Schrödinger equation derived from complex Gaussian

propagator with no prior physical assumptions

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Feynman based his path integral approach on a Lagrangian formula for the physical

action, transposed into the complex number system by a rule that is arbitrary;

Gaussian integrals then lead to the Schrödinger equation. Here we dispense with the

action formula, and work directly with the complex-valued Gaussian as an object of

interest in its own right. We show that there is no need to postulate any physics to

start with, because the Gaussian, with all its terms made fully complex, and acting as

a propagator, describes the relevant physics anyway, provided only that the system is

normalized in the standard way. Thus we derive the nonrelativistic Schrödinger

equation including scalar and vector potentials, using no prior physical assumptions,

but rather as a process of irregular motion that has the essential characteristics of the

random walk in R except that it is executed under the rules of C. Thus Schrödinger's

equation is an expression of the laws of large numbers via the Gaussian, and does not

have to be viewed as an irreducible postulate of empirical physics.

**Key words** 

Time-dependent Schrödinger equation; Feynman action formula; Path integrals; Gaussian

processes; Random walks

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### 1 Introduction

We are often reminded (e.g. Ref. [1]) that the Schrödinger equation was postulated empirically, being justified only *post hoc* by the fact that it agrees with quantum mechanical observations. This equation might appear less strange if it could be reasoned from something more basic.

We take the Gaussian density function acting as a propagator, but instead of using it in the real number system **R**, we let it act in the complex number system **C**. Why **C**? **C** has a more complete logic [2] (see Discussion, **3.1**). This is such a general quality that in appealing to it, consistency would require us to make *all* terms complex, not just some arbitrary selection of variables. We do so. However certain restrictions will appear when the system is normalized.

The motivation for this model was to work up a better understanding of Feynman's approach to the path integral [3, 4]. One feels that Feynman's action formula, though mysterious in itself, must contain essential physical insight since it leads to the Schrödinger equation, while his use of Gaussian integrals seems no more than a mathematical tool. Yet the Gaussian seems a good place to start, because the real Gaussian is familiar and well understood. So before going to the action formula, it is reasonable to prepare the ground by looking at (a) the Gaussian in **R**, then (b) the Gaussian in **C**, then (c) the complex Gaussian as a propagator, and then (d) the complex Gaussian propagator that is norm-conserving. One might guess that (d) could be a fairly complicated object. How close does (d) get us to Schrödinger's equation, before we even introduce Feynman's action formula? Could (d) then be coupled with something other than the action formula, and still reach Schrödinger's equation, perhaps by a reduced version that might capture what is essential or special about the action formula, or if not, what exactly would be lacking?

To put it another way, we are familiar with the Gaussian in the context of randomness, so what is the smallest thing we need to add to "randomness" (version (d)) to get the Schrödinger equation?

The exercise went further than expected.

The surprising result is that the action formula is not needed at all. After developing the Gaussian as in (d) above, there will be no call on any information from physics (de Broglie relations, Newton's laws, force fields, energy conservation, classical action with or without Feynman's arbitrary formulation). We reach Schrödinger's equation (nonrelativistic, with scalar and vector potentials), together with unambiguous demonstration of the midpoint rule for vector potential.

In effect we use the intuitively appealing path integral method in a slightly extended way, by taking a more general starting point than usual, and calculating the normalization constant correct to first order in the time increment, not merely zero order as in Feynman [3, 4]. The need for any input of physical information is thereby removed.

Since the derivation depends only on the Gaussian, the physical content of Feynman's action formula, and of the Schrödinger equation itself, is seen to reside in the Gaussian when the latter is closely considered. That is the central result of this article.

Derivation of these results is given in **2**. In **3**, the Discussion, we consider how to interpret them. The real Gaussian is well known to be associated with a random walk in many manifestations. What does it mean if the Gaussian is complex instead of real? We suggest reasons why it may still be understood as a random process, but one that is executed according to the rules of **C** instead of **R**. We hasten to add that this picture is consistent with the usual textbook teaching of quantum mechanics [1], including Born's statistical

interpretation (and is unlike the type of random walk described by Nelson and others.) We highlight what is gained here by comparing with previous justifications (Schrödinger [5], Feynman [3, 4], Kac [6], Nelson [7], Jauch [8] and others). Conclusions are summarised in 4.

## 2 Derivation

A preliminary overview of the subsections is as follows:

- 2.1 Real Gaussian: We recall the form of the real Gaussian and its normalization constant.
- 2.2 A "convenient" complex Gaussian: We want all the terms in the Gaussian to be complex in general, with the Gaussian acting to propagate another function  $\psi$ . We will see that the normalization requirement for  $\psi$  puts constraints on the composition of the Gaussian propagator; we are to find those constraints. However the general case gives rise to many possibilities that cannot be dealt with in a single concise manner. As a first stage we will arrive at the essential equations, including normalization, using the most "convenient" case for the composition of the Gaussian. This part of the derivation follows the path integral method [3, 4]. The other cases will be dealt with later (see 2.4).
- 2.3 Correction to first order in the time increment: Feynman [3] and Feynman and Hibbs [4] calculated the normalization constant correct to zero order in the small time increment  $\varepsilon$ , but in their working they take Taylor expansions up to first order in  $\varepsilon$ . On the face of it, this seems inconsistent. To ensure consistency therefore, we make the normalization constant correct to first order in  $\varepsilon$ . This introduces factors into the normalization constant that are precursors to scalar potential and the midpoint rule for the vector potential, which are not specified in the un-normalized Gaussian.

