MULTIPLICATIVE STRUCTURE IN THE STABLE SPLITTING OF $\Omega SU(n)$

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ABSTRACT. We show that the Schubert filtration on the affine Grassmannian of $SL_n(\mathbb{C})$, known to topologists as the Bott filtration on $\Omega SU(n)$, stably splits as an $\mathbb{A}_\infty$ but not as an $\mathbb{E}_2$ filtration. We further prove that the splitting becomes $\mathbb{E}_2$ after smashing with complex cobordism. As a limiting case, we study the coherence of Snaith's construction of periodic complex cobordism. We determine that it is a construction of MUP as an $\mathbb{E}_2$-ring spectrum, but not as an $\mathbb{E}_3$-ring spectrum.

CONTENTS

1. INTRODUCTION

The space $\Omega SU(n)$ of based loops in $SU(n)$ is well-studied by both algebraic topologists and geometric representation theorists.

In algebraic topology, it was a long-standing conjecture of Mahowald, eventually proven by Mitchell and Richter [CM88, Theorem 2.1], that the suspension spectrum

$$\Sigma^\infty_+ \Omega SU(n) \simeq \mathbb{S} \vee \Sigma^\infty \mathbb{CP}^{n-1} \vee \cdots$$

splits as an infinite wedge sum. On the other hand, since $\Omega SU(n) \simeq \Omega^2 BSU(n)$ is a double loop space, its suspension spectrum is naturally an $\mathbb{E}_2$-ring spectrum (or, in other words, the affine Grasmannian of $SL_n(\mathbb{C})$ admits a factorization structure). It is the main purpose of our work here to study the interaction of the $\mathbb{E}_2$-ring structure with the splitting of Mitchell and Richter. Roughly speaking, we will prove that the splitting respects the underlying $\mathbb{A}_\infty$-ring structure, but does not respect the $\mathbb{E}_2$-ring structure before base change to complex cobordism.

As we will recall in much more detail in Sections 2 and 3 below, the theory of the Belinson-Drinfeld Grassmannian provides a natural $\mathbb{E}_2$ filtration of the space $\Omega SU(n)$. This *Schubert filtration* is indexed by the coweights of $SU(n)$, but restriction along the diagonal produces a coarser, integer-graded $\mathbb{E}_2$ filtration that Mitchell names the *Bott filtration* of $\Omega SU(n)$. Suspending, we obtain a Bott filtration of $\Sigma^\infty_+ \Omega SU(n)$ that forms an $\mathbb{E}_2$-algebra object in the symmetric monoidal category of filtered spectra. What we prove is as follows:

Theorem 1.1. *As an $\mathbb{A}_\infty$-algebra object in filtered spectra, the Bott filtration of $\Sigma^\infty_+ \Omega SU(n)$ is equivalent to its associated graded.*

Theorem 1.2. *As an $\mathbb{E}_2$-algebra object in filtered spectra, the Bott filtration of $\Sigma^\infty_+ \Omega SU(n)$ is not equivalent to its associated graded.*

Theorem 1.3. *As an $\mathbb{E}_2$-algebra object in filtered MU-module spectra, the Bott filtration of $MU \wedge \Sigma^\infty_+ \Omega SU(n)$ is equivalent to its associated graded.*

The final result above, regarding MU-module spectra, can be seen as a once-looped analogue of work of Kitchloo [Kit01]. Kitchloo studied a splitting, due to Miller [Mil85], of $\Sigma^\infty_+ SU(n)$. His theorem is that, *for complex-oriented E*, the corresponding direct sum decomposition of $E_*(SU(n))$ is multiplicative.

Our proof of Theorem 1.3 is by obstruction theory. We show in Section 7 that all obstructions to an $\mathbb{E}_2$-equivalence vanish. On the other hand, we prove Theorem 1.2 by explicitly calculating a non-zero obstruction in Section 9.

It remains to discuss Theorem 1.1, the $\mathbb{A}_\infty$ splitting. **SOMETHING ABOUT STIEFEL MANIFOLDS**

Our original interest in the subject arose from the limiting case $n \to \infty$. There, one considers a stable splitting of $\Omega SU \simeq BU$ due to Snaith [Sna79]. In particular, Snaith shows that

$$\Sigma^\infty_+ BU \simeq \bigvee MU(n),$$

where $MU(n)$ is the Thom spectrum of the canonical bundle over $BU(n)$. Inverting the Bott element, Snaith uses his splitting to note that $\Sigma^\infty_+ BU[\beta^{-1}]$ is equivalent to the periodic complex bordism spectrum MUP as a homotopy commutative ring spectrum.

It was our hope that Snaith's equivalence of homotopy commutative ring spectra could be promoted to an $\mathbb{E}_\infty$-equivalence, and so used to profitably study the $\mathbb{E}_\infty$-ring structure on MUP. Indeed, in the motivic setting Gepner and Snaith [GS09] use $\Sigma^\infty_+ BGL[\beta^{-1}]$ to *define* an $\mathbb{E}_\infty$-ring structure on $PMGL$. Though it is not made obvious in the literature, it was pointed out to us by Jacob Lurie that Snaith [Sna77] made a power operations computation precluding his equivalence from being $\mathbb{E}_\infty$. In the final Section 10 of this paper, we refine Snaith's results:

Theorem 1.4. *There is an equivalence of $\mathbb{E}_2$-ring spectra*

$$MUP \simeq \Sigma^\infty_+ BU[\beta^{-1}],$$

but $MUP \not\simeq \Sigma^\infty_+ BU[\beta^{-1}]$ as $\mathbb{E}_3$-ring spectra.

We end with two open questions regarding natural extensions of our work:

What is the structure of the equivariant splitting?

What is the proper motivic analogue of our result?

2. Filtered and Graded Ring Spectra

It will be important for us to have a precise language for discussing filtered and graded spectra, what it means to be split, what it means to take associated graded, and the multiplicative aspects of these constructions. Here we review a framework from [Lur15] for studying graded and filtered objects. The reader is referred to [Lur15] for a more thorough treatment and all proofs.

Let $\mathcal{D}$ be an ∞-category which we will regard as the diagram category. Our filtered objects will be valued in the functor category $\mathrm{Sp}^{\mathcal{D}}$. This will be no more difficult than just ordinary spectra because limits, colimits, and smash products will be considered pointwise.

Denote by $\mathbb{Z}_{\geq 0}$ the poset of non-negative integers, denoted $[n]$, thought of as an ordinary category where $\mathrm{Hom}([a],[b])$ is a singleton if $a \leq b$, and empty otherwise. Denote by $\mathbb{Z}_{\geq 0}^{ds}$ the corresponding discrete category. We may then take nerves to obtain ∞-categories $N(\mathbb{Z}_{\geq 0})$ and $N(\mathbb{Z}_{\geq 0}^{ds})$, which will serve as the indexing sets for filtered and graded spectra. The reader is warned that our numbering conventions are opposite the ones in [Lur15].

Definition 2.1. Let $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ denote the functor category $\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{ds}, \mathrm{Sp}^{\mathcal{D}})$. We shall refer to $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ as the category of graded objects in $\mathrm{Sp}^{\mathcal{D}}$. Its objects can be thought of as sequences $X_0, X_1, X_2, \cdots \in \mathrm{Sp}^{\mathcal{D}}$.

