LINEAR WAVE THEORY

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LINEAR WAVE THEORY

DESCRIPTION

This application yields first-order approximations for various parameters of wave motion as predicted by the wave theory bearing the same name (also known as small amplitude, sinusoidal, or Airy theory). It provides estimates for engineering quantities such as water surface elevation, general wave properties, particle kinematics, and pressure as functions of wave height and period, water depth, and position in the wave form.

INTRODUCTION

The effects of water waves are of major importance in the field of coastal engineering. Waves are a major factor in determining geometry and composition of beaches and significantly influence planning and design of harbors, waterways, shore protection measures, coastal structures, and other coastal works.

In general, actual water-wave phenomena are complex and difficult to describe mathematically because of nonlinearities, three-dimensional characteristics, and apparent random behavior. The most elementary wave theory, referred to as small-amplitude or linear wave theory, was developed by Airy (1845). This nomenclature derives from the simplifying assumptions of its derivation. Additionally, it represents a first approximation resulting from a formal perturbation procedure for waves of finite amplitude.

GENERAL ASSUMPTIONS AND LIMITATIONS

A typical representation of a wave is depicted in Figure 2-1-1.

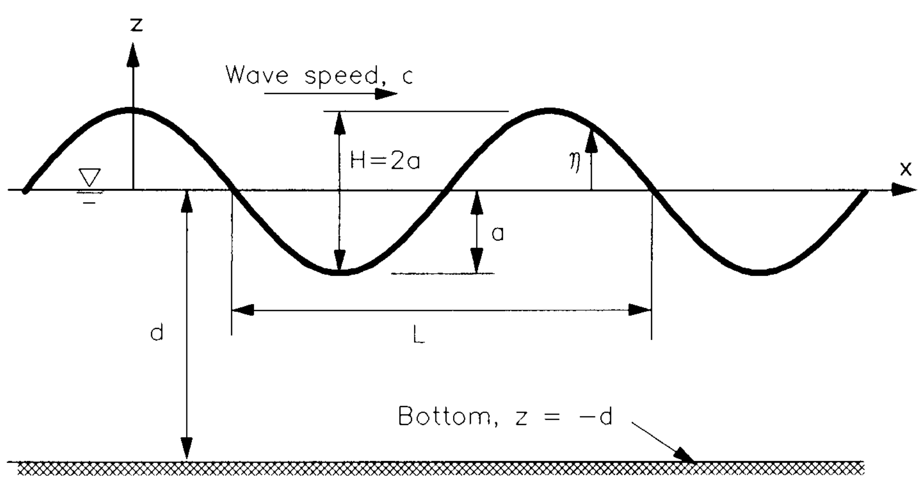

Figure 2-1-1. Progressive Wave

Common terminology for wave discussions includes:

d = still-water depth

η = free surface elevation relative to still water ($z = 0$)

a = wave amplitude

H = wave height = $2a$ for small-amplitude waves

L = wavelength

T = wave period

c = velocity of wave propagation (celerity) = L/T

k = wave number = $2\pi/L$

ω = wave angular frequency = $2\pi/T$

u , w = horizontal and vertical components of velocity vector $\vec{u}$

ϕ = velocity potential $\vec{u} = \vec{\nabla}\phi$

g = acceleration of gravity

ρ = density of water

General assumptions and limitations used to derive wave theories are:

- Waves are two-dimensional (2-D) in the x-z plane.
- Waves propagate in a permanent form over a smooth horizontal bed of constant depth in the positive x-direction.
- There is no underlying current.
- Fluid is inviscid and incompressible, having no surface tension.
- Flow is irrotational.
- Coriolis effect is neglected.

Governing Equation

The assumption of irrotational flow leads to the existence of a velocity potential. An ideal fluid must satisfy the mass continuity equation and can be expressed in primitive variables as:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

or, in terms of the velocity potential:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{2}$$

Equation *2* is the 2-D Laplace equation.

BOUNDARY CONDITIONS

The governing equation describes a boundary value problem. The various boundary conditions of the problem domain affect the form and complexity of the solution of the Laplace equation. The boundary conditions can be summarized as:

Bottom Boundary Condition (BBC). Fluid must not pass through the seafloor. Therefore at the sea bottom, the vertical component of the water particle velocity must vanish.

$$\frac{\partial \phi}{\partial z} = 0 \qquad at \quad z = -d \tag{3}$$

Kinematic Free Surface Boundary Condition (KFSBC). The fluid particle velocity normal to the free surface is equal to the velocity of the free surface itself. This condition implies that a water particle on the free surface remains there.

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial z} = 0 \qquad at \quad z = \eta \tag{4}$$

Dynamic Free Surface Boundary Condition (DFSBC). Expressed as the Bernoulli equation, the pressure at the free surface is constant. This requirement follows from an assumption that atmospheric pressure above the fluid is itself constant and no surface tension is present at the air-water interface.

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + g\eta = f(t) \qquad at \quad z = \eta \tag{5}$$

Periodicity. The wave is periodic in time and space.

$$\begin{gathered} \phi(x,t) = \phi(x+L,t) \\ \phi(x,t) = \phi(x,t+T) \end{gathered} \tag{6}$$

ADDITIONAL SIMPLIFYING ASSUMPTIONS

The partial differential equations are difficult to solve because of problems associated with the free surface boundary conditions. They are nonlinear and occur at location $z = \eta$, which is initially unknown. Linear wave theory derives from applying simplifying assumptions to the free surface boundary conditions. The still-water wave height H is assumed to be very small relative to both the wavelength L and the still-water depth d :

$$H \ll L \qquad \text{and} \qquad H \ll d$$

The result of these assumptions is that the nonlinear terms which contain products of terms of order of H are then negligible in comparison with remaining linear terms of order H . Also, the free surface boundary conditions may now be applied at the still-water level $z = 0$. The simplified or linearized free surface boundary conditions then reduce to:

Kinematic Free Surface Boundary Condition (KFSBC).

$$\frac{\partial\phi}{\partial z} - \frac{\partial\eta}{\partial t} = 0 \qquad at \quad z = 0 \tag{7}$$

Dynamic Free Surface Boundary Condition (DFSBC).

$$\frac{\partial\phi}{\partial t} + g\eta = 0 \qquad at \quad z = 0 \tag{8}$$

Solution of the Governing Equation

The solution for the velocity potential is obtained by applying the method of separation of variables and the given boundary conditions. The resulting solution is stated below as well as the linear dispersion relation:

$$\phi = \frac{\pi H}{kT}\frac{\cosh(ks)}{\sinh(kd)}\sin\theta \qquad \text{or} \qquad \phi = \frac{gH}{2\omega}\frac{\cosh(ks)}{\cosh(kd)}\sin\theta \tag{9}$$

$$\omega^2 = gk\tanh(kd) \qquad \text{or} \qquad c^2 = \frac{g}{k}\tanh(kd) \tag{10}$$

where

$\Theta = k(x - ct) = kx - wt =$ wave phase angle

$s = z + d$ measured upwards from the seabed

The dispersion relation describes the manner in which a field of propagating waves consisting of many frequencies would separate or "disperse" due to the different celerities of the various frequency components (Dean and Dalrymple, 1984). The linear dispersion relation is a transcendental function but may be readily evaluated using a Pade approximation (Hunt, 1979):

$$c^2 = gd\left[y + \left(1 + \sum_{n=1}^{9} d_n y^n\right)^{-1}\right]^{-1} \tag{11}$$

where

$$y = \frac{\omega^2 d}{g}$$

d_n = constants

$d_1 = 0.66667$	$d_4 = 0.06320$	$d_7 = 0.00171$
$d_2 = 0.35550$	$d_5 = 0.02174$	$d_8 = 0.00039$
$d_3 = 0.16084$	$d_6 = 0.00654$	$d_9 = 0.00011$

This approximation has an accuracy better than 0.01 percent over the range $0 \le y \le \infty$.

