

6. Quadratic congruences.

A. Ushakov

MA503, October 6, 2021

Contents

Today we discuss quadratic congruences and their applications in cryptography.

- Quadratic congruence $x^2 \equiv_{pq} a$.
- Finding all zeros of $x^2 \equiv_{pq} a$.
- Finding all zeros of $x^2 \equiv_p a$.
- Remote coin flipping. Example. Security.
- General quadratic congruence.
- Quadratic residues.
- Quadratic residues modulo p and primitive roots.
- Euler's criterion.
- Legendre symbol.
- Quadratic reciprocity law.
- Quadratic residues modulo a prime p .
- Quadratic residues modulo a composite n .
- Goldwasser–Micali cryptosystem. Example.

Quadratic congruence $x^2 \equiv_{pq} a$

Proposition

If p is an odd prime, then for every a such that $\gcd(p, a) = 1$ the congruence $x^2 \equiv_p a$ has no solutions or 2 solutions of the form $\pm s$.

- By Lagrange theorem $x^2 - a = 0$ has at most 2 zeros in the field \mathbb{Z}_p .
- Furthermore, if s is a solution, then $-s \not\equiv_p s$ is a solution too.

Proposition

Let p, q be distinct odd primes, $n = pq$. Then for every a such that $\gcd(n, a) = 1$ the congruence $x^2 \equiv_n a$ has either 0 or 4 solutions.

$$x^2 \equiv_{pq} a \Leftrightarrow \begin{cases} x^2 \equiv_p a \\ x^2 \equiv_q a \end{cases}$$

- Assume that $x^2 \equiv_{pq} a$ has a solution.
- Then $x^2 \equiv_p a$ has two solutions $x \equiv_p \pm s$,
- and $x^2 \equiv_q a$ has two solutions $x \equiv_q \pm t$.

Finally, using Chinese remainder theorem to solve four systems

$$\begin{cases} x \equiv_p \pm s \\ x \equiv_q \pm t \end{cases}$$

we get four distinct solutions of the form $\{\alpha_1, -\alpha_1, \alpha_2, -\alpha_2\}$.

Finding all zeros of $x^2 \equiv_{pq} a$

Lemma

Let p, q be distinct odd primes, $n = pq$. Let $\pm\alpha_1, \pm\alpha_2$ be the solutions of $x^2 \equiv_n a$. Then $\gcd(n, \alpha_1 - \alpha_2) = p$ or q .

$$\begin{aligned}\alpha_1^2 \equiv_n a \text{ and } \alpha_2^2 \equiv_n a &\Rightarrow 0 \equiv_n \alpha_1^2 - \alpha_2^2 = (\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \\ &\Rightarrow n \mid (\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2).\end{aligned}$$

But $\pm\alpha_1, \pm\alpha_2$ are **distinct modulo n** and, hence,

$$n \nmid \alpha_1 - \alpha_2 \quad \text{and} \quad n \nmid \alpha_1 + \alpha_2$$

Hence, one of the primes, say p , divides $\alpha_1 - \alpha_2$ and q divides $\alpha_1 + \alpha_2$. Therefore,

$$\gcd(n, \alpha_1 - \alpha_2) = p < n.$$

Corollary (Reduction: factorization \rightarrow solving $x^2 \equiv_n a$)

Given n (which is a product of two unknown distinct primes p, q) if we can find all solutions of $x^2 \equiv_n a$ (for any a), then we can find p and q , i.e., solve factorization problem for n .

Since factorization for numbers of the form pq is computationally hard, the problem to find all solutions of $x^2 \equiv_n a$ is hard.

Finding all zeros of $x^2 \equiv_p a$

Let p be a prime such that $p \equiv_4 3$. Let $1 \leq a \leq p-1$. Assume that $x^2 \equiv_p a$ has a solution $1 \leq \alpha < p$, i.e., $\alpha^2 \equiv_p a$.

Lemma

The numbers $x = \pm a^{(p+1)/4}$ are the solutions of $x^2 \equiv_p a$.

