Trigonometry Cram Sheet

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1 Definition

Triangle ABC has a right angle at C and sides of length a, b, c. The trigonometric functions of angle A are defined as follows:

1. $\sin A = \dfrac{a}{c} = \dfrac{\text{opposite}}{\text{hypotenuse}}$
2. $\cos A = \dfrac{b}{c} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
3. $\tan A = \dfrac{a}{b} = \dfrac{\text{opposite}}{\text{adjacent}}$
4. $\csc A = \dfrac{c}{a} = \dfrac{\text{hypotenuse}}{\text{opposite}}$
5. $\sec A = \dfrac{c}{b} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$
6. $\cot A = \dfrac{b}{a} = \dfrac{\text{adjacent}}{\text{opposite}}$

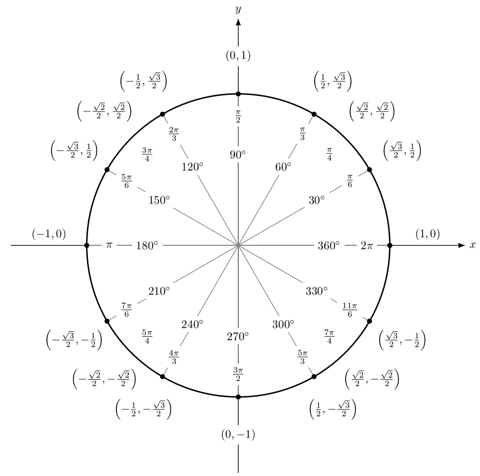
1.1 Extensions to Angles $> 90°$

A point P in the Cartesian plane has coordinates (x, y), where x is considered as positive along OX and negative along OX', while y is considered as positive along OY' and negative along OY. The distance from origin O to point P is positive and denoted by $r = \sqrt{x^2 + y^2}$. The angle A described *counterclockwise* from OX is considered *positive*. If it is described *clockwise* from OX it is considered *negative*.

For an angle A in any quadrant, the trigonometric functions of A are defined as follows:

1. $\sin A = \dfrac{y}{r}$
2. $\cos A = \dfrac{x}{r}$
3. $\tan A = \dfrac{y}{x}$
4. $\csc A = \dfrac{r}{y}$
5. $\sec A = \dfrac{r}{x}$
6. $\cot A = \dfrac{x}{y}$

1.2 The Unit Circle



1.3 Degrees and Radians

A *radian* is that angle θ subtended at center O of a circle by an arc MN equal to the radius r. Since 2π radians $= 360°$ we have:

$1\,\text{radian} = 180°/\pi = 57.29577951308232\ldots°$

$1° = \pi/180\,\text{radians} = 0.017453292519943\ldots$ radians

1.4 Signs and Variations

Quadrant	$\sin A$	$\cos A$	$\tan A$
I	$+$ $(0, 1)$	$+$ $(1, 0)$	$+$ $(0, \infty)$
II	$+$ $(1, 0)$	$-$ $(0, -1)$	$-$ $(-\infty, 0)$
III	$-$ $(0, -1)$	$-$ $(-1, 0)$	$+$ $(0, \infty)$
IV	$-$ $(-1, 0)$	$+$ $(0, 1)$	$-$ $(-\infty, 0)$

Quadrant	$\cot A$	$\sec A$	$\csc A$
I	$+$ $(\infty, 0)$	$+$ $(1, \infty)$	$+$ $(\infty, 1)$
II	$-$ $(0, -\infty)$	$-$ $(\infty, -1)$	$+$ $(1, \infty)$
III	$+$ $(\infty, 0)$	$-$ $(-1, \infty)$	$-$ $(\infty, -1)$
IV	$-$ $(0, -\infty)$	$+$ $(\infty, 1)$	$-$ $(-1, \infty)$

2 Properties and General Forms

2.1 Properties

2.1.1 $\sin x$

Domain: $\{x | x \in \mathbb{R}\}$ or $(-\infty, +\infty)$

Range: $\{y | -1 \leq y \leq 1\}$ or $[-1, 1]$

Period: 2π

VA: none

x-intercepts: $k\pi$ where $k \in \mathbb{Z}$

Parity: odd

2.1.2 $\cos x$

Domain: $\{x | x \in \mathbb{R}\}$ or $(-\infty, +\infty)$

Range: $\{y | -1 \leq y \leq 1\}$ or $[-1, 1]$

Period: 2π

VA: none

x-intercepts: $\frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

Parity: even

2.1.3 $\tan x$

Domain: $\left\{x | x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$ or $\bigcup_{k \in \mathbb{Z}} \left(\frac{(k-1)\pi}{2}, \frac{(k+1)\pi}{2}\right)$

Range: $\{y | y \in \mathbb{R}\}$ or $(-\infty, +\infty)$

Period: π

VA: $x = \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

x-intercepts: $k\pi$ where $k \in \mathbb{Z}$

Parity: odd

2.1.4 $\csc x$

Domain: $\{x | x \neq k\pi, k \in \mathbb{Z}\}$or $\bigcup_{k \in \mathbb{Z}} (k\pi, (k+1)\pi)$

Range: $\{y | y \leq 1 \cup y \geq 1\}$ or $(-\infty, -1] \cup [1, +\infty)$

Period: 2π

VA: $x = k\pi$ where $k \in \mathbb{Z}$

x-intercepts: none

Parity: odd

2.1.5 $\sec x$

Domain: $\left\{x | x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$ or $\bigcup_{k \in \mathbb{Z}} \left(\frac{(k-1)\pi}{2}, \frac{(k+1)\pi}{2}\right)$

