A NOTE ON BAND SURGERY AND THE SIGNATURE OF A KNOT

ALLISON H. MOORE AND MARIEL VAZQUEZ

ABSTRACT. Band surgery is an operation which transforms a knot or link in the three-sphere into another knot or link. We prove that if two quasi-alternating knots K and K' of the same square-free determinant are related by a band surgery, then the absolute value of the difference in their signatures is either 0 or 8. This obstruction follows from a more general theorem about the difference in the Heegaard Floer d-invariants for two L-spaces of the same square-free first homology order, and that are related by distance one Dehn fillings. These results imply that $T(2, 5)$ is the only torus knot $T(2, m)$ with m square-free that admits a chirally cosmetic banding, i.e. a band surgery operation to its mirror image. Band surgery on knots has important applications to the study of reconnection events in polymer chains, particularly the local action of recombination enzymes on circular DNA molecules. In this note we report on preliminary numerical data based on Monte Carlo simulations of non-coherent banding on knots in the simple cubic lattice. For lattice knots of crossing number at most eight, we observe chirally cosmetic bandings for only three knots: 5_1, 8_8 and 8_{20}. The relative likelihood of these events in the numerical experiments is negligible, thus suggesting that chirality cosmetic bandings are rare.

1. INTRODUCTION

Band surgery is an operation on knots or links in the three-sphere. Let L be a link, and let $b : I \times I \to S^3$ be an embedding of the unit square with $L \cap b(I \times I) = b(I \times \partial I)$. Let L' denote the link obtained by replacing $b(I \times \partial I)$ in L with $b(\partial I \times I)$. Then we say that the link L' results from *band surgery* along L. Figure 1 illustrates band surgeries relating two pairs of knots. When L and L' are oriented links and band surgery respects their orientations, the operation is called *coherent* band surgery, otherwise its is called *non-coherent*. Coherent band surgery converts a knot into a two-component link, whereas non-coherent band surgery converts a knot to another knot. In this article we are concerned only with non-coherent band surgery operations relating unoriented knots.

For quasi-alternating knots, we find a new obstruction to band surgery via two knot invariants, the determinant $\det(K)$ and the signature $\sigma(K)$ of a knot. The determinant of a knot is an odd integer, whereas the signature of a knot is an even integer, and both are determined by the Seifert module of the knot [GL78, Tro62]. A quasi-alternating knot is a generalization of an alternating knot due to Ozsváth and Szabó [OS05b]. All alternating knots are quasi-alternating.

Theorem 1.1. *Let K and K' be a pair of quasi-alternating knots and suppose that $\det(K) = m = \det(K')$ for some square-free integer m. If there exists a band surgery relating K and K', then $|\sigma(K) - \sigma(K')|$ is 0 or 8.*

Two pairs of knots for which the above theorem applies are shown in Figure 1. Theorem 1.1 is reminiscent of Murasugi's well-known statement that $|\sigma(L) - \sigma(L')| \leq 1$ when L and L' are related
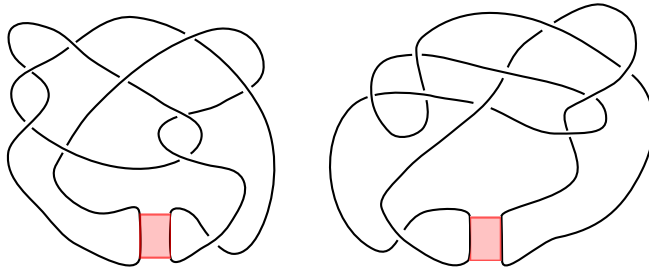
FIGURE 1. Non-coherent band surgery indicated by the shaded area relates the knots $7_1 = T(2,7)$ and 5_2^*, the mirror of 5_2 (left), and the knots 6_2 and 7_2 (right). Diagrams corresponding with these band moves were found by Zeković [Zek15]. In contrast, Theorem 1.1 implies that there cannot exist a band surgery between 7_1 and 5_2 or between 6_2 and 7_2^*, based on the values of the determinant and signature for these knots. See example 4.5.

by a coherent band surgery [Mur65, Lemma 7.1]. In general though, non-coherent band surgery may change the signature of a knot by arbitrary amounts. For example, the torus knot $T(2,m)$ has signature $1-m$ and is related by a single band surgery operation to the unknot, which has zero signature. Murasugi also proved that for any knot K, the signature controls the determinant of a knot modulo 4 [Mur65, Theorem 5.6]. More precisely, if $|\sigma(K)| \equiv 0$ (resp. 2) modulo 4, then $|\det(K)| \equiv 1$ (resp. 3) modulo 4. This immediately implies

$$|\sigma(K) - \sigma(K')| \equiv |\det(K) - \det(K')| \pmod 4,$$

for *any* pair of knots, which provides an explanation for the significance of the numbers 0 and 8 in Theorem 1.1. The determinant of a knot, or more generally the first homology of its branched double cover, can often provide an effective obstruction to the existence of a band surgery relating a pair of knots (see for example [AK14, KM14, Kan16]). However, when a pair of knots K, K' have branched double covers with isomorphic first homology (which implies $\det(K) = \det(K')$), such obstructions do not apply. Theorem 1.1 fills this gap in the case where m is square-free.

A special case occurs when a band surgery relates a knot K with its mirror image K^*. This is called a *chirally cosmetic banding*. For example, a chirally cosmetic banding relates the torus knot $T(2,5)$ with its mirror image $T(2,-5)$, as shown in Figure 2. In fact, it is an immediate corollary of Theorem 1.1 that $T(2,5)$ is the *only* nontrivial torus knot of the form $T(2,m)$, for m square-free that admits a chirally cosmetic banding. In section 4, we compare the square-free condition of Theorem 1.1 with the constructive examples of chirally cosmetic bandings given in [IJM17], for which the determinant is a square. We also report on the results of a numerical study of chirally cosmetic banding for knots embedded in the simple cubic lattice $\mathbb{Z}^3$. The relative probability of band surgery transitions between pairs of knots are computed with a Markov chain Monte Carlo (MCMC) algorithm with multiple Markov chain (MMC) sampling. In the numerical experiments, the relative likelihood of the observed chirally cosmetic bandings is negligible (see section 4.2). Chirally cosmetic bandings are observed in the numerical experiments for only three knot types: 5_1, 8_8, 8_{20}. Both the theoretical and numerical results suggest that chirally cosmetic banding is a rare phenomenon in general.

Band surgery on knots has important applications to the study of reconnection events in polymer chains, particularly the local action of recombination enzymes on circular DNA molecules. Knots and links in DNA are frequently modeled as self avoiding polygons in $\mathbb{Z}^3$ and studied via numerical
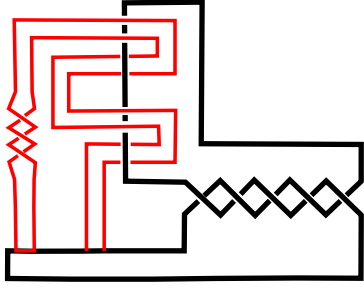
FIGURE 2. A band move taking the torus knot $T(2,5) = 5_1$ to its mirror $T(2,-5) = 5_1^*$. The existence of this band move is due to Zeković [Zek15].

experiments similar to the ones we conduct in section 4 (e.g. Stolz et al. in [SYB$^+$17]). The biological motivation for these simulations is briefly discussed in section 4.3.

Theorem 1.1 follows as a corollary of Theorem 1.2, a more general statement about the Heegaard Floer d-invariants of pairs of L-spaces related by integral surgery along a null-homologous knot.

Theorem 1.2. *Let Y and Y' be L-spaces with $H_1(Y) = H_1(Y') = \mathbb{Z}/m$, where m is an odd square-free integer. If Y' is obtained by a distance one surgery along any knot K in Y, then*

$$|d(Y, \mathfrak{t}) - d(Y', \mathfrak{t}')| = 2 \text{ or } 0$$

where $\mathfrak{t}$ and $\mathfrak{t}'$ denote the unique self-conjugate Spin^c structures on Y and Y'.

Heegaard Floer homology is a powerful package of three and four-manifold invariants due to Ozsváth and Szabó [OS03]. An *L-space* is a rational homology sphere whose Heegaard Floer homology is as simple as possible. One particularly useful invariant from the Heegaard Floer package is the *d-invariant* of the pair $(Y, \mathfrak{t})$, where Y is an oriented rational homology sphere and t is an element of Spin$^c(Y) \cong H^2(Y; \mathbb{Z})$. The d-invariant is a rational number, defined as the minimal grading of a certain submodule of the Heegaard Floer module $HF^+(Y, \mathfrak{t})$ [OS03]. The reader is referred to [OS03] for an introduction to these invariants, and all required background for this note is in sections 2 and 3.

The *distance* between two Dehn surgery slopes refers to their minimal geometric intersection number. A surgery slope that intersects the meridian of a knot exactly once is called a *distance one surgery*, or an *integral* surgery. Given two knots or links related by band surgery, the Montesinos trick [Mon76] implies that their branched double covers may be obtained by distance one Dehn fillings of a three-manifold with torus boundary. To prove Theorem 1.2 we will apply the Heegaard Floer mapping cone formula of Ozsváth and Szabó [OS11], and a theorem of Ni and Wu [NW15, Proposition 1.6] which describes the d-invariants of a manifold obtained by integral surgery along a null-homologous knot in an L-space in terms of certain integer-valued knot invariants due to Rasmussen [Ras03].

