Trigonometry Cram Sheet

October 27, 2015

Contents

1 Definition

Triangle ABC has a right angle at C and sides of length a, b, c. The trigonometric functions of angle A are defined as follows:

1. $\sin A = \dfrac{a}{c} = \dfrac{\text{opposite}}{\text{hypotenuse}}$
2. $\cos A = \dfrac{b}{c} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
3. $\tan A = \dfrac{a}{b} = \dfrac{\text{opposite}}{\text{adjacent}}$
4. $\csc A = \dfrac{c}{a} = \dfrac{\text{hypotenuse}}{\text{opposite}}$
5. $\sec A = \dfrac{c}{b} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$
6. $\cot A = \dfrac{b}{a} = \dfrac{\text{adjacent}}{\text{opposite}}$

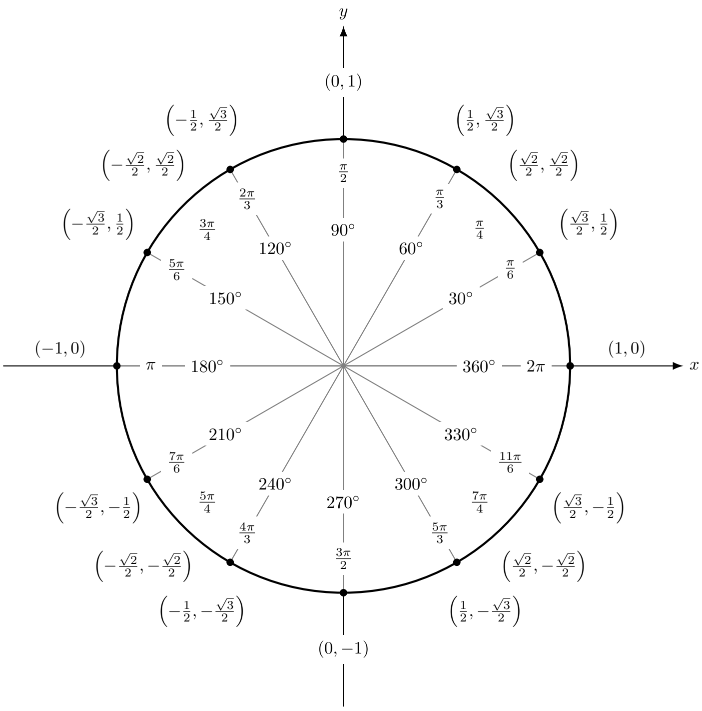
1.1 Extensions to Angles $> 90°$

A point P in the Cartesian plane has coordinates (x, y), where x is considered as positive along OX and negative along OX', while y is considered as positive along OY' and negative along OY. The distance from origin O to point P is positive and denoted by $r = \sqrt{x^2 + y^2}$. The angle A described *counterclockwise* from OX is considered *positive*. If it is described *clockwise* from OX it is considered *negative*.

For an angle A in any quadrant, the trigonometric functions of A are defined as follows:

1. $\sin A = \dfrac{y}{r}$
2. $\cos A = \dfrac{x}{r}$
3. $\tan A = \dfrac{y}{x}$
4. $\csc A = \dfrac{r}{y}$
5. $\sec A = \dfrac{r}{x}$
6. $\cot A = \dfrac{x}{y}$

1.2 The Unit Circle



1.3 Degrees and Radians

A *radian* is that angle θ subtended at center O of a circle by an arc MN equal to the radius r. Since 2π radians $= 360°$ we have:

$1\,\text{radian} = 180°/\pi = 57.29577951308232\ldots°$

$1° = \pi/180\,\text{radians} = 0.017453292519943\ldots\ \text{radians}$

1.4 Signs and Variations

Quadrant	$\sin A$	$\cos A$	$\tan A$
I	$+$ $(0, 1)$	$+$ $(1, 0)$	$+$ $(0, \infty)$
II	$+$ $(1, 0)$	$-$ $(0, -1)$	$-$ $(-\infty, 0)$
III	$-$ $(0, -1)$	$-$ $(-1, 0)$	$+$ $(0, \infty)$
IV	$-$ $(-1, 0)$	$+$ $(0, 1)$	$-$ $(-\infty, 0)$

