# LINEAR WAVE THEORY

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# LINEAR WAVE THEORY

## DESCRIPTION

This application yields first-order approximations for various parameters of wave motion as predicted by the wave theory bearing the same name (also known as small-amplitude, sinusoidal, or Airy theory). It provides estimates for common items of interest such as water surface elevation, general wave properties, particle kinematics, and pressure as functions of wave height and period, water depth, and position in the wave form.

## INPUT

The coordinate system and terminology used to define linear wave motion are shown in Figure 2-1-1.

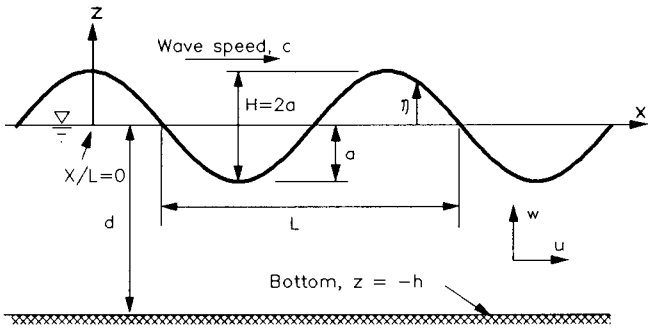


**Figure 2-1-1. Small-Amplitude Wave System**

All data input for this application is done on one screen. The following list describes the necessary input parameters with their corresponding units and range of data recognized by this application:

| Item | Symbol | Units | Data Range | | |
|---|---|---|---|---|---|
| Wave height | *H* | ft, m | 0.1 | to | 200.0 |
| Wave period | *T* | sec | 1.0 | to | 1000.0 |
| Water depth | *d* | ft, m | 0.1 | to | 5000.0 |
| Vertical coordinate | *z* | ft, m | -5100.0 | to | 100.0 |
| Horizontal coordinate as a fraction of wavelength | *X/L* | | 0.0 | to | 1.0 |

## OUTPUT

Results from this application are displayed on one screen. Those data include the original input values (in final units) and the following parameters:

| Item | Symbol | English Units | Metric Units |
|---|---|---|---|
| Wavelength | $L$ | ft | m |
| Wave celerity | $C$ | ft/sec | m/sec |
| Group velocity | $C_g$ | ft/sec | m/sec |
| Energy density | $E$ | ft-lb/ft$^2$ | N-m/m$^2$ |
| Energy flux | $P$ | ft-lb/sec-ft | N-m/sec-m |
| Ursell number | $U_r$ | | |
| Surface elevation | $\eta$ | ft | m |
| Horizontal particle displacement | $\xi$ | ft | m |
| Vertical particle displacement | $\zeta$ | ft | m |
| Horizontal particle velocity | $u$ | ft/sec | m/sec |
| Vertical particle velocity | $w$ | ft/sec | m/sec |
| Horizontal particle acceleration | $\partial u/\partial t$ | ft/sec$^2$ | m/sec$^2$ |
| Vertical particle acceleration | $\partial w/\partial t$ | ft/sec$^2$ | m/sec$^2$ |
| Pressure | $p$ | lb/ft$^2$ | N/m$^2$ |

## PROCEDURE

The bulleted items in the following lists indicate potentially optional instruction steps. Any application in ACES may be executed in a given session without quitting the program. The bulleted items provide instructions for accessing the application from various menu areas of the ACES program. Ignore bulleted instruction that are not applicable.

**Single Case Mode**

- Press [F1] on the Main Menu to select Single Case Mode.
- Fill in the highlighted input fields on the General Specifications screen (or leave the default values). Press [F1] when all data on this screen are correct.
- Press [F2] on the Functional Area Menu to select Wave Theory.
- Press [F1] on the Wave Theory Application Menu to select Linear Wave Theory.

1. Fill in the highlighted input fields on the Linear Wave Theory screen. Respond to any corrective instructions appearing at the bottom of the screen. Press [F1] when all data on this screen are correct.
2. All input and output data are displayed on the screen in the final system of units.
3. Press one of the following keys to select the appropriate action:

   [F1] Return to Step 1 for a new case.

   [F3] Send a summary of this case to the print file or device.

   [F10] Exit this application and return to the Wave Theory Application Menu.

**Multiple Case Mode**

- Press [F2] on the Main Menu to select Multi Case Mode.
- Fill in the highlighted input fields on the General Specifications screen (or leave the default values). Press [F1] when all data on this screen are correct.
- Press [F2] on the Functional Area Menu to select Wave Theory.
- Press [F1] on the Wave Theory Application Menu to select Linear Wave Theory.

1. Move the cursor to select a variable on the Linear Wave Theory screen (the selected variable name blinks). The current set of values for the variable is displayed on the right portion of the screen. When all variable sets are correct, go to Step 3.
2. Enter a set of values for the subject variable by following one of the input methods:

   a. Press [R] to select random method. Enter up to 20 values constituting a set for this variable (one in each field) on the right side of the screen. The set of 20 values originally displayed (first execution) in these fields contains the "delimiting" value, which "delimits" or "ends" the set. The "delimiting" value is *not* included as a member in the set unless it is the sole member.

   b. Press [I] to select incremental method. Fill in the fields for minimum, maximum, and increment values for this variable on the right side of the screen. In this method, the members of the set include all values from the minimum to the maximum (both inclusive) at the specified increment.

   The units field should also be specified for the variable regardless of input method. All members of a set of values for a subject variable are assigned the specified units. When all data are correct for the subject variable, press

[F10] to return to Step 1. Errors are reported at the bottom of the screen and are corrected by pressing [F1] to allow respecification of the data for the subject variable.

3. Press [F1] to process the cases resulting from the combinations of the sets of data for all variables. The summary of each case will be sent to the print file or device. The screen will display the total number of cases to be processed as well as report progress. Errors are reported at the bottom of the screen and are corrected by pressing [F1] to allow respecification of variable sets.
4. Press one of the following keys to select the appropriate action:

   [F1] Return to Step 1 to specify new sets.

   [F10] Exit this application and return to the Wave Theory Application Menu.

## EXAMPLE PROBLEM

### Input

All data input for this application is done on one screen. The values and corresponding units selected for this example are shown below.

| Item | Symbol | Value | Units |
|---|---|---|---|
| Wave height | $H$ | 6.30 | ft |
| Wave period | $T$ | 8.00 | sec |
| Water depth | $d$ | 20.00 | ft |
| Vertical coordinate | $z$ | -12.00 | ft |
| Horizontal coordinate as a fraction of wavelength | $X/L$ | 0.75 | |

### Output

Results from this application are displayed on one screen. Those data include the original input values and the following parameters (refer to Figure 2-1-2 for location of the parameters):

| Item | Symbol | Value | Units |
|---|---|---|---|
| Wavelength | $L$ | 189.90 | ft |
| Wave celerity | $C$ | 23.74 | ft/sec |

| | | | |
|---|---|---|---|
| Group velocity | $C_g$ | 20.87 | ft/sec |
| Energy density | $E$ | 317.45 | ft-lb/ft$^2$ |
| Energy flux | $P$ | 6625.07 | ft-lb/sec-ft |
| Ursell number | $U_r$ | 28.40 | |
| Surface elevation | $\eta$ | 0.00 | ft |
| Horizontal particle displacement | $\xi$ | 4.59 | ft |
| Vertical particle displacement | $\zeta$ | 0.00 | ft |
| Horizontal particle velocity | $u$ | 0.00 | ft/sec |
| Vertical particle velocity | $w$ | -0.93 | ft/sec |
| Horizontal particle acceleration | $\partial u/\partial t$ | -2.83 | ft/sec$^2$ |
| Vertical particle acceleration | $\partial w/\partial t$ | 0.00 | ft/sec$^2$ |
| Pressure | $p$ | 767.83 | lb/ft$^2$ |

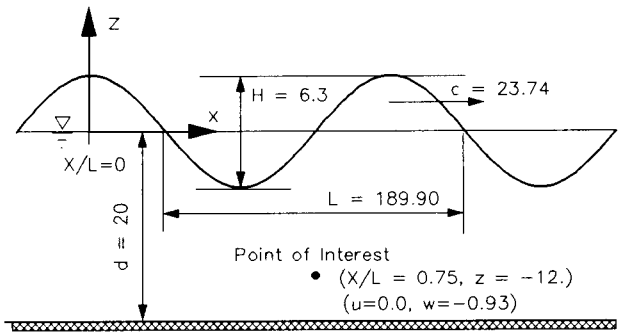


**Figure 2-1-2. Linear Wave Theory Example Output**

## REFERENCES AND BIBLIOGRAPHY

Airy, G. B. 1845. "Tides and Waves," *Encyclopaedia Metropolitana*, Vol. 192, pp. 241-396.

