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An elementary derivation of $E=mc^2$

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The equality $E = mc^2$ is derived in a fashion suitable for presentation in an elementary physics course for nonscience majors. It assumes only 19th-century physics and knowledge of the photon.

Einstein's original derivation of the relation between the inertia and the energy content of a body[1] assumes the knowledge of the relativistic Doppler effect. It was done after his seminal paper on special relativity, but 17 years before Compton's experiments of 1922. Had it been done before 1905 and had the particle properties of the photon already been known at that time, the following derivation could conceivably have been carried out.

One starts with the following four simple assumptions.

(1) The Newtonian formulas for the kinetic energy and the linear momentum of a free body of mass m and speed v, $mv^2/2$ and mv. Correspondingly, one assumes $(v/c)^2 \ll 1$ throughout.

(2) The laws of conservation of energy and momentum, but not the law of conservation of mass believed before 1905 to be valid.

(3) The Doppler effect, which has been known since the first half of the 19th century: Radiation (whether it be sound waves or electromagnetic waves) of speed c and frequency ν (when the observer is at rest relative to the source) will be perceived to have that frequency altered by a factor $1 + v/c$ $(1 - v/c)$ when the observer moves relative to the source with speed v and in a direction toward it (away from it).

(4) Electromagnetic radiation (in particular visible light) as produced by a source at rest consists of quanta (photons) that have particle properties: Radiation of frequency ν consists of photons of energy $h\nu$ and momentum $h\nu/c$. h and c are *constants.*

If these properties of photons had been known to a 19th-century physicist, these statements would have been the accepted truth of the day. Accept them therefore as the basis for an analysis of the following physical process that could be considered as a thought experiment, but which is not beyond realization.

A source of radiation emits two photons simultaneously while remaining at rest in some (Newtonian!) inertial reference frame R. Conservation of momentum requires these two photons to have equal and opposite momenta, and therefore equal frequencies ν. Therefore, they also have equal energies $h\nu$. Conservation of energy requires that the internal energy of the source diminishes by an amount

$$\Delta E = 2h\nu. \qquad (1)$$

Assume now that this process is viewed from a different reference frame R', which is moving uniformly relative to the rest frame of the source and in such a way that the source is seen to move with speed v in the same direction as one of the photons. Conservation of momentum then requires the momentum of the source before emission p_i' to be equal to the momentum of the source after emission p_f' together with the two momenta of the photons:

$$p_i' = p_f' + \left(\frac{h\nu}{c}\right)\left(1 + \frac{v}{c}\right) - \left(\frac{h\nu}{c}\right)\left(1 - \frac{v}{c}\right).$$

The loss in source momentum, $\Delta p'$, is therefore

$$\Delta p' = (2h\nu/c^2)v. \qquad (2)$$

But the source in reference frame R is at rest both before and after emission; in frame R' it must therefore have the same speed v both before and after emission. Now, according to assumption (1) above, the Newtonian formula for momentum is the product of mass times speed. The momentum loss of the source is thus found to *require* a change in mass Δm times v; and that change of mass is found to be

$$\Delta m = 2h\nu/c^2. \qquad (3)$$

Conservation of energy further requires that the initial energy of the source E_i' be equal to its final energy E_f', together with the energies of the two photons:

$$E_i' = E_f' + h\nu(1 + v/c) + h\nu(1 - v/c),$$

or

$$\Delta E' = 2h\nu = \Delta E. \qquad (4)$$

Thus the energy loss ΔE of the source is the same in both reference frames, R and R'.

Inserting Eq. (4) into Eq. (3), the change of mass is found to be

$$\Delta m = \Delta E/c^2. \qquad (5)$$

One is thus forced to conclude that the emission of the two photons reduces the mass of the source, and this mass loss amounts to an energy loss of $\Delta E = \Delta mc^2$. The equivalence of inertial mass loss and energy loss has thus been derived from the above four assumptions.

One can go one step further and assume that *all* of the mass of the source is used up by emitting photons of large enough frequency; it must be so that $h\nu = m_i c^2/2$, where m_i is the initial mass of the source. That mass then disappears, and its energy is present in the two photons that have total energy $E = m_i c^2$. Therefore, the mass m_i must have been associated with that amount of energy.

This concludes the elementary derivation. One can add to it several layers of sophistication. The simplest one is to permit the source (as seen from R') to move at an arbitrary angle α relative to one of the photons. This adds a factor $\cos\alpha$ in the Doppler effect. Momentum conservation in R' then requires two equations, one for the parallel and one for the perpendicular components of the momenta. The end result is of course the same.

Another modification would be to assume the relativistic Doppler effect and the relativistic expressions for the linear momentum of the source. One can still maintain $\alpha = 0$ at first. This results in the relativistic relation between the (rest) mass and the total energy (mass energy plus kinetic energy) of the source. If a finite angle α is also assumed, one returns to the assumptions underlying Einstein's original derivation, and our assumption (4) is no longer necessary to derive Eq. (5).

