V. SUMMARY

We described a simple experiment, suitable for student laboratory use, on the transition from order to chaos. Period doubling bifurcations and chaotic motion are clearly visible in both the near Hamiltonian as well as the dissipative case. The generation of new modes by tangent bifurcations can also be observed. The chaotic trajectories observed in the strongly dissipative case appear to be strange attractors, as confirmed by numerical integration of the differential equation describing the system.

ACKNOWLEDGMENTS

Over the years that this experiment was built and data taken, several students participated in the project. We wish to thank I. Sezan and E. Gottschalk for their contributions. The work received partial support from DOE No. DE-AC02-84ER-13146. We also wish to express our thanks to O. Manley of the DOE for his interest and encouragement.

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An alternate derivation of relativistic momentum

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An alternate derivation of the expression for relativistic momentum is given which does not rely on the symmetric glancing collision first introduced by Lewis and Tolman in 1909 and used by most authors today. The collision in the alternate derivation involves a non-head-on elastic collision of one body with an identical one initially at rest, in which the two bodies after the collision move symmetrically with respect to the initial axis of the collision. Newtonian momentum is found not to be conserved in this collision and the expression for relativistic momentum emerges when momentum conservation is imposed. In addition, kinetic energy conservation can be verified in the collision. Alternatively, the collision can be used to derive the expression for relativistic kinetic energy without resorting to a work-energy calculation. Some consequences of a totally inelastic collision between these two bodies are also explored.

I. INTRODUCTION

The concept of relativistic momentum and its relation to relativistic mass $\mathbf{p} = m\mathbf{u} = \gamma m_0\mathbf{u}$ are at the foundation of relativistic dynamics. In most elementary treatments of relativity,[1] the expression for relativistic momentum is found by imposing momentum conservation on a particular elastic collision of two identical bodies. In this collision viewed in one frame, one body makes a symmetric, glancing collision with another body that only moves in the $\pm y$ direction. The same collision is then viewed in another frame, in which the first body only moves in the $\pm y'$ direction and the second makes the symmetric, glancing collision. Symmetry of the problem in the two frames, transformation of components of velocity, and the limit of a very slight collision then lead to the correct expression for relativistic momentum.

In a field such as special relativity, in which there are usually many examples and paradoxes to illustrate particular points, it is rather strange to find only one example for finding the expression for relativistic momentum from momentum conservation.[2] Perhaps this would not be surprising if the collision used were one which is natural or encountered often. But the fact is that this particular collision is an artificial one, cooked up solely to find the expression for relativistic momentum, and not used in subsequent discussions. Moreover, in the usual treatment of the problem, the expression for the relativistic momentum (or relativistic mass) emerges only in the limit of zero angle of deflection of the collision, i.e., in the limit in which no collision takes place. These features lead one to ask if there might not be another way to find the expression for relativistic momentum. One possible alternative is the subject of this paper.

II. AN ELASTIC COLLISION

Consider the following elastic collision between two identical bodies. In the S frame before the collision, one body has an initial velocity u_i in the $+x$ direction and the

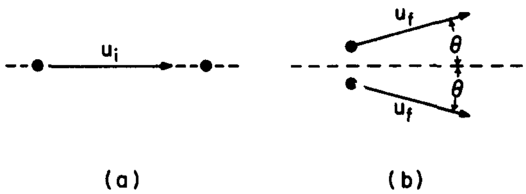

Fig. 1. Elastic collision of two identical bodies as viewed in the S frame, in which one body is initially at rest. Because the bodies are considered as point particles, the impact parameter necessary to avoid a head-on collision is not shown in the diagram. (a) Initial value. (b) Final value.