Indeed, $x^2 = \left(a^{(p+1)/4}\right)^2 = a^{(p+1)/2} \equiv_p (\alpha^2)^{(p+1)/2} \equiv_p \alpha^{p+1} \equiv_p \alpha^{p-1} \alpha^2 \equiv_p a$.

Corollary

If $p \equiv_4 3$, then we can efficiently solve quadratic congruences $x^2 \equiv_p a$.

But what if $p \equiv_4 1$? That gives two cases:

- If $p \equiv_8 5$, then either $x \equiv_p \pm a^{\frac{n+3}{8}}$ or $x \equiv_p \pm a^{\frac{n+3}{8}} 2^{\frac{n-1}{4}}$
- If $p \equiv_8 1$, then apply Tonelli (randomized) algorithm.

Thus, there are efficient algorithms to compute \sqrt{a} modulo any prime p .

Protocol for remote coin flipping

Consider a simple game of 2 people, Alice and Bob. Alice throws a fair coin and if

- the coin lands **heads up** then Alice wins;
- the coin lands **tails up** then Bob wins.

This game is easy to play if Alice and Bob are in the same place. But what if Alice is in US and Bob is in Europe and they use a phone line for communication? In such a setting they can use the following protocol.

-
- **Alice** chooses large distinct primes p and q satisfying $p \equiv_4 3$.
 - **Alice** computes $n = pq$ and sends n to Bob.
 - **Bob** chooses a random $1 \leq \alpha_1 \leq n - 1$ such that $\gcd(\alpha_1, n) = 1$.
 - **Bob** sends $a = \alpha_1^2 \% n$ to Alice.
 - **Alice** finds all solutions of $x^2 \equiv_n a$ by solving modulo p and q

$$\begin{cases} x^2 \equiv_p a \\ x^2 \equiv_q a \end{cases} \Rightarrow \begin{cases} x \equiv_p \pm a^{(p+1)/4}, \\ x \equiv_q \pm a^{(q+1)/4}. \end{cases}$$

That gives four systems of linear congruences. Using CRT, Alice finds four solutions $\pm\alpha_1, \pm\alpha_2$ and sends a random solution $z \in \{\pm\alpha_1, \pm\alpha_2\}$ to Bob.

- **Bob** checks if $z = \pm\alpha_1$ and, if so, he announces Alice a winner. Otherwise Bob computes p and q (he can do that efficiently when $z \neq \pm\alpha_1$), announces himself a winner, and sends p, q to Alice as a proof.

Protocol for remote coin flipping: example

- **Alice** chooses $p = 43$ and $q = 71$, and send $n = 43 \cdot 71 = 3053$ to Bob.
- **Bob** chooses random $x = 192$ and sends $a = 192^2 \% 3053 = 228$ to Alice.
- **Alice** solves $x^2 \equiv_{43} 228 \equiv_{43} 13$ by computing the power

$$13^{(p+1)/4} = 13^{11} \equiv_{43} 20.$$

and gets ± 20 . **Alice** solves $x^2 \equiv_{71} 228 \equiv_{71} 15$ by computing the power

$$15^{(q+1)/4} = 15^{18} \equiv_{71} 21$$

and get ± 21 . Then she solves four systems of linear congruences

$$\begin{cases} x \equiv_{43} 20 \\ x \equiv_{71} 21 \end{cases} \quad \begin{cases} x \equiv_{43} -20 \\ x \equiv_{71} 21 \end{cases} \quad \begin{cases} x \equiv_{43} 20 \\ x \equiv_{71} -21 \end{cases} \quad \begin{cases} x \equiv_{43} -20 \\ x \equiv_{71} -21 \end{cases}$$

Solving each subsystem she finds 4 solutions of $x^2 \equiv_{3053} 228$:

$$x \equiv_{3053} 192, 2861, 1399, 1654.$$

Note that $-192 \equiv_{3053} 2861$ and $-1399 \equiv_{3053} 1654$. Assume that Alice chooses $z = 1399$ and sends it to Bob.

- Bob compares Alice's choice 1399 with his original number $1399 \equiv \pm 192?$ and since comparison fails he announces himself a winner. To prove that he is indeed a winner he computes $\gcd(3053, 1399 - 192) = \gcd(3053, 1207) = 71$ and sends 71 and $3053/71 = 43$ to Alice.

