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Angle Modulation: $u(t) = A_c \cos(w_c t + \varphi(t))$

PM: $\varphi(t) = K_p\, m(t)$

FM: $\varphi(t) = 2\pi K_f \int_{-\infty}^{t} m(\tau)\, d\tau$

$$f_i(t) = f_c + \frac{1}{2\pi} \frac{d}{dt} \varphi(t)$$

so FM modulates $f_i - f_c$

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Using the Bessel fncns — a tone

$\downarrow$ angle modulation

<u>band-unlimited</u>

must be <u>approx</u> bandlimited

FM $\rightarrow$ band unlimited

$\downarrow$ BPF

band limited (distorted)

tone thru PM $\Rightarrow$ $u(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta) \cos((\omega_c + n\omega_0)t)$

$\uparrow$ decays in n

"<u>Rule of Thumb</u>" if we have mod. index β, freq. of tone f_0, the effective BW is

$$B = 2(\beta + 1) f_0$$ $\leftarrow$ both FM and PM

centered around f_c

centered around f_c

for a tone this gives us 98% of signal power

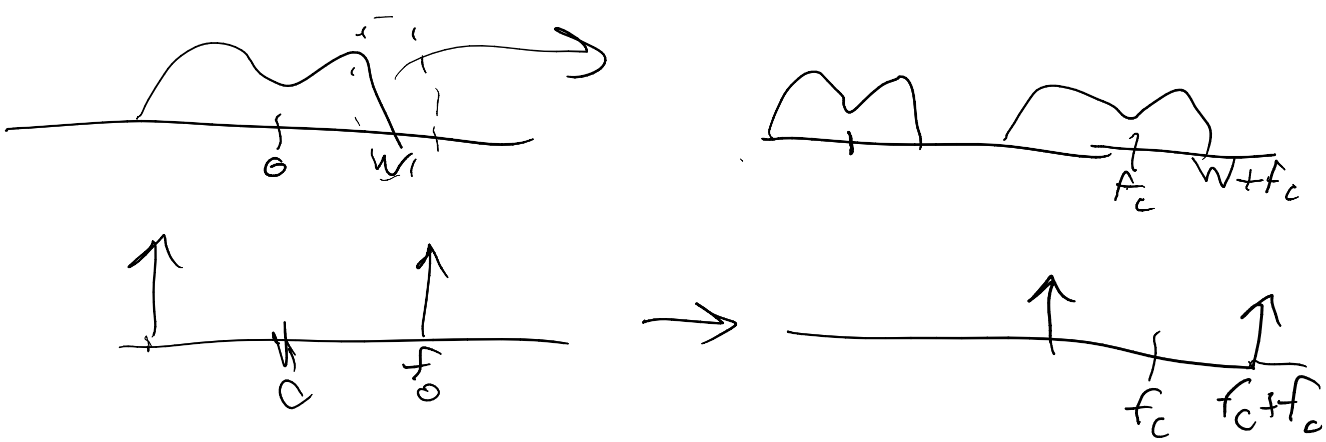
Carson's Rule

For message signal with BW W, the BW of the angle-modulated signal is approximately

$$\boxed{B_c = 2(\beta+1)W}$$

centered about f_c

even in AM, had tone freq. $f_o \sim$ to W



Angle Mod TX

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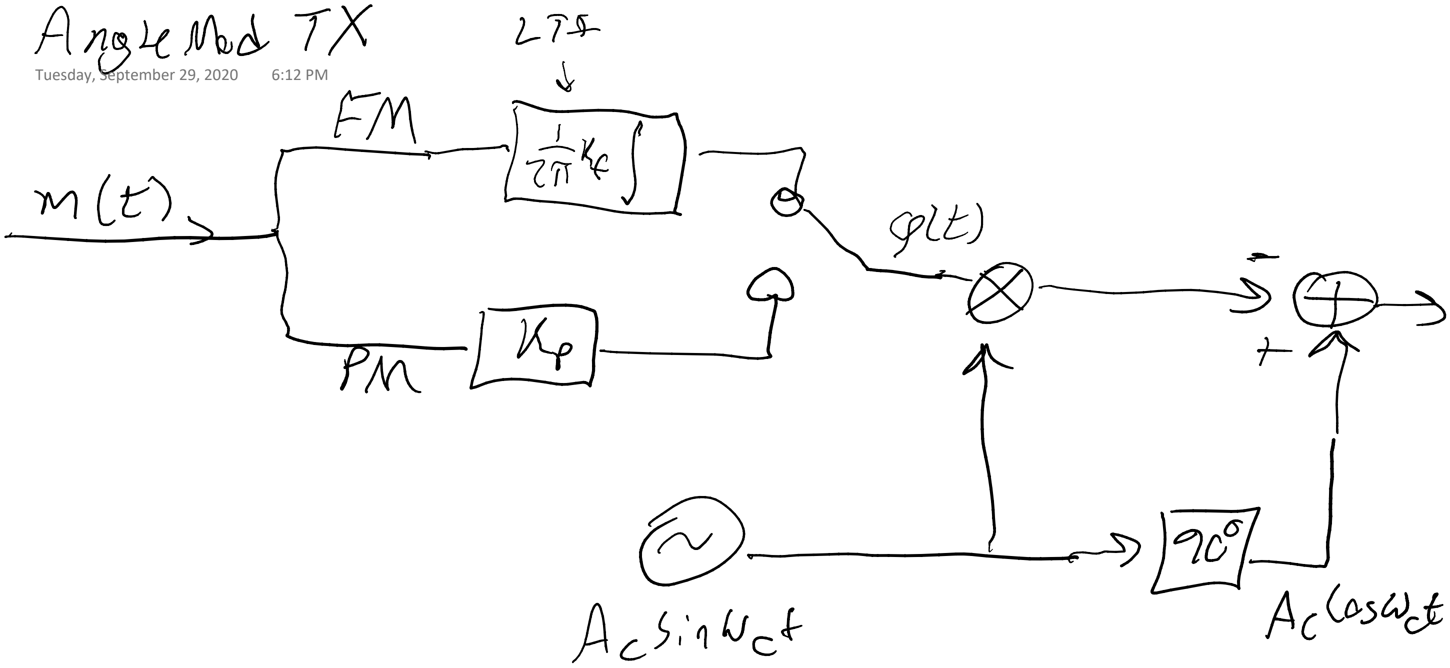