Quadrant	$\cot A$	$\sec A$	$\csc A$
I	$+$ $(\infty, 0)$	$+$ $(1, \infty)$	$+$ $(\infty, 1)$
II	$-$ $(0, -\infty)$	$-$ $(\infty, -1)$	$+$ $(1, \infty)$
III	$+$ $(\infty, 0)$	$-$ $(-1, \infty)$	$-$ $(\infty, -1)$
IV	$-$ $(0, -\infty)$	$+$ $(\infty, 1)$	$-$ $(-1, \infty)$

2 Properties and General Forms

2.1 Properties

2.1.1 $\sin x$

Domain: $\{x|x \in \mathbb{R}\}$ or $(-\infty, +\infty)$

Range: $\{y|-1 \leq y \leq 1\}$ or $[-1, 1]$

Period: 2π

VA: none

x-intercepts: $k\pi$ where $k \in \mathbb{Z}$

Parity: odd

2.1.2 $\cos x$

Domain: $\{x|x \in \mathbb{R}\}$ or $(-\infty, +\infty)$

Range: $\{y|-1 \leq y \leq 1\}$ or $[-1, 1]$

Period: 2π

VA: none

x-intercepts: $\frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

Parity: even

2.1.3 $\tan x$

Domain: $\left\{x|x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$ or $\bigcup_{k\in\mathbb{Z}} \left(\frac{(k-1)\pi}{2}, \frac{(k+1)\pi}{2}\right)$

Range: $\{y|y \in \mathbb{R}\}$ or $(-\infty, +\infty)$

Period: π

VA: $x = \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

x-intercepts: midway between asymptotes

Parity: odd

2.1.4 $\csc x$

Domain: $\{x|x \neq k\pi, k \in \mathbb{Z}\}$or $\bigcup_{k\in\mathbb{Z}} (k\pi, (k+1)\pi)$

Range: $\{y|y \leq 1 \cup y \geq 1\}$ or $(-\infty, -1] \cup [1, +\infty)$

Period: 2π

VA: $x = k\pi$ where $k \in \mathbb{Z}$

x-intercepts: none

Parity: odd

2.1.5 $\sec x$

Domain: $\left\{x|x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$ or $\bigcup_{k\in\mathbb{Z}} \left(\frac{(k-1)\pi}{2}, \frac{(k+1)\pi}{2}\right)$

Range: $\{y|y \leq 1 \cup y \geq 1\}$ or $(-\infty, -1] \cup [1, +\infty)$

Period: 2π

VA: $x = \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

x-intercepts: none

Parity: even

2.1.6 $\cot x$

Domain: $\{x|x \neq k\pi, k \in \mathbb{Z}\}$ or $\bigcup_{k\in\mathbb{Z}} (k\pi, (k+1)\pi)$

Range: $\{y|y \in \mathbb{R}\}$ or $(-\infty, +\infty)$

Period: π

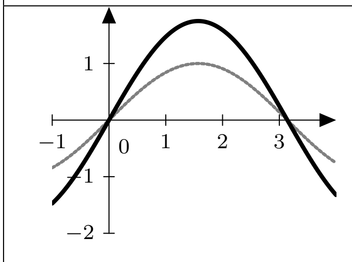
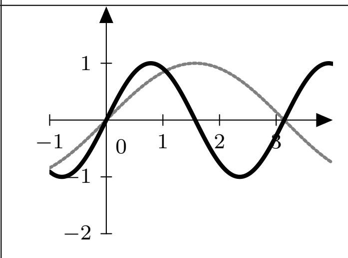
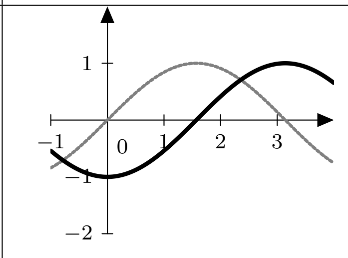
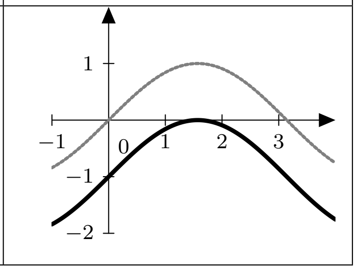
VA: $x = k\pi$ where $k \in \mathbb{Z}$

x-intercepts: midway between asymptotes

Parity: odd

2.2 General Forms of Trigonometric Functions

Given some trigonometric function $f(x)$, its general form is represented as $y = Af(B(x - C)) + D$, where its amplitude is $|A|$, its period is $\frac{2\pi}{|B|}$ or $\frac{\pi}{|B|}$ (for tangent and cotangent), its phase shift is C, and its vertical translation is D units upward (if $D > 0$) or D units downward (if $D < 0$). The maximum and minimum value for $\sin x$ and $\cos x$ is $A + D$ and $-A + D$ respectively.