Definition 2.2. Let $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ denote the functor category $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathrm{Sp}^{\mathcal{D}})$. We shall refer to $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ as the category of filtered objects in $\mathrm{Sp}^{\mathcal{D}}$. Its objects can be thought of as sequences $Y_0 \to Y_1 \to Y_2 \to \cdots \in \mathrm{Sp}^{\mathcal{D}}$ filtering $\mathrm{colim}_i\, Y_i$.

The obvious map $N(\mathbb{Z}_{\geq 0}^{ds}) \to N(\mathbb{Z}_{\geq 0})$ induces a restriction functor $\mathrm{res} : \mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ which can be thought of as forgetting the maps in the filtered object. The restriction fits into an adjunction

$$I : \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) \rightleftharpoons \mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}) : \mathrm{res}$$

where the left adjoint $I : \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ is given by left Kan extension. The functor I can be described explicitly as taking a graded object $X_0, X_1, X_2, \cdots$ to the filtered object

$$I(X_0, X_1, \cdots) = (X_0 \to X_0 \oplus X_1 \to X_0 \oplus X_1 \oplus X_2 \to \cdots).$$

2.1. **Monoidal structures.** By [Lur15, Corollary 2.3.9], the categories $\mathbf{Gr}(\mathrm{Sp})$ and $\mathbf{Fil}(\mathrm{Sp})$ may be given symmetric monoidal structures via the Day convolution. Then, via the identifications $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) = \mathbf{Gr}(\mathrm{Sp})^{\mathcal{D}}$ and $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}) = \mathbf{Fil}(\mathrm{Sp})^{\mathcal{D}}$, the categories $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ and $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ may be given symmetric monoidal structures pointwise on $\mathcal{D}$. In both cases, we denote the resulting operation by $\otimes$. Explicitly, the filtered tensor product

$$(X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots) \otimes (Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \cdots)$$

of two filtered spectra is computed as

$$X_0 \otimes Y_0 \longrightarrow \mathrm{colim}\begin{pmatrix} \begin{tikzcd}[ampersand replacement=\&] X_0 \wedge Y_1 \& \\ X_0 \wedge Y_0 \arrow[u] \arrow[r] \& X_1 \wedge Y_0 \end{tikzcd} \end{pmatrix} \longrightarrow \mathrm{colim}\begin{pmatrix} \begin{tikzcd}[ampersand replacement=\&] X_0 \wedge Y_2 \& \& \\ X_0 \wedge Y_1 \arrow[u] \arrow[r] \& X_1 \wedge Y_1 \& \\ X_0 \wedge Y_0 \arrow[u] \arrow[r] \& X_1 \wedge Y_0 \arrow[u] \arrow[r] \& X_2 \wedge Y_0 \end{tikzcd} \end{pmatrix} \longrightarrow \cdots .$$

For graded spectra, the analogous formula is:

$$(A_0, A_1, A_2, \cdots) \otimes (B_0, B_1, B_2, \cdots) \simeq \left(A_0 \wedge B_0, (A_1 \wedge B_0) \vee (A_0 \wedge B_1), \cdots, \bigvee_{i+j=n} A_i \wedge B_j, \cdots \right).$$

The unit $\mathbb{S}_{\mathcal{D}}^{gr}$ of $\otimes$ in $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ is the constant diagram at S^0 in degree 0 and $*$ otherwise; the unit $\mathbb{S}_{\mathcal{D}}^{fil}$ in $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$ is $I\,\mathbb{S}_{\mathcal{D}}^{gr}$. We may then talk about $\mathbb{E}_n$-algebras in $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ and $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$.

There is also an associated graded functor $\mathrm{gr} : \mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ such that the composite $\mathrm{gr} \circ I : \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ is an equivalence. This can be thought of pointwise by the formula

$$\mathrm{gr}(X_0 \to X_1 \to X_2 \to \cdots) = X_0, X_1/X_0, X_2/X_1, \cdots .$$

The functors I and gr can be given symmetric monoidal structures such that the composite $\mathrm{gr} \circ I : \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}) \to \mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ is a symmetric monoidal equivalence. It follows in particular that they extend to functors between the categories of $\mathbb{E}_n$-algebras in $\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}})$ and $\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}})$. Thus, given an $\mathbb{E}_n$-algebra Y in filtered spectra, we obtain a canonical $\mathbb{E}_n$ structure on its associated graded $\mathrm{gr}(Y)$. Conversely, given $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}))$, we obtain $IX \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}))$.

Definition 2.3. An object $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}))$ is called $\mathbb{E}_n$*-split* if there exists $Y \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}(\mathrm{Sp}^{\mathcal{D}}))$ and an equivalence $X \simeq IY$ in $\mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Fil}(\mathrm{Sp}^{\mathcal{D}}))$.

Given an $\mathbb{E}_n$-split filtered spectrum X, we can recover the underlying graded spectrum by taking the associated graded.

3. Filtrations on Loop Spaces of Stiefel Manifolds

I believe Mitchell shows in [Mit86] that the filtration is filtered $\mathbb{A}_\infty$.

Definition 3.1. Let $\mathbb{C}((t))$ denote $\mathbb{C}[[t]][t^{-1}]$. A *lattice* L is a finitely-generated $\mathbb{C}[[t]]$-submodule of $\mathbb{C}((t))^n$ such that $L \otimes_{\mathbb{C}[[t]]} \mathbb{C}((t)) = \mathbb{C}((t))^n$. A lattice L is *special* if $\Lambda^n L = \mathbb{C}[[t]] \subset \mathbb{C}((t))$. More generally, if R is any $\mathbb{C}$-algebra, an R-family of special lattices is a finitely generated projective $R[[t]]$-submodule of $R((t))^n$ satisfying similar properties.

The set of special lattices carries a natural topology modeling $\Omega SU(n)$. The notion of R-families of special lattices gives a functor of points description of the algebro-geometric affine Grassmannian, which is an Ind-scheme over $\mathbb{C}$. As with any affine Grassmannian, there is a natural filtration of this Ind-schemes by genuine schemes, indexed by the coweights of $SU(n)$:

4. A General Splitting Machine

Let $[n]$ denote the linearly ordered set of integers $0 \le i \le n$. Define $\mathbf{Fil}_n = \mathrm{Fun}([n], \mathrm{Sp}^{\mathcal{J}})$ and $\mathbf{Cofil}_n = \mathrm{Fun}([n]^{\mathrm{op}}, \mathrm{Sp}^{\mathcal{J}})$. These categories admit functors to $\mathrm{Sp}^{\mathcal{J}}$ by taking colimit and limit, respectively. Let $\mathcal{C}_n = \mathbf{Fil}_n \times_{\mathrm{Sp}^{\mathcal{J}}} \mathbf{Cofil}_n$. Finally, let $\mathbf{Gr}_n = \mathrm{Fun}([n]^{\mathrm{ds}}, \mathrm{Sp}^{\mathcal{J}})$ where $[n]^{\mathrm{ds}}$ denotes the underlying discrete category. We have the following lemma:

Lemma 4.1. *For all integers $n > 0$, there is a fully faithful functor $i_n : \boldsymbol{Gr}_{n+1} \to \mathcal{C}_n$.*

Proof. An element of $\mathcal{C}_n$ is given by a sequence of functors

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \simeq Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0$$

where the middle □

We may then take inverse limits to get a category $\mathcal{C}_\infty = \mathbf{Fil}(\mathrm{Sp}^{\mathcal{J}}) \times_{\mathrm{Sp}^{\mathcal{J}}} \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$ and a functor $i : \mathbf{Gr} \to \mathcal{C}_\infty$.