<u>2.4 Check alternatives:</u> With equations in hand for the "convenient" case, we check all the alternative cases for normalizability.

2.5 Schrödinger equation: Finally we write the admissible (i.e. normalizable) equations for the propagator and the corresponding differential equation. The latter takes the form of the Schrödinger equation. No assumptions based on physical observations will have been made in the course of the derivation.

We now set this out in detail.

## 2.1 Real Gaussian

The general form of the Gaussian distribution for one variable is

$$P(x) = \frac{1}{K} \exp\left(\frac{(x-\mu)^2}{2\sigma^2}\right),\tag{1}$$

where P is the density at the point x, the mean and variance are  $\mu$  and  $\sigma^2$  respectively, and K is the normalization constant [9].

An equation in this form can represent a time-dependent process,

$$P(x,t) = \frac{1}{K} \exp\left(\frac{(x-ut)^2}{2Dt}\right),\tag{2}$$

where t is the time, u is the drift velocity, and D is the diffusion constant. The mean is now ut and the variance is Dt. Putting  $\int Pdx$  over all space equal to 1, we have

$$K = [2\pi Dt]^{1/2}. (3)$$

A Gaussian in the form of Eq. (2) may be written as a propagator [10],

$$\Pi(\eta, \varepsilon; x, t) = \frac{1}{K} \exp\left(-\frac{(\eta - u(x, t)\varepsilon)^2}{2D(x, t)\varepsilon}\right),\tag{4}$$

where at a point (x,t),  $\Pi$  dictates the step length  $\eta$  in the small time interval  $\varepsilon$ . The drift u and the diffusivity D are allowed to vary with x and t. Acting at each point on a density function P(x,t), the propagator determines how the local density will change with time. K retains the same form as above, now written

$$K = [2\pi D(x,t)\varepsilon]^{1/2}.$$
 (5)

# 2.2 A "convenient" complex Gaussian

We intend to see what happens when all the parameters and variables in the Gaussian propagator are allowed to be complex. This makes for a complicated picture, so it is convenient to start with one particular case. Specifically, we replace the real diffusivity D by the pure imaginary diffusivity iD (with D real). Also we keep D constant (no dependence on x or t). In this way Eq. (4) is replaced by

$$\Pi(\eta, \varepsilon; x, t) = \frac{1}{K} \exp \frac{i(\eta - u(x, t)\varepsilon)^2}{2D\varepsilon}.$$
(6)

We remark that Eq. (4) has been justified using random variable theory [10], which applies only to real systems because of its axioms. In looking at the complex counterpart, Eq. (6), we recognize that the same justification does not apply, but at this stage we are merely exploring an interesting equation. It will turn out to have connection with statistics, to be considered afterwards (see Discussion).

Eq. (6) is related to the propagator in Feynman's path integral approach [3, 4]. Taking up that approach, we are to calculate  $\psi(x,t+\varepsilon)$  as the sum of contributions transferred from values of  $\psi$  situated nearby at a slightly earlier time, that is, from  $\psi(x+\eta,t)$ . Those transfers are calculated from the propagator  $\Pi(\eta,\varepsilon;x+\eta,t)$ , hence

$$\psi(x,t+\varepsilon) = \int \Pi(\eta,\varepsilon;x+\eta,t)\psi(x+\eta,t)d\eta. \tag{7}$$

From Eqs. (6) and (7),

$$\psi(x,t+\varepsilon) = \int \frac{1}{K} \exp \frac{i(\eta - u(x+\eta,t)\varepsilon)^2}{2D\varepsilon} \psi(x+\eta,t) d\eta.$$
 (8)

To abbreviate some notation in Eq. (8), let

$$u = u(x,t), \tag{9}$$

$$u_{+} = u(x + \eta, t). \tag{10}$$

The square in the exponential term is  $\eta^2 - 2\eta u_+ \varepsilon + u_+^2 \varepsilon^2$ , so, with no approximations,

$$\exp\frac{i(\eta - u_{+}\varepsilon)^{2}}{2D\varepsilon} = \exp\frac{i\eta^{2}}{2D\varepsilon} \exp\frac{i\eta u_{+}}{D} \exp\frac{iu_{+}^{2}\varepsilon}{2D}.$$
 (11)

When  $\eta$  and  $\varepsilon$  approach zero, the terms  $\eta u_+/D$  and  $u_+^2 \varepsilon/2D$  also approach zero because  $u_+$  and D are finite. Hence we may write Taylor expansions of the second and third exponentials on RHS in Eq. (11),

$$\exp{-\frac{i\eta u_{+}}{D}} = 1 - \frac{i\eta u_{+}}{D} - \frac{\eta^{2} u_{+}^{2}}{2D^{2}},$$
(12)

$$\exp\frac{iu_{+}^{2}\varepsilon}{2D} = 1 + \frac{iu_{+}^{2}\varepsilon}{2D},\tag{13}$$

keeping terms only to second order in  $\eta$  and first order in  $\varepsilon$  (as we will do throughout, following Feynman [3] and Feynman and Hibbs [4]).

