathbf{Gr} \to \mathbf{Gr}$ is a symmetric monoidal equivalence. It follows in particular that they extend to functors between the categories of $\mathbb{E}_n$-algebras in **Gr** and **Fil**. Thus, given an $\mathbb{E}_n$ algebra Y in filtered spectra, we obtain a canonical $\mathbb{E}_n$ structure on its associated graded $\operatorname{gr}(Y)$. Conversely, given $X \in \operatorname{Alg}_{\mathbb{E}_n}(\mathbf{Gr})$, we obtain $IX \in \operatorname{Alg}_{\mathbb{E}_n}(\mathbf{Fil})$.

Definition 2.3. An object $X \in \operatorname{Alg}_{\mathbb{E}_n}(\mathbf{Fil})$ is called *$\mathbb{E}_n$-split* if there exists $Y \in \operatorname{Alg}_{\mathbb{E}_n}(\mathbf{Gr})$ and an equivalence $X \simeq IY$ of $\mathbb{E}_n$ filtered spectra.

Given an $\mathbb{E}_n$-split filtered spectrum X, we can recover the underlying graded spectrum by taking the associated graded.

2.1. **Square zero rings.** We will now discuss square zero extensions in our framework. For this, it will be convenient to work with the category $\mathbf{Gr}_u$ of *unital* graded spectra in the strong sense that the unit induces an equivalence in grading 0. Note that there is a fully faithful functor $T : \mathrm{Sp} \to \mathbf{Gr}_u$ which sends a spectrum A to the graded spectrum

$$S^0 \longrightarrow A \longrightarrow * \longrightarrow * \longrightarrow \cdots .$$

Its essential image is the full subcategory $i : \mathbf{Gr}_u^{\leq 1} \to \mathbf{Gr}_u$ consisting of unital graded spectra X such that X_i is contractible for $i > 1$. The inclusion i admits a left adjoint $L^{\leq 1}$ which can be thought of as truncating above grading 1. This localization is visibly compatible with the monoidal structure, and so $\mathbf{Gr}_u^{\leq 1}$ inherits a symmetric monoidal structure such that $L^{\leq 1}$ is symmetric monoidal and the inclusion i is lax monoidal. Furthermore, the restricted functor $\bar{T} : \mathrm{Sp} \to \mathbf{Gr}_u^{\leq 1}$ may be promoted to a symmetric monoidal equivalence where Sp is given the cocartesian monoidal structure. By Proposition 2.4.3.9 of [Lur17], any $Y \in \mathbf{Gr}_u \leq 1$ admits an essentially unique $\mathbb{E}_n$-algebra structure for any integer $0 \leq n \leq \infty$.

Definition 2.4. Let $Y \in \mathbf{Gr}_u \leq 1$. We will refer to $iY \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}_u)$ and $\operatorname{colim} iY \in \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})$ with their induced monoidal structures as having the *square zero* $\mathbb{E}_n$ structure.

In particular, for any $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}_u)$, we have a map $X \to iL^{\leq 1}X$ which by Remark 7.3.2.13 of [Lur17] is a map of $\mathbb{E}_n$-algebras when $iL^{\leq 1}X$ is given the induced monoidal structure. However, this is exactly the square zero structure by uniqueness. We may then take colimits to obtain a canonical $\mathbb{E}_n$ map $\operatorname{colim} X \to \operatorname{colim} iL^{\leq 1}X$. We may summarize this discussion informally by saying that any $\mathbb{E}_n$-split ring spectrum X has an $\mathbb{E}_n$ map to the square zero extension determined by X_1.

A nice feature of square zero $\mathbb{E}_n$ rings is that it is easy to understand their space of units.

Proposition 2.5. *Let $0 \leq n \leq \infty$ be an integer and $X \in Sp$. Give the spectrum $S^0 \vee X$ the square zero $\mathbb{E}_n$ structure. There is a canonical equivalence*

$$GL_1(S^0 \vee X) \simeq GL_1(S^0) \times \Omega^\infty X.$$

Proof. TBD...suffices to take $n = \infty$ by uniqueness. □

3. The (Segal-Mitchell-Richter?) Filtration on $\Omega SU(n)$

I believe Mitchell shows in [Mit86] that the filtration is filtered $\mathbb{A}_\infty$. We need to check this.

