21 DERIVATION OF KIRCHHOFF'S LAWS

21.1 Theoretical basis

21.1.1 The laws of classical electromagnetism

The force $\boldsymbol{F}$ on a charge q subjected to an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$ is given by

$$\boldsymbol{F} = q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

where $\boldsymbol{v}$ is the velocity of the charge. The fields $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfy Maxwell's equations:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \qquad [21.1]$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad [21.2]$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad [21.3]$$

$$c^2 \nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}}{\varepsilon_0} + \frac{\partial \boldsymbol{E}}{\partial t} \qquad [21.4]$$

where ρ is the (total) charge per unit volume and $\boldsymbol{j}$ is the (total) current per unit area.

21.1.2 The macroscopic Maxwell equations

The charge density ρ is a field which is zero everywhere except within the sharp peaks associated with the charged particles. The $\boldsymbol{E}$ and $\boldsymbol{B}$ fields have corresponding complexity at the inter-particle scale. If we give up our interest in what happens at this scale and spatially smooth (instantaneously) all the fields over volumes a suitable number of inter-particle spacings across, the *macroscopic* Maxwell equations are obtained. If, in addition, we add the fact that all the charged particles can be classified as either bound or free, the smoothed charge and current densities can be expressed (correct to dipole terms) as:

$$\rho_{\text{free}} + \rho_{\text{fixed}} - \nabla \cdot \boldsymbol{P} \qquad [21.5]$$

and

$$\boldsymbol{j}_{\text{free}} + \frac{\partial \boldsymbol{P}}{\partial t} + \nabla \times \boldsymbol{M} \qquad [21.6]$$

respectively where ρ_{free} and ρ_{fixed} are the smoothed free and fixed charge densities (which are equal and opposite in the interior of a good conductor), $\boldsymbol{P}$ is the polarisation, $\boldsymbol{j}_{\text{free}}$ is the smoothed free current density, and $\boldsymbol{M}$ the magnetisation.

In what follows all the fields are to be considered as spatially smoothed.

21.1.3 *E* and *B* in terms of potentials

It follows from the third and second macroscopic Maxwell equations that it is possible to write

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\boldsymbol{E} = -\nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t}$$

where $\boldsymbol{A}$ is a macroscopic (spatially smoothed) vector field (the magnetic potential) and ϕ is a macroscopic scalar field (the electric potential).

21.1.4 The potentials in terms of the charge and current distributions

Choice of the Lorentz gauge

$$\nabla \cdot \boldsymbol{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}$$

leads to similar second order differential equations for the macroscopic potentials ϕ and $\boldsymbol{A}$ which have solutions

$$\phi(\boldsymbol{r};\, t) = \frac{1}{4\pi\varepsilon_o} \int_{\text{all space}} \frac{\rho\,(\boldsymbol{r}', t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{|\boldsymbol{r} - \boldsymbol{r}'|}\, \mathrm{d}^3\boldsymbol{r}'$$

and

$$\boldsymbol{A}(\boldsymbol{r};\, t) = \frac{1}{4\pi\varepsilon_o c^2} \int_{\text{all space}} \frac{\boldsymbol{j}\,(\boldsymbol{r}', t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{|\boldsymbol{r} - \boldsymbol{r}'|}\, \mathrm{d}^3\boldsymbol{r}'.$$

The macroscopic potentials at $(\boldsymbol{r}, t)$ are determined by the *retarded* values of ρ and $\boldsymbol{j}$, that is the values they had at the (earlier) time $t - |\boldsymbol{r} - \boldsymbol{r}'|/c$. This is because, travelling at the speed of light, it takes a time $|\boldsymbol{r} - \boldsymbol{r}'|/c$ for information about a charge at $\boldsymbol{r}'$ to reach the point $\boldsymbol{r}$.