Organization. In section 2 we introduce notation and establish some prerequisite topological and homological information. Section 3 contains the necessary background in Heegaard Floer homology along with the proof of Theorem 1.2. In section 4 we prove Theorem 1.1 and further consider the case of chirally cosmetic pairs. We report the results of preliminary numerical studies on non-coherent banding and discuss the biological significance of this work.

2. PRELIMINARIES

2.1. **Homological preliminaries.** In this section we state some homological preliminaries and set notation. We assume all singular (co)homology groups have coefficients in $\mathbb{Z}$. Let K be a knot in a rational homology sphere Y, and let $M = Y - N(K)$ denote the complement of K in Y. A *slope* is an isotopy class of an unoriented simple closed curve on the boundary of M. The *Dehn filling* $M(\eta)$ is the closed, oriented three-manifold obtained by gluing a solid torus V to M by a homeomorphism which identifies a meridian of ∂V to the slope η on ∂M. When K is null-homologous, there is a standard choice of a meridian and longitude on the bounding torus ∂M. A slope η may be written in terms of this basis as $\eta = (a, b)$ and then rational Dehn surgery with surgery coefficient a/b along K in Y is well-defined and denoted by $Y_{a/b}(K)$. The *distance* between two surgery slopes on ∂M is the minimal geometric intersection number of the two curves, denoted $\Delta(\eta, \nu)$ for any pair of slopes η, ν. In this note we are particularly concerned with surgery slopes which intersect the meridian of K exactly once, i.e. *distance one surgeries*, or *integral* surgeries. When K is null-homologous, a homological argument (see for example [Gai18, Lemma 8.1]) shows that $H_1(M) \cong H_1(Y) \oplus \mathbb{Z}$, where the $\mathbb{Z}$-summand is generated by the meridian of K. This implies that $H_1(Y_{a/b}(K)) \cong H_1(Y) \oplus \mathbb{Z}/a$ by a standard Mayer-Vietoris argument.

Lemma 2.1. *Suppose that $|H_1(Y)| = |H_1(Y')|$ is square-free. If Y' is obtained from Y by a distance one surgery on a knot K, then K is null-homologous in Y and the surgery coefficient is ± 1.*

This essentially follows from the argument in [LM17, Theorem 2.4]. Indeed, Lemma 2.1 may be stated and proved in even greater generality: if $H_1(Y)$ may be written as a sum $\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2 \oplus \cdots \oplus \mathbb{Z}/d_k$, where each integer d_i is square-free, and $|H_1(Y)| = |H_1(Y')|$, then by an adaptation of [LM17, Theorem 2.4], we may similarly prove that K is null-homologous. We provide here a proof of the simpler case that each d_i is a distinct prime for brevity.

Proof. We assume that

$$H_1(Y) \cong \mathbb{Z}/p_1 \oplus \mathbb{Z}/p_2 \oplus \cdots \oplus \mathbb{Z}/p_k,$$

where the integers p_i are distinct primes. The manifold M is the complement of K in Y, and $H_1(M) \cong \mathbb{Z} \oplus H$ for some finite group H. The rational longitude λ_M is the unique slope in ∂M characterized by the property that its image $i_*(\lambda_M)$ under the map induced by the inclusion $\partial M \hookrightarrow M$ is of finite order in $H_1(M)$. We select (μ, λ_M) as a basis of $H_1(\partial M)$, where μ is some curve dual to λ_M. In this basis, filling slopes α and β yielding Y and Y', respectively, may be written as

$$\begin{aligned} \alpha &= p\mu + q\lambda_M \\ \beta &= r\mu + s\lambda_M \end{aligned}$$

for some integers $p, r \geq 0$ and q, s relatively prime to p, r. It turns out that the rational longitude controls the order of the first homology of any Dehn filling of M. (For a more detailed explanation, see [Wat12] or [LM17, Lemma 2.3].) Indeed, for any filling slope $\eta \neq \lambda_M$ we have

$$(1) \qquad\qquad |H_1(M(\eta)| = \mathrm{ord}_H\, i_*(\lambda_M) \cdot |H| \cdot \Delta(\eta, \lambda_M)$$

where $\mathrm{ord}_H \, i_*(\lambda_M)$ denotes the (necessarily finite) order of the rational longitude in H. Because we assume that $|H_1(Y)| = |H_1(Y')|$, equation (1) implies that

$$
\begin{aligned}
|H_1(Y)| = |H_1(M(\alpha))| &= \mathrm{ord}_H \, i_*(\lambda_M) \cdot |H| \cdot \Delta(\alpha, \lambda_M) \\
&= \mathrm{ord}_H \, i_*(\lambda_M) \cdot |H| \cdot \Delta(\beta, \lambda_M) \\
&= |H_1(M(\beta))| = |H_1(Y')|.
\end{aligned}
$$

This implies that $p = \Delta(\alpha, \lambda_M) = \Delta(\beta, \lambda_M) = r$. We also have the assumption that the distance $\Delta(\alpha, \beta)$ is one. Therefore

$$
1 = \Delta(\alpha, \beta) = p(q - s),
$$

and so $p = 1$. By (1) we have that

$$
|H_1(Y)| = |H_1(M(\alpha))| = \mathrm{ord}_H \, i_*(\lambda_M) \cdot |H|.
$$

Now, $i_*(\lambda_M)$ generates a subgroup of H, hence $\mathrm{ord}_H \, i_*(\lambda_M)$ divides $|H|$. But $|H_1(Y)|$ is square-free, therefore $\mathrm{ord}_H \, i_*(\lambda_M) = 1$. Because $\mathrm{ord}_H \, i_*(\lambda_M) = 1$, the rational longitude is null-homologous in M, as well as in the filled manifold $M(\alpha)$. Moreover, in $M(\alpha)$ the rational longitude is homologous to the core of the filling torus, which is K. Thus we have that K is null-homologous in $M(\alpha) = Y$.

Now we take the standard basis (μ, λ) on $H_1(\partial M)$, where μ is the meridian of K, and λ denotes the preferred longitude. Because $H_1(Y_{a/b}(K)) = H_1(Y) \oplus \mathbb{Z}/a$, the filling slopes α and β may be written as μ and $\mu + q\lambda$, for some integer q. The condition that the slope yielding Y' is distance one from the meridian implies the slope must in fact be $\mu \pm \lambda$, i.e. the surgery coefficient is $a/b = \pm 1$. $\qquad \square$

For the remainder of the article, we will assume that K is a null-homologous knot in Y with meridian μ and longitude λ. We will be primarily concerned with slopes of the form $n\mu + \lambda$, for $n \in \mathbb{Z}$, which are framing curves on the boundary of a neighborhood of K. These slopes intersect the meridian once transversely and inherit an orientation from K. The notation $Y_n(K)$ will denote the result of integral surgery along K, and K_n will denote the core of the surgery $Y_n(K)$.

The set $\mathrm{Spin}^c(Y)$ is the space of nowhere vanishing vector fields on Y modulo homotopy outside of a ball. There is a non-canonical correspondence $\mathrm{Spin}^c(Y) \cong H^2(Y)$. We write $\mathfrak{t}$ to denote an element of $\mathrm{Spin}^c(Y)$, and write $\underline{\mathrm{Spin}}^c(Y, K)$ to denote the relative Spin^c structures on $(M, \partial M)$. Note that $\underline{\mathrm{Spin}}^c(Y, K)$ has an affine identification with $H^2(Y, K)$, which by excision is isomorphic with $H^2(M, \partial M) \cong H_1(M)$. In particular, because K is null-homologous and $H_1(M) \cong H_1(Y) \oplus \mathbb{Z}$, we may label an element ξ of $\underline{\mathrm{Spin}}^c(Y, K)$ by $\xi = (\mathfrak{t}, s)$, where $\mathfrak{t} \in \mathrm{Spin}^c(Y)$ and $s \in \mathbb{Z}$. Note that when $H_1(Y)$ has odd order, there is a unique self-conjugate Spin^c structure $\mathfrak{t}_0$, distinguished from the others by the requirement that it satisfies $c_1(\mathfrak{t}_0) = 0 \in H^2(Y)$.

2.2. **Band surgery.** Given a link L in S^3, let $b : I \times I \to S^3$ be an embedding of the unit square such that $L \cap b(I \times I) = b(I \times \partial I)$. Two links L and L' are related by a *band surgery* if $L' = (L - b(I \times \partial I)) \cup b(\partial I \times I)$. See the example in Figure 1. Band surgeries that are compatible with specified orientations on L and L' are called coherent; otherwise a band surgery is called non-coherent. In this note we are only concerned with non-coherent band surgeries between a pair of knots K and K', which are necessarily unoriented knots. Elsewhere in the literature non-coherent band surgery is called incoherent band surgery or unoriented banding (e.g. [AK14, Kan16, IJM17]).