Corollary 4.1.1. *The functor i is fully faithful.*

Proof. □

at the end, restrict connectivity so that it's monoidal

5. Multiplicative Aspects of Weiss Calculus

In this section, we will briefly review notions of Weiss calculus to set notation and then prove a statement about its multiplicative properties. The reader is referred to [Wei95] for proofs and additional details. We note that the discussion there is in the case of real vector spaces, but the results work just the same in the complex case. We shall also work in the language of ∞-categories rather than topological categories, and Remark 5.2 justifies this passage.

Let $\mathcal{J}$ be the ∞-category which is the nerve of the topological category whose objects are finite dimensional complex vector spaces equipped with a Hermitian inner product and whose morphisms are spaces of linear isometries.

The theory of Weiss calculus studies functors out of $\mathcal{J}$ in a way analogous to Goodwillie calculus, by understanding successive "polynomial approximations" to these functors. Here, we will discuss only the stable setting where we apply the theory to the functor category $\mathrm{Sp}^{\mathcal{J}}$. The central definition is:

Definition 5.1. A functor $F \in \mathrm{Sp}^{\mathcal{J}}$ is polynomial of degree at most n if the natural map

$$F(V) \to \lim_{U} F(U \oplus V)$$

is an equivalence, where the limit is indexed over the ∞-category of nonzero subspaces $U \subset \mathbb{C}^{n+1}$.

As in Goodwillie calculus, the inclusion of the full subcategory $\mathrm{Poly}^{\leq n}(\mathrm{Sp}^{\mathcal{J}}) \subset \mathrm{Sp}^{\mathcal{J}}$ of functors which are polynomial of degree at most n admits a left adjoint

$$P_n : \mathrm{Sp}^{\mathcal{J}} \rightleftharpoons \mathrm{Poly}^{\leq n}(\mathrm{Sp}^{\mathcal{J}}) : j_n.$$

The unit η_n of this adjunction provides for each $F \in \mathrm{Sp}^{\mathcal{J}}$ a natural transformation $F \to P_n F$ which we will refer to as the *degree n polynomial approximation* of F.

Remark 5.2. This universal property was not explicitly stated in [Wei95], but it follows formally from Weiss's results as follows: the functor P_n and the transformation η_n can be defined explicitly as in [Wei95] by iteratively applying the functor $\tau_n : \mathrm{Sp}^{\mathcal{J}} \to \mathrm{Sp}^{\mathcal{J}}$ defined by the formula

$$\tau_n F(V) = \lim_{U} F(U \oplus V)$$

with the limit indexed as in Definition 5.1. The facts required of the functors P_n in the proof of Theorem 6.1.1.10 in [Lur16] are precisely the content of Theorem 6.3 of [Wei95].

Given this universal property, Proposition 5.4 of [Wei95] ensures the existence of a natural Taylor tower

$$F \longrightarrow \cdots \longrightarrow P_n F \xrightarrow{p_{n-1}} P_{n-1} F \longrightarrow \cdots \longrightarrow P_0 F$$

living under any functor $F \in \mathrm{Sp}^{\mathcal{J}}$. The fiber $D_n F$ of p_{n-1} has the special property that it is polynomial of degree at most n and $P_{n-1} D_n F \simeq 0$. Such a functor is called *n-homogeneous*; such functors are completely classified by the following theorem:

Theorem 5.3 ([Wei95, Theorem 7.3])**.** *Let $F \in Sp^{\mathcal{J}}$. Then F is an n-homogeneous functor if and only if there exists a spectrum Θ with an action of the unitary group $U(n)$ such that*

$$F(V) = (\Theta \wedge S^{nV})_{hU(n)}.$$

The observation of Goodwillie, as exploited by [Aro01], is that this provides a canonical way to split certain functorial filtrations whose successive quotients are homogeneous. More precisely, we have the following theorem:

Theorem 5.4 ([Aro01])**.** *Suppose $F \in Sp^{\mathcal{J}}$ is a functor together with an increasing filtration*

$$0 = F^{(0)} \longrightarrow F^{(1)} \longrightarrow F^{(2)} \longrightarrow \cdots F$$

by functors $F^{(i)} \in Sp^{\mathcal{J}}$ with the property that the successive quotients $F^{(n)}/F^{(n-1)}$ are n-homogeneous for all integers $n > 0$. Then, each functor $F^{(n)}$ is polynomial of degree at most n and each composite $F^{(n-1)} \longrightarrow F^{(n)} \xrightarrow{\eta_{n-1}} P_{n-1} F^{(n)}$ is an equivalence.

In [Aro01], this result is applied to the functor $F \in \mathrm{Sp}^{\mathcal{J}}$ defined by the formula

$$F_V(W) = \Sigma^{\infty}_{+} \Omega \mathcal{J}(V, V \oplus W)$$

where $V \in \mathcal{J}$ is a fixed finite dimensional complex vector space. Arone provides a filtration $F_V^{(0)}(W) \subset F_V^{(1)}(W) \subset \cdots F_V(W)$ which is functorial in both V and W, and which satisfies the constraints of Theorem 5.4 for fixed V. This provides a stable splitting of the space $\Omega \mathcal{J}(V, V \oplus$

W). Letting $W = \mathbb{C}$ and $V = \mathbb{C}^{n-1}$, we obtain splittings of the loop groups $\Omega SU(n)$, and for higher dimension W, this provides splittings of the loop spaces of Stiefel manifolds.

In order to upgrade the results of [Aro01] to structured multiplicative splittings, we must understand the multiplicative properties of the polynomial approximation functors. More precisely, for a functor $F \in \mathrm{Sp}^{\mathcal{J}}$, we aim to understand the Taylor tower of $F \wedge F$ in terms of the tower for F. The results in this section are likely known to experts, but the authors were not able to locate it in the literature. They thank Jacob Lurie for suggesting that Proposition 5.8 is true.

The idea is to consider all the polynomial approximations at once. To do so, we first set some additional notation:

Definition 5.5. Let $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$ denote the functor category $\mathrm{Fun}(\mathbb{Z}^{op}_{\geq 0}, \mathrm{Sp}^{\mathcal{J}})$. We shall refer to $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$ as the category of cofiltered objects in $\mathrm{Sp}^{\mathcal{J}}$. Its objects can be thought of as towers of functors $Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots \in \mathrm{Sp}^{\mathcal{J}}$.

