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Citation: American Journal of Physics **64**, 392 (1996); doi: 10.1119/1.18253
View online: https://doi.org/10.1119/1.18253
View Table of Contents: https://aapt.scitation.org/toc/ajp/64/4
Published by the American Association of Physics Teachers

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Elementary derivation of Kepler's laws

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(Received 7 February 1995; accepted 26 July 1995)

A simple derivation of all three so-called Kepler laws is presented in which the orbits, bound and unbound, follow directly and immediately from conservation of energy and angular momentum. The intent is to make this crowning achievement of Newtonian mechanics easily accessible to students in introductory physics courses. The method is also extended to simplify the derivation of the Rutherford scattering law. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

The so-called Kepler laws of planetary motion have been of central interest for Newtonian mechanics ever since the appearance of Newton's *Principia*.[1] They are discussed in most introductory textbooks of physics[2,3] and continue to be a subject of lively interest in the pages of the *American Journal of Physics*.[4] This interest is not surprising because the understanding of planetary motion has been one of the oldest challenges in many human cultures and continues to excite the sense of wonder among young scientists today.

The purpose of the present article is to give a new elementary derivation of all three of the Kepler laws intended to make their physics accessible to first year university students taking introductory mechanics. I have used this derivation in my own introductory classes for more than a decade and find that it, and the many associated problems, are a highlight of the introduction which I give to physics. In contrast, most first-year textbooks give a description of Kepler's laws but apparently regard their derivation as too difficult. Perhaps the derivation given here can then fill an important gap.

The elementary proof, given in the next section follows directly, in a few easy steps, from conservation of energy and angular momentum which, in turn, follow from $F=ma$ and the central nature of the universal gravitational force, $F=GmM/r^2$. These conservation laws, on which we build, are usually covered thoroughly, and often even elegantly, in first year textbooks.

In succeeding sections, beyond the proof, we provide further discussion of bound elliptic orbits and extend the treatment to the unbound Kepler orbits and to the Rutherford scattering law.

II. ELEMENTARY PROOF OF KEPLER'S LAWS

A. Kepler's first law (the law of orbits): All planets move in elliptical orbits having the Sun at one focus

For a planet of mass m in a bound orbit (negative total energy E), around the Sun of mass M, we have the constant total energy, E

$$E \equiv mv^2/2 - GM\,m/r, \tag{1}$$

where r is the distance of the planet from the Sun and v its velocity. $(-E/m)$ is a positive constant of the motion. Because the force is central we also have conserved angular momentum, l

$$l \equiv mvh, \tag{2}$$

where $h(\equiv r \sin\phi$, with ϕ the angle between **v** and **r**) is the perpendicular distance from the planet's instantaneous velocity vector to the Sun (see Fig. 1). From the definition of h we have $h \leq r$. (l/m) is also a positive constant of the motion. Using Eq. (2) in Eq. (1), we obtain

$$\frac{[(l/m)^2/2(-E/m)]}{h^2} - \frac{[GM/(-E/m)]}{r} = -1. \tag{3}$$

The relationship (3) between r and h, both taken from a common center of force defines an ellipse. In the next section we show that for an ellipse of semimajor axis a and semiminor axis b we have

$$\frac{b^2}{h^2} - \frac{2a}{r} = -1 \quad (h \leq r). \tag{4}$$

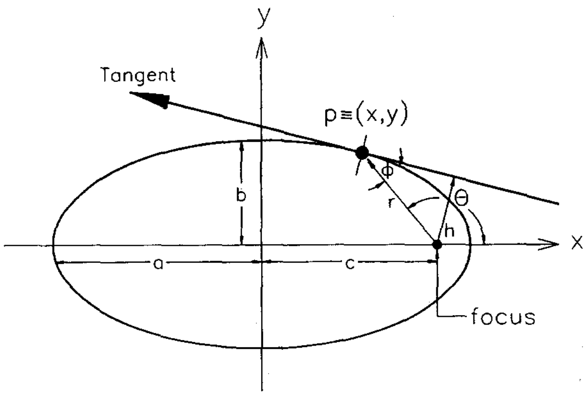

Fig. 1. The geometry for the bound elliptical orbit of a planet at point p around the Sun at the focus. The ellipse parameters (a, b, and c) are shown as well as three alternative pairs of coordinates: x and y, r and θ, r and h, where h is the perpendicular distance from the focus to the tangent at point p. The ellipse shown has $b=a/2$.