Common Variables of Interest

The solution for ϕ and the linear dispersion relation provide a foundation for deriving expressions for other common variables of interest. These equations are listed below and can be found in many texts including Sarpkaya and Isaacson (1981) and the SPM (1984).

Wavelength:
$$L = cT \tag{12}$$

Group velocity:
$$C_g = \frac{1}{2}\left[1 + \frac{2kd}{\sinh(2kd)}\right]c \tag{13}$$

Water surface elevation:
$$\eta = \frac{H}{2}\cos\theta \tag{14}$$

Average energy density:
$$E = \frac{1}{8}\rho g H^2 \tag{15}$$

Energy flux:
$$P = E C_g \tag{16}$$

Pressure:
$$p = -\rho g z + \frac{1}{2}\rho g H \frac{\cosh(ks)}{\cosh(kd)}\cos\theta \tag{17}$$

Horizontal particle displacement:
$$\xi = -\frac{H}{2}\frac{\cosh(ks)}{\sinh(kd)}\sin\theta \tag{18}$$

Vertical particle displacement:
$$\zeta = \frac{H}{2}\frac{\sinh(ks)}{\sinh(kd)}\cos\theta \tag{19}$$

Horizontal particle velocity:
$$u = \frac{\pi H}{T}\frac{\cosh(ks)}{\sinh(kd)}\cos\theta \tag{20}$$

Vertical particle velocity:
$$w = \frac{\pi H}{T}\frac{\sinh(ks)}{\sinh(kd)}\sin\theta \tag{21}$$

Horizontal particle acceleration:
$$\frac{\partial u}{\partial t} = \frac{2\pi^2 H}{T^2}\frac{\cosh(ks)}{\sinh(kd)}\sin\theta \tag{22}$$

Vertical particle acceleration:
$$\frac{\partial w}{\partial t} = -\frac{2\pi^2 H}{T^2}\frac{\sinh(ks)}{\sinh(kd)}\cos\theta \tag{23}$$

A common parameter for the applicability of various wave theories is the Stokes (1847) or Ursell (1953) parameter, which is defined as:

$$U_r = \frac{H L^2}{d^3} \tag{24}$$

This parameter is reported for convenience.

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Cnoidal Wave Theory

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CNOIDAL WAVE THEORY

DESCRIPTION

This application yields various parameters of wave motion as predicted by first-order (Isobe, 1985) and second-order (Hardy and Kraus, 1987) approximations for cnoidal wave theory. It provides estimates for common items of interest such as water surface elevation, general wave properties, kinematics, and pressure as functions of wave height and period, water depth, and position in the wave form.

INTRODUCTION

The accurate description of waves in the nearshore region is an important element in the design and analysis process. The complexity of nonlinear wave theories has apparently discouraged their common application, especially for reconnaissance level investigations; yet they often provide significant improvements over linear wave theory descriptions. Today's common desktop microcomputers offer adequate computational power to implement many nonlinear wave theories.

Solutions for wave attributes described by this theory are expressed in terms of the Jacobian elliptic function *cn*, thus providing an explanation for the theory's name. The original source for the theory was a paper by the Dutch mathematicians Korteweg and de Vries (1895). More recent derivations for cnoidal wave theory include those of Keulegan and Patterson (1940), Keller (1948), Laitone (1960), Chappelear (1962), Fenton (1979), and Isobe and Kraus (1983) among others. Differences between the derivations consist primarily of choices in perturbation parameter, definition of celerity, and order of the solution. The following discussion is taken principally from Hardy and Kraus (1987), and Isobe and Kraus (1983).

GENERAL ASSUMPTIONS AND LIMITATIONS

A representative cnoidal wave profile is illustrated in Figure 2-2-1.

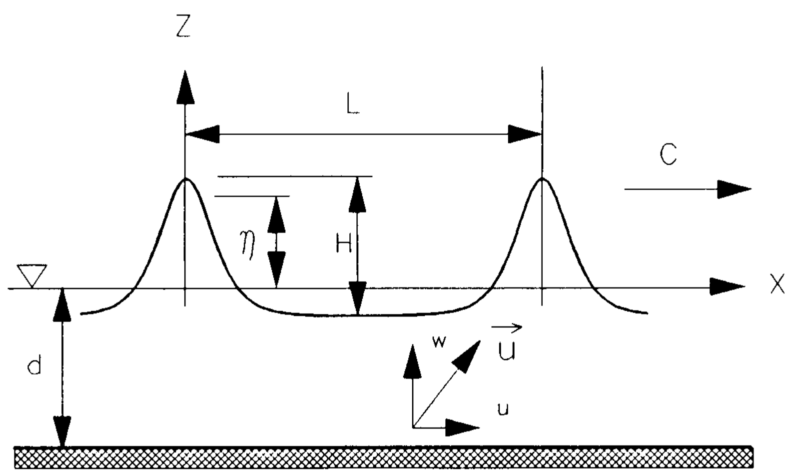

Figure 2-2-1. Cnoidal Wave Profile

The assumptions and terminology for general wave theories will be reviewed first, followed by specific assumptions and treatments associated with cnoidal wave theory. Terms common to wave discussions include:

d = still-water depth

η = free surface elevation relative to still water ($z = 0$)

a = wave amplitude

H = wave height = $2a$ for small-amplitude waves

L = wavelength

T = wave period

c = velocity of wave propagation (celerity) = L/T

$\vec{u}$ = water particle velocity vector

u , w = horizontal and vertical components of velocity vector $\vec{u}$

ϕ = velocity potential $\vec{u} = \vec{\nabla}\phi$

ψ = stream function

g = acceleration of gravity

ρ = density of water

p_B = Bernoulli constant

General assumptions and limitations used to derive wave theories include:

- Waves are two-dimensional in the x-z plane.
- Waves propagate in a permanent form over a smooth horizontal bed of constant depth in the positive x-direction.
- There is no underlying current.
- Fluid is inviscid and incompressible, having no surface tension.
- Flow is irrotational.

GOVERNING EQUATION

The assumption of irrotational flow leads to the existence of a velocity potential. An ideal fluid must satisfy the mass continuity equation and can be expressed in terms of the velocity potential as:

$$\nabla^2 \phi = 0 \tag{1}$$

This is the 2-D Laplace equation.

BOUNDARY CONDITIONS

The governing equation describes a boundary value problem. The various boundary conditions of the problem domain affect the form and complexity of the solution of the Laplace equation. The boundary conditions can be summarized as:

Bottom Boundary Condition (BBC). Fluid must not pass through the seafloor. Therefore, at the sea bottom, the vertical component of the water particle velocity must vanish.

$$\phi_z = 0 \qquad at \quad z = -d \tag{2}$$

Kinematic Free Surface Boundary Condition (KFSBC). The fluid particle velocity normal to the free surface is equal to the velocity of the free surface itself. This condition implies that a water particle on the free surface remains there. It also implies that the wave does not break.

$$\eta_t + \phi_x \eta_x - \phi_z = 0 \qquad at \quad z = \eta \tag{3}$$

Dynamic Free Surface Boundary Condition (DFSBC). Expressed as the Bernoulli equation, the pressure at the free surface is constant. This requirement follows from an assumption that atmospheric pressure above the fluid is itself constant and no surface tension is present at the air-water interface .

$$-\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + gz = \frac{p_b}{\rho} \qquad at \quad z = \eta \tag{4}$$

Periodicity. The wave is periodic in time and space.

$$\phi(x,t) = \phi(x+L,t) \qquad \phi(x,t) = \phi(x,t+T) \tag{5}$$

CNOIDAL WAVE THEORY CONSIDERATIONS

Cnoidal wave theory is valid in relatively shallow water. For deriving waves of finite amplitude, the simplifying assumptions which characterize small-amplitude (linear) wave theory $(H \ll L)$ and $(H \ll d)$ are not appropriate. For shallow water, the second assumption $(H \ll d)$ is invalid. Consequently, the nonlinear terms of the free surface boundary conditions are retained in the solution of the boundary value problem.

