Symmetries of Rose Windows

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August 26, 2021

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- Vertices, Edges, Paths
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Introduction

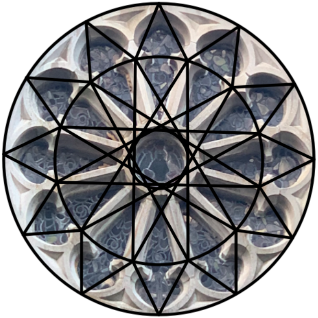
Group Theory Definitions

- Vertex: Each point where 2 edges meet on a given polygon
- Edge: The lines of a polygon which joins two vertices together
- Path: A sequence of edges that begins at a vertex, and travels from vertex to vertex along edges of the graph

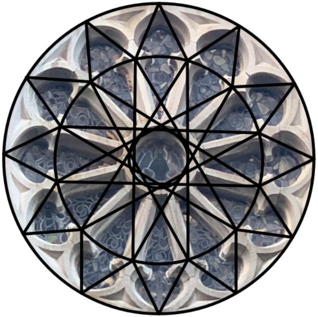
Rose Windows

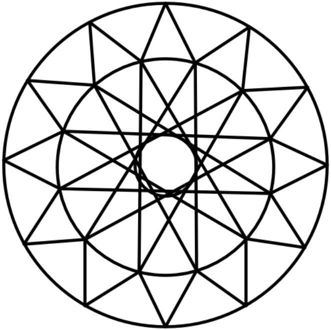
- Notation
- Parts
- Examples
- Restrictions

Connect the dots



Connect the dots





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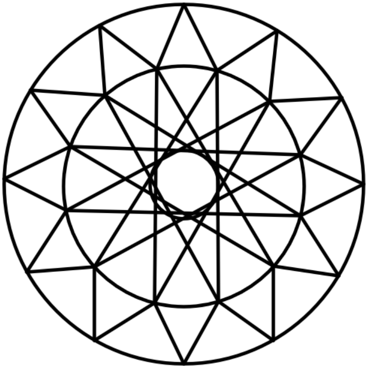
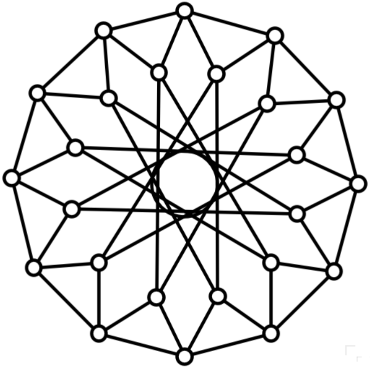
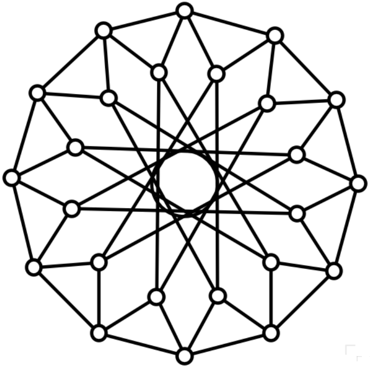
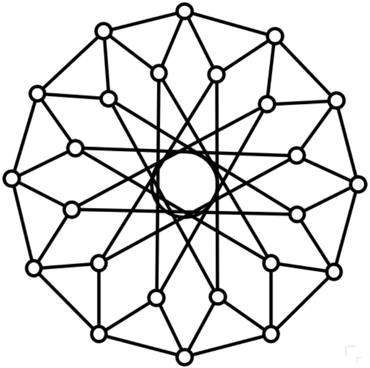


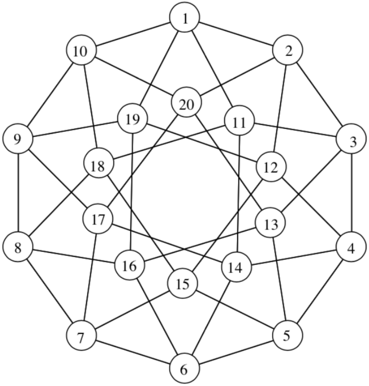


Connect the dots



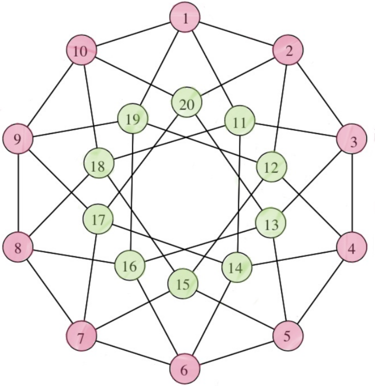
Connect the dots





Notation of Rose Windows

Rose window: $R_n(a, r)$

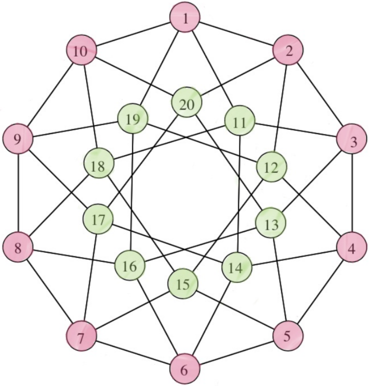


$R_n(a, r)$

Notation of Rose Windows

Rose window: $R_n(a, r)$

- $n =$ the number of vertices on each level

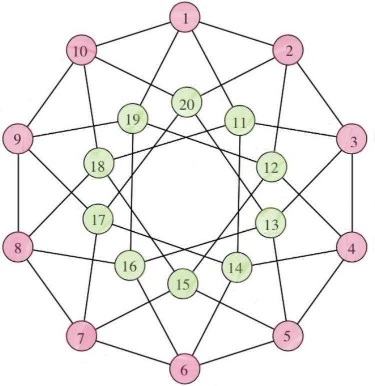


$R_{10}(a, r)$

Notation of Rose Windows

Rose window: $R_n(a, r)$

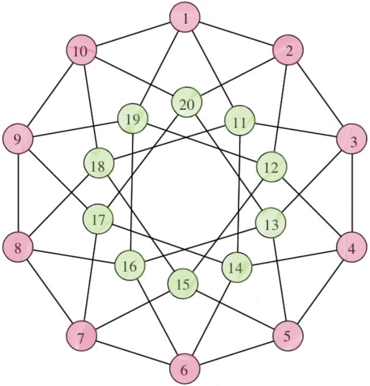
- $n =$ the number of vertices on each level
- $vertices = 2n$



$R_{10}(a, r)$

Notation of Rose Windows

Rose window: $R_n(a, r)$ The vertices are labeled:

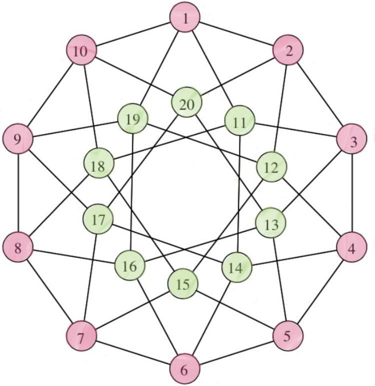


$R_{10}(a, r)$

Notation of Rose Windows

Rose window: $R_n(a, r)$ The vertices are labeled:

- $1, \dots, n$ which are vertices $A_1, A_2, \dots, A_n$

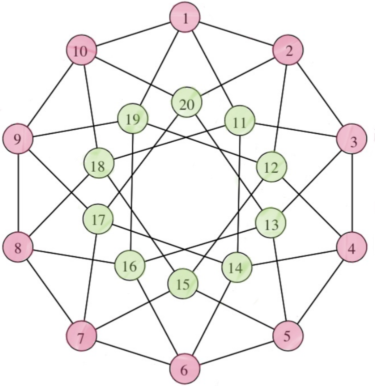


$R_{10}(a, r)$

Notation of Rose Windows

Rose window: $R_n(a, r)$ The vertices are labeled:

- $1, \ldots, n$ which are vertices $A_1, A_2, \ldots, A_n$
- $n+1, n+1, \ldots, 2n$ which are vertices $B_1, B_2, \ldots, B_n$