$$\underbrace{-\varphi(t) A_c \sin w_c t + A_c \cos w_c t}$$

narrow band FM formula
derived using $\cos\theta \approx 1$
$\sin\theta \approx \theta$

Less narrow band signal

$$u(t) = A_c \cos(w_c t + \Phi(t))$$

pick an integer n s.t. $\Phi(t)/n$ is small

(relative to 2π
so that
$\sin \Phi/n \approx \Phi/n$)

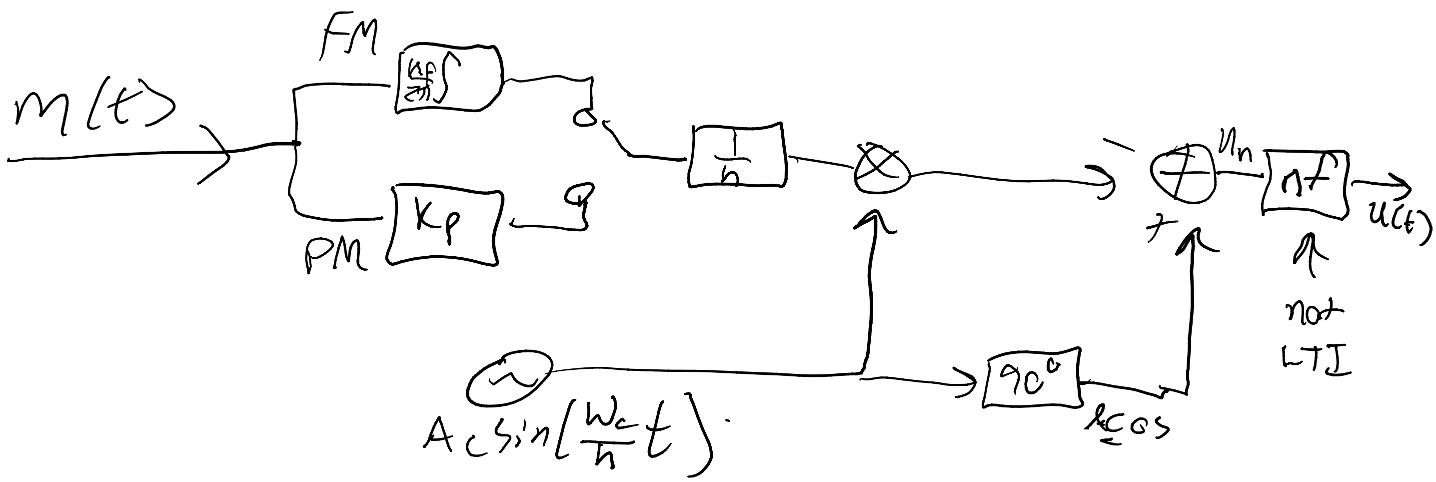
you can always do this so long as m bounded

then $u_n(t) = A_c \cos\left(\frac{w_c}{n} t + \frac{\Phi(t)}{n}\right) \leftarrow$ narrowband signal

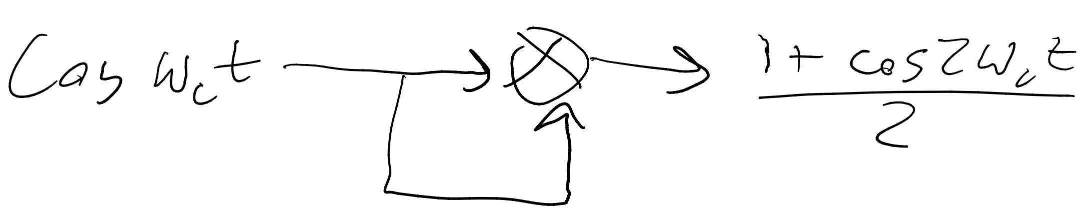
Can generate these using the scheme above

