Topic 5

Events and Probability

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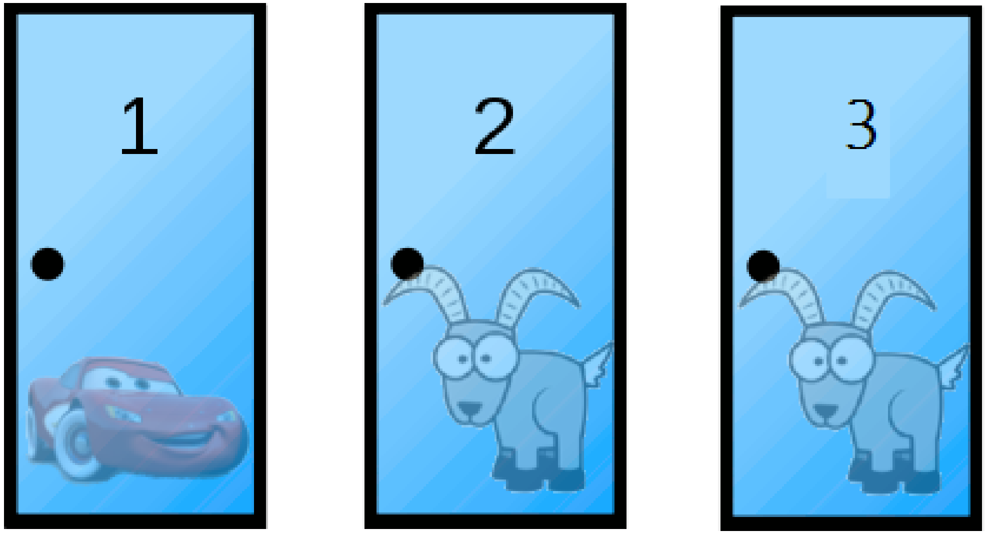
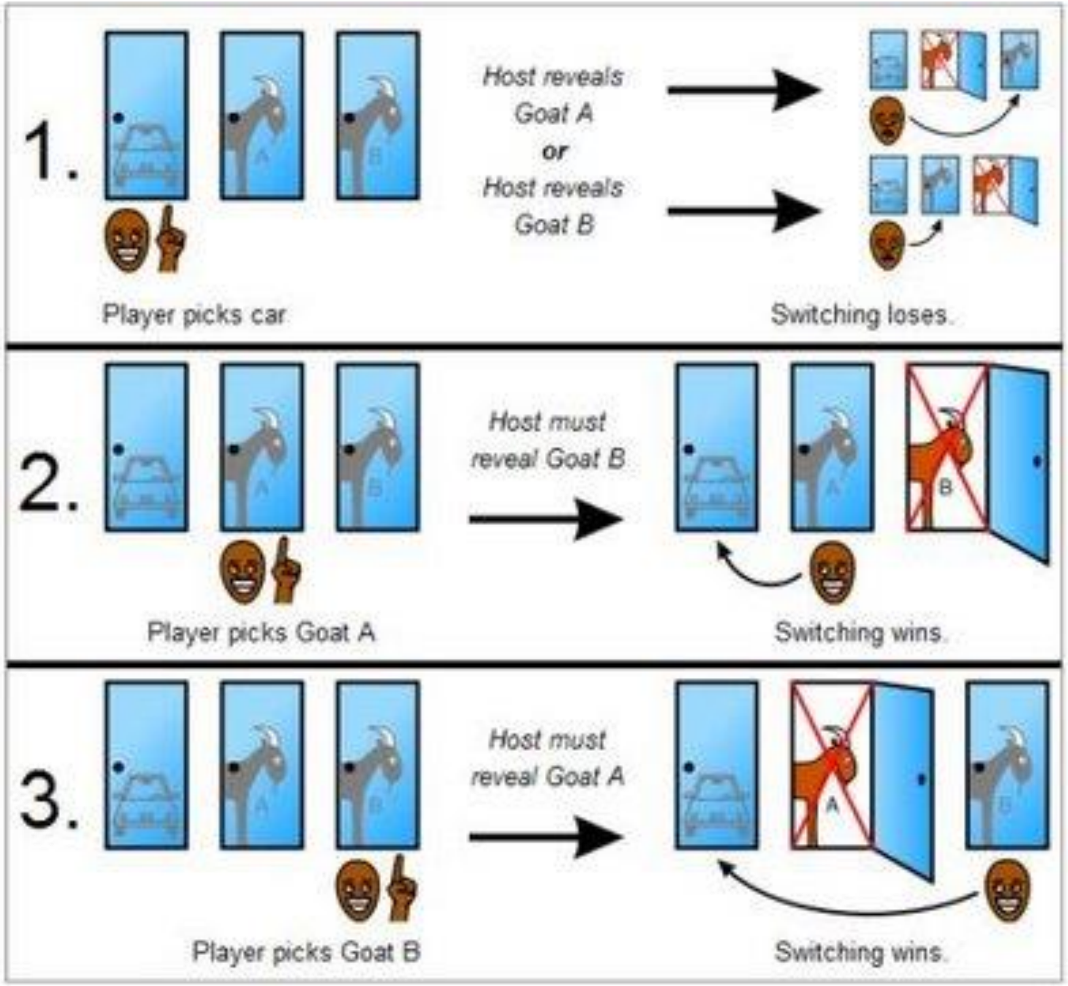
Monty Hall Problem