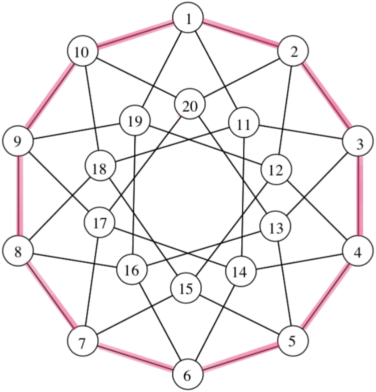


$R_{10}(a, r)$

Parts of Rose Windows

Rose window: $R_n(a, r)$

- Rim: A_i - A_{i+1}

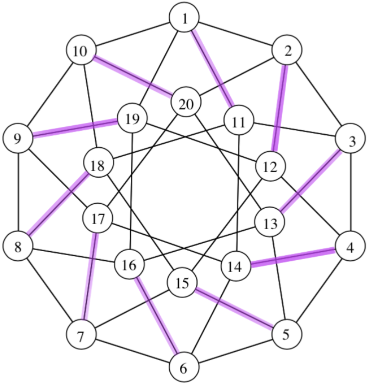


$R_{10}(a, r)$

Parts of Rose Windows

Rose window: $R_n(a, r)$

- In-Spoke: A_i - B_i

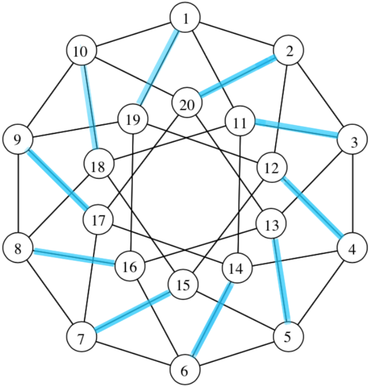


$R_{10}(a, r)$

Parts of Rose Windows

Rose window: $R_n(a, r)$

- Out-Spoke: B_i - A_{i+a}

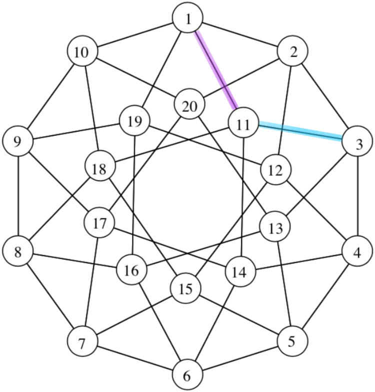


$R_{10}(a, r)$

Parts of Rose Windows

Rose window: $R_n(a, r)$

- Out-Spoke: B_i - A_{i+a}
- $a =$ spread of A_n vertices (outside vertices) via the in-spoke, out-spoke

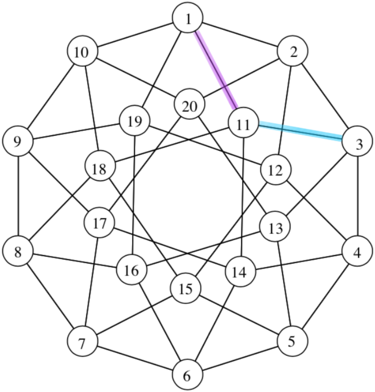


$R_{10}(a, r)$

Parts of Rose Windows

Rose window: $R_n(a, r)$

- Out-Spoke: B_i - A_{i+a}
- $a =$ spread of A_n vertices (outside vertices) via the in-spoke, out-spoke
- $a = 2$

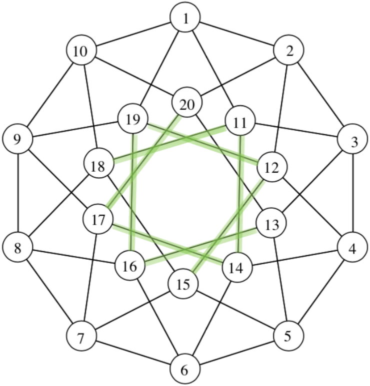


$R_{10}(2, r)$

Parts of Rose Windows

Rose window: $R_n(a, r)$

- Hub: B_i - B_{i+r}

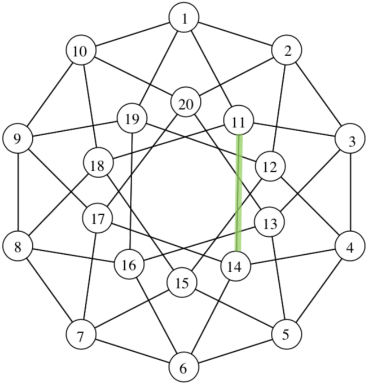


$R_{10}(2, r)$

Parts of Rose Windows

Rose window: $R_n(a, r)$

- Hub: B_i - B_{i+r}
- $r =$ spread of the B_n vertices (inside vertices) via the hub

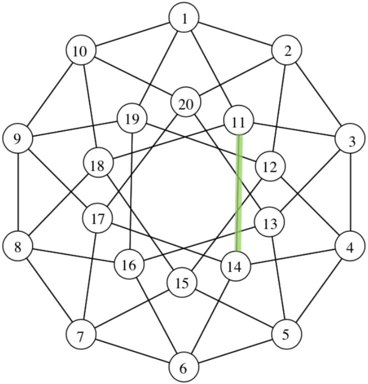


$R_{10}(2, r)$

Parts of Rose Windows

Rose window: $R_n(a, r)$

- Hub: B_i - B_{i+r}
- $r =$ spread of the B_n vertices (inside vertices) via the hub
- $r = 3$

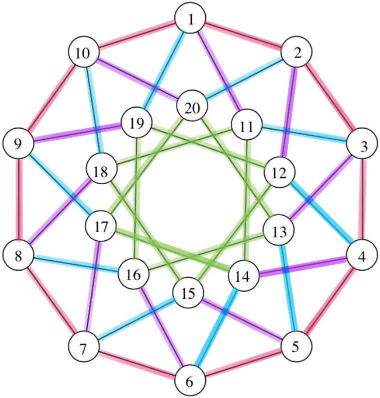


$R_{10}(2, 3)$

Parts of Rose Windows

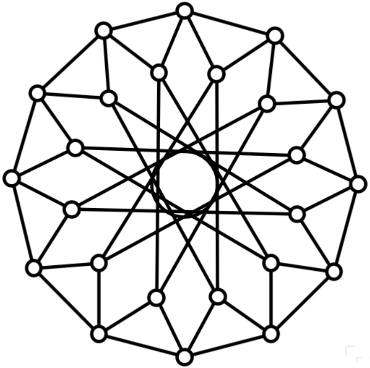
Rose window: $R_n(a, r)$

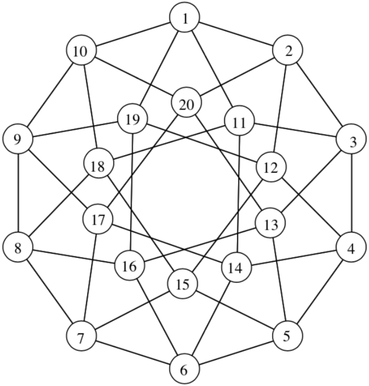
- Rim: A_i - A_{i+1} (pink)
- In-Spoke: A_i - B_i (purple)
- Out-Spoke: B_i - A_{i+a} (blue)
- Hub: B_i - B_{i+r} (green)



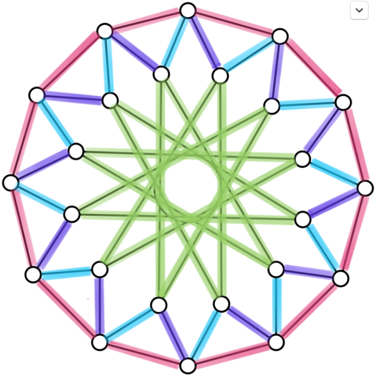
$R_{10}(2, 3)$

Connect the dots

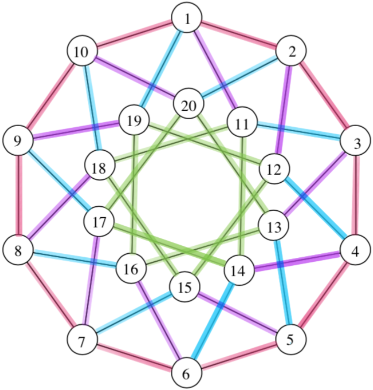




Parts of a Rose Window



$R_{12}(1, 5)$



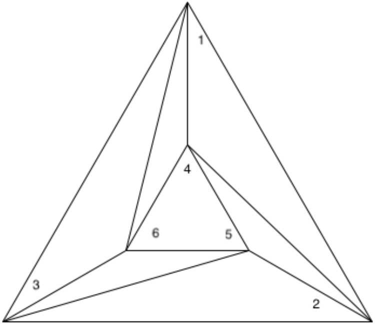
$R_{10}(2, 3)$

Examples of small n

- $n = 3$
- $n = 4$

n = 3

This is what a Rose window of $R_3(1,1)$



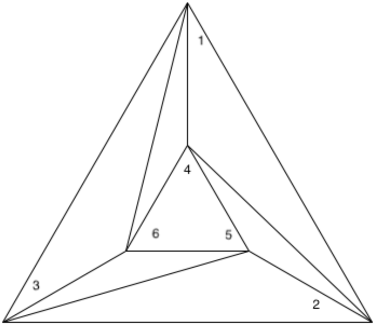
n = 3

- $1, ..., n$ which are vertices $A_1, A_2, ..., A_n$
- $n+1, n+1, ..., 2n$ which are vertices $B_1, B_2, ..., B_n$