Dean, R. G., and Dalrymple, R. A. 1984. *Water Wave Mechanics for Engineers and Scientists*, Prentice-Hall, Englewood Cliffs, NJ, pp. 41-86.

Hunt, J. N. 1979. "Direct Solution of Wave Dispersion Equation," *Journal of Waterway, Port, Coastal and Ocean Division*, American Society of Civil Engineers, Vol. 105, No. WW4, pp. 457-459.

Sarpkaya, T., and Isaacson, M. 1981. *Mechanics of Wave Forces on Offshore Structures*, Van Nostrand Reinhold, New York, pp. 150-168.

*Shore Protection Manual*. 1984. 4th ed., 2 Vols., US Army Engineer Waterways Experiment Station, Coastal Engineering Research Center, US Government Printing Office, Washington, DC, Chapter 2, pp. 6-33.

Stokes, G. G. 1847. "On the Theory of Oscillatory Waves," *Transactions of the Cambridge Philosophical Society*, Vol. 8, pp. 441-455.

Ursell, F. 1953. "The Long-Wave Paradox in the Theory of Gravity Waves," *Proceedings of the Cambridge Philosophical Society*, Vol. 49, pp. 685-694.

# CNOIDAL WAVE THEORY

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# CNOIDAL WAVE THEORY

## DESCRIPTION

This application yields various parameters of wave motion as predicted by first-order (Isobe, 1985) and second-order (Hardy and Kraus, 1987) approximations for cnoidal wave theory. It provides estimates for common items of interest such as water surface elevation, general wave properties, kinematics, and pressure as functions of wave height and period, water depth, and position in the wave form.

## INPUT

The coordinate system and terminology used to define cnoidal wave motion is shown in Figure 2-2-1.

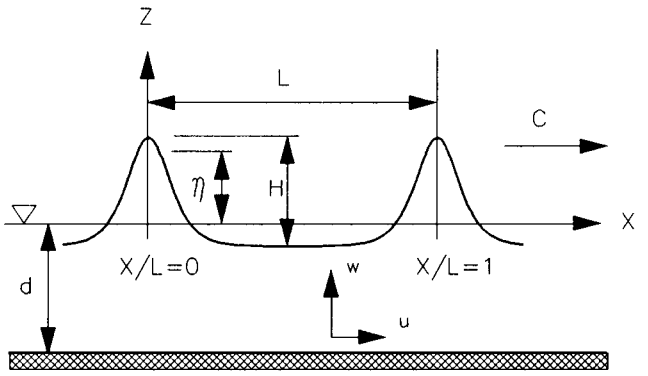


**Figure 2-2-1. Progressive Cnoidal Wave System**

All data input for this application is done on one screen. The following list describes the necessary input parameters with their corresponding units and range of data recognized by this application:

| Item | Symbol | Units | Data Range | | |
|---|---|---|---|---|---|
| Wave height | *H* | ft, m | 0.1 | to | 200.0 |
| Wave period | *T* | sec | 1.0 | to | 1000.0 |
| Water depth | *d* | ft, m | 0.1 | to | 5000.0 |
| Vertical coordinate | *z* | ft, m | -5100.0 | to | 100.0 |

| | | | | |
|---|---|---|---|---|
| Horizontal coordinate as a fraction of wavelength | $X/L$ | 0.0 | to | 1.0 |
| Order approximation | | 1 | or | 2 |

## OUTPUT

Results from this application are written to one screen. In addition, there is an option (available in Single Case Mode only) to send data to plot output file 1 (default name **PLOTDAT1.OUT**). This application also generates three screen plots. The three plots are described in the *Procedure* section of this document. The screen output and the content of plot output file 1 are described below (refer to the **Example Problem** section for a paradigm).

### Screen Output

Results which are displayed on one screen include the original input values (in final units) and the following parameters:

| Item | Symbol | English Units | Metric Units |
|---|---|---|---|
| Wavelength | $L$ | ft | m |
| Wave celerity | $C$ | ft/sec | m/sec |
| Energy density | $E$ | ft-lb/ft$^2$ | N-m/m$^2$ |
| Energy flux | $P$ | ft-lb/sec-ft | N-m/sec-m |
| Ursell number | $HL^2/d^3$ | | |
| Surface elevation | $\eta$ | ft | m |
| Horizontal velocity | $u$ | ft/sec | m/sec |
| Vertical velocity | $w$ | ft/sec | m/sec |
| Horizontal acceleration | $\partial u/\partial t$ | ft/sec$^2$ | m/sec$^2$ |
| Vertical acceleration | $\partial w/\partial t$ | ft/sec$^2$ | m/sec$^2$ |
| Pressure | $p$ | lb/ft$^2$ | N/m$^2$ |

### Plot Output File 1

Plot output file 1 contains water surface and velocity values across two wavelengths. The format of the file is described below.

| Field | Columns | Format | Data |
|---|---|---|---|
| 1 | 1-10 | F10.3 | ($X/L$) horizontal coordinate as a fraction of wavelength |
| 2 | 11-20 | F10.3 | ($\eta$) water surface elevation |
| 3 | 21-30 | F10.3 | ($u$) horizontal component of the water velocity |
| 4 | 31-40 | F10.3 | ($w$) vertical component of the water velocity |

## PROCEDURE

The bulleted items in the following lists indicate potentially optional instruction steps. Any application in ACES may be executed in a given session without quitting the program. The bulleted items provide instructions for accessing the application from various menu areas of the ACES program. Ignore bulleted instruction steps that are not applicable.

**Single Case Mode**

- Press [F1] on the Main Menu to select Single Case Mode.
- Fill in the highlighted input fields on the General Specifications screen (or leave the default values). Press [F1] when all data on this screen are correct.
- Press [F2] on the Functional Area Menu to select Wave Theory.
- Press [F2] on the Wave Theory Application Menu to select Cnoidal Wave Theory.

1. Fill in the highlighted input fields on the Cnoidal Wave Theory screen. Respond to any corrective instructions appearing at the bottom of the screen. Press [F1] when all data on this screen are correct.
2. All input and output data are displayed on the screen in the final system of units.
3. Press one of the following keys to select the appropriate action:

   [F1] Return to Step 1 for a new case.

   [F2] Invoke the Plot Menu screen (see the following section titled **Cnoidal Wave Theory Plot Menu**).

   [F3] Send a summary of this case to the print file or device.

   [F4] Generate a file containing plot data (Plot Output File 1).

[F10] Exit this application and return to the Wave Theory Application Menu.

### Cnoidal Wave Theory Plot Menu

This application generates three plots. The plots may be accessed from the **CNOIDAL WAVE THEORY PLOT MENU**, which appears when the **Plot Data** option ([F2] key) on the data output screen is selected. To access a plot, move the cursor (using the arrow keys) to the desired selection on the menu and press [F1]. (Appendix C describes options to customize plots.) Available selections are:

- Water Surface Elevation (see Figure 2-2-3)
- Horizontal Velocity (see Figure 2-2-4)
- Vertical Velocity (see Figure 2-2-5)
- ALL PLOTS

  **NOTE:** This option will make all the plots available for viewing. Use the **NEXT** option of the graphics package (Appendix C) to view each plot successively.
- EXIT MENU

### Multiple Case Mode

- Press [F2] on the Main Menu to select Multi Case Mode.
- Fill in the highlighted input fields on the General Specifications screen (or leave the default values). Press [F1] when all data on this screen are correct.
- Press [F2] on the Functional Area Menu to select Wave Theory.
- Press [F2] on the Wave Theory Application Menu to select Cnoidal Wave Theory.

1. Move the cursor to select a variable on the Cnoidal Wave Theory screen (the selected variable name blinks). The current set of values for the variable is displayed on the right portion of the screen. When all variable sets are correct, go to Step 3.
2. Enter a set of values for the subject variable by following <u>one</u> of the input methods:

a. Press [R] to select random method. Enter up to 20 values constituting a set for this variable (one in each field) on the right side of the screen. The set of 20 values originally displayed (first execution) in these fields contains the "delimiting" value, which "delimits" or "ends" the set. The "delimiting" value is *not* included as a member in the set unless it is the sole member.

b. Press [I] to select incremental method. Fill in the fields for minimum, maximum, and increment values for this variable on the right side of the screen. In this method, the members of the set include all values from the minimum to the maximum (both inclusive) at the specified increment.