And then use "frequency multiplier" by factor n

(find in lab)



n = 2



Demod

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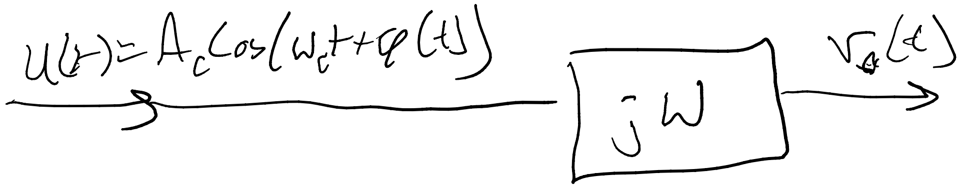
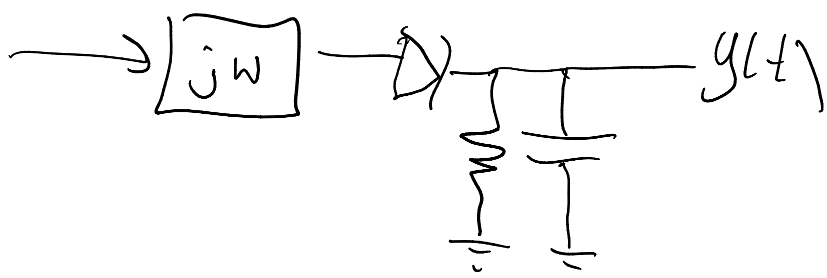


$$v_o(t) = A_c\,(w_c + \varphi'(t))\,(-\sin\omega_c t)$$

$$\underset{FM}{=} -\underbrace{A_c(\omega_c + m(t))}_{\text{Conventional AM}} \sin\omega_c t$$

FM demod



FM has very inexpensive demod
(no mixer)

PM — backwards IQ demod (like for SSB)
(can do for FM too)

Noise

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AWGN

pretty much same

filtered noise

$$u(t)+n_0(t) \longrightarrow \boxed{BPF_{B_C}} \xrightarrow{r(t)\approx u(t)+n(t)} \boxed{\text{demod}} \xrightarrow{y(t)}$$

$$n(t) = n_I(t)\cos\omega_c t - n_Q(t)\sin\omega_c t$$

$$r(t) = u(t) + n(t)$$

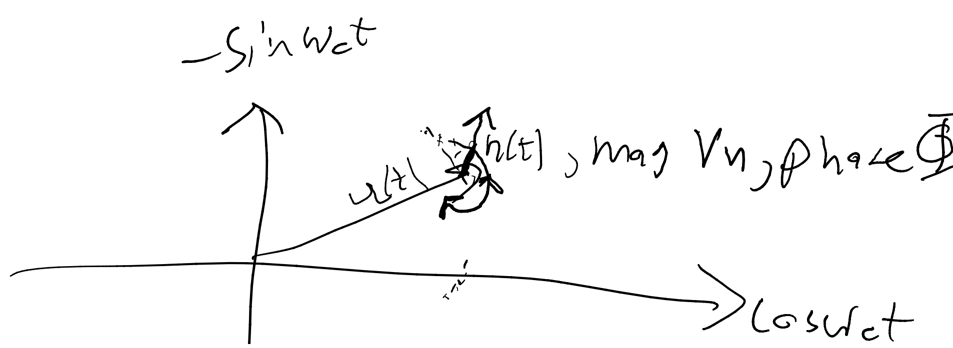
$\downarrow$ polar form

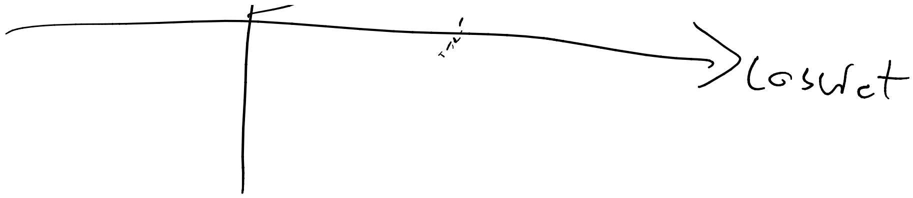
$$= u(t) + \sqrt{n_I^2(t)+n_Q^2(t)}\,\cos\left(\omega_c t + \arctan\left(\frac{n_Q(t)}{n_I(t)}\right)\right)$$

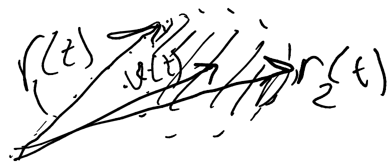
$$= u(t) + V_n(t)\cos(\omega_c t + \Phi_n(t))$$

$\uparrow$

narrowly vibrating phasor







"drunken phasor"

our signal + gaussian
- distr. phasor

Noise power smaller - $\sigma \downarrow$ and the radius of likely uncertainty $\downarrow$

distinct from AM in that the phase is as vulnerable as the amplitude

$$|r(t)| \approx A_c + V_n(t)\cos(\Phi_n(t) - \varphi(t))$$

$$\angle r(t) \approx \varphi(t) + \arctan\left(\frac{V_n(t)\sin(\Phi_n - \varphi(t))}{A_c + V_n(t)\cos(\Phi_n(t) - \varphi(t))}\right)$$

taylor arctan $x \approx x$

$$\approx \varphi(t) + \frac{V_n \sin(\Phi_n(t) - \varphi(t))}{A_c + V_n(t)\cos(\Phi_n(t) - \varphi(t))}$$

high SNR

V_n

$(\Phi_n(t) \quad \varphi(t))$

high SNR

$V_n << A_c \approx \varphi(t) + \frac{V_n}{A_c} \sin(\Phi_n(t) - \varphi(t))$

$= |r(t)| \cos(\omega_c t + \angle r(t))$

$$r(t) \approx \left(A_c + V_n(t) \cos(\Phi_n(t) - \varphi(t))\right)$$

$$\times \cos\left(\omega_c t + \varphi(t) + \frac{V_n(t)}{A_c} \sin(\Phi_n(t) - \varphi(t))\right)$$