Protocol for remote coin flipping: security

If Bob announces himself a winner, he has to prove that by showing prime factors of n .

- It does not give Bob any new information if he receives $z = \pm\alpha_1$ (recall that he generated α_1 himself). In that case Bob has to factor n , which is computationally hard.
- If $z \neq \pm\alpha_1$, then $z = \pm\alpha_2$. In that case Bob has all four solutions $\{\pm\alpha_1, \pm\alpha_2\}$ and can compute prime factors of n by computing $\gcd(\alpha_1 \pm \alpha_2, n)$.

Theorem

Dishonest Bob can not cheat (efficiently) in the coin flipping protocol unless he can (efficiently) solve the integer factorization problem for n .

General quadratic congruence

Definition

Let p be an odd prime. Let $a, b, c \in \mathbb{Z}$, where $p \nmid a$. The congruence $ax^2 + bx + c \equiv_p 0$ is called a **quadratic congruence**.

$$\begin{aligned} ax^2 + bx + c \equiv_p 0 &\Leftrightarrow x^2 + \frac{b}{a}x + \frac{c}{a} \equiv_p 0 \\ &\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} \equiv_p 0 \\ &\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 \equiv_p \frac{b^2}{4a^2} - \frac{c}{a}. \end{aligned}$$

Hence, to solve a quadratic congruence, we can solve a congruence $y^2 \equiv_n d$, where $d = \frac{b^2}{4a^2} - \frac{c}{a}$ for y and then subtract $\frac{b}{2a}$, i.e., compute

$$\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} - \frac{b}{2a} = \frac{1}{2a} \left(\sqrt{b^2 - 4ac} - b \right).$$

Does it look familiar?

Everything reduces to $\sqrt{b^2 - 4ac}$ modulo p .

(It is important that we can divide by 2 and by a modulo p in computations above.)

Quadratic residues

Definition

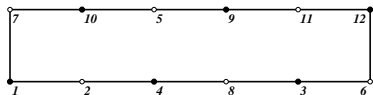
A unit $a \in \mathbb{Z}$ modulo $n \in \mathbb{N}$ is called a **quadratic residue** of n if $x^2 \equiv_n a$ has a solution. Otherwise, it is called a **quadratic nonresidue** of n .

$Q_n = \left\{ a \in U_n \mid x^2 \equiv_n a \text{ has a solution} \right\}$ – the set of all residues of n .

For instance, to find all quadratic residues of 13 we compute squares of units $1, \dots, 6$:

$$\begin{array}{llll} 1^2 \equiv_{13} 1 & \Rightarrow & \sqrt{1} = 1 \text{ or } 12 & \text{mod } 13 \\ 2^2 \equiv_{13} 4 & \Rightarrow & \sqrt{4} = 2 \text{ or } 11 & \text{mod } 13 \\ 3^2 \equiv_{13} 9 & \Rightarrow & \sqrt{9} = 3 \text{ or } 10 & \text{mod } 13 \\ 4^2 \equiv_{13} 3 & \Rightarrow & \sqrt{3} = 4 \text{ or } 9 & \text{mod } 13 \\ 5^2 \equiv_{13} 12 & \Rightarrow & \sqrt{12} = 5 \text{ or } 8 & \text{mod } 13 \\ 6^2 \equiv_{13} 10 & \Rightarrow & \sqrt{10} = 6 \text{ or } 7 & \text{mod } 13 \end{array}$$

We do not need to compute squares of $7, \dots, 12$, they are the same. Therefore 1, 3, 4, 9, 10, 12 are quadratic residues of 13.



$r = 2$ is a primitive root of 13. Consider the graph of powers of 2. The quadratic residues of 13 have black dots and nonresidues have white dots.

Observe that **every second power of r is a residue**. This is true for every odd prime p .