From eqns. (11), (12) and (13),

$$\exp\frac{i(\eta - u_{+}\varepsilon)^{2}}{2D\varepsilon} = \left[1 - \frac{i\eta u_{+}}{D} - \frac{\eta^{2} u_{+}^{2}}{2D^{2}} + \frac{iu_{+}^{2}\varepsilon}{2D}\right] \exp\frac{i\eta^{2}}{2D\varepsilon}.$$
(14)

These authors [3, 4] point out that a term such as the exponential containing  $\eta^2/\varepsilon$  on RHS of Eq. (8) oscillates rapidly with small  $\varepsilon$  except near  $\eta = 0$ , and the rapid oscillations would contribute little to the integration on  $\eta$  due to cancellations (given relative smoothness of  $\psi$  and other factors). Since appreciable contributions to the integral are then expected only for small  $\eta$ , Taylor expansion of  $\psi(x + \eta, t)$  is justified.

Substituting Eq. (14) in Eq. (8) and making Taylor expansions of  $\psi(x,t+\varepsilon)$  and  $\psi(x+\eta,t)$ ,

$$\psi(x,t) + \varepsilon \frac{\partial \psi}{\partial t} = \int d\eta \frac{1}{K} \left[ 1 - \frac{i\eta u_{+}}{D} - \frac{\eta^{2} u_{+}^{2}}{2D^{2}} + \frac{iu_{+}^{2} \varepsilon}{2D} \right] \left( \exp \frac{i\eta^{2}}{2D\varepsilon} \right) \times \left[ \psi(x,t) + \eta \frac{\partial \psi}{\partial x} + \frac{1}{2} \eta^{2} \frac{\partial^{2} \psi}{\partial x^{2}} \right]$$
(15)

Recalling eqns. (9), (10), with Taylor expansion for small  $\eta$ ,

$$u_{+} = u(x + \eta, t) = u(x, t) + \eta \frac{\partial u(x, t)}{\partial x} + \dots = u + \eta \frac{\partial u}{\partial x} + \dots$$
 (16)

(Second order term is not shown as it will go to higher order when multiplied later.) Using the integrals

$$\int_{-\infty}^{\infty} \exp \frac{i\eta^2}{2D\varepsilon} d\eta = \left(2\pi i D\varepsilon\right)^{1/2},\tag{17}$$

$$\int_{-\infty}^{\infty} \eta \exp \frac{i\eta^2}{2D\varepsilon} d\eta = 0, \tag{18}$$

$$\int_{-\infty}^{\infty} \eta^2 \exp \frac{i\eta^2}{2D\varepsilon} d\eta = \left(2\pi i D\varepsilon\right)^{1/2} i D\varepsilon,\tag{19}$$

$$\int_{-\infty}^{\infty} \eta^4 \exp \frac{i\eta^2}{2D\varepsilon} d\eta = (2\pi i D\varepsilon)^{1/2} (-3) D^2 \varepsilon^2, \tag{20}$$

we find that the two terms in Eq. (15) that involve  $u_{+}^{2}$ , when expanded by Eq. (16), cancel each other (to first order in  $\varepsilon$ ) upon integration,

$$\int d\eta \left[ -\frac{\eta^2 u_+^2}{2D^2} + \frac{iu_+^2 \varepsilon}{2D} \right] \left( \exp \frac{i\eta^2}{2D\varepsilon} \right) = \int d\eta u_+^2 \left[ -\frac{\eta^2}{2D^2} + \frac{i\varepsilon}{2D} \right] \left( \exp \frac{i\eta^2}{2D\varepsilon} \right)$$

$$= \int d\eta \left( u + \eta \frac{\partial u}{\partial x} \right)^2 \left[ -\frac{\eta^2}{2D^2} + \frac{i\varepsilon}{2D} \right] \left( \exp \frac{i\eta^2}{2D\varepsilon} \right)$$

$$= 0. \tag{21}$$

The only other term in Eq. (15) that involves  $u_{+}$  is  $-i\eta u_{+}/D$ . Expanding again with Eq. (16),

$$-\frac{i\eta u_{+}}{D} = -\frac{i\eta u}{D} - \frac{i\eta^{2}}{D} \frac{\partial u}{\partial x}.$$
 (22)