Vertices:

- $A_1 : 1$
- $A_2 : 2$
- $A_3 : 3$
- $B_1 : 4$
- $B_2 : 5$
- $B_3 : 6$

$R_3(1,1)$



n = 3

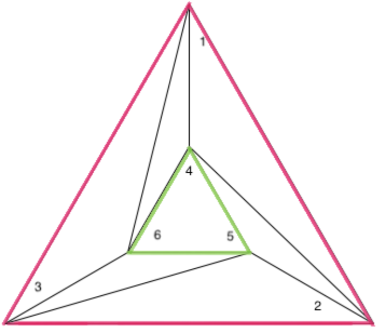
Rim: A_i - A_{i+1} (pink)

- $A_1 - A_2$
- $A_2 - A_3$
- $A_3 - A_1$

Hub: B_i - B_{i+r} (green)

- $B_1 - B_2$
- $B_2 - B_3$
- $B_3 - B_1$

$R_3(1,1)$



n = 3

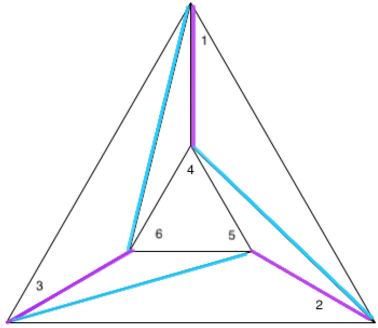
In-Spoke: A_i - B_i (purple)

- $A_1 - B_1$
- $A_2 - B_2$
- $A_3 - B_3$

Out-Spoke: B_i - A_{i+a} (blue)

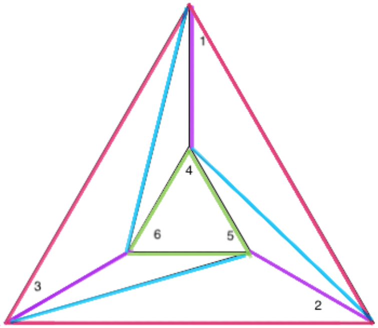
- $B_1 - A_2$
- $B_2 - A_3$
- $B_3 - A_1$

$R_3(1,1)$



n = 3

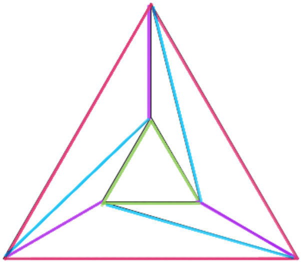
This is a Rose window of $R_3(1,1)$



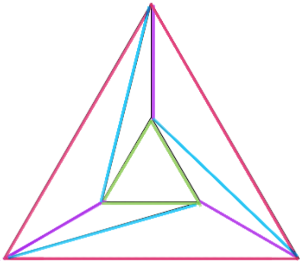
n = 3

You can alter (a, r) to create different paths inside the polygon.

$R_3(2, 1)$

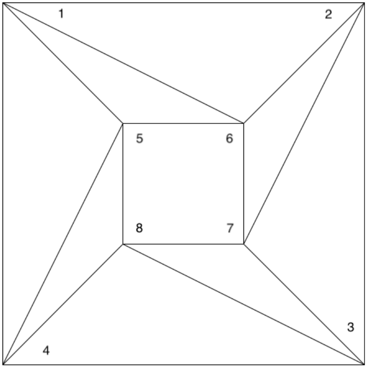


$R_3(1, 2)$



$n = 4$

This is a Rose window of $R_4(3, 1)$



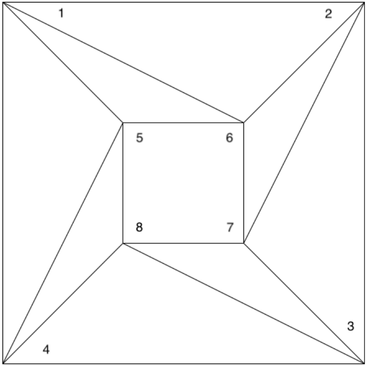
n = 4

- $1, ..., n$ which are vertices $A_1, A_2, ..., A_n$
- $n+1, n+1, ..., 2n$ which are vertices $B_1, B_2, ..., B_n$

Vertices:

- $A_1 : 1$
- $A_2 : 2$
- $A_3 : 3$
- $A_4 : 4$
- $B_1 : 5$
- $B_2 : 6$
- $B_3 : 7$
- $B_4 : 8$

$R_4(3, 1)$



n = 4

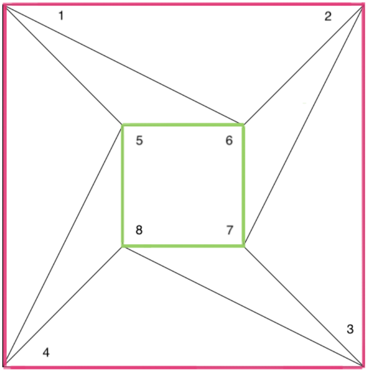
Rim: A_i - A_{i+1} (pink)

- $A_1 - A_2$
- $A_2 - A_3$
- $A_3 - A_4$
- $A_4 - A_1$

Hub: B_i - B_{i+r} (green)

- $B_1 - B_2$
- $B_2 - B_3$
- $B_3 - B_4$
- $B_4 - B_1$

$R_4(3,1)$



n = 4

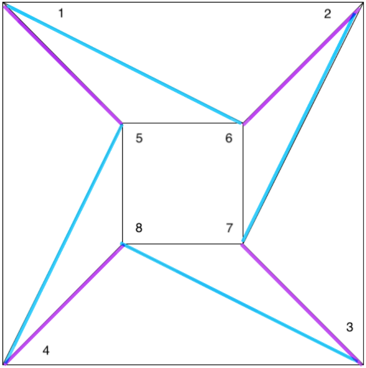
In-Spoke: A_i - B_i (purple)

- $A_1 - B_1$
- $A_2 - B_2$
- $A_3 - B_3$
- $A_4 - B_4$

Out-Spoke: B_i - A_{i+a} (blue)

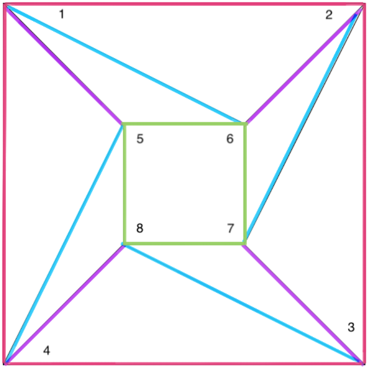
- $B_1 - A_4$
- $B_2 - A_1$
- $B_3 - A_2$
- $B_4 - A_3$

$R_4(3,1)$



n = 4

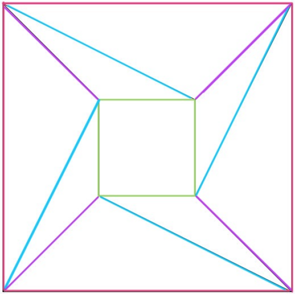
This is a Rose window of $R_4(3,1)$



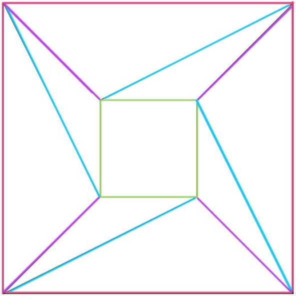
n = 4

You can alter the a and r values, which will change the location of the spokes.