The units field should also be specified for the variable regardless of input method. All members of a set of values for a subject variable are assigned the specified units. When all data are correct for the subject variable, press [F10] to return to Step 1. Errors are reported at the bottom of the screen and are corrected by pressing [F1] to allow respecification of the data for the subject variable.

3. Press [F1] to process the cases resulting from the combinations of the sets of data for all variables. The summary of each case will be sent to the print file or device. The screen will display the total number of cases to be processed as well as report progress. Errors are reported at the bottom of the screen and are corrected by pressing [F1] to allow respecification of variable sets.
4. Press one of the following keys to select the appropriate action:

   [F1] Return to Step 1 to specify new sets.

   [F10] Exit this application and return to the Wave Theory Application Menu.

**NOTE:** Multiple Case Mode does not generate any plot output files or plots.

## EXAMPLE PROBLEMS

### Example 1 - First-Order Approximation

#### Input

All data input for this application is done on one screen. The values and corresponding units selected for this first example are shown below.

| Item | Symbol | Value | Units |
|---|---|---|---|
| Wave height | $H$ | 10.00 | ft |
| Wave period | $T$ | 15.00 | sec |
| Water depth | $d$ | 25.00 | ft |
| Vertical coordinate | $z$ | -12.50 | ft |
| Horizontal coordinate as a fraction of wavelength | $X/L$ | 0.50 | |

### Output

Results from this application are written to one screen and, if requested, to plot output file 1 (default name **PLOTDAT1.OUT**). In addition, three screen plots are generated. Each of these outputs for the example problem is presented below.

### Screen Output

Results from this application are displayed on one screen. Those data include the original input values and the following parameters (see Figure 2-2-2):

| Item | Symbol | Value | Units |
|---|---|---|---|
| Wavelength | $L$ | 455.74 | ft |
| Wave celerity | $C$ | 30.38 | ft/sec |
| Energy density | $E$ | 621.52 | ft-lb/ft$^2$ |
| Energy flux | $P$ | 17625.85 | ft-lb/sec-ft |
| Ursell number | $HL^2/d^3$ | 132.93 | |
| Surface elevation | $\eta$ | -2.14 | ft |
| Horizontal velocity | $u$ | -2.43 | ft/sec |
| Vertical velocity | $w$ | 0.00 | ft/sec |
| Horizontal acceleration | $\partial u/\partial t$ | 0.00 | ft/sec$^2$ |
| Vertical acceleration | $\partial w/\partial t$ | 0.01 | ft/sec$^2$ |
| Pressure | $p$ | 643.23 | lb/ft$^2$ |

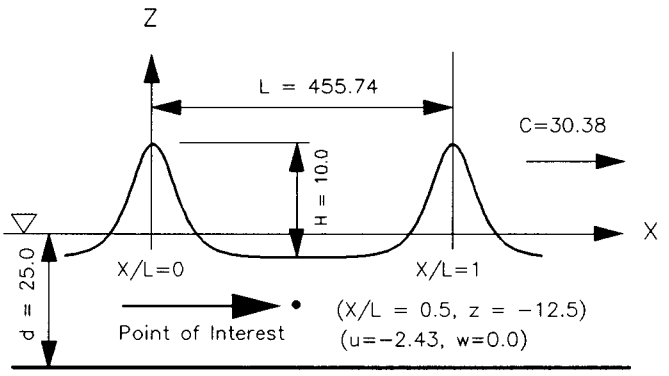


**Figure 2-2-2. Cnoidal Theory (First-Order Approximation) Example Output**

## Plot Output File 1

Table 2-2-1 below is a partial listing of plot output file 1 generated (if requested) by this application for the example problem.

Table 2-2-1
Partial Listing of Plot Output File 1 for Example Problem 1

| X/L | ETA (ft) | U (ft/sec) | W (ft/sec) |
|---|---|---|---|
| -1.000 | 7.860 | 8.917 | 0.000 |
| -0.992 | 7.805 | 8.854 | 0.430 |
| -0.984 | 7.641 | 8.668 | 0.841 |
| -0.976 | 7.376 | 8.367 | 1.216 |
| -0.968 | 7.020 | 7.963 | 1.541 |
| -0.960 | 6.588 | 7.474 | 1.807 |
| -0.952 | 6.096 | 6.916 | 2.008 |
| ⇓ | ⇓ | ⇓ | ⇓ |
| 0.952 | 6.096 | 6.916 | -2.008 |
| 0.960 | 6.588 | 7.474 | -1.807 |
| 0.968 | 7.020 | 7.963 | -1.541 |
| 0.976 | 7.376 | 8.367 | -1.216 |
| 0.984 | 7.641 | 8.668 | -0.841 |
| 0.992 | 7.805 | 8.854 | -0.430 |

## Screen Plots

This application generates three screen plots. Figures 2-2-3 through 2-2-5 are plots of the water surface elevation (ETA) and the horizontal (U) and vertical (W) water velocity as a function of the horizontal coordinate (X/L).

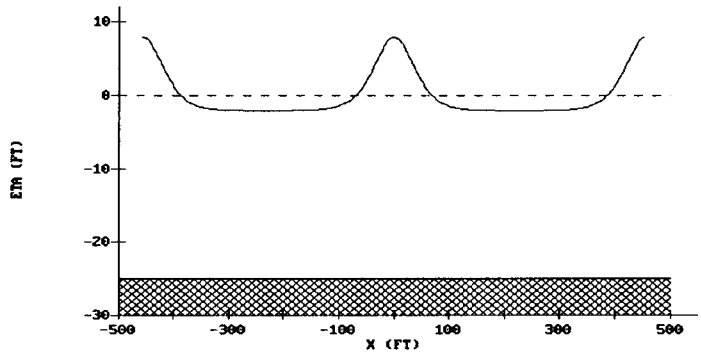


Figure 2-2-3. Water Surface Elevation

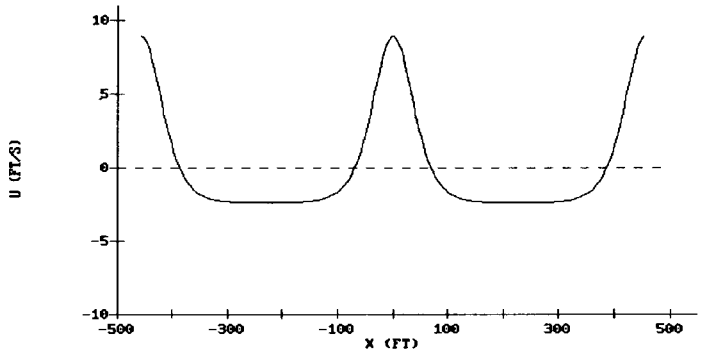


Figure 2-2-4. Horizontal Water Velocity

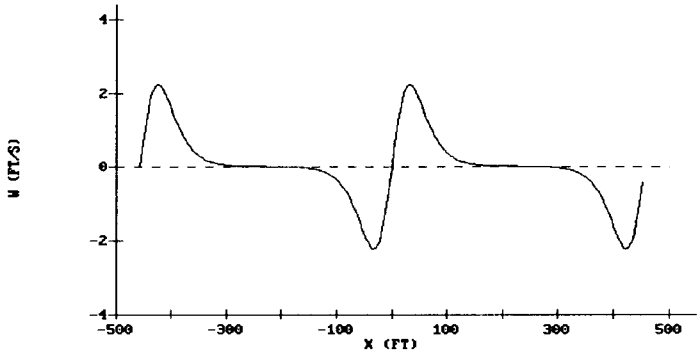


Figure 2-2-5. Vertical Water Velocity

### Example 2 - Second-Order Approximation

#### Input

All data input for this application is done on one screen. The values and corresponding units selected for this second example are shown below.

| Item | Symbol | Value | Units |
|---|---|---|---|
| Wave height | $H$ | 10.00 | ft |
| Wave period | $T$ | 15.00 | sec |
| Water depth | $d$ | 25.00 | ft |
| Vertical coordinate | $z$ | -12.50 | ft |
| Horizontal coordinate as a fraction of wavelength | $X/L$ | 0.50 | |

#### Output

Results from this application are written to one screen and, if requested, to plot output file 1 (default name **PLOTDAT1.OUT**). In addition, three screen plots are generated. Each of these outputs for the example problem is presented below.