Conjecture 3.1. *The filtration is $\mathbb{E}_2$.*

I guess now we know this is just true. We should probably thank Jacob for bringing to our attention that this conjecture of Mahowald is actually well-known by geometric representation theorists.

4. An $\mathbb{A}_\infty$-splitting by Weiss Calculus

In this section, we extend the methods of [Aro01] to produce $\mathbb{A}_\infty$ stable splittings of Stiefel manifolds.

4.1. **Weiss Calculus.** Let $\mathcal{J}$ be the ∞-category which is the nerve of the topological category whose objects are finite dimensional complex vector spaces equipped with a Hermitian inner product and whose morphisms are spaces of linear isometries.

4.2. **General splitting machinery.** Let $[n]$ denote the linearly ordered set of integers $0 \leq i \leq n$. Define $\mathbf{Fil}_n = \mathrm{Fun}([n], \mathrm{Sp}^{\mathcal{J}})$ and $\mathbf{Cofil}_n = \mathrm{Fun}([n]^{\mathrm{op}}, \mathrm{Sp}^{\mathcal{J}})$. These categories admit functors to $\mathrm{Sp}^{\mathcal{J}}$ by taking colimit and limit, respectively. Let $\mathcal{C}_n = \mathbf{Fil}_n \times_{\mathrm{Sp}^{\mathcal{J}}} \mathbf{Cofil}_n$. Finally, let $\mathbf{Gr}_n = \mathrm{Fun}([n]^{\mathrm{ds}}, \mathrm{Sp}^{\mathcal{J}})$ where $[n]^{\mathrm{ds}}$ denotes the underlying discrete category. We have the following lemma:

Lemma 4.1. *For all integers $n > 0$, there is a fully faithful functor $i_n : \boldsymbol{Gr}_{n+1} \to \mathcal{C}_n$.*

Proof. An element of $\mathcal{C}_n$ is given by a sequence of functors

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \simeq Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0$$

where the middle □

We may then take inverse limits to get a category $\mathcal{C}_\infty = \mathbf{Fil}(\mathrm{Sp}^{\mathcal{J}}) \times_{\mathrm{Sp}^{\mathcal{J}}} \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$ and a functor $i : \mathbf{Gr} \to \mathcal{C}_\infty$.

Corollary 4.1.1. *The functor i is fully faithful.*

Proof. □

at the end, restrict connectivity so that it's monoidal

5. An $\mathbb{E}_2$-splitting in Complex Cobordism

In this brief section, we remark that the $\mathbb{A}_\infty$-splitting

$$\Sigma^\infty_+ \Omega SU(n) \simeq ???$$

becomes $\mathbb{E}_2$ after smashing with MU. More precisely, we show that there is an equivalence of $\mathbb{E}_2$-MU-algebras

$$MU \wedge \Sigma^\infty_+ \Omega SU(n) \simeq ???.$$

The $\mathbb{A}_\infty$-MU-algebra equivalence constructed in Section 4 is realized by a map of $\mathbb{A}_\infty$-$\mathbb{S}$-algebras

$$\Sigma^\infty \Omega SU(n) \longrightarrow ???. \tag{1}$$

Our task is to show that (1) may be refined to a morphism of $\mathbb{E}_2$-ring spectra. We do so by obstruction theory–the key fact powering our proof is that

$$MU_* \left(\Omega SU(n)\right) \cong 0$$

whenever $* = 0$. **FIND A REFERENCE**. In fact, inspired by [CM15], we prove the following more general result:

Theorem 5.1. *Suppose that R is an $\mathbb{E}_2$-ring spectrum with no homotopy groups in odd degrees. Then any $\mathbb{A}_\infty$-ring homomorphism*

$$\Sigma^\infty_+ \Omega SU(n) \to R$$

lifts to a morphism of $\mathbb{E}_2$-ring spectra.

Proof. By taking connective covers, one learns that any $\mathbb{A}_\infty$-ring homomorphism

$$\Sigma^\infty_+ \Omega SU(n) \to R$$

must factor through the natural $\mathbb{E}_2$-algebra map $\tau_{\geq 0} R \to R$. Thus, without loss of generality we will assume that R is (-1)-connected.