amplitude	period	phase shift	vertical translation
			

3 Identities

3.1 Basic Identities

3.1.1 Reciprocal Identities

$\csc\theta = \dfrac{1}{\sin\theta};\ \sin\theta = \dfrac{1}{\csc\theta}$

$\sec\theta = \dfrac{1}{\cos\theta};\ \cos\theta = \dfrac{1}{\sec\theta}$

$\cot\theta = \dfrac{1}{\tan\theta};\ \tan\theta = \dfrac{1}{\cot\theta}$

$\sin\theta\csc\theta = \cos\theta\sec\theta = \tan\theta\cot\theta = 1$

3.1.2 Ratio Identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta};\ \cos\theta = \dfrac{\sin\theta}{\tan\theta};\ \sin\theta = \cos\theta\tan\theta$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta};\ \sin\theta = \dfrac{\cos\theta}{\cot\theta};\ \cos\theta = \sin\theta\cot\theta$

3.1.3 Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1;\ \sin^2\theta = 1 - \cos^2\theta;\ \cos^2\theta = 1 - \sin^2\theta$

$\tan^2\theta + 1 = \sec^2\theta;\ \tan^2\theta = \sec^2\theta - 1;\ \sec^2\theta - \tan^2\theta = 1$

$\cot^2\theta + 1 = \csc^2\theta;\ \cot^2\theta = \csc^2\theta - 1;\ \csc^2\theta - \cot^2\theta = 1$

3.1.4 Co-function Identities

$\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$

$\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta$

$\csc\left(\dfrac{\pi}{2} - \theta\right) = \sec\theta$

$\sec\left(\dfrac{\pi}{2} - \theta\right) = \csc\theta$

$\cot\left(\dfrac{\pi}{2} - \theta\right) = \tan\theta$

3.1.5 Parity Identities

$\sin(-A) = -\sin A$

$\cos(-A) = \cos A$

$\tan(-A) = -\tan A$

$\csc(-A) = -\csc A$

$\sec(-A) = \sec A$

$\cot(-A) = -\cot A$

3.2 Sum and Difference

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

$\cot(\alpha \pm \beta) = \dfrac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$

3.3 Double Angle

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$

$\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$

3.4 Half Angle

Let $\mathcal{Q}_n$, where $n \in \{1, 2, 3, 4\}$, denote the set of all angles within the n^{th} quadrant of the Cartesian plane.

$$\sin\frac{\alpha}{2} = \begin{cases} \sqrt{\frac{1-\cos\alpha}{2}} & \frac{\alpha}{2} \in (\mathcal{Q}_1 \cup \mathcal{Q}_2) \\ -\sqrt{\frac{1-\cos\alpha}{2}} & \frac{\alpha}{2} \in (\mathcal{Q}_3 \cup \mathcal{Q}_4) \end{cases}$$

$$\cos\frac{\alpha}{2} = \begin{cases} \sqrt{\frac{1+\cos\alpha}{2}} & \frac{\alpha}{2} \in (\mathcal{Q}_1 \cup \mathcal{Q}_4) \\ -\sqrt{\frac{1+\cos\alpha}{2}} & \frac{\alpha}{2} \in (\mathcal{Q}_2 \cup \mathcal{Q}_3) \end{cases}$$

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha} = \csc\alpha - \cot\alpha$$

3.5 Multiple Angle

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

$\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$

$\sin 4\alpha = 4\sin\alpha\cos\alpha - 8\sin^3\alpha\cos\alpha$

$\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$

$\tan 4\alpha = \dfrac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$

$$\sin(n\alpha) = \sum_{i=0}^{n} \binom{n}{i} \cos^i\alpha \sin^{n-i}\alpha \sin\left(\frac{(n-i)\pi}{2}\right)$$

$$\cos(n\alpha) = \sum_{i=0}^{n} \binom{n}{i} \cos^i\alpha \sin^{n-i}\alpha \cos\left(\frac{(n-i)\pi}{2}\right)$$