#### Screen Output

Results from this application are displayed on one screen. Those data include the original input values, and the following parameters (see Figure 2-2-6):

| Item | Symbol | Value | Units |
|---|---|---|---|
| Wavelength | $L$ | 445.78 | ft |
| Wave celerity | $C$ | 29.72 | ft/sec |
| Energy density | $E$ | 614.59 | ft-lb/ft$^2$ |
| Energy flux | $P$ | 17154.86 | ft-lb/sec-ft |
| Ursell number | $HL^2/d^3$ | 127.18 | |
| Surface elevation | $\eta$ | -2.03 | ft |
| Horizontal velocity | $u$ | -2.70 | ft/sec |
| Vertical velocity | $w$ | 0.00 | ft/sec |
| Horizontal acceleration | $\partial u/\partial t$ | 0.00 | ft/sec$^2$ |
| Vertical acceleration | $\partial w/\partial t$ | 0.01 | ft/sec$^2$ |
| Pressure | $p$ | 673.11 | lb/ft$^2$ |

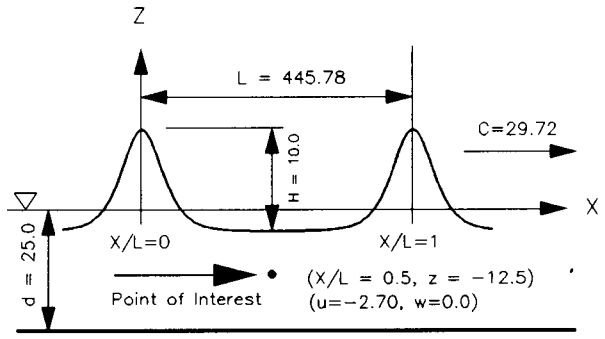


Figure 2-2-6. Cnoidal Theory (Second-Order Approximation) Example Output

## Plot Output File 1

Table 2-2-2 below is a partial listing of plot output file 1 generated (if requested) by this application for the second example problem.

Table 2-2-2
Partial Listing of Plot Output File 1 for Example Problem 2

| X/L | ETA (ft) | U (ft/sec) | W (ft/sec) |
|---|---|---|---|
| -1.000 | 7.972 | 6.601 | 0.000 |
| -0.992 | 7.907 | 6.564 | 0.177 |
| -0.984 | 7.716 | 6.455 | 0.355 |
| -0.976 | 7.408 | 6.277 | 0.533 |
| -0.968 | 7.000 | 6.033 | 0.708 |
| -0.960 | 6.510 | 5.730 | 0.879 |
| -0.952 | 5.959 | 5.374 | 1.040 |
| -0.944 | 5.369 | 4.975 | 1.185 |
| -0.936 | 4.761 | 4.542 | 1.309 |
| ⇓ | ⇓ | ⇓ | ⇓ |
| 0.944 | 5.369 | 4.975 | -1.185 |
| 0.952 | 5.959 | 5.374 | -1.040 |
| 0.960 | 6.510 | 5.730 | -0.879 |
| 0.968 | 7.000 | 6.033 | -0.708 |
| 0.976 | 7.408 | 6.277 | -0.533 |
| 0.984 | 7.716 | 6.455 | -0.355 |
| 0.992 | 7.907 | 6.564 | -0.177 |

## Screen Plots

This application generates three screen plots. Figures 2-2-7 through 2-2-9 are plots of the water surface elevation (ETA) and the horizontal (U) and vertical (W) water velocity as a function of the horizontal coordinate (X/L).

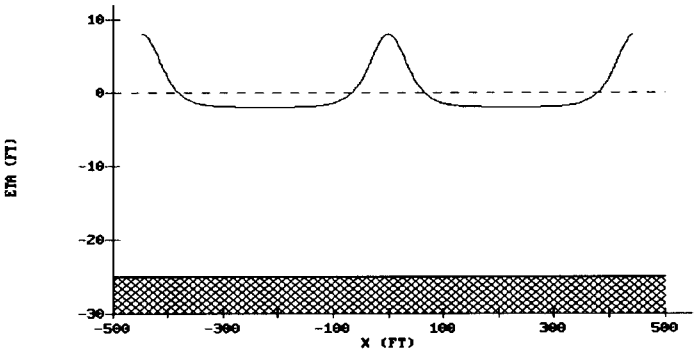


Figure 2-2-7. Water Surface Elevation

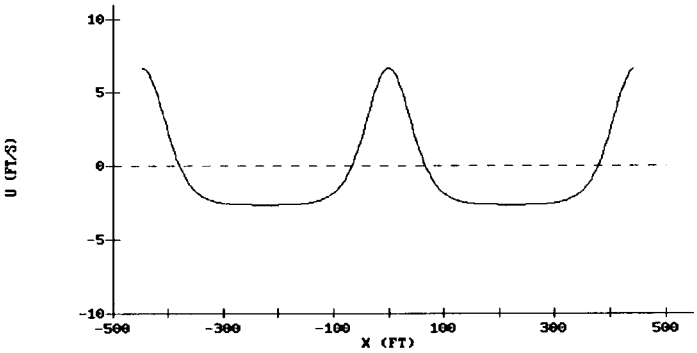


Figure 2-2-8. Horizontal Water Velocity

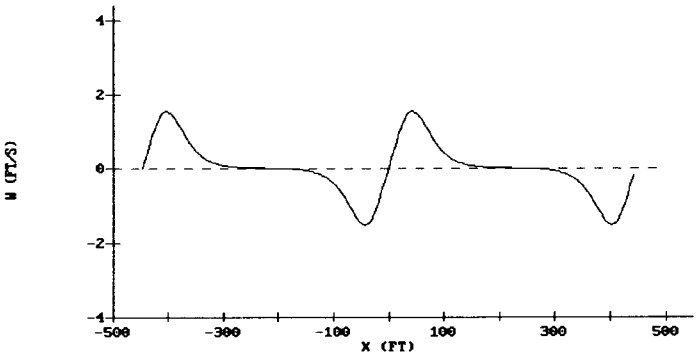


Figure 2-2-9. Vertical Water Velocity

## REFERENCES AND BIBLIOGRAPHY

Abramowitz, M., and Stegun, I. A. 1972. *Handbook of Mathematical Functions*, Dover Publications, New York, 1046 pp.

Chappelear, J. E. 1962. "Shallow Water Waves," *Journal of Geophysical Research*," Vol. 67, No. 12, pp. 4693-4704.

Davis, H. T. 1962. *Introduction to Nonlinear Differential and Integral Equations*, Dover Publications, New York, 596 pp.

Fenton, J. D. 1979. "A High Order Cnoidal Wave Theory," *Journal of Fluid Mechanics*, Vol. 94, pp. 129-161.

Hardy, T. A., and Kraus, N. C. 1987. "A Numerical Model for Shoaling and Refraction of Second-Order Cnoidal Waves over an Irregular Bottom," Miscellaneous Paper CERC-87-9, US Army Engineer Waterways Experiment Station, Vicksburg, MS.

Isobe, M. 1985. "Calculation and Application of First-Order Cnoidal Wave Theory," *Coastal Engineering*, Vol. 9, pp. 309-325.

Isobe, M., and Kraus, N. C. 1983. "Derivation of a Second-Order Cnoidal Wave Theory," Hydraulics Laboratory Report No. YNU-HY-83-2, Department of Civil Engineering, Yokohama National University, 43 pp.

Keller, J. B. 1948. "The Solitary Wave and Periodic Waves in Shallow Water," *Communication of Pure and Applied Mathematics*, Vol. 1, pp. 323-339.

Keulegan, G. H., and Patterson, G. W. 1940. "Mathematical Theory of Irrotational Translation Waves," *Journal of Research of the National Bureau of Standards*, Vol. 24, pp. 47-101.

Korteweg, D. J., and de Vries, G. 1895. "On the Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves," *Philosophy Magazine*, Series 5, Vol. 39, pp. 422-443.

Laitone, E. V. 1960. "The Second Approximation to Cnoidal and Solitary Waves," *Journal of Fluid Mechanics*, Vol. 9, pp. 430-444.

