# Contents

# 1   Definition

Triangle $ABC$ has a right angle at $C$ and sides of length $a$, $b$, $c$. The trigonometric functions of angle $A$ are defined as follows:

1. $\sin A = \dfrac{a}{c} = \dfrac{\text{opposite}}{\text{hypotenuse}}$

2. $\cos A = \dfrac{b}{c} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

3. $\tan A = \dfrac{a}{b} = \dfrac{\text{opposite}}{\text{adjacent}}$

4. $\csc A = \dfrac{b}{a} = \dfrac{\text{hypotenuse}}{\text{opposite}}$

5. $\sec A = \dfrac{c}{a} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$

6. $\cot A = \dfrac{b}{a} = \dfrac{\text{adjacent}}{\text{opposite}}$

## 1.1   Extensions to Angles $> 90°$

A point $P$ in the Cartesian plane has coordinates $(x, y)$, where $x$ is considered as positive along $OX$ and negative along $OX'$, while $y$ is considered as positive along $OY'$ and negative along $OY$. The distance from origin $O$ to point $P$ is positive and denoted by $r = \sqrt{x^2 + y^2}$. The angle $A$ described *counterclockwise* from $OX$ is considered *positive*. If it is described *clockwise* from $OX$ it is considered *negative*.

For an angle $A$ in any quadrant, the trigonometric functions of $A$ are defined as follows:

1. $\sin A = \dfrac{y}{r}$

2. $\cos A = \dfrac{x}{r}$

3. $\tan A = \dfrac{y}{x}$

4. $\csc A = \dfrac{r}{y}$

5. $\sec A = \dfrac{r}{x}$

6. $\cot A = \dfrac{x}{y}$

## 1.2   Degrees and Radians

A *radian* is that angle $\theta$ subtended at center $O$ of a circle by an arc $MN$ equal to the radius $r$. Since $2\pi$ radians $= 360°$ we have:

$1\,\text{radian} = 180°/\pi = 57.29577951308232\ldots°$

$1° = \pi/180\,\text{radians} = 0.017453292519943\ldots\text{radians}$

## 1.3   Signs and Variations

| Quadrant | $\sin A$ | $\cos A$ | $\tan A$ |
|---|---|---|---|
| I | + $(0,1)$ | + $(1,0)$ | + $(0,\infty)$ |
| II | + $(1,0)$ | − $(0,-1)$ | − $(-\infty,0)$ |
| III | − $(0,-1)$ | − $(-1,0)$ | + $(0,\infty)$ |
| IV | − $(-1,0)$ | + $(0,1)$ | − $(-\infty,0)$ |

| Quadrant | $\cot A$ | $\sec A$ | $\csc A$ |
|---|---|---|---|
| I | + $(\infty,0)$ | + $(1,\infty)$ | + $(\infty,1)$ |
| II | − $(0,-\infty)$ | − $(\infty,-1)$ | + $(1,\infty)$ |
| III | + $(\infty,0)$ | − $(-1,\infty)$ | − $(\infty,-1)$ |
| IV | − $(0,-\infty)$ | + $(\infty,1)$ | − $(-1,\infty)$ |