The equality of Eqs. (3) and (4) not only completes the proof of Kepler's first law, but also immediately gives the orbit parameters, a and b, in terms of the constants of the motion, $(-E/m)$ and (l/m)

$$a=GM/2(-E/m), \quad b=(l/m)/(-2E/m)^{1/2}. \tag{5}$$

B. Kepler's second law (the law of areas): A line joining a planet to the Sun sweeps out equal areas in equal times

This law is the only one of the three commonly proved in introductory physics textbooks. Referring to Fig. 1, the time derivative of the area, A, swept out is

$$\frac{dA}{dt}=\tfrac{1}{2}vh=\frac{l}{2m}, \tag{6}$$

which is constant. Thus this law is directly associated with the conservation of angular momentum.

C. Kepler's third law (the law of periods): The square of the period of any planet about the Sun is proportional to the cube of the planet's mean distance from the Sun

Using the second law the period, T, of the planet must be equal to the total area of the ellipse, divided by the constant, dA/dt. The total area of an ellipse is πab. Therefore,

$$T=2\pi ab/(l/m)=(2\pi/\sqrt{GM})a^{3/2}$$

or (7)

$$T^2=(4\pi^2/GM)a^3.$$

Interpreting the semimajor axis as the mean distance from the Sun, the result Eq. (7) proves the third law. The constant, $(4\pi^2/GM)$, which applies to all planets of our solar system is, about, $3.0\times10^{-34}\ y^2/m^3$.

III. SOME PROPERTIES OF ELLIPSES

Referring to Fig. 1, the familiar equations for the ellipse relate to the coordinates x and y,

$$x^2/a^2+y^2/b^2=1, \tag{8}$$

or, alternatively, to the polar coordinates r and θ,

$$r=a(1-\epsilon^2)/(1+\epsilon\cos\theta), \tag{9}$$

where the eccentricity, ϵ, is defined by $c\equiv(a^2-b^2)^{1/2}\equiv\epsilon a$, with c being the distance from the center of the ellipse to its focus.

Next we provide the derivation, beginning with Eq. (8), of the unfamiliar relationship, Eq. (4), defining an ellipse, required for the proof of Sec. II. We do this in two steps. First we show that the ellipse Eq. (8) satisfies Eq. (4). Second we show that every orbit which satisfies Eq. (4) is an ellipse identical to Eq. (8).

To carry out the first step we note that the coordinate r, shown on Fig. 1, is defined by

$$r\equiv[y^2+(x-c)^2]^{1/2}. \tag{10}$$

We substitute for y^2 from Eq. (8),

$$y^2=b^2(1-x^2/a^2)=(1/a^2)(a^2-c^2)(a^2-x^2),$$

to obtain

$$r=a^{-1}(a^2-cx), \tag{11}$$

which is also an equation for the ellipse in terms of the coordinates r and x.

To find h in terms of x (or y) we start with the general formula for the perpendicular distance, h, from an arbitrary point (x_1,y_1) to an arbitrary straight line, $y=y'x+y_0$, where $y'(\equiv dy/dx)$ is the slope of the line and y_0 is its y intercept. The formula is

$$h=(1+y'^2)^{-1/2}(y_0-y_1+y'x_1). \tag{12}$$

Here $(x_1,y_1)\equiv(c,0)$ and the y intercept is $y_0=y-y'x$. The straight line (see Fig. 1) is the tangent to the ellipse at the position (x,y) of the planet so that

$$y'\equiv\frac{dy}{dt}=-\left(\frac{x}{y}\right)\left(\frac{b^2}{a^2}\right), \tag{13}$$

which follows directly from Eq. (8). Substituting into the square of Eq. (13) for y', y_0, y_1, and x_1, we obtain

$$\begin{aligned}
h^2&=(1+x^2b^4/y^2a^4)^{-1}(y+x^2b^2/ya^2-xcb^2/ya^2)^2\\
&=(y^2a^4+x^2b^4)^{-1}(y^2a^2+b^2x^2-b^2xc)^2\\
&=b^2[a^4(1-x^2/a^2)+x^2(a^2-c^2)]^{-1}\\
&\quad\times[a^2(1-x^2/a^2)+x^2-xc]^2\\
&=b^2(a^4-x^2c^2)^{-1}(a^2-xc)^2\\
&=b^2(a^2+xc)^{-1}(a^2-xc),
\end{aligned} \tag{14}$$

which, is a relationship which holds for the ellipse, Eq. (8).