*Shore Protection Manual*. 1984. 4th ed., 2 Vols., US Army Engineer Waterways Experiment Station, Coastal Engineering Research Center, US Government Printing Office, Washington, DC, Chapter 2, pp. 44-55.

Stokes, G. G. 1847. "On the Theory of Oscillatory Waves," *Transactions of the Cambridge Philosophical Society*, Vol. 8, pp. 441-455.

# FOURIER SERIES WAVE THEORY

## TABLE OF CONTENTS

# FOURIER SERIES WAVE THEORY

## DESCRIPTION

This application yields various parameters for progressive waves of permanent form, as predicted by Fourier series approximation. It provides estimates for common engineering parameters such as water surface elevation, integral wave properties, and kinematics as functions of wave height, period, water depth, and position in the wave form which is assumed to exist on a uniform co-flowing current. Stokes first and second approximations for celerity (i.e., values of the mean Eulerian current or mean mass transport rate) may be specified. Fourier series of up to 25 terms may be selected to approximate the wave. In addition to providing kinematics at a given point in the wave, this application provides graphical presentations of kinematics over two wavelengths (at a given z coordinate), and the vertical profile of selected kinematics under the wave crest. The methodology is based upon a series of papers by J. D. Fenton (Reinecker and Fenton, 1981; Fenton, 1988a; Fenton, 1988b; Fenton, 1990) and R. J. Sobey (Sobey, Goodwin, Thieke, and Westberg, 1987). LINPACK routines (Dongarra et al., 1979) are used to solve the set of up to 60 simultaneous equations to determine the Fourier coefficients for the series.

## PROCEDURE

This section provides instructions for running this application in the **Single Case** mode. The **Multiple Case** mode is not available.

- Press F1 on the Main Menu to select Single Case Mode.
- Fill in the highlighted input fields on the General Specifications screen (or leave the default values). Press F1 when all data on this screen are correct.
- Press F2 on the Functional Area Menu to select Wave Theory.
- Press F3 on the Wave Theory Application Menu to select Fourier Series Wave Theory.

## Input

The coordinate system and terminology used to define Fourier series wave motion are shown in Figure 2-3-1.

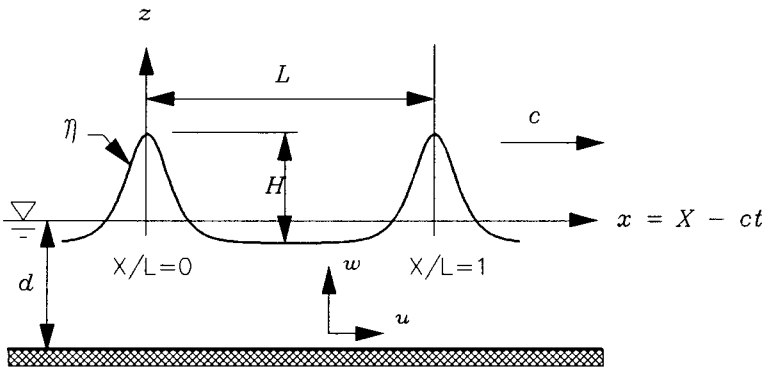


Figure 2-3-1. Progressive Fourier Series Wave System

Initial data input for this application is done on one screen. The following list describes the necessary input parameters with their corresponding units and range of data recognized by this application:

| Item | Symbol | Units | Data Range | | |
|---|---|---|---|---|---|
| Wave height | $H$ | ft, m | 0.1 | to | 200.0 |
| Wave period | $T$ | sec | 1.0 | to | 1000.0 |
| Water depth | $d$ | ft, m | 0.1 | to | 5000.0 |
| Celerity Definition | | | E(Euler) | or | S(Stokes) |
| Mean Velocity | $u$ | fps, mps | 1.0 | to | 10.0 |
| Number of terms in Fourier Series | | | 1 | to | 25 |
| Number of steps in Wave Height ramping | | | 1 | to | 10 |

In addition to the above input the user has the option to request kinematics (horizontal and vertical velocity and acceleration, pressure, and water surface elevation) at a selected point of interest. This option is presented to the user only after computations are performed using the above initial input values. The option to get kinematics is offered on the *second* screen displaying output and is described in a later section.

When the required input data on the screen are correct, press one of the following keys to select the next appropriate action:

F1 Perform computations.

F10 Exit this application and return to the Wave Theory Application Menu.

**Output**

Results from this application are displayed on two screens. In addition, there is an option to send data to plot output file 1 (default name **PLOTDAT1.OUT**). This application also generates nine screen plots. These various outputs are described in the following sections.

**Screen Output**

*First Screen*

Results which are displayed on the first screen include the original input values (in final units) and the following parameters:

| Item | Symbol | English Units | Metric Units |
|---|---|---|---|
| Celerity | | ft/sec | m/sec |
| Wavelength | | ft | m |
| Mean Eulerian Fluid Velocity | | ft/sec | m/sec |
| Mean Mass Transport Velocity | | ft/sec | m/sec |
| Mean Velocity relative to Wave | | ft/sec | m/sec |
| Volume Flux | | $ft^2/sec$ | $m^2/sec$ |
| Bernoulli Constant | | $ft^2/sec^2$ | $m^2/sec^2$ |
| Impulse | $I$ | $lb\text{-}sec/ft^2$ | $N\text{-}sec/m^2$ |
| Kinetic Energy | $E_K$ | $ft\text{-}lb/ft^2$ | $N\text{-}m/m^2$ |
| Potential Energy | $E_P$ | $ft\text{-}lb/ft^2$ | $N\text{-}m/m^2$ |
| Energy density | $E_K+E_P$ | $ft\text{-}lb/ft^2$ | $N\text{-}m/m^2$ |
| Mean Square of Bed Velocity | $U_b{}^2$ | $ft^2/sec_2$ | $m^2/sec^2$ |
| Radiation Stress | $S_{xx}$ | $ft\text{-}lb/ft^2$ | $N\text{-}m/m^2$ |
| Wave Power (Energy flux) | $F$ | ft-lb/sec-ft | N-m/sec-m |
| Volume Flux | $Q$ | $ft^2/sec$ | $m^2/sec$ |
| Bernoulli Constant | $R$ | $ft^2/sec^2$ | $m^2/sec^2$ |

After viewing the first output screen press one of the following keys to select the next appropriate action:

- [F1] Return to the input screen for a new case.
- [F2] Send a summary of this case to the print file or device.
- [F5] Invoke the next screen to view additional output results and request kinematics at a particular point of interest.
- [F10] Exit this application and return to the Wave Theory Application Menu.

*Second Screen*

Results displayed on the second screen include the wave surface elevations at the crest and trough, the dimensionless Fourier coefficients, and *if desired*, the following kinematics at a selected point of interest.

| | | | |
|---|---|---|---|
| Velocity (horizontal and vertical) | $U, W$ | ft/sec | m/sec |
| Acceleration (horizontal and vertical) | $a_x, a_z$ | ft/sec$^2$ | m/sec$^2$ |
| Pressure | $p$ | lb/ft$^2$ | N/m$^2$ |
| Water surface elevation | $\eta$ | ft | m |

After viewing the second output screen, press one of the following keys to select the next appropriate action:

- [y] Invoke the *requestor* for entering the coordinates where kinematics are desired.
- [F1] Return to Previous Screen.
- [F10] Exit this application and return to the Wave Theory Application Menu.

When [y] is pressed, a *requestor* is displayed requesting the point of interest where kinematics are desired. Enter a horizontal $(X/L)$ and vertical $Z$ coordinate where kinematics are desired. The horizontal coordinate is entered as a fraction of the wavelength where 1.0 is the crest and 0.5 is the trough. The vertical coordinate is entered as a distance below the wave surface. After the coordinates are entered press [Alt][F1] to accept the values and make the computations, or press [Alt][F10] to exit the *requestor* and return to the previous screen.

After the kinematic computations have been performed and displayed press one of the following keys to select the next appropriate action:

- [F1] Return to Previous Screen.
- [F2] Invoke the Plot Menu screen (see section **Screen Plots**).
- [F3] Print the Kinematics.
- [F4] Generate a file containing plot data (see section **Plot Output File 1**).
- [F5] Invoke *requestor* for new kinematics location.
- [F10] Exit this application and return to the Wave Theory Application Menu.