other body is at rest, as shown in Fig. 1(a). The S' frame is defined to be the frame in which the two bodies have velocities which are equal in magnitude and opposite in direction before the collision, specifically $+v$ and $-v$, as shown in Fig. 2(a). The S' frame therefore moves to the right with respect to the S frame with velocity v. The relation between the initial velocity u_i in S and v in S' is then given by relativistic velocity addition,

$$u_i = \frac{u'_x + v}{1 + u'_x v/c^2} = \frac{2v}{1 + v^2/c^2}. \tag{1}$$

For later use we will need the initial relativistic factor $\gamma_i(v)$, where $\gamma_i = (1 - u_i^2/c^2)^{-1/2}$. This is easily calculated to be

$$\gamma_i = \frac{(1 + v^2/c^2)}{(1 - v^2/c^2)}. \tag{2}$$

Next consider the collision itself. This is not assumed to be a head-on collision, as no relevant information could be obtained from that. Rather, we select a collision in the S' frame in which one of the bodies moves along the $+y'$ axis after the collision[3] (it does not matter which one). As in the usual discussion, momentum is assumed to be a vector in the direction of the velocity, the magnitude of the momentum depending only on the (rest) mass of the body and its velocity. Since the total momentum in the S' frame is zero before the collision, momentum conservation implies that the total momentum is zero after the collision. This means that the other body must move along the $-y'$ axis after the collision with the same speed as the first body. Further, since the collision is elastic, and therefore reversible, the magnitude of the velocity of either body in the S' frame after the collision must also be v, the same as before the collision. (If it were argued that the velocity after the collision was smaller than before the collision, then the collision run backwards would give a velocity after the collision which is larger than that before, a contradiction.) The motion of the bodies in the S' frame after the collision is shown in Fig. 2(b).

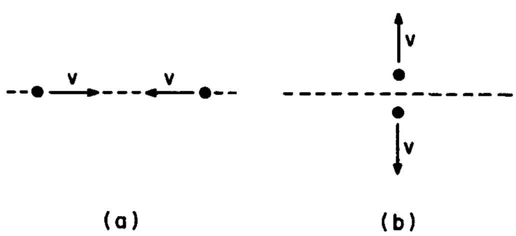

Fig. 2. The same elastic collision viewed in the S' frame, in which the initial velocities of the two bodies are equal in magnitude and opposite in direction. The collision is chosen to be one which is not head-on, specifically that one in which the two bodies after the collision move along lines that are perpendicular to the incident velocities. (a) Initial value. (b) Final value.

Now let us view the motion of the two bodies after the collision in the S frame, which is shown in Fig. 1(b). In the S frame the bodies move with equal speeds at the same angle with respect to the x axis. The components of the velocity of the upper body are given from relativistic velocity addition as

$$u_{fx} = v,$$
$$u_{fy} = v(1 - v^2/c^2)^{1/2}, \tag{3}$$

since $u'_{fx} = 0$ and $u'_{fy} = v$. The velocity components of the lower body differ only in the sign of the y component. For later use we will need the relativistic factor $\gamma_f(v)$, where $\gamma_f = (1 - u_f^2/c^2)^{-1/2}$, with $u_f^2 = u_{fx}^2 + u_{fy}^2$. This is found to be

$$\gamma_f = 1/(1 - v^2/c^2). \tag{4}$$

III. RELATIVISTIC MOMENTUM

We are now prepared to see what can be inferred about momentum if momentum conservation is asserted to hold in the S frame. First, let us ask if momentum is conserved if the Newtonian expression for the momentum is used. From Eq. (1) the total initial momentum is

$$P_i = m_0 u_i = 2m_0 v/(1 + v^2/c^2). \tag{5}$$

From Eq. (3) the total final momentum is only in the $+x$ direction and is given by

$$P_f = 2m_0 u_{fx} = 2m_0 v. \tag{6}$$

Clearly Eq. (5) is not equal to Eq. (6) so Newtonian momentum is not conserved in the S frame.