The category $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$ is the natural target for the Weiss tower. The following construction makes this precise:

Construction 5.6. We now construct a functor

$$\mathrm{Tow} : \mathrm{Sp}^{\mathcal{J}} \to \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$$

with the property that it sends a functor $F \in \mathrm{Sp}^{\mathcal{J}}$ to its Taylor tower

$$\mathrm{Tow}(F) = P_0 F \longleftarrow P_1 F \longleftarrow P_2 F \longleftarrow \cdots .$$

Recall that the P_n functors are given as left adjoints of the fully faithful inclusions $\mathrm{Poly}^{\leq n}(\mathrm{Sp}^{\mathcal{J}}) \subset \mathrm{Sp}^{\mathcal{J}}$. We proceed by telling a parametrized version of this story that includes all n simultaneously. The proper framework for such a story is the formalism of *relative adjunctions*; these are developed in the ∞-categorical context in [Lur16], Section 7.3.2.

Consider the category $\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\geq 0}$ together with the full subcategory $(\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\geq 0})_{\mathrm{poly}} \subset \mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\geq 0}$ on the pairs $(F, [n])$ such that $F \in \mathrm{Poly}^{\leq n}(\mathrm{Sp}^{\mathcal{J}})$. Via projection, these fit into a diagram

$$\begin{tikzcd}
\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\geq 0} \arrow[dr, "q"'] & & (\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\geq 0})_{\mathrm{poly}} \arrow[ll, "i"'] \arrow[dl, "p"] \\
& \mathbb{Z}^{op}_{\geq 0} &
\end{tikzcd}$$

This will be relevant to us because the category of sections of q are precisely $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$. The sections of p can be thought of those cofiltered functors such that the nth piece is polynomial of degree at most n. We will denote this category of sections of p by $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})_{\mathrm{poly}}$.

On the fibers over an integer $[n] \in \mathbb{Z}^{op}_{\geq 0}$, we see the inclusion $\mathrm{Sp}^{\mathcal{J}} \leftarrow \mathrm{Poly}^{\leq n}(\mathrm{Sp}^{\mathcal{J}})$. It is in this sense that the current picture is a parametrized version of the ordinary polynomial approximations. We now claim that i admits a left adjoint $P^{\mathrm{total}} : \mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\geq 0} \to (\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\geq 0})_{\mathrm{poly}}$ *relative* to $\mathbb{Z}^{op}_{\geq 0}$. The strategy is to use Proposition 7.3.2.6 of [Lur16], which tells us that we need to check the following three statements:

(1) The functors p and q are locally Cartesian categorical fibrations.
(2) For each $[n] \in \mathbb{Z}^{op}_{\geq 0}$, the functor on fibers $i|_{p^{-1}[n]} : p^{-1}[n] \to q^{-1}[n]$ admits a right adjoint.
(3) The functor i carries locally p-Cartesian morphisms of $(\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\geq 0})_{\mathrm{poly}}$ to locally q-Cartesian morphisms of $\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}^{op}_{\geq 0}$.

Condition (2) is clear from the existence of polynomial approximations in Weiss calculus. To see conditions (1) and (3), we first note that q is in fact a Cartesian fibration because it is a projection from a product. Moreover, the q-Cartesian morphisms are precisely those morphisms which are equivalences on the $\mathrm{Sp}^{\mathcal{J}}$ coordinate. We now observe that for any pair $(F,[m]) \in \mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op}$ such that $F \in \mathrm{Poly}^{\leq m}(\mathrm{Sp}^{\mathcal{J}})$ and morphism $\sigma : [n] \to [m]$, any q-Cartesian edge lying over σ with target $(F,[m])$ is also in the full subcategory $(\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op})_{\mathrm{poly}}$. This implies that p is also a Cartesian fibration and that the inclusion i carries p-Cartesian edges to q-Cartesian edges. Since any Cartesian fibration is a categorical fibration ([Lur17, Proposition 3.3.1.7]), conditions (1) and (3) are verified.

We now wish to look at the adjunction at the level of sections of q and p. Considering functors from $\mathbb{Z}_{\geq 0}^{op}$ into Diagram 5.6, we obtain a new diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op}) \arrow[rr, bend left, "P_*^{\mathrm{total}}"] \arrow[dr, "q_*"'] & & \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, (\mathrm{Sp}^{\mathcal{J}} \times \mathbb{Z}_{\geq 0}^{op})_{\mathrm{poly}}) \arrow[ll, "i_*"] \arrow[dl, "p_*"] \\
& \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathbb{Z}_{\geq 0}^{op}) &
\end{tikzcd}$$

which exhibits P^{total} as a left adjoint of i_* relative to $\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathbb{Z}_{\geq 0}^{op})$. Proposition 7.3.2.5 of [Lur16] ensures that there is an adjunction at the level of fibers above $\mathrm{id} \in \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathbb{Z}_{\geq 0}^{op})$:

$$\mathrm{Tow}^* : \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}}) \rightleftharpoons \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})_{\mathrm{poly}} : j.$$

Finally, observe that the unique functor $r : \mathbb{Z}_{\geq 0}^{op} \to *$ induces an adjunction

$$r^* : \mathrm{Sp}^{\mathcal{J}} \rightleftharpoons \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}}) : \lim$$

where r^* is the constant functor and lim is the same as right Kan extension along r. We now compose these adjunctions, denoting $\mathrm{Tow} = \mathrm{Tow}^* \circ r^*$ to obtain:

$$\mathrm{Tow} : \mathrm{Sp}^{\mathcal{J}} \rightleftharpoons \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})_{\mathrm{poly}} : \lim .$$

It remains to check that Tow actually recovers the Weiss tower. TBD

This concludes the construction of Tow.

The next task is to understand the monoidal structure on Tow. The idea is that we would like to express $\mathrm{Tow}(F \wedge F)$ in terms of $\mathrm{Tow}(F)$ and a "Day convolution" monoidal structure on $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$. However, there is trouble defining the convolution as in Section 2.1 because smash product does not preserve *limits* of spectra in each variable separately. The situation becomes better if one restricts to the category $\mathrm{Sp}_{\mathrm{fin}}$ of *finite* spectra. The full subcategory $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathrm{Sp}_{\mathrm{fin}}^{\mathcal{J}}) \subset \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})$ is closed under the convolution product defined in Section 2.1, and therefore inherits a symmetric monoidal structure. By Spanier-Whitehead duality, this induces a symmetric monoidal structure on $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, (\mathrm{Sp}_{\mathrm{fin}}^{\mathcal{J}})^{op})$, which in turn induces a symmetric monoidal structure on its opposite, $\mathbf{Cofil}(\mathrm{Sp}_{\mathrm{fin}}^{\mathcal{J}})$. This can be described explicitly as sending

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One would now hope that Tow restricts to a symmetric monoidal functor when we replace $\mathrm{Sp}^{\mathcal{J}}$ by $\mathrm{Sp}_{\mathrm{fin}}^{\mathcal{J}}$. However, we would like to understand functors which do not necessarily take values in finite spectra. Furthermore, it is not even clear that the derivatives of a functor $\mathcal{J} \to \mathrm{Sp}$ factoring through $\mathrm{Sp}_{\mathrm{fin}}$ will still factor through $\mathrm{Sp}_{\mathrm{fin}}$. We may simply restrict to this situation.