Monty Hall Problem



Basic Counting Techniques

- **Multiplication Principle**
  - Without Repetition: $n!$
  - With Repetition: n^r
- **Permutation** - groupings+arrangments
  - Distinct Objects
    - Arrange in a line/row
      - Select all (n): $n! = {}^nP_n$
      - Select r: ${}^nP_r = \frac{n!}{(n-r)!}$
    - Arrange in a circle: $(n-1)!$
  - Identical Objects: $\frac{n!}{n_1!n_2!\ldots n_r!}$
- **Combination** - groupings only
  - Without Repetition: ${}^nC_r = \frac{n!}{r!(n-r)!}$
  - With Repetition: $\frac{(n+r-1)!}{r!(n-1)!}$

Cases

1) Ranking (eg. 1st, 2nd, 3rd,…) - **Permutation**
2) Position
a) President, Vice President, Treasurer, Secretary – **Permutation**
b) Committee - **Combination**

Experiment, Outcomes and Sample Space

Experiment

- A **process** by which an observation (or measurement) is obtained

Outcomes

- All the **possible observations** of the experiment

Sample Space

- The **set of all possible outcomes** of a statistical experiment and presented by the symbol

Set Point

- The **elements** of a sample space

Examples

<u>Example 1:</u>

Toss a coin once ← Experiment

Head, Tail ← Outcomes

S={Head, Tail} ← Sample Space

<u>Example 2:</u>

A roll of dice ← Experiment

1, 2, 3, 4, 5, 6 ← Outcomes

S={1, 2, 3, 4, 5, 6} ← Sample Space

<u>Example 3:</u>

Birth of a baby ← Experiment

Boy, Girl ← Outcomes

S={Boy, Girl} ← Sample Space

5.2.1 Multiplication Principles

- If an experiment consists of three steps and if the first step can result in m outcomes, the second step in n outcomes and the third step in k outcomes, then

Total outcomes for the experiment = $m.n.k$

EXAMPLE

A certain shoe comes in 2 different styles with each style available in 4 distinct colors. If the store wishes to display pairs of these shoes showing all of its various styles and colors, how many different pairs would the store have on displays?

$$2 \times 4 = 8$$

style color

Example

If an experiment consists of throwing a dice and then drawing a letter at random from the English alphabet, how many set points are in the sample space?