# 2   Properties

## 2.1   $\sin x$

**Domain:** $\{x|x \in \mathbb{R}\}$

**Range:** $\{y| -1 \leq y \leq 1\}$

**Period:** $2\pi$

**VA:** none

$x$**-intercepts:** $k\pi$ where $k \in \mathbb{Z}$

**Parity:** odd

## 2.2   $\cos x$

**Domain:** $\{x|x \in \mathbb{R}\}$

**Range:** $\{y| -1 \leq y \leq 1\}$

**Period:** $2\pi$

**VA:** none

$x$**-intercepts:** $\frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

**Parity:** even

## 2.3   $\tan x$

**Domain:** $\left\{x|x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$

**Range:** $\{y|y \in \mathbb{R}\}$

**Period:** $\pi$

**VA:** $x = \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

$x$**-intercepts:** midway between asymptotes

**Parity:** odd

## 2.4   $\cot x$

**Domain:** $\{x|x \neq k\pi, k \in \mathbb{Z}\}$

**Range:** $\{y|y \in \mathbb{R}\}$

**Period:** $\pi$

**VA:** $x = k\pi$ where $k \in \mathbb{Z}$

$x$**-intercepts:** midway between asymptotes

**Parity:** odd

# 3   Identities

## 3.1   Basic Identities

### 3.1.1   Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta}; \ \sin\theta = \frac{1}{\csc\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}; \ \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}; \ \tan\theta = \frac{1}{\cot\theta}$$

$$\sin\theta\csc\theta = \cos\theta\sec\theta = \tan\theta\cot\theta = 1$$

### 3.1.2   Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}; \ \cos\theta = \frac{\sin\theta}{\tan\theta}; \ \sin\theta = \cos\theta\tan\theta$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}; \ \sin\theta = \frac{\cos\theta}{\cot\theta}; \ \cos\theta = \sin\theta\cot\theta$$

### 3.1.3   Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1; \ \sin^2\theta = 1 - \cos^2\theta; \ \cos^2\theta = 1 - \sin^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta; \ \tan^2\theta = \sec^2\theta - 1; \ \sec^2\theta - \tan^2\theta = 1$$

$$\cot^2\theta + 1 = \csc^2\theta; \ \cot^2\theta = \csc^2\theta - 1; \ \csc^2\theta - \cot^2\theta = 1$$

### 3.1.4   Co-function Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = -\sin\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = -\tan\theta$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = -\csc\theta$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \sec\theta$$

$$\cot\left(\frac{\pi}{2} - \theta\right) = -\cot\theta$$

### 3.1.5   Parity Identities

$$\sin(-A) = -\sin A$$

$$\cos(-A) = \cos A$$

$$\tan(-A) = -\tan A$$

$$\csc(-A) = -\csc A$$

$$\sec(-A) = \sec A$$

$$\cot(-A) = -\cot A$$

## 3.2 Sum and Difference

$$\sin\left(\alpha \pm \beta\right) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos\left(\alpha \pm \beta\right) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan\left(\alpha \pm \beta\right) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\cot\left(\alpha \pm \beta\right) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$$

## 3.3 Double Angle

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

## 3.4 Half Angle

Let $\mathcal{Q}_n$, where $n \in \{1, 2, 3, 4\}$, denote the set of all points within the $n$th quadrant of the Cartesian plane.

$$\sin\frac{\alpha}{2} = \begin{cases} \sqrt{\frac{1-\cos\alpha}{2}} & \frac{\alpha}{2} \in (\mathcal{Q}_1 \cup \mathcal{Q}_2) \\ -\sqrt{\frac{1-\cos\alpha}{2}} & \frac{\alpha}{2} \in (\mathcal{Q}_3 \cup \mathcal{Q}_4) \end{cases}$$

$$\cos\frac{\alpha}{2} = \begin{cases} \sqrt{\frac{1+\cos\alpha}{2}} & \frac{\alpha}{2} \in (\mathcal{Q}_1 \cup \mathcal{Q}_4) \\ -\sqrt{\frac{1+\cos\alpha}{2}} & \frac{\alpha}{2} \in (\mathcal{Q}_2 \cup \mathcal{Q}_3) \end{cases}$$

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha} = \csc\alpha - \cot\alpha$$

## 3.5 Multiple Angle

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

$$\sin 4\alpha = 4\sin\alpha\cos\alpha - 8\sin^3\alpha\cos\alpha$$

$$\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$$

$$\tan 4\alpha = \frac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$$

$$\sin\left(n\alpha\right) = \sum_{i=0}^{n}\binom{n}{i}\cos^i\alpha\sin^{n-i}\alpha\sin\left(\frac{(n-i)\pi}{2}\right)$$

$$\cos\left(n\alpha\right) = \sum_{i=0}^{n}\binom{n}{i}\cos^i\alpha\sin^{n-i}\alpha\cos\left(\frac{(n-i)\pi}{2}\right)$$