It is clear that the composite $\mathbb{A}_\infty$-ring homomorphism

$$\Sigma^\infty_+ \Omega SU(n) \longrightarrow R \longrightarrow \tau_{\leq 0} R \simeq H\pi_0(R)$$

may be lifted to an $\mathbb{E}_2$-ring homomorphism factoring through $\tau_{\le 0}\Sigma^\infty_+\Omega SU(n) \simeq H\mathbb{Z}$. Suppose now for $q > 0$ that we have chosen an $\mathbb{E}_2$-ring homomorphism

$$\Sigma^\infty_+\Omega SU(n) \longrightarrow \tau_{\le q-1}R$$

lifting the given $\mathbb{A}_\infty$-algebra map

$$\Sigma^\infty_+\Omega SU(n) \longrightarrow R \longrightarrow \tau_{\le q-1}R.$$

We will show that there is no obstruction to the existence of a further $\mathbb{E}_2$-lift

$$\Sigma^\infty_+\Omega SU(n) \longrightarrow \tau_{\le q}R.$$

According to [CM15, Theorem 4.1], there is a diagram of principal fibrations

$$\begin{tikzcd}
\mathbb{E}_2\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\le q}R) \arrow[r] \arrow[d] & \mathbb{A}_\infty\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\le q}R) \arrow[d] \\
\mathbb{E}_2\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\le q-1}R) \arrow[r] \arrow[d] & \mathbb{A}_\infty\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\le q-1}R) \arrow[d] \\
\mathcal{S}_*(BSU(n), K(\pi_q R, q+3)) \arrow[r] & \mathcal{S}_*(SU(n), K(\pi_q R, q+2))
\end{tikzcd}$$

For q odd, $\tau_{\le q-1}R \simeq \tau_{\le q}R$, so there is no obstruction. Let us therefore assume that q is even.

Since the cohomology of $BSU(n)$ is even-concentrated with coefficients in any abelian group, we have that $\pi_0\,\mathcal{S}_*(BSU(n), K(\pi_q R, q+3)) \cong H^{q+3}(BSU(n); \pi_q R)$ is zero. It follows then that the given class

$$x \in \pi_0\mathbb{E}_2\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\le q-1}R)$$

admits some lift

$$\widetilde{x} \in \mathbb{E}_2\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\le q}R).$$

We may need to modify $\widetilde{x}$ to match our chosen $\mathbb{A}_\infty$-ring homomorphism. This is always possible so long as the map

$$\pi_1(\mathcal{S}_*(BSU(n), K(\pi_q R, q+3))) \longrightarrow \pi_1(\mathcal{S}_*(SU(n), K(\pi_q R, q+2)))$$

is surjective. Said in other terms, this is just the map

$$H^{2q+2}(BSU(n); \pi_q R) \longrightarrow H^{2q+1}(SU(n); \pi_q R) \cong H^{2q+2}(\Sigma SU(n); \pi_q R)$$

induced by the natural map $\Sigma SU(n) \to BSU(n)$. It is a classical fact that this map is surjective (it follows from a calculation with the bar spectral sequence, using the fact that the cohomology of $SU(n)$ is exterior). **Maybe you can check this Allen**. $\square$

6. Obstructions to a General $\mathbb{E}_2$-splitting

We will now show that the $\mathbb{A}_\infty$ splitting

$$\Sigma^\infty_+\Omega SU(n) \simeq ???$$

cannot be promoted to an $\mathbb{E}_2$-splitting before smashing with complex-cobordism. According to ????, such a splitting would yield an $\mathbb{E}_2$-ring homomorphism $\Sigma^\infty_+\Omega SU(n) \to \Sigma^\infty_+\mathbb{CP}^{n-1}$, where $\Sigma^\infty_+\mathbb{CP}^{n-1}$ is given the square-zero multiplication. Furthermore, the precomposition with the inclusion $\Sigma^\infty_+\mathbb{CP}^{n-1} \longrightarrow \Sigma^\infty_+\Omega SU(n)$ must yield the identity map.