A band surgery can be described as a 2-string tangle replacement. In such a tangle replacement, the pairs (S^3, L) and (S^3, L') are decomposed as

$$(S^3, L) = (B_o, t_o) \cup (B, t) \qquad \text{and} \qquad (S^3, L') = (B_o, t_o) \cup (B, t')$$

where S^3 is the union of the two three-balls B and B_o, where the sphere $\partial B = \partial B_o$ intersects each of L and L' transversely in four points, and where $t = (B \cap L)$, $t' = (B \cap L')$ and $t_o = (B_o \cap L) = (B_o \cap L')$. The pairs (B, t), (B, t') and (B_o, t_o) are two-string tangles. Note that the "outside" tangle (B_o, t_o) is shared by both of L and L'. After possibly isotoping L and L', any band surgery can be described by a standard local two-string tangle replacement in a knot diagram, where $(B, t) = (\asymp) = (0)$ and $(B, t') = ()() = (\infty)$.

The Montesinos trick [Mon76] implies that the branched double covers of knots (or links) related by a band surgery are obtained from distance one Dehn fillings of a three-manifold M with torus boundary. This three-manifold M is the double cover of the ball B_o branched over t_o. Alternatively, M can be described as $\Sigma(L) - N(K)$, where here the knot K is the lift in the branched cover of the properly embedded arc arising as the core of the band in (B, t). The filling slope α yielding $\Sigma(L)$ is the meridian of K, and the filling slope β yielding $\Sigma(L')$ is distance one from α, meaning α and β intersect geometrically once.

Remark 2.2. Alternatively, one may consider a pair of links related by a single band surgery to be obtained from the horizontal $(\asymp)$ and vertical $()()$ smoothings of a crossing $(\nearrow\!\!\!\!\nwarrow)$ in a diagram of a some link. That is, the three links $\{(\nearrow\!\!\!\!\nwarrow), (\asymp), ()()\}$ form an unoriented skein triple. Any two links related by a single band surgery may also be viewed as the numerator and denominator closures of some fixed tangle, though these are not perspectives explored in the current article.

3. Heegaard Floer invariants and the proof of Theorem 1.2

3.1. **Heegaard Floer background and the mapping cone formula.** We will assume some familiarity with Heegaard Floer homology, referring the reader to [OS03, OS11] for more information. We take all Heegaard Floer complexes with coefficients in the field $\mathbb{F} = \mathbb{Z}/2$. One of the main components of the Heegaard Floer package we will use is the d-invariant $d(Y, \mathfrak{t})$, or *correction term*, which is a rational number associated to a Spin^c rational homology sphere $(Y, \mathfrak{t})$. More specifically, given a rational homology sphere Y, the Heegaard Floer module $HF^+(Y)$ splits over Spin^c structures, and in each summand we have

$$(2) \qquad\qquad HF^+(Y, \mathfrak{t}) = \mathbb{F}[U, U^{-1}]/U \cdot \mathbb{F}[U] \oplus HF^+_{red}(Y, \mathfrak{t}).$$

The first summand is abbreviated $\mathcal{T}^+_d$ and referred to as the *tower*. The second summand is a torsion $\mathbb{F}[U]$-module. An *L-space* is a rational homology sphere whose Heegaard Floer homology $HF^+(Y)$ is a free $\mathbb{F}[U]$-module with rank $|H^2(Y; \mathbb{Z})|$. That is, the torsion summand in (2) vanishes, leaving only a tower in each Spin^c structure. The d-invariant $d := d(Y, \mathfrak{t})$ is defined to be the minimal Maslov grading of the tower. The d-invariants switch sign under orientation-reversal [OS03].

Associated to (Y, K) and each relative Spin^c structure ξ in $\underline{\text{Spin}^c}(Y, K)$ is the knot Floer chain complex $C_\xi = CFK^\infty(Y, K, \xi)$ [Ras03, OS04, OS11]. The complex is $\mathbb{Z}$-filtered over $\mathbb{F}[U, U^{-1}]$ with a second filtration induced by the action of the variable U. We denote these two filtrations as $(i, j) = $ *(algebraic, Alexander)*. The chain complex also has a homological Maslov grading that we suppress in the notation. Multiplication by U decreases the Maslov grading by two and the

Alexander filtration by one, and the action of $PD[\mu]$ on ξ shifts the Alexander filtration by one, i.e. $C_{\xi+PD[\mu]} = C_\xi[(0, -1)]$.

As in [OS11], for each ξ in $\underline{\mathrm{Spin}}^c(Y, K)$ there are complexes $A_\xi^+ = C_\xi\{\max\{i, j\} \geq 0\}$ and $B_\xi^+ = C_\xi\{i \geq 0\}$. The complex B_ξ^+ is $CF^+(Y, G_{Y,K}(\xi))$. The map $G_{Y,K} : \underline{\mathrm{Spin}}^c(Y, K) \to \mathrm{Spin}^c(Y)$ is defined in [OS11, Section 2.2], and sends a relative Spin^c structure to a Spin^c structure in the target manifold indicated by the subscript. The complex A_ξ^+ represents the Heegaard Floer complex of a large surgery $Y_N(K)$ in a certain Spin^c structure, where $Y_N(K)$ is obtained by Dehn surgery along a framing curve $N\mu + \lambda$ with $N >> 0$. There are analogous complexes in the 'hat' version of Heegaard Floer homology. In particular we have $\widehat{A}_\xi = C_\xi\{\max\{i, j\} = 0\}$ and $\widehat{B}_\xi = C_\xi\{i = 0\} \cong \widehat{CF}(Y, G_{Y,K}(\xi))$. The complexes are related by chain maps

$$
(3) \qquad
\begin{aligned}
v_\xi^+ &: A_\xi^+ \to B_\xi^+, & h_\xi^+ &: A_\xi^+ \to B_{\xi+PD[K_n]}^+, \\
\widehat{v}_\xi &: \widehat{A}_\xi \to \widehat{B}_\xi, & \widehat{h}_\xi &: \widehat{A}_\xi \to \widehat{B}_{\xi+PD[K_n]},
\end{aligned}
$$

where K_n is the push-off of K inside $Y - N(K)$ using framing $n\mu + \lambda$. In [OS11, Theorem 4.1], it is shown that the maps v_ξ^+ and h_ξ^+ correspond with a negative definite cobordism $W_N' : Y_N(K) \to Y$ equipped with the Spin^c structures $\mathfrak{v}$ and $\mathfrak{h} = \mathfrak{v} + PD[\widehat{F}]$, respectively, which extend a given Spin^c structure on $Y_N(K)$. Here, $[\widehat{F}]$ generates $H_2(W_N', Y)$ and is represented by a capped-off Seifert surface for K.

The complexes A_ξ^+ and B_ξ^+ each contain a non-torsion summand, i.e. a tower. On homology, each of the maps v_ξ^+ and h_ξ^+ induces an endomorphism of the towers, which is multiplication by U_ξ^V or U_ξ^H for integers $V_\xi \geq 0$ and $H_\xi \geq 0$. This defines the knot invariants V_ξ and H_ξ, which appeared originally as the *local h-invariants* in [Ras03]. We remark for later use that when $V_\xi > 0$, the corresponding map $\widehat{v}_\xi$ is identically zero, and similarly $H_\xi > 0$ implies $\widehat{h}_\xi$ is zero.

Recall that because K is null-homologous, $\underline{\mathrm{Spin}}^c(Y, K) \cong \mathrm{Spin}^c(Y) \oplus \mathbb{Z}$, where the $\mathbb{Z}$-summand is generated by the meridian of K [Gai18, Lemma 8.1]. We write $\xi = (\mathfrak{t}, s)$ for $\mathfrak{t} \in \mathrm{Spin}^c(Y)$ and $s \in \mathbb{Z}$. Fixing $\mathfrak{t} \in \mathrm{Spin}^c(Y)$, we have the subgroup $\{\mathfrak{t}\} \oplus \mathbb{Z} \cong \mathbb{Z}$. Because the following properties have appeared in various forms throughout the literature, we provide just a sketch of the proof.

Property 3.1. *Let K be a null-homologous knot in a rational homology sphere Y and let $\mathfrak{t}$ be a self-conjugate Spin^c structure on Y. Then the invariants $V_{(\mathfrak{t},s)}$ and $H_{(\mathfrak{t},s)}$ satisfy:*

    (1) $V_{(\mathfrak{t},s)} \geq V_{(\mathfrak{t},s+1)} \geq V_{(\mathfrak{t},s)} - 1,$

    (2) $V_{(\mathfrak{t},s)} = H_{(\mathfrak{t},-s)},$

    (3) $H_{(\mathfrak{t},s)} \geq V_{(\mathfrak{t},s)}$ *for $s \geq 0$.*

Proof sketch. Since $\underline{\mathrm{Spin}}^c(Y, K) \cong \mathrm{Spin}^c(Y) \oplus \mathbb{Z}$, we have that $\xi + PD[\mu] = (\mathfrak{t}, s) + PD[\mu] = (\mathfrak{t}, s+1)$. When we fix a self-conjugate Spin^c structure on $\mathfrak{t}$ on Y, the statements above relating $V_{(\mathfrak{t},s)}$ and $H_{(\mathfrak{t},s)}$ are analogous to those for the invariants V_s and H_s, with $s \in \mathbb{Z}$, for the case of a knot in an integer homology sphere. In particular, Property (1) follows as a direct analogue of [Ras03, Property 7.6] (see also [NW15, Lemma 2.4]). Next, because conjugation changes the sign of the first Chern class and the H^2-action, we may verify using the formulas of [OS08, Section 4.3] that $\mathfrak{h}_{(\mathfrak{t},s)}$ and $\mathfrak{v}_{(\mathfrak{t},s)}$ are conjugate Spin^c structures on the four-manifold cobordism W_N'. This implies Property (2). Finally, (1) and (2) imply (3). See also [HLZ15, Lemmas 2.3-2.5]. $\qquad \square$

The proof of Theorem 1.2 below will require the mapping cone formula for the Heegaard Floer homology of the integral surgery $Y_n(K)$. We give only a terse review here, sending the reader to Ozsváth and Szabó for details [OS08, OS11]. The generalization of the mapping cone formula specific to rational surgeries on null-homologous knots in L-spaces is also reviewed in [NW15, Gai18]. Note that for our applications, we only require the formulation for integral surgery and 'hat' homology.