Next we ask if a modification of the definition of momentum can be found which allows momentum to be conserved in the S frame. Assuming that momentum is in the direction of the velocity and that the magnitude of the momentum for a given body depends only on the velocity of that body, we are led to look for an expression for momentum of the form

$$\mathbf{p} = m_0 f(u)\mathbf{u} = m_0 g(\gamma)\mathbf{u}, \tag{7}$$

where $f(0) = g(1) = 1$ is required for the appropriate reduction of relativistic momentum to Newtonian momentum in the low velocity limit. Applying Eq. (7) to the conservation of momentum in the S frame gives the initial total momentum and final total momentum as

$$P_i = 2m_0 g(\gamma_i)v/(1 + v^2/c^2),$$
$$P_f = 2m_0 g(\gamma_f)v. \tag{8}$$

Equating the initial and final momenta in Eq. (8) and using Eqs. (2) and (4) then gives

$$[g(\gamma_i)/g(\gamma_f)] = (1 + v^2/c^2) = (\gamma_i/\gamma_f). \tag{9}$$

By inspection one possible $g(\gamma)$ that is consistent with momentum conservation in the S frame is $g(\gamma) = \gamma$, which leads to the standard expression for relativistic momentum

$$\mathbf{p} = \gamma m_0 \mathbf{u} = m_0 \mathbf{u}/(1 - u^2/c^2)^{1/2}. \tag{10}$$

Although the above derivation shows that momentum conservation can be saved by the modification of the New-

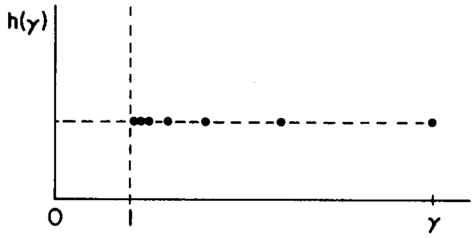

Fig. 3. A sequence of points for which the function $h(\gamma)$ has the same value. Since the initial point in the sequence is at an arbitrary γ and the sequence has points arbitrarily close to $\gamma = 1$, the function $h(\gamma)$ at any γ is the same as $h(1)$, i.e., $h(\gamma) = 1$.

tonian momentum given in Eq. (10), the derivation does not prove that Eq. (10) is the only choice. The following discussion cleans up that detail. Consider Eq. (9) rewritten in the following form:

$$[g(\gamma_i)/\gamma_i] = [g(\gamma_f)/\gamma_f]. \tag{11}$$

If γ_i and γ_f were independent variables in Eq. (11), then it would follow directly that each side of Eq. (11) must be a constant, namely 1, since $g(1) = 1$. But γ_i and γ_f are not independent. Specifically from Eqs. (2) and (4), γ_i and γ_f are related by

$$2\gamma_f - \gamma_i = [2/(1 - v^2/c^2)] - [(1 + v^2/c^2)/(1 - v^2/c^2)] = 1, \tag{12}$$

which means that γ_f can be expressed in terms of γ_i as

$$\gamma_f = (1 + \gamma_i)/2. \tag{13}$$

Letting $h(\gamma) \equiv g(\gamma)/\gamma$ and using Eq. (13) to eliminate γ_f, Eq. (11) can be rewritten as

$$h[\gamma_i] = h[(1 + \gamma_i)/2] \tag{14}$$

for any γ_i. Equation (14) states that the value of the function $h(\gamma)$ at any point γ will be equal to the value that function has one-half of the way between 1 and γ. By iteration, it will have the same value at a point one-fourth the distance from 1 to γ, the same as at a point one-eighth the same distance, and so on, as illustrated in Fig. 3. By continuously halving the interval the value of $h(\gamma)$ at any point γ will be equal to the value the function has at a point arbitrarily close to 1. Since the nonrelativistic limit imposes the constraint $h(1) = 1$, the value of $h(\gamma)$ at any γ must also be equal to 1. Since $g(\gamma) = \gamma h(\gamma)$, $g(\gamma) = \gamma$ is the only solution and the expression for relativistic momentum, given by Eq. (10), is unique. As in the case with the usual derivations of relativistic momentum,[1] this derivation does not, of course, prove that momentum conservation holds in other collisions.

