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MA 503 Homework 1.

1.1. Let $a = 1485$ and $b = 1745$

(1) find $gcd(1485, 1785)$

$b > a$

Euclidean formula
$gcd(a,b)$
$b = q_1 \cdot a + r_1$.

$1745 = 1 \cdot 1485 + 260$

$1485 = 5 \times 260 + 185$

$260 = 185 \times 1 + 75$

$185 = 2 \times 75 + 35$

$75 = 35 \times 2 + 5$

$35 = 7 \times 5 + 0.$

$\therefore gcd(1485, 1785) = 5$

(2) find $\alpha, \beta \in \mathbb{Z}$ satisfying $1485 \cdot \alpha + 1785 \cdot \beta = gcd(1485, 1745)$

$5 = 75 - 35 \times 2$

$5 = 75 - (185 - 2 \times 75) \times 2$

$= 5 \cdot 75 - 185 \times 2$

$\Rightarrow 5 \cdot (260 - 185) - 185 \cdot 2$

$\Rightarrow 5 \cdot 260 - 7 \cdot 185$

$\Rightarrow 5 \cdot 260 - 7(1485 - 5 \times 260)$

$= 40 \cdot 260 - 7(1485)$

$= 40(1745 - 1485) - 7 \cdot (1485)$

$= 40(1745) - 47(1485)$

$= -47(1485) + 40(1745)$

$\therefore$ this is in the form $\alpha(1485) + 1785 \cdot \beta$

$\therefore \alpha = -47 \quad \beta = 40$

(3) Compute lcm (1485, 1745)

$$\text{lcm}(1485, 1745) = \frac{1485 * 1745}{\gcd(1485, 1745)}$$

$$= \frac{1485 \cdot 1745}{5}$$

$$\Rightarrow \underline{\underline{518,265}}$$

Exercise 1.2

The fibonacci numbers $\{f_i\}$ are defined recurrently by

$$\begin{cases} f_1 = 1, \\ f_2 = 1; \\ f_3 = f_1 + f_2 \\ \vdots \\ f_n = f_{n-1} + f_{n-2} \end{cases}$$

Use Euclidean lemma to show $\gcd(f_n, f_{n+1}) = 1$

$\Rightarrow$ Since we know $f_1 = 1$, $f_2 = 1$, $f_3 = 2$,
we can conclude that $\gcd(f_1, f_2) = 1$

$$\boxed{\begin{aligned} &\gcd(a+b, b) \\ &= \gcd(a, b) \end{aligned}}$$

Suppose $\gcd(f_n, f_{n+1}) = 1$, we will show
$\gcd(f_{n+1}, f_{n+2}) = 1$.

Consider $\gcd(f_{n+1}, f_{n+2}) = \gcd(f_{n+1}, f_{n+1} + f_n)$

$\because f_{n+2} = f_{n+1} + f_n$

then $\gcd(f_{n+1}, f_{n+1} + f_n) = \gcd(f_{n+1}, f_n) = 1$
(gcd property)

Hence, $\gcd(f_n, f_{n+1}) = 1$ for all $n > 0$

1.3] Use Mathematical Induction to prove that

$6 \mid 7^n - 1$ for every $n \in \mathbb{N}$.

Proof

$\Rightarrow$

$n=1$ $7^1 - 1 = 7-1 = 6$ is divide by 6

Inductive step: assuming statement is true for all cases of n,

ie,

$7^n - 1$ is divisible by 6

then for case $n+1$

$7^{n+1} - 1 = 7 \times 7^n - 1 = 7 \cdot (7^n - 1) + 6$

is also divisible by 6 since both term are divisible by 6.

Hence the proof.

or

Induct^n step:

$7^{n+1} - 1 = 7 \cdot 7^n - 1$

$\Rightarrow (6+1) 7^n - 1$

$\Rightarrow 6 \cdot 7^n + (7^n - 1)$

by hypothesis,

$(7^n - 1)$ is divisible by 6

& the sum is also divisible by 6.

1.4] Prove that a decimal no $a_n a_{n-1} \cdots a_1 a_0$ is divisible by 11 iff the alternating sum of digits:

$a_n - a_{n-1} + a_{n-2} - a_{n-3} + a_{n-4} - \cdots$ is divisible by 11.