$R_4(3,1)$



$R_4(1,1)$



Restrictions on r

While you can alter (a, r), not every value will work.

r must be co-prime to n.

Co-Prime: Two numbers are co-prime if the only common factors they have is 1.

Restrictions on r

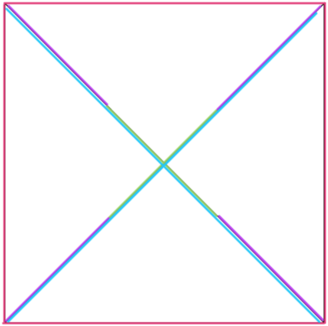
When r is co-prime to n, the B vertices will not all connect together.

This is not the case with a since the rim ensures that all A vertices are connected.
Rim: A_i - A_{i+1}

Restrictions on r

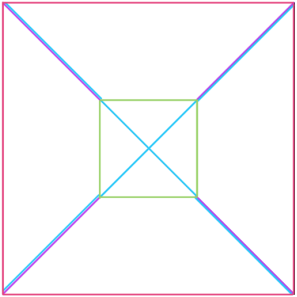
This shows that when r is co-prime, the B vertices don't connect.

$R_4(2, 2)$



Restrictions on r

Here is an example where a is co-prime, and the rose window holds: $R_4(2,1)$



Definitions

Abelian Group: (commutative) $ab = ba$

Non-Abelian Group: $ab \neq ba$

Group: A set of elements combined with an operation

Dihedral Group: A group of symmetries of a regular polygon which includes rotations and reflections

Dihedral Groups

D_4 contains two abelian subgroups

D_4 is a non-abelian group

D_4

	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_0	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	R_0	D'	D	H	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	H	D'	D
R_{270}	R_{270}	R_0	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	R_0	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	R_0	R_{270}	R_{90}
D	D	V	D'	H	R_{270}	R_{90}	R_0	R_{180}
D'	D'	H	D	V	R_{90}	R_{270}	R_{180}	R_0

Types of Symmetry

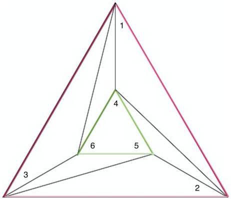
There are 3 types of symmetry that can be applied to rose window graphs

- ρ
- μ
- σ

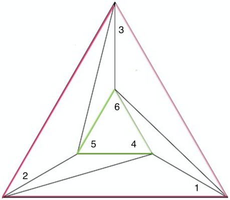
ρ

ρ is a rotation of a rose graph in a clockwise fashion

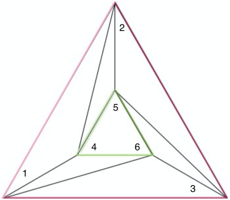
$$\rho : A_i - A_{i+1}, \quad B_i - B_{i+1}$$



ρ_0



ρ_1

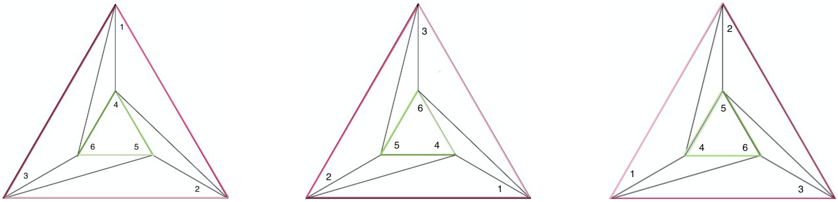


ρ_2

$n = 3$

$$\rho : A_i - A_{i+1}, \quad B_i - B_{i+1}$$

$A_1 - A_2 \quad B_1 - B_2$ | $A_2 - A_3 \quad B_2 - B_3$ | $A_3 - A_1 \quad B_3 - B_1$

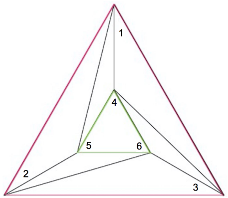


ρ_0 ρ_1 ρ_2

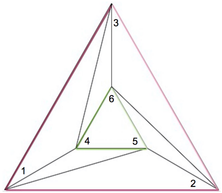
μ

μ is a reflection of a rose graph about a certain vertex

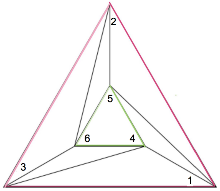
$$\mu : A_i - A_{-i}, \quad B_i - B_{-a-i}$$



μ_1



μ_2

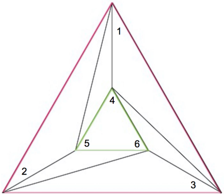


μ_3

$n = 3$

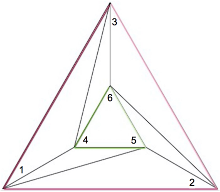
$$\mu : A_i - A_{-i}, \qquad B_i - B_{-a-i}$$

μ_1



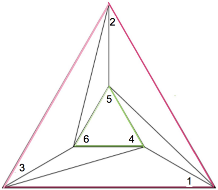
$A_1 - A_1 \quad B_1 - B_1$
$A_2 - A_3 \quad B_2 - B_3$
$A_3 - A_2 \quad B_3 - B_2$

μ_2



$A_1 - A_3 \quad B_1 - B_3$
$A_2 - A_2 \quad B_2 - B_2$
$A_3 - A_1 \quad B_3 - B_1$

μ_3



$A_1 - A_2 \quad B_1 - B_2$
$A_2 - A_1 \quad B_2 - B_1$
$A_3 - A_3 \quad B_3 - B_3$

σ

There is a third type of symmetry called σ (sigma). We will come back to this later.

Definitions

Cayley Table: Arranges all the possible products of a group's elements.

Group: A set of elements combined with an operation

Dihedral Group: A group of symmetries of a regular polygon which includes rotations and reflections

Dihedral Groups

If you just consider the symmetries formed only by ρ and μ, those symmetries form the exact same sets as the dihedral group.

This is partly because with the rose window you are looking at the outside polygon, the inside one will follow along.

$n = 3$

This table is formed from the symmetries of R_3.

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_3	μ_1	μ_2
ρ_2	ρ_2	ρ_0	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	ρ_0	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_1	ρ_2	ρ_0
μ_3	μ_3	μ_1	μ_2	ρ_2	ρ_0	ρ_1

$n = 3$

This is the Cayley table of D_3.

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_3	μ_1	μ_2
ρ_2	ρ_2	ρ_0	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	ρ_0	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_1	ρ_2	ρ_0
μ_3	μ_3	μ_1	μ_2	ρ_2	ρ_0	ρ_1

$n = 3$

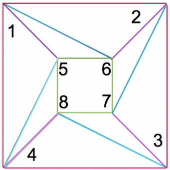
You can see here that $R_3 = D_3$ since they are the same table

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_3	μ_1	μ_2
ρ_2	ρ_2	ρ_0	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	ρ_0	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_1	ρ_2	ρ_0
μ_3	μ_3	μ_1	μ_2	ρ_2	ρ_0	ρ_1

n = 4

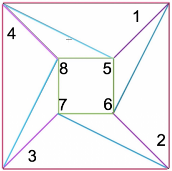
$$\rho : A_i - A_{i+1}, B_i - B_{i+1}$$

$A_1 - A_2 \quad B_1 - B_2$



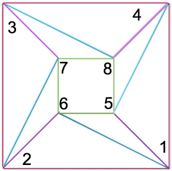
ρ_0

$A_2 - A_3 \quad B_2 - B_3$



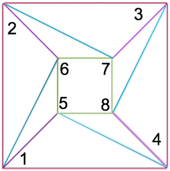
ρ_1

$A_3 - A_4 \quad B_3 - B_4$



ρ_2

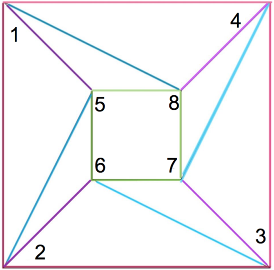
$A_4 - A_1 \quad B_4 - B_1$

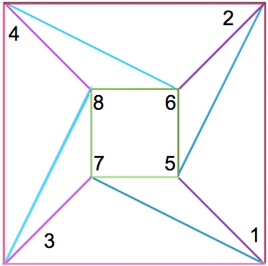


ρ_3

n = 4

$$\mu : A_i - A_{-i}, B_i - B_{-a-i}$$





$A_1 - A_1$ $B_1 - B_1$
$A_2 - A_4$ $B_2 - B_4$
$A_3 - A_3$ $B_3 - B_3$
$A_4 - A_2$ $B_4 - B_2$

$A_1 - A_3$ $B_1 - B_3$
$A_2 - A_2$ $B_2 - B_2$
$A_3 - A_1$ $B_3 - B_1$
$A_4 - A_4$ $B_4 - B_4$

$n = 4$

This table is formed from the symmetries of R_4.