Definition 5.7. Let $\overline{\mathrm{Sp}}^{\mathcal{J}}$ denote the full subcategory of functors $F \in \mathrm{Sp}^{\mathcal{J}}$ such that the functors $P_n F$ factor through $\mathrm{Sp}_{\mathrm{fin}}^{\mathcal{J}}$ for all n. Additionally, let $\overline{\mathrm{Sp}}_{\mathrm{fin}}^{\mathcal{J}}$ denote the full subcategory of $F \in \overline{\mathrm{Sp}}^{\mathcal{J}}$ for which F itself factors through $\mathrm{Sp}_{\mathrm{fin}}^{\mathcal{J}}$.

Proposition 5.8. *The Weiss tower functor Tow restricts to a symmetric monoidal functor*

$$\mathit{Tow}_{\mathit{fin}} : \overline{\mathit{Sp}}^{\mathcal{J}} \to \boldsymbol{\mathit{Cofil}}(\overline{\mathit{Sp}}^{\mathcal{J}}_{\mathit{fin}}).$$

Proof. Combining Construction 5.6 with Definition 5.7, we obtain the diagram

$$\begin{tikzcd}
\mathrm{Sp}^{\mathcal{J}} \arrow[r, shift left] & \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}}) \arrow[l, shift left] & \mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}})_{\mathrm{poly}} \arrow[l, hook', "j"] \\
\overline{\mathrm{Sp}}^{\mathcal{J}} \arrow[u, hook] & \mathbf{Cofil}(\overline{\mathrm{Sp}}^{\mathcal{J}}) \arrow[u, hook] \arrow[l, "\lim"] & \mathbf{Cofil}(\overline{\mathrm{Sp}}^{\mathcal{J}}_{\mathrm{fin}})_{\mathrm{poly}} \arrow[u, hook] \arrow[l, hook', "\overline{j}"]
\end{tikzcd}$$

where the hooked arrows denote inclusions of full subcategories. Recall that j admits a left adjoint Tow_* and lim admits a left adjoint r^*. It is clear that these left adjoints restrict to functors $\overline{\mathrm{Tow}}_*$ and r^* on the bottom row. Consequently, setting $\mathrm{Tow}_{\mathrm{fin}} = \overline{\mathrm{Tow}}_* \circ r^*$, we obtain an adjunction

$$\mathrm{Tow}_{\mathrm{fin}} : \overline{\mathrm{Sp}}^{\mathcal{J}} \rightleftharpoons \mathbf{Cofil}(\overline{\mathrm{Sp}}^{\mathcal{J}}_{\mathrm{fin}})_{\mathrm{poly}} : \overline{j} \circ \lim\,.$$

We now show that there are natural symmetric monoidal structures on both of these categories such that the right adjoint $\overline{j} \circ \lim$ has a symmetric monoidal structure. This is because it admits a different factorization

$$\mathbf{Cofil}(\overline{\mathrm{Sp}}^{\mathcal{J}}_{\mathrm{fin}})_{\mathrm{poly}} \xrightarrow{\overline{j}'} \mathbf{Cofil}(\overline{\mathrm{Sp}}^{\mathcal{J}}_{\mathrm{fin}}) \xrightarrow{\lim} \overline{\mathrm{Sp}}^{\mathcal{J}}.$$

The category $\mathbf{Cofil}(\overline{\mathrm{Sp}}^{\mathcal{J}}_{\mathrm{fin}})$ inherits a symmetric monoidal structure from $\mathbf{Cofil}(\mathrm{Sp}^{\mathcal{J}}_{\mathrm{fin}})$ as a full subcategory which is closed under the tensor product. As such, the functor lim is symmetric monoidal because the the tensor product commutes with limits of finite spectra separately in each variable.

bserve that a finite limit of functors which are polynomial of degree at most n is itself polynomial of degree at most n. This implies that the full subcategory $\overline{j}'$ is closed under the symmetric monoidal structure. It follow that the left adjoint $\overline{\mathrm{Tow}}_*$ naturally admits the structure of a oplax symmetric monoidal functor. Moreover, it is clear that the constant functor $\overline{\mathrm{Sp}}^{\mathcal{J}}_{\mathrm{fin}} \to \mathbf{Cofil}(\overline{\mathrm{Sp}}^{\mathcal{J}}_{\mathrm{fin}})$ is symmetric monoidal, and so precomposing by it yields an oplax monoidal functor $\mathrm{Tow}_{\mathrm{fin}} : \overline{\mathrm{Sp}}^{\mathcal{J}}_{\mathrm{fin}} \to \mathbf{Cofil}(\overline{\mathrm{Sp}}^{\mathcal{J}}_{\mathrm{fin}})_{\mathrm{poly}}$.

Concretely, the oplax structure can be described on the nth filtered piece as follows: since $\mathrm{Tow}_{\mathrm{fin}}(F) \in \mathbf{Cofil}(\overline{\mathrm{Sp}}^{\mathcal{J}}_{\mathrm{fin}})_{\mathrm{poly}}$, the filtered piece $(\mathrm{Tow}_{\mathrm{fin}}(F) \otimes \mathrm{Tow}_{\mathrm{fin}}(F))_n$ is polynomial of degree at most n. It follows that the natural map from $F \wedge F$ factors through a map $\varphi_n : P_n(F \wedge F) \to (\mathrm{Tow}_{\mathrm{fin}}(F) \otimes \mathrm{Tow}_{\mathrm{fin}}(F))_n$. To see that $\mathrm{Tow}_{\mathrm{fin}}$ is a symmetric monoidal functor, it suffices to show that each φ_n is an equivalence. TODO

□

Remark 5.9. The functor $F(W) = \Sigma^\infty_+ \Omega\, \mathcal{J}(V, V+W)$ described after Theorem 5.4 does not take values in finite spectra (note that the V has been suppressed in the notation). However, it is filtered by functors $F^{(n)}(W)$ which do take values in finite complexes for all $n < \infty$. Moreover, $P_n F(W) = F^{(n)}(W)$.

Remark 5.10. Proposition 5.8 is written in the language of Weiss calculus as that is the present application, but the proof works equally well in Goodwillie calculus.

6. Stable $\mathbb{A}_\infty$ Splittings

The main result of [Aro01] shows that the Mitchell-Richter filtration on $\Omega SU(n)$ (and more generally, for loops on a Stiefel manifold) stably splits. The key insight is that this filtration has extra structure: it is a particular value of a *functor* which has a natural filtration. The tool that allows for the exploitation of this structure is Weiss's theory of orthogonal or unitary calculus.

In this section, we extend the methods of [Aro01] to produce $\mathbb{A}_\infty$ stable splittings of Stiefel manifolds. We will begin this section by reviewing the theory of calculus introduced in [Wei95]. We then make a statement about the multiplicativity of the construction which ...