Solving Eq. (11) for x yields

$$x=(a^2/c)(1-r/a), \tag{15}$$

and similarly, solving Eq. (14) for x yields

$$x=(a^2/c)(1-h^2/b^2)/(1+h^2/b^2). \tag{16}$$

Finally, equating Eq. (15) and Eq. (16) yields the conclusion of the first step of our proof, namely, that the ellipse Eq. (8) satisfies the relationship Eq. (4) between r and h.

We need a second step in the proof that the relationship Eq. (4) defines an ellipse of semimajor axis a and semiminor axis b because, although Eq. (4) appears at first sight to be a standard orbit equation, connecting the two coordinates which define a plane, h is not a coordinate in the sense that r

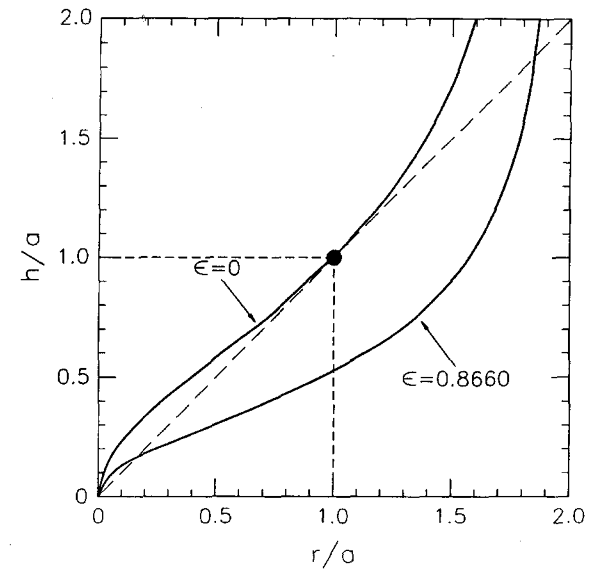

Fig. 2. The elliptical orbit, Eq. (4), in terms of the coordinates r and h for two different values of the eccentricity: $\epsilon=0$ (a circle) and $\epsilon=0.8660$ (the ellipse of Fig. 1 for which $b=a/2$). Since $h\leq r$, the elliptical motion pertains to that part of the curve which lies in the lower half of the quadrant, that is, below the dashed line of $r=h$.

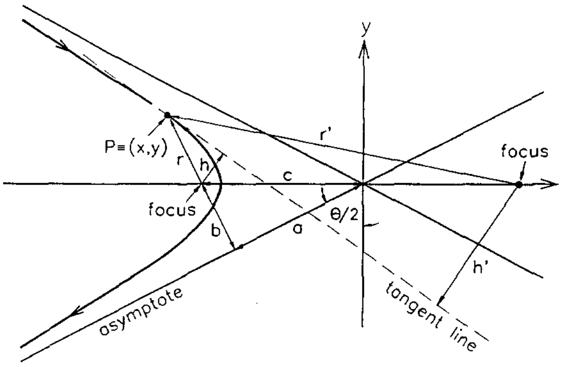

Fig. 3. The geometry of the hyperbola pertaining to unbound Kepler orbits of planets and to the orbit of alpha particles in Rutherford scattering. The hyperbola (heavy line) is confined between two asymptotes. For the unbound Kepler orbit the Sun is at the focus $(x,y)\equiv(-c,0)$ and the orbit parameters a, b, and c are indicated. As in Fig. 1, above, we have chosen $b=a/s$. For the planet at point P its distance, r, from the Sun as well as the perpendicular distance, h, to its tanget line are also shown. For alpha–nucleus scattering the same hyperbola applies but the nucleus is at the other focus, $(x,y)\equiv(c,0)$ for which the alpha–nucleus distance is r' and the perpendicular distance to the tanget line is h'. r' and h' are indicated on the figure as well as the scattering angle θ.

or x or y are. Rather h [and Eq. (4)] depend explicitly on the first derivative of a function contained in an equation relating the pair of actual coordinates of each point of the orbit, which is assumed to lie in a two-dimensional plane. Since the differential Eq. (4) is of first order its general solution is a uniquely determined one-parameter family of curves. However, since both h and r are defined with reference to a fixed point—the center of force—it is clear that if a particular orbit, Eq. (8), satisfies the relationship Eq. (4), then so also does every other orbit that can be obtained from it by means of a rigid rotation of the particular orbit about the force center. The family of all orbits obtained by rotating the particular orbit is a one-parameter family of curves (the parameter is the rotation angle), each of which satisfies the relationship Eq. (4). Therefore, every curve which satisfies Eq. (4) is an ellipse of semimajor axis a and semiminor axis b.