demod $\rightarrow$ PM: get phase
FM: get f_i

$$y_{PM}(t) = \underbrace{\varphi(t)}_{\text{signal}} + \underbrace{\frac{V_n(t)}{A_c} \sin(\Phi_n(t) - \varphi(t))}_{\text{noise}}$$

noise $= Y_n = \frac{V_n}{A_c} \sin \Phi - \varphi$

$$y_{FM}(t) = \frac{1}{2\pi} \frac{d}{dt} \left(\varphi + \frac{V_n}{A_c} \sin(\Phi_n - \varphi)\right)$$

$$= k_f m(t) + \frac{1}{2\pi} Y_n'(t)$$

signal + noise

To get y = signal + noise, used a high SNR assumption

functions of n_I, n_Q

$$Y_n = \frac{V_n}{A_c}\sin(\Phi - \varphi)$$

functions of the scheme

$A_c \nearrow$ decreases noise power

in AM, increase A_c, SNR $\nearrow$ because $\nearrow P_S$ but noise power is fixed

here $\nearrow A_c \to \searrow N$ but has no effect on S

but still $\nearrow A_c \to$ SNR $\nearrow$

Same effect thru different route

$$Y_n(t) = \frac{V_n(t)}{A_c} \sin\left(\Phi_n(t) - \varphi(t)\right)$$

$$= \frac{1}{A_c}\left(V_n(t) \sin\Phi_n \cos\varphi - V_n \cos\Phi_n \sin\varphi\right)$$

$$= \frac{1}{A_c}\left(n_Q(t)\cos\varphi - n_I(t)\sin\varphi\right)$$

we had a narrowband approx for FM/PM, can apply

$B_c \ggg W$ ↖ BW of $m(t)$ So we can

^ BW of noise

assume $\varphi(t) \approx \varphi$ approx constant compared to Φ_n

$$Y_n \approx \frac{n_Q(t)}{A_c}\cos\varphi - \frac{n_I(t)}{A_c}\sin\varphi$$

input noise (after filter) $I: n_I$, $Q: n_Q$, output $I: \frac{n_Q}{A_c}$, $Q: \frac{n_I}{A_c}$

$\uparrow A_c \searrow P_N$

only relationship we have now

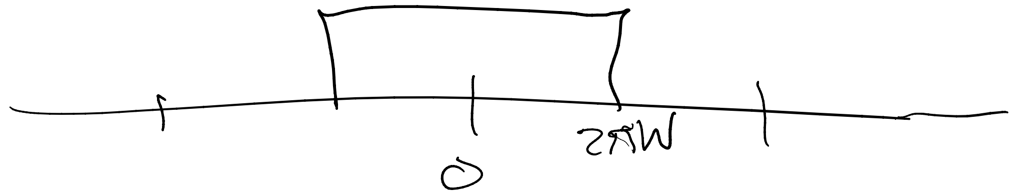
after demod, at bb

PSD

I/Q have same PSD as Y_n

$$S_{Y_n}(\omega) = S_{\frac{n_Q}{A_c}}(\omega) = S_{\frac{n_I}{A_c}}(\omega) = \begin{cases} N_0/A_c^2, & |\omega| \le 2\pi W \\ 0, & \text{else} \end{cases}$$

Noise PSD



For PM

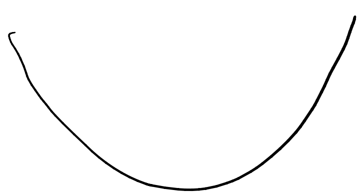
for FM,

noise is $\frac{1}{2\pi} Y_n'(t)$

$\downarrow$ PSD

$$\left|\frac{1}{2\pi} j\omega\right|^2 S_{Y_n}(\omega)$$

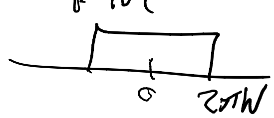
$$\frac{1}{4\pi^2}\omega^2 S_{Y_n}(\omega)$$



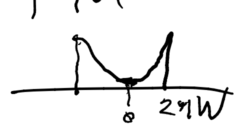
quadratic PSD

baseband case:

PM



FM



$$P_n = \frac{1}{2\pi}\int S_n(w)\,dw = \begin{cases} \dfrac{2N_0 W}{A_c^2}, & PM \\ \dfrac{2}{3}\dfrac{W^3 N_0}{A_c^2}, & FM \end{cases}$$

Signal power $$P_s = \begin{cases} k_p^2 P_m, & PM \\ k_f^2 P_m, & FM \end{cases}$$

$$\boxed{\begin{aligned} SNR_{PM} &= \frac{k_p^2 A_c^2}{2}\,\frac{P_m}{N_0 W} \\ SNR_{FM} &= \frac{3k_f^2 A_c^2}{2W^2}\,\frac{P_m}{N_0 W} \end{aligned}}$$

$\leftarrow$ better in narrowband case (as $W\nearrow$, $SNR\searrow$ bad!)