## 3.6 Power Reduction

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$\cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$$

$$\sin^4\theta = \frac{3 - 4\cos 2\theta + \cos 4\theta}{8}$$

$$\cos^4\theta = \frac{3 + 4\cos 2\theta + \cos 4\theta}{8}$$

$$\sin^5\theta = \frac{10\sin\theta - 5\sin 3\theta + \sin 5\theta}{16}$$

$$\cos^5\theta = \frac{10\cos\theta + 5\cos 3\theta + \cos 5\theta}{16}$$

## 3.7 Product to Sum

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin\left(\alpha+\beta\right) + \sin\left(\alpha-\beta\right)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin\left(\alpha+\beta\right) - \sin\left(\alpha-\beta\right)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos\left(\alpha+\beta\right) + \cos\left(\alpha-\beta\right)\right]$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos\left(\alpha+\beta\right) - \cos\left(\alpha-\beta\right)\right]$$

## 3.8 Sum to Product

$$\sin\theta \pm \sin\varphi = 2\sin\frac{\theta \pm \varphi}{2}\cos\frac{\theta \mp \varphi}{2}$$

$$\cos\theta \pm \cos\varphi = \pm 2\sin\frac{\theta + \varphi}{2}\cos\frac{\theta - \varphi}{2}$$

## 3.9 Other Related Identities

- If $x + y + z = \pi$, then $\sin 2x + \sin 2y + \sin 2z = 4\sin x \sin y \sin z$.
- *Triple Tangent Identity.* If $x + y + z = \pi$, then $\tan x + \tan y + \tan z = \tan x \tan y \tan z$.
- *Triple Cotangent Identity.* If $x + y + z = \frac{\pi}{2}$, then $\cot x + \cot y + \cot z = \cot x \cot y \cot z$.