7. $\mathbb{E}_2$ Splittings in Complex Cobordism

In this brief section, we remark that the $\mathbb{A}_\infty$-splitting

$$\Sigma^\infty_+ \Omega SU(n) \simeq ???$$

becomes $\mathbb{E}_2$ after smashing with MU. More precisely, we show that there is an equivalence of $\mathbb{E}_2$-MU-algebras

$$MU \wedge \Sigma^\infty_+ \Omega SU(n) \simeq ???.$$

The $\mathbb{A}_\infty$-MU-algebra equivalence constructed in Section 6 is realized by a map of $\mathbb{A}_\infty$-$\mathbb{S}$-algebras

$$\Sigma^\infty \Omega SU(n) \longrightarrow ???. \tag{1}$$

Our task is to show that (1) may be refined to a morphism of $\mathbb{E}_2$-ring spectra. We do so by obstruction theory–the key fact powering our proof is that

$$MU_*\left(\Omega SU(n)\right) \cong 0$$

whenever $*$ is odd. **FIND A REFERENCE**. In fact, inspired by [CM15], we prove the following more general result:

Theorem 7.1. *Suppose that R is an $\mathbb{E}_2$-ring spectrum with no homotopy groups in odd degrees. Then any $\mathbb{A}_\infty$-ring homomorphism*

$$\Sigma^\infty_+ \Omega SU(n) \to R$$

lifts to a morphism of $\mathbb{E}_2$-ring spectra.

Proof. By taking connective covers, one learns that any $\mathbb{A}_\infty$-ring homomorphism

$$\Sigma^\infty_+ \Omega SU(n) \to R$$

must factor through the natural $\mathbb{E}_2$-algebra map $\tau_{\geq 0} R \to R$. Thus, without loss of generality we will assume that R is (-1)-connected.

It is clear that the composite $\mathbb{A}_\infty$-ring homomorphism

$$\Sigma^\infty_+ \Omega SU(n) \longrightarrow R \longrightarrow \tau_{\leq 0} R \simeq H\pi_0(R)$$

may be lifted to an $\mathbb{E}_2$-ring homomorphism factoring through $\tau_{\leq 0}\Sigma^\infty_+ \Omega SU(n) \simeq H\mathbb{Z}$. Suppose now for $q > 0$ that we have chosen an $\mathbb{E}_2$-ring homomorphism

$$\Sigma^\infty_+ \Omega SU(n) \longrightarrow \tau_{\leq q-1} R$$

lifting the given $\mathbb{A}_\infty$-algebra map

$$\Sigma^\infty_+ \Omega SU(n) \longrightarrow R \longrightarrow \tau_{\leq q-1} R.$$

We will show that there is no obstruction to the existence of a further $\mathbb{E}_2$-lift

$$\Sigma^\infty_+ \Omega SU(n) \longrightarrow \tau_{\leq q} R.$$

According to [CM15, Theorem 4.1], there is a diagram of principal fibrations

$$\begin{tikzcd}
\mathbb{E}_2\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\leq q}R) \arrow[r] \arrow[d] & \mathbb{A}_\infty\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\leq q}R) \arrow[d] \\
\mathbb{E}_2\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\leq q-1}R) \arrow[r] \arrow[d] & \mathbb{A}_\infty\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\leq q-1}R) \arrow[d] \\
\mathcal{S}_*(BSU(n), K(\pi_q R, q+3)) \arrow[r] & \mathcal{S}_*(SU(n), K(\pi_q R, q+2))
\end{tikzcd}$$

For q odd, $\tau_{\leq q-1}R \simeq \tau_{\leq q}R$, so there is no obstruction. Let us therefore assume that q is even.

Since the cohomology of $BSU(n)$ is even-concentrated with coefficients in any abelian group, we have that $\pi_0\,\mathcal{S}_*(BSU(n), K(\pi_q R, q+3)) \cong H^{q+3}(BSU(n); \pi_q R)$ is zero. It follows then that the given class

$$x \in \pi_0\mathbb{E}_2\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\leq q-1}R)$$

admits some lift

$$\widetilde{x} \in \mathbb{E}_2\text{-Ring}(\Sigma^\infty_+\Omega SU(n), \tau_{\leq q}R).$$

We may need to modify $\widetilde{x}$ to match our chosen $\mathbb{A}_\infty$-ring homomorphism. This is always possible so long as the map

$$\pi_1(\mathcal{S}_*(BSU(n), K(\pi_q R, q+3))) \longrightarrow \pi_1(\mathcal{S}_*(SU(n), K(\pi_q R, q+2)))$$

is surjective. Said in other terms, this is just the map

$$H^{2q+2}(BSU(n); \pi_q R) \longrightarrow H^{2q+1}(SU(n); \pi_q R) \cong H^{2q+2}(\Sigma SU(n); \pi_q R)$$

induced by the natural map $\Sigma SU(n) \to BSU(n)$. It is a classical fact that this map is surjective (it follows from a calculation with the bar spectral sequence, using the fact that the cohomology of $SU(n)$ is exterior). **Maybe you can check this Allen**. $\square$

8. Square Zero Algebras

We will now discuss square zero extensions in our framework. For this, it will be convenient to work with the category $\mathbf{Gr}_u$ of *unital* graded spectra in the strong sense that the unit map induces an equivalence in grading 0. Note that there is a fully faithful functor $T : \mathrm{Sp} \to \mathbf{Gr}_u$ which sends a spectrum A to the graded spectrum

$$S^0, A, *, *, \cdots .$$

Its essential image is the full subcategory $i : \mathbf{Gr}_u^{\leq 1} \to \mathbf{Gr}_u$ consisting of unital graded spectra X such that X_k is contractible for $k > 1$. In this section, we analyze graded spectra in this subcategory $\mathbf{Gr}_u^{\leq 1}$. Our goal is to show any such graded spectrum admits an essentially unique $\mathbb{E}_n$-algebra structure for any $0 \leq n \leq \infty$. This goal is realized in Proposition 8.1.

The inclusion i fits into an adjunction

$$L^{\leq 1} : \mathbf{Gr}_u \rightleftharpoons \mathbf{Gr}_u^{\leq 1} : i$$

where the left adjoint $L^{\leq 1}$ can be thought of as truncating above grading 1. The localization $L^{\leq 1}$ is visibly compatible with the monoidal structure in the sense that for any $f : X \to Y$ in $\mathbf{Gr}_u$ such that $L^{\leq 1}f$ is an equivalence and any $Z \in \mathbf{Gr}_u$, the natural map $L^{\leq 1}(X \wedge Z) \to L^{\leq 1}(Y \wedge Z)$ is an equivalence. We are now in the situation of Proposition 2.2.1.9 of [Lur16], and so we may conclude that $\mathbf{Gr}_u^{\leq 1}$ inherits a symmetric monoidal structure such that $L^{\leq 1}$ is symmetric monoidal and the inclusion i is lax monoidal. This monoidal structure can be described explicitly by the formula

$$X \otimes_{\mathbf{Gr}_u^{\leq 1}} Y = L^{\leq 1}(X \otimes_{\mathbf{Gr}_u} Y).$$

We may then apply Remark 7.3.2.13 of [Lur16] to obtain an adjunction at the level of algebras for any integer $0 \leq n \leq \infty$:

$$L_{\mathrm{alg}}^{\leq 1} : \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}_u) \rightleftharpoons \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}_u^{\leq 1}) : i_{\mathrm{alg}}.$$