$$\beta_f = \frac{k_f}{W}\max|m(t)|$$

$$\beta_p = k_p \max|m(t)|$$

$$P_R = A_c^2/2$$

$$SNR_{PM} = P_R\left(\frac{\beta_p}{\max(|m(t)|)}\right)^2 \frac{P_M}{N_0 W}$$

$$SNR_{PM} = P_R \left(\frac{\quad}{\max(|m(t)|)} \right)^2 \frac{\quad}{N_0 W}$$

$$\boxed{SNR_{FM} = P_R \left(\frac{\beta f}{\max(|m(t)|)} \right)^2 \frac{P_m}{N_0 W}}$$

Same formula in terms of mod. index

$\nearrow \beta$ increases SNR

$\boxed{\nearrow \beta \text{ also increases } B_C \text{ by Carson}}$

ex. Power/BW tradeoff

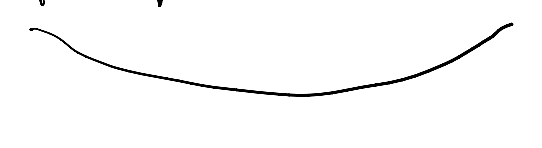
(issue, $\nearrow \beta$ too much, not narrowband any more, maybe approx falls apart...)

---

in FM, most noise is at higher freqs,

= Pre-emphasis filter amplifies high-freq before mod.

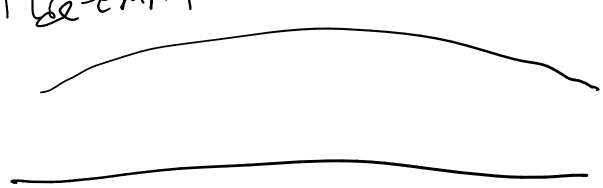
$H_{pre\text{-}emph.}$



SNR ~ const

after demod

$H_{de\text{-}emph}$



de emphasis filter

- in FM, higher freq. see more noise

Wrap-up of Analog

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Bw-efficiency: SSB-AM > DSB-AM >> FM / PM

VSB-AM

Power-efficiency: FM / PM >> AM > conventional AM

highest SNR

Ease of implementation: $\underbrace{\text{Conv. AM} > \text{FM}}_{\text{car radios!}}$ > everything else

(I, Q demod, vel dan ixer)

A&D

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- Review of sampling theory / DTFT
- Quantization (discrete-time vs. digital)
- Coding Intro
- Intro to decision theory

→ dive into Modulation schemes

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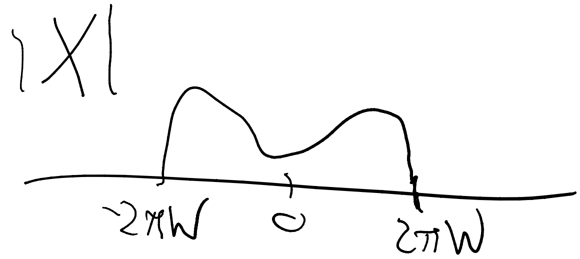
Say $x(t)$ cts-time signal, $X(\Omega)=0$ for $|\Omega| \geq 2\pi W$

cts-time freq.



Nyquist Sampling THM: The sampled signal $x[n]=x(nT_s)$ can "perfectly reconstruct" the signal $x(t)$ so long as

$$\frac{1}{T_s} = \boxed{f_s \geq 2W}$$

$$\hookrightarrow x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\mathrm{sinc}\left(\frac{t}{T_s} - n\right)$$

is the perfect reconstruction

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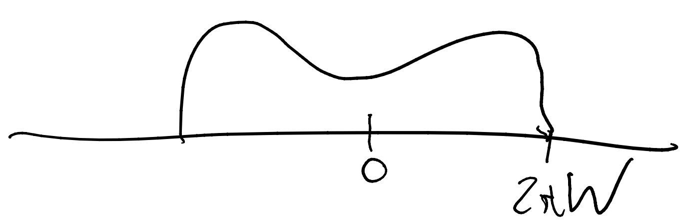
Def (DTFT)