For each $\mathfrak{t} \in \mathrm{Spin}^c(Y)$ and $0 \leq i < n$, sum up the complexes $\widehat{A}_\xi = \widehat{A}_{(\mathfrak{t},s)}$ and $\widehat{B}_\xi = \widehat{B}_{(\mathfrak{t},s)}$ into

$$\widehat{\mathbb{A}}_{(\mathfrak{t},i)} = \bigoplus_{s \in \mathbb{Z}} (s, \widehat{A}_{(\mathfrak{t},i+ns)})$$

and

$$\widehat{\mathbb{B}}_{(\mathfrak{t},i)} = \bigoplus_{s \in \mathbb{Z}} (s, \widehat{B}_{(\mathfrak{t},i)}),$$

where the first component of each tuple records the index of the summand. Using the maps $\widehat{v}_\xi$ and $\widehat{h}_\xi$ from equation (3) above, we define the map $\widehat{\mathbb{D}}_{(\mathfrak{t},i)} : \widehat{\mathbb{A}}_{(\mathfrak{t},i)} \to \widehat{\mathbb{B}}_{(\mathfrak{t},i)}$ by

$$\widehat{\mathbb{D}}_{(\mathfrak{t},i),n}(s, a_s) = (s, \widehat{v}_{(\mathfrak{t},i+ns)}(a_s)) + (s+1, \widehat{h}_{(\mathfrak{t},i+ns)}(a_s)).$$

The mapping cone complex of $\widehat{\mathbb{D}}_{(\mathfrak{t},i),n}$ is denoted $\widehat{\mathbb{X}}_{(\mathfrak{t},i),n}$. Let us omit the surgery coefficient 'n' and write the summand of the cone corresponding to the equivalence class of $\xi = (\mathfrak{t}, i)$ more concisely as $\widehat{\mathbb{X}}_\xi$.

Theorem 3.2 (Ozsváth and Szabó, [OS11]). *Let $\xi \in \underline{\mathrm{Spin}}^c(Y, K)$. Then there is a relatively-graded isomorphism of groups*

(4) $$H_*(\widehat{\mathbb{X}}_\xi) \cong \widehat{HF}(Y_n(K), G_{Y_n(K),K_n}(\xi)).$$

Given a Spin^c structure in $\mathrm{Spin}^c(Y_n(K)) \cong \mathrm{Spin}^c(Y) \oplus \mathbb{Z}/n$, there is a unique Spin^c structure on Y which extends over the two-handle cobordism $W_n(K) : Y \to Y_n(K)$ and this defines the projection from $\mathrm{Spin}^c(Y_n(K))$ to $\mathrm{Spin}^c(Y)$. This projection appears in the following result of Ni and Wu, which will allow us to describe the d-invariants of surgeries in terms of the invariants $V_{\mathfrak{t},i}$, which crucially, are non-negative integers.

Proposition 3.3 (Proposition 1.6 in [NW15]). *Fix an integer $n > 0$ and a self-conjugate Spin^c structure $\mathfrak{t}$ on an L-space Y. Let K be a null-homologus knot in Y. Then, there exists a bijective correspondence $i \leftrightarrow \mathfrak{t}_i$ between $\mathbb{Z}/n$ and the Spin^c structures on $\mathrm{Spin}^c(Y_n(K))$ that extend $\mathfrak{t}$ over $W_n(K)$ such that*

(5) $$d(Y_n(K), \mathfrak{t}_i) = d(Y, \mathfrak{t}) + d(L(n, 1), i) - 2N_{\mathfrak{t},i}$$

where $N_{\mathfrak{t},i} = \max\{V_{\mathfrak{t},i}, V_{\mathfrak{t},n-i}\}$. Here, we assume that $0 \leq i < n$.

Proposition 3.3 was originally proved for knots in the three-sphere, but it generalizes immediately to the case of a null-homologous knot in an L-space [NW15], which is the version stated here (see also [LMV17]). The term on the right includes the d-invariants of the lens space $L(n, 1)$, which are made explicit in section 3.3.

The following proposition will be useful in the proofs of the main results. A proof of Proposition 3.4 is given in [LMV17, Proposition 2.10].

Proposition 3.4 (Lidman-Moore-Vazquez [LMV17]). *Let K be a null-homologous knot in a $\mathbb{Z}/2$-homology sphere Y. Let $\mathfrak{t}$ be the self-conjugate $Spin^c$ structure on Y, and let $\mathfrak{t}_0$ be the $Spin^c$ structure on $Y_n(K)$ described in Proposition 3.3 above. Then $\mathfrak{t}_0$ is also self-conjugate on $Y_n(K)$.*

3.2. Proof of Theorem 1.2.

Theorem 1.2. *Let Y and Y' be L-spaces with $H_1(Y) = H_1(Y') = \mathbb{Z}/m$, where m is an odd square-free integer. If Y' is obtained by a distance one surgery along any knot K in Y, then*

$$(6) \qquad |d(Y, \mathfrak{t}) - d(Y', \mathfrak{t}')| = 2 \ or \ 0$$

where $\mathfrak{t}$ and $\mathfrak{t}'$ denote the unique self-conjugate $Spin^c$ structures on Y and Y'.

Proof. Suppose that Y and Y' are L-spaces, with $H_1(Y) = H_1(Y') = \mathbb{Z}/m$ for m an odd square-free integer, and that Y' is obtained by a distance one surgery along some knot K in Y. By Lemma 2.1, K is null-homologous in Y and the surgery coefficient is ± 1.

Let us first consider when the surgery coefficient is exactly $n = +1$, that is, the filling slope yielding Y' is $\mu + \lambda$. Since K is a null-homologous knot in an L-space Y along which there exists an integral surgery to Y', we may apply Proposition 3.3. In particular, fixing the unique self-conjugate $Spin^c$ structure $\mathfrak{t}$ on Y, there is one $Spin^c$-structure $\mathfrak{t}_0$ on $Spin^c(Y') \cong Spin^c(Y) \cong \mathbb{Z}/m$ that extends $\mathfrak{t}$ over $W_1(K)$. By Proposition 3.4, $\mathfrak{t}_0$ is self-conjugate on Y'. But there is a unique self-conjugate $Spin^c$ structure $\mathfrak{t}'$ in Y'. Therefore $\mathfrak{t}_0 = \mathfrak{t}'$ and by equation (5) we have

$$(7) \qquad d(Y', \mathfrak{t}') = d(Y, \mathfrak{t}) + d(L(1,1), 0) - 2N_{\mathfrak{t},0}.$$

Because $L(1,1)$ is the three-sphere, the term $d(L(1,1), 0)$ vanishes, leaving

$$(8) \qquad d(Y, \mathfrak{t}) - d(Y', \mathfrak{t}') = 2N_{\mathfrak{t},0},$$

where $N_{\mathfrak{t},0} = \max\{V_{\mathfrak{t},0}, V_{\mathfrak{t},1}\}$. We claim that $V_{\mathfrak{t},0}$ is at most one.

Consider the mapping cone formula for the Heegaard Floer chain complex of Y'. Recall that $Y' = Y_n(K)$, where we have assumed that $n = +1$. By assumption, both Y and Y' are L-spaces. Therefore $H_*(\widehat{B}_\xi) \cong \mathbb{F}$, and $H_*(\widehat{\mathbb{X}}_\xi) \cong \mathbb{F}$ for any $\xi \in \underline{Spin}^c(Y, K)$.

By [OS05a, Proposition 2.1], any sufficiently large surgery $Y_N(K)$ will also be an L-space. Therefore we also have that

$$(9) \qquad H_*(\widehat{A}_\xi) \cong \mathbb{F} \text{ for all } \xi \in \underline{Spin}^c(Y, K).$$

Thus, the Heegaard Floer homology of $Y_n(K)$ is completely determined by the numbers V_ξ and H_ξ for each $\xi \in \underline{Spin}^c(Y, K)$.

Since the mapping cone splits over $Spin^c(Y)$, we may restrict our attention to the unique self-conjugate $Spin^c$ structure $\mathfrak{t}$ on Y. We have that the 'hat version' of the mapping cone formula specialized to $n = +1$ surgery along K and restricted to $\mathfrak{t}$ on Y is given by

where we have written $\xi = (\mathfrak{t}, s)$ in the above diagram for clarity.