$\underline{6}$ x $\underline{26}$ = 156

Dice Alphabet

EXAMPLE

How many set of meals consisting of a soup, sandwich, dessert, and a drink are possible if we can select from 4 soups, 3 kinds of sandwiches, 5 desserts and 4 drinks?

$$4 \times 3 \times 5 \times 4 = 240$$

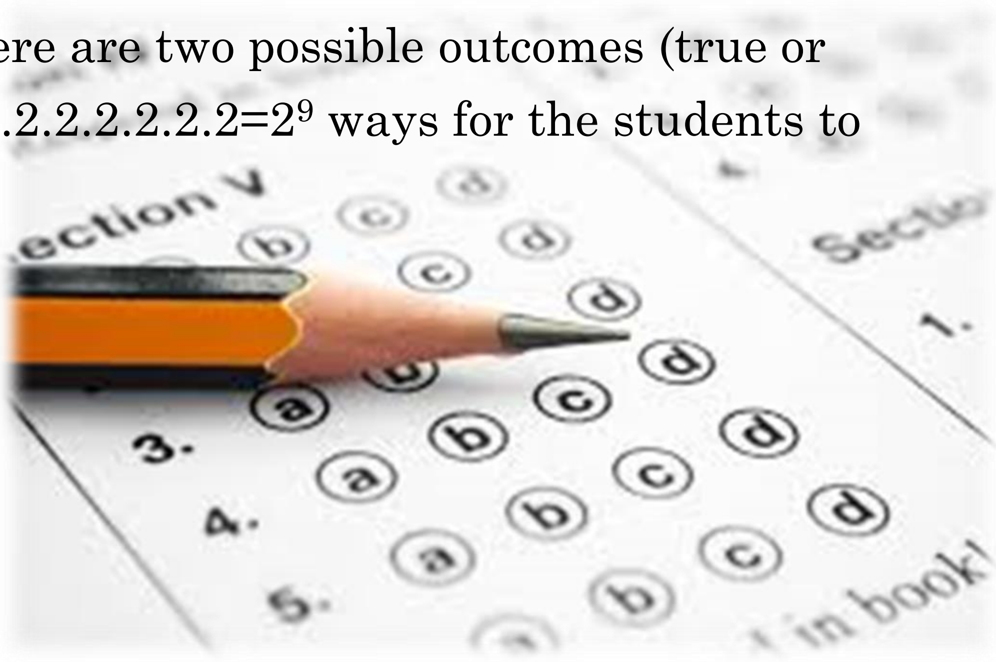
Soups sandwiches desserts drinks

Try This

If a test consists of 9 true-false questions, in how many different ways can a student marks the test paper with one answer to each questions.

Solution:

Each of the questions, there are two possible outcomes (true or false). So, there are 2.2.2.2.2.2.2.2.2=2^9 ways for the students to answer the questions.



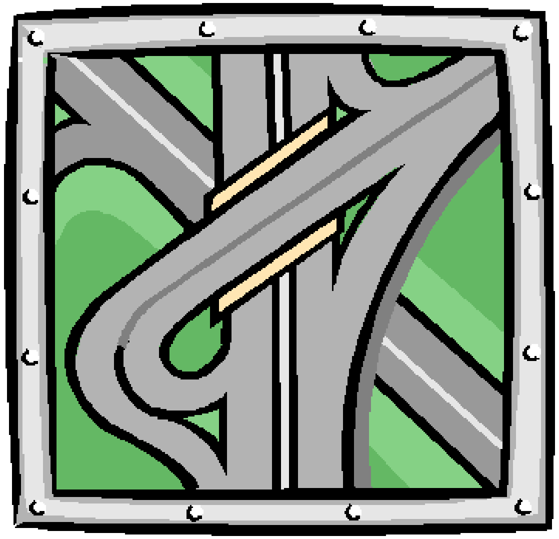
Try This

There are six roads between A and B and four roads between B and C. Find the number of ways that one can drive:

a. from A to C by way of B

b. round-trip from A to C by way of B

c. round-trip from A to C by way of B without using the same road more than once



Solution:

a. $A \longrightarrow B \longrightarrow C$

$= 6 \times 4 = 24$ ways

b. $A \rightleftarrows B \rightleftarrows C$

$= 6 \times 4 \times 4 \times 6 = 576$

c. $6 \times 4 \times 3 \times 5 = 360$

5.2.2 Permutations

- An **arrangement** of all or part of a set of objects.
- The number of permutations of n distinct objects is,

$$n! = {}^{n}P_{n}$$

Example

Consider 3 letters which is a, b and c. How many possible ways that it can be arranged?