Quadratic residues modulo p and primitive roots

Lemma

Let p be an odd prime. Let r be a primitive root of p and $a \equiv_p r^k$. Then a is a quadratic residue of p if and only if k is even.

$$\begin{aligned} a \text{ is a quadratic residue} &\Leftrightarrow x^2 \equiv_p a \text{ is satisfied for some } x = r^i \\ &\Leftrightarrow r^{2i} \equiv_p r^k \\ &\Leftrightarrow 2i \equiv_{p-1} k \\ &\Leftrightarrow \gcd(2, p-1) = 2 \text{ divides } k \\ &\Leftrightarrow k \text{ is even.} \end{aligned}$$

± 1 are the only solutions of $x^2 \equiv_p 1$.

$$(a) \quad a \equiv_p r^k \Rightarrow a^{(p-1)/2} \equiv_p \pm 1;$$

$$(b) \quad k \text{ is even} \Rightarrow a^{(p-1)/2} \equiv_p 1;$$

$$(c) \quad k \text{ is odd} \Rightarrow a^{(p-1)/2} \equiv_p -1.$$

$$(a) \quad (a^{(p-1)/2})^2 = a^{p-1} \equiv_p 1 \text{ and, hence, } a^{(p-1)/2} \text{ is a solution of } x^2 \equiv_p 1.$$

$$(b) \quad k = 2j \Rightarrow a^{(p-1)/2} \equiv_p (r^{2j})^{(p-1)/2} \equiv_p r^{j(p-1)} \equiv_p (r^{p-1})^j \equiv_p 1^j = 1.$$

$$(c) \quad k = 2j+1 \Rightarrow a^{(p-1)/2} \equiv_p (r^{2j+1})^{(p-1)/2} \equiv_p r^{j(p-1)+(p-1)/2} \equiv_p r^{(p-1)/2} \equiv_p -1.$$

Quadratic residues modulo p : Euler's criterion

The next theorem gives an easy way to check if a is a quadratic residue of p .

Theorem (Euler's criterion)

Let p be an odd prime and $\gcd(p, a) = 1$. Then

$$(EC1) \quad a \in Q_p \iff a^{(p-1)/2} \equiv_p 1.$$

$$(EC2) \quad a \notin Q_p \iff a^{(p-1)/2} \equiv_p -1.$$

For instance,

- 2 is a quadratic residue modulo 7, because $2^{\frac{7-1}{2}} \equiv_7 1$.
- 2 is a quadratic residue modulo 17, because $2^{\frac{17-1}{2}} \equiv_{17} 1$.
- 3 is a quadratic nonresidue modulo 17, because $3^{\frac{17-1}{2}} \equiv_{17} -1$.

Legendre symbol

Definition

Let p be an odd prime and $a \in \mathbb{Z}$. The **Legendre symbol** (a/p) is defined by

$$(a/p) = \begin{cases} 0 & \text{if } a \text{ is not a unit modulo } p; \\ 1 & \text{if } a \text{ is a quadratic residue of } p; \\ -1 & \text{if } a \text{ is a quadratic nonresidue of } p. \end{cases}$$

Legendre symbol is just a piece of notation. It is introduced for convenience!

(Easy properties of the Legendre symbol)

- | | |
|-----------------------------|---|
| (1) $(1/p) = 1$. | (4) $(ab/p) = (a/p)(b/p)$. |
| (2) $(a^2/p) = 1$. | (5) $(-1/p) = -1^{(p-1)/2} = \begin{cases} 1 & \text{if } p \equiv_4 1; \\ -1 & \text{if } p \equiv_4 3. \end{cases}$ |
| (3) $(a/p) = a^{(p-1)/2}$. | |

Proposition

- (\mathbb{Q}_p, \cdot) is a group.
- $a \mapsto (a/p)$ is a homomorphism from U_p to $(\{-1, 1\}, \cdot)$.

Quadratic reciprocity law

Furthermore, the following holds.