Range: $\{y | y \leq 1 \cup y \geq 1\}$ or $(-\infty, -1] \cup [1, +\infty)$

Period: 2π

VA: $x = \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

x-intercepts: none

Parity: even

2.1.6 $\cot x$

Domain: $\{x | x \neq k\pi, k \in \mathbb{Z}\}$ or $\bigcup_{k \in \mathbb{Z}} (k\pi, (k+1)\pi)$

Range: $\{y | y \in \mathbb{R}\}$ or $(-\infty, +\infty)$

Period: π

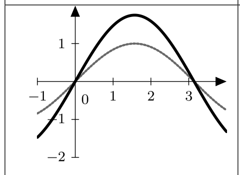
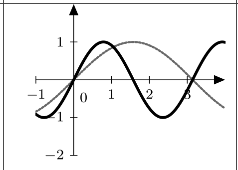
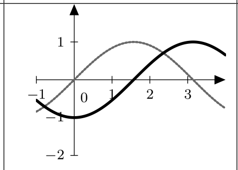
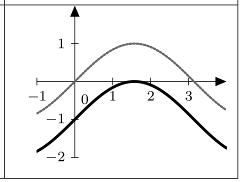
VA: $x = k\pi$ where $k \in \mathbb{Z}$

x-intercepts: $\frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

Parity: odd

2.2 General Forms of Trigonometric Functions

Given some trigonometric function $f(x)$, its general form is represented as $y = Af(B(x - C)) + D$, where its amplitude is $|A|$, its period is $\frac{2\pi}{|B|}$ or $\frac{\pi}{|B|}$ (for tangent and cotangent), its phase shift is C, and its vertical translation is D units upward (if $D > 0$) or D units downward (if $D < 0$). The maximum and minimum value for $\sin x$ and $\cos x$ is $A + D$ and $-A + D$ respectively.

amplitude	period	phase shift	vertical translation
			

3 Identities

3.1 Basic Identities

3.1.1 Reciprocal Identities

$\csc\theta = \frac{1}{\sin\theta}$; $\sin\theta = \frac{1}{\csc\theta}$

$\sec\theta = \frac{1}{\cos\theta}$; $\cos\theta = \frac{1}{\sec\theta}$

$\cot\theta = \frac{1}{\tan\theta}$; $\tan\theta = \frac{1}{\cot\theta}$

$\sin\theta\csc\theta = \cos\theta\sec\theta = \tan\theta\cot\theta = 1$

3.1.2 Ratio Identities

$\tan\theta = \frac{\sin\theta}{\cos\theta}$; $\cos\theta = \frac{\sin\theta}{\tan\theta}$; $\sin\theta = \cos\theta\tan\theta$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$; $\sin\theta = \frac{\cos\theta}{\cot\theta}$; $\cos\theta = \sin\theta\cot\theta$

3.1.3 Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$; $\sin^2\theta = 1 - \cos^2\theta$; $\cos^2\theta = 1 - \sin^2\theta$

$\tan^2\theta + 1 = \sec^2\theta$; $\tan^2\theta = \sec^2\theta - 1$; $\sec^2\theta - \tan^2\theta = 1$

$\cot^2\theta + 1 = \csc^2\theta$; $\cot^2\theta = \csc^2\theta - 1$; $\csc^2\theta - \cot^2\theta = 1$

3.1.4 Co-function Identities

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$

$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$

$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$

3.1.5 Parity Identities

$\sin(-A) = -\sin A$

$\cos(-A) = \cos A$

$\tan(-A) = -\tan A$

$\csc(-A) = -\csc A$

$\sec(-A) = \sec A$

$\cot(-A) = -\cot A$

3.2 Sum and Difference

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$$

3.3 Double Angle

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

3.4 Half Angle

Let $\mathcal{Q}_n$, where $n \in \{1, 2, 3, 4\}$, denote the set of all angles within the n^{th} quadrant of the Cartesian plane.

$$\sin\frac{\alpha}{2} = \begin{cases} \sqrt{\frac{1-\cos\alpha}{2}} & \text{if } \frac{\alpha}{2} \in (\mathcal{Q}_1 \cup \mathcal{Q}_2) \\ -\sqrt{\frac{1-\cos\alpha}{2}} & \text{if } \frac{\alpha}{2} \in (\mathcal{Q}_3 \cup \mathcal{Q}_4) \end{cases}$$

$$\cos\frac{\alpha}{2} = \begin{cases} \sqrt{\frac{1+\cos\alpha}{2}} & \text{if } \frac{\alpha}{2} \in (\mathcal{Q}_1 \cup \mathcal{Q}_4) \\ -\sqrt{\frac{1+\cos\alpha}{2}} & \text{if } \frac{\alpha}{2} \in (\mathcal{Q}_2 \cup \mathcal{Q}_3) \end{cases}$$

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1 + \cos\alpha} = \frac{1 - \cos\alpha}{\sin\alpha} = \csc\alpha - \cot\alpha$$

3.5 Multiple Angle

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

$\sin 4\alpha = 4\sin\alpha\cos\alpha - 8\sin^3\alpha\cos\alpha$

$\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$

$$\tan 4\alpha = \frac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$$

$$\sin(n\alpha) = \sum_{i=0}^{n} \binom{n}{i} \cos^i\alpha \sin^{n-i}\alpha \sin\left(\frac{(n-i)\pi}{2}\right)$$

$$\cos(n\alpha) = \sum_{i=0}^{n} \binom{n}{i} \cos^i\alpha \sin^{n-i}\alpha \cos\left(\frac{(n-i)\pi}{2}\right)$$

3.6 Power Reduction

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$\cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$$

$$\sin^4\theta = \frac{3 - 4\cos 2\theta + \cos 4\theta}{8}$$

$$\cos^4\theta = \frac{3 + 4\cos 2\theta + \cos 4\theta}{8}$$

$$\sin^5\theta = \frac{10\sin\theta - 5\sin 3\theta + \sin 5\theta}{16}$$

$$\cos^5\theta = \frac{10\cos\theta + 5\cos 3\theta + \cos 5\theta}{16}$$

3.7 Product to Sum

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha+\beta) - \cos(\alpha-\beta)\right]$$

3.8 Sum to Product

$$\sin\theta \pm \sin\varphi = 2\sin\frac{\theta\pm\varphi}{2}\cos\frac{\theta\mp\varphi}{2}$$

$$\cos\theta + \cos\varphi = 2\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}$$

$$\cos\theta - \cos\varphi = -2\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}$$

3.9 Linear Combinations

For some purposes it is important to know that any linear combination of sine waves of the same period or frequency but different phase shifts is also a sine wave with the same period or frequency, but a different phase shift.