**Plot Output File 1**

Plot output file 1 contains two sections. The first section includes horizontal and vertical velocity and acceleration, pressure, and water surface elevation at Z as a function of the horizontal coordinate (X/L). The second section contains the horizontal velocity, vertical acceleration, and pressure under the wave crest.

Section 1 of the plot output file 1

Kinematics at Z (across two wavelengths)

| Field | Columns | Format | Data |
|---|---|---|---|
| 1 | 1-10 | F10.3 | counter |
| 2 | 11-20 | F10.3 | $(X/L)$ horizontal coordinate as a fraction of wavelength |
| 3 | 21-30 | F10.3 | $(\eta)$ water surface elevation |
| 4 | 31-40 | F10.3 | $(U)$ horizontal component of water velocity |
| 5 | 41-50 | F10.3 | $(W)$ vertical component of water velocity |
| 6 | 51-60 | F10.3 | $(p)$ pressure |
| 7 | 61-70 | F10.3 | $(a_x)$ horizontal acceleration |
| 8 | 71-80 | F10.3 | $(a_z)$ vertical acceleration |

Section 2 of the plot output file 1

Kinematics Under Wave Crest

| Field | Columns | Format | Data |
|---|---|---|---|
| 1 | 1-10 | F10.3 | counter |
| 2 | 11-20 | F10.3 | $(Z)$ vertical coordinate under wave crest |
| 3 | 21-30 | F10.3 | $(\eta)$ water surface elevation |
| 4 | 31-40 | F10.3 | $(U)$ horizontal component of water velocity |
| 5 | 41-50 | F10.3 | $(p)$ pressure |
| 6 | 51-60 | F10.3 | $(a_z)$ vertical acceleration |

### Screen Plots

This application generates nine plots. The plots may be accessed from the **FOURIER SERIES WAVE THEORY PLOT MENU (KINEMATICS)**, which appears when the **Plot Data** option (F2 key) on the *second* data output screen is selected. To access a plot, move the cursor (using the arrow keys) to the desired selection on the menu and press F1. (Appendix C describes options to customize plots.) Available selections are:

(at Z) Across Two Wavelengths

- Horizontal Velocity (see Figure 2-3-2 of Example Problem)
- Vertical Velocity (see Figure 2-3-3 of Example Problem)
- Horizontal Acceleration (see Figure 2-3-4 of Example Problem)
- Vertical Acceleration (see Figure 2-3-5 of Example Problem)
- Pressure (see Figure 2-3-6 of Example Problem)
- Water Surface Elevation (see Figure 2-3-7 of Example Problem)

Under Wave Crest

- Horizontal Velocity (see Figure 2-3-8 of Example Problem)
- Vertical Acceleration (see Figure 2-3-9 of Example Problem)
- Pressure (see Figure 2-3-10 of Example Problem)

- ALL PLOTS

  **NOTE:** This option will make all the plots available for viewing. Use the **NEXT** option of the graphics package (Appendix C) to view each plot successively.

- EXIT MENU

## EXAMPLE PROBLEM

### Input

Initial data input for this application is done on one screen. The following list describes the necessary input parameters with their corresponding units and range of data recognized by this application:

| Item | Symbol | Value | Units |
|---|---|---|---|
| Wave height | $H$ | 4.50 | ft |
| Wave period | $T$ | 9.00 | sec |
| Water depth | $d$ | 22.00 | ft |
| Celerity Definition | | E(Euler) | |
| Mean Velocity | $u$ | 0.00 | ft/s |
| Number of terms if Fourier Series | | 16 | |
| Number of steps in Wave Height ramping | | 5 | |

### Output

Results from this application are displayed on two screens. In addition, there is an option to send data to plot output file 1 (default name **PLOTDAT1.OUT**). This application also generates nine screen plots. The plots are described in the *Procedure* section of this document. The screen output and the content of plot output file 1 are described below (refer to the **Example Problem** section for a paradigm).

## Screen Output

*First Screen*

Results which are displayed on the first screen include the original input values (in final units) and the following parameters:

| Item | Symbol | Value | Units |
|---|---|---|---|
| Celerity | | 25.620 | ft/sec |
| Wavelength | | 230.581 | ft |
| Mean Eulerian Fluid Velocity | | 0.000 | ft/sec |
| Mean Mass Transport Velocity | | 0.140 | ft/sec |
| Mean Velocity relative to Wave | | 25.620 | ft/sec |
| Volume Flux | | 3.088 | $ft^2/sec$ |
| Bernoulli Constant | | 329.518 | $ft^2/sec^2$ |
| Impulse | $I$ | 0.61411E+01 | $lb\text{-}sec/ft^2$ |
| Kinetic Energy | $E_K$ | 78.667 | $ft\text{-}lb/ft^2$ |
| Potential Energy | $E_P$ | 77.306 | $ft\text{-}lb/ft^2$ |
| Energy density | $E_K+E_P$ | 155.97 | $ft\text{-}lb/ft^2$ |
| Mean Square of Bed Velocity | $U_b^2$ | 2.6479 | $ft^2/sec_2$ |
| Radiation Stress | $S_{xx}$ | 198.62 | $ft\text{-}lb/ft^2$ |
| Wave Power (Energy flux) | $F$ | 3577.6 | ft-lb/sec-ft |
| Volume Flux | $Q$ | 560.55 | $ft^2/sec$ |
| Bernoulli Constant | $R$ | 1037.3 | $ft^2/sec^2$ |

*Second Screen*

Results displayed on the second screen include the wave surface elevations at the crest and trough, the dimensionless Fourier coefficients, and *if desired*, the following kinematics at the selected point of interest.

| Point of interest | $(x/L, Z)$ | Horizontal 0.000 | Vertical -5.000 ft | |
|---|---|---|---|---|
| Velocity (horizontal and vertical) | $U, W$ | 3.150 | 0.000 | ft/sec |
| Acceleration (horizontal and vertical) | $a_x, a_z$ | 0.000 | -1.302 | $ft/sec^2$ |
| Pressure | $p$ | 473.223 | | $lb/ft^2$ |
| Water surface elevation | $\eta$ | 2.803 | | ft |

## Plot Output File 1

Table 2-3-1 below is a partial listing of plot output file 1 generated (if requested) by this application for the example problem.

Table 2-3-1
Partial Listing of Plot Output File 1

Section 1 of the plot output file 1

Kinematics at Z (across 2 wavelengths)

| | X/L | ETA (ft) | U(x/L,z) (ft/s) | W(x/L,z) (ft/s) | PRESSURE ($lb/ft^2$) | $a_x$(x/L,z) ($ft/s^2$) | $a_z$(x/L,z) ($ft/s^2$) |
|---|---|---|---|---|---|---|---|
| 1 | -1.000 | 2.803 | 3.150 | 0.000 | 473.223 | 0.000 | -1.302 |
| 2 | -0.992 | 2.795 | 3.143 | 0.107 | 472.899 | 0.177 | -1.297 |
| 3 | -0.984 | 2.773 | 3.122 | 0.213 | 471.927 | 0.352 | -1.280 |
| 4 | -0.976 | 2.737 | 3.087 | 0.317 | 470.318 | 0.525 | -1.252 |
| 5 | -0.968 | 2.686 | 3.039 | 0.419 | 468.082 | 0.693 | -1.215 |
| 6 | -0.960 | 2.623 | 2.978 | 0.517 | 465.238 | 0.856 | -1.167 |
| 7 | -0.952 | 2.547 | 2.904 | 0.612 | 461.808 | 1.012 | -1.110 |
| 8 | -0.944 | 2.460 | 2.819 | 0.701 | 457.820 | 1.160 | -1.044 |
| 9 | -0.936 | 2.362 | 2.722 | 0.786 | 453.303 | 1.300 | -0.971 |
| 10 | -0.928 | 2.255 | 2.615 | 0.865 | 448.294 | 1.429 | -0.892 |
| ⇓ | ⇓ | ⇓ | ⇓ | ⇓ | ⇓ | ⇓ | ⇓ |
| 241 | 0.920 | 2.140 | 2.499 | -0.937 | 442.830 | -1.548 | -0.807 |
| 242 | 0.928 | 2.255 | 2.615 | -0.865 | 448.295 | -1.429 | -0.892 |
| 243 | 0.936 | 2.362 | 2.722 | -0.786 | 453.305 | -1.300 | -0.971 |
| 244 | 0.944 | 2.460 | 2.819 | -0.701 | 457.821 | -1.160 | -1.044 |
| 245 | 0.952 | 2.547 | 2.904 | -0.612 | 461.810 | -1.012 | -1.110 |
| 246 | 0.960 | 2.623 | 2.978 | -0.517 | 465.239 | -0.856 | -1.167 |
| 247 | 0.968 | 2.686 | 3.039 | -0.419 | 468.083 | -0.693 | -1.215 |
| 248 | 0.976 | 2.737 | 3.088 | -0.317 | 470.318 | -0.525 | -1.252 |
| 249 | 0.984 | 2.773 | 3.122 | -0.213 | 471.928 | -0.352 | -1.280 |
| 250 | 0.992 | 2.795 | 3.143 | -0.107 | 472.899 | -0.177 | -1.297 |
| 251 | 1.000 | 2.803 | 3.150 | 0.000 | 473.223 | 0.000 | -1.302 |