Since the counit $Li \to \mathrm{id}$ before lifting to algebras is an equivalence, we have that the counit $L_{\mathrm{alg}}^{\leq 1} i_{\mathrm{alg}} \to \mathrm{id}$ is also an equivalence. This implies in particular that i_{alg} is fully faithful. We are now in position to prove the main proposition of this section:

Proposition 8.1. *Let $0 \leq n \leq \infty$ be an integer. Then, there is a sequence of equivalences of categories*

$$Sp \xrightarrow{\bar{T}} \boldsymbol{Gr}_u^{\leq 1} \longrightarrow Alg_{\mathbb{E}_n}(\boldsymbol{Gr}_u^{\leq 1}) \longrightarrow Alg_{\mathbb{E}_n}(\boldsymbol{Gr}_u) \times_{\boldsymbol{Gr}_u} \boldsymbol{Gr}_u^{\leq 1}$$

where the first functor $\bar{T}$ is obtained by restricting the codomain of the functor $T : Sp \to \boldsymbol{Gr}_u$. In particular, for any $X \in \boldsymbol{Gr}_u^{\leq 1}$, the graded spectrum $iX \in \boldsymbol{Gr}_u$ has an essentially unique $\mathbb{E}_n$-algebra structure.

Proof. The third arrow is defined by i_{alg}, and is an equivalence because i_{alg} is fully faithful, so it remains to consider the first two arrows.

We have already seen that the functor $\bar{T} : \mathrm{Sp} \to \mathbf{Gr}_u^{\leq 1}$ is an equivalence of categories. However, it may be promoted to a symmetric monoidal equivalence when Sp is given the cocartesian monoidal structure - that is, the monoidal structure defined by $\vee$, the coproduct. This monoidal structure has a very special property: by Proposition 2.4.3.9 of [Lur16], there is for each n an equivalence $\mathrm{Sp} \simeq \mathrm{Alg}_{\mathbb{E}_n}^{\vee}(\mathrm{Sp})$, where the superscript $\vee$ indicates that we are considering algebras under the wedge. Informally, this says that any $Y \in \mathrm{Sp}$ admits an essentially unique $\mathbb{E}_n$-algebra structure under the coproduct. It follows that the same holds for any $X \in \mathbf{Gr}_u^{\leq 1}$, and so there is an equivalence $\mathbf{Gr}_u^{\leq 1} \to \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}_u^{\leq 1})$, as desired. $\square$

Terminology 8.2. Let $0 \leq n \leq \infty$ be an integer. By taking composing with the colimit functor, Proposition 8.1 provides a functor

$$\omega_n : \mathrm{Sp} \to \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})$$

which we will refer to as the square zero extension. It sends a spectrum X to a ring with underlying spectrum $S^0 \vee X$. We will call any $\mathbb{E}_n$-algebra structure produced via Proposition 8.1 or ω_n a *square zero* $\mathbb{E}_n$ structure.

Remark 8.3. For any $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathbf{Gr}_u)$, we have a map $X \to i_{alg} L_{alg}^{\leq 1} X$ of $\mathbb{E}_n$-algebras. Taking colimits, we obtain a map $\operatorname{colim} X \to \operatorname{colim} i_{alg} L_{alg}^{\leq 1} X$ of $\mathbb{E}_n$ ring spectra. We may summarize this informally by saying that any $\mathbb{E}_n$-split ring spectrum X has an $\mathbb{E}_n$ map to the square zero extension determined by its degree one component X_1.

We will need to understand structured maps into square zero extensions. This amounts to understanding the space of units. In classical algebra, given a commutative ring A and an A-module M, the group of units of the square zero extension are given by the formula

$$(A \oplus M)^{\times} \simeq A^{\times} \times M.$$

A similar formula holds in our context for suspension spectra of connected spaces.

Proposition 8.4. *Let $0 \leq n \leq \infty$ be an integer and $X \in \mathcal{S}$ a connected space. There is a canonical equivalence*

$$GL_1(\omega_n(\Sigma^{\infty} X)) \simeq GL_1(S^0) \times QX$$

of $\mathbb{E}_n$-algebras in spaces, where QX is our notation for $\Omega^{\infty}\Sigma^{\infty} X$.

Proof. The functors ω_n are compatible under restriction, so it suffices to prove the statement for $n = \infty$. For this case, we will show that there is a splitting

$$gl_1(\omega_\infty(\Sigma^\infty X)) \simeq gl_1(S^0) \vee \Sigma^\infty X$$

of spectra, where gl_1 denotes the spectrum of units of an $\mathbb{E}_\infty$-ring introduced in [May77]. We first look at what happens on homotopy. Recall that for any $\mathbb{E}_\infty$ ring spectrum Y, we have the formula

$$\pi_*(gl_1(Y)) \simeq (\pi_*(Y))^\times$$

where on the right hand side, we consider $\pi_*(Y)$ as a graded ring. In our case, this yields an identification

$$\pi_*(gl_1(\omega_\infty(\Sigma^\infty X))) \simeq (\pi_*(S^0) \oplus \pi_*(\Sigma^\infty X))^\times \simeq \pi_*(S^0)^\times \times \pi_*(\Sigma^\infty X)$$

where we have used that on homotopy, $\omega_\infty(\Sigma^\infty X)$ is a square zero extension. To conclude the proof, it suffices to show that the two factors on the right hand side can be realized by maps of spectra.

The first factor is realized simply by gl_1 of the unit map $S^0 \to \omega_\infty(\Sigma^\infty X)$. In fact, it is not difficult to see directly that this map is split.

For the second factor, observe that since $\omega_\infty(\Sigma^\infty X)$ is an $\mathbb{E}_\infty$-ring, it receives a canonical $\mathbb{E}_\infty$ map

$$\Sigma^\infty_+ QX \longrightarrow \omega_\infty(\Sigma^\infty X)$$

from the free $\mathbb{E}_\infty$ ring on $\Sigma^\infty X$ which extends the canonical map of spectra $\Sigma^\infty X \to \omega_\infty(\Sigma^\infty X)$. Now, note that there is an adjunction [May77]

$$\Sigma^\infty_+\Omega^\infty : \mathrm{Sp} \rightleftharpoons \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp}) : gl_1$$

under which the above map may be identified with a map

$$b : \Sigma^\infty X \to gl_1(\omega_\infty(\Sigma^\infty X))$$

of spectra. **NEED TO SAY A TINY BIT MORE**

Finally, we may take the map $a \vee b : gl_1(S^0) \vee \Sigma^\infty X \to gl_1(\omega_\infty(\Sigma^\infty X))$ and the above comments show that it is an equivalence, as desired. $\square$

9. Obstructions to a General $\mathbb{E}_2$ Splitting

Let $3 < n \leq \infty$ be an integer. We will now show that the $\mathbb{A}_\infty$ splitting

$$\Sigma^\infty_+\Omega SU(n) \simeq ???$$

cannot be promoted to an $\mathbb{E}_2$-splitting before smashing with MU.