Definition

The two-argument form of the arctangent function, denoted by $\tan^{-1}(y,x)$ gathers information on the signs of the inputs in order to return the appropriate quadrant of the computed angle. Thus, it is defined as:

$$\tan^{-1}(y,x) = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x>0, \\ \tan^{-1}\left(\frac{y}{x}\right)+\pi & \text{if } x<0 \text{ and } y\ge 0, \\ \tan^{-1}\left(\frac{y}{x}\right)-\pi & \text{if } x<0 \text{ and } y<0, \\ +\frac{\pi}{2} & \text{if } x=0 \text{ and } y>0, \\ -\frac{\pi}{2} & \text{if } x=0 \text{ and } y<0, \\ \text{undefined} & \text{if } x=0 \text{ and } y=0. \end{cases}$$

3.9.1 Sine and Cosine

In the case of a non-zero linear combination of a sine and cosine wave (which is just a sine wave with a phase shift of $\frac{\pi}{2}$), we have:

$$a\sin x + b\cos x = c\sin(x+\theta)$$

where $c = \pm\sqrt{a^2+b^2}$ and θ satisfies the equations $c\cos\theta = a$ and $c\sin\theta = b$, or $\theta = \tan^{-1}(b,a)$.

3.9.2 Arbitrary Phase Shift

More generally, for an arbitrary phase shift, we have:

$$a\sin x + b\sin(x+\theta) = c\sin(x+\varphi)$$

where $c = \pm\sqrt{a^2+b^2+2ab\cos\theta}$, and φ satisfies the equations $c\cos\varphi = a + b\cos\theta$ and $c\sin\varphi = b\sin\theta$ or $\varphi = \tan^{-1}(b\sin\theta, a+b\cos\theta)$.

3.10 Other Related Identities

- If $x+y+z=\pi$, then $\sin 2x + \sin 2y + \sin 2z = 4\sin x\sin y\sin z$.
- *Triple Tangent Identity.* If $x+y+z=\pi$, then $\tan x + \tan y + \tan z = \tan x\tan y\tan z$.
- *Triple Cotangent Identity.* If $x+y+z=\frac{\pi}{2}$, then $\cot x + \cot y + \cot z = \cot x\cot y\cot z$.
- *Ptolemy's Theorem.* If $w+x+y+z=\pi$, then $\sin(w+x)\sin(x+y) = \sin w\sin y + \sin x\sin z$.
- $\cot x\cot y + \cot y\cot z + \cot z\cot x = 1$
- $a\cos x + b\sin x = \sqrt{a^2+b^2}\cos\left(x - \tan^{-1}(b,a)\right)$
- *Tangent of an Average.* $\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta} = -\frac{\cos\alpha-\cos\beta}{\sin\alpha-\sin\beta}$
- $\tan x + \sec x = \tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$
- $\displaystyle\sum_{i=0}^{n}\sin(\varphi+i\alpha) = \frac{\sin\frac{(n+1)\alpha}{2}\sin\left(\varphi+\frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}}$

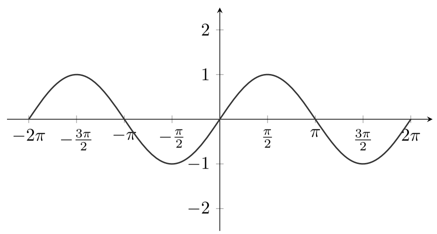
- $\displaystyle\sum_{i=0}^{n}\cos(\varphi+i\alpha) = \frac{\sin\frac{(n+1)\alpha}{2}\cos\left(\varphi+\frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}}$
- $\displaystyle\sum_{n=1}^{\infty}\prod_{m=1}^{n}\cos\frac{m\pi}{2n+1} = 1$

3.11 Identities without Variables

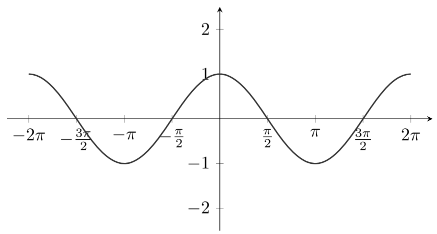
- *Morrie's Law.* $\cos 20^\circ\cdot\cos 40^\circ\cdot\cos 80^\circ = \frac{1}{8}$
- $\sin 20^\circ\cdot\sin 40^\circ\cdot\sin 80^\circ = \frac{\sqrt{3}}{8}$
- $\cos 24^\circ + \cos 48^\circ + \cos 96^\circ + \cos 168^\circ = \frac{1}{2}$
- $\cos\frac{2\pi}{21} + \cos\left(2\cdot\frac{2\pi}{21}\right) + \cos\left(4\cdot\frac{2\pi}{21}\right) + \cos\left(5\cdot\frac{2\pi}{21}\right) + \cos\left(8\cdot\frac{2\pi}{21}\right) + \cos\left(10\cdot\frac{2\pi}{21}\right) = \frac{1}{2}$
- $\cos\frac{\pi}{5} = \cos 36^\circ = \frac{1}{4}(\sqrt{5}+1) = \frac{1}{2}\varphi$
- $\sin\frac{\pi}{10} = \sin 18^\circ = \frac{1}{4}(\sqrt{5}-1) = \frac{1}{2}\varphi^{-1}$
- $\sin^2 18^\circ + \sin^2 30^\circ = \sin^2 36^\circ$