Now suppose that $V_{(\mathfrak{t},0)} \geq 2$. Property 3.1(1) implies that $V_{(\mathfrak{t},1)} \geq 1$. Property 3.1(2) implies that $H_{(\mathfrak{t},0)} \geq 2$, and Property 3.1(3) then implies that $H_{(\mathfrak{t},1)} \geq 1$. Together these imply that the maps $\widehat{v}_{(\mathfrak{t},0)}, \widehat{v}_{(\mathfrak{t},1)}, \widehat{h}_{(\mathfrak{t},0)}$ and $\widehat{h}_{(\mathfrak{t},1)}$ are zero. In particular, $\widehat{A}_{(\mathfrak{t},0)}$ and $\widehat{A}_{(\mathfrak{t},1)}$ are in the kernel of $(\widehat{\mathbb{D}}_\xi)_*$, and so

$$\mathrm{rank\ ker}((\widehat{\mathbb{D}}_\xi)_*) \geq 2.$$

By [Gai17, Lemma 12], the map induced by $\widehat{\mathbb{D}}_{(\mathfrak{t},i)}$ on homology is surjective. Thus the long exact triangle

(10)
$$H_*(\widehat{\mathbb{A}}_\xi) \xrightarrow{(\widehat{\mathbb{D}}_\xi)_*} H_*(\widehat{\mathbb{B}}_\xi)$$
$$H_*(\widehat{\mathbb{X}}_\xi)$$

implies that $\mathrm{ker}((\widehat{\mathbb{D}}_\xi)_*) \cong H_*(\widehat{\mathbb{X}}_\xi)$. But now (9) and (10) imply that

$$\mathrm{rank\ ker}((\widehat{\mathbb{D}}_\xi)_*) = \mathrm{rank\ } H_*(\mathbb{X}_\xi^+) = \mathrm{rank\ } \widehat{HF}(Y', \mathfrak{t}') \geq 2,$$

which contradicts that Y' is an L-space. Therefore $V_{\mathfrak{t},0}$ is at most one. This verifies the claim.

Finally, since $N_{\mathfrak{t},0} = \max\{V_{\mathfrak{t},0}, V_{\mathfrak{t},1}\} = V_{\mathfrak{t},0}$, this completes the proof of the theorem in the case that the surgery coefficient is $n = +1$.

If instead the surgery coefficient is $n = -1$, then we may reverse the roles of Y and Y'. In particular, we consider $n = +1$ surgery along a null-homologous knot K_1 in Y' yielding the manifold Y (here K_1 is the core of the previous surgery). The above argument applies with the corresponding change in notation. $\square$

Remark 3.5. In fact, Theorem 1.2 may be stated in more generality. Suppose that Y' is obtained from Y by a distance one surgery along a knot K. If we have that

$$H_1(Y) \cong \mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2 \oplus \cdots \oplus \mathbb{Z}/d_k \cong H_1(Y')$$

where each d_i is square-free and odd, and then as remarked above in section 2, Lemma 2.1 can be extended to show that K is null-homologous in Y. The rest of the proof of Theorem 1.2 proceeds as written.

3.3. **Lens spaces and d-invariants.** The d-invariants of lens spaces can be computed with the following recursive formula of Ozsváth and Szabó.

Theorem 3.6 (Proposition 4.8 in [OS03]). *Let $p > q > 0$ be relatively prime integers. Then, there exists an identification $\mathrm{Spin}^c(L(p,q)) \cong \mathbb{Z}/p$ such that*

(11)
$$d(L(p,q), i) = -\frac{1}{4} + \frac{(2i + 1 - p - q)^2}{4pq} - d(L(q,r), j)$$

for $0 \leq i < p + q$. Here, r and j are the reductions of p and $i \pmod{q}$ respectively.

It is well-known (see for example [OS06, Section 3.4]), that the self-conjugate Spin^c structures on the lens space $L(p,q)$ correspond with the set

(12)
$$\mathbb{Z} \cap \{\frac{p + q - 1}{2}, \frac{q - 1}{2}\}.$$

TABLE 1. Nomenclature conversion chart. Here K^* denotes the mirror of K. Our convention on nomenclature for mirroring agrees with that of [BSV13], and the slogan that "positive knots have negative signature" [Tra88, CG88]. In the last line n can be any odd integer.

Brasher	Rolfsen	Knotplot	
3_1	3_1^*	3_1	$T(2,3)$
5_1	5_1^*	5_1	$T(2,5)$
5_2	5_2^*	5_2	
6_2	6_2^*	6_2	
7_1	7_1^*	7_1	$T(2,7)$
8_8	8_8	8_8^*	
8_{19}	8_{19}	8_{19}^*	
8_{20}	8_{20}^*	8_{20}	
n_1	n_1^*	n_1	$T(2,n)$

Note that when p is odd, there is a unique self conjugate Spin^c structure.

Corollary 3.7. *Suppose that $m > 0$ is a square-free odd integer. There exists a distance one surgery along any knot K in $L(m,1)$ yielding $-L(m,1)$ if and only if $m = 1$ or $m = 5$.*

Proof. Suppose first that $m > 0$ is a square-free odd integer and there exists a distance one surgery along any knot K in $L(m,1)$. Because the d-invariants change sign under orientation-reversal [OS03], Theorem 1.2 implies that $d(L(m,1), \mathfrak{t}_0) = 0$ or 1. By equation (12), the Spin^c structure $\mathfrak{t}_0$ corresponds with 0, hence by equation (11) we have

$$d(L(m,1), 0) = \frac{m-1}{4}.$$

The d-invariant is equal 0 for $m = 1$ and equal 1 for $m = 5$.

For the reverse direction, note that the branched double cover of $T(2,m)$ is the lens space $L(m,1)$, and that $L(1,1)$ is the three-sphere. The Montesinos trick implies that any banding along $T(2,m)$ lifts to a distance one Dehn surgery in $L(m,1)$. In the case $m = 5$, a chirally cosmetic banding from the torus knot $T(2,5)$ to its mirror image $T(2,-5)$ exists [Zek15] and is pictured in Figure 2. This band move lifts to a distance one filling taking $L(2,5)$ to $-L(2,5)$. In the case $m = 1$, the relevant banding is a Reidemeister-I move along an unknot. $\square$

4. BAND SURGERY, CHIRALLY COSMETIC BANDINGS, AND APPLICATIONS TO DNA RECOMBINATION

4.1. Theorem 1.1 as a corollary to Theorem 1.2.
Recall that the double cover $\Sigma(K)$ of the three-sphere branched over a knot K is a rational homology sphere with $H_1(\Sigma(K))$ of odd order. Ozsváth and Szabó [OS03] defined an integer-valued knot invariant

(13) $$\delta(K) = 2d(\Sigma(K), \mathfrak{t}_0),$$

where $\mathfrak{t}_0$ is the Spin^c structure induced by the unique spin structure on $\Sigma(K)$. For alternating knots, and for knots of small crossing number, Manolescu and Owens proved that $\delta(K)$ is determined by the signature.

Theorem 4.1 (Theorems 1.2 and 1.3 in [MO07]). *If the knot K is alternating or admits a diagram with nine or fewer crossings, then $\delta(K) = -\sigma(K)/2$.*
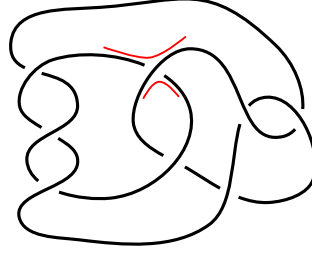
FIGURE 3. A non-coherent band surgery taking 8_{19} to 3_1^* is obtained by the unoriented skein resolution indicated in red.

Lisca and Owens [LO15, Theorem 1] later generalized Theorem 4.1 to quasi-alternating knots. The class QA of quasi-alternating knots generalizes alternating knots [OS05b] as follows.

Definition 4.2. The set QA of quasi-alternating links is the smallest set of links satisfying the following properties:

(1) The unknot is in QA.

(2) The set is closed under the following operation: If L is a link which admits a diagram with a crossing such that

  (a) both resolutions $L_h = (\asymp)$ and $L_v = ()()$ are in QA

  (b) $\det(L_v) \neq 0$, $\det(L_h) \neq 0$, and

  (c) $\det(L) = \det(L_v) + \det(L_h)$,

  then L is in QA.

Therefore, we have that the signature of a knot is determined by $\delta(K) = 2d(\Sigma(K), \mathfrak{t}_0)$ for alternating knots, quasi-alternating knots, and all knots of up to nine-crossings. In this context, Theorem 1.1 follows as a corollary to Theorem 1.2.