The free surface boundary conditions can be simplified by adopting a moving coordinate system having the same velocity as the wave celerity. The unsteady terms (time derivatives) are then eliminated. However, this procedure requires an additional assumption for an initial definition of celerity. The two most common assumptions were originally proposed by Stokes (1847) and are stated below:

Stokes Definition 1 (avg horizontal velocity = 0)	**Stokes Definition 2** (avg mass flux = 0)
$\frac{1}{L}\int_0^L u \, dx = 0$	$\frac{1}{L}\int_0^L \int_{-d}^{\eta} u \, dz \, dx = 0$

(6)

The second definition of wave celerity was selected for this derivation. The tabulated approximations resulting from the derivation will be expressed relative to a fixed coordinate system.

Wave theories are often derived in any of three sets of variables: primitive variables (u, w), velocity potential ϕ, or stream function ψ. The assumption of 2-D, irrotational flow of an incompressible fluid leads to the following relationships between the three variables:

$$\psi_z = \phi_x = u \qquad -\psi_x = \phi_z = w \tag{7}$$

Use of the stream function automatically satisfies the KFSBC (Equation *3*) and consequently simplifies its form. The governing equation and boundary equations can then be restated in terms of stream function as:

Laplace Equation:
$$\nabla^2 \psi = 0 \tag{8}$$

BBC:
$$\psi = 0 \qquad at \quad z = -d \tag{9}$$

KFSBC:
$$\psi = q \qquad at \quad z = \eta \tag{10}$$

DFSBC:
$$\frac{1}{2}(\psi_x^2 + \psi_z^2) + g\eta = \frac{p_b}{\rho} \qquad at \quad z = \eta \tag{11}$$

Other Conditions:
$$\bar{\eta} = 0 \tag{12}$$

$$\eta(0) - \eta\left(\frac{L}{2}\right) = H \tag{13}$$

Celerity Definition:
$$c = -\frac{q}{d} \tag{14}$$

The variable q in the above expressions represents the unit flow rate. Expressions for the periodic nature of the wave have been recast into alternate forms. Equation *12* requires the average value of η taken over a wavelength to be zero, while Equation *13* defines the wave height.

In shallow water, the vertical and horizontal length scales are of different orders of magnitude. For convenience, the variables are nondimensionalized using the horizontal length scale (L), vertical length scale (H), and velocity scale $\sqrt{gd}$:

$$X = \frac{x}{L} \qquad Z = \frac{z}{d} \qquad N = \frac{\eta}{d} \qquad \Psi = \frac{\psi}{d\sqrt{gd}}$$
$$Q = \frac{q}{d\sqrt{gd}} \qquad P = \frac{p}{\rho g d} \qquad P_B = \frac{p_B}{\rho g d} \tag{15}$$

The complete boundary value problem restated in terms of these nondimensional variables is:

Governing Equation: $$\Psi_{ZZ} + \left(\frac{d}{L}\right)^2 \Psi_{XX} = 0 \tag{16}$$

BBC: $$\Psi = 0 \qquad at \quad z = -1 \tag{17}$$

KFSBC: $$\Psi = q \qquad at \quad z = N \tag{18}$$

DFSBC: $$\frac{1}{2}\left[\Psi_Z^2 + \left(\frac{d}{L}\right)^2 \Psi_X^2\right] + N = P_B \qquad at \quad z = N \tag{19}$$

Other Conditions: $$\overline{N} = 0 \tag{20}$$

$$N(0) - N\left(\frac{1}{2}\right) = \frac{H}{d} \tag{21}$$

Solution of the Governing Equations

The boundary value problem consists of a set of nonlinear partial differential equations. The principal approach used to solve finite-amplitude wave behavior is the perturbation method. Dependent variables in the problem are assumed to be functions of an auxiliary parameter, δ, and are expanded in a power series of a small perturbation parameter, ϵ.

Waves of permanent form can be described by three independent variables, H, L, and d, from which two independent nondimensional quantities can be formed: (H/L) and (H/d). One of these ratios is often selected as the perturbation parameter in finite-amplitude wave theories. In this derivation of cnoidal wave theory, the second quantity is selected as the perturbation parameter:

$$\epsilon = \frac{H}{d} \tag{22}$$

The modulus of the elliptic integral, κ, is selected as the auxiliary parameter.

The nondimensional variables of the boundary value problem are expanded as power series about ϵ:

$$\Psi(X, Z, \kappa, \epsilon) = \sum_{n=0}^{\infty} \Psi_n(X, Z, \kappa)\epsilon^n = \Psi_0 + \epsilon\Psi_1 + \epsilon^2\Psi_2 + \epsilon^3\Psi_3 + \ldots \tag{23}$$

$$N(X, \kappa, \epsilon) = \sum_{n=1}^{\infty} N_n(X, \kappa)\epsilon^n = \epsilon N_1 + \epsilon^2 N_2 + \epsilon^3 N_3 + \ldots \tag{24}$$

$$Q(\kappa,\epsilon)=\sum_{n=0}^{\infty}Q_n(\kappa)\epsilon^n=Q_0+\epsilon Q_1+\epsilon^2Q_2+\epsilon^3Q_3+\dots \tag{25}$$

$$P_B(\kappa,\epsilon)=\sum_{n=0}^{\infty}P_n(\kappa)\epsilon^n=P_0+\epsilon P_1+\epsilon^2P_2+\epsilon^3P_3+\dots \tag{26}$$

$$\left(\frac{d}{L}\right)^2(\kappa,\epsilon)=\sum_{n=1}^{\infty}\delta_n(\kappa)\epsilon^n=\epsilon\delta_1+\epsilon^2\delta_2+\epsilon^3\delta_3+\dots \tag{27}$$

The series expansions for N and $(d/L)^2$ (Equations *24* and *27*) do not have *zero* terms because they are always smaller than ϵ.

The expanded forms for variables (Equations *23-27*) are inserted into the Equations *16-21*. For example, substituting the expanded form for Ψ in the governing Equation *16* yields:

$$\begin{gathered}\left(\Psi_{0ZZ}+\epsilon\Psi_{1ZZ}+\epsilon^2\Psi_{2ZZ}+\epsilon^3\Psi_{3ZZ}+\dots\right)\\+(\epsilon\delta_1+\epsilon^2\delta_2+\epsilon^3\delta_3+\dots)\times\left(\Psi_{0XX}+\epsilon\Psi_{1XX}+\epsilon^2\Psi_{2XX}+\epsilon^3\Psi_{3XX}+\dots\right)=0\end{gathered} \tag{28}$$

and at the bottom boundary (Equation *17*):

$$\Psi_0+\epsilon\Psi_1+\epsilon^2\Psi_2+\epsilon^3\Psi_3+\dots=0 \tag{29}$$

Equation *28* also provides some useful initial information. With the wave height equal to zero $(\epsilon=0)$, uniform flow conditions exist as a result of the moving coordinate system. Further, it can be seen that:

$$\Psi_0=b_{00}Z \qquad \Rightarrow \qquad \Psi_{0XX}=\Psi_{0ZZ}=0 \tag{30}$$

The term b_{00} is a constant to be determined later.