4 Graphs

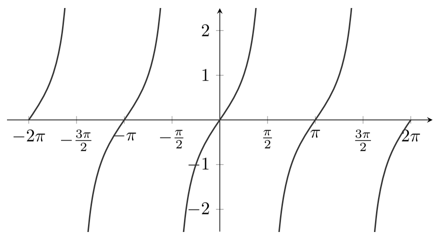
4.1 $y = \sin x$



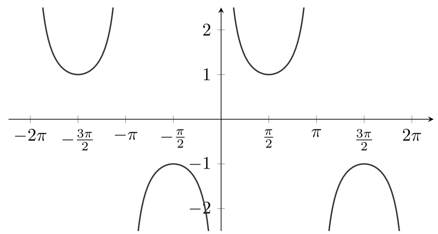
4.2 $y = \cos x$



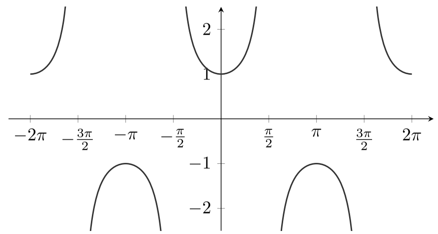
4.3 $y = \tan x$



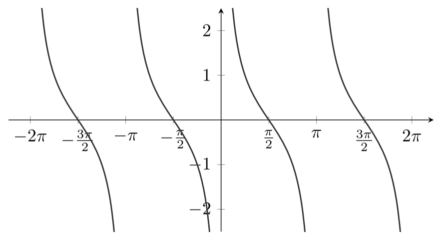
4.4 $y = \csc x$



4.5 $y = \sec x$



4.6 $y = \cot x$



5 Tables

5.1 Exact Values of Trigonometric Functions

$A°$	A rad	$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
$0°$	0	0	1	0	∞	1	∞
$15°$	$\pi/12$	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$2-\sqrt{3}$	$2+\sqrt{3}$	$\sqrt{6}-\sqrt{2}$	$\sqrt{6}+\sqrt{2}$
$30°$	$\pi/6$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
$45°$	$\pi/4$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$60°$	$\pi/3$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
$75°$	$5\pi/12$	$\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$2+\sqrt{3}$	$2-\sqrt{3}$	$\sqrt{6}+\sqrt{2}$	$\sqrt{6}-\sqrt{2}$
$90°$	$\pi/2$	1	0	$\pm\infty$	0	$\pm\infty$	1
$105°$	$7\pi/12$	$\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$-\left(2+\sqrt{3}\right)$	$-\left(2-\sqrt{3}\right)$	$-\left(\sqrt{6}+\sqrt{2}\right)$	$\sqrt{6}-\sqrt{2}$
$120°$	$2\pi/3$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
$135°$	$3\pi/4$	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
$150°$	$5\pi/6$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
$165°$	$11\pi/12$	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$-\left(2-\sqrt{3}\right)$	$-\left(2+\sqrt{3}\right)$	$-\left(\sqrt{6}-\sqrt{2}\right)$	$\sqrt{6}+\sqrt{2}$
$180°$	π	0	-1	0	$\mp\infty$	-1	$\pm\infty$
$195°$	$13\pi/12$	$-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$2-\sqrt{3}$	$2+\sqrt{3}$	$-\left(\sqrt{6}-\sqrt{2}\right)$	$-\left(\sqrt{6}+\sqrt{2}\right)$
$210°$	$7\pi/6$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
$225°$	$5\pi/4$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
$240°$	$4\pi/3$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
$255°$	$17\pi/12$	$-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$2+\sqrt{3}$	$2-\sqrt{3}$	$-\left(\sqrt{6}+\sqrt{2}\right)$	$-\left(\sqrt{6}-\sqrt{2}\right)$
$270°$	$3\pi/2$	-1	0	$\pm\infty$	0	$\mp\infty$	-1
$285°$	$19\pi/12$	$-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$-\left(2+\sqrt{3}\right)$	$-\left(2-\sqrt{3}\right)$	$\sqrt{6}+\sqrt{2}$	$-\left(\sqrt{6}-\sqrt{2}\right)$
$300°$	$5\pi/3$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
$315°$	$7\pi/4$	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
$330°$	$11\pi/6$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2
$345°$	$23\pi/12$	$-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$-\left(2+\sqrt{3}\right)$	$-\left(2+\sqrt{3}\right)$	$\sqrt{6}-\sqrt{2}$	$-\left(\sqrt{6}+\sqrt{2}\right)$
$360°$	2π	0	1	0	$\mp\infty$	1	$\mp\infty$