3.6 Power Reduction

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$\cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$$

$$\sin^4\theta = \frac{3 - 4\cos 2\theta + \cos 4\theta}{8}$$

$$\cos^4\theta = \frac{3 + 4\cos 2\theta + \cos 4\theta}{8}$$

$$\sin^5\theta = \frac{10\sin\theta - 5\sin 3\theta + \sin 5\theta}{16}$$

$$\cos^5\theta = \frac{10\cos\theta + 5\cos 3\theta + \cos 5\theta}{16}$$

3.7 Product to Sum

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha+\beta) - \cos(\alpha-\beta)\right]$$

3.8 Sum to Product

$$\sin\theta \pm \sin\varphi = 2\sin\frac{\theta\pm\varphi}{2}\cos\frac{\theta\mp\varphi}{2}$$

$$\cos\theta + \cos\varphi = 2\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}$$

$$\cos\theta - \cos\varphi = -2\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}$$

3.9 Linear Combinations

For some purposes it is important to know that any linear combination of sine waves of the same period or frequency but different phase shifts is also a sine wave with the same period or frequency, but a different phase shift.

The two-argument form of the arctangent function, denoted by $\tan^{-1}(y,x)$ gathers information on the signs of the inputs in order to return the appropriate quadrant of the computed angle. Thus, it is defined as:

$$\tan^{-1}(y,x) = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x > 0, \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \tan^{-1}\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

3.9.1 Sine and Cosine

In the case of a non-zero linear combination of a sine and cosine wave (which is just a sine wave with a phase shift of $\frac{\pi}{2}$), we have:

$$a\sin x + b\cos x = c\sin(x+\theta)$$

where $c = \pm\sqrt{a^2+b^2}$ and θ satisfies the equations $c\cos\theta = a$ and $c\sin\theta = b$.

3.9.2 Arbitrary Phase Shift

More generally, for an arbitrary phase shift, we have:

$$a\sin x + b\sin(x+\theta) = c\sin(x+\varphi)$$

where $c = \sqrt{a^2+b^2+2ab\cos\theta}$, and φ satisfies $c\sin(x+\theta) = a + b\cos\theta$ and $c\sin x = b\sin\theta$ or $\varphi = \tan^{-1}(b\sin\theta, a + b\cos\theta)$.

3.10 Other Related Identities

- If $x+y+z=\pi$, then $\sin 2x + \sin 2y + \sin 2z = 4\sin x\sin y\sin z$.
- *Triple Tangent Identity.* If $x+y+z=\pi$, then $\tan x + \tan y + \tan z = \tan x\tan y\tan z$.
- *Triple Cotangent Identity.* If $x+y+z=\frac{\pi}{2}$, then $\cot x + \cot y + \cot z = \cot x\cot y\cot z$.
- *Ptolemy's Theorem.* If $w+x+y+z=\pi$, then $\sin(w+x)\sin(x+y) = \sin w\sin y + \sin x\sin z$.
- $\cot x\cot y + \cot y\cot z + \cot z\cot x = 1$
- $\tan x + \sec x = \tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$
- $\displaystyle\sum_{i=0}^{n}\sin(\varphi+i\alpha) = \frac{\sin\frac{(n+1)\alpha}{2}\sin\left(\varphi+\frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}}$
- $\displaystyle\sum_{i=0}^{n}\cos(\varphi+i\alpha) = \frac{\sin\frac{(n+1)\alpha}{2}\cos\left(\varphi+\frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}}$
- $\displaystyle\sum_{n=1}^{\infty}\prod_{m=1}^{n}\cos\frac{m\pi}{2n+1} = 1$

3.11 Identities without Variables

- $\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ = \dfrac{1}{8}$
- $\sin 20^\circ \cdot \sin 40^\circ \cdot \sin 80^\circ = \dfrac{\sqrt{3}}{8}$
- $\cos 24^\circ + \cos 48^\circ + \cos 96^\circ + \cos 168^\circ = \dfrac{1}{2}$
- $\cos \frac{2\pi}{21} + \cos\left(2 \cdot \frac{2\pi}{21}\right) + \cos\left(4 \cdot \frac{2\pi}{21}\right) + \cos\left(5 \cdot \frac{2\pi}{21}\right) + \cos\left(8 \cdot \frac{2\pi}{21}\right) + \cos\left(10 \cdot \frac{2\pi}{21}\right) = \frac{1}{2}$
- $\cos \dfrac{\pi}{5} = \cos 36^\circ = \frac{1}{4}(\sqrt{5} + 1) = \frac{1}{2}\varphi$
- $\sin \dfrac{\pi}{10} = \sin 18^\circ = \frac{1}{4}(\sqrt{5} - 1) = \frac{1}{2}\varphi^{-1}$
- $\sin^2 18^\circ + \sin^2 30^\circ = \sin^2 36^\circ$