## 3.10 Identities without Variables

$$\cos 20° \cdot \cos 40° \cdot \cos 80° = \frac{1}{8}$$

$$\sin 20° \cdot \sin 40° \cdot \sin 80° = \frac{\sqrt{3}}{8}$$

$$\cos 24° + \cos 48° + \cos 96° + \cos 168° = \frac{1}{2}$$

# 4   Tables

## 4.1   Exact Values of Trigonometric Functions

| $A°$ | $A$ rad | $\sin A$ | $\cos A$ | $\tan A$ | $\cot A$ | $\sec A$ | $\csc A$ |
|---|---|---|---|---|---|---|---|
| $0°$ | $0$ | $0$ | $1$ | $0$ | $\infty$ | $1$ | $\infty$ |
| $15°$ | $\pi/12$ | $\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$ | $\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$ | $2-\sqrt{3}$ | $2+\sqrt{3}$ | $\sqrt{6}-\sqrt{2}$ | $\sqrt{6}+\sqrt{2}$ |
| $30°$ | $\pi/6$ | $\frac{1}{2}$ | $\frac{1}{2}\sqrt{3}$ | $\frac{1}{3}\sqrt{3}$ | $\sqrt{3}$ | $\frac{2}{3}\sqrt{3}$ | $2$ |
| $45°$ | $\pi/4$ | $\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}\sqrt{2}$ | $1$ | $1$ | $\sqrt{2}$ | $\sqrt{2}$ |
| $60°$ | $\pi/3$ | $\frac{1}{2}\sqrt{3}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{1}{3}\sqrt{3}$ | $2$ | $\frac{2}{3}\sqrt{3}$ |
| $75°$ | $5\pi/12$ | $\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$ | $\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$ | $2+\sqrt{3}$ | $2-\sqrt{3}$ | $\sqrt{6}+\sqrt{2}$ | $\sqrt{6}-\sqrt{2}$ |
| $90°$ | $\pi/2$ | $1$ | $0$ | $\pm\infty$ | $0$ | $\pm\infty$ | $1$ |
| $105°$ | $7\pi/12$ | $\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$ | $-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$ | $-\left(2+\sqrt{3}\right)$ | $-\left(2-\sqrt{3}\right)$ | $-\left(\sqrt{6}+\sqrt{2}\right)$ | $\sqrt{6}-\sqrt{2}$ |
| $120°$ | $2\pi/3$ | $\frac{1}{2}\sqrt{3}$ | $\frac{1}{2}$ | $-\sqrt{3}$ | $-\frac{1}{3}\sqrt{3}$ | $-2$ | $\frac{2}{3}\sqrt{3}$ |
| $135°$ | $3\pi/4$ | $\frac{1}{2}\sqrt{2}$ | $-\frac{1}{2}\sqrt{2}$ | $-1$ | $-1$ | $-\sqrt{2}$ | $\sqrt{2}$ |
| $150°$ | $5\pi/6$ | $\frac{1}{2}$ | $-\frac{1}{2}\sqrt{3}$ | $-\frac{1}{3}\sqrt{3}$ | $-\sqrt{3}$ | $-\frac{2}{3}\sqrt{3}$ | $2$ |
| $165°$ | $11\pi/12$ | $\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$ | $-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$ | $-\left(2-\sqrt{3}\right)$ | $-\left(2+\sqrt{3}\right)$ | $-\left(\sqrt{6}-\sqrt{2}\right)$ | $\sqrt{6}+\sqrt{2}$ |
| $180°$ | $\pi$ | $0$ | $-1$ | $0$ | $\mp\infty$ | $-1$ | $\pm\infty$ |
| $195°$ | $13\pi/12$ | $-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$ | $-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$ | $2-\sqrt{3}$ | $2+\sqrt{3}$ | $-\left(\sqrt{6}-\sqrt{2}\right)$ | $-\left(\sqrt{6}+\sqrt{2}\right)$ |
| $210°$ | $7\pi/6$ | $\frac{1}{2}$ | $-\frac{1}{2}\sqrt{3}$ | $\frac{1}{3}\sqrt{3}$ | $\sqrt{3}$ | $-\frac{2}{3}\sqrt{3}$ | $-2$ |
| $225°$ | $5\pi/4$ | $-\frac{1}{2}\sqrt{2}$ | $-\frac{1}{2}\sqrt{2}$ | $1$ | $1$ | $-\sqrt{2}$ | $-\sqrt{2}$ |
| $240°$ | $4\pi/3$ | $-\frac{1}{2}\sqrt{3}$ | $-\frac{1}{2}$ | $\sqrt{3}$ | $\frac{1}{3}\sqrt{3}$ | $-2$ | $-\frac{2}{3}\sqrt{3}$ |
| $255°$ | $17\pi/12$ | $-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$ | $-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$ | $2+\sqrt{3}$ | $2-\sqrt{3}$ | $-\left(\sqrt{6}+\sqrt{2}\right)$ | $-\left(\sqrt{6}-\sqrt{2}\right)$ |
| $270°$ | $3\pi/2$ | $-1$ | $0$ | $\pm\infty$ | $0$ | $\mp\infty$ | $-1$ |
| $285°$ | $19\pi/12$ | $-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$ | $\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$ | $-\left(2+\sqrt{3}\right)$ | $-\left(2-\sqrt{3}\right)$ | $\sqrt{6}+\sqrt{2}$ | $-\left(\sqrt{6}-\sqrt{2}\right)$ |
| $300°$ | $5\pi/3$ | $-\frac{1}{2}\sqrt{3}$ | $\frac{1}{2}$ | $-\sqrt{3}$ | $-\frac{1}{3}\sqrt{3}$ | $2$ | $-\frac{2}{3}\sqrt{3}$ |
| $315°$ | $7\pi/4$ | $-\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}\sqrt{2}$ | $-1$ | $-1$ | $\sqrt{2}$ | $-\sqrt{2}$ |
| $330°$ | $11\pi/6$ | $-\frac{1}{2}$ | $\frac{1}{2}\sqrt{3}$ | $-\frac{1}{3}\sqrt{3}$ | $-\sqrt{3}$ | $\frac{2}{3}\sqrt{3}$ | $-2$ |
| $345°$ | $23\pi/12$ | $-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$ | $\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$ | $-\left(2+\sqrt{3}\right)$ | $-\left(2+\sqrt{3}\right)$ | $\sqrt{6}-\sqrt{2}$ | $-\left(\sqrt{6}+\sqrt{2}\right)$ |
| $360°$ | $2\pi$ | $0$ | $1$ | $0$ | $\mp\infty$ | $1$ | $\mp\infty$ |