5.2 Relations Between Trig Functions

	$\sin\theta=u$	$\cos\theta=u$	$\tan\theta=u$	$\csc\theta=u$	$\sec\theta=u$	$\cot\theta=u$
$\sin\theta$	u	$\sqrt{1-u^2}$	$\dfrac{u}{\sqrt{1+u^2}}$	$\dfrac{1}{u}$	$\dfrac{\sqrt{u^2-1}}{u}$	$\dfrac{1}{\sqrt{1+u^2}}$
$\cos\theta$	$\sqrt{1-u^2}$	u	$\dfrac{1}{\sqrt{1+u^2}}$	$\dfrac{\sqrt{u^2-1}}{u}$	$\dfrac{1}{u}$	$\dfrac{u}{\sqrt{1+u^2}}$
$\tan\theta$	$\dfrac{u}{\sqrt{1-u^2}}$	$\dfrac{\sqrt{1-u^2}}{u}$	u	$\dfrac{1}{\sqrt{u^2-1}}$	$\sqrt{u^2-1}$	$\dfrac{1}{u}$
$\csc\theta$	$\dfrac{1}{u}$	$\dfrac{1}{\sqrt{1-u^2}}$	$\dfrac{\sqrt{1+u^2}}{u}$	u	$\dfrac{u}{\sqrt{u^2-1}}$	$\sqrt{1+u^2}$
$\sec\theta$	$\dfrac{1}{\sqrt{1-u^2}}$	$\dfrac{1}{u}$	$\sqrt{1+u^2}$	$\dfrac{u}{\sqrt{u^2-1}}$	u	$\dfrac{\sqrt{1+u^2}}{u}$
$\cot\theta$	$\dfrac{\sqrt{1-u^2}}{u}$	$\dfrac{u}{\sqrt{1-u^2}}$	$\dfrac{1}{u}$	$\sqrt{u^2-1}$	$\dfrac{1}{\sqrt{u^2-1}}$	u

6 Inverse Trigonometric Functions

If $x=\sin y$, then $y=\sin^{-1}x$, i.e. the angle whose sine is x or arcsine of x, is a multiple-valued function of x which is a collection of single-valued functions called *branches*. Similarly, the other inverse trigonometric functions are multiple-valued.

For many purposes, a particular branch is required. This is called the *principal branch* and the values for this branch are called *principal values.*

6.1 Principal Values

Since none of the six trigonometric functions are one-to-one, they are restricted in order to have inverse functions. Therefore the ranges of the inverse functions are proper subsets of the domains of the original functions.

Principal values for $x\geq 0$	Principal values for $x<0$
$0\leq\sin^{-1}x\leq\frac{\pi}{2}$	$-\frac{\pi}{2}\leq\sin^{-1}x<0$
$0\leq\cos^{-1}x\leq\frac{\pi}{2}$	$\frac{\pi}{2}<\cos^{-1}x\leq\pi$
$0\leq\tan^{-1}x<\frac{\pi}{2}$	$-\frac{\pi}{2}<\tan^{-1}x<0$
$0<\cot^{-1}x\leq\frac{\pi}{2}$	$\frac{\pi}{2}<\cot^{-1}x<\pi$
$0\leq\sec^{-1}x<\frac{\pi}{2}$	$\frac{\pi}{2}<\sec^{-1}x\leq\pi$
$0<\csc^{-1}x\leq\frac{\pi}{2}$	$-\frac{\pi}{2}\leq\csc^{-1}x<0$

6.2 Identities

In all cases it is assumed that principal values are used.

6.2.1 Reciprocal Identities

$$\sin^{-1}\frac{1}{x}=\csc^{-1}x$$

$$\cos^{-1}\frac{1}{x}=\sec^{-1}x$$

$$\csc^{-1}\frac{1}{x}=\sin^{-1}x$$

$$\sec^{-1}\frac{1}{x}=\cos^{-1}x$$

$$\tan^{-1}\frac{1}{x}=\begin{cases}\frac{\pi}{2}-\tan^{-1}x=\cot^{-1}x & \text{if } x>0\\ -\frac{\pi}{2}-\tan^{-1}x=\cot^{-1}x-\pi & \text{if } x<0\end{cases}$$

$$\cot^{-1}\frac{1}{x}=\begin{cases}\frac{\pi}{2}-\cot^{-1}x=\tan^{-1}x & \text{if } x>0\\ \frac{3\pi}{2}-\cot^{-1}x=\pi+\tan^{-1}x & \text{if } x<0\end{cases}$$

6.2.2 Negative Identities

$$\sin^{-1}(-x)=-\sin^{-1}x$$

$$\cos^{-1}(-x)=\pi-\cos^{-1}x$$

$$\tan^{-1}(-x)=-\tan^{-1}x$$

$$\csc^{-1}(-x)=-\csc^{-1}x$$

$$\sec^{-1}(-x)=\pi-\sec^{-1}x$$

$$\cot^{-1}(-x)=\pi-\cot^{-1}x$$

6.2.3 Complementary Identities

$$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$$

$$\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$$

$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$

6.2.4 Sum and Difference

$\sin^{-1}\alpha \pm \sin^{-1}\beta = \sin^{-1}\left(\alpha\sqrt{1-\beta^2} \pm \beta\sqrt{1-\alpha^2}\right)$

$\cos^{-1}\alpha \pm \cos^{-1}\beta = \cos^{-1}\left(\alpha\beta \mp \sqrt{(1-\alpha^2)(1-\beta^2)}\right)$

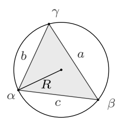
$\tan^{-1}\alpha \pm \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha \pm \beta}{1 \mp \alpha\beta}\right)$

7 Relationships Between Sides and Angles

7.1 Law of Sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Extended Law of Sines



$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R,$$

where R is the circumradius of the triangle.