4 Tables

4.1 Exact Values of Trigonometric Functions

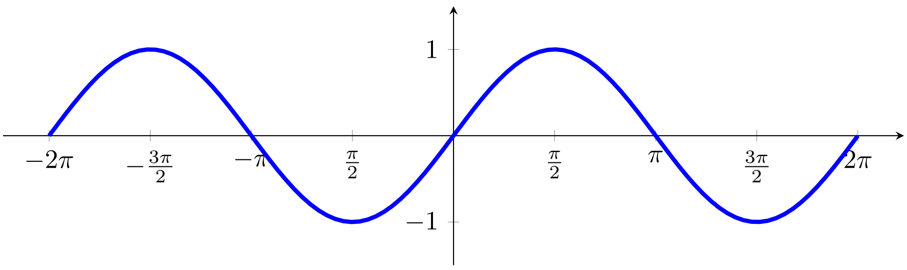
$A°$	A rad	$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
$0°$	0	0	1	0	∞	1	∞
$15°$	$\pi/12$	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$2-\sqrt{3}$	$2+\sqrt{3}$	$\sqrt{6}-\sqrt{2}$	$\sqrt{6}+\sqrt{2}$
$30°$	$\pi/6$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
$45°$	$\pi/4$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$60°$	$\pi/3$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
$75°$	$5\pi/12$	$\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$2+\sqrt{3}$	$2-\sqrt{3}$	$\sqrt{6}+\sqrt{2}$	$\sqrt{6}-\sqrt{2}$
$90°$	$\pi/2$	1	0	$\pm\infty$	0	$\pm\infty$	1
$105°$	$7\pi/12$	$\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$-\left(2+\sqrt{3}\right)$	$-\left(2-\sqrt{3}\right)$	$-\left(\sqrt{6}+\sqrt{2}\right)$	$\sqrt{6}-\sqrt{2}$
$120°$	$2\pi/3$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
$135°$	$3\pi/4$	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
$150°$	$5\pi/6$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
$165°$	$11\pi/12$	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$-\left(2-\sqrt{3}\right)$	$-\left(2+\sqrt{3}\right)$	$-\left(\sqrt{6}-\sqrt{2}\right)$	$\sqrt{6}+\sqrt{2}$
$180°$	π	0	-1	0	$\mp\infty$	-1	$\pm\infty$
$195°$	$13\pi/12$	$-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$2-\sqrt{3}$	$2+\sqrt{3}$	$-\left(\sqrt{6}-\sqrt{2}\right)$	$-\left(\sqrt{6}+\sqrt{2}\right)$
$210°$	$7\pi/6$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
$225°$	$5\pi/4$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
$240°$	$4\pi/3$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
$255°$	$17\pi/12$	$-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$2+\sqrt{3}$	$2-\sqrt{3}$	$-\left(\sqrt{6}+\sqrt{2}\right)$	$-\left(\sqrt{6}-\sqrt{2}\right)$
$270°$	$3\pi/2$	-1	0	$\pm\infty$	0	$\mp\infty$	-1
$285°$	$19\pi/12$	$-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$-\left(2+\sqrt{3}\right)$	$-\left(2-\sqrt{3}\right)$	$\sqrt{6}+\sqrt{2}$	$-\left(\sqrt{6}-\sqrt{2}\right)$
$300°$	$5\pi/3$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
$315°$	$7\pi/4$	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
$330°$	$11\pi/6$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2
$345°$	$23\pi/12$	$-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$-\left(2+\sqrt{3}\right)$	$-\left(2+\sqrt{3}\right)$	$\sqrt{6}-\sqrt{2}$	$-\left(\sqrt{6}+\sqrt{2}\right)$
$360°$	2π	0	1	0	$\mp\infty$	1	$\mp\infty$