$$X(\omega)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

1) $X(\omega)$ is 2π-periodic

2) $X:\mathbb{R}\to\mathbb{C}$

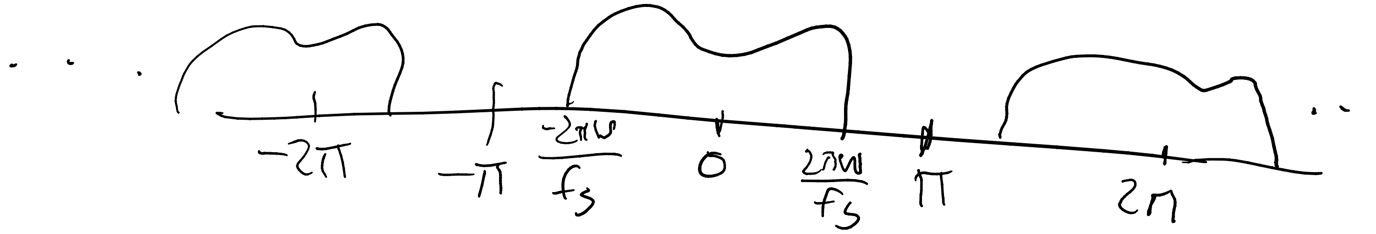
$X(\Omega)$



$\downarrow f_s > 2W$

$$\boxed{\omega_d = 2\pi f/f_s = \Omega/f_s} \leftarrow$$

$X(\omega)$



if $f_s < 2W$, these overlap, and I get <u>aliasing</u>

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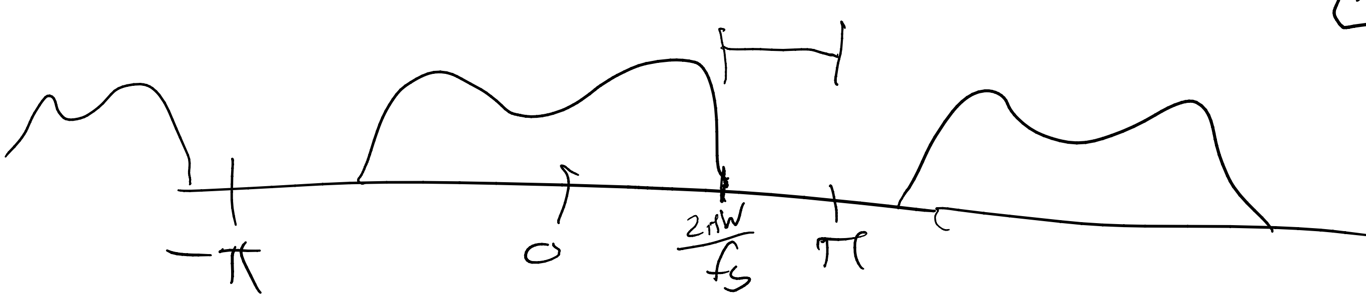
1) <u>Nyquist Rate</u>. the slowest you can sample a bandlimited signal and still avoid aliasing ($f_s = 2W$)

2) <u>Nyquist Frequency</u>: If I sample at f_s, what is the highest freq. of a signal that will <u>not</u> get aliased (cutoff of the anti-aliasing filter)

$(f_c = f_s/2)$

3) <u>Guard Band</u>

guard band: in ω_d is $\pi - \frac{2\pi W}{f_s}$ or in f is $\boxed{f_s - 2W}$



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Quantization

Ex Scalar quantization

Represent my signal using N possible values

$$\hat{x}_i, \; i = 1, 2, \ldots, N. \quad \text{Commonly } N = 2^k$$

Discrete-time signal with values in $\mathbb{R}$

partitioning $\mathbb{R}$ into N intervals, $R_i, i = 1, \ldots, N$

s.t. the quantization map is given by

$$Q(x[n]) = \hat{x}[n] = \hat{x}_i \quad \text{for } x[n] \in R_i$$

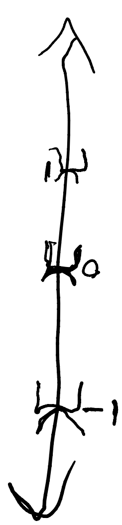
$N = 4$

$R_1 = (-\infty, -1)$

$R_2 = [-1, 0)$

$R_3 = [0, 1)$

$R_4 = [1, \infty)$



$\hat{x}_1 = -2$ $\quad$ $\hat{x}_3 = 0$

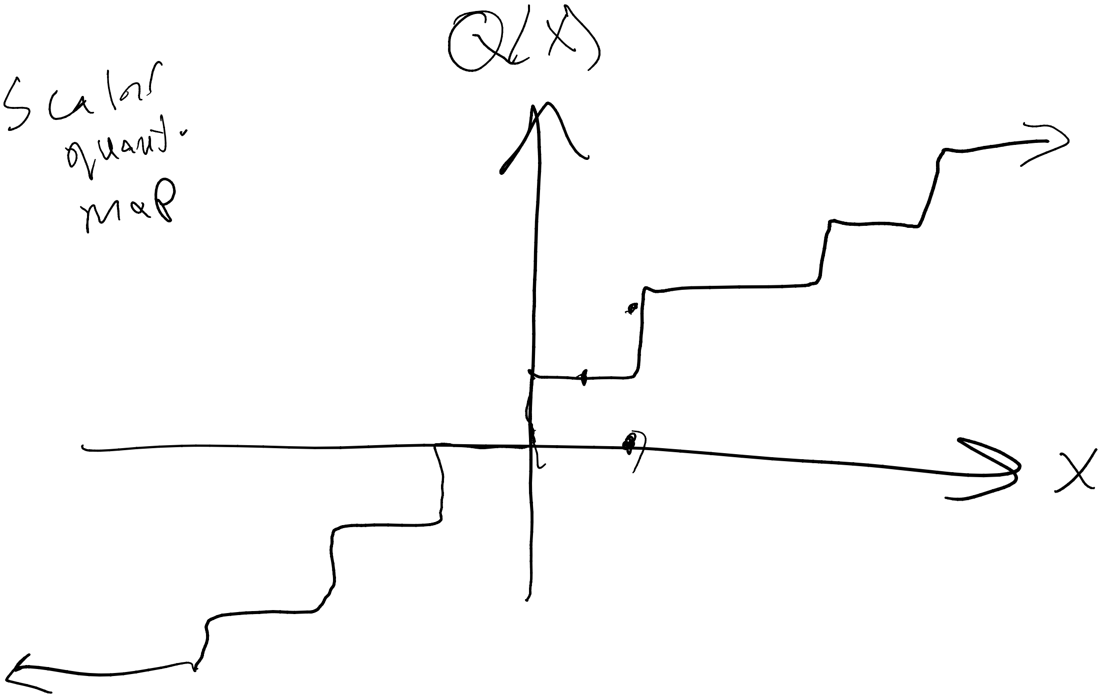
$\hat{x}_2 = -1$ $\quad$ $\hat{x}_4 = 1$

W.P. $\hat{x}$ ... 00 $\quad$ $\hat{x}$ ... 10

ex. rep. $\hat{x}_1$ as 00 $\hat{x}_3$ as 10

$\hat{x}_2$ as 01 $\hat{x}_4$ as 11

Scalar quant. map



Q:

I have N-level quantization scheme, how many bits do I need to rep. 1 value

A: $\lceil \log_2 N \rceil$

we are wasteful when $N \neq 2^k$

Quantization $\longrightarrow$ Distortion

Distortion

(quantization noise)

1) How does this affect the SNR?