The $\boldsymbol{E}$ and $\boldsymbol{B}$ fields derived from the potentials have parts that remain in the vicinity of the sources ρ and $\boldsymbol{j}$ (induction fields) and parts that are travelling waves which can carry energy away from the sources (radiation fields). Mathematically the radiation fields arise from the retardation $|\boldsymbol{r} - \boldsymbol{r}'|/c$ in the expressions for the potentials and disappear if it is ignored.

21.2 Circuits

Electromagnetic energy supplied to an arrangement of conductors, insulators, and magnetic material is either stored, degraded into heat, or radiated away. We define a circuit as *an arrangement which can be adequately described without allowing for radiation.*

As mentioned above, radiation fields arise from retardation and ignoring them is permissible only if it is permissible to ignore the retardation. This is the case if the extent of the arrangement is very much smaller than the wavelength of radiation at the frequency being considered, or there is something special about the arrangement which makes any significantly retarded fields negligible.

In the first case, given the dimensions of real hardware, this means systems which are excited at frequencies less than a few GHz (wavelengths longer than a few cm).

The most striking example of the second case is the transmission line, which can be treated successfully as a circuit (via the telegraph equations). Transmission lines are almost by definition more than a wavelength long so some of the contributions to the fields at a particular cross section will arise from charges and currents which are more than a wavelength away. However the charges and currents at any point along a line are equal and opposite on the two conductors and their fields are dipole fields which fall off rapidly with distance becoming negligible in magnitude before their retardation becomes significant.

21.3 Linear conduction

In most conducting materials the current density j_{free} exhibits a very close to linear dependence on the force $\boldsymbol{F}$ applied to the mobile charges (usually electrons). Further, this near-linear dependence is usually found to be nearly *isotropic* (same in all directions in the material), and nearly *homogeneous* (same at all places in the material) as well.

Given this observed behaviour, the theoretical concept of a conductor with a precisely linear, isotropic, and homogeneous (LIH) dependence of j_{free} on $\boldsymbol{F}$ will be useful. We express such a model as:-

$$j_{\text{free}} = \frac{\sigma}{q}\boldsymbol{F} \qquad \text{(LIH conductor, Ohms law)}$$

where σ is a positive scalar constant called the (electrical) *conductivity* and q is the charge carried by each mobile particle.

We picture the conduction as follows. The effect of $\boldsymbol{F}$ alone would be to accelerate the charges and cause a current which increased continuously. That observed currents are steady we attribute to the momentum being gained by the charges from the action of $\boldsymbol{F}$ being destroyed at the same rate by collisions with the fixed atoms of the material. The proportionality arises from the details of the mechanisms by which the momentum is gained and lost.

21.4 Kirchhoff's laws in stationary circuits with linear conductors

21.4.1 Kirchhoff's first law

From the macroscopic Maxwell equations the conservation equation for the free charge ρ_{free} is easily derived as

$$-\int_S j_{\text{free}} \cdot \mathrm{d}\boldsymbol{S} = \int_\tau \frac{\partial \rho_{\text{free}}}{\partial t}\mathrm{d}\tau$$

where S is the surface of the volume τ. This may be written

$$\sum_{r=1}^{n} I_r = \frac{\partial q}{\partial t} \qquad [21.7]$$

where I_r is one of the n currents entering τ and q is the charge in τ.

This statement of the conservation of charge is the general form of Kirchhoff's first law. When the charge q is negligible (or constant) it states that the sum of the currents entering τ is zero.

21.4.2 Kirchhoff's second law

When attempting to work out the behaviour of an electromagnetic system using Maxwell's equations, it is not usually helpful to imagine the system divided into driving and driven parts - we have to seek self consistent solutions for the whole system. In the realm of circuits however it is often possible to identify a part of a system which has a strong influence on the rest but is not itself significantly influenced in return, e.g. a system consisting of a signal generator and an amplifier under test. We can take a cause and effect view.