Theorem (Square roots of 2 modulo p)

Let p be an odd prime, then

$$(2/p) = \begin{cases} 1 & \text{if } p \equiv_8 1 \text{ or } p \equiv_8 7; \\ -1 & \text{if } p \equiv_8 3 \text{ or } p \equiv_8 5. \end{cases}$$

Theorem (Quadratic reciprocity law)

Let p and q be distinct odd primes. Then

$$(p/q) = \begin{cases} (q/p) & \text{if } p \equiv_4 1 \text{ or } q \equiv_4 1; \\ -(q/p) & \text{if } p \equiv_4 3 \text{ and } q \equiv_4 3. \end{cases}$$

$$(29/53) = (53/29) = (24/29) = (6/29)$$

$$= (2/29)(3/29) = -1 \cdot (29/3) = -(2/3) = 1.$$

$$(-46/17) = (-63/17) = (9/17)(-7/17) = (10/17) = (2/17)(5/17)$$

$$= 1 \cdot (17/5) = (2/5) = -1.$$

Quadratic residues modulo prime powers

Theorem

$a \in \mathbb{Q}_{p^s} \iff a \in \mathbb{Q}_p$ for an odd prime p .

Theorem

Let a be an odd integer. The following holds.

- a is a quadratic residue modulo 2.
- a is a quadratic residue modulo 4 $\iff a \equiv_4 1$.
- a is a quadratic residue modulo 2^s $\iff a \equiv_8 1$.



Quadratic residues modulo general composite numbers

Let $n = p_1^{a_1} \dots p_m^{a_m}$ and $a \in U_n$.

Theorem

a is a quadratic residue of $n \Leftrightarrow a$ is a quadratic residue of $p_i^{a_i}$ for each $1 \leq i \leq m$.

For instance, $a = 6$ is not a quadratic residue of $n = 65 = 5 \cdot 13$ because

$$\begin{aligned}(6/5) &= (1/5) = 1, \\ (6/13) &= (2/13)(3/13) = -1(13/3) = -(1/3) = -1.\end{aligned}$$

$a = 108$ is a quadratic residue of $n = 143 = 11 \cdot 13$ because

$$\begin{aligned}(108/11) &= (9/11) = 1, \\ (108/13) &= (4/13) = 1.\end{aligned}$$

$a = 106$ is not a quadratic residue of $n = 143 = 11 \cdot 13$ because

$$\begin{aligned}(106/11) &= (7/11) = -(11/7) = -(4/7) = -1, \\ (106/13) &= (2/13) = -1.\end{aligned}$$

In general, the problem to decide if a is a quadratic residue of a composite n is computationally hard!

It is easy only if we know prime factorization of n .

Goldwasser–Micali cryptosystem

*Goal: Bob wants to encrypt and send Alice one bit m (called **plaintext**).*

Key generation (performed by Alice):

- Choose randomly two distinct prime numbers p and q .
- Compute $n = pq$ called **public modulus**.
- Choose a random integer a such that $(a/p) = (a/q) = -1$.

Finally, Alice publishes the numbers n and a , together called **Alice's public key**.

Encryption (performed by Bob):

- Choose a random $1 < r < n$ and use (n, a) to compute

$$c = \begin{cases} r^2 \% n & \text{if } m = 0, \\ ar^2 \% n & \text{if } m = 1. \end{cases}$$

The number c (called **ciphertext**) is then sent to Alice.

Decryption (performed by Alice):

- Alice computes (c/p) and $m = \begin{cases} 0 & \text{if } (c/p) = 1, \\ 1 & \text{if } (c/p) = -1. \end{cases}$

By design, $m = 0 \Leftrightarrow (c/p) = 1$. Hence, Alice recovers the original value of m .

Goldwasser–Micali cryptosystem

*Goal: Bob wants to encrypt and send Alice one bit m (called **plaintext**).*

Key generation (performed by Alice):

- Choose randomly two distinct prime numbers p and q .
- Compute $n = pq$ called **public modulus**.
- Choose a random integer a such that $(a/p) = (a/q) = -1$.

Finally, Alice publishes the numbers n and a , together called **Alice's public key**.

Encryption (performed by Bob):

- Choose a random $1 < r < n$ and use (n, a) to compute

$$c = \begin{cases} r^2 \% n & \text{if } m = 0, \\ ar^2 \% n & \text{if } m = 1. \end{cases}$$

The number c (called **ciphertext**) is then sent to Alice.

Decryption (performed by Alice):

- Alice computes (c/p) and $m = \begin{cases} 0 & \text{if } (c/p) = 1, \\ 1 & \text{if } (c/p) = -1. \end{cases}$

By design, $m = 0 \Leftrightarrow (c/p) = 1$. Hence, Alice recovers the original value of m .