With Eq. (21) and Eq. (22), Eq. (15) becomes

$$\psi(x,t) + \varepsilon \frac{\partial \psi}{\partial t} = \int d\eta \frac{1}{K} \left[ 1 - \frac{i\eta u}{D} - \frac{i\eta^2}{D} \frac{\partial u}{\partial x} \right] \left( \exp \frac{i\eta^2}{2D\varepsilon} \right) \times \left[ \psi(x,t) + \eta \frac{\partial \psi}{\partial x} + \frac{1}{2} \eta^2 \frac{\partial^2 \psi}{\partial x^2} \right]$$

$$= \int d\eta \psi \frac{1}{K} \left[ 1 - \frac{i\eta u}{D} - \frac{i\eta^2}{D} \frac{\partial u}{\partial x} \right] \exp \frac{i\eta^2}{2D\varepsilon}$$

$$+ \frac{\partial \psi}{\partial x} \frac{1}{K} \left[ \eta - \frac{i\eta^2 u}{D} \right] \exp \frac{i\eta^2}{2D\varepsilon}$$

$$+ \frac{\partial^2 \psi}{\partial x^2} \frac{1}{K} \left[ \frac{1}{2} \eta^2 \right] \exp \frac{i\eta^2}{2D\varepsilon}.$$
(23)

To evaluate the normalization factor K, we compare the leading  $\psi(x,t)$  terms on the two sides. On the left-hand side there is simply  $\psi(x,t)$ . On the right-hand side,  $\psi(x,t)$  is multiplied by the expression on the LHS of the following equation, which, from Eq. (17), is evaluated as

$$\frac{1}{K} \int_{-\infty}^{\infty} \exp \frac{i\eta^2}{2D\varepsilon} d\eta = \frac{1}{K} (2\pi i D\varepsilon)^{1/2}.$$
 (24)

In order that both sides of Eq. (23) agree in the limit as  $\varepsilon$  approaches zero, K must be chosen so that the expression in Eq. (24) equals 1; that is,

$$K = \left(2\pi i D\varepsilon\right)^{1/2}.\tag{25}$$

The equation is now correct to zero order in  $\varepsilon$ . This is the normalization factor given in the classic path integral formulation [3, 4] using the foregoing justification.

But since we have been taking Taylor expansions to first order in  $\varepsilon$ , we should not be satisfied to have the normalization constant specified only to zero order in  $\varepsilon$  -- consistency requires that we develop it to first order in  $\varepsilon$ .

### 2.3 Correction to first order in time increment

To do this, we insert into the normalization constant a first-order term in  $\varepsilon$ , making it as general as possible. If this precaution were to be unnecessary, the added term would prove to be zero. So we amend Eq. (25) to

$$K = (2\pi i D\varepsilon)^{1/2} (1 + \varepsilon T(x, t)), \tag{26}$$

where T is independent of  $\psi$  (otherwise the propagator is not Gaussian), but may be some complex function of x and t, and  $\varepsilon$  is small. The choice of sign for T is arbitrary.

Resuming in the manner of the Feynman exposition [3, 4], we use Eq. (26) and substitute

$$\frac{1}{K} = \left(2\pi i D\varepsilon\right)^{-1/2} \left(1 - \varepsilon T(x, t)\right) \tag{27}$$

into Eq. (6) for the propagator, which becomes

$$\Pi(\eta, \varepsilon; x, t) = \left(2\pi i D\varepsilon\right)^{-1/2} \left(1 - \varepsilon T(x, t)\right) \exp \frac{i(\eta - u(x, t)\varepsilon)^2}{2D\varepsilon}.$$
 (28)

To develop the differential equation, we substitute Eq. (27) into Eq. (23). Included is a term involving u which will later yield the vector potential A. Feynman warned that caution must be used with the integral  $\int A dx$ , possibly due to the individual trajectory of a single particle being undifferentiable ("like Brownian motion"), and introduced the midpoint rule to get the integral to correspond with the known physics [3]. However, a single particle trajectory is unlike a density function, and we treat the latter as well behaved (Riemann integrable). We note Feynman's caution, but address it, not only to the technique of forming the integral itself, but to normalizing the integral after it is formed.

We proceed then to the integrations in Eq. (23), using Eq. (27) for 1/K, and take from Eq. (19) that the  $\eta^2$  terms in the integrand produce  $iD\varepsilon$  terms in the integral,

$$\varepsilon \frac{\partial \psi}{\partial t} = \frac{(iD\varepsilon)}{2} \frac{\partial^2 \psi}{\partial x^2} - (iD\varepsilon) \frac{iu}{D} \frac{\partial \psi}{\partial x} - (iD\varepsilon) \frac{i}{D} \frac{\partial u}{\partial x} \psi - \varepsilon T \psi. \tag{29}$$

Dividing through by  $\varepsilon$ , we get the differential equation

$$\frac{\partial \psi}{\partial t} = \frac{iD}{2} \frac{\partial^2 \psi}{\partial x^2} + u \frac{\partial \psi}{\partial x} + \frac{\partial u}{\partial x} \psi - T\psi, \tag{30}$$

and its complex conjugate

$$\frac{\partial \psi^*}{\partial t} = -\frac{iD}{2} \frac{\partial^2 \psi^*}{\partial x^2} + u \frac{\partial \psi^*}{\partial x} + \frac{\partial u}{\partial x} \psi^* - T^* \psi^*. \tag{31}$$