7.2 Law of Cosines

$\cos\alpha = \dfrac{b^2+c^2-a^2}{2bc}; \quad a = \sqrt{b^2+c^2-2bc\cos\alpha}$

$\cos\beta = \dfrac{a^2+c^2-b^2}{2ac}; \quad b = \sqrt{a^2+c^2-2ac\cos\beta}$

$\cos\gamma = \dfrac{a^2+b^2-c^2}{2ab}; \quad c = \sqrt{a^2+b^2-2ab\cos\gamma}$

7.3 Law of Tangents

$\dfrac{a-b}{a+b} = \dfrac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}$

$\dfrac{b-c}{b+c} = \dfrac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)}$

$\dfrac{c-a}{c+a} = \dfrac{\tan\frac{1}{2}(\gamma-\alpha)}{\tan\frac{1}{2}(\gamma+\alpha)}$

7.4 Law of Cotangents

Let s be the semi-perimeter, that is, $s = \frac{a+b+c}{2}$, and r be the radius of the inscribed circle, then:

$$\frac{\cot\frac{\alpha}{2}}{s-a} = \frac{\cot\frac{\beta}{2}}{s-b} = \frac{\cot\frac{\gamma}{2}}{s-c} = \frac{1}{r}$$

and furthermore that the inradius is given by:

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

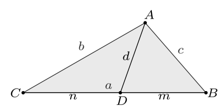
7.5 Mollweide's Formula

Each of these identities uses all six parts of the triangle—the three angles and the lengths of the three sides.

$\dfrac{a+b}{c} = \dfrac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\gamma}{2}}$

$\dfrac{a-b}{c} = \dfrac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\gamma}{2}}$

7.6 Stewart's Theorem



Let D be a point in $\overline{BC}$ of $\triangle ABC$. If $|BD| = m$, $|CD| = n$, and $|AD| = d$, then $b^2m + c^2n = a\left(d^2 + mn\right)$.

7.7 Angles in Terms of Sides

Let $s = \frac{a+b+c}{2}$ be the semiperimeter of the triangle, then:

$\alpha = \sin^{-1}\left(\dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}\right)$

$\beta = \sin^{-1}\left(\dfrac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}\right)$

$\gamma = \sin^{-1}\left(\dfrac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}\right)$

8 Solving Triangles

A general form triangle has six main characteristics: three linear (side lengths a, b, c) and three angular (α, β, γ). The classical plane trigonometry problem is to specify three of the six characteristics and determine the other three. A triangle can be uniquely determined in this sense when given any of the following:

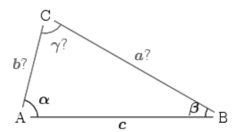
- Three sides (SSS)
- Two sides and the included angle (SAS)
- Two sides and an angle not included between them (SSA), if the side length adjacent to the angle is shorter than the other side length.
- A side and the two angles adjacent to it (ASA)
- A side, the angle opposite to it and an angle adjacent to it (AAS).

For all cases in the plane, at least one of the side lengths must be specified. If only the angles are given, the side lengths cannot be determined, because any similar triangle is a solution.

Notes

- To find an unknown angle, the law of cosines is safer than the law of sines. The reason is that the value of sine for the angle of the triangle does not uniquely determine this angle. For example, if $\sin\beta = 0.5$, the angle β can be equal either 30° or 150°. Using the law of cosines avoids this problem: within the interval from 0° to 180° the cosine value unambiguously determines its angle. On the other hand, if the angle is small (or close to 180°), then it is more robust numerically to determine it from its sine than its cosine because the arc-cosine function has a divergent derivative at 1 (or −1).
- We assume that the relative position of specified characteristics is known. If not, the mirror reflection of the triangle will also be a solution. For example, three side lengths uniquely define either a triangle or its reflection.

8.1 AAS/ASA Triangle



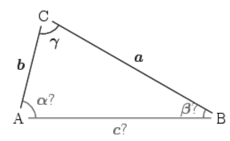
The known characteristics are the side c and the angles α, β. The third angle $\gamma = 180° - \alpha - \beta$.

Two unknown side can be calculated from the law of sines:

$$a = \frac{c\sin\alpha}{\sin\gamma}; \quad b = \frac{c\sin\beta}{\sin\gamma}.$$

The procedure for solving an AAS triangle is same as that for an ASA triangle: First, find the third angle by using the angle sum property of a triangle, then find the other two sides using the law of sines.

8.2 SAS Triangle



Here the lengths of sides a, b and the angle γ between these sides are known. The third side can be determined from the law of cosines:

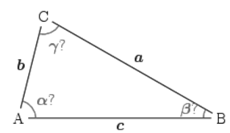
$$c = \sqrt{a^2+b^2-2ab\cos\gamma}$$

Now we use law of cosines to find the second angle:

$$\alpha = \cos^{-1}\frac{b^2+c^2-a^2}{2bc}$$

Finally, $\beta = 180° - \alpha - \gamma$.

8.3 SSS Triangle

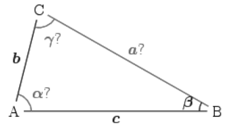


Let three side lengths a, b, c be specified. To find the angles α, β, the law of cosines can be used:

$$\alpha = \cos^{-1}\frac{b^2+c^2-a^2}{2bc}; \quad \beta = \cos^{-1}\frac{a^2+c^2-b^2}{2ac}.$$

Then angle $\gamma = 180° - \alpha - \beta$.

8.4 SSA Triangle

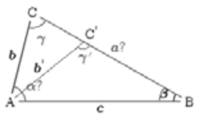


This case is not solvable in all cases; a solution is guaranteed to be unique only if the side length adjacent to the angle is shorter than the other side length. Assume that two sides b, c and the angle β are known. The equation for the angle γ can be implied from the law of sines:

$$\sin\gamma = \frac{c\sin\beta}{b}$$

We denote further $D = \frac{c\sin\beta}{b}$ (equation's right side). There are four possible cases:

1. If $D > 1$, no such triangle exists because the side b does not reach line BC. For the same reason a solution does not exist if the angle $\beta \geq 90°$ and $b \leq c$.
2. If $D = 1$, a unique solution exists: $\gamma = 90°$, i.e., the triangle is right-angled.
3. If $D < 1$, two alternatives are possible.