(Table 2-3-1 Continued on the Next Page)

(Table 2-3-1 Concluded)

Section 2 of the plot output file 1

Kinematics Under Wave Crest

| | Z (ft) | ETA (ft) | U(x/L,z) (ft/s) | PRESSURE (lb/ft$^2$) | $a_z$(x/L,z) (ft/s$^2$) |
|---|---|---|---|---|---|
| 1 | -22.000 | 2.803 | 2.682 | 1539.861 | 0.000 |
| 2 | -21.901 | 2.803 | 2.682 | 1533.514 | -0.007 |
| 3 | -21.802 | 2.803 | 2.682 | 1527.168 | -0.014 |
| 4 | -21.702 | 2.803 | 2.682 | 1520.823 | -0.021 |
| 5 | -21.603 | 2.803 | 2.683 | 1514.480 | -0.028 |
| 6 | -21.504 | 2.803 | 2.683 | 1508.138 | -0.035 |
| 7 | -21.405 | 2.803 | 2.683 | 1501.797 | -0.042 |
| 8 | -21.306 | 2.803 | 2.683 | 1495.458 | -0.049 |
| 9 | -21.206 | 2.803 | 2.683 | 1489.120 | -0.056 |
| 10 | -21.107 | 2.803 | 2.684 | 1482.784 | -0.063 |
| ⇓ | ⇓ | ⇓ | ⇓ | ⇓ | ⇓ |
| 243 | 2.009 | 2.803 | 3.667 | 47.563 | -1.995 |
| 244 | 2.108 | 2.803 | 3.676 | 41.609 | -2.006 |
| 245 | 2.207 | 2.803 | 3.685 | 35.658 | -2.017 |
| 246 | 2.307 | 2.803 | 3.694 | 29.709 | -2.028 |
| 247 | 2.406 | 2.803 | 3.703 | 23.762 | -2.039 |
| 248 | 2.505 | 2.803 | 3.712 | 17.818 | -2.050 |
| 249 | 2.604 | 2.803 | 3.722 | 11.875 | -2.061 |
| 250 | 2.703 | 2.803 | 3.731 | 5.935 | -2.072 |
| 251 | 2.803 | 2.803 | 3.740 | -0.003 | -2.083 |

## Screen Plots

This application generates nine screen plots. Figures 2-3-2 through 2-3-10 are plots of the horizontal and vertical velocity and acceleration, pressure, and water surface elevation at Z as a function of the horizontal coordinate (X/L). Also plotted is the horizontal velocity, vertical acceleration, and pressure under the wave crest.