We already have normalization at time zero. We also require that Eqs. (30, 31) conserve the norm over time,

$$\frac{d}{dt} \int_{-\infty}^{\infty} (\psi^* \psi) dx = 0. \tag{32}$$

To examine this, we expand the expression

$$\frac{d}{dt} \int_{-\infty}^{\infty} (\psi^* \psi) dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} (\psi^* \psi) dx = \int_{-\infty}^{\infty} \left( \psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t} \psi \right) dx$$

$$= \int_{-\infty}^{\infty} \psi^* \left( \frac{iD}{2} \frac{\partial^2 \psi}{\partial x^2} + u \frac{\partial \psi}{\partial x} + \frac{\partial u}{\partial x} \psi - T \psi \right) dx + \int_{-\infty}^{\infty} \left( -\frac{iD}{2} \frac{\partial^2 \psi^*}{\partial x^2} + \frac{\partial \psi^*}{\partial x} u + \psi^* \frac{\partial u}{\partial x} - \psi^* T^* \right) \psi dx$$

$$= \int_{-\infty}^{\infty} \left\{ \psi^* \frac{iD}{2} \frac{\partial^2 \psi}{\partial x^2} - \frac{iD}{2} \frac{\partial^2 \psi^*}{\partial x^2} \psi \right\} + \left\{ \psi^* u \frac{\partial \psi}{\partial x} + 2 \psi^* \frac{\partial u}{\partial x} \psi + \frac{\partial \psi^*}{\partial x} u \psi \right\} + \left\{ -\psi^* T \psi - \psi^* T^* \psi \right\} dx$$
(33)

where we used Eqs. (30, 31) in the second-last line, and rearranged terms for the last line.

Taking the first curly bracket on RHS of Eq. (33),

$$\int_{-\infty}^{\infty} \left\{ \psi^* \frac{iD}{2} \frac{\partial^2 \psi}{\partial x^2} - \frac{iD}{2} \frac{\partial^2 \psi^*}{\partial x^2} \psi \right\} dx = \frac{iD}{2} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left\{ \psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \psi \right\} dx$$

$$= \frac{iD}{2} \left[ \psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \psi \right]_{-\infty}^{\infty}$$

$$= 0, \tag{34}$$

the definite integral being zero with square-integrable  $\psi$ .

Taking the second curly bracket on RHS, we notice it contains the derivative of the triple product, which we separate,

$$\int_{-\infty}^{\infty} \psi^* u \frac{\partial \psi}{\partial x} + \psi^* \frac{\partial u}{\partial x} \psi + \frac{\partial \psi^*}{\partial x} u \psi dx + \int_{-\infty}^{\infty} \psi^* \frac{\partial u}{\partial x} \psi dx$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial x} (\psi^* u \psi) dx + \int_{-\infty}^{\infty} \psi^* \frac{\partial u}{\partial x} \psi dx$$

$$= \left[ \psi^* u \psi \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \psi^* \frac{\partial u}{\partial x} \psi dx$$

$$= \int_{-\infty}^{\infty} \psi^* \frac{\partial u}{\partial x} \psi dx, \tag{35}$$

the quantity  $\left[\psi^*u\psi\right]_{-\infty}^\infty$  being zero, again because  $\psi$  is square-integrable.

For the last two terms on RHS, we write the real and imaginary parts of T separately,

T = a + ib,  $T^* = a - ib$ , with a and b real. Then

$$\int_{-\infty}^{\infty} -(\psi^* T \psi + \psi^* T^* \psi) dx = -\int_{-\infty}^{\infty} \psi^* (a + ib + a - ib) \psi dx$$
$$= -\int_{-\infty}^{\infty} \psi^* (2a) \psi dx. \tag{36}$$

Since Eqs. (34), (35) and (36) must sum to zero in order to satisfy Eq. (32), we must have

$$\int_{-\infty}^{\infty} \psi^* \frac{\partial u}{\partial x} \psi dx - \int_{-\infty}^{\infty} \psi^* (2a) \psi dx = 0,$$
(37)

and hence

$$a(x,t) = \frac{1}{2} \frac{\partial u(x,t)}{\partial x},\tag{38}$$

where it is necessary that the equality holds at all x,t. There is no restriction on b(x,t) because it always cancels.

Thus T is not necessarily zero. Re T is zero only if u is constant. And T is determined only up to a phase factor. A "free" phase factor in the normalization constant does not of course occur in real systems, but it arises in the present complex system because of the absolute square term in relevant equations.

To correct the normalization constant to first order in  $\varepsilon$ , we substitute

$$T(x,t) = \frac{1}{2} \frac{\partial u(x,t)}{\partial x} + ib(x,t)$$
(39)

into the propagator Eq. (28), which is then written

$$\Pi(\eta, \varepsilon; x, t) = (2\pi i D\varepsilon)^{-1/2} \exp{-i\varepsilon b(x, t)} \exp{-\frac{1}{2}\varepsilon \frac{\partial u(x, t)}{\partial x}} \exp{\frac{i(\eta - u(x, t)\varepsilon)^2}{2D\varepsilon}}$$
(40)

for small  $\varepsilon$ . The differential equation (30) and its complex conjugate (31) become

$$\frac{\partial \psi}{\partial t} = \frac{iD}{2} \frac{\partial^2 \psi}{\partial x^2} + u(x,t) \frac{\partial \psi}{\partial x} + \frac{1}{2} \frac{\partial u(x,t)}{\partial x} \psi - ib(x,t) \psi, \tag{41}$$

$$\frac{\partial \psi^*}{\partial t} = -\frac{iD}{2} \frac{\partial^2 \psi^*}{\partial x^2} + u(x,t) \frac{\partial \psi^*}{\partial x} + \frac{1}{2} \frac{\partial u(x,t)}{\partial x} \psi^* + ib(x,t) \psi^*. \tag{42}$$