The free surface boundary conditions require special treatment because they occur at a location which is initially unknown. Instead of expansion with a power series, variables requiring evaluation at the free surface are expanded using a Taylor series about the known still-water level $(Z=0)$. For example, the Taylor series expansion for Ψ at the free surface is:

$$\Psi(X,N)=\Psi(X,0)+N\Psi_Z(X,0)+\frac{N^2}{2}\Psi_{ZZ}(X,0)+\dots \tag{31}$$

Continuing with the perturbation technique, for a given equation, terms are grouped by equal powers of ϵ, and terms of an individual order on the left side must equal terms of like order on the right side. For example, again consider the governing Equation *28*:

$$\epsilon \text{ terms:} \qquad \Psi_{1ZZ}=0 \tag{32}$$

ϵ^2 terms: $$\Psi_{2ZZ} + \delta_1 \Psi_{1XX} = 0 \tag{33}$$

ϵ^3 terms: $$\Psi_{3ZZ} + \delta_1 \Psi_{2XX} + \delta_2 \Psi_{1XX} = 0 \tag{34}$$

This procedure is also applied to the remaining five boundary condition equations (*17-21*). The result is a set of equations for each order of ϵ. For example, the set of equations of second-order (ϵ^2) terms is:

$$\left[\begin{array}{c} \Psi_{2ZZ} + \delta_1 \Psi_{1XX} = 0 \\ \Psi_2 = 0 \qquad at \quad Z = -1 \\ \Psi_2 + b_{00} N_2 + N_1 \Psi_{1Z} = Q_2 \qquad at \quad Z = N \\ N_2 + b_{00}(\Psi_{2Z} + N_1 \Psi_{1ZZ}) + \frac{1}{2}(\Psi_{1Z})^2 = P_2 \qquad at \quad Z = N \\ \overline{N}_2 = 0 \\ N_2(0) - N_2\left(\frac{1}{2}\right) = 0 \end{array}\right] \tag{35}$$

The solution of n^{th} order equations requires information from the equations of $(n+1)^{th}$ order. In total, the algebraic manipulations are quite lengthy and are beyond the intent of this report. A short summary of the procedure follows. The equations of 0^{th} order determine P_0 and Q_0 once b_{00} is determined. The equations of 1^{st} order determine b_{00} such that a nontrivial solution of N_1 and Ψ_1 exists. In similar fashion, the 2^{nd} order equations determine N_1 and Ψ_1 if a nontrivial solution of N_2 and Ψ_2 exists. The process continues upward to the desired order of solution. This particular ACES application will provide results for 1^{st} and 2^{nd} order solutions.

The solution for the cnoidal theory contains elliptic integrals and Jacobian elliptic functions which arise from the choice of κ as the auxiliary parameter and from the solution of certain nonlinear differential equations. Standard numerical methods described in Abramowitz and Stegun (1972) are employed for approximating these quantities.

RESULTS FROM THE THEORY

The resulting approximations for the critical elements of the wave description are summarized below. Integral properties, water particle kinematics, and other traditional quantities of interest are included in the tabulations. All quantities are relative to a fixed frame of reference.

First-Order Solutions

Dispersion relation: $$\frac{16\kappa^2 K^2}{3} = \frac{gHT^2}{d^2} \tag{36}$$

Celerity: $$c = \sqrt{gd}(C_0 + \epsilon C_1) \tag{37}$$

$$C_0 = 1 \tag{37.1}$$

$$C_1 = \frac{1 + 2\lambda - 3\mu}{2} \tag{37.2}$$

Wavelength: $$L = cT \tag{38}$$

Water surface elevation: $$\eta = d(A_0 + A_1 \,\mathrm{cn}^2\theta) \tag{39}$$

$$A_0 = \epsilon(\lambda - \mu) \tag{39.1}$$

$$A_1 = \epsilon \tag{39.2}$$

Average energy density: $$E = \rho g H^2 E_0 \tag{40}$$

$$E_0 = \frac{-\lambda + 2\mu + 4\lambda\mu - \lambda^2 - 3\mu^2}{3} \tag{40.1}$$

Energy flux: $$F = \rho g H^2 \sqrt{gd}\, F_0 \tag{41}$$

$$F_0 = E_0 \tag{41.1}$$

Pressure: $$p = p_b - \frac{\rho}{2}[(u - c)^2 + w^2] - g\rho(z + d) \tag{42}$$

$$p_b = \rho g d(P_0 + \epsilon P_1) \tag{42.1}$$

$$P_0 = \frac{3}{2} \tag{42.2}$$

$$P_1 = \frac{1 + 2\lambda - 3\mu}{2} \tag{42.3}$$

Horizontal velocity: $$u = \sqrt{gd}(B_{00} + B_{10} \,\mathrm{cn}^2\theta) \tag{43}$$

$$B_{00} = \epsilon(\lambda - \mu) \tag{43.1}$$

$$B_{10} = \epsilon \qquad (43.2)$$

Vertical velocity: $$w = \sqrt{gd}\,\frac{4Kd\,\mathrm{csd}}{L}\left(\frac{z+d}{d}\right)B_{10} \qquad (44)$$

Horizontal acceleration: $$\frac{\partial u}{\partial t} = \sqrt{gd}\,B_{10}\frac{4K}{T}\mathrm{csd} \qquad (45)$$

Vertical acceleration: $$\frac{\partial w}{\partial t} = \sqrt{gd}\,\frac{4Kd}{L}\left(\frac{z+d}{d}\right)B_{10}\frac{2K}{T}(\mathrm{sn}\theta\,\mathrm{dn}\theta - \mathrm{cn}\theta\,\mathrm{dn}\theta + \kappa^2\mathrm{sn}\theta\,\mathrm{cn}\theta) \qquad (46)$$

The following general symbols also require definition:

$$\kappa' = \sqrt{1-\kappa^2} \qquad (47)$$

$$\lambda = \frac{\kappa'^2}{\kappa^2} \qquad (48)$$

$$\mu = \frac{E}{\kappa^2 K} \qquad (49)$$

$$\theta = 2K\left[\left(\frac{x}{L}\right) - \left(\frac{t}{T}\right)\right] \qquad (50)$$

K and E_e are the complete elliptic integrals of the first and second kind, respectively; $\mathrm{sn}\theta$, $\mathrm{cn}\theta$, $\mathrm{dn}\theta$ are the Jacobian elliptic functions, and

$$csd = \mathrm{cn}\theta\,\mathrm{sn}\theta\,\mathrm{dn}\theta \qquad (51)$$

Second-Order Approximations

Dispersion relation: $$\frac{16\kappa^2K^2}{3} = \frac{gHT^2}{d^2}\left[1 - \epsilon\left(\frac{1+2\lambda}{4}\right)\right] \qquad (52)$$

Celerity: $$c = \sqrt{gd}(C_0 + \epsilon C_1 + \epsilon^2 C_2) \qquad (53)$$

$$C_0 = 1 \qquad (53.1)$$

$$C_1 = \frac{1+2\lambda-3\mu}{2} \qquad (53.2)$$

$$C_2 = \frac{-6 - 16\lambda + 5\mu - 16\lambda^2 + 10\lambda\mu + 15\mu^2}{40} \tag{53.3}$$

Water surface elevation: $$\eta = d(A_0 + A_1 \,\mathrm{cn}^2\theta + A_2 \,\mathrm{cn}^4\theta) \tag{54}$$

$$A_0 = \epsilon(\lambda - \mu) + \epsilon^2\left(\frac{-2\lambda + \mu - 2\lambda^2 + 2\lambda\mu}{4}\right) \tag{54.1}$$

$$A_1 = \epsilon - \frac{3}{4}\epsilon^2 \tag{54.2}$$

$$A_2 = \frac{3}{4}\epsilon^2 \tag{54.3}$$

Average energy density: $$E = \rho g H^2 (E_0 + \epsilon E_1) \tag{55}$$

$$E_0 = \frac{-\lambda + 2\mu + 4\lambda\mu - \lambda^2 - 3\mu^2}{3} \tag{55.1}$$

$$E_1 = \frac{1}{30}(\lambda - 2\mu - 17\lambda\mu + 3\lambda^2 - 17\lambda^2\mu + 2\lambda^3 + 15\mu^3) \tag{55.2}$$

Energy flux: $$F = \rho g H^2 \sqrt{gd}(F_0 + \epsilon F_1) \tag{56}$$

$$F_0 = E_0 \tag{56.1}$$

$$F_1 = \frac{1}{30}(-4\lambda + 8\mu + 53\lambda\mu - 12\lambda^2 - 60\mu^2 + 53\lambda^2\mu - 120\lambda\mu^2 - 8\lambda^3 + 75\mu^3) \tag{56.2}$$