$\Rightarrow$ Let take a number

$a_n 10^n + a_{n-1} 10^{n-1} + a_{n-2} 10^{n-2} + \ldots a_1 10^1 + a_0 10^0$

Now $10^n \equiv_{11} (-1)^n$ for $n \in \mathbb{N}$, Hence

$a_n 10^{10} + a_{n-1} 10^{n-1} \cdots a_0 10^0 \equiv_{11}$

$a_n (-1)^n + a_{n-1} (-1)^{n-1} \ldots a_0 \cdot 1$ no

n (any n-digit number

$\Rightarrow$ $x \equiv_{11} a_n - a_{n-1} + a_{n-2} - a_{n-3} \cdots$

with example,

Let us assume a no 7293 to check for its divisibility by 11, we need to add alternate no.

so

$7 \quad 2 \quad 9 \quad 3$

$7 + 9 = 16 \quad 2 + 3 = 5$

difference = 11 & this divisible by 11.

1.5] Compute the remainder of division of 3^{100} by 7

$\Rightarrow$ $\frac{3^{100}}{7}$

Lets write 3^{100} in terms of to nearest multiple of 7.

$\therefore \frac{(3^3)^{33} \cdot 3}{7}$

$\Rightarrow \frac{27^{33} \cdot 3}{7}$

$\Rightarrow \frac{(28-1)^{33} \cdot 3}{7} \Rightarrow \frac{(-1)^{33} \cdot 3}{7} \Rightarrow \frac{-3}{7}$

since 28 is a multiple of 7

since we cannot have a negative remainder that is -3, to find positive ~~number~~ remainder we just add 7 to it.

$\therefore -3+7 = +4$ $\therefore$ the remainder is +4

1.6] Prove that $6 \mid n(n+1)(2n+1)$ for every $n \in \mathbb{N}$ by checking $[n]_6 \cdot [n+1]_6 \cdot [2n+1]_6 = [0]$ for each congruence class $[n]_6$.

~~For~~ Proof by example

$\rightarrow$ By using mathematical induction, lets subs. $n=1,2,3,4$ ..

for $n=1$

$n(n+1)(2n+1) = 1(1+1)(2 \cdot 1+1)$

$= 6$ divible by 6.

for $n=2 \Rightarrow$ $2(2+1)(2 \cdot 2+1) = 30$

$n=3 \Rightarrow$ $3(3+1)(3 \cdot 2+1) = 84$

$n=4 \Rightarrow$ $4(4+1)(2 \cdot 4+1) = 180$

again divisible by 6.

$\therefore$ for every value of n remainder is 0.

Since we know from the example any value of n substituted in $n(n+1)(2n+1)$ will be even.

it satisfies mod 2 & mod 3.

and since its divisible by 2, 3
we know that if a no is divided by 2, 3 it will be divisible by 6.

Hence, the proof.

1-7] Consider the set of all complex number C equipped with standard multiplication. which of the following are closed under?

(1) $\mathbb{R}$ → Yes.

(2) The set of purely imaginary no $\mathbb{R}i = \{ai \mid a \in \mathbb{R}\}$ → Yes

(3) $\{1, -1, i, -i\}$ → NO

(4) $\mathbb{N}$ → Yes

(5) $\{a + b\sqrt{2}i \mid a, b \in \mathbb{Q}\}$ → Yes

(6) $\{-1, 0, 1\}$ → Yes.

1-8] Define on $X = \{a, b, c\}$ using table

$\cdot$	a	b	c
a	b	a	c
b	b	c	a
c	c	c	c

commutative
$a \cdot b = b \cdot a$

(1) is commutative? → NO.

$\because$ $a \cdot b = a$ & $b \cdot a = b$

$a \cdot b \neq b \cdot a$ so its not commutative

(2) Is associative? → NO.

$\because$ $(a \cdot b) \cdot c \neq a \cdot (b \cdot c)$

$(a \cdot b) = a$
$(a \cdot b) \cdot c = c$

& $a(b \cdot c) = a$
$a \cdot (b \cdot c)$
$= a \cdot a = \underline{b}$

$\boxed{\therefore\ c \neq b}$ → reason

(3) Is closed on $\{a, b\}$? → NO

$\because$ closed subset has 4 condition $a \cdot a$, $a \cdot b$, $b \cdot a$ & $b \cdot b$.
Here $b \cdot b = c$, so the condition fails.

(4) we say that $x \in X$ is the multiplicative identity if $xy = y$ for every $y \in X$? Do we have a multiplicative identity for our operation? → NO.

it doesnot meet the condition $a \cdot b = b \cdot a = a$ & so.