Goldwasser–Micali cryptosystem

*Goal: Bob wants to encrypt and send Alice one bit m (called **plaintext**).*

Key generation (performed by Alice):

- Choose randomly two distinct prime numbers p and q .
- Compute $n = pq$ called **public modulus**.
- Choose a random integer a such that $(a/p) = (a/q) = -1$.

Finally, Alice publishes the numbers n and a , together called **Alice's public key**.

Encryption (performed by Bob):

- Choose a random $1 < r < n$ and use (n, a) to compute

$$c = \begin{cases} r^2 \% n & \text{if } m = 0, \\ ar^2 \% n & \text{if } m = 1. \end{cases}$$

The number c (called **ciphertext**) is then sent to Alice.

Decryption (performed by Alice):

- Alice computes (c/p) and $m = \begin{cases} 0 & \text{if } (c/p) = 1, \\ 1 & \text{if } (c/p) = -1. \end{cases}$

By design, $m = 0 \Leftrightarrow (c/p) = 1$. Hence, Alice recovers the original value of m .

Goldwasser–Micali cryptosystem

*Goal: Bob wants to encrypt and send Alice one bit m (called **plaintext**).*

Key generation (performed by Alice):

- Choose randomly two distinct prime numbers p and q .
- Compute $n = pq$ called **public modulus**.
- Choose a random integer a such that $(a/p) = (a/q) = -1$.

Finally, Alice publishes the numbers n and a , together called **Alice's public key**.

Encryption (performed by Bob):

- Choose a random $1 < r < n$ and use (n, a) to compute

$$c = \begin{cases} r^2 \% n & \text{if } m = 0, \\ ar^2 \% n & \text{if } m = 1. \end{cases}$$

The number c (called **ciphertext**) is then sent to Alice.

Decryption (performed by Alice):

- Alice computes (c/p) and $m = \begin{cases} 0 & \text{if } (c/p) = 1, \\ 1 & \text{if } (c/p) = -1. \end{cases}$

By design, $m = 0 \Leftrightarrow (c/p) = 1$. Hence, Alice recovers the original value of m .

Goldwasser–Micali cryptosystem (worked out example)

Key generation

- Choose random primes $p = 7$, $q = 13$.
- We can choose $a = 5$ because $(5/7) = (5/13) = -1$.

Encryption: To encrypt $m = 1$ Bob

- picks a random $r = 11$
- sends $5 \cdot 11^2 \equiv_{91} 59 = c$ to Alice.

Decryption: To decrypt the message Alice computes $(59/7) = (5/7) = (7/5) = (2/5) = -1$ and, hence, $m = 1$.

*Goldwasser–Micali cryptosystem is an example of a **probabilistic encryption** because the same message m has many different ciphers (defined by the choice of r).*

E.g., in the example above, Bob could generate $r = 2$ in which case he'd get the cipher $c = 20$.

Goldwasser–Micali cryptosystem: basic security

By design, a passive eavesdropper obtains two public values: n and c . Furthermore,

$$\begin{aligned} m = 0 &\Leftrightarrow x^2 \equiv_n c \text{ has a solution} \\ &\Leftrightarrow c \text{ is a quadratic residue modulo } n = pq. \end{aligned}$$

Hence, security of the Goldwasser–Micali cryptosystem relies on computational hardness of the problem to decide if a is a quadratic residue of n , or not.

Goldwasser–Micali cryptosystem: message expansion

The Goldwasser–Micali public key cryptosystem is not practical, because each bit of the plaintext is encrypted with a number modulo N . For it to be secure, it is necessary that Eve be unable to factor the number $N = pq$, so in practice N should be (at least) a 1000-bit number. Thus, if Alice wants to send k bits of plaintext to Bob, her ciphertext will be $1000k$ bits long. Thus the Goldwasser–Micali public key cryptosystem has a **message expansion** ratio of 1000, since the ciphertext is 1000 times as long as the plaintext. In general, the Goldwasser–Micali public key cryptosystem expands a message by a factor of $\log_2(N)$. There are other probabilistic public key cryptosystems whose message expansion is much smaller.