Pressure: $$p = p_b - \frac{\rho}{2}[(u - c)^2 + w^2] - g\rho(z + d) \tag{57}$$

$$p_b = \rho g d(P_0 + \epsilon P_1 + \epsilon^2 P_2) \tag{57.1}$$

$$P_0 = \frac{3}{2} \tag{57.2}$$

$$P_1 = \frac{1 + 2\lambda - 3\mu}{2} \tag{57.3}$$

$$P_2 = \frac{-1 - 16\lambda + 15\mu - 16\lambda^2 + 30\lambda\mu}{40} \tag{57.4}$$

Horizontal velocity:

$$u = \sqrt{gd}\Big[(B_{00} + B_{10}\,\mathrm{cn}^2\theta + B_{20}\,\mathrm{cn}^4\theta) - \frac{1}{2}\left(\frac{z+d}{d}\right)^2 (B_{01} + B_{11}\,\mathrm{cn}^2\theta + B_{21}\,\mathrm{cn}^4\theta)\Big] \tag{58}$$

$$B_{00} = \epsilon(\lambda - \mu) + \epsilon^2\left(\frac{\lambda - \mu - 2\lambda^2 + 2\mu^2}{4}\right) \tag{58.1}$$

$$B_{10} = \epsilon + \epsilon^2\left(\frac{1 - 6\lambda + 2\mu}{4}\right) \tag{58.2}$$

$$B_{20} = -\epsilon^2 \tag{58.3}$$

$$B_{01} = \frac{3\lambda}{2}\epsilon^2 \tag{58.4}$$

$$B_{11} = 3\epsilon^2(1 - \lambda) \tag{58.5}$$

$$B_{21} = -\frac{9}{2}\epsilon^2 \tag{58.6}$$

Vertical velocity:

$$w = \sqrt{gd}\,\frac{4Kd\,\mathrm{csd}}{L} \times\left[\left(\frac{z+d}{d}\right)(B_{10} + 2B_{20}\,\mathrm{cn}^2\theta) - \frac{1}{6}\left(\frac{z+d}{d}\right)^3 (B_{11} + 2B_{21}\,\mathrm{cn}^2\theta)\right] \tag{59}$$

Horizontal acceleration:

$$\frac{\partial u}{\partial t} = \sqrt{gd}\,\left\{\left[B_{10} - \frac{1}{2}\left(\frac{z+d}{d}\right)^2 B_{11}\right]\frac{4K}{T}\,\mathrm{csd} + \left[B_{20} - \frac{1}{2}\left(\frac{z+d}{d}\right)^2 B_{21}\right]\frac{8K}{T}\,\mathrm{cn}^2\theta\,\mathrm{csd}\right\} \tag{60}$$

Vertical acceleration:

$$\frac{\partial w}{\partial t} = \sqrt{gd}\,\frac{4Kd}{L}\left\{\frac{8K}{T}\,\mathrm{csd}^2\left[\left(\frac{z+d}{d}\right)B_{20} - \frac{1}{6}\left(\frac{z+d}{d}\right)^3 B_{21}\right] + \left[\left(\frac{z+d}{d}\right)(B_{10} + 2B_{20}\,\mathrm{cn}^2\theta) - \frac{1}{6}\left(\frac{z+d}{d}\right)^3 (B_{11} + 2B_{21}\,\mathrm{cn}^2\theta)\right] \times \left[\mathrm{sn}\theta\,\mathrm{dn}\theta - \mathrm{cn}\theta\,\mathrm{dn}\theta + \kappa^2\mathrm{sn}\theta\,\mathrm{cn}\theta\right]\right\} \tag{61}$$

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FOURIER SERIES WAVE THEORY

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FOURIER SERIES WAVE THEORY

DESCRIPTION

This application yields various parameters for progressive waves of permanent form, as predicted by Fourier series approximation. It provides estimates for common engineering parameters such as water surface elevation, integral wave properties, and kinematics as functions of wave height, period, water depth, and position in the wave form which is assumed to exist on a uniform co-flowing current. Stokes first and second approximations for celerity (i.e., values of the mean Eulerian current or mean mass transport rate) may be specified. Fourier series of up to 25 terms may be selected to approximate the wave. In addition to providing kinematics at a given point in the wave, this application provides graphical presentations of kinematics over two wavelengths (at a given z coordinate), and the vertical profile of selected kinematics under the wave crest. The methodology is based upon a series of papers by J. D. Fenton (Reinecker and Fenton, 1981; Fenton, 1988a; Fenton, 1988b; Fenton, 1990) and R. J. Sobey (Sobey, Goodwin, Thieke, and Westberg, 1987). LINPACK routines (Dongarra et al., 1979) are used to solve the set of up to 60 simultaneous equations to determine the Fourier coefficients for the series.

INTRODUCTION

The accurate description of the steady wave is an important element in the design and analysis process. While linear wave theory has traditionally (and often correctly) been used for a large group of wave applications, nonlinear wave theories often provide significant improvements over linear wave theory descriptions, particularly where wavelengths are long or short relative to water depth. Recent developments and technology enable engineers to easily employ higher order wave theories to increase the accuracy of their estimates of wave properties. Today's common desktop microcomputers offer adequate computational power to implement many nonlinear wave theories.

Common practice includes the use of Stokes' theory in deep water, cnoidal theory in shallow water, and Fourier series wave theory in deep, transitional, and shallow water. Contemporary versions of Stokes' and cnoidal theory are readily accessible at fifth order, and Fourier series theory at any order, though 15^{th} to 20^{th} order appears adequate to resolve even highly nonlinear waves. All are capable of incorporating a uniform underlying current. Stokes and cnoidal theory have limited regions of physical validity, and at higher orders, can be quite accurate. The domain of applicability for Fourier approximation wave theory includes that of both Stokes and cnoidal theory. It is based upon a numerical approach, and substitutes computational effort for the limited domain of the above perturbation methods. It is easily tractable at any desired order (subject to machine precision), and has become a well-established, robust, and accurate engineering tool for the steady wave problem.

Several variations of Fourier series wave theory have been presented in recent years: (Chappelear, 1961; Dean, 1965, and 1974; Schwartz, 1974; Dalrymple, 1974; Cokelet, 1977; Chaplin, 1980; Reinecker and Fenton, 1981; Le Mehaute, et al., 1984; Fenton, 1988a, and 1988b). Differences between the various approaches are well summarized by Sobey (Sobey, et al., 1987), and can be generally categorized by choice of field variable (stream function versus velocity potential), form of the terms in the cosine series and behavior in deep water, inclusion of a current (none, uniform, or shear), and formulation, extensibility, treatment, and efficiency of numerical approaches for the problem. The methodology of Fenton (1988a and 1988b) is utilized here to formulate the problem and determine the Fourier series coefficients, while revised derivations for wave properties have been taken from Sobey (1988) and Klopman (1990).

General Assumptions and Terminology

A representative wave profile is illustrated in Figure 2-3-1.

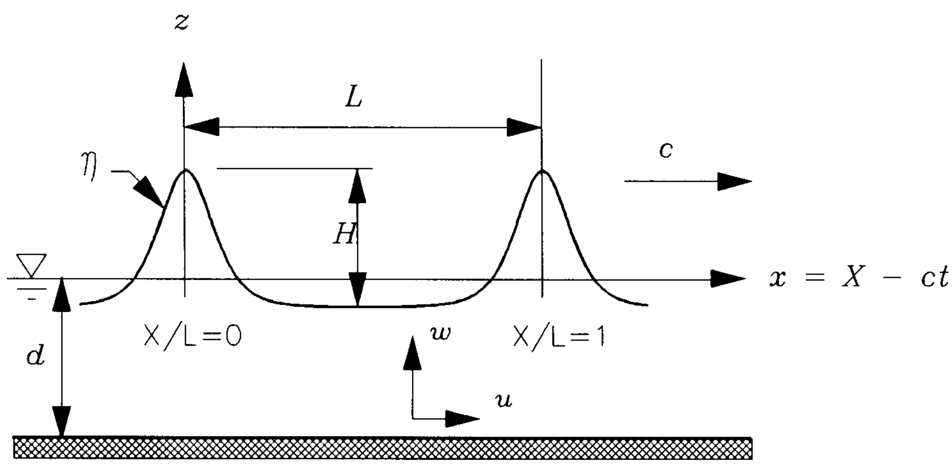

Figure 2-3-1. Wave in Steady Reference Frame (which Moves at Wave Speed c)

Assumptions applicable to the above progressive wave include:

- The wave is two-dimensional in the x-z plane.
- It propagates in a permanent form over a smooth horizontal bed of constant depth in the positive x-direction.
- There is a uniform underlying current.
- The fluid is inviscid and incompressible, having no surface tension.
- The flow is irrotational.