Recall now that there is an adjunction **CITE**

$$\Sigma^\infty_+ : \textbf{Double Loop Spaces} \leftrightarrows \mathbb{E}_2\text{-Rings} : GL_1.$$

Using this, we may form the adjoint double loop map

$$\Omega SU(n) \to GL_1(\Sigma^\infty_+\mathbb{CP}^{n-1}).$$

I'll wait to write this until we work out the previous sections, since I think it will be helpful to reference abstract nonsense about filtered stuff we prove earlier.

Suppose there is a map $\Sigma^\infty_+\Omega SU(n) \to \Sigma^\infty_+\mathbb{CP}^{n-1}$. This is adjoint to a double loop map $\Omega SU(n) \to GL_1(\Sigma^\infty_+\mathbb{CP}^{n-1})$ which lands in the component $SL_1(\Sigma^\infty_+\mathbb{CP}^{n-1}) \simeq \Omega^2\Sigma^2\mathbb{CP}^{n-1}$. BLAH BLAH

7. Snaith's Construction of Periodic Complex Bordism

A classical theorem of Snaith [Sna81] gives an equivalence of homotopy commutative ring spectra

$$\Sigma^\infty_+ BU[\beta^{-1}] \simeq MUP.$$

The equivalence arises from considering the total MU-Chern class map $BU \to GL_1(MUP)$. It is known from [Sna77] that the total Chern class in integral homology is not an infinite loop map. It follows from the existence of an $\mathbb{E}_\infty$ map $MUP \to H\mathbb{Z}P$ from periodic complex bordism to periodic integral homology that Snaith's equivalence is not an equivalence of $\mathbb{E}_\infty$ ring spectra. The following theorem refines this observation:

Theorem 7.1. *The equivalence $\Sigma^\infty_+ BU[\beta^{-1}] \simeq MUP$ is $\mathbb{E}_2$ but not $\mathbb{E}_3$.*

Proof. Proof goes here □

Comment now about GepnerSnaith. We should cite at some point here or the introduction all of [Sna77], [GS09], and the Snaith book with the original splitting.

8. Miscellaneous stuff here

It would be nice to at some point deal with showing the associated graded E_2 structure of BU is the thom spectrum VMU(n). I've directly pasted in some writing from a previous argument I claimed, but it definitely uses that $\coprod BU(n)$ is an E_2 algebra over $\mathbb{Z}_{\geq 0}$ which I never got straight an actual proof of.

Proposition 8.1. *The associated graded of $\Sigma^\infty_+ BU$ is E_2 equivalent to the Thom spectrum $\bigvee MU(n)$.*

Proof. Let $R = BU$ with its natural filtration, and let $R^\oplus = \coprod BU(n)$ with its natural filtration. Let M be the (E_∞) filtered spectrum which is $MU(n)$ in degree n, and all maps are 0. In other words, $\bigvee MU(n)$ with its natural filtration is $I(res(M))$.

We begin with a filtered E_∞ map $z : R^\oplus \to I(res(M))$ coming from the zero section. Then, $R^\oplus$ comes with the structure of an E_2 algebra over $\mathbb{Z}^{fil}_{\geq 0}$. In fact, $I(res(M))$ has a trivial structure as an E_∞-algebra over $\mathbb{Z}^{fil}_{\geq 0}$ via the augmentation $\mathbb{Z}^{fil}_{\geq 0} \to S^{0,fil} \to I(res(M))$. We may then tensor z along the augmentation to get a map of E_2 filtered spectra $z' : R \to I(res(M))$.

There is a canonical equivalence $I(res(M)) \otimes \mathbb{A} \simeq M$ because M is in the image of $\mathbb{A} \otimes I(-)$ (that is, all the maps in the filtration of M were zero). As such, M acquires a canonical structure as an $\mathbb{A}$ algebra such that the map $M \otimes \mathbb{A} \to M$ is a map of E_2 rings (in fact I think it's E_∞?).

Finally, we observe that we may tensor z' with $\mathbb{A}$ and compose with the multiplication map to get an E_2 map $R \otimes A \to M \otimes A \to M$ which is the right thing up to homotopy, so it's an equivalence. □

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What is $\Sigma^\infty_+\Omega SU(n)[\beta^{-1}]$, by the way? Is it related to a periodic version of the $X(n)$-filtration of MU??

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