Solution:

So, $abc, acb, bac, cab, bca, cba = 6$

or

$n! = 3.2.1 = 6$

The number of permutations of ***n*** distinct objects taken ***r*** at a time is

$$ {}^{n}P_{r} = \frac{n!}{(n-r)!} $$

Example:

Consider 3 letters which is *a*, *b* and *c*. How many possible ways that it can be arranged if to form a two letters word?

Solution:

$$ {}^{3}P_{2} = \left(\frac{3!}{(3-2)!} \right) $$

$$ = \frac{3!}{1!} $$

$$ = 6 $$

Example

Two lottery tickets are drawn from 20 for a first and a second prize. Find the number of permutations in the space S.

Solution:

$$
\begin{aligned}
{}^{20}P_2 &= \left(\frac{20!}{(20-2)!}\right) \\
&= \frac{20!}{18!} \\
&= 380
\end{aligned}
$$

Example

There are 20 members in a club. In how many ways that the selection can be done if one president and one vice president are to be selected?

Solution:

$$^{20}P_2 = 380$$

Try This

How many different arrangements start with the letter D can be made from the letters of the word MODULE ?

Solution:

$$^{5}P_{5} = 120$$

The number of permutations of $\boldsymbol{n}$ distinct objects arranged in a circle is $(n\ \text{-}1)!$

Example:

In how many ways can 5 students be arranged in a circle?

Solution:

$(n\text{-}1)!=(5\text{-}1)!$

$=24$

The number of distinct permutations of things of which n_1 are of one kind, n_2 of a 2nd kind…,n of a k^{th} kind is

$$\boxed{\frac{n!}{n_1!n_2!\ldots n_k!}}$$

Example:

How many different ways can 3 red, 4 yellow and 2 blue bulbs be arranged in a string of Christmas tree lights with 9 sockets.

Solution:

$$\frac{9!}{3!4!2!} = 1260$$

The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the 2nd, and so forth is $\binom{n}{n_{1,} n_2 \ldots n_r} = \dfrac{n!}{n_1! n_2! \ldots n_r!}$ where $n_1 + n_2 + \ldots + n_r = n.$

Example:

In how many ways can 7 scientists be assigned to one triple and two double hotel rooms?

Solution:

$$\binom{7}{3.2.2} = \frac{7!}{3!2!2!}$$

$$= 210$$

Try This

How many different words can be made from the letter of the word ***PROBABILITY*** ?

Solution:

$$\frac{11!}{2!2!} = 9979200$$

5.2.3 Combinations

- A grouping of all or part of a set of objects.
- The number of groupings of ***n*** distinct objects taken ***r*** at a time is

$$^{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- ** selection without replacement

Example

Faculty of Computing and Informatics MMU offers 6 selective subjects to Beta students. The students are required to choose 2 elective subjects. If the students randomly select 2 elective subjects out of 6 subjects, how many possible combinations are there?

Solution:

So, $n = 6, r = 2$

Thus,

$$
\begin{aligned}
{}^{6}C_{2} &= \binom{6}{2} \\
&= \frac{6!}{2!(6-2)!} \\
&= \frac{6!}{2!4!} \\
&= 15
\end{aligned}
$$

there are 15 ways for the students to select 2 elective subjects.

Example

A bag contains six white marbles and five red marbles.
Find the number of ways three marbles can be drawn from the bag if

a) they can be any color
b) one must be white and two red
c) they must all be of the same color