4.2 Relations Between Trig Functions

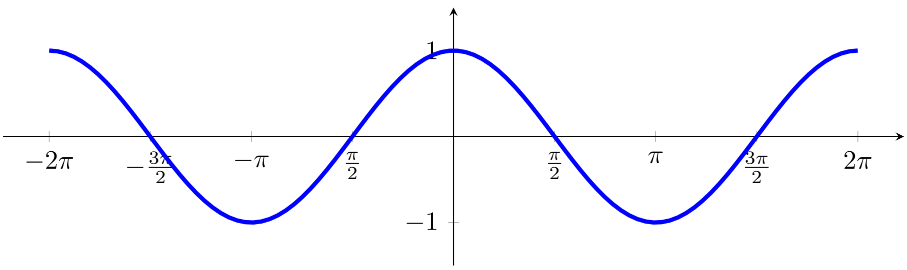
	$\sin\theta = u$	$\cos\theta = u$	$\tan\theta = u$	$\csc\theta = u$	$\sec\theta = u$	$\cot\theta = u$
$\sin\theta$	u	$\sqrt{1-u^2}$	$\frac{u}{\sqrt{1+u^2}}$	$\frac{1}{u}$	$\frac{\sqrt{u^2-1}}{u}$	$\frac{1}{\sqrt{1+u^2}}$
$\cos\theta$	$\sqrt{1-u^2}$	u	$\frac{1}{\sqrt{1+u^2}}$	$\frac{\sqrt{u^2-1}}{u}$	$\frac{1}{u}$	$\frac{u}{\sqrt{1+u^2}}$
$\tan\theta$	$\frac{u}{\sqrt{1-u^2}}$	$\frac{\sqrt{1-u^2}}{u}$	u	$\frac{1}{\sqrt{u^2-1}}$	$\sqrt{u^2-1}$	$\frac{1}{u}$
$\csc\theta$	$\frac{1}{u}$	$\frac{1}{\sqrt{1-u^2}}$	$\frac{\sqrt{1+u^2}}{u}$	u	$\frac{u}{\sqrt{u^2-1}}$	$\sqrt{1+u^2}$
$\sec\theta$	$\frac{1}{\sqrt{1-u^2}}$	$\frac{1}{u}$	$\sqrt{1+u^2}$	$\frac{u}{\sqrt{u^2-1}}$	u	$\frac{\sqrt{1+u^2}}{u}$
$\cot\theta$	$\frac{\sqrt{1-u^2}}{u}$	$\frac{u}{\sqrt{1-u^2}}$	$\frac{1}{u}$	$\sqrt{u^2-1}$	$\frac{1}{\sqrt{u^2-1}}$	u

5 Graphs

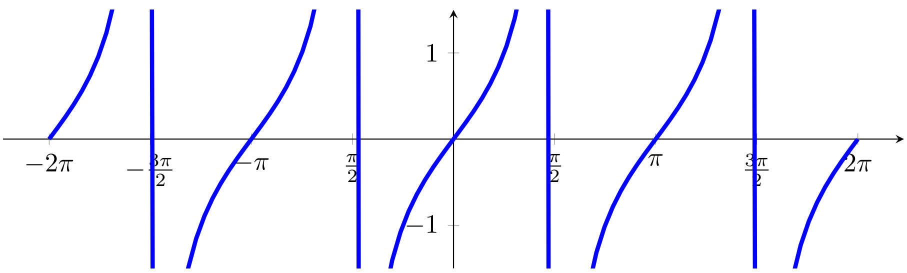
5.1 $y = \sin x$



5.2 $y = \cos x$



5.3 $y = \tan x$



6 Inverse Trigonometric Functions

If $x = \sin y$, then $y = \sin^{-1} x$, i.e. the angle whose sine is x or arcsine of x, is a multiple-valued function of x which is a collection of single-valued functions called *branches*. Similarly, the other inverse trigonometric functions are multiple-valued.

For many purposes, a particular branch is required. This is called the *principal branch* and the values for this branch are called *principal values*.

6.1 Principal Values

Principal values for $x \geq 0$	Principal values for $x < 0$
$0 \leq \sin^{-1} x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \sin^{-1} x < 0$
$0 \leq \cos^{-1} x \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1} x \leq \pi$
$0 \leq \tan^{-1} x < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1} x < 0$
$0 < \cot^{-1} x \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \cot^{-1} x < \pi$
$0 \leq \sec^{-1} x < \frac{\pi}{2}$	$\frac{\pi}{2} < \sec^{-1} x \leq \pi$
$0 < \csc^{-1} x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \csc^{-1} x < 0$

6.2 Identities

In all cases it is assumed that principal values are used.