IV. RELATIVISTIC KINETIC ENERGY

This collision can also be used as an illustration in a discussion of relativistic kinetic energy. If the expression for relativistic kinetic energy is derived in the usual manner, using a work-energy calculation,[4] then one can check that relativistic kinetic energy is conserved for this collision in the S frame. Specifically, the initial and final kinetic energies are

$$K_i = (\gamma_i - 1)m_0c^2,$$
$$K_f = 2(\gamma_f - 1)m_0c^2. \tag{15}$$

Using Eqs. (2) and (4) it is easily verified that K_i and K_f are both equal to $2m_0v^2/(1 - v^2/c^2)$, showing that kinetic energy is conserved in this collision. By adding $2m_0c^2$ to the initial and final kinetic energies, total energy conservation is confirmed as well.

Alternatively, the expression for relativistic kinetic energy can be determined directly from this collision without making use of a work-energy calculation. As with the derivation of momentum, what we want to do is find the expression for relativistic kinetic energy such that kinetic energy is conserved in the S frame (it is of course trivially conserved in the S' frame). In addition the relativistic kinetic energy must reduce to the Newtonian kinetic energy in the limit of small velocities. Since kinetic energy should be a scalar function of the velocity, the trial form for relativistic kinetic energy may be taken to be

$$K = F(u)m_0c^2 = G(\gamma)m_0c^2. \tag{16}$$

The factor of c^2 is included in Eq. (16) in order to make the functions F and G dimensionless. For zero velocity, $K = 0$, which means that $F(0) = G(1) = 0$. For small velocity, G can be expanded in a Taylor series about $\gamma = 1$, giving

$$G(\gamma) = G(1) + G'(1)(\gamma - 1) + \cdots, \tag{17}$$

where the first term vanishes since $G(1) = 0$. Since $\gamma - 1 \approx u^2/(2c^2)$ for small u and $K = m_0u^2/2$ in the same limit, a comparison of Eqs. (16) and (17) shows that $G'(1) = 1$. Therefore the nonrelativistic limit of $G(\gamma)$ is given by

$$G(\gamma) \approx (\gamma - 1), \quad (\gamma - 1) \ll 1. \tag{18}$$

Now let us impose relativistic kinetic energy conservation on the collision in the S frame, using the trial form Eq. (16). The initial and final kinetic energies are then

$$K_i = G(\gamma_i)m_0c^2,$$
$$K_f = 2G(\gamma_f)m_0c^2. \tag{19}$$

Equating the initial and final kinetic energies gives $G(\gamma_i) = 2G(\gamma_f)$. Using Eq. (13) to express γ_f in terms of γ_i then gives the condition which the function G must satisfy

$$G[(\gamma_i + 1)/2] = [G(\gamma_i)]/2. \tag{20}$$

Equation (20) states that starting with a value of $G(\gamma)$ at

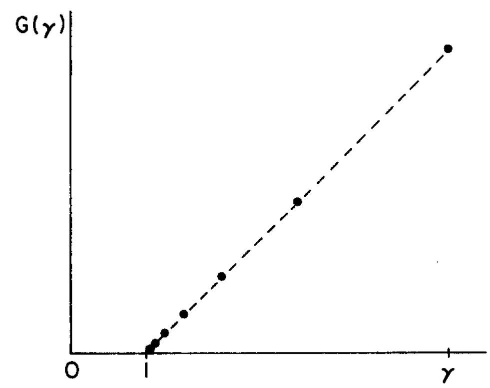

Fig. 4. A sequence of points for the function $G(\gamma)$ which must fall on a straight line through the point $\gamma = 1$, $G = 0$. Since the initial point in the sequence is arbitrary and the sequence has points arbitrarily close to $\gamma = 1$, where the nonrelativistic limit requires the slope to be equal to 1, the function $G(\gamma)$ is given by $G(\gamma) = \gamma - 1$ exactly.