In this spirit we make explicit the contribution by the driving part of a system to the forces exerted on the charges in the driven part. We express the force on each mobile charge in the driven part, which we shall refer to as the circuit, as the sum of the external applied force and the force due to all the charge and current in the circuit (except that due to itself), i.e.

$$\boldsymbol{F}(\boldsymbol{r},t) = \boldsymbol{F}_{\text{ex}}(r,t) + \boldsymbol{F}_{\text{ct}}(r,t)\,.$$

The average drift velocity of the charges in the circuit, $\boldsymbol{v}_{\text{drift}}(\boldsymbol{r};t)$, is given by

$$j_{\text{free}}(\boldsymbol{r};t) = \mathrm{n}(\boldsymbol{r})\, \mathrm{q}\, \boldsymbol{v}_{\text{drift}}(\boldsymbol{r};t)$$

where $\mathrm{n}(\boldsymbol{r})$ is the number of mobile charges per unit volume. Then following from section 21.3 we can write

$$\frac{n(\boldsymbol{r})\ q\ \boldsymbol{v}_{\text{drift}}(\boldsymbol{r},t)}{\sigma(\boldsymbol{r})} = \frac{\boldsymbol{F}_{\text{ex}}(\boldsymbol{r},t) + \boldsymbol{F}_{\text{ct}}(\boldsymbol{r},t)}{q}\ .$$

Now $\quad \dfrac{\boldsymbol{F}_{\text{ct}}}{q} = \boldsymbol{E}_{\text{ct}} + (\boldsymbol{v}_{\text{drift}} + \boldsymbol{v}_{\text{cond}}) \wedge \boldsymbol{B}_{\text{ct}}$

where $\boldsymbol{E}_{\text{ct}}$ and $\boldsymbol{B}_{\text{ct}}$ are the electric and magnetic fields at $\boldsymbol{r}$ due to the charge and current in the circuit and $\boldsymbol{v}_{\text{cond}}$ is the contribution to the velocity of the charges from any motion of the conductor. (We have assumed that the random thermal motion of the charges gives rise to effects only at the level of Johnson noise, see 20.2, and have ignored it.)

Using $\quad \boldsymbol{E}_{\text{ct}} = -\nabla\phi_{\text{ct}} - \dfrac{\partial \boldsymbol{A}_{\text{ct}}}{\partial t}$

yields

$$\frac{\boldsymbol{F}_{\text{ex}}}{q} = \frac{\partial \boldsymbol{A}_{\text{ct}}}{\partial t} - (\boldsymbol{v}_{\text{drift}} + \boldsymbol{v}_{\text{cond}}) \wedge \boldsymbol{B}_{\text{ct}} + \frac{\boldsymbol{j}_{\text{free,ct}}}{\sigma} + \nabla\phi_{\text{ct}}\ .$$

Next we restrict the discussion to stationary circuits ($\boldsymbol{v}_{\text{cond}} = 0$) and form the line integral of this equation from point a to point b in the circuit along a path $\boldsymbol{l}$ which is parallel to $\boldsymbol{v}_{\text{drift}}$ (and therefore to $\boldsymbol{j}_{\text{free}}$). For such paths the term in $\boldsymbol{v}_{\text{drift}} \wedge \boldsymbol{B}_{\text{ct}} \cdot \mathrm{d}\boldsymbol{l}$ disappears (because $\boldsymbol{v}_{\text{drift}}$ is along $\mathrm{d}\boldsymbol{l}$) and we have

$$\int_a^b \frac{\boldsymbol{F}_{\text{ex}} \cdot \mathrm{d}\boldsymbol{l}}{e} = \int_a^b \frac{\partial \boldsymbol{A}_{\text{ct}}}{\partial t} \cdot \mathrm{d}\boldsymbol{l} + \int_a^b \frac{\boldsymbol{j}_{\text{free,ct}}}{\sigma} \cdot \mathrm{d}\boldsymbol{l} + (\phi_b - \phi_a)_{\text{ct}} \tag{21.8}$$

This is the general form of Kirchoff's second law for circuits with linear conductors. It describes a balance between (on the left hand side) a measure of the external forces applied to a circuit and (on the right hand side) a measure of the response of the circuit.

