Replace summation notation in Eq. 4 with vector notation.

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E} - \nabla^2 \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E} - \nabla^2 \vec{E}) \tag{1}$$

Replace curl in Eq. 7 with Levi-Cevita contravariant.

$$\epsilon^{i,j,k} \hat{x}_i \nabla_j (\vec{\nabla} \times \vec{E})_k = \vec{\nabla}(\vec{\nabla} \cdot \vec{E} - \nabla^2 \vec{E}) \tag{2}$$

Substitute RHS of Eq. 8 into Eq. 6.

$$\left(\delta^l{}_j \delta^m{}_k - \delta^l{}_k \delta^m{}_h\right) \hat{x}_i \nabla_j \nabla^m E^n = \vec{\nabla}(\vec{\nabla} \cdot \vec{E} - \nabla^2 \vec{E}) \tag{3}$$

Simplify Eq. 5.

$$\hat{x}_m \nabla_n \nabla^m E^n - \hat{x}_n \nabla_m \nabla^m E^n = \vec{\nabla}(\vec{\nabla} \cdot \vec{E} - \nabla^2 \vec{E}) \tag{4}$$

Simplify Eq. 3.

$$\left(\delta^l{}_j \delta^m{}_k \hat{x}_i \nabla_j \nabla^m E^n\right) - \left(\delta^l{}_k \delta^m{}_h \hat{x}_i \nabla_j \nabla^m E^n\right) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E} - \nabla^2 \vec{E}) \tag{5}$$

Replace curl in Eq. 2 with Levi-Cevita contravariant.

$$\epsilon^{i,j,k} \epsilon_{n,j,k} \hat{x}_i \nabla_j \nabla^m E^n = \vec{\nabla}(\vec{\nabla} \cdot \vec{E} - \nabla^2 \vec{E}) \tag{6}$$

Eq. 7 is an identity.

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E} - \nabla^2 \vec{E}) \tag{7}$$

Thus we see that LHS of Eq. 1 is equal to RHS. Eq. 8 is an identity.

$$\epsilon^{i,j,k} \epsilon_{n,j,k} = \delta^l{}_j \delta^m{}_k - \delta^l{}_k \delta^m{}_h \tag{8}$$