2) How does this affect the spectrum? ← <u>harder</u>

<u>Def</u> <u>Squared Error Distortion</u> for <u>quantizer</u> Q and signal $x[n]$ is given by

$$\left(x[n] - Q(x[n])\right)^2 \equiv \tilde{x}^2$$

↑ this varies with n

<u>Def</u> The <u>average distortion</u> (or <u>mean square error</u>) is given by

$$\boxed{D = E[\tilde{x}^2]}$$

Ex. $X(t)$ white Gaussian 0-mean

$$S_X(w) = \begin{cases} 2, & \left|\frac{w}{2\pi}\right| \le 100\,\text{Hz} \\ 0, & \text{else} \end{cases}$$

Sample X at Nyquist ($f_s = 200$ Hz)

Let's say I use 8-level quantizer

$R_1 = (-\infty, -60)$, $R_2 = (-60, -40)$, ... $R_7 = (40, 60)$, $R_8 = (60, \infty)$

$\hat{x}_1 = -70$, $\hat{x}_2 = -50$, $\hat{x}_3 = -30$, ..., $\hat{x}_8 = 70$

Rate: 3 bits/sample (8 level)

200 samples/second

$$\boxed{R = 600 \text{ bits/second}}$$

$$D = E[X - Q(X)] = \int_{-\infty}^{\infty} (x - Q(x))^2 f_X(x)\,dx$$

$f_X(x)$ is pdf of X which has $S_X(\omega) = \begin{cases} 2, |f| < 100 \\ 0, \text{else} \end{cases}$

we know $R_X(0) = \sigma^2 = E[X^2]$

$$\sigma^2 = R_X(\tau)\Big|_{\tau=0} = \int_{-\infty}^{\infty} \frac{1}{2\pi} S_X(\omega) e^{j\omega\tau}\, d\omega\Big|_{\tau=0}$$

$$= \frac{1}{2\pi} \int_{-2\pi(100)}^{2\pi(100)} 2\, d\omega = 400$$

$$f_X(x) = \frac{1}{\sqrt{2\pi(400)}} e^{-x^2/800}$$

$$D = \int_{\mathbb{R}} (x - Q(x))^2 f_X(x)$$

$$= \sum_{i=1}^{8} \int_{R_i} (x - \hat{x}_i)^2 f_X(x)\, dx$$

Closed form

closed form

$\longrightarrow$ MATLAB

$$\boxed{D \approx 33.38}$$

$$\boxed{D \approx 33.38}$$

$$\text{mean-squared error} = \boxed{\text{quantization noise}}$$

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Def for r.v. X, quantizer Q,
the signal-to-quantization noise ratio (SQNR)
is
$$SQNR = \frac{E[X^2]}{E[(X-Q(X))^2]}$$
← power of a r.v.

Ex. $E[X^2] = 400$, $D = 33.38$, $SQNR \approx 11.98$

$SQNR_{dB} = 10\log_{10}(11.98) = 10.78\,dB$

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Find the <u>best</u> quantizer

Optimal = greatest possible SQNR

or

for fixed power

lowest possible D

<u>Uniform case</u>

Every region same size i.e. $R_i = (a_i, a_{i+1})$

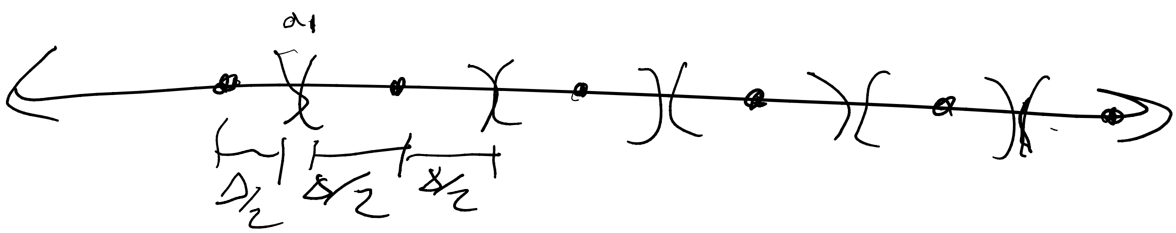
then $a_{i+1} - a_i = \Delta$

for $i = 1, \ldots, N-1$ because $R_1 = (-\infty, a_0)$

$R_N = (a_{N-1}, \infty)$

we choose $\hat{X}_i = a_i - \Delta/2$, for $i = 1, \ldots N-1$

$\hat{X}_N = a_{N-1} + \Delta/2$



$D = \int^{a_1} \ldots \int^{\infty}$

$$D = \int_{-\infty} (x-(a_1-\Delta/2))^2 f_X(x)\,dx + \int_{a_{N-1}} (x-(a_{N-1}+\Delta/2))^2 f_X(x)\,dx$$