Solution:

a) $^{11}C_3 = 165$

b) $^{6}C_1 \times {}^{5}C_2 = 60$

c) $^{6}C_3 + {}^{5}C_3 = 30$

Example

A certain class has 10 male and 8 female students. How many are there to form a committee of members consisting of:

i) 5 people

ii) 5 people with the same gender

Solution:

i) $^{18}C_5 = 8560$

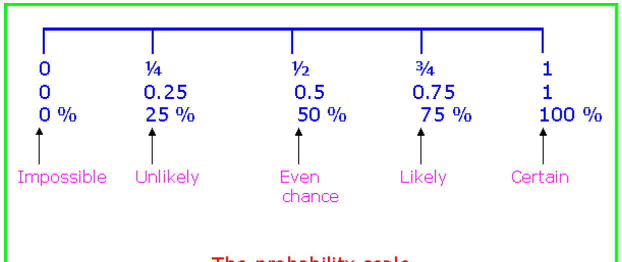
ii) $^{10}C_5 + {}^{8}C_5 = 308$

5.3 Concept of Probability

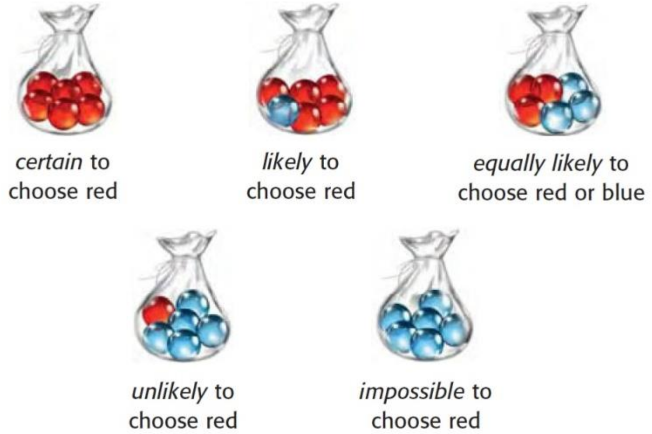
5.3.1 Probability of An Event

- The probability of event A to occur,

$$P(A) = \frac{\text{Number of ways that event A can occur}}{\text{Total number of possible outcomes}}$$
$$= \frac{n(A)}{n(S)}$$



The probability scale



EXAMPLE

i) Find the probability of obtaining a head and probability of obtaining a tail for one toss of a coin.

ii) Find the probability of obtaining an even number in one roll of a dice.

SOLUTION:

i) P (head) = 0.5   and P (tail) = 0.5

ii) P (even) = 0.5

5.3.2 Axioms of Probability

- Let event A be an event in a finite sample space S,
- Axiom 1: $0 \leq P(A) \leq 1$ for each event A in S
- Axiom 2: $\mathbf{P(\,\boldsymbol{S}\,) = 1}$

Example

A mixture of candies contains 6 mints, 4 coffees, and 3 chocolates. If a person makes a random selection of one of these candies, find the probability of getting

a) a mint

b) a coffee or a chocolate

Solution:

a) $P(\text{a mint}) = \frac{6}{13}$

b) $P(\text{a coffee} \cup \text{a chocolate}) = \frac{3+4}{13} = \frac{7}{13}$

Example

There are 3 of 20 tires in storage are defective and 4 of them are randomly chosen for inspection (that is, each tire has the same chance of being selected), what is the probability that only one of the defective tires will be included?

Solution:

$$\frac{{}^{3}C_{1} \times {}^{17}C_{3}}{{}^{20}C_{4}} = \frac{\binom{3}{1}\binom{17}{3}}{\binom{20}{4}} \quad \text{or} \quad P(3\overline{D}1D) = \left(\frac{17\times16\times15\times3}{20\times19\times18\times17}\right)4$$

$$= 0.4211$$

Try This

In a shipment of 15 Pre-Calculus books, 7 have missing cover. Out of a sample of 4, what is the probability that at least 3 have the missing cover?

Solution: 0.2308

Try This

Determine the probability that the sum of the dice is 2 or 9 if two balance dice are rolled?