## 4.2 Relations Between Trig Functions

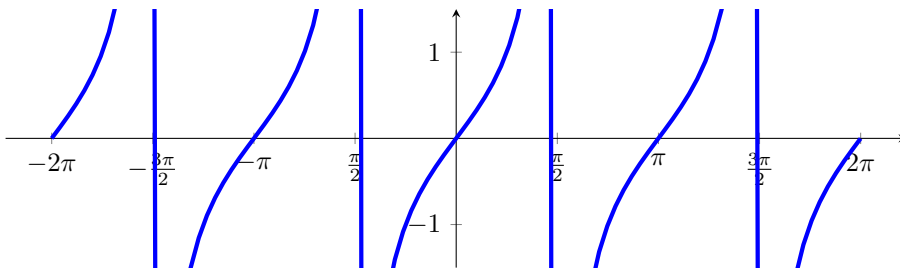| | $\sin\theta = u$ | $\cos\theta = u$ | $\tan\theta = u$ | $\csc\theta = u$ | $\sec\theta = u$ | $\cot\theta = u$ |
|---|---|---|---|---|---|---|
| $\sin\theta$ | $u$ | $\sqrt{1-u^2}$ | $\dfrac{u}{\sqrt{1+u^2}}$ | $\dfrac{1}{u}$ | $\dfrac{\sqrt{u^2-1}}{u}$ | $\dfrac{1}{\sqrt{1+u^2}}$ |
| $\cos\theta$ | $\sqrt{1-u^2}$ | $u$ | $\dfrac{1}{\sqrt{1+u^2}}$ | $\dfrac{\sqrt{u^2-1}}{u}$ | $\dfrac{1}{u}$ | $\dfrac{u}{\sqrt{1+u^2}}$ |
| $\tan\theta$ | $\dfrac{u}{\sqrt{1-u^2}}$ | $\dfrac{\sqrt{1-u^2}}{u}$ | $u$ | $\dfrac{1}{\sqrt{u^2-1}}$ | $\sqrt{u^2-1}$ | $\dfrac{1}{u}$ |
| $\csc\theta$ | $\dfrac{1}{u}$ | $\dfrac{1}{\sqrt{1-u^2}}$ | $\dfrac{\sqrt{1+u^2}}{u}$ | $u$ | $\dfrac{u}{\sqrt{u^2-1}}$ | $\sqrt{1+u^2}$ |
| $\sec\theta$ | $\dfrac{1}{\sqrt{1-u^2}}$ | $\dfrac{1}{u}$ | $\sqrt{1+u^2}$ | $\dfrac{u}{\sqrt{u^2-1}}$ | $u$ | $\dfrac{\sqrt{1+u^2}}{u}$ |
| $\cot\theta$ | $\dfrac{\sqrt{1-u^2}}{u}$ | $\dfrac{u}{\sqrt{1-u^2}}$ | $\dfrac{1}{u}$ | $\sqrt{u^2-1}$ | $\dfrac{1}{\sqrt{u^2-1}}$ | $u$ |

