Quiz 3

$$P_{X,Y}(x,y) = \begin{cases} 10x & 0 \le x \le y^2,\ 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the MMSE estimate of X based on Y.

$$\hat{X}_{MMSE}(y) = E[X \mid Y=y]$$

$$f_Y(y) = \int_0^{y^2} 10x \, dx = 5y^4 \quad (0 \le y \le 1)$$

$$f_{X|Y}(x,y) = \frac{10x}{5y^4} = \frac{2x}{y^4} \quad (0 \le x \le y^2,\ 0 \le y \le 1)$$

$$\therefore \hat{X}_{MMSE}(y) = E[X \mid Y=y] = \int_0^{y^2} x\left(\frac{2x}{y^4}\right) dx = \frac{2}{y^4}\int_0^{y^2} x^2 \, dx$$

$$= \frac{2}{3}\frac{y^6}{y^4} - 0$$

$$= \frac{2}{3}y^2$$

b) Find the corresponding value of the mean squared error of this estimate.

$$e^2 = E[(\hat{X}_{MMSE}(y) - X)^2] = \int_0^1 dy \int_0^{y^2} \left(\frac{2}{3}y^2 - x\right)^2 \overbrace{f_{X,Y}(x,y)}^{10x} \, dx$$

$$= \int_0^1 dy \int_0^{y^2} \left(\frac{40}{9}y^4 x - \frac{40}{3}y^2 x^2 + 10x^3\right) dx$$

$$= \int_0^1 dy \left(\frac{20}{9}y^8 - \frac{40}{9}y^8 + \frac{5}{2}y^8\right) = \int_0^1 \frac{5}{18}y^8 \, dy$$

$$= \frac{5}{18}\left(\frac{1}{9}\right)y^9 \Big]_0^1$$

$$= \frac{5}{162}$$

c) Find the linear MMSE of X based on Y.

$$\hat{X}_{LMMSE}(y) = a_0 + a_1 Y$$

where $a_0 = E[X] - a_1 E[Y]$, $a_1 = \dfrac{E[XY] - E[X]E[Y]}{E[Y^2] - (E[Y])^2}$

$$f_X(x) = \int_{\sqrt{x}}^{1} 10x \, dy = 10xy \Big]_{\sqrt{x}}^{1} = 10x - 10x^{3/2}$$

$$\therefore E[X] = \int_0^1 x(10x - 10x^{3/2}) \, dx = \frac{10}{21}$$

$$E[X^2] = \int_0^1 x^2(10x - 10x^{3/2}) \, dx = \frac{5}{18}$$

$$E[Y] = \int_0^1 y(5y^4) \, dy = \frac{5}{6}$$

$$E[Y^2] = \int_0^1 y^2(5y^4) \, dy = \frac{5}{7}$$

$$E[XY] = \int_0^1 \int_0^{y^2} xy(10x) \, dx \, dy = \int_0^1 \frac{10}{3} y^7 \, dy = \frac{5}{12}$$

$$\therefore a_1 = \frac{\frac{5}{12} - \left(\frac{10}{21}\right)\left(\frac{5}{6}\right)}{\frac{5}{7} - \left(\frac{5}{6}\right)^2} = 1$$

$$a_0 = \frac{10}{21} - 1\left(\frac{5}{6}\right) = -\frac{5}{14}$$

$$\therefore \hat{X}_{LMMSE}(y) = -\frac{5}{14} + y$$

d) Find the corresponding value of mean squared the linear estimate.

$$e^2 = (1-\rho^2)\,Var(x)$$

where $$\rho = \frac{E[XY] - E[X]E[Y]}{\sqrt{(E[X^2] - (E[X])^2)(E[Y^2] - (E[Y])^2)}}$$

$$Var(x) = E[X^2] - (E[x])^2$$

$$\therefore\ e^2 = \left(1 - \frac{\left(\frac{5}{12} - \left(\frac{10}{21}\right)\left(\frac{5}{6}\right)\right)^2}{\left(\frac{5}{18} - \left(\frac{10}{21}\right)^2\right)\left(\frac{5}{7} - \left(\frac{5}{6}\right)^2\right)}\right)\left(\frac{5}{18} - \left(\frac{10}{21}\right)^2\right)$$

$$= \frac{55}{1764}$$