   (a) If $b < c$, the angle γ may be acute: $\gamma = \sin^{-1} D$ or obtuse: $\gamma' = 180° - \gamma$. The picture above shows the point C, the side b and the angle γ as the first solution, and the point C', side b' and the angle γ' as the second solution.

   (b) If $b \geq c$ then $\beta \geq \gamma$ (the larger side corresponds to a larger angle). Since no triangle can have two obtuse angles, γ is acute angle and the solution $\gamma = \sin^{-1} D$ is unique.

Once γ is obtained, the third angle $\alpha = 180° - \beta - \gamma$.

The third side can then be found from the law of sines:

$$a = \frac{b\sin\alpha}{\sin\beta}$$

8.5 Right Triangle

Solving right triangles is simply using the definitions of the trigonometric functions and the Pythagorean theorem to determine the other parts. The right angle $\gamma = 90°$ is always assumed to be given.

9 Polar Coordinates

A point P can be located by rectangular coordinates (x, y) or polar coordinates (r, θ).

The angle θ is a *directed angle*, that is, it is positive if it is measured counterclockwise from the initial side to the terminal side, and negative if it is measured clockwise.

The value r is a *directed distance*, it is positive if the point P lies on the terminal side of θ and negative if P is on the extension of the terminal side.

9.1 Properties

- Every ordered pair of polar coordinates (r, θ) locates a unique point in the plane.
- However, a point P on the plane may be specified by an infinite number of ordered pairs (r, θ).
- The pole O may be specified by the ordered pair $(0, \theta)$ where $\theta \in \mathbb{R}$.
- Let $P(r, \theta)$ be a point in the polar plane. Then $(r, \theta + 2k\pi)$ are also coordinates of the point P for any $k \in \mathbb{Z}$.
- It can also be shown that $((-1)^n r, \theta + n\pi)$ are also coordinates of P, where $n \in \mathbb{Z}$.

9.2 Coordinate Transformation

Polar to Rectangular

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Rectangular to Polar

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(y, x) \end{cases}$$

where $\tan^{-1}(y, x)$ is the two-argument form of the arctangent function (see section 3.9).

10 Special Polar Graphs

Theorem

A polar graph is:

1. **symmetric with respect to the polar axis** if an equivalent equation is obtained when (r, θ) is replaced by either $(r, -\theta)$ or $(-r, \pi - \theta)$.
2. **symmetric with respect to the $\frac{\pi}{2}$-axis** if an equivalent equation is obtained when (r, θ) is replaced by either $(r, \pi - \theta)$ or $(-r, -\theta)$.

3. **symmetric with respect to the pole** if an equivalent equation is obtained when (r, θ) is replaced by either $(-r, \theta)$ or $(r, \pi + \theta)$.

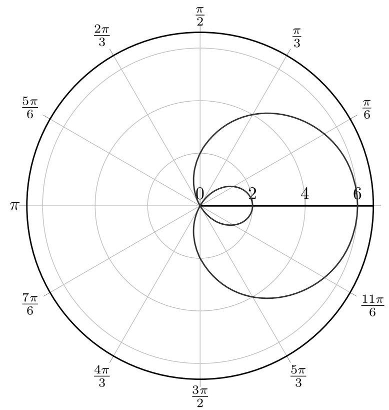
10.1 Limaçon of Pascal

A polar equation of the form $r = a + b\cos\theta$ or $r = a + b\sin\theta$ has a polar graph which is called a *limaçon*.

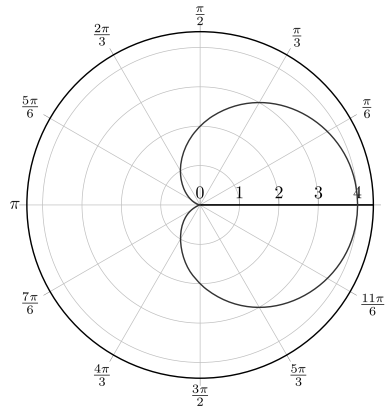
Let OQ be a line joining origin O to any point in Q on a circle of diameter b passing through O. Then the curve is the locus of all points in P such that $|PQ| = a$.

Types of Limaçons

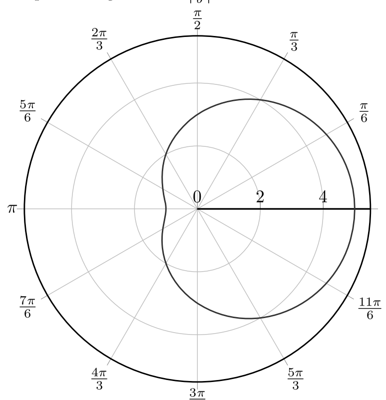
- Looped limaçon if $0 < \left|\frac{a}{b}\right| < 1$



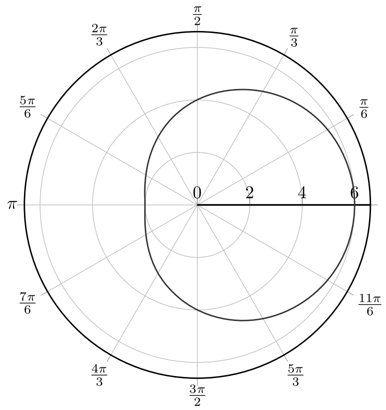
- Cardioid if $\left|\frac{a}{b}\right| = 1$