Theorem 1.1. *Let K and K' be a pair of quasi-alternating knots and suppose that $\det(K) = m = \det(K')$ for some square-free integer m. If there exists a band surgery relating K and K', then $|\sigma(K) - \sigma(K')|$ is 0 or 8.*

Proof. If K and K' are quasi-alternating knots, then their branched double covers $\Sigma(K)$ and $\Sigma(K')$ are L-spaces [OS05b] and $H_1(\Sigma(K)) \cong \mathbb{Z}/m \cong H_1(\Sigma(K'))$ since $\det(K) = m = \det(K')$. If K and K' differ by a band move, then $\Sigma(K')$ is obtained by a distance one filling along a knot in $\Sigma(K)$ (and vice versa). Therefore,

$$|d(\Sigma(K), \mathfrak{t}_0) - d(\Sigma(K'), \mathfrak{t}_0')| = 2 \text{ or } 0$$
$$\Rightarrow \quad |\delta(K) - \delta(K')| = 4 \text{ or } 0$$
$$\Rightarrow \quad |\sigma(K) - \sigma(K')| = 8 \text{ or } 0$$

where the first line is equation (6), the second is the definition of δ in equation (13), and the third follows from Theorem 4.1. $\square$

Corollary 4.3. *Excluding 8_{19}, let K and K' be knots of eight or fewer crossings with $\det(K) = m = \det(K')$ for m a square-free integer. If there exists a banding from K to K', then $|\sigma(K) - \sigma(K')| = 8$ or 0.*

TABLE 2. **Non-coherent banding for pairs of knots of the same determinant.** All knots of up to 8 crossings are organized by determinant and signature. A straight dashed line indicates that Theorem 1.1 implies no banding relates K and K'. A solid line indicates that a band relating K and K' has been reported either here or in the literature. Curly brackets indicate a non-alternating, quasi-alternating knot, and square brackets indicate a knot that is not quasi-alternating. All other knots are alternating. Bold indicates an achiral knot.

$\det(K)$	3	5	7	9	11	13
-6	$[8_{19}]$		7_1			
-4		5_1				7_3
-2	3_1		5_2		6_2 7_2	
0		$\mathbf{4_1}$		$\{8_{20}\}$ $\{8_{20}^*\}$	$\mathbf{6_3}$	$8_1, 8_1^*$
				6_1 $\quad 6_1^*$		
2	3_1^*		5_2^*		6_2^* 7_2^*	
4		5_1^*				7_3^*
6	$[8_{19}^*]$		7_1^*			

$\det(K)$	15	17	19	21	23	25
-4				8_5		
-2	7_4 $\{8_{21}\}$	7_5 $\quad 8_2$	7_6 $\quad 8_4$		8_6 $\quad 8_7$	
0		$\mathbf{8_3}$		$7_7, 7_7^*$		8_8 $\,8_8^*$ $\quad \mathbf{8_9}$
2	7_4^* $\{8_{21}^*\}$		7_6^* $\quad 8_4^*$		8_6^* $\quad 8_7^*$	
4		7_5^* $\quad 8_2^*$		8_5^*		

$\det(K)$	27	29	31	33	35	37	45
-4				8_{15}			
-2			8_{14}		8_{16}		
0	$8_{10}, 8_{10}^*, 8_{11}, 8_{11}^*$	$\mathbf{8_{12}}, 8_{13}, 8_{13}^*$				$\mathbf{8_{17}}$	$\mathbf{8_{18}}$
2			8_{14}^*		8_{16}^*		
4				8_{15}^*			

Proof. All knots of up to eight crosings are alternating with the exception of 8_{19}, 8_{20} and 8_{21}. Of these, 8_{20} and 8_{21} are quasi-alternating. The branched double cover of any alternating or quasi-alternating knot is an L-space. However, 8_{19} is not quasi-alternating (in fact, it is homology-thick) and its branched double cover is not an L-space, hence it is excluded. Since Theorem 4.1 holds for all knots of up to nine crossings we obtain the statement. $\square$

Example 4.4. A band surgery relates the pair 3_1^* and 8_{19}. See Figure 3.

Example 4.5. Consider the knots 6_2 and 7_2, for which $\det(6_2) = 11 = \det(7_2)$ and $\sigma(6_2) = -2 = \sigma(7_2)$. A single band surgery relates 6_2 and 7_2 (and hence 6_2^* and 7_2^*) [Zek15]. Theorem 1.1 implies there is no band surgery relating 6_2 and 7_2^*. Similarly, consider 7_1 and 5_2. Now, $\det(7_1) = \det(5_2) = 7$ and $\sigma(7_1) = -6$ and $\sigma(5_2) = -2$, so there cannot be a band move relating 7_1 and 5_2. However, Figure 1 illustrates the existence of a band surgery relating 7_1 and 5_2^* [Zek15].

(A) The knots 8_8 and 8_{20}.   (B) Chirally cosmetic banding along 8_8^* and 8_{20}.
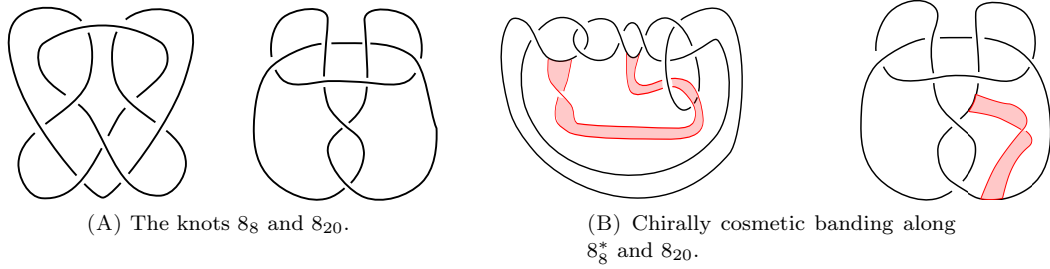
FIGURE 4. (A) The knots 8_8 and 8_{20} are symmetric unions, and as such their determinants are squares. (B) Chirally cosmetic bandings relating the pair 8_8^* and 8_8 (left) and the pair 8_{20} and 8_{20}^* (right). The banding exhibited for 8_8^* was discovered via computer simulation as described in section 4.3. The banding for 8_{20} induces a "4-move."

For the benefit of the reader we have provided a nomenclature conversion to Rolfsen [Rol90] and KnotPlot in Table 1 for the knots mentioned in this article.

Example 4.6. Table 2 lists all knots up to eight crossings partitioned by determinant and organized by signature. The table indicates those pairs (K, K') for which Theorem 1.1 guarantees that a band relating them does not exist (dashed lines), and those pairs of knots for which a band is known to exist (solid lines). For knots of up to seven crossings, all known bandings were previously reported in [Kan16]. Note that the existence of a band is recorded in Table 2 *only* for pairs of knots with the same determinant.

4.2. **Chirally cosmetic bandings between prime knots with ≤ 8 crossings are rare.** A *chirally cosmetic* banding refers to a non-coherent band surgery taking a knot to its mirror image. The results in the previous section immediately imply the following corollary.

Corollary 4.7. *The only nontrivial torus knot $T(2, m)$, with m square free, admitting a chirally cosmetic banding is $T(2, 5)$.*

Proof. The statement follows from Corollary 3.7 because the branched double cover of $T(2, m)$ is the lens space $L(m, 1)$. Alternatively, we may use Theorem 1.1. The torus knot $T(2, m)$ has determinant m and signature $1 - m$. The absolute value of the difference in signature between $T(2, m)$ and $T(2, -m)$ is 8 when $m = 5$, and 0 when $m = 1$. The case of $m = 5$ is pictured in Figure 2. The case of $m = 1$ is the unknot. $\square$

The knot K in $L(5, 1)$ along which there exists a distance one filling yielding $-L(5, 1)$ descends under the covering involution to the core of the band move from $T(2, 5)$ to $T(2, -5)$. As observed in [IJM17], the complement of this knot is the hyperbolic knot complement known as the "figure-eight sibling" and is well-known to be amphichieral [MP06, Wee85]. The special symmetries of this manifold suggest an explanation for the uniqueness of $T(2, 5)$; see [IJM17] for a discussion of this perspective.

In [IJM17], several constructions of knots admitting chirally cosmetic bandings are described, in addition to 5_1. These include the knot 9_{27}, which has determinant 49, Whitehead doubles of achiral knots (i.e. certain satellites), which have determinant 1, and a general construction of certain

TABLE 3. **Chirally cosmetic bandings are rare.** Here we report on the relative likelihood of chirally cosmetic bandings for knots of up to eight crossings. The second column indicates the probability of cosmetic banding for each knot K in column 1. The probabilities were computed out of 3×10^6 total band moves performed on each knot type. Confidence intervals listed in column 3 were computed with the ratio estimation technique described in [SYB+17]. Column four indicates the total number of cosmetic bandings observed for each starting knot type. The last column indicates the length range observed in sampled polygons.

Knot	$P(K - K^*) \times 10^{-5}$	Confidence interval	Observed	$[\ell_{min}, \ell_{max}]$
5_1	3.467	[3.159, 3.785]	104	[42, 2016]
5_1^*	2.800	[2.517, 3.084]	84	[42, 1938]
8_8	0	—	0	[100, 2334]
8_8^*	0.003	[0.002, 0.065]	1	[104, 2894]
8_{20}	42.833	[41.698, 43.986]	1285	[70, 2370]
8_{20}^*	46.400	[45.129, 47.650]	1392	[68, 2196]

symmetric unions. A symmetric union is a connected sum of a knot and its mirror image, modified by a tangle replacement such that the resulting diagram admits an axis of mirror symmetry (see Figure 4A). The determinant of a symmetric union is always a square [KT57, Moo16].