	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	δ_2	δ_1	μ_1	μ_2
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	δ_1	δ_2	μ_2	μ_1
μ_1	μ_1	δ_1	μ_2	δ_2	ρ_0	ρ_2	ρ_1	ρ_3
μ_2	μ_2	δ_2	μ_1	δ_1	ρ_2	ρ_0	ρ_3	ρ_1
δ_1	δ_1	μ_2	δ_2	μ_1	ρ_3	ρ_1	ρ_0	ρ_2
δ_2	δ_2	μ_1	δ_1	μ_2	ρ_1	ρ_3	ρ_2	ρ_0

n = 4

This is the Cayley table of D_4.

	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_0	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	R_0	D'	D	H	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	H	D'	D
R_{270}	R_{270}	R_0	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	R_0	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	R_0	R_{270}	R_{90}
D	D	V	D'	H	R_{270}	R_{90}	R_0	R_{180}
D'	D'	H	D	V	R_{90}	R_{270}	R_{180}	R_0

$n = 4$

You can see here that $R_4 = D_4$ since they are the same table

	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	δ_2	δ_1	μ_1	μ_2
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	δ_1	δ_2	μ_2	μ_1
μ_1	μ_1	δ_1	μ_2	δ_2	ρ_0	ρ_2	ρ_1	ρ_3
μ_2	μ_2	δ_2	μ_1	δ_1	ρ_2	ρ_0	ρ_3	ρ_1
δ_1	δ_1	μ_2	δ_2	μ_1	ρ_3	ρ_1	ρ_0	ρ_2
δ_2	δ_2	μ_1	δ_1	μ_2	ρ_1	ρ_3	ρ_2	ρ_0

Types of Symmetry

We've seen that

$$R_n = D_n$$

so the symmetries of rose window graphs are dihedral groups.

Types of Symmetry

Lets establish that σ has the same kind of symmetry property as ρ and μ

Definitions

Definitions:

Isomorphism: A relationship between two sets such that preserves the binary relationships between elements of the sets at preserved; $A \cong B$

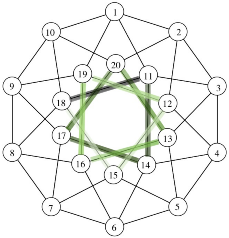
N-Cycle: Permutation cycle

Co-Prime: Two numbers are co-prime if the only common factors they have is 1.

Definitions

N-Cycle of 1 means that you will go through all of the elements in that set once before repeating any of the elements.

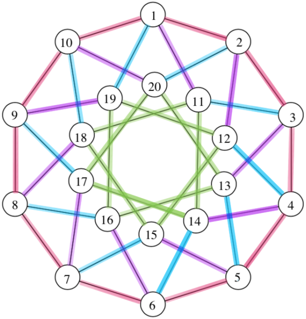
In the sense of Rose Window Graphs that means all the vertices (elements) must all be connected and you only pass through the same vertex once before every vertex is covered.

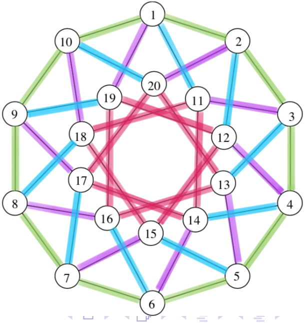


Isomorphism

Before we continue it is important to include the isomorphic equation essential to a rose window, If $(n, r) = 1$ then,

$$R_n(a, r) \cong R_n(ar^{-1}, r^{-1})$$





Isomorphism

Before we continue it is important to include the isomorphic equation essential to a rose window, If $(n, r) = 1$ then,

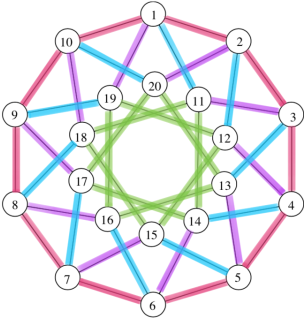
$$R_n(a, r) \cong R_n(ar^{-1}, r^{-1})$$

the isomorphism is being given by

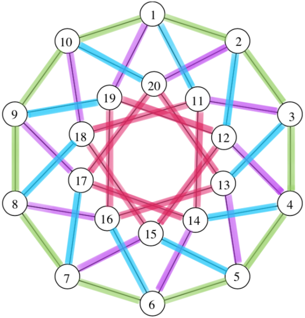
$$A_i \to B_{-ir^{-1}}, B_i \to A_{-ir^{-1}}$$

Isomorphism

$$A_i \to B_{-ir}, B_i \to A_{-ir}$$

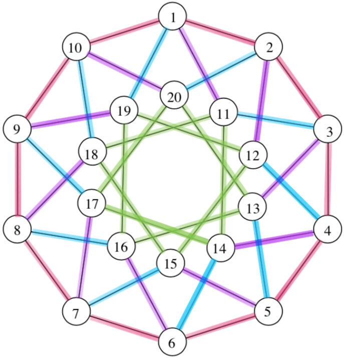


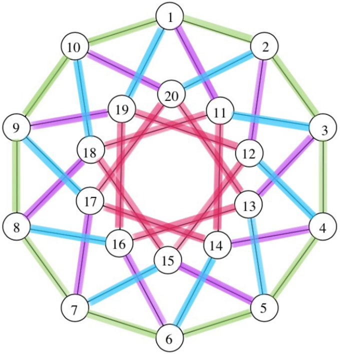
$$A_i \to B_{-ir^{-1}}, B_i \to A_{-ir^{-1}}$$



σ

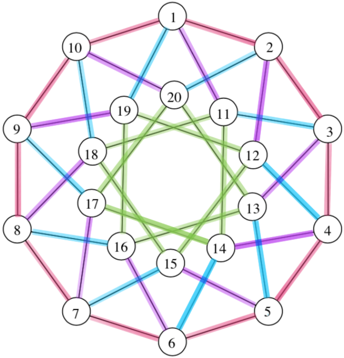
Lemma: There is a symmetry σ that sends the rims to the hubs and the hubs to the rims

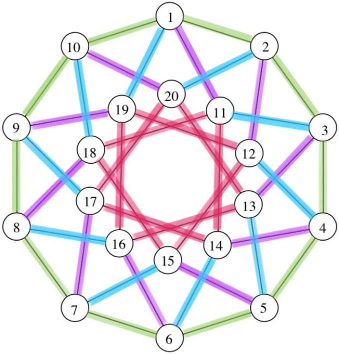




σ

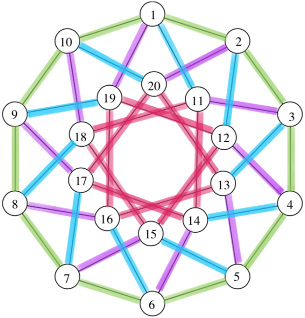
Lemma: There is a symmetry σ that sends the rims to the hubs and the hubs to the rims if and only if $r^2 = \pm 1$ and $ra = \pm a$.



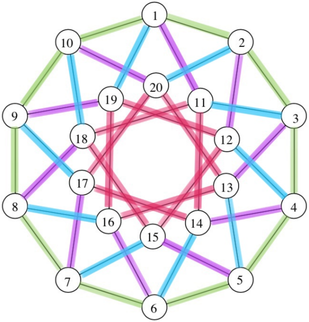


Isomorphism

Isomorphism result:



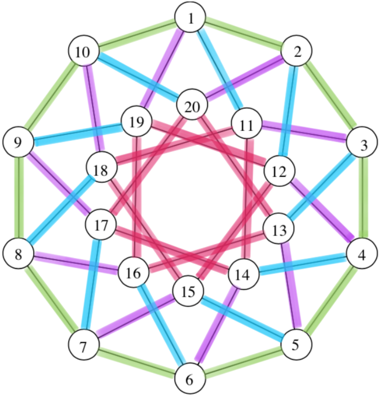
Lemma result:



σ proof $\rightarrow$

Assume: $r^2 = \pm 1$ and $ra = \pm a$

Then since $ra = \pm a$, we can assume that $ra = -a$.