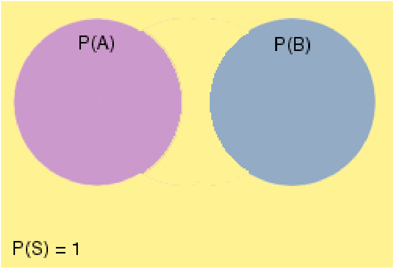
Solution: 5/36

5.3.3 The Addition Rule

- If events A and B are:

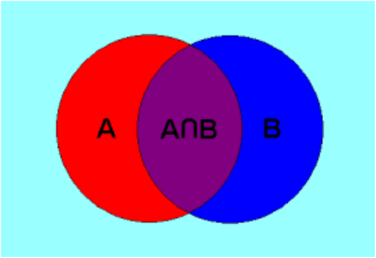
a) Mutually exclusive/disjoint,

- $P(A \cup B) = P(A) + P(B)$



b) Mutually non-exclusive/joint,

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Example

The probability that a student will rate a lecturer during academic evaluation very poor, poor, fair, good, very good or excellent are 0.07, 0.12, 0.17, 0.32, 0.21 and 0.11. What are the probabilities that he/she will rate the lecturer very poor, poor, fair or good?

Solution:

$$P(\text{very poor} \cup \text{poor} \cup \text{fair} \cup \text{good})$$
$$= 0.07 + 0.12 + 0.17 + 0.32$$
$$= 0.68$$

Example

Given the contingency table below, find the following:

	Full Time(FT)	Part Time(PT)	Total
Male (M)	3808	2535	6343
Female (F)	4321	3598	7919
Total	8129	6133	14262

a) $P(FT) = \dfrac{8129}{14262}$

b) $P(FT \cup F) = P(FT) + P(F) - P(FT \cap F)$

$$= \frac{8129}{14262} + \frac{7919}{14262} - \frac{4321}{14262} = \frac{11727}{14262}$$

c) $P(F \cap M) = 0$

d) $P(F \cup M) = P(F) + P(M)$

$$= \frac{6343}{14262} + \frac{7919}{14262} = 1$$

e) $P(PT \cap M) = \dfrac{2535}{14262}$

Important Keywords

English Sentence	Inequality Sign
x is at least 5	$x \geq 5$
x is at most 5	$x \leq 5$
x is between 5 and 7	$5 < x < 7$
x is from 5 to 7	$5 \leq x \leq 7$
x is no more than 5	$x \leq 5$
x is no less than 5	$x \geq 5$
x is less than 5	$x < 5$
x is greater/more than 5	$x > 5$

EXAMPLE

In a certain residential suburb, 60% of all households subscribe metropolitan newspaper published in a nearby city, 80% subscribe to the ocal afternoon paper, and 50% of all households subscribe to both papers. If a household is selected at random, what is the probability that it subscribe to

i) at least one of the two newspaper

ii) exactly one of the two newspaper

SOLUTION:

Let A = metropolitan newspaper and B = local afternoon paper

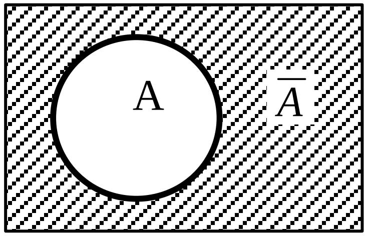
Given $P(A) = 0.6$, $P(B) = 0.8$, $P(A \cap B) = 0.5$

i) $P(A \cup B) = P(A) + P(B) – P(A \cap B) = 0.6 + 0.8 - 0.5 = 0.9$

ii) $P(\text{exactly one}) = 0.1 + 0.3 = 0.4$

5.3.4 The Complement Rule

- Let event A be an event in sample space S,
- **P(A') = 1- P(A)** where P(S) = 1



Try This

In a certain college town, 25% of the students failed Mathematics, 15% failed chemistry and 10% failed both Mathematics and Chemistry. A student is selected at random. What is the probability that he failed neither Mathematics nor Chemistry?

Solution: 0.70

5.4 Concept of Conditional Probability

5.4.1 Conditional Probability

- We will use the notation P (B | A) to represent the probability of an event B will occur given that another event A has already occurred:

$$\underbrace{P(B}_{\text{occurred later}}|\underbrace{A)}_{\text{occurred first}} = \frac{P(B \cap A)}{P(A)}$$

Note that:

If event A, that event B will…

If event A, …event B…

Given event A, that event B…

Event A - first

Event B - second (later)

a) **$P(B \cap A) = P(A \cap B)$**

b) **$P(B|A) \neq P(A|B)$**

Try This

In Ali's school, 15% of the students failed Mathematics, 25% of the students failed English and 10% of the students failed both Mathematics and English.

a) If Ali failed English, find the probability that he will fail Mathematics.
b) If Ali failed Mathematics, find the probability that he will fail English.
c) What is the probability that Ali failed either Mathematics or English?