It is interesting to note that, excluding 5_1, the examples of knots admitting chirally cosmetic bandings presented in [IJM17] have determinant a square. Corollary 4.7 and the observation that most known constructions of knots admitting chirally cosmetic bandings are symmetric unions suggest that this phenomenon is uncommon. We test this hypothesis with a numerical approach. We use computer simulations to explore the existence of non-coherent banding amongst knots with up to 8 crossings. The set of knot types considered in this study are the 63 non-trivial prime knots with crossing number at most 8. When a knot K is chiral, both K and K^* are included as distinct starting knot types. For each starting knot type, we perform 3×10^6 reconnection events, uniformly sampled from the space of lattice embeddings using the methods described in the next section.

We find chirally cosmetic bandings to be exceedingly rare. Chirally cosmetic bandings were observed in the numerical experiments for only three knot types: $5_1, 8_8, 8_{20}$, with the transition probabilities and numbers observed listed in Table 3. The band move for 5_1 was previously reported in [Zek15] and shown in Figure 2. We manually identified the band move for 8_{20}, which may be realized as a "4-move" in a symmetric union diagram, where a 4-move refers to the replacement of a positive 2-twist with a negative 2-twist. It is also known that 8_8 admits a symmetric union presentation [Lam17] (see Figure 4A). It is not known to the authors whether a 4-move move relates 8_8 with its mirror. See [IJM17] for a definition of 4-moves, or Figure 4B for an example.

4.3. **Numerical methods and applications to DNA recombination.**
We designed a numerical experiment in order to assess the occurrence and relative likelihood of chirally cosmetic bandings amongst knots of up to 8 crossings. For each knot type K, we use the BFACF algorithm [MS93] to generate large ensembles of cubic lattice representations of type K. Then we algorithmically search for and perform non-coherent band surgery operations. This was done by a preliminary adaptation of the computational suite of software developed for the case of coherent band surgery in work of the second author (see [SYB+17]).

A *conformation* of a knot K refers to an embedding of K into the simple cubic lattice $\mathbb{Z}^3$. The BFACF algorithm, used to uniformly sample random conformations, is a Markov Chain Monte Carlo algorithm that samples self-avoiding knotted polygons (or more generally, self-avoiding walks with fixed endpoints) in $\mathbb{Z}^3$. This algorithm has the dual benefit of preserving the isotopy type of a knotted polygonal chain and being ergodic in the state-space of self-avoiding polygons of fixed knot type. Thus, it is particularly well suited for studying knots quantitatively. The algorithm acts by performing a sequence of small perturbations along the self-avoiding polygon. The probabilities of these perturbations depend on two fugacity parameters, q and z [MS93, Sza09]. We here fix q to be 1, and vary z to range in $[0.117, 0.2125]$. While the ergodicity classes of the BFACF Markov chain correspond with the isotopy classes of knots, the algorithm does not preserve the length of the chain, defined to be the number of unit steps in the embedding of K in $\mathbb{Z}^3$ (notice column 5 in Table 3). For further detail on the BFACF algorithm see [MS93, Chapter 9].

To simulate non-coherent band surgery, BFACF is first run on each non-trivial prime knot with at most eight crossings[1]. We employ a composite Markov chain (CMC) Monte Carlo process in order to efficiently randomize sampling of conformations. The CMC variation of the BFACF algorithm iterates on parallel Markov chains with different fugacity parameters, executing an exchange of states (i.e. conformations) when certain criteria are met. Given an ensemble of conformations with a fixed length and knot type, the algorithm searches for two edges along the polygon that are candidates for non-coherent banding. These occur when a conformation contains four vertices occupying the four corners of a unit square, and are referred to as *reconnection sites*. Because, *a priori*, pairs of reconnection sites may occur anywhere along the chain, we do not impose any filters on arc length between pairs of reconnection sites. In order to identify the isotopy type of the resulting knots, and produce transition probabilities for the results, we use an implementation of the HOMFLY-PT polynomial [FYH+85, PT87] based on published algorithms [GMGCL99, Jen89]. In cases where the HOMFLY-PT may lead to ambiguous knot identification, we apply KnotFinder [CL18]. Block mean analysis and ratio estimation are implemented to correct for bias and dependence between samples. These notions are described in [Fis96] and their implementation follows that of [SYB+17] and [Sch12, Chapters 3, 6]. More details on the composite Markov chain method of the BFACF algorithm and block mean analysis and ratio estimation methods used here may be found in the Supplementary Materials of [SYB+17].

Amongst knots of eight crossings or fewer, the HOMFLY-PT polynomial distinguishes all knots from their mirror images, with the exception of 8_{17}. When considered as an oriented knot, 8_{17} is negative achiral, i.e. it is isotopic to the reverse of its mirror image. Here we ignore orientation because we are interested in non-coherent band surgery, hence the knot 8_{17} is treated as achiral. For knots of up to 10 crossings, there is a short list of knots with different crossing numbers which fail to be distinguished by their HOMFLY-PT polynomials. In particular, the pairs $(5_1, 10_{132})$ and $(8_8, 10_{129})$ are on this list. It is then possible that a transition to a 10 crossing knot by non-coherent banding could be incorrectly identified as a transition to 5_1 or 8_8. Therefore, for every transition involving 5_1 and 5_1^* or 8_8 and 8_8^* identified by our algorithm, we performed a secondary identification of the product knots using the KnotFinder feature of KnotInfo (which is back-ended by KnotScape [CL18]), and excluded any transitions to 10 crossing knots. Indeed, using the HOMFLY-PT polynomial, our algorithm identified 16 possible transitions from 8_8 to 8_8^* and 8 possible transitions from 8_8^* to 8_8. Only one of these 24 transitions realizes a chirally cosmetic

---

[1]Although we only report transition probabilities of chirally cosmetic bandings in this note, we collected data on other transitions for a broader future study.

banding along 8_8 (see Figure 4B); the remaining 23 transitions corresponded with band moves relating 8_8 and 10_{129} or their mirrors.

Results pertaining to band surgery on knots and links are also of interest in biology and physics. In the last thirty years, questions rooted in biology have lead mathematicians to model the action of site-specific recombination enzymes on circular DNA using tangles and coherent or non-coherent band surgery (reviewed in [SESC95]). The reactions mediated by these enzymes are thought of as local reconnection events and modeled mathematically by band surgery on knots and links. A reconnection site on a single DNA molecule is a short, non-palindromic word in the nucleotide sequence which induces an orientation along the DNA circle. When two such sites occur along the same molecule, the sites are said to be in *direct* repeat if they induce the same orientation along the chain and in *inverted* repeat if they induce opposite orientations. Biological reactions involving local reconnection occur naturally, and have been reported on extensively in biological experiments, computational studies and mathematically [SESC95, VS04, VCS05, SIG$^+$13, SYB$^+$17]. Site-specific recombinases mediate the integration and excision of genetic material [GW71], the resolution of dimers formed during replication [SSS88], and regulate gene expression via inversions of the nucleotide sequence [HJ90]. For a more detailed discussion of this biological model we direct the reader to [LMV17, Section 5]. Local reconnection processes also appear in other natural processes. Noteworthy is reconnection of knotted or linked fluid vortices which follow similar topological simplification patterns as those observed for DNA [KKI16, KI13].

Broadly speaking, we are interested in reconnection events within the cell. In particular, we aim to understand how the topological chirality of DNA knots and links affects the action of site-specific recombinases, or *conversely*, how the action of site-specific recombinases affects observable changes in chirality in DNA knots and links. To this end, we pose the following question:

Question 4.8. *Suppose that a pair of knots K and K' are related by reconnection (i.e. band surgery). What is the likelihood that K and K' are of different chirality?*

When K is a chiral knot and K' is the mirror of K, it is clear what we mean by *different chirality*. We have reported here that knots admitting chirally cosmetic bandings are rare and that the relatively likelihoods of these transitions are statistically unlikely. In a forthcoming numerical study we quantify what it means for arbitrary knots to have different chirality and report on the results of extended numerical studies to show that changes in chirality are quite prevalent during reconnection, in stark contrast with the situation for the chiral pair of a knot and its mirror image.

REFERENCES

[AK14]   Tetsuya Abe and Taizo Kanenobu. Unoriented band surgery on knots and links. *Kobe J. Math.*, 31(1-2):21–44, 2014.

[BSV13]   Reuben Brasher, Rob G. Scharein, and Mariel Vazquez. New biologically motivated knot table. *Biochemical Society Transactions*, 41(2):606–611, 2013.

[CG88]   Tim D. Cochran and Robert E. Gompf. Applications of Donaldson's theorems to classical knot concordance, homology 3-spheres and property P. *Topology*, 27(4):495–512, 1988.

[CL18]   Jae Choon Cha and Charles Livingston. Knotinfo: Table of Knot Invariants. http://www.indiana.edu/ knotinfo, 2018.

[Fis96]   George S. Fishman. *Monte Carlo*. Springer Series in Operations Research. Springer-Verlag, New York, 1996. Concepts, algorithms, and applications.

[FYH$^+$85] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, and A. Ocneanu. A new polynomial invariant of knots and links. *Bull. Amer. Math. Soc. (N.S.)*, 12(2):239–246, 1985.