Each term in [21.8] has the units work/charge, a combination which occurs frequently in circuits and is given the name Volt (after Alessandro Volta, who in 1802 made the first battery and first produced continuous currents).

The left hand side is the work done by the source of the applied force when it causes unit charge to traverse the path ab. It is called the *applied electromotive force* or *emf*. (An unfortunate name because its units are not those of force.)

Of the three terms on the right hand side of equation [21.4] the first is called the *inductive voltage drop*, and depends on the free, polarisation, and magnetisation currents (that are caused in the circuit by the emf), the second is called the *resistive voltage drop* and describes the hindering of the free current in the circuit, and the third term, which depends on the free, fixed, and polarisation charge distributions set up in the circuit by the emf, is called the *capacitive voltage drop*. Of the three terms only the third involves the electric potential. In some circumstances contributions to the first term will be called *induced emfs*.

An example of the balance between an applied emf and a capacitive voltage drop is the following. Consider a length of neutral copper wire to which a longitudinal emf is applied by some external source. The effect of the emf is to push the free charges towards the ends of the wire. The resulting charge distribution, which appears almost instantaneously following the application of the force, is such that the effect of its electric field cancels that of the applied force throughout the wire. The integral of this electric field along the wire is the voltage drop.

Note, it is commonly stated that the electric field inside a perfect conductor is zero, this is not always the case. What happens is that the free charges in a perfect conductor rapidly rearrange themselves to maintain the net *force* on them at zero. If the emf applied to the free electrons in our copper wire is due to its motion in a constant magnetic field the only $\boldsymbol{E}$ field in the system is that due to the charge distribution which is created and this is not zero. Only if the applied emf is due solely to an applied $\boldsymbol{E}$ field (which it often is) is the total $\boldsymbol{E}$ field zero.

21.5 Sources of applied emf

We have resolved the forces experienced by charges in circuits into externally applied and local components. We view the electromotive force applied to a circuit as what makes things happen. The applied force is transmitted throughout the circuit by mutual repulsion of the free charges in its conductors. Sources of emf may be relatively localised (e.g. in the electrolyte of a chemical cell) or distributed (e.g. along a winding in a dynamo). We discuss briefly some sources of emf.

21.5.1 Chemical cells

An example of how electrochemical emfs can arise is the following:- In an electrolyte between two plane parallel electrodes, gradients of the densities of singly charged positive and negative ions are maintained by some means. The current density normal to the plates, the z direction, has conduction and diffusion components and is given by

$$j_z = ne\mu_n E_z - D_n(-e)\frac{\partial n}{\partial z} + pe\mu_p E_z - D_p e\frac{\partial p}{\partial z}$$

where n and p are the number densities of the negative and positive ions, μ_n and μ_p are their mobilities, D_n and D_p are their diffusion coefficients, and E_z is the electric field. As the electrolyte is a good conductor it will be neutral everywhere i.e. $n = p$. Assuming $\frac{\partial n}{\partial z}$ is constant ($= \Delta n/d$) where d is the distance between the plates, using the Einstein relation

$$D = \frac{\mu k T}{e}$$

and setting the current to zero in equilibrium we obtain the open circuit voltage between the electrodes

$$E_z d = \frac{-kT(\mu_n - \mu_p)}{e(n\mu_n + p\mu_p)}\Delta n$$

which is equal to the emf. Δn is maintained by chemical reactions at the surfaces of the plates. The familiar zinc carbon (Léclanché) cell has an emf of about 1.5 V.

21.5.2 Dynamos

Refer back to the note at the end of section 21.4. We consider two types of dynamo.

(i) An open loop of perfectly conducting wire with a magnet moving in its vicinity, no load connected.

The emf applied to the charges in the wire is due to the electric field produced by the changing magnetic field. Charges are pushed to the ends of the wire until their electric field cancels the applied field. The electric field inside the perfectly conducting wire is then zero. (There is no force on the charges along the wire due to the magnetic field as a consequence of any motion they have along the wire.)