Terminology relevant to the following discussion includes:

(X, Z) = coordinates in Fixed (non-translating) Reference Frame

$(x, z) = (X - ct, Z)$ = coordinates in Steady Reference Frame (which moves at wave speed c)

d = still-water depth $(z = 0)$

$\eta(x)$ = water surface elevation

H = wave height

L = wavelength

T = wave period

$k = \frac{2\pi}{L}$ = wave number

c = wave speed

$\bar{u}$ = mean value of horizontal fluid velocity for a constant value of z, over one wavelength

$\overline{u_1}$ = time-mean Eulerian current (corresponding to Stokes *First* approximation for c)

$\overline{u_2}$ = depth-averaged mass transport velocity (corresponding to Stokes *Second* approximation for c)

$\psi(x, z)$ = stream function

Q = constant volume flow rate per unit width under the steady wave

$q = \bar{u}d - Q$ = constant volume flow rate per unit width due to wave

R = Bernoulli constant

$r = R - gd$ = Bernoulli constant (separated for convenience in treatment of deep water)

(U, W) = velocity components in Fixed Reference Frame

$(u, w) = (U - c, W)$ = velocity components in Steady Reference Frame

g = acceleration of gravity

ρ = density of water

GOVERNING EQUATION AND BOUNDARY CONDITIONS

The steady wave problem is a boundary value problem consisting of the two-dimensional Laplace equation as the governing field equation, boundary conditions at the free surface and bottom, and the assertion that the wave is periodic in space and time.

The wave problem has traditionally been formulated in three sets of variables: primitive variables (u, w), velocity potential ϕ, or stream function ψ. The assumption of two-dimensional, irrotational flow of an incompressible fluid leads to the following relationships between the three variables:

$$\psi_z = \phi_x = u \qquad -\psi_x = \phi_z = w \tag{1}$$

The formulation in stream function, and the adoption of a steady (i.e., moving at wave speed c) coordinate system (as depicted in Figure 2-3-1) simplifies the form of some of the boundary conditions for the problem. The governing equation and boundary conditions formulated in stream function for the selected coordinate system can be summarized as follows:

Governing (Laplace) Equation. The governing field equation is the two-dimensional Laplace equation. It represents the conservation of mass and momentum within the fluid field.

$$\frac{\partial^2 \psi(x,z)}{\partial x^2} + \frac{\partial^2 \psi(x,z)}{\partial z^2} = 0 \qquad (2)$$

Bottom Boundary Condition (BBC). The seabed is considered impermeable. Therefore, at the sea bottom, the flow rate through the bottom boundary must be 0.

$$\psi(x,-d) = 0 \qquad at \quad z = -d \qquad (3)$$

Kinematic Free Surface Boundary Condition (KFSBC). The fluid particle velocity normal to the free surface is equal to the velocity of the free surface itself. This condition implies that all water particles on the free surface remain there. Therefore, at the free surface, the flow rate through the boundary must also be 0. It also implies that the wave does not break.

$$\psi(x,\eta) = -Q \qquad at \quad z = \eta \qquad (4)$$

Dynamic Free Surface Boundary Condition (DFSBC). Expressed as the Bernoulli equation, this boundary condition asserts that the pressure at the free surface is constant (equal to atmospheric pressure). This requirement follows from the preliminary assumptions that atmospheric pressure above the free surface is itself constant and that surface tension may be neglected at the air-sea interface.

$$\frac{1}{2}\left\{\left(\frac{\partial \psi(x,\eta)}{\partial x}\right)^2 + \left(\frac{\partial \psi(x,\eta)}{\partial z}\right)^2\right\} + g\eta(x) = R \qquad at \quad z = \eta \qquad (5)$$

Periodicity. The wave is periodic in time and space.

$$\psi(x,t) = \psi(x+L,t) \qquad \psi(x,t) = \psi(x,t+T) \qquad (6)$$

Additional Considerations Relative to the Dispersion Relation

The adoption of a moving coordinate system (having the same velocity as the wave phase speed) simplifies the form of the KFSBC by eliminating a time derivative. However, this simplification yields some additional complexities which have been emphasized by Fenton (1988a, 1988b, 1990) and Sobey et al. (1987). The exact wave speed is correctly predicted by the wave theory only for the singular case of zero underlying current. It is more rational to assert that the wave theory predicts the wave phase speed relative to the underlying current, which is seldom zero in a realistic environment. A principal consequence of an underlying current is a Doppler shift of the apparent wave period (relative to a stationary observer or gage). Hence, the underlying current velocity must also be known in order to resolve the wave problem.

Some confusion results from consideration of the above concerns relative to the traditional definitions of Stokes (1847) for wave speed. Concerning the first definition, consider a given time-mean Eulerian current ($\overline{u_1}$) which is a time-mean horizontal fluid velocity at any point (X, Z) wholly in the fluid (within the fixed, non-moving frame of reference). This corresponds to the current recorded by a stationary meter. By considering $-\bar{u}$ to be the mean fluid velocity (i.e., at a fixed z coordinate, but averaged over one wavelength) in the steady (moving) frame of reference, the relationship between the wave speed and the time-mean Eulerian current applied to Stokes' first definition is:

$$c = \bar{u} + \overline{u_1} \tag{7}$$

The special case of $\overline{u_1} = 0$ corresponds exactly to Stokes first definition of wave speed where $c = \bar{u}$.

An alternate estimate of the wave phase speed is a function of the depth-averaged mass transport velocity (Stokes' drift velocity) $(\overline{u_2})$. By considering Q to be the volume flow rate (per unit width) under the wave in the steady (moving) frame of reference, the depth-averaged fluid velocity is then $-Q/d$. The relationship between wave speed and the mass transport velocity is suggested by Stokes' second definition as:

$$c = \frac{Q}{d} + \overline{u_2} \tag{8}$$

The special case of $\overline{u_2} = 0$ corresponds to Stokes' second definition of wave speed where $c = Q/d$. It is a condition that is commonly imposed in wave flumes or any situation where mass transport is actually or assumed to be restricted. It is not, however, generally true in open water sea states.

The above two equations include the principle of an underlying current in the dispersion relation and can represent the shift in wave period due to the current. A final consideration in formulating the dispersion relationship involves the boundary condition requiring the wave to be periodic in space and time. This condition implicitly requires the wave speed to satisfy the familiar relation $c = L/T$. The dispersion relation is declared in the wave problem by selecting the definition of wave speed, and providing the appropriate value for either the mean Eulerian velocity, or the Stokes drift velocity:

Stokes' *First* Approximation of Wave Speed	Stokes' *Second* Approximation of Wave Speed	
$c = \frac{L}{T} = \frac{2\pi}{kT} = \bar{u} + \overline{u_1}$	$c = \frac{L}{T} = \frac{2\pi}{kT} = \frac{Q}{d} + \overline{u_2}$	(9)
(specify mean Eulerian velocity $\overline{u_1}$)	(specify Stokes' drift velocity $\overline{u_2}$)	

Solution and Method

Fourier series wave theory derives its name from an approximate (but potentially very accurate) solution to the governing wave equation using a Fourier cosine series in kx, as follows:

$$\psi(x, z) = -\bar{u}(d + z) + \left(\frac{g}{k^3}\right)^{\frac{1}{2}} \sum_{j=1}^{N} B_j \frac{\sinh jk(d+z)}{\cosh jkd} \cos jkx \tag{10}$$

where

B_j = dimensionless Fourier coefficients

The solution is obtained by numerically computing the N Fourier coefficients to satisfy a system of simultaneous equations that consist of the two free-surface boundary conditions (evaluated at $N+1$ evenly spaced points on the free surface between a crest and following trough) as well as the dispersion relation when given the wave height, wave period, water depth, and either the mean Eulerian velocity, or the Stokes drift velocity. The set of dimensionless variables and equations as described by Fenton (1988a and 1988b) are listed in Tables 2-3-1 through 2-3-3.