Solution: a) 2/5 b) 2/3 c) 0.30

Try This

The events A and B are such that $P(A) = 0.7$, $P(B \cup A) = 0.9$ and $P(B | A) = 0.45$.

a) Find $P(B)$
b) Find $P(A | B)$

Solution: a) 0.515 b) 0.6117

Try This

The events A and B are such that $P(A) = 0.7$, $P(B \cup A) = 0.9$ and $P(B \mid A) = 0.45$.

a) Find $P(B)$
b) Find $P(A \mid B)$

Solution: a) 0.515 b) 0.6117

Try This

The probability that a randomly selected employee from a company drive to his/her working place is 0.32 and the joint probability that an employee selected is a female who does not drive is 0.08. If an employee selected at random who does not drive, what is the probability that the employee is a female?

Solution: 0.1176

Try This

Find a) $P(F)$ b) $P(M)$ c) $P(PT \cap F)$ d) $P(FT \cap F)$

e) $P(PT \cap M)$ f) $P(FT \cap M)$ g) $P(PT \mid F)$ h) $P(FT \mid F)$

i) $P(PT \mid M)$ j) $P(FT \mid M)$

	Full Time(FT)	Part Time(PT)	Total
Male (M)	3808	2535	6343
Female (F)	4321	3598	7919
Total	8129	6133	14262

Try This

If the probability that a communication system will have high fidelity is 0.81 and the probability that it will have high fidelity and selectivity is 0.18, what is the probability that a system with high fidelity will also have selectivity?

Solution:2/9

Try This

A random sample of 200 adults are classified below by sex and their level of education attained.

Education	Male	Female
Elementary	38	45
Secondary	28	50
College	22	17
Total	88	112

If a person is picked at random from this group, find the probability that

a) the person is a male, given that the person has a secondary education

b) the person does not have a college degree, given that a person is a female

Solution: (a) 28/88 (b) 95/112

5.4.2 Independent Event

- **Event A and event B are INDEPENDENT events only if**
- $P(A|B) = P(A)$ **or** $P(B|A) = P(B)$ **or** $P(A \cap B) = P(A) \cdot P(B)$
- The occurrence of one event **does not affect** the probability of the occurrence of the other event.

EXAMPLE

Are events 'Machine I (A)' and 'Defective (D)' are independent?

	Defective (D)	Good (G)	Total
Machine I (A)	9	51	60
Machine II (B)	6	34	40
Total	15	85	100

SOLUTION: Yes, they are independent

EXAMPLE

The probability that a randomly selected student from a class wears contact lens is 0.20 and the joint probability that a student selected at random is a girl who does not wear contact lens is 0.03. If the class having the same number of male and female students. Are events "does not wear contact lens" and "the student selected is a girl" independent?

SOLUTION: Not independent

Try This

There are total of 400 Beta students registered in Software Engineering for both campuses. The table below represented the tabulation of students according to their gender and campus.

	Melaka (ML)	Cyberjaya (C)	Total
Female (F)		60	180
Male (M)			
Total		140	

a) Complete the table.

b) If one person is selected at random from these 400 students, find the probability that

i) This person is a male given that he is from Cyberjaya campus

ii) This person is female or from Melaka campus

c) Are the events "Melaka" and "female" independent?

Solution: bi) 4/7 bii) 4/5 c) No

Try This

A survey was conducted on the online services provided by a bank where one thousand customers of the bank were interviewed. The responses of these customers are shown in the following probability distribution table.

	Good (G)	Average (A)	Poor (P)
Male (M)	0.36	0.20	0.02
Female (F)	0.24	0.08	0.10

Are events "Female" and "Poor" independent?

Solution: Not independent