- Dimpled limaçon if $1 < \left|\frac{a}{b}\right| < 2$

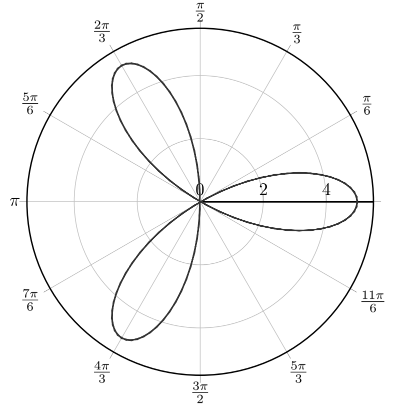


- Convex limaçon if $\left|\frac{a}{b}\right| \geq 2$

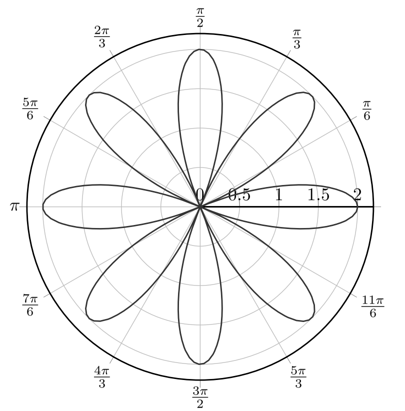


10.2 Rose

A rose with n leaves has a polar equation $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$ where a is a constant and n is an odd integer.



For an even integer n, the polar graph of an equation $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$ is a rose with $2n$ leaves.

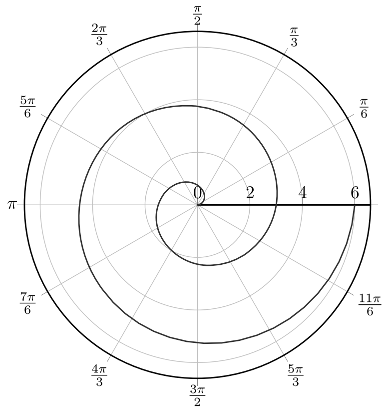


Properties

- The length of one leaf in the polar graph of a rose is $|a|$.
- If n is odd, then the graph of the polar equation $r = a\cos(n\theta)$ is symmetric with respect to the polar axis.
- If n is odd, then the graph of the polar equation $r = a\sin(n\theta)$ is symmetric with respect to the $\frac{\pi}{2}$-axis.
- A rose with an even number of leaves is symmetric with respect to the polar axis, the $\frac{\pi}{2}$-axis, and the pole.

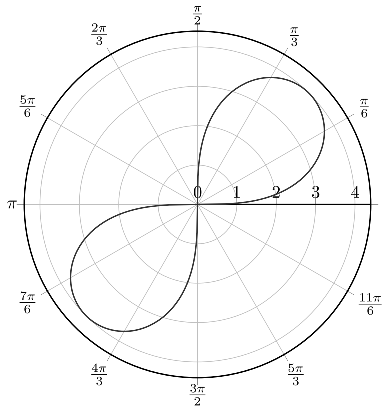
10.3 Spiral of Archimedes

The polar graph of a polar equation $r = a\theta$ where $\theta > 0$ and $a \in \mathbb{R}$ is called a *spiral*.



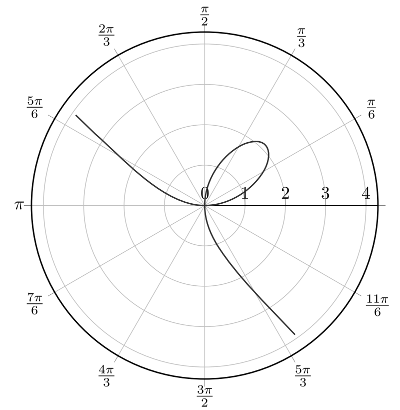
10.4 Lemniscate of Bernoulli

A polar equation $r^2 = a\cos 2\theta$ or $r^2 = a\sin 2\theta$ has a polar graph that is called a *lemniscate*.



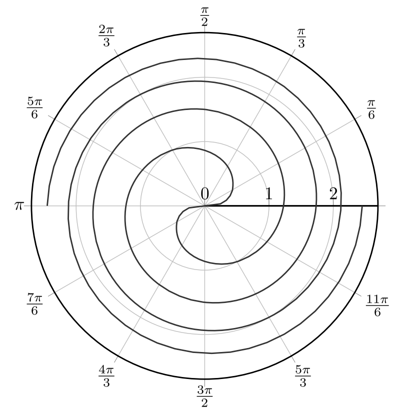
10.5 Folium of Descartes

A *folium* is a plane curve proposed by Descartes to challenge Fermat's extremum-finding techniques. It has a polar equation $r = \frac{3a\sec\theta\tan\theta}{1+\tan^3\theta}$.



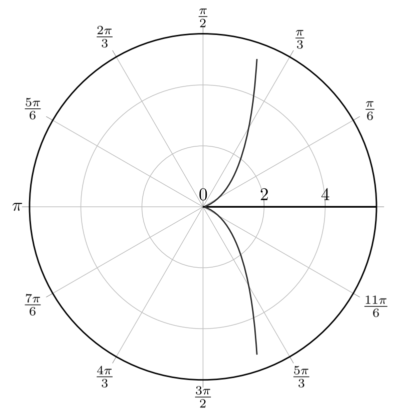
10.6 Spiral of Fermat

The *Fermat's spiral*, also known as the parabolic spiral, has a polar equation $r^2 = a^2\theta$. The resulting spiral is symmetric with respect to the origin.



10.7 Cissoid of Diocles

Given an origin O and a point P on the curve, let S be the point where the extension of the line OP intersects the line $x = 2a$ and R be the intersection of the circle of radius a and center $(a, 0)$ with the extension of OP. Then the *cissoid of Diocles* is the curve which satisfies $OP = RS$. It has a polar equation $r = 2a\sin\theta\tan\theta$.



10.8 Epispiral

The *epispiral* is a plane curve with a polar equation $r = a\sec(n\theta)$. Then there are n sections if n is odd (in blue), or $2n$ sections if n is even (in red). A slightly more symmetric version considers instead $r = a|\sec(n\theta)|$.