Applying the integral form of Maxwell 2 to a closed path $\boldsymbol{l}$ along the wire and across the gap between the ends we have

$$\int_{\text{wire}} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} + \int_{\text{gap}} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t}\cdot \mathrm{d}\boldsymbol{S}$$

where $\boldsymbol{E}$ is the *total* electric field (due to the changing $\boldsymbol{B}$ field and the charge distribution set up), S is the surface bounded by $\boldsymbol{l}$ and the directions of the line integral and $\mathrm{d}\boldsymbol{S}$ are related by a right hand screw (the normal convention). The left hand side reduces to the line integral of $\boldsymbol{E}$ in the gap – which is the voltage drop between them, and therefore equal to the emf which would be applied to any circuit connected to the ends of the wire. (The electric fields due to the changing magnetic field and the charges on the ends of the wire (which oppose each other in the wire) are in the same direction in the gap, we do not know (or care) what their relative contributions in the gap are.)

Summarising we have

$$\text{emf} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t}\cdot \mathrm{d}\boldsymbol{S},$$

the emf is the rate of change of magnetic flux linking the coil as the field changes.

(ii) An open loop of perfectly conducting wire moving at constant velocity $\boldsymbol{v}$ in a static, non uniform, magnetic field.

Here we can evaluate the emf directly from the definition given in section 21.4.2 i.e.

$$\text{emf} = \int_{\text{wire}} (\boldsymbol{v}\wedge\boldsymbol{B})\cdot \mathrm{d}\boldsymbol{l},$$

If the gap in the loop is small, this is the same as

$$= \int_{\text{wire}} \boldsymbol{B}\cdot(\mathrm{d}\boldsymbol{l}\wedge\boldsymbol{v}),$$

i.e. the emf is the rate of change of magnetic flux through the loop due to its motion.

Thus in both situations, *which are totally different*, the emf is given by the rate of change of magnetic flux through the loop. (This is sometimes called "the flux rule".)

The ac dynamos (alternators) used to generate the emf of the mains power supply are of the second kind.

21.5.3 Signal generators

Signal generators are electronic oscillator circuits which convert emfs arising from the mains alternator or batteries into alternating emfs whose frequencies, amplitudes, and often also waveforms, can be selected by the user. Oscillators are discussed in chapter 17.

21.5.4 Solar cells

A solar cell is a semiconductor device in which there is a region of strong electric field where photogenerated electron-hole pairs are separated. The *I-V* characteristic of a PN junction type cell is

$$\mathrm{I} = I_o\left(\mathrm{e}^{\frac{eV}{kT}} - 1\right) - I_p$$

where I_o is a constant and I_p is the photo generated current. On open circuit ($I = 0$) the emf V is given by

$$V = \frac{kT}{e}\ln\left(\frac{I_p}{I_o}\right)$$

On short circuit ($V = 0$) the current is $-I_p$.

The load is usually chosen to maximise the power delivered, typically up to 80% of (I_p V) can be obtained.

21.5.5 Thermal emfs

As a consequence of electrons being involved in both electric currents and the transport of heat *thermoelectric* effects occur.

Thomson emf The tendency of mobile charges to diffuse away from the hot end of a uniform conducting bar is opposed by the build up of an electric field. The strength of the field depends on the temperature gradient.

Peltier effect The tendency of mobile charges to cross the boundary between two different, originally neutral, conductors in contact is opposed by the build up of a potential difference. The name given to the fact that this *contact potential* is temperature dependent is the Peltier effect.

Thermocouple (Seebeck emf) A length of wire of one material is joined to a length of wire of a different material to form a ring. A break is made in one of the wires. When the two joints are held at different temperatures the combination of the Thomson and Peltier emfs gives rise to a potential difference (the Seebeck emf) at the break. It is rather small; for one of the strongest combinations of metals; copper and constantan, it is only 40 μV per K of temperature difference between the two joints. Of the order of 1 mV per K may be observed with a combination of a metal and a semiconductor.

21.6 References

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