It is noted that (4) and (40) are precise counterparts, as each represents a Gaussian propagator for which the norm is conserved: the former conserves the norm in  $\mathbf{R}$ , and the latter conserves the norm in the sense of  $\mathbf{C}$ .

Feynman [3] did not discuss how to maintain constancy of the norm to first order in  $\varepsilon$  (norm conservation over time), but did not need to, because in his approach the necessary information is carried in from observational evidence: that is, in the Lagrangian, which gives the scalar and vector potentials, and in the midpoint rule found by trial-and-error to ensure the integrations match the known physics. We found the correct formulation purely by normalizing the Gaussian-propagated system, without reference to observational physics. We remark for later reference (see Discussion) that the term containing the factor 1/2 arose at Eq. (38), which comes about because the norm is calculated from the two terms of a complex conjugate pair.

## 2.4 Checking alternatives

We have constructed the normalization constant when the Gaussian propagator Eq. (6) was restricted, by a "convenient" choice, to have real D and u, with D constant. We now go back over the equations critical for normalization to see how they stand when those restrictions are relaxed in any way possible.

Let us look at the terms in Eq. (33) with the stated variables being complex instead of real. Thus D is to be replaced by  $\operatorname{Re}D+i\operatorname{Im}D$ , and  $D^*$  by  $\operatorname{Re}D-i\operatorname{Im}D$ . Also u is replaced by  $\operatorname{Re}u+i\operatorname{Im}u$ , and  $u^*$  by  $\operatorname{Re}u-i\operatorname{Im}u$ ; while T and  $T^*$  remain as before. Then if we collect the terms comprising real components of these variables, the integrations come to zero, as they did before, for the real components only; but the corresponding terms with imaginary components would fail to cancel because of altered signs in the complex conjugate terms. Also if D is not constant with respect to x, D cannot be taken outside the integral sign for the integration over x. If any one or more of these modifications were to be made, residual terms would be left, involving  $\psi$ , its derivatives, their complex conjugates, and imaginary components of parameters, unable to be simplified as a general case and therefore not identically zero for all square-integrable  $\psi$ . In such cases, the norm would not be conserved. Could this be rectified by setting T to cancel the unwanted terms? No, because that would make T a function of  $\psi$ , which would mean that propagation is not by a Gaussian, therefore not admissible for the present discussion.

As for D, it must be constant in space, but there is nothing in our equations that says it must be constant in time. That is the only relaxation in the restrictions that we have reason to identify. We will comment further on this in 2.5.

Since our aim is to consider the Gaussian in a *fully* complex form, we must also consider how the independent variables (the space and time coordinates) are to be regarded when they too

are written as complex numbers. Explicitly, each space and time coordinate would be represented by a complex plane instead of just an axis of real numbers. Usually we are only interested in  $\Pi$  and  $\psi$  over the real axes, so for most purposes the complex Gaussian is adequately represented with the space and time coordinates written as real, whilst keeping in mind that they could be treated as complex should that be needed.

# 2.5 Schrödinger equation

A few notational changes will align the equations with standard usage. We substitute D = 1/m, (43)

noting that the diffusivity D is a measure of the tendency to spread, while the mass (inertia) m is a measure of the tendency to resist displacement, these terms being mutually reciprocal. This is equivalent to  $D = \hbar/m$ , where it is implied that Planck's constant  $\hbar = 1$ . Also our use of u has implied the x component of the vector  $\mathbf{u}$ ; we now denote this component as  $u_x$ , with

$$u_x = A_x D = \frac{A_x}{m}. (44)$$

This equation deals with x components; we may write similar equations for u and A components in the y and z directions in considering three dimensions. Finally we replace b by

$$b = \frac{\mathbf{A}^2}{2m} + \phi,$$
where  $\mathbf{A}^2 = A_x^2 + A_y^2 + A_z^2$ . (45)

These substitutions make the Schrödinger equation a little more complicated, but are aligned with common usage, and it will turn out that the Hamiltonian is made simpler when written in terms of  $\phi$  instead of b. The substitutions imply no new conditions and no change in substantive meaning, because  $A_x$  is defined at Eq. (44) in terms of  $u_x$  and D, which are given as input parameters from the beginning, at Eq. (6) ( $u_x$  is shown there as u); analogous input

parameters  $u_y$  and  $u_z$  would define  $A_y$  and  $A_z$ ; while the latitude enjoyed by b is now carried by  $\phi$ .