σ proof $\rightarrow$

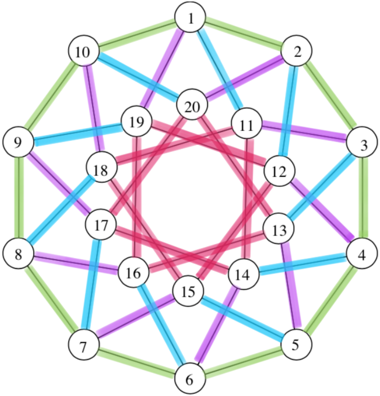
Assume: $r^2 = \pm 1$ and $ra = \pm a$

Then since $ra = \pm a$, we can assume that $ra = -a$.

Consider the isomorphism which is given by

$$A_i \rightarrow B_{-ir^{-1}} \quad B_i \rightarrow A_{-ir^{-1}}$$

.



σ proof $\rightarrow$

Assume:

$$r^2 = \pm 1 \text{ and } ra = \pm a$$

Then since $ra = \pm a$, we can assume that $ra = -a$. Consider the isomorphism which is given by

$$A_i \rightarrow B_{-ir^{-1}} \quad B_i \rightarrow A_{-ir^{-1}}$$

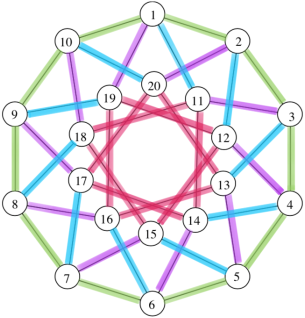
Therefore σ can allow

$$A_i \rightarrow B_{ri}, \qquad B_i \rightarrow A_{ri}$$

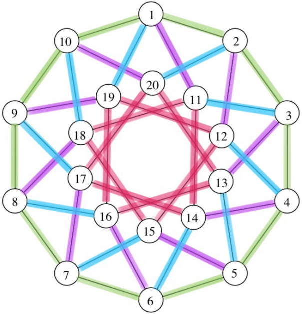
to be given to accomplish the interchanging of the rim and hub edges.

σ proof $\rightarrow$

$A_i \rightarrow B_{-ir^{-1}} \quad B_i \rightarrow A_{-ir^{-1}}$



$A_i \rightarrow B_{ri}, \qquad B_i \rightarrow A_{ri}$



σ proof $\rightarrow$

Assumption: $r^2 = \pm 1$ and $ra = \pm a$

Recall: $A_i \rightarrow B_{i(-r^{-1})}, \quad B_i \rightarrow A_{i(-r^{-1})}$ Goal: $A_i \rightarrow B_{ri}, \quad B_i \rightarrow A_{ri}$

σ proof $\rightarrow$

Assumption: $r^2 = \pm 1$ and $ra = \pm a$

Recall: $A_i \rightarrow B_{i(-r^{-1})}, \quad B_i \rightarrow A_{i(-r^{-1})}$ Goal:
$A_i \rightarrow B_{ri}, \quad B_i \rightarrow A_{ri}$

Let:

$r = -1 \qquad -r^{-1} : -\frac{1}{r} = 1$ so $A_i \rightarrow B_{ri}, B_i \rightarrow A_{ri}$

$r = 1 \qquad -r^{-1} : -\frac{1}{r} = -1$ so $A_i \rightarrow B_{-ri}, B_i \rightarrow A_{-ri}$

σ proof $\rightarrow$

Assumption: $r^2 = \pm 1$ and $ra = \pm a$

Recall: $A_i \rightarrow B_{i(-r^{-1})}, \quad B_i \rightarrow A_{i(-r^{-1})}$ Goal: $A_i \rightarrow B_{ri}, \quad B_i \rightarrow A_{ri}$

Let:

$r = -1 \qquad -r^{-1} : -\frac{1}{r} = 1 \qquad$ so $\qquad A_i \rightarrow B_{ri}, B_i \rightarrow A_{ri}$

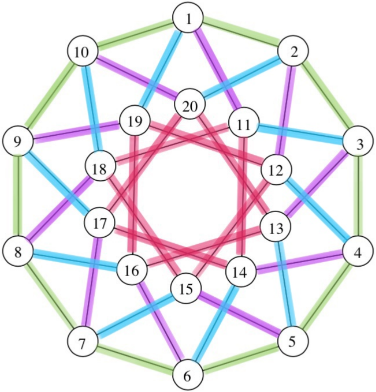
$r = 1 \qquad -r^{-1} : -\frac{1}{r} = -1 \qquad$ so $\qquad A_i \rightarrow B_{-ri}, B_i \rightarrow A_{-ri}$

Since $ar = \pm a$, then $r = \pm 1$. This allows $-r = r$

So $A_i \rightarrow B_{ri}, B_i \rightarrow A_{ri}$ and $r^2 = \pm 1$ and $ra = \pm a$

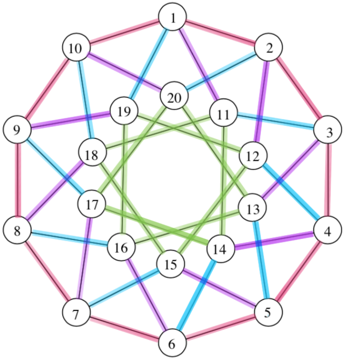
σ proof $\rightarrow$

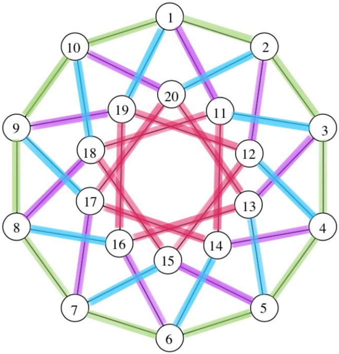
So $A_i \rightarrow B_{ri}, B_i \rightarrow A_{ri}$ and $r^2 = \pm 1, \quad ra = \pm a$



σ proof $\leftarrow$

Assume: σ is the symmetry which interchanges rim and hub edges.





σ proof $\leftarrow$

Assume: σ is the symmetry which interchanges rim and hub edges.

The hub edges must form an n-cycle and (r,n) must be 1.

σ proof $\leftarrow$

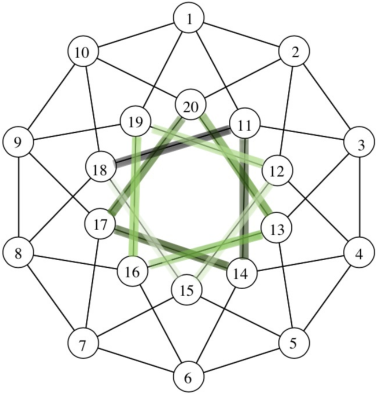
Recall:

n-cycle = 1

All of the elements in that set are present once before repeating any of the elements.

$(r, n) = 1$

r and n are co-prime, so their only common factor is 1.

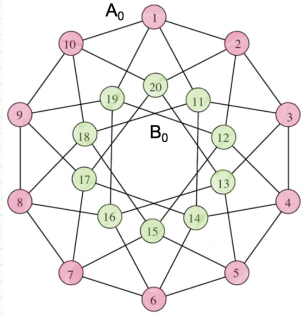


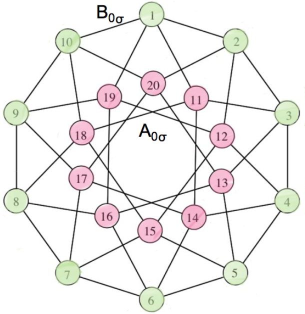
σ proof $\leftarrow$

By applying the correct element of D_n, we can assume that

$$A_{0\sigma} = B_0 \text{ and } B_{0\sigma} = A_0$$

.





σ proof $\leftarrow$

By applying the correct element of D_n, we can assume that

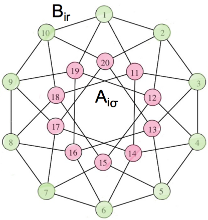
$$A_{0\sigma} = B_0 \text{ and } B_{0\sigma} = A_0$$

.