# 5 Graphs

## 5.1 $y = \sin x$
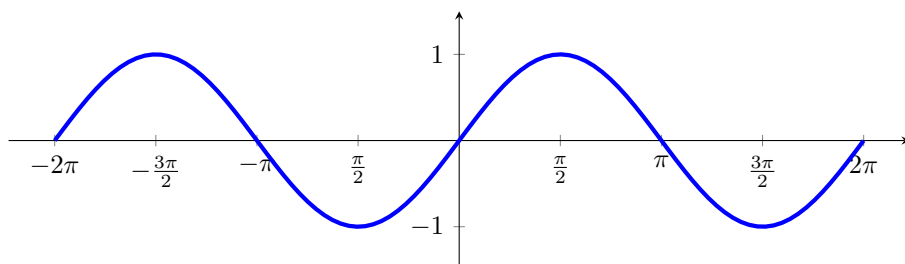


## 5.2 $y = \cos x$
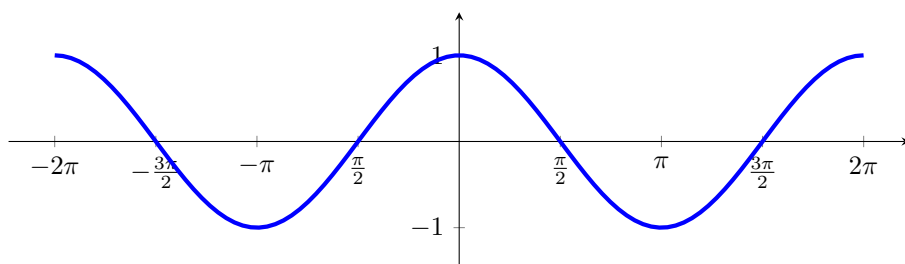
**5.3** $y = \tan x$

# 6 Inverse Trigonometric Functions

If $x = \sin y$, then $y = \sin^{-1} x$, i.e. the angle whose sine is $x$ or inverse sine of $x$, is a multiple-valued function of $x$ which is a collection of single-valued functions called *branches*. Similarly, the other inverse trigonometric functions are multiple-valued.

For many purposes, a particular branch is required. This is called the *principal branch* and the values for this branch are called *principal values*.

## 6.1 Principal Values

| Principal values for $x \geq 0$ | Principal values for $x < 0$ |
|---|---|
| $0 \leq \sin^{-1} x \leq \frac{\pi}{2}$ | $-\frac{\pi}{2} \leq \sin^{-1} x < 0$ |
| $0 \leq \cos^{-1} x \leq \frac{\pi}{2}$ | $\frac{\pi}{2} < \cos^{-1} x \leq \pi$ |
| $0 \leq \tan^{-1} x < \frac{\pi}{2}$ | $-\frac{\pi}{2} < \tan^{-1} x < 0$ |
| $0 < \cot^{-1} x \leq \frac{\pi}{2}$ | $\frac{\pi}{2} < \cot^{-1} x < \pi$ |
| $0 \leq \sec^{-1} x < \frac{\pi}{2}$ | $\frac{\pi}{2} < \sec^{-1} x \leq \pi$ |
| $0 < \csc^{-1} x \leq \frac{\pi}{2}$ | $-\frac{\pi}{2} \leq \csc^{-1} x < 0$ |

## 6.2 Identities

In all cases it is assumed that principal values are used.

### 6.2.1 Reciprocal Identities

$$\csc^{-1} x = \sin^{-1} \left( \frac{1}{x} \right)$$

$$\sec^{-1} x = \cos^{-1} \left( \frac{1}{x} \right)$$

$$\cot^{-1} x = \tan^{-1} \left( \frac{1}{x} \right)$$

### 6.2.2 Parity Identities

$$\sin^{-1} (-x) = -\sin^{-1} x$$

$$\cos^{-1} (-x) = \pi - \cos^{-1} x$$

$$\tan^{-1} (-x) = -\tan^{-1} x$$

$$\csc^{-1} (-x) = -\csc^{-1} x$$

$$\sec^{-1} (-x) = \pi - \sec^{-1} x$$

$$\cot^{-1} (-x) = \pi - \cot^{-1} x$$

### 6.2.3 Pythagorean Identities (sort of)

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$$

### 6.2.4 Sum and Difference

$$\sin^{-1} \alpha \pm \sin^{-1} \beta = \sin^{-1} \left( \alpha \sqrt{1 - \beta^2} \pm \beta \sqrt{1 - \alpha^2} \right)$$