Incorporating these changes into Eqs. (41, 42),

$$\frac{\partial \psi}{\partial t} = \frac{i}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{A_x}{m} \frac{\partial \psi}{\partial x} + \frac{1}{2m} \left( \frac{\partial A_x}{\partial x} \right) \psi - \frac{iA_x^2}{2m} \psi - i\phi \psi. \tag{46}$$

We may write similar equations for components in the y and z directions, leading to

$$\frac{\partial \psi}{\partial t} = \left(\frac{i}{2m}\nabla^2 + \frac{1}{m}\mathbf{A}\cdot\nabla + \frac{1}{2m}\nabla\cdot\mathbf{A} - \frac{i}{2m}\mathbf{A}^2 - i\phi\right)\psi,\tag{47}$$

which is familiar as the Schrödinger equation [11] with  $\phi$  and  $\mathbf{A}$  representing scalar and vector potentials, as may characterise an electromagnetic field. (As indicated earlier,  $\psi$  varies with time and space coordinates, as do the potentials.) Formal generalisation of the Schrödinger equation to three dimensions using the path integral derivation has been shown [12].

This is equivalent to the operator equation for the Hamiltonian,

$$H = \frac{1}{2m} (\mathbf{p} - \mathbf{A})^2 + \phi. \tag{48}$$

We remark in passing that the classical Gaussian, Eq. (4), permits asymmetry in that the steps may tend predominantly in a particular direction, as if steps are decided by a coin-toss using a coin with bias; net tendency is expressed by the variable u(x,t) in Eq. (4) which thus describes a flow or drift. With the same equation written as Eq. (6) for the complex case, u(x,t) again describes a net tendency or bias, but due to the algebra of complex functions the bias is manifested in Schrödinger's equation not as simple flow, but as the vector potential  $\mathbf{A}(\mathbf{r},t)$ , which is related to u(x,t) through Eq. (44).

For norm conservation in time, calculation is required to first order in the time increment  $\varepsilon$ , not just zero order. The necessary term T(x,t) in the normalization constant produced a real component that has the same effect as the midpoint rule for the vector potential, and an imaginary component that yields the scalar potential term. To preserve the norm in the complex case, the diffusivity D in the real case is replaced by a pure imaginary factor iD (with D real) in the complex case.

So we see that the form of the Schrödinger equation Eq. (46) is consistent with a process determined by a complex Gaussian propagator Eq. (6), with norm conservation.

No rationale for charge having different possible values has been developed here.

Accordingly the derived equations do not refer to charge having any other value than 1. In this respect they are of the same form as appears in other non-empirical derivations [8].

We noted earlier that D, and hence m, could vary in time without violating norm conservation. This would however go against the conservation of mass, so the latter must be regarded as a separate law, not explained within the present non-relativistic arguments. We follow common practice in writing m, not m(t), acknowledging that this implies that mass is constant in time, but is not here proven to be so. This remark also applies to previous derivations.

## 3 Discussion

# 3.1 Comparison with random walk

It has been long known that algebraic solutions for cubic and quartic equations must involve square roots of quantities that become negative for particular values of the coefficients [13]. If we refused to deal with such cases, there would be an artificial restriction on which situations we could consider, even when solutions are real and innocuous.

For this and similar reasons [2], it may be said that C has a more complete logic than R (as briefly suggested in the Introduction), which is of course borne out by the many applications of the complex number system.

We have used this as a general justification for using **C**, avoiding *ad hoc* selection of any specific complex variable "just because it works". Although it is somewhat cumbersome to take all variables as complex, norm conservation restricts the equation to the standard form.

When irregular motion is calculated using ordinary numbers  $\mathbf{R}$ , the random-walk equation is the known result. As a new result, we have made the comparable calculation in complex numbers  $\mathbf{C}$ .

In both cases, the movements are taken "not too large". Thus in **R**, the regime has finite variance, so it equates to Gaussian after many iterations (central limit theorem). In **C**, the complex Gaussian ensures that movements are again "not too large".

That is all that is needed to lead precisely to the form of Schrödinger's equation, without having made any physical assumptions. Physical laws emerge as consequences of the laws of large numbers.

(It may avoid possible misunderstanding to note that a random walk in C is sometimes described as though it has two degrees of freedom in the Argand diagram, with steps in the real direction being independent of steps in the imaginary direction. That picture does not respect the one-dimensional character of a complex number, so it is not appropriate for our discussion.)

## 3.2 Born postulate

The continuous density functions, whether real or complex, progress deterministically in time. Use of the term "random" then needs to be justified. This is clear in the real case (the classic random walk), given that the equations have been deduced from random variable theory [10]. But the axioms of random variable theory are defined so as to be inapplicable to complex values, so the statistical interpretation is not so immediate. However, in the complex case, having arrived at the Schrödinger equation, we find that  $|\psi|^2$  does after all have a statistical interpretation. Originally postulated by Born to account for quantum mechanical observations [11], this interpretation was later presented as a theorem within axiomatic approaches [14-17], tending to remove the Born rule as a separate postulate of quantum mechanics. Further, it has been argued as the only consistent way to interpret complex amplitudes [18, 19].