some point γ, the value of G at a point one-half the way between 1 and γ is one-half the original value for G. As illustrated in Fig. 4, this places the new point on a graph of G vs γ on a straight line going from $\gamma = 1, G = 0$ to the starting values of γ and G. The interval can be halved again and again, with each new point falling on the same straight line. By iteration the process produces a sequence of points arbitrarily close to $\gamma = 1$. The straight line for the sequence must be of the form

$$G(\gamma) = (\text{constant})(\gamma - 1), \tag{21}$$

where the constant could depend on the initial values of γ and $G(\gamma)$ used to start the sequence. However, comparison of Eqs. (21) and (18) in the nonrelativistic region $(\gamma - 1) \ll 1$ shows that the constant must be equal to 1. Therefore,

$$G(\gamma) = \gamma - 1 \tag{22}$$

is the exact expression for $G(\gamma)$ for any γ. When Eq. (22) is substituted in Eq. (16), this gives the correct relativistic kinetic energy $K = (\gamma - 1) m_0 c^2$.

V. AN INELASTIC COLLISION

If the collision is modified by making the bodies stick together after the collision, i.e., a totally inelastic collision, then one can proceed in a standard way to demonstrate the increase in rest mass of the composite body and the conversion of kinetic energy to rest-mass energy. For the inelastic collision, the initial and final momenta are, using Eqs. (1) and (2),

$$P_i = \gamma_i m_0 u_i = 2 m_0 v / (1 - v^2/c^2),$$
$$P_f = M_0 v / (1 - v^2/c^2)^{1/2}, \tag{23}$$

where M_0 is the rest mass of the composite body. Equating the two expressions for momentum then leads to the rest mass of the composite body

$$M_0 = 2 m_0 / (1 - v^2/c^2)^{1/2}, \tag{24}$$

which is larger than $2m_0$, the initial rest mass. In the S' frame the initial kinetic energy is $2(\gamma - 1) m_0 c^2$, where $\gamma = 1/(1 - v^2/c^2)^{1/2}$, and the final kinetic energy is zero. Therefore from Eq. (24) the increase in rest-mass during the collision $2(\gamma - 1) m_0$ is equal to the decrease in kinetic energy divided by c^2. The conversion of kinetic energy to rest-mass can also be verified in the S frame, where the initial kinetic energy is given in Eq. (15) and the final kinetic energy is $(\gamma - 1) M_0 c^2$.

Finally, the conservation of total energy (or equivalently conservation of relativistic mass) is valid for the inelastic collision, since when we add the kinetic energy to the rest-mass energy, a decrease in the former is compensated by an increase in the latter, keeping the sum constant. Therefore, total energy conservation holds for both the elastic and inelastic collisions, but kinetic energy and rest mass are conserved only for the elastic collision.

VI. CONCLUSIONS

The collision used in this example gives an alternate way of finding the expression for relativistic momentum from the one usually presented. This collision is somewhat more natural than the symmetric glancing collision usually used, since in one frame it involves one body incident on another one at rest. The derivation of relativistic momentum does not require a limit in which no collision takes place, as in the standard derivation. The same collision can be used either to check kinetic energy conservation or to find the expression for relativistic kinetic energy. A modification of the final state of the collision to the case in which the bodies stick together allows a verification that the increase in the rest mass of the system is equal to the decrease in kinetic energy divided by c^2.

This derivation of relativistic momentum is, however, not without some subtleties. The form of relativistic momentum, Eq. (7) must be introduced at the start and the manipulations to find the modifying factor are relatively simple only if the modifying factor is taken as a function of γ. Finally, the proof of the uniqueness of the expression for relativistic momentum involves a different kind of mathematical reasoning than is usually encountered. However, once understood, the same reasoning can be used to find the expression for relativistic kinetic energy without first going through a work-energy calculation.