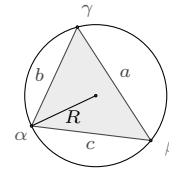$$\cos^{-1} \alpha \pm \cos^{-1} \beta = \cos^{-1} \left( \alpha\beta \mp \sqrt{(1 - \alpha^2)(1 - \beta^2)} \right)$$

$$\tan^{-1} \alpha \pm \tan^{-1} \beta = \tan^{-1} \left( \frac{\alpha \pm \beta}{1 \mp \alpha\beta} \right)$$

# 7 Relationships Between Sides and Angles

## 7.1 Law of Sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

**Extended Law of Sines**



$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R,$$

where $R$ is the circumradius of the triangle.

## 7.2 Law of Cosines

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}; \ a = \sqrt{b^2 + c^2 - 2bc \cos \alpha}$$

$$\cos \beta = \frac{a^2 + c^2 - b^2}{2ac}; \ b = \sqrt{a^2 + c^2 - 2ac \cos \beta}$$

$$\cos \gamma = \frac{a^2 + b^2 - c^2}{2ab}; \ c = \sqrt{a^2 + b^2 - 2ab \cos \gamma}$$

## 7.3 Law of Tangents

$$\frac{a - b}{a + b} = \frac{\tan \frac{1}{2} (\alpha - \beta)}{\tan \frac{1}{2} (\alpha + \beta)}$$

$$\frac{b - c}{b + c} = \frac{\tan \frac{1}{2} (\beta - \gamma)}{\tan \frac{1}{2} (\beta + \gamma)}$$

$$\frac{c - a}{c + a} = \frac{\tan \frac{1}{2} (\gamma - \alpha)}{\tan \frac{1}{2} (\gamma + \alpha)}$$

## 7.4 Law of Cotangents

Let $s$ be the semi-perimeter, that is, $s = \frac{(a+b+c)}{2}$, and $r$ be the radius of the inscribed circle, then

$$\frac{\cot(\frac{\alpha}{2})}{s-a} = \frac{\cot(\frac{\beta}{2})}{s-b} = \frac{\cot(\frac{\gamma}{2})}{s-c} = \frac{1}{r}$$

and furthermore that the inradius is given by $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.
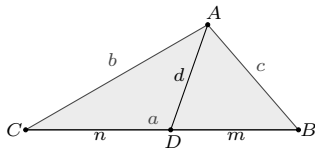
## 7.5 Mollweide's Formula

Each of these identities uses all six parts of the triangle—the three angles and the lengths of the three sides.

$$\frac{a+b}{c} = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\gamma}{2}\right)}$$

$$\frac{a-b}{c} = \frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\gamma}{2}\right)}$$

## 7.6 Stewart's Theorem



Let $D$ be a point in $\overline{BC}$ of $\triangle ABC$. If $|BD| = m$, $|CD| = n$, and $|AD| = d$, then $b^2 m + c^2 n = a\left(d^2 + mn\right)$.

# 8 Solving Triangles

## 8.1 AAA Triangle

1. Write "no solution" as your answer.

## 8.2 AAS/ASA Triangle

1. Solve for the missing angle.

2. Use the Law of Sines twice for the other two sides.

## 8.3 SAS Triangle

1. Use the Law of Cosines for the other non-included angle.

2. Use the Law of Sines for the missing side.

3. Solve for the missing angle.

## 8.4 SSS Triangle

1. Use the Law of Cosines twice for the two angles.

2. Solve for the missing angle.

## 8.5 SSA Triangle

When $\alpha$ is acute

| $0 < a < h$ | $a = h$ | $a > b$ | $h < a < b$ |
|---|---|---|---|
| |  |  |  |
| 0 | 1 | 1 | 2 |

When $\alpha$ is obtuse

| $0 < a \le b$ | $a > b$ |
|---|---|
| | |
| 0 | 1 |