This article was motivated by the criticism[2] of a faulty derivation in my recent book.[3]

[1]A. Einstein, "Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?," Ann. Phys. (Leipzig) **18**, 639–641 (1905).
[2]V. F. Weisskopf, Book Review of Ref. 3, Found. Phys. **19**, 221–223 (1989).
[3]F. Rohrlich, *From Paradox to Reality* (Cambridge U.P., Cambridge, 1987). Corrected in the paperback edition (1989).

The power radiated by a slowly moving accelerated acoustic point dipole

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This article considers the acoustic analog of a nonrelativistic accelerated point dipole. Having derived the source density of an acoustic point dipole, the velocity potential is calculated to find the energy flux contributing to the radiated power. A low-speed approximation simplifies the calculations considerably, and the radiated power is obtained. The result is finally applied to the special case of a small rigid body and to simple harmonic oscillations of a sphere.

I. INTRODUCTION

This article provides an example where calculations from one area of physics (acoustics) can provide insight into a different area (electromagnetism). When performing calculations in electromagnetism, it often proves useful to consider the acoustic analog first. This gives a rough idea of which phenomena to expect and provides a check of the electromagnetic calculations. The acoustic case is simpler than the electromagnetic, since the electric and vector potentials of electromagnetism are replaced by one scalar field: the velocity potential. I believe that students will find this approach useful for the reasons presented above, and that it will broaden their background.

A good example is Larmors' formula for the radiated power from an accelerated point charge moving at a speed much smaller than the speed of light. Using SI units, Larmors' formula becomes[1]

$$W_{\rm rad} = q^2a^2/6\pi\epsilon_0c^3. \tag{1}$$

Here, $W_{\rm rad}$ is the radiated power, q is the electric charge, a is the acceleration, ϵ_0 is the permittivity of free space, and c is the speed of light. The analogous formula for an accelerated source of acoustic radiation with constant source strength (electromagnetic analog conservation of charge) moving at a speed much smaller than the speed of sound in the medium under consideration is

$$W_{\rm rad} = \sigma Q^2a^2/12\pi c^3. \tag{2}$$

Here, σ is the density of the medium, Q is the source strength (electromagnetic analog charge), a is the acceleration, and c is the speed of sound. This formula is easily obtained by the procedure presented in this article. Thus we see that by considering the simpler acoustic analog, we can understand the behavior of the power radiated by an accelerated electric charge at low speeds.

In this article, the power radiated by an accelerated dipole moving at a speed much smaller than the speed of sound is considered as a representative example of the analog of acoustics and electromagnetism. In Sec. II, the source density of an acoustic point dipole is found and the velocity potential calculated. In Sec. III, the energy flux contributing to the radiated power is obtained. In this instance, a low-speed approximation simplifies the calculations considerably. In Sec. IV, an expression for the radiated power is found and applied to the special case of a small rigid body in Sec. V, and to simple harmonic oscillations of a sphere in Sec. VI.

The motivation for considering the dipole rather than the monopole is that the former provides a less trivial example of the method. The dipole is also more interesting since it has not, to the best of my knowledge, been discussed before.

II. THE VELOCITY POTENTIAL

Denote the acoustic velocity potential by $\psi(\mathbf{r},t)$, the source density by $\rho(\mathbf{r},t)$, and the velocity of sound by c. The formal solution to the inhomogeneous wave equation

$$\frac{1}{c}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi = \rho \tag{3}$$

can, by means of a Green's function, be found to be[2,3]

$$\psi(\mathbf{r},t) = \int_{-\infty}^{t} dt' \int_{R^3} d^3r' \times\delta(t-t'-\tfrac{1}{c}|\mathbf{r}-\mathbf{r}'|)\frac{\rho(\mathbf{r}',t')}{4\pi|\mathbf{r}-\mathbf{r}'|}, \tag{4}$$

where $\delta(x)$ is the Dirac delta function and R^3 indicates integration over all space.

The source density of a dipole is

$$\rho(\mathbf{r},t) = Q(t)\{\delta(\mathbf{r}-\mathbf{s}(t)) - \delta(\mathbf{r}-[\mathbf{s}(t)-\mathbf{d}(t)])\}, \tag{5}$$

where $Q(t)$ is the source strength, $\mathbf{s}(t)$ is the trajectory of the positive pole, and $\mathbf{d}(t)$ is the distance between the poles with direction oriented from the negative to the positive pole. If this dipole is seen from a sufficient distance, we can make a point dipole approximation and expand (5) in a Taylor series about $\mathbf{d}(t)=0$, keeping only the first non-vanishing term

$$\rho(\mathbf{r},t) = -\mathbf{P}(t)\cdot\nabla\delta(\mathbf{r}-\mathbf{s}(t)), \tag{6}$$

where $\mathbf{P}(t) = Q(t)\mathbf{d}(t)$ is the dipole moment. By using a