Although the proof that the relationship Eq. (4) defines an ellipse is somewhat tedious it is not inaccessible to bright first year students. What matters from the teaching point of view is that the physics dominates: in a few short steps, directly related to the conservation of energy and angular momentum, one is led to an equation defining an ellipse.

In order not to deflect from the physics interest it is my practise to present the derivation given in this section as a handout intended to help those students who wish to know more about ellipses and who want to relate the unfamiliar form of Eq. (4) to something that they know.

It is interesting to plot the ellipse for pairs of coordinates other than the usual x and y of Fig. 2. For example, the x and r ellipse of Eq. (11) or Eq. (15) is a straight line for which the elliptical motion lies between the maximum and minimum values of r, that is, between $(1-\epsilon)a$ and $(1+\epsilon)a$. Similarly, for the x and h equation the ellipse lies between the limits $(1-\epsilon)a\leq h\leq(1+\epsilon)h$, or, correspondingly, $-a\leq x\leq a$. The h and r ellipse, Eq. (4), which is of importance to us in this paper, is shown on Fig. 2, for two values of the eccentricity, $\epsilon=0$ and $\epsilon=0.8660$. The latter corresponds to a choice of $b=a/2$, which was also the choice for the ellipse of Fig. 1. By the definition of h, the only values of h which can have any physical meaning are those for which $h\leq r$. Thus the elliptical motion takes place in the half quadrant for which, $h\leq r$, that is, below the dashed line of Fig. 2. For $\epsilon=0$ we have a circle and, indeed, on Fig. 2, the only physical point is $h=r=a$.

IV. UNBOUND KEPLER ORBITS

It is well known that when the total energy E is a positive constant the orbit of the Kepler problem is hyperbolic. This fact is often stated in introductory physics texts. We now prove it by the same simple methods used for elliptic orbits above.

For positive E we rewrite Eq. (3) as

$$\frac{[(l/m)^2/2(E/m)]}{h^2}-\frac{[GM/(E/m)]}{r}=1. \tag{17}$$

For this orbit equation we note that the relevant hyperbola, shown on Fig. 3, obeys the equation

$$\frac{b^2}{h^2}-\frac{2a}{r}=1, \tag{18}$$

which can be contrasted with the ellipse, Eq. (4). Again, the equality of Eqs. (17) and (18) proves that the orbit is hyperbolic and gives the orbit parameters, a and b, to be

$$a=GM/2(E/m),\quad b=(l/m)/(2E/m)^{1/2}. \tag{19}$$

To complete the proof we must derive the hyperbola Eq. (18), in terms of the coordinates r and h, from the usual equation of a hyperbola, in terms of the coordinates x and y:

$$x^2/a^2-y^2/b^2=1. \tag{20}$$

In the (x,y) plane of Fig. 3 the hyperbola lies between two asymptotes whose directions are determined by the choice of b and a. (For the hyperbola illustrated in Fig. 3 we have chosen $b=a/2$, as we did for the ellipse of Fig. 2 above.) There are, of course, two equal hyperbolas, both of which satisfy Eq. (20). We have drawn only the left-side hyperbola on Fig. 3, and not its image mirrored in the y axis.

This choice between the two hyperbolas is arbitrary but the choice for a system of hyperbola plus focus is not. When we refer to the physics we place the center-of-force at the focus: $(x,y)=(-c,0)$ or $(+c,0)$ with $c^2 \equiv a^2+b^2$. For the left hyperbola shown on Fig. 3 the choice of $(-c,0)$ as the focus corresponds to gravitational attraction—the orbit is "pulled around" the center-of-force. Placing the focus at the other position, $(+c,0)$, would correspond to the unphysical orbit of antigravity—with the orbit "pushed away" from the center-of-force. Although this latter case has no relevance to the Kepler problem it does provide the orbit for Rutherford scattering (see Sec. VI, below). The orbits for the left-side hyperbola, with the two possible foci both obey the same hyperbola equation in the x and y coordinates, that is, Eq. (20), but for the r and h coordinates the two cases obey different equations. We are interested here in the case of the Kepler problem, with the left-side hyperbola taken together with the left-side focus.