On the basis that a statistical interpretation is justified for  $|\psi|^2$ , and given the above-noted comparisons, it seems appropriate to sum up the present model as an irregular motion characterized by a complex Gaussian propagator, or, in another form of words, as a random process executed under the rules of the complex number system.

# 3.3 Comparisons with previous justifications of Schrödinger's equation

Previous approaches to Schrödinger's equation have started with considerable information taken from observational physics.

Thus Schrödinger [5] drew on the conservation of energy, the de Broglie relations and Hamilton's analogy between waves and particles to construct wavefunctions. Feynman [3] used the Lagrangian to express action and develop path integrals. In both arguments a very large amount of physical knowledge is encompassed in these inputs.

While the present approach was obviously guided by the previous ones, we did not use any physical information as input: the physical content here appears entirely as output, after a purely algebraic examination of the complex Gaussian when acting as a propagator.

There is also a distinction with the use of complex numbers. Both Schrödinger [5] and Feynman [3, 4] introduced complex phase factors in the wavefunctions and path integrals arbitrarily, giving no *a priori* justification for doing so. We overcome that arbitrariness by allowing all the variables to be complex, and then recovering the Schrödinger equation by taking norm conservation into account. Our use of **C** is holistic. It can be claimed that its use in this general way is not arbitrary, but can be justified by its more symmetrical and complete logic, so that the complex number system has greater power and reach ("it is just a better number system").

Further to the primary sources [3-5], we single out three classic contributions of different kinds [6-8].

Kac [6] recognized that the diffusion equation is related to the Schrödinger equation by analytic continuation in the time variable. However he did not say why complex numbers should be used at all -- apart from the empirical success of doing so -- or why time should be the variable distinguished in this way.

Nelson [7] attempted to reformulate quantum mechanics in terms of real statistical processes.

Nelson was aware that an unsatisfactory feature of the model was its predication on continuous trajectories for the particles. Nelson also assumed Newtonian mechanics. Neither of those features is assumed here.

In a group-theoretic approach, Jauch [8] put forward as a theorem of unitary operators that Galilean kinematics constrains the Hamiltonian to take the form shown in (48). A difficulty

with this has been pointed out recently. Brown and Holland [20] showed that it depends on a seemingly innocuous but nontrivial assumption at one particular step. They maintain that unless a satisfactory *a priori* justification is provided for this step, the Jauch theorem has an obscure foundation; other derivations made along the same lines are open to the same question ([20] and references therein). Much more generally, on the complex Hilbert space formalism, Mackey [21] indicated that it rests on postulates that have never had prior justification: its use was "arbitrary ... based on the practical consideration that it was known to work".

In summary, reliance on prior physical assumptions in some form, and arbitrary use of **C**, are consistent characteristics of previous approaches to the Schrödinger equation ([22] and references therein), which are avoided in the present approach.

Midpoint rule: In a further detail, the midpoint rule for the vector potential emerges unambiguously in the present approach, instead of by trial-and-error as in [3] and [12]. Thus when Feynman [3] and Schulman [12] formulate the path integral with a vector potential A(x,t), they state it is not clear whether A should be evaluated at the initial point of each step, or the final point, or somewhere between. Using trial and error, they choose the midpoint, giving a term with the factor 1/2, and afterwards justify that choice by getting agreement with the Schrödinger equation and the Hamiltonian. In contrast, we obtain the same term for the midpoint rule (in Eq. (46)) for a formal reason, namely that the norm is necessarily calculated from co-contributions of the two terms of a complex conjugate pair (Eq. (32) et seq., leading to (38)). Of course the underlying reason is that the norm is here calculated in the sense of  $\mathbf{C}$ , not  $\mathbf{R}$ .

### **4 Conclusion**

It is concluded that the content of Schrödinger's equation is equivalent to that of the Gaussian propagator, normalized in the sense of **C**. No physical assumptions are needed (such as the Lagrangian, or action formula).

The process may be understood as being stochastic (in effect, a type of "random walk") executed under the rules of the complex number system. More precisely, the density  $|\psi|^2$  at any instant is real and stochastic, while the evolution of  $|\psi|^2$  from time to time is worked out from  $\psi$  and  $\psi^*$  at the level of  $\mathbb{C}$ .

This of course is the teaching in quantum mechanics [11]. What is novel here is that the particles are not viewed as intrinsically strange; rather the unfamiliarity lies in counting with the more complete and powerful number system **C**. And if quantum strangeness is displaced from the unique character of particles to the way they are counted, then there is nothing unique at all, because nature is only using the same system of numbers we devised for ourselves in many applications.

Feynman's path integral technique was in part foreshadowed by Einstein's work on Brownian motion [23]. Just as the latter approach showed that a random walk in **R** explains the diffusion equation, we have found, by carrying over the same form of argument from **R** to **C**, that the analogous process in **C** fits the Schrödinger equation in comparable manner.

In both cases (random walk in **R**, and in **C**), no specific physical properties, and certainly no mysterious properties, need to be assigned to the individual units. In both cases, we see physical laws emerging as effects of large numbers in a population of units. Thus Schrödinger's equation does not have to be viewed as an irreducible postulate of empirical physics.

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