This example therefore serves as an alternate method of deriving relativistic momentum and, if desired, relativistic kinetic energy. It is motivated by the feeling that there should be more than one way to derive a particular result, particularly when the standard derivation is based on an uncommon collision. This example is not intended to give any deeper insight into the meaning of relativistic momentum and energy, as, for example, would be provided at a later stage by a discussion of momentum and energy as components of a four vector.

[1]The symmetric glancing collision was first introduced by Lewis and Tolman as a means of finding the expression for relativistic momentum [G. N. Lewis and R. C. Tolman, Philos. Mag. **18**, 510 (1909)]. The sample of references below illustrate how widely this same collision is used by present-day authors in pedagogical derivations of relativistic momentum. There are some variations in how the collision is analyzed, however. For example, some authors use the frame in which both bodies collide symmetrically, while others assume a trial form for the momentum as in Eq. (7); A. B. Arons, *Developments of Concepts of Physics* (Addison-Wesley, Reading, MA, 1965), p. 917; A. Beiser, *Perspectives of Modern Physics* (McGraw-Hill, New York, 1969), p. 34; R. M. Eisberg and L. S. Lerner, *Physics* (McGraw-Hill, New York, 1981), p. 659; R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures, Vol. I.* (Addison-Wesley, Reading, MA, 1963), p. 16-7; A. P. French, *Special Relativity* (Norton, New York, 1968), p. 170; T. M. Helliwell, *Introduction to Special Relativity* (Mudd, Claremont, CA, 1972), p. 74; C. Kacser, *Introduction to the Special Theory of Relativity* (Prentice-Hall, Englewood Cliffs, NJ, 1967), p. 148; C. Kittel, W. D. Knight, and M. A. Ruderman, *Mechanics* (McGraw-Hill, New York, 1965), p. 382; R. Resnick, *Introduction to Special Relativity* (Wiley, New York, 1968), p. 112.; E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (Freeman, San Francisco, CA, 1966), p. 105; R. T. Weidner and R. L. Sells, *Elementary Modern Physics* (Allyn and Bacon, Boston, MA 1980), p. 57.

[2]A few authors use a collinear collision in which one body hits and sticks to another body at rest, giving a totally inelastic collision. The analysis then proceeds by way of an argument given by Tolman [R. C. Tolman, Philos. Mag. **23**, 375 (1912)]. By assuming both momentum conservation and relativistic mass conservation, the expression for relativistic momentum emerges with less effort than in the derivations quoted in Ref. 1. [For the ultimate in brevity see L. C. Baird, Am. J. Phys. **48**, 779 (1980) and a discussion of that derivation by P. D. Gupta, Am. J. Phys. **49**, 890 (1981).] One of the problems with these derivations is that there is no *a priori* reason to assume that relativistic mass is conserved. Even if relativistic mass conservation is assumed in addition to momentum conservation, that does not justify an additional assumption that the velocity-dependent factor which modifies the momentum is the same as that which modifies the relativistic mass. However, by making a Lorentz transformation perpendicular to the line of the collision, it is possible to

justify relativistic mass conservation from momentum conservation [W. A. Blanpied, *Modern Physics* (Holt, Rinehart, and Winston, New York, 1971), p. 223]. Unfortunately, the justification is involved and the simplicity gained by using the inelastic collision is lost. The following references use the simple inelastic collision and the assumptions of both relativistic mass conservation and momentum conservation, with the same relativistic factor applying to both of these quantities; W. C. Michels, M. Correll, and A. L. Patterson, *Foundations of Physics* (Van Nostrand, Princeton, NJ, 1968), p. 227; R. Resnick and D. Halliday, *Basic Concepts in Relativity and Early Quantum Theory* (Wiley, New York, 1985), 2nd ed., p. 95; A. Shadowitz, *Special Relativity* (Saunders, Philadelphia, PA, 1969), p. 85; J. G. Taylor, *Special Relativity* (Oxford, London, 1975), p. 48; R. T. Weidner, *Physics* (Allyn and Bacon, Boston, MA, 1985), p. 873.

