Quiz 5

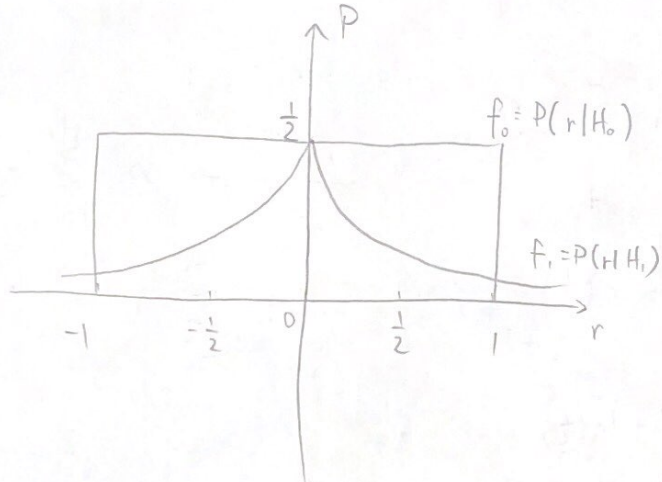
Allister Liu

$$f_0(r) = \begin{cases} \frac{1}{2} & -1 \le r \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_1(r) = \frac{1}{2} e^{-|r|}$$

Assume $P_0 = \frac{1}{3}$, $P_1 = \frac{2}{3}$

a)



b) $$P_{err} = P_0 \cdot P_F + P_1 \cdot P_D$$

$$= \frac{1}{3} \cdot 2 \int_0^{\frac{1}{2}} \frac{1}{2}\, dr + \frac{2}{3} \cdot 2 \int_{\frac{1}{2}}^{\infty} \frac{1}{2} e^{-r}\, dr$$

$$= \frac{1}{3} \cdot 2 \left(\frac{1}{2} - 0\right) + \frac{2}{3} \cdot 2 \left(-0 + e^{-\frac{1}{2}}\right)$$

$$= \frac{1}{2} + \frac{2}{3} e^{-1/2}$$

c) Likehood Ratio Test.

$$\frac{P(r|H_1)}{P(r|H_0)} = \frac{f_1(r)}{f_0(r)} = \frac{\frac{1}{2}e^{-|r|}}{\frac{1}{2}} = e^{-|r|} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P_0}{P_1} = \frac{1}{2}$$

$$\therefore \frac{f_1(r)}{f_2(r)} = \underset{H_2}{\overset{H_1}{\gtrless}} \frac{1}{2} \Rightarrow \frac{\frac{1}{2}e^{-|r|}}{\begin{cases} \frac{1}{2} & -1 \le r \le 1 \\ 0 & \text{else} \end{cases}} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}$$

for $-1 \le r \le 1$:

$$\frac{f_1}{f_0} = e^{-|r|} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}$$

$$\therefore \ln(e^{-|r|}) \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\tfrac{1}{2})$$

$$|r| \underset{H_1}{\overset{H_0}{\gtrless}} -\ln(\tfrac{1}{2})$$

for $|r| > 1$:

$$\frac{\frac{1}{2}e^{-|r|}}{0} \to \infty \underset{H_2}{\overset{H_1}{\gtrless}} \frac{1}{2}$$

$$\therefore |r| \underset{H_2}{\overset{H_1}{\gtrless}} 1$$

$\therefore$ decision rule:

$$\begin{cases} r < -1 & \to H_1 \\ -1 < r < -\ln(\frac{1}{2}) & \to H_0 \\ -\ln(\frac{1}{2}) < r < \ln(\frac{1}{2}) & \to H_1 \\ \ln(\frac{1}{2}) < r < 1 & \to H_0 \\ 1 < r & \to H_1 \end{cases}$$

thresholds: ± 1,

$\pm \ln(\frac{1}{2})$.

d) Find P_0 such that the decision rule that minimizes the probability of error always decides the same hypothesis, regardless of the observation.

$$P_1 \cdot P(r|H_1) \gtrless P_0 \cdot P(r|H_0)$$

$$P(r|H_1) \gtrless \frac{P_0}{P_1} \cdot P(r|H_0)$$

$$P(r|H_1) \gtrless \frac{P_0}{1-P_0} P(r|H_0)$$

$$\therefore \frac{1}{2} e^{-|r|} \gtrless \frac{P_0}{1-P_0} \left(\begin{cases} \frac{1}{2} & -1 \leq r \leq 1 \\ 0 & \text{else} \end{cases} \right)$$

$$\therefore e^{-1} \gtrless \frac{P_0}{1-P_0}.$$

$$e^{-1} - P_0 e^{-1} \gtrless P_0.$$

$$e^{-1} \gtrless P_0 (1+e^{-1})$$

$$\therefore P_0 = \frac{e^{-1}}{1+e^{-1}}$$

$$\therefore 0 < P_0 \leq \frac{e^{-1}}{1+e^{-1}}$$

$$\therefore 0 < P_0 \leq \frac{1}{e+1}$$