$$+ \sum_{i=1}^{N-2} \int_{a_1+(i-1)\Delta}^{a_1+\Delta} (x-(a_1+(i-1)\Delta+\Delta/2))^2 f_X(x)\,dx$$

parameters = 2, a_1, Δ

if I know I want N-level quantizer

To find optimal quantizer,

minimize $D(a_1, \Delta)$ in $\mathbb{R}^2$

have a functional form, easy to search the 2-d space using SGD for example

<u>Non-uniform case</u> $R_i = (a_i, a_{i+1})$

$$D = \sum \int_{R_i} (x - \hat{x}_i)^2 f_x(x)\, dx$$

$a_1, \ldots, a_{N-1}$ are all parameters

$\hat{x}_1, \ldots, \hat{x}_N$ are all parameters

$2N-1$ parameters

$$\frac{\partial D}{\partial a_i} = f_x(a_i)\left((a_i - \hat{x}_i)^2 - (a_i - \hat{x}_{i+1})^2\right) = 0$$

$$a_i = \frac{1}{2}(\hat{x}_i + \hat{x}_{i+1})$$

So analytically, the a_i are best chosen as the midpoints,

can show analytically that $\frac{\partial D}{\partial \hat{x}_i} = 0 \rightarrow \hat{x}_i = \dfrac{\int_{a_{i-1}}^{a_i} x \cdot f_x(x)\,dx}{\int_{a_{i-1}}^{a_i} f_x(x)\,dx}$

Centroid

but still, this only constrains a $2N-1$ parameter optimization problem

$\rightarrow$ use SGD still, but constrained

- v

→ use SGD still, but constrained to a_i → midpts of $\hat{x}$, and $\hat{x}$ being centroids of R_i

Optimal quantizer "findable" by SGD if X, N are given, <u>easier</u> in uniform case

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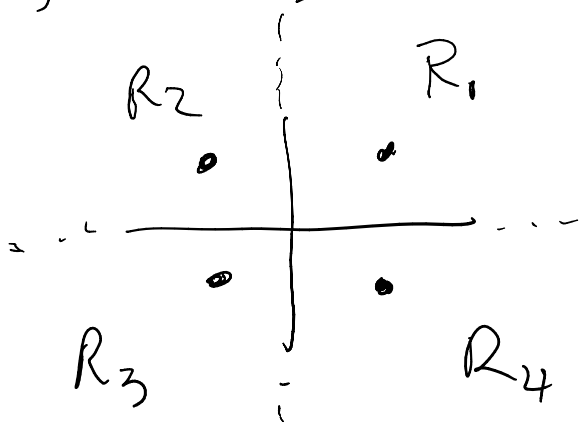
Scalar quant: $x[n] \xrightarrow{Q} \hat{x}[n] \in \mathbb{R}$

Vector quantization: map multiple time instances to one quantized value

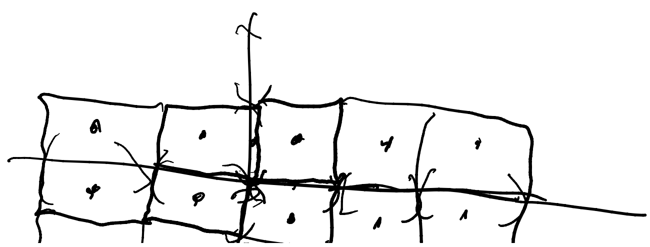
$$\underbrace{(x[n_1], \ldots, x[n_k])}_{\in \mathbb{R}^k} \xrightarrow{Q} \hat{x}_i \in \mathbb{R}^k$$

Ex. $k=2$, $(x[n], x[n+1]) \rightarrow \begin{cases} (1,1) \\ (-1,1) \\ (-1,-1) \\ (1,-1) \end{cases}$ if $x_n \gtrless 0$ and $x_{n+1} \lessgtr 0$

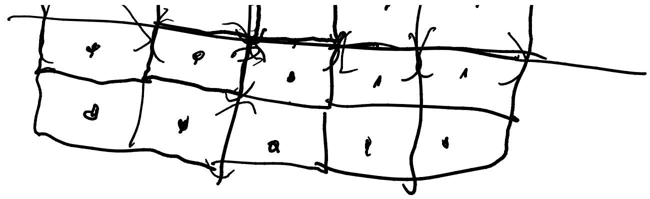


if I'm quantizing pairwise

uniform scalar $\longrightarrow$ what is uniform vector?



Cartesian Product of two uniform



of
two uniform
scalar, i.e. rectangles

My decision regions can have arbitrary shape in arbitrary dimension — pretty crazy!

optimality - still use SGD in very high dim.

→ still, centroids of decision regions are optimal

Quantizing in $\mathbb{R}^n$ with K quant. regions (K values)

$$\begin{pmatrix} x[n] \\ \vdots \\ x[n+m] \end{pmatrix} \to \hat{x}_i \in \mathbb{R}^m, \; i=1,\dots,K$$

m signal values → K quantized values

$\lceil \log_2 K \rceil$ bits required to rep. m signal vals

$$\boxed{R = \frac{\log_2 K}{m} \text{ bits/sample}}$$