By a correct choice of r such that, we can assume

$$A_{1\sigma} = B_r, A_{2\sigma} = B_{2r}, ..., A_{i\sigma} = B_{ir}$$

.

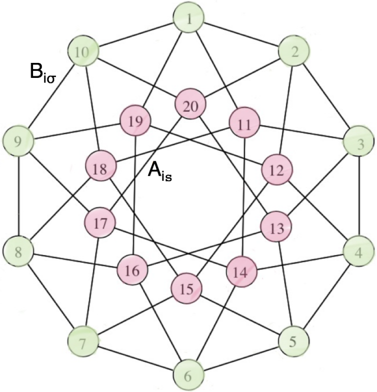


σ proof $\leftarrow$

Now $B_{1\sigma}$ is a rim vertex and A_s is a hub vertex. In general

$$B_{i\sigma} = A_{is}$$

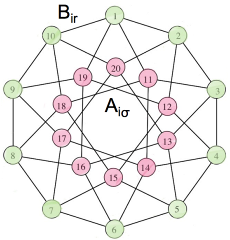
s denotes that A is a hub vertex
σ denotes the symmetry was applied to that vertex



σ proof $\leftarrow$

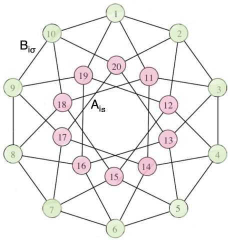
The symmetry is applied:

$A : A_{i\sigma} = B_{ir}$



The notation is X_{ir} if the vertices were sent to the outer level

$B : B_{i\sigma} = A_{is}$

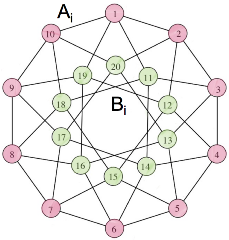


The notation is X_{is} if the vertices were sent to the inner level

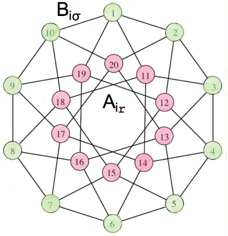
σ proof $\leftarrow$

This means that σ^2 sends $A_0 \to A_0$ and $A_1 \to A_{rs}$.

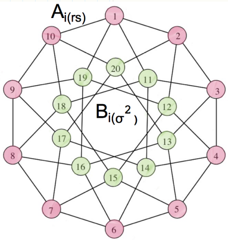
$A_i, \ B_i$



$A_{ir}, \ B_{i\sigma}$



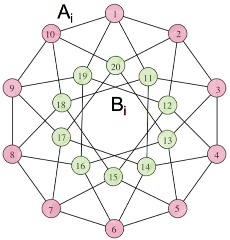
$A_{i(rs)}, \ B_{i(\sigma^2)}$

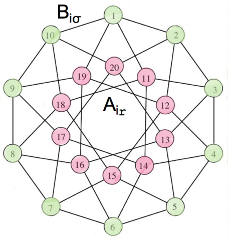


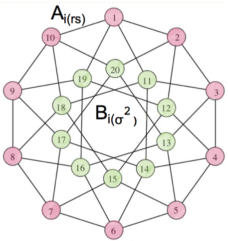
σ proof $\leftarrow$

This means that σ^2 sends $A_0 \to A_0$ and $A_1 \to A_{rs}$. Thus $rs = \pm 1$.

$A_i, \; B_i$ $\quad$ $A_{ir}, \; B_{i\sigma}$ $\quad$ $A_{i(rs)}, \; B_{i(\sigma^2)}$



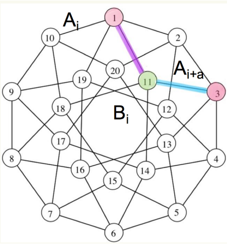




σ proof $\leftarrow$

This means that σ^2 sends $A_0 \to A_0$ and $A_1 \to A_{rs}$. Thus $rs = \pm 1$. Now consider the spoke edges that join

$$A_i \text{ to } B_i \text{ to } A_{i+a}$$



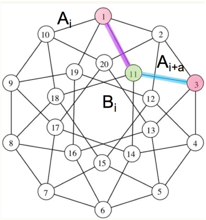
σ proof $\leftarrow$

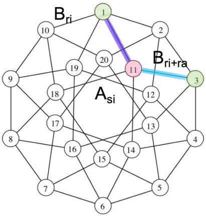
This means that σ^2 sends $A_0 \to A_0$ and $A_1 \to A_{rs}$. Thus $rs = \pm 1$. Now consider the spoke edges that join

$$A_i \text{ to } B_i \text{ to } A_{i+a}$$

σ sends those vertices to

$$B_{ri} \text{ to } A_{si} \text{ to } B_{ri+ra}$$





σ proof $\leftarrow$

One of the following systems must hold for all $i \in \mathbb{Z}_n$:

1. $ri = si$; $\quad ri + ra + a = si$
2. $ri + ra = si$; $\quad ri + a = si$

e

σ proof $\leftarrow$

One of the following systems must hold for all $i \in \mathbb{Z}_n$:

1. $ri = si$; $\quad ri + ra + a = si$
2. $ri + ra = si$; $\quad ri + a = si$

Let $i = 1$ in system 1: it yields $ra = -a$ and $r^2 = \pm 1$

Substitute ri for si.

$$ri + ra + a = si$$
$$si + ra + a = si$$
$$ra + a = 0$$
$$ra = -a$$

If $ra = -a$, then $r^2 = \pm 1$

σ proof $\leftarrow$

One of the following systems must hold for all $i \in \mathbb{Z}_n$:

1. $ri = si$; $\quad ri + ra + a = si$
2. $ri + ra = si$; $\quad ri + a = si$

Let $i = 1$ in system 1: it yields $ra = -a$ and $r^2 = \pm 1$
In system 2, it is followed that $ra = a$, so $a = (s - r)i$ for all i.

$$ri + ra = si$$
$$ri = si - ra$$
$$\text{Substitute } si - ra \text{ for } ri$$
$$(si - ra) + a = si$$
$$si - ra + a = si$$
$$ra = a.$$

$$ri + a = si$$
$$a = si - ri$$
$$a = (s - r)i.$$

σ proof $\leftarrow$

One of the following systems must hold for all $i \in \mathbb{Z}_n$:

1. $ri = si$; $\quad ri + ra + a = si$
2. $ri + ra = si$; $\quad ri + a = si$

Let $i = 1$ in system 1: it yields $ra = -a$ and $r^2 = \pm 1$
In system 2, it is followed that $ra = a$, so $a = (s - r)i$ for all i.
Letting $i = 0$ shoes that $a = 0$, which is not allowed.

σ proof $\rightarrow$ Summary

Assume: $r^2 = \pm 1$ and $ra = \pm a$

Isomorphism of a rose window graph says:

$$A_i \rightarrow B_{-ir^{-1}}, \quad B_i \rightarrow A_{-ir^{-1}}$$

By using the assumption, the isomorphism can manipulated and become:

$$A_i \rightarrow B_{ri}, \quad B_i \rightarrow A_{ri}$$

This interchanges the hub and rim edges

σ proof $\leftarrow$ Summary

Assume: σ interchanges rim and hub edges.

Choose a n and r such that (r, n) is co-prime.