[Gai17]   Fyodor Gainullin. The mapping cone formula in Heegaard Floer homology and Dehn surgery on knots in S^3. *Algebr. Geom. Topol.*, 17(4):1917–1951, 2017.

[Gai18]   Fyodor Gainullin. Heegaard Floer homology and knots determined by their complements. *Algebr. Geom. Topol.*, 18(1):69–109, 2018.

[GL78]   C. McA. Gordon and R. A. Litherland. On the signature of a link. *Invent. Math.*, 47(1):53–69, 1978.

[GMGCL99] G. Gouesbet, S. Meunier-Guttin-Cluzel, and C. Letellier. Computer evaluation of Homfly polynomials by using Gauss codes, with a skein-template algorithm. *Appl. Math. Comput.*, 105(2-3):271–289, 1999.

[GW71]   M E Gottesman and R A Weisberg. Prophage insertion and excision. *Cold Spring Harbor Monograph Archive*, 2:113–138, 1971.

[HJ90]   Karen A. Heichman and Reid C. Johnson. The Hin invertasome: Protein-mediated joining of distant recombination sites at the enhancer. *Science*, 249(4968):511–517, 1990.

[HLZ15]   Jennifer Hom, Tye Lidman, and Nicholas Zufelt. Reducible surgeries and Heegaard Floer homology. *Math. Res. Lett.*, 22(3):763–788, 2015.

[IJM17]   Kazuhiro Ichihara, In Dae Jong, and Hidetoshi Masai. Cosmetic banding on knots and links. Preprint, arXiv:1602.01542 [math.GT], 2017.

[Jen89]   R. J. Jenkins. *Knot Theory, Simple Weaves, and an Algorithm for Computing the HOMFLY Polynomial*. 1989. Thesis (Master's) – Carnegie Mellon University.

[Kan16]   Taizo Kanenobu. Band surgery on knots and links, III. *J. Knot Theory Ramifications*, 25(10):1650056, 12, 2016.

[KI13]   Dustin Kleckner and William T. M. Irvine. Creation and dynamics of knotted vortices. *Nat Phys*, 9(4):253–258, 04 2013.

[KKI16]   D Kleckner, L H Kauffman, and W T M Irvine. How superfluid vortex knots untie. *Nat Phys*, 03 2016.

[KM14]   Taizo Kanenobu and Hiromasa Moriuchi. Links which are related by a band surgery or crossing change. *Bol. Soc. Mat. Mex. (3)*, 20(2):467–483, 2014.

[KT57]   Shin'ichi Kinoshita and Hidetaka Terasaka. On unions of knots. *Osaka Math. J.*, 9:131–153, 1957.

[Lam17]   Christoph Lamm. The search for non-symmetric ribbon knots. arXiv:1710.06909 [math.GT], 2017.

[LM17]   Tye Lidman and Allison H. Moore. Cosmetic surgery in L-spaces and nugatory crossings. *Trans. Amer. Math. Soc.*, 369(5):3639–3654, 2017.

[LMV17]   Tye Lidman, Allison H. Moore, and Mariel Vazquez. Distance one lens space fillings and band surgery on the trefoil knot. arXiv:1710.07418 [math.GT], 2017.

[LO15]   Paolo Lisca and Brendan Owens. Signatures, Heegaard Floer correction terms and quasi-alternating links. *Proc. Amer. Math. Soc.*, 143(2):907–914, 2015.

[MO07]   Ciprian Manolescu and Brendan Owens. A concordance invariant from the Floer homology of double branched covers. *Int. Math. Res. Not. IMRN*, (20):Art. ID rnm077, 21, 2007.

[Mon76]   José M. Montesinos. Three-manifolds as 3-fold branched covers of S^3. *Quart. J. Math. Oxford Ser. (2)*, 27(105):85–94, 1976.

[Moo16]   Allison H. Moore. Symmetric unions without cosmetic crossing changes. In *Advances in the mathematical sciences*, volume 6 of *Assoc. Women Math. Ser.*, pages 103–116. Springer, [Cham], 2016.

[MP06]   Bruno Martelli and Carlo Petronio. Dehn filling of the "magic" 3-manifold. *Comm. Anal. Geom.*, 14(5):969–1026, 2006.

[MS93]   Neal Madras and Gordon Slade. *The self-avoiding walk*. Probability and its Applications. Birkhäuser Boston, Inc., Boston, MA, 1993.

[Mur65]   Kunio Murasugi. On a certain numerical invariant of link types. *Trans. Amer. Math. Soc.*, 117:387–422, 1965.

[NW15]   Yi Ni and Zhongtao Wu. Cosmetic surgeries on knots in S^3. *J. Reine Angew. Math.*, 706:1–17, 2015.

[OS03]   Peter Ozsváth and Zoltán Szabó. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. *Adv. Math.*, 173(2):179–261, 2003.

[OS04]   Peter Ozsváth and Zoltán Szabó. Holomorphic disks and knot invariants. *Adv. Math.*, 186(1):58–116, 2004.

[OS05a]   Peter Ozsváth and Zoltán Szabó. On knot Floer homology and lens space surgeries. *Topology*, 44(6):1281–1300, 2005.

[OS05b]   Peter Ozsváth and Zoltán Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194(1):1–33, 2005.

[OS06]   Brendan Owens and Sašo Strle. Rational homology spheres and the four-ball genus of knots. *Adv. Math.*, 200(1):196–216, 2006.

[OS08]   Peter S. Ozsváth and Zoltán Szabó. Knot Floer homology and integer surgeries. *Algebr. Geom. Topol.*, 8(1):101–153, 2008.

[OS11]   Peter S. Ozsváth and Zoltán Szabó. Knot Floer homology and rational surgeries. *Algebr. Geom. Topol.*, 11(1):1–68, 2011.

[PT87]   Józef H. Przytycki and PawełTraczyk. Conway algebras and skein equivalence of links. *Proc. Amer. Math. Soc.*, 100(4):744–748, 1987.

[Ras03]   Jacob Andrew Rasmussen. *Floer homology and knot complements*. ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–Harvard University.

[Rol90]   Dale Rolfsen. *Knots and links*, volume 7 of *Mathematics Lecture Series*. Publish or Perish, Inc., Houston, TX, 1990. Corrected reprint of the 1976 original.

[Sch12]   Matthew Frank Schmirler. Strand Passage and Knotting Probabilities in an Interacting Self-Avoiding Polygon Model. Master's thesis, University of Saskatchewan, Saskatoon, Saskatchewan, Canada, 2012.

[SESC95]   De Witt Sumners, Claus Ernst, Sylvia J. Spengler, and Nicholas R. Cozzarelli. Analysis of the mechanism of DNA recombination using tangles. *Quarterly Reviews of Biophysics*, 28(3):253–313, 1995.

[SIG+13]   Koya Shimokawa, Kai Ishihara, Ian Grainge, David J. Sherratt, and Mariel Vazquez. FtsK-dependent XerCD-*dif* recombination unlinks replication catenanes in a stepwise manner. *Proc Natl Acad Sci USA*, 110(52):20906–20911, 2013.

[SSS88]   CJ Stirling, G Stewart, and DJ Sherratt. Multicopy plasmid stability in Escherichia coli requires host-encoded functions that lead to plasmid site-specific recombination. *Molecular and General Genetics MGG*, 214(1):80–84, 1988.

[SYB+17]   R. Stolz, M. Yoshida, R. Brasher, M. Flanner, K. Ishihara, D. Sherratt, K. Shimokawa, and M. Vazquez. Pathways of DNA unlinking: a story of stepwise simplification. *Scientific Reports*, 7(1):12420, 2017.

[Sza09]   Michael Lorne Szafron. *Knotting statistics after a local strand passage in unknotted self-avoiding polygons in* $\mathbf{Z}^3$. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–The University of Saskatchewan (Canada).

[Tra88]   PawełTraczyk. Nontrivial negative links have positive signature. *Manuscripta Math.*, 61(3):279–284, 1988.

[Tro62]   H. F. Trotter. Homology of group systems with applications to knot theory. *Ann. of Math. (2)*, 76:464–498, 1962.

[VCS05]   M. Vazquez, S. D. Colloms, and D. Sumners. Tangle analysis of Xer recombination reveals only three solutions, all consistent with a single three-dimensional topological pathway. *J. Mol. Biol.*, 346(2):493–504, 2005.

[VS04]   Mariel Vazquez and De Witt Sumners. Tangle analysis of Gin site-specific recombination. *Math. Proc. Cambridge Philos. Soc.*, 136(3):565–582, 2004.

[Wat12]   Liam Watson. Surgery obstructions from Khovanov homology. *Selecta Math. (N.S.)*, 18(2):417–472, 2012.

[Wee85]   Jeffrey Renwick Weeks. *Hyperbolic structures on three-manifolds (Dehn surgery, knot, volume)*. ProQuest LLC, Ann Arbor, MI, 1985. Thesis (Ph.D.)–Princeton University.

[Zek15]   Ana Zeković. Computation of Gordian distances and H_2-Gordian distances of knots. *Yugosl. J. Oper. Res.*, 25(1):133–152, 2015.