To begin the proof of Eq. (18), we note that from Eq. (20) we have

$$y=b(x^2/a^2-1)^{1/2}, \tag{21}$$

$$y' \equiv \frac{dy}{dx}=\left(\frac{b^2}{a^2}\right)\left(\frac{x}{y}\right), \tag{22}$$

and the y intercept for the tangent line at point P is

$$y_0=y-y'x. \tag{23}$$

Choosing the left-side focus, $(x_1,y_1)=(-c,0)$ we find at once, in place of Eq. (10)

$$r \equiv [y^2+(x+c)^2]^{1/2}=-a^{-1}(a^2+xc). \tag{24}$$

Similarly, we find, using Eq. (12)

$$h^2=-b^2(a^2+xc)/(a^2-xc), \tag{25}$$

which can be compared to Eq. (13) for the ellipse. Combining Eqs. (24) and (25) yields the desired hyperbola, Eq. (18). The second step of the proof is, of course, the same as it was for the ellipse.

V. ACCESSIBLE ORBIT PROBLEMS

With the simple relationship between the orbit parameters and the constants of the motion, derived above, a myriad of interesting problems can immediately be tackled by the students. The point is that any two pieces of information pertaining to a, b, c, l/m, E/m, T, etc., completely specify the orbit. A few examples are as follows.

(i) If you fire a cannonball horizontally at the North Pole with an initial velocity of $v=0.98\times10^4$ m/s, sketch the orbit and find the orbit parameters. (Assume the earth is spherically symmetric and neglect air friction.) Find the period of the motion. The solution for this problem is an elliptical orbit whose major axis lies along the earth's axis and is greater than the earth radius.

(ii) If your friendly computer "Hal" launches you from your spaceship into outerspace at a distance from the Sun of 3.1×10^{11} m, with a speed of 8.2×10^4 m/s and a direction of motion such that your perpendicular distance is 1.86×10^{11} m from the Sun, find out what will be your distance of closest approach to the Sun. Also, find out if your orbit is bound or unbound.

(iii) If a comet were to strike the earth in such a way that its orbital velocity instantly increased by 10% but the direction of the velocity remained unchanged by the collision, find the effect on the earth's orbit (originally assumed to be circular) and its period.

Further, a great deal of celestial mechanics becomes accessible and transparent.

VI. THE RUTHERFORD SCATTERING LAW

In spite of its importance the Rutherford scattering law is not a subject normally covered in introductory physics textbooks, perhaps because the concepts of cross sections are usually optional or omitted. Indeed, the concept of a differential cross section, needed for the Rutherford law is quite sophisticated for a first year course. However, because of my own personal predilections in physics I like to say more about atomic and nuclear cross sections in my introductory class than is the standard fare. When I then also give a full treatment, as above, of the Kepler laws it is very tempting to go further and derive the Rutherford scattering law. The connection between the crowning achievement of Newtonian mechanics and the foundations of modern subatomic physics is very compelling. This is an approach to which I was led by the PSSC courses of two decades ago and which has been admirably presented by French in his textbook, *Newtonian Mechanics*.[3]

The purpose of this section is to show that the derivation of the Rutherford law benefits fully from the simplification of the Kepler laws introduced above. If the Coulomb force is $F=kQ_1Q_2/r^2$, in an obvious notation, then the total energy, E, of an alpha particle in its orbit around a gold nucleus is given by

$$E=mv^2/2+kQ_1Q_2/r. \tag{26}$$

Using the conserved angular momentum, $l \equiv mvh$, we find, instead of Eq. (3)

$$\frac{l^2/2mE}{h^2}+\frac{kQ_1Q_2/E}{r}=1. \tag{27}$$

This orbit is that for the Kepler problem with a repulsive force. We have the left-side hyperbola with the right-side focus, as shown on Fig. 3, where the coordinates r' and h' are also indicated. In terms of r' and h' the hyperbola equation is

$$\frac{b^2}{(h')^2}+\frac{2a}{r'}=1, \tag{28}$$

which is to be compared with the ellipse Eq. (4) and Eq. (18) which pertains to the hyperbola-focus system for attractive forces. Again, comparing Eqs. (28) and (27) gives us the orbit parameters:

$$a=kQ_1Q_2/2E, \quad b=(l^2/2mE)^{1/2}. \tag{29}$$

The proof of Eq. (28) follows closely the derivation of Sec. IV above. Equations (21)–(23) still apply. However, with the choice of $(x_1,y_1)=(+c,0)$, we find

$$r'=a^{-1}[a^2+xc], \tag{30}$$

and

$$(h')^2=-b^2(a^2-xc)/(a^2+xc), \tag{31}$$

which are to be compared with the corresponding variables of Secs. III and IV. Combining Eqs. (30) and (31) now yields Eq. (28).