Table 2-3-1
Vector of Dimensionless Problem Variables *(Fenton, 1988)*

Variable	Dimensionless Form	Variable	Dimensionless Form
depth	$z_1 = kd$	free surface elevation at crest ($x = 0$)	$z_{10} = k\eta_0$
wave height	$z_2 = kH$	free surface elevation at $\left(x = \frac{L}{2N}\right)$	$z_{11} = k\eta_1$
wave period	$z_3 = T(gk)^{1/2}$	free surface elevation at $\left(x = \frac{L}{N}\right)$	$z_{12} = k\eta_2$
wave speed	$z_4 = c(k/g)^{1/2}$	...	...
mean Eulerian velocity	$z_5 = \overline{u_1}(k/g)^{1/2}$	free surface elevation at trough $\left(x = \frac{L}{2}\right)$	$z_{N+10} = k\eta_N$
Stokes' drift velocity	$z_6 = \overline{u_2}(k/g)^{1/2}$	Fourier series coefficient	$z_{N+11} = \beta_1$
mean fluid velocity	$z_7 = \bar{u}(k/g)^{1/2}$	...	$z_{N+12} = \beta_2$
volume flow rate per unit width	$z_8 = q(k^3/g)^{1/2}$	...	...
Bernoulli constant	$z_9 = r(k/g)$	...	$z_{2N+10} = \beta_N$

Table 2-3-1 lists the name and order of dimensionless variables for use in the system of equations. Table 2-3-2 lists the relationships being formalized and their order as a system of equations to be solved simultaneously. Table 2-3-3 identifies some special forms of the equations used for numerical stability or speed in deepwater environments. Deepwater simplifications are automatically employed when $(d/L > 3/2)$ with L initially estimated by a linear wave theory approximation.

Table 2-3-2
System of Equations (Fenton, 1988)

Relation	Equation
H to d	$f_1(z_{1,2}) = kH - (H/d)kd = 0$
H to T	$f_2(z_{2,3}) = kH - \left(\frac{H}{gT^2}\right)(T(gk)^{1/2})^2 = 0$
$c = \frac{L}{T} = \frac{2\pi}{kT}$	$f_3(z_{3,4}) = c(k/g)^{1/2}T(gk)^{1/2} - 2\pi = 0$
$c = \bar{u} + \overline{u_1}$	$f_4(z_{4,5,7}) = \overline{u_1}(k/g)^{1/2} + \bar{u}(k/g)^{1/2} - c(k/g)^{1/2} = 0$
$c = \frac{Q}{d} + \overline{u_2}$	$f_5(z_{1,4,6,7,8}) = \overline{u_2}(k/g)^{1/2} + \bar{u}(k/g)^{1/2} - c(k/g)^{1/2} - \frac{q(k^3/g)^{1/2}}{kd} = 0$
apply selected approx for c $u_c = \overline{u_1}$ or $\overline{u_2}$	$f_6(z_{2,5or6}) = u_c(k/g)^{1/2} - \frac{u_c}{(gH)^{1/2}}(gH)^{1/2} = 0$
$\overline{\eta(x)} = 0$	$f_7(z_{10,N+10,m+10}) = k\eta_0 + k\eta_N + 2\sum_{m=1}^{N-1} k\eta_m = 0$
$H = \eta_0 - \eta_N$	$f_8(z_{2,10,N+10}) = k\eta_0 - k\eta_N - kH = 0$
KFSBC: solved at $m = 0 \ldots N$ points on free surface	$f_{m+9}(z_{1,7,8,m+10,N+j+10}) = -q\left(\frac{k^3}{g}\right)^{1/2} - k\eta_m\bar{u}\left(\frac{k}{g}\right)^{1/2} + \sum_{j=1}^{N} B_j\left[\frac{\sinh j(kd + k\eta_m)}{\cosh jkd}\right]\cos\frac{jm\pi}{N} = 0$
DFSBC: solved at $m = 0 \ldots N$ points on free surface	$f_{N+10+m}(z_{1,7,9,10,m+10,N+j+10}) = k\eta_m - \frac{rk}{g} + \frac{1}{2}\left(-\bar{u}\left(\frac{k}{g}\right)^{1/2} + \sum_{j=1}^{N} jB_j\left[\frac{\cosh j(kd + k\eta_m)}{\cosh jkd}\right]\cos\frac{jm\pi}{N}\right)^2 + \frac{1}{2}\left(\sum_{j=1}^{N} jB_j\left[\frac{\sinh j(kd + k\eta_m)}{\cosh jkd}\right]\sin\frac{jm\pi}{N}\right)^2 = 0$

Table 2-3-3 System of Equations (Special Variations for Deep Water) *(Fenton, 1988)*	
Relation	Equation
proxy	$f_1(z_1) = kd + 1 = 0$
$c = \frac{Q}{d} + \overline{u_2}$	$f_5(z_{4,6,7}) = \overline{u_2}(k/g)^{1/2} + \bar{u}(k/g)^{1/2} - c(k/g)^{1/2} = 0$
KFSBC: solved at $m = 0 \ldots N$ points on free surface	$f_{m+9}(z_{7,8,m+10,N+j+10}) = -q\left(\frac{k^3}{g}\right)^{1/2} - k\eta_m \bar{u}\left(\frac{k}{g}\right)^{1/2} + \sum_{j=1}^{N} B_j e^{jk\eta_m} \cos\frac{jm\pi}{N} = 0$
DFSBC: solved at $m = 0 \ldots N$ points on free surface	$f_{N+10+m}(z_{7,9,m+10,N+j+10}) = k\eta_m - \frac{rk}{g} + \frac{1}{2}\left(-\bar{u}\left(\frac{k}{g}\right)^{1/2} + \sum_{j=1}^{N} jB_j e^{jk\eta_m}\cos\frac{jm\pi}{N}\right)^2 + \frac{1}{2}\left(\sum_{j=1}^{N} jB_j e^{jk\eta_m}\sin\frac{jm\pi}{N}\right)^2 = 0$

Tables 2-3-1 and 2-3-2 describe a system of $2N + 10$ equations:

$$\mathbf{f}(\mathbf{z}) = \{f_i(\mathbf{z}),\ i = 1 \ldots 2N + 10\} = \mathbf{0} \tag{11}$$

The system of equations is solved iteratively using Newton's method:

$$\left[\frac{\partial f_i}{\partial z_j}\right]^{(n)} \cdot (\mathbf{z}^{(n+1)} - \mathbf{z}^{(n)}) = -\mathbf{f}(\mathbf{z}^{(n)}) \tag{12}$$

where

n = Iteration index

$\mathbf{z}^{(n)}$ = Values of $(z_i,\ i = 1 \ldots 2N + 10)$ at iteration n

$\mathbf{z}^{(n+1)}$ = Values of $(z_i,\ i = 1 \ldots 2N + 10)$ at iteration $n + 1$

$\mathbf{f}(\mathbf{z}^{(n)})$ = Values of $\{f_i(\mathbf{z}),\ i = 1 \ldots 2N + 10\}$ evaluated at iteration n