6.2.1 Reciprocal Identities

$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$$

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

$$\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$$

6.2.2 Parity Identities

$$\sin^{-1}(-x) = -\sin^{-1} x$$

$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$

$$\tan^{-1}(-x) = -\tan^{-1} x$$

$$\csc^{-1}(-x) = -\csc^{-1} x$$

$$\sec^{-1}(-x) = \pi - \sec^{-1} x$$

$$\cot^{-1}(-x) = \pi - \cot^{-1} x$$

6.2.3 Pythagorean Identities (sort of)

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$$

6.2.4 Sum and Difference

$$\sin^{-1}\alpha \pm \sin^{-1}\beta = \sin^{-1}\left(\alpha\sqrt{1-\beta^2} \pm \beta\sqrt{1-\alpha^2}\right)$$

$$\cos^{-1}\alpha \pm \cos^{-1}\beta = \cos^{-1}\left(\alpha\beta \mp \sqrt{(1-\alpha^2)(1-\beta^2)}\right)$$

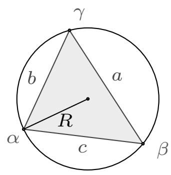
$$\tan^{-1}\alpha \pm \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha \pm \beta}{1 \mp \alpha\beta}\right)$$

7 Relationships Between Sides and Angles

7.1 Law of Sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Extended Law of Sines



$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R,$$

where R is the circumradius of the triangle.

7.2 Law of Cosines

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc};\ a = \sqrt{b^2+c^2-2bc\cos\alpha}$$

$$\cos\beta = \frac{a^2+c^2-b^2}{2ac};\ b = \sqrt{a^2+c^2-2ac\cos\beta}$$

$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab};\ c = \sqrt{a^2+b^2-2ab\cos\gamma}$$

7.3 Law of Tangents

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}$$

$$\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)}$$

$$\frac{c-a}{c+a} = \frac{\tan\frac{1}{2}(\gamma-\alpha)}{\tan\frac{1}{2}(\gamma+\alpha)}$$

7.4 Law of Cotangents

Let s be the semi-perimeter, that is, $s = \frac{(a+b+c)}{2}$, and r be the radius of the inscribed circle, then

$$\frac{\cot\left(\frac{\alpha}{2}\right)}{s-a} = \frac{\cot\left(\frac{\beta}{2}\right)}{s-b} = \frac{\cot\left(\frac{\gamma}{2}\right)}{s-c} = \frac{1}{r}$$

and furthermore that the inradius is given by $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

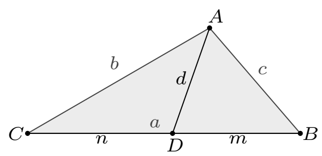
7.5 Mollweide's Formula

Each of these identities uses all six parts of the triangle—the three angles and the lengths of the three sides.

$$\frac{a+b}{c} = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\gamma}{2}\right)}$$

$$\frac{a-b}{c} = \frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\gamma}{2}\right)}$$

7.6 Stewart's Theorem



Let D be a point in $\overline{BC}$ of $\triangle ABC$. If $|BD| = m$, $|CD| = n$, and $|AD| = d$, then $b^2m + c^2n = a\left(d^2 + mn\right)$.

8 Solving Triangles

8.1 AAA Triangle

1. Write "no solution" as your answer.

8.2 AAS/ASA Triangle

1. Solve for the missing angle.
2. Use the Law of Sines twice for the other two sides.

8.3 SAS Triangle

1. Use the Law of Cosines for the other non-included angle.
2. Use the Law of Sines for the missing side.
3. Solve for the missing angle.

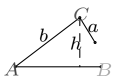
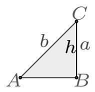
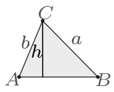
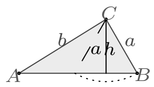
8.4 SSS Triangle

1. Use the Law of Cosines twice for the two angles.
2. Solve for the missing angle.

8.5 SSA Triangle

This is the ambiguous case. There could be either only one solution, two solutions, or even none at all.

When α is acute

$0 < a < h$	$a = h$	$a > b$	$h < a < b$
			
0	1	1	2

When α is obtuse

$0 < a \leq b$	$a > b$
0	1