When the alpha particle is far from the gold nucleus the potential energy vanishes and therefore $E=T_\alpha$, the initial kinetic energy of the alpha particle. The orbit parameter b is in fact, from Fig. 3, the usual "impact parameter." From the geometry of Fig. 3 we see that

$$b=a\cot(\theta/2), \tag{32}$$

where θ is the scattering angle, and then

$$\frac{db}{d\theta}=-\left(\frac{a}{2}\right)\sin^{-2}\left(\frac{\theta}{2}\right). \tag{33}$$

To complete the derivation of the Rutherford scattering law we need to introduce the definitions pertaining to the differential scattering cross section, $d\sigma/d\Omega$. Here we follow the conventional treatment whose elements are the following. The partial cross section element, $d\sigma$, is defined to be proportional to the fraction of alpha particles whose impact parameters lie in between b and $b+db$

$$d\sigma\equiv-2\pi\, b\, db, \tag{34}$$

where the minus sign indicates that we start with large impact parameters (small scattering angles) and work downward in b. $d\Omega$ is the area on the unit sphere between θ and $\theta+d\theta$.

$$d\Omega\equiv2\pi\sin\theta\, d\theta=4\pi\sin(\theta/2)\cos(\theta/2)d\theta. \tag{35}$$

Therefore, using Eq. (33),

$$\begin{aligned}\frac{d\sigma}{d\Omega}&=-\frac{b}{2\sin(\theta/2)\cos(\theta/2)}\frac{db}{d\theta}\\&=a^2/[4\sin^4(\theta/2)]\\&=[kQ_1Q_2/4T_\alpha\sin^2(\theta/2)]^2,\end{aligned} \tag{36}$$

which is the Rutherford scattering law.

VII. CONCLUSION

A derivation is given of Kepler's laws in which the physics is easy and immediate. Any complications reside in the mathematics of ellipses and even these are within the grasp of the students who have, in the past decade or more, come into my introductory physics class. Therefore, the derivations presented here fulfilled their intent of making the whole Keplerian problem easily accessible to physics students in first year. The intellectual payoff is large for the effort involved and that is the essence of introducing physics to willing students.

ACKNOWLEDGMENTS

The author is indebted to many of his colleagues, particularly to Professor Douglas Beder, for encouraging this excursion into the Kepler problem. He is especially indebted to a referee who pointed out the special nature of the variable h, the perpendicular distance, and, while doing so also provided the proof given in step 2 of Sec. III. This is a very evident example of where the referee system is constructive.

[1]I. Newton, *Principia* (1687); F. Cajori's revision of A. Motte's translation (1729), (University of California, Berkeley, CA 1960).

[2]Examples are: R. Resnick, D. Halliday, and K. S. Krane, *Physics*, 4th ed. (Wiley, New York, 1992) or H. Benson, *University Physics* (Wiley, New York, 1991).

[3]A. P. French, *Newtonian Mechanics* (Norton, New York, 1971). This excellent textbook gives a very thorough analysis of the whole Kepler problem including many interesting historical facts.

[4]J. Sivardière, "A simple look at the Kepler motion," Am. J. Phys. **56**, 132–135 (1988) which contains many earlier references. See also W. Hauser, "On planetary motion," Am. J. Phys. **53**, 905–907 (1985). Almost every volume of the *American Journal of Physics* contains several articles pertaining to the Kepler problem.

QUANTUM FIELD THEORY AND DIRAC THEORY

Almost by accident, Dirac's theory of the electron gave the same results as quantum field theory for processes involving only electrons, positrons, and/or photons. But quantum field theory is more general—it can account for processes like nuclear beta decay that could not be understood along the lines of Dirac's theory. There is nothing in quantum field theory that requires particles to have any particular spin. The electron does happen to have the spin that Dirac's theory required, but there are other particles with other spins and those other particles have antiparticles and this has nothing to do with the negative energies about which Dirac speculated. Yet the *mathematics* of Dirac's theory has survived as an essential part of quantum field theory; it must be taught in every graduate course in advanced quantum mechanics. The formal structure of Dirac's theory has thus survived the death of the principles of relativistic wave mechanics that Dirac followed in being led to his theory.

Steven Weinberg, *Dreams of a Final Theory* (Pantheon Books, New York, 1992), p. 152.