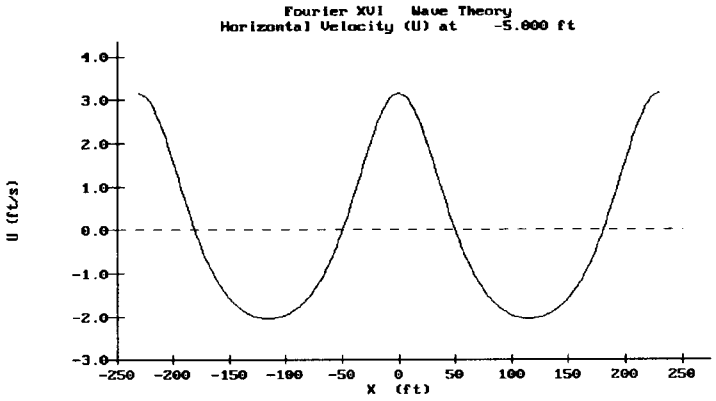


Figure 2-3-2. Horizontal Water Velocity at Z

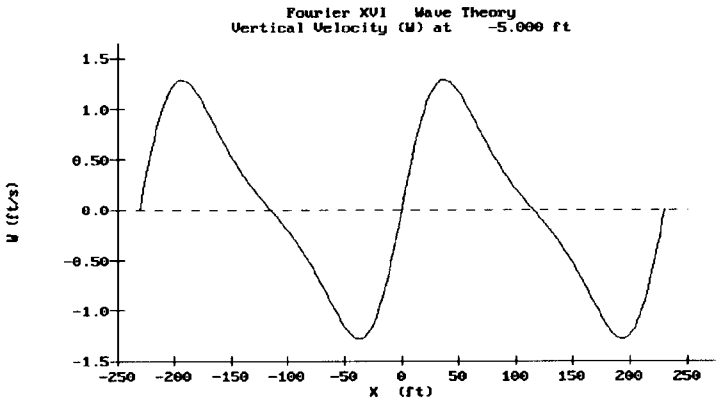


Figure 2-3-3. Vertical Water Velocity at Z

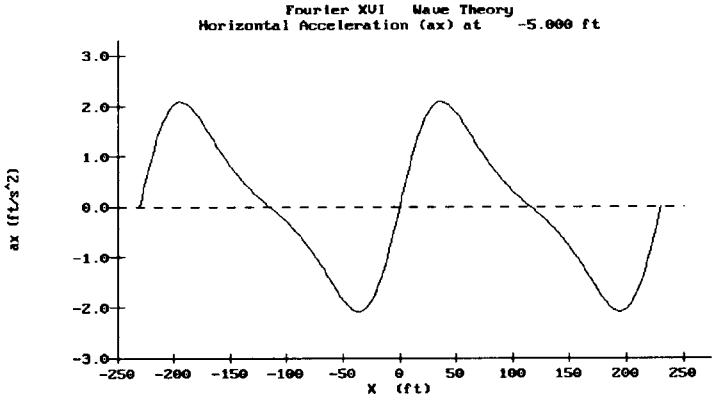


Figure 2-3-4. Horizontal Acceleration at Z

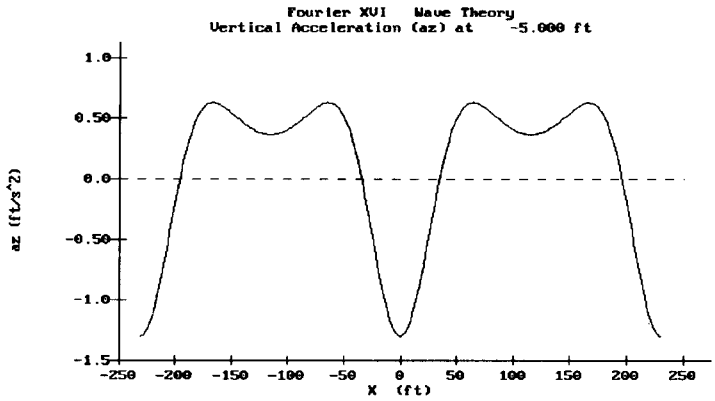


Figure 2-3-5. Vertical Acceleration at Z

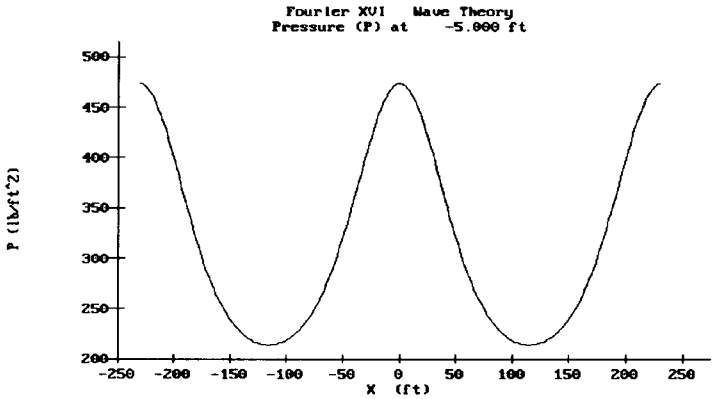


Figure 2-3-6. Pressure at Z

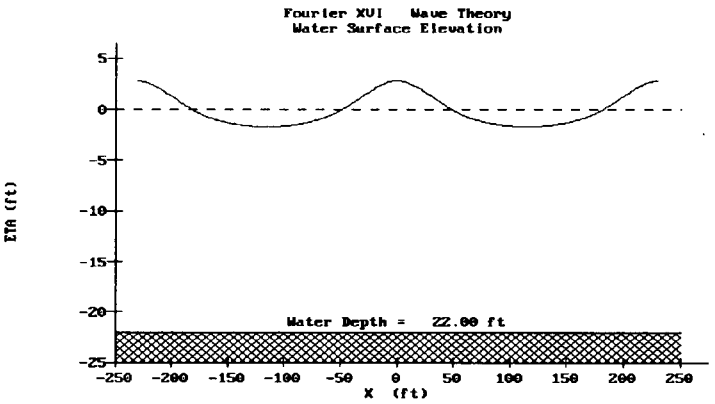


Figure 2-3-7. Water Surface Elevation

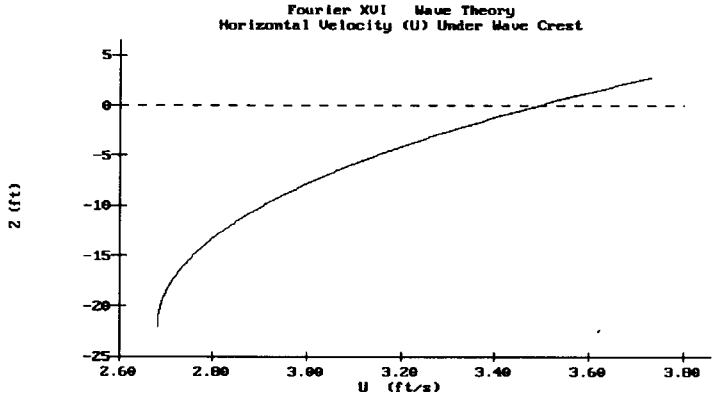


Figure 2-3-8. Horizontal Water Velocity Under Wave Crest

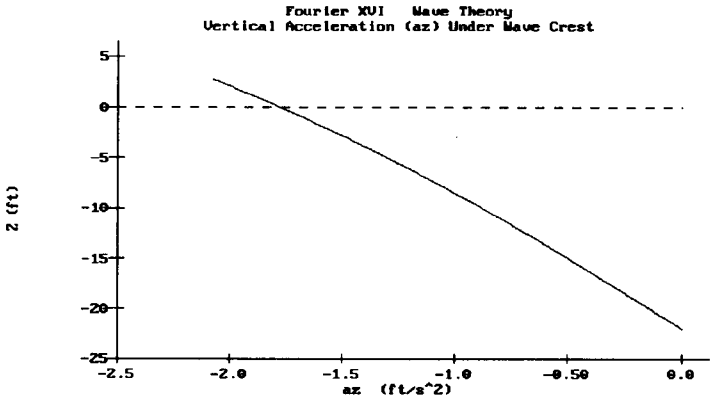


Figure 2-3-9. Vertical Acceleration Under Wave Crest

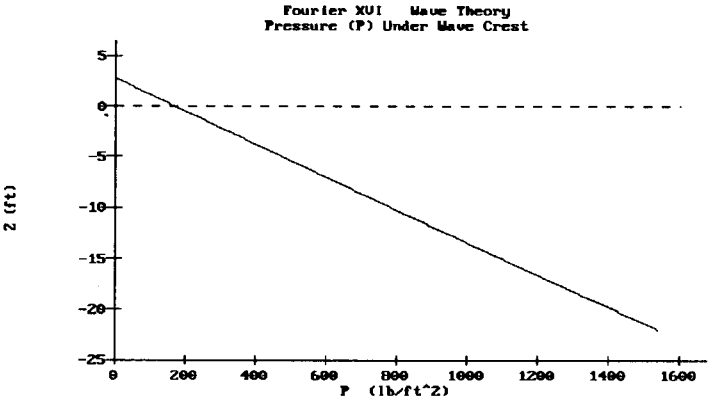


Figure 2-3-10. Pressure Under Wave Crest

## REFERENCES AND BIBLIOGRAPHY

Chaplin, J. R., 1980. "Developments of Stream-Function Wave Theory," *Coastal Engineering*, Vol. 3, pp. 179-205.

Chappelear, J. E., 1961. "Direct Numerical Calculation of Wave Properties," *Journal of Geophysical Research*, Vol. 66, No. 2, pp. 501-508.

Cokelet, E. D., 1977. "Steep Gravity Waves in Water of Arbitrary Uniform Depth," *Proceedings of the Royal Society of London*, Series A, Vol. 286, pp. 183-230.

Dalrymple, R. A., 1974. "A Finite Amplitude Wave on a Linear Shear Current," *Journal of Geophysical Research*, Vol. 79, No. 30, pp 4498-4504.

Dalrymple, R. A., and Solana, P., 1986. "Nonuniqueness in Stream Function Wave Theory," *Journal of Waterway, Port, Coastal and Ocean Division*, American Society of Civil Engineers, Vol. 112, No. 2, pp. 333-337.

Dean, R. G., 1965. "Stream Function Representation of Nonlinear Ocean Waves," *Journal of Geophysical Research*, Vol. 70, No. 18, pp. 4561-4572.

Dean, R. G., 1974. "Evaluation and Development of Water Wave Theories for Engineering Application," Special Report No. 1, Coastal Engineering Research Center, 2 Vols.

Dongarra, J. J., Moler, C. B., Bunch, J. R., and Stewart, G. W., 1979. LINPACK User's Guide, S. I. A. M., Philadelphia.

Fenton, J. D., 1988a. "The Numerical Solution of Steady Water Wave Problems," *Computers and Geoscience*, Vol. 14, No. 3, pp. 357-368.

Fenton, J. D., 1988b. Discussion of "Nonuniqueness in Stream Function Wave Theory," by R. A. Dalrymple and P. Solana, *Journal of Waterway, Port, Coastal and Ocean Division*, American Society of Civil Engineers, Vol. 114, No. 1, pp. 110-112.

Fenton, J. D., 1990. "Nonlinear Wave Theories," Ocean Engineering Science, The Sea, Vol. 9, Part A, Edited by Le Mehaute, B., and Hanes, D., John Wiley and Sons, New York, pp. 3-25.

Klopman, G., 1990. "A Note on Integral Properties of Periodic Gravity Waves in the Case of a Non-zero Mean Eulerian Velocity," *Journal of Fluid Mechanics*, Vol. 211, pp. 609-615.

Le Mehaute, B., Lu, C. C., and Ulmer, E. W., 1984. "Parametized Solution to Nonlinear Wave Problem," *Journal of Waterway, Port, Coastal and Ocean Division*, American Society of Civil Engineers, Vol. 110, No. 3, pp. 309-320.

Reinecker, M. M., and Fenton, J. D., 1981. "A Fourier Approximation Method for Steady Water Waves," *Journal of Fluid Mechanics*, Vol. 104, pp. 119-137.

Schwartz, L. W., 1974. "Computer Extension and Analytical Continuation of Stokes' Expansion for Gravity Waves," *Journal of Fluid Mechanics*, Vol. 62, Part 3, pp. 553-578.

Sobey, R. J., 1988. Discussion of "Nonuniqueness in Stream Function Wave Theory," by R. A. Dalrymple and P. Solana, *Journal of Waterway, Port, Coastal and Ocean Division*, American Society of Civil Engineers, Vol. 114, No. 1, pp. 112-114.

Sobey, R. J., Goodwin, P., Thieke, R. J., Westberg, R. J. Jr., 1987. "Application of Stokes, Cnoidal, and Fourier Wave Theories," *Journal of Waterway, Port, Coastal and Ocean Division,* American Society of Civil Engineers, Vol. 113, No. 6, pp. 565-587.

Stokes, G. G., 1847. "On the Theory of Oscillatory Waves," *Transactions of the Cambridge Philosophical Society,* Vol. 8, pp. 441-455.