[3]That such a collision is possible is seen by the following argument, in which collisions with various impact parameters are viewed in the S' frame. If the collision is head-on the left moving body reverses direction, i.e., it goes off at 0° with respect to the $+x'$ axis. If the collision is a very slight glancing one, then the left moving body continues to move almost undisturbed, i.e., it goes off at 180° with respect to the $+x'$ axis. Between head-on and glancing are impact parameters for which the left moving body goes off at any angle between 0° and 180°. We select for this example that particular collision from among all possible collisions in which the left moving body goes off at 90° with respect to the $+x'$ axis.

[4]The derivation of relativistic kinetic energy from the work-energy theorem can be found in any of the books listed in Ref. 1.

Floating metal ring in an alternating magnetic field

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The levitation of a metal ring in an alternating magnetic field, often used as a demonstration of induced current in introductory courses, is shown to be due entirely to some rather small phase differences. The fields and phases are directly measured and used to calculate the force, with the result agreeing with the weight of the ring. All theory and measurement are within the reach of students taking a first course in electromagnetism.

I. INTRODUCTION

Electromagnetic levitation of a metal ring (sometimes called Thompson's ring[1,2]) is frequently used as a demonstration of induced current. An aluminum ring (~ 1 g) is placed over the protruding iron core of an electromagnet connected to 50 Hz ac (Fig. 1). When the current is turned on, the ring floats in air, the height of levitation increasing with the magnitude of the current. Levitation of this nature is believed to be first observed by Ampere,[1,2] and is mentioned in various textbooks.[3–8]

The force responsible for levitation comes of course from the interaction between the magnetic field **B** and the current I induced in the ring by the changing flux. This qualitative explanation, which is the one generally advanced, is incomplete. The "natural" and naive assumption is that the magnetic fields are all in phase. Then the horizontal magnetic field B_h (which is responsible for the vertical levitating force) would be in phase with the vertical component B_v, and hence with the flux Φ through the ring. Thus B_h is 90° out of phase with the emf ϵ, and also with I if the ring is regarded as noninductive. The force, proportional to $I B_h$, then averages to zero. Thus levitation is due to phase differences neglected in the above naive treatment. These phase differences arise from (a) the phase lag caused by the inductance of the ring[9] and (b) (less obviously, but in fact more importantly) the nonuniformity of the phase of the magnetic field. This paper works out the force in terms of these phase differences, and presents measurements of the fields and phases for a quantitative estimate of the force. The estimate agrees with the weight of the ring. Surprisingly, the relevant phase differences in our experimental setup are quite small (a few degrees) and might easily have been neglected. Both the theory and the measurement make no reference to the magnetic properties of the iron rod other than its approximate linearity and are accessible to undergraduates taking a first course in electromagnetism.

The current induced in the ring is quite large (about 45 A at the floating position when the current in the primary coil is 3.5A), so the fields are significantly perturbed by the presence of the ring. We shall work out the force both in

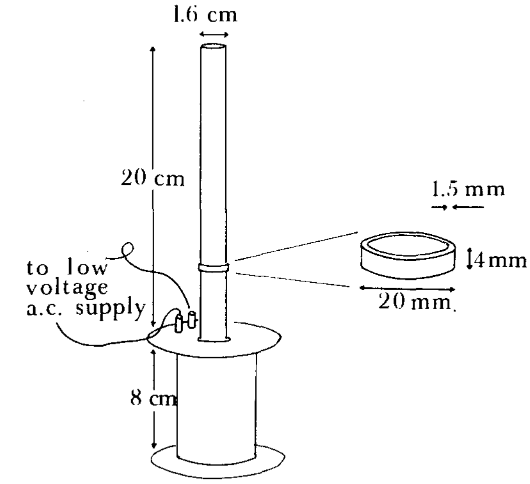

Fig. 1. The apparatus.