$\left[\frac{\partial f_i}{\partial z_j}\right]^{(n)}$ = Jacobian matrix evaluated at iteration n as:

$$\frac{\partial f_i}{\partial z_j} \approx \frac{f_i(z_1,\cdots,z_j+\Delta_j,\cdots,z_{2N+10}) - f_i(z_1,\cdots,z_j,\cdots,z_{2N+10})}{\Delta_j} \tag{13}$$

Note: The Jacobian matrix may be evaluated analytically or numerically. As indicated by Equation (*13*), in this implementation the Jacobian matrix is evaluated numerically. Each of $2N+10$ equations were evaluated a total of $2N+10$ times with the value of Δ_j as below:

$$\Delta_j = z_j/100 \quad for \quad z_j > 10^{-4}$$

$$\Delta_j = 10^{-5} for \quad z_j \leq 10^{-4}$$

Iterations continue until the solution converges to the following criteria:

$$\sum_{j=1}^{2N+10} \left| z_j^{(n+1)} - z_j^{(n)} \right| < 10^{-6} \tag{14}$$

within $(n_{max} \leq 9)$ iterations. Double precision LINPACK (Dongarra et al; 1979) routines are employed to solve the matrix Equation (*12*) at each iteration. As currently implemented in ACES, the square matrix is dimensioned to a maximum rank of *60*, and the solved matrix rank is $2N+10$, which allows a maximum of $(N_{max} = 25)$ terms in the Fourier series approximation.

Solution Aids and Insights

Several researchers (Dalrymple and Solana, 1986), (Fenton, 1988) and (Sobey, 1988) have noted that Fourier theory formulations can potentially admit the odd harmonics $(L, L/3, L/5, \ldots L/N)$ singularly, or in combinations, particularly in shallow water. Sobey (1988) notes that two features of the formulation (symmetry about crest, and periodicity at lateral boundary conditions) permit the manifestation of these apparent multicrested solutions. Several strategies for achieving the fundamental solution (rather than higher odd harmonics) have been discussed in the cited literature, and the method employed by Fenton (1988) is implemented in this version. A "ramping" mechanism is provided to solve the problem by initially (internally) solving for a lower wave height, and approaching the specified wave height in a specified (as input) number of evenly spaced steps. The approach eliminates the problem at the expense of additional iterations to achieve the final fundamental solution for the specified wave height.

Maximum Wave Checks

Most wave theories (both analytic and numerically based) are capable of yielding valid mathematical solutions to physically implausible data; particularly with regards to wave steepness and depth-related breaking. In part, this is a consequence of some of the assumptions imposed upon the boundary problem formulation. As an aid in restricting solutions to an observed physically valid domain, empirical data and formulations are often employed to estimate the validity of the given wave. The following expression (Fenton, 1990) is used for estimating the greatest wave as a function of both wavelength and depth:

$$H_{max} = d \left\{ \frac{.141063\frac{L}{d} + .0095721\left(\frac{L}{d}\right)^2 + .0077829\left(\frac{L}{d}\right)^3}{1 + .078834\frac{L}{d} + .0317567\left(\frac{L}{d}\right)^2 + .0093407\left(\frac{L}{d}\right)^3} \right\} \tag{15}$$

In the limits, the leading term in the numerator of the above expression provides the familiar steepness limit for short waves ($H_{max}/L \rightarrow .141063$), and as ($\lim L/d \rightarrow \infty$), the ratio of coefficients of the cubic terms provides the familiar ratio ($.0077829(L/d)^3/.0093407(L/d)^3 \rightarrow .83322$). This simple empirical test is applied using the given water depth, and solved wavelength as a rough filter for implausible wave specifications.

DERIVED RESULTS

Traditional engineering quantities of interest about the wave are derived from the solution of the governing equation. Since the solution is expressed as a Fourier series, many of the derived quantities will also be functions of the series. Formulas for kinematics, integral properties, and other relevant items are included in the following tabulations. All quantities are relative to the stationary (non-moving) frame of reference.

Kinematics and Other Derived Variables

Velocities:

Horizontal: $$u(x,z) = \frac{\partial \psi}{\partial z} = -\bar{u} + \left(\frac{g}{k}\right)^{\frac{1}{2}} \sum_{j=1}^{N} jB_j \frac{\cosh jk(d+z)}{\cosh jkd} \cos jkx \qquad (16)$$

Vertical: $$w(x,z) = -\frac{\partial \psi}{\partial x} = \left(\frac{g}{k}\right)^{\frac{1}{2}} \sum_{j=1}^{N} jB_j \frac{\sinh jk(d+z)}{\cosh jkd} \sin jkx \qquad (17)$$

Accelerations:

Horizontal: $$a_x(x,y) = \frac{du}{dt} = u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \qquad (18)$$

Vertical: $$a_z(x,y) = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = u\frac{\partial u}{\partial z} - w\frac{\partial u}{\partial x} \qquad (19)$$

where

$$\frac{\partial u}{\partial x} = -(gk)^{\frac{1}{2}} \sum_{j=1}^{N} j^2 B_j \frac{\cosh jk(d+z)}{\cosh jkd} \sin jkx$$

$$\frac{\partial u}{\partial z} = (gk)^{\frac{1}{2}} \sum_{j=1}^{N} j^2 B_j \frac{\sinh jk(d+z)}{\cosh jkd} \cos jkx$$

Pressure: $$p(x,y) = \rho r - \rho g z - \frac{1}{2}\rho(u^2 + w^2) \qquad (20)$$

Water Surface: $$\eta(x) = \sum_{j=1}^{N-1} f_j \cos jkx + \frac{1}{2} f_N \cos Nkx \qquad (21)$$

where

$$f_j = \frac{2}{N}\left\{\frac{1}{2}\eta_0 + \sum_{m=1}^{N-1} \eta_m \cos\frac{jm\pi}{N} + \frac{1}{2}\eta_N \cos j\pi\right\}$$

Notes:

$$X = x + ct \quad , \quad Z = z \qquad r = R - gd$$
$$U(X, Z, t) = u(x, z) + c$$
$$W(X, Z, t) = w(x, z)$$
$$P(X, Z, t) = p(x, z)$$

Integral Properties

Potential Energy: (per unit horizontal area)

$$E_P = \overline{\int_0^{\eta} (\rho g Z) dZ} \tag{22}$$

$$E_P = \frac{1}{2} \rho g \overline{\eta^2}$$

Momentum: (per unit horizontal area) (Impulse)

$$I = \overline{\int_{-d}^{\eta} (\rho U) dZ} \tag{23}$$

$$I = \rho (cd - Q)$$

Kinetic Energy: (per unit horizontal area)

$$E_K = \overline{\int_{-d}^{\eta} \left(\frac{1}{2} \rho (U^2 + W^2) \right) dZ} \tag{24}$$

$$E_K = \frac{1}{2} (cI - \rho \overline{u_1} Q)$$

Mean Square of Bed Velocity:

$$\overline{U_b^2} = \frac{1}{L} \int_0^L U^2 (X, -d, t) dX \tag{25}$$

$$\overline{U_b^2} = 2(R - gd) - c^2 + 2\overline{u_1} c$$

Energy Flux: (per unit length of crest) (Wave Power)

$$F = \overline{\int_{-d}^{\eta} \left(P + \frac{1}{2} \rho (U^2 + W^2) + \rho g Z \right) U dZ} \tag{26}$$

$$F = (3E_K - 2E_P - 2\overline{u_1} I) c + \frac{1}{2} \overline{U_b^2} (I + \rho c d)$$

Radiation Stress:

$$S_{xx} = \overline{\int_{-d}^{\eta} (P + \rho U^2) dZ} - \frac{1}{2} \rho g d^2 \tag{27}$$

$$S_{xx} = 4E_K - 3E_P + \rho d \overline{U_b^2} - 2\overline{u_1} I$$

Notes:

$$X = x + ct \quad , \quad Z = z \qquad P(X, Z, t) = p(x, z)$$
$$U(X, Z, t) = u(x, z) + c \qquad \overline{\int (\;\;)} \Rightarrow \textit{averaged over one wavelength}$$
$$W(X, Z, t) = w(x, z)$$

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