Suppose that such a splitting existed. By Remark 8.3, we would obtain an $\mathbb{E}_2$-ring homomorphism $\Sigma^\infty_+\Omega SU(n) \to \Sigma^\infty_+\mathbb{CP}^{n-1}$, where $\Sigma^\infty_+\mathbb{CP}^{n-1}$ is given the square-zero multiplication. Furthermore, the precomposition with the inclusion $\Sigma^\infty_+\mathbb{CP}^{n-1} \longrightarrow \Sigma^\infty_+\Omega SU(n)$ must yield the identity map. In particular, the map sends the generator of $\pi_2(\Sigma^\infty \mathbb{CP}^{n-1})$ to the generator of $\pi_2(\Sigma^\infty\Omega SU(n))$.

Recall now that there is an adjunction [May77]

$$\Sigma^\infty_+ : \textbf{Double Loop Spaces} \rightleftharpoons \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp}) : GL_1.$$

Using this, we may form the adjoint $\mathbb{E}_2$ map

$$\Omega SU(n) \to GL_1(\Sigma^\infty_+\mathbb{CP}^{n-1}).$$

The right hand side is identified as an $\mathbb{E}_2$ algebra by Proposition 8.4. In particular, we obtain an $\mathbb{E}_2$ composite

$$\phi : \Omega SU(n) \to GL_1(\Sigma^\infty_+ \mathbb{CP}^{n-1}) \simeq GL_1(S^0) \times Q\,\mathbb{CP}^{n-1} \to Q\,\mathbb{CP}^{n-1}$$

which has the additional property that it is an isomorphism on π_2.

We now show that such a map ϕ cannot exist due to the operations that exist in the homotopy of an $\mathbb{E}_2$ algebra.

Observation 9.1. Let $Y \in \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})$, and suppose we are given a map $S^2 \to Y$. This extends to an $\mathbb{E}_2$ map $\Omega^2 S^4 \to Y$. We may precompose with the map $S^5 \to \Omega^2 S^4$ adjoint to the Hopf map $S^7 \to S^4$. This procedure determines a natural operation

$$\nu^u : \pi_2(Y) \to \pi_5(Y)$$

in the homotopy of any $\mathbb{E}_2$-algebra in spaces.

Remark 9.2. The notation is meant to hint at the fact that if $Y = \Omega^\infty X$ comes from a spectrum, then the operation ν^u is given by multiplication by the element $\nu \in \pi_3(\mathbb{S})^\wedge_2$ from the 2-primary homotopy groups of the sphere spectrum. Thus, ν^u is an unstable version of ν that is already seen in any $\mathbb{E}_2$ algebra in spaces.

Finally, we show that ϕ cannot be compatible with ν^u on homotopy by directly computing ν^u on either side.

For $n > 3$, observe that the natural map $\Omega SU(n) \to BU$ is an isomorphism in homology up to degree 7. This implies that $\pi_5(\Omega SU(n)) \simeq \pi_5(BU) \simeq 0$ because BU is even. Hence, ν^u is trivial on the generator of $\pi_2(\Omega SU(n))$.

Similarly, the map $Q\,\mathbb{CP}^{n-1} \to Q\,\mathbb{CP}^\infty$ is an isomorphism on π_5 for $n > 3$. However, it was computed in [Liu63, Theorem II.8] that $\pi_5(\mathbb{CP}^\infty) = \mathbb{Z}/2$ generated by ν times the degree 2 generator. Hence, by Remark 9.2, if $\beta \in \pi_2(Q\,\mathbb{CP}^{n-1})$ denotes the generator, then $\nu^u(\beta) \in \pi_5(Q\,\mathbb{CP}^{n-1})$ is nontrivial. This implies that there can be no $\mathbb{E}_2$ map ϕ which induces an isomorphism on π_2 and concludes the proof.

Remark 9.3. Taking the limit as $n \to \infty$, we obtain the statement that the map $BU \to Q\,\mathbb{CP}^\infty$ implementing the splitting principle does not lift to an $\mathbb{E}_2$ map. This map is well-studied: among other places, it appears as the first connecting map in the Weiss tower for the functor $V \mapsto BU(V)$. As such, it can be seen as a "BU-analog" to the Kahn-Priddy map.

10. Snaith's Construction of Periodic Complex Bordism

A classical theorem of Snaith [Sna81] gives an equivalence of homotopy commutative ring spectra

$$\Sigma^\infty_+ BU[\beta^{-1}] \simeq MUP.$$

The equivalence arises from considering the total MU-Chern class map $BU \to GL_1(MUP)$. It is known from [Sna77] that the total Chern class in integral homology is not an infinite loop map. It follows from the existence of an $\mathbb{E}_\infty$ map $MUP \to H\,\mathbb{Z}\,P$ from periodic complex bordism to periodic integral homology that Snaith's equivalence is not an equivalence of $\mathbb{E}_\infty$ ring spectra. The following theorem refines this observation:

Theorem 10.1. *The equivalence $\Sigma^\infty_+ BU[\beta^{-1}] \simeq MUP$ is $\mathbb{E}_2$ but not $\mathbb{E}_3$.*

Proof. Proof goes here □

Comment now about GepnerSnaith. We should cite at some point here or the introduction all of [Sna77], [GS09], and the Snaith book with the original splitting.

11. Miscellaneous stuff here

It would be nice to at some point deal with showing the associated graded E_2 structure of BU is the thom spectrum VMU(n). I've directly pasted in some writing from a previous argument I claimed, but it definitely uses that $\coprod BU(n)$ is an E_2 algebra over $\mathbb{Z}_{\geq 0}$ which I never got straight an actual proof of.

Proposition 11.1. *The associated graded of $\Sigma^\infty_+ BU$ is E_2 equivalent to the Thom spectrum $\bigvee MU(n)$.*

Proof. Let $R = BU$ with its natural filtration, and let $R^\oplus = \coprod BU(n)$ with its natural filtration. Let M be the (E_∞) filtered spectrum which is $MU(n)$ in degree n, and all maps are 0. In other words, $\bigvee MU(n)$ with its natural filtration is $I(res(M))$.

We begin with a filtered E_∞ map $z : R^\oplus \to I(res(M))$ coming from the zero section. Then, $R^\oplus$ comes with the structure of an E_2 algebra over $\mathbb{Z}^{fil}_{\geq 0}$. In fact, $I(res(M))$ has a trivial structure as an E_∞-algebra over $\mathbb{Z}^{fil}_{\geq 0}$ via the augmentation $\mathbb{Z}^{fil}_{\geq 0} \to S^{0,fil} \to I(res(M))$. We may then tensor z along the augmentation to get a map of E_2 filtered spectra $z' : R \to I(res(M))$.

There is a canonical equivalence $I(res(M)) \otimes \mathbb{A} \simeq M$ because M is in the image of $\mathbb{A} \otimes I(-)$ (that is, all the maps in the filtration of M were zero). As such, M acquires a canonical structure as an $\mathbb{A}$ algebra such that the map $M \otimes \mathbb{A} \to M$ is a map of E_2 rings (in fact I think it's E_∞?).

Finally, we observe that we may tensor z' with $\mathbb{A}$ and compose with the multiplication map to get an E_2 map $R \otimes A \to M \otimes A \to M$ which is the right thing up to homotopy, so it's an equivalence. □

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What is $\Sigma^\infty_+ \Omega SU(n)[\beta^{-1}]$, by the way? Is it related to a periodic version of the $X(n)$-filtration of MU??

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