If σ is applied to an vertex, its corresponding vertex:
 Is sent outside, an r added: $A_{i\sigma} : B_1 \rightarrow B_{1r}$
 Is sent inside, an s is added: $B_{i\sigma} : A_1 \rightarrow A_{1s}$

By manipulating the edge equations, the subscripts (i, a, r, s) show that $r^2 = \pm 1$ and $ra = \pm a$

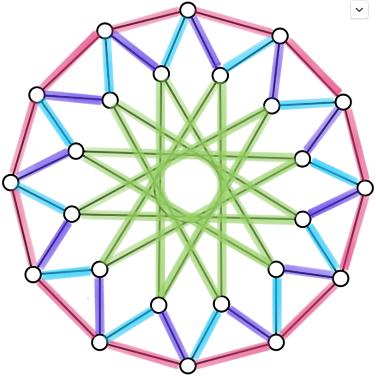
This interchanges the hub and rim edges

Examples

- From around campus
- Portions of a window
- How they hold with the Symmetry

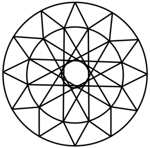
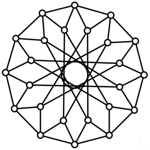
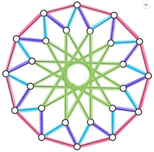
Examples from around Campus

Chapel Window



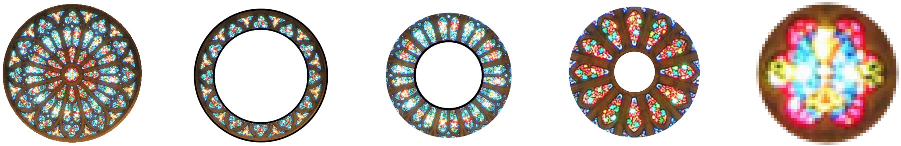
Chapel Window



From Around Campus

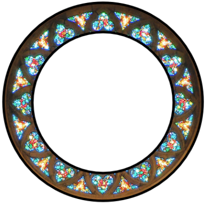
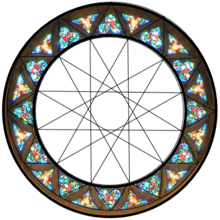
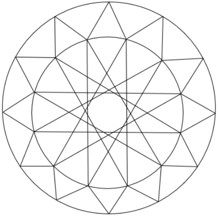
The rose window from All Saint's is comprised of 4 levels.



From Around Campus

Portions of a Window

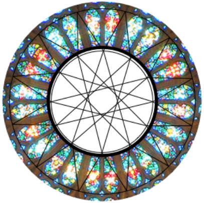
This is the outer most rose window graph of the Rose Window. It is denoted as $R_{12}(1,5)$

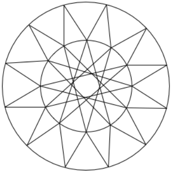




Portions of a Window

Portions of a Window

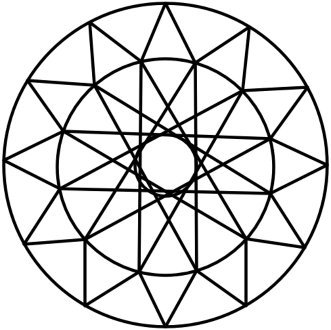
This is the second layer of the of the Rose Window. It is denoted as $R_{12}(1,5)$



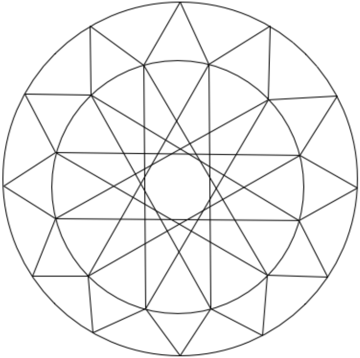


Portions of a Window

Rose Window from Chapel

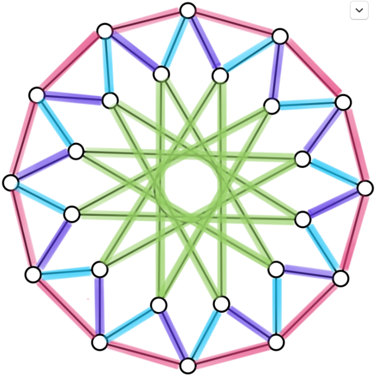


Rose Window from All Saint's



Showing they Hold

- Rim: A_i - A_{i+1} (pink)
- In-Spoke: A_i - B_i (purple)
- Out-Spoke: B_i - A_{i+a} (blue)
- Hub: B_i - B_{i+r} (green)
- $\rho : A_i - A_{i+1}, \quad B_i - B_{i+1}$
- $\mu : A_i - A_{-i}, \quad B_i - B_{-a-i}$

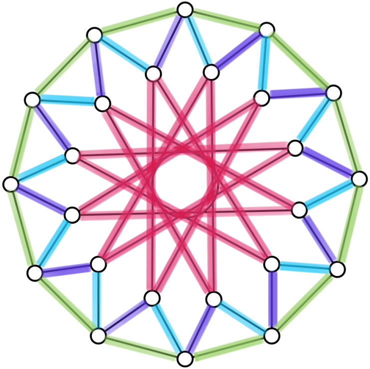


$R_{12}(1,5)$

Showing they Hold: σ

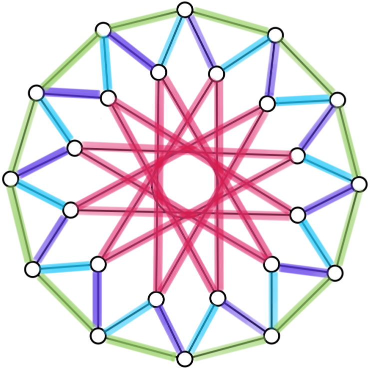
Isomorphism:

$A_i \rightarrow B_{-ir^{-1}}, \quad B_i \rightarrow A_{-ir^{-1}}$



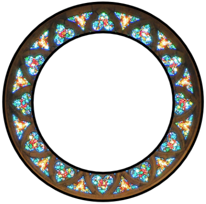
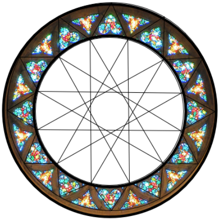
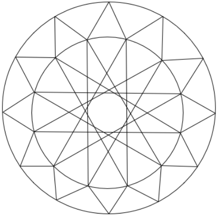
By assumption:

$A_i \rightarrow B_{ri}, \quad B_i \rightarrow A_{ri}$



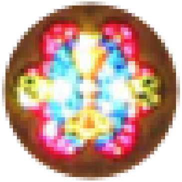
Portions of a Window

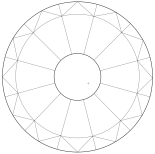
This is the outer most rose window graph of the Rose Window. It is denoted as $R_{12}(1,5)$

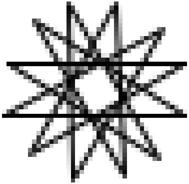




Showing they Hold

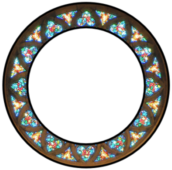
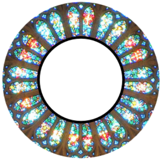
The other layers of the rose window graph of the Rose Window follow suit in the sense that are denoted as $R_{12}(1,5)$

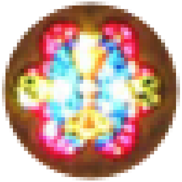


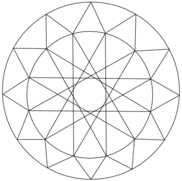
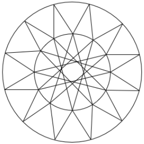
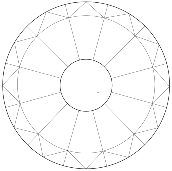
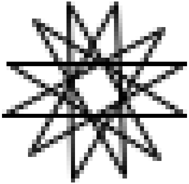




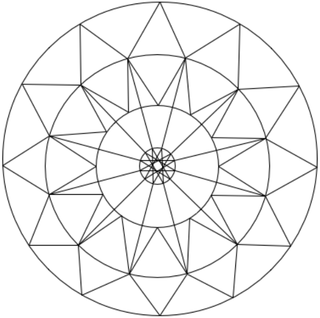
Showing they Hold

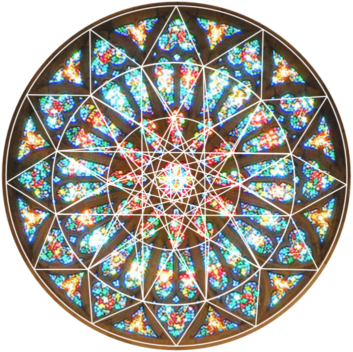


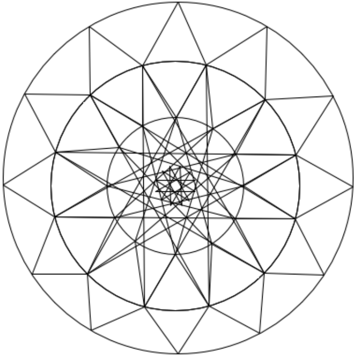





Rose Window



Rose Window





Rose Window