

## Homework 6

Please note that you have to typeset your assignment (using L<sup>A</sup>T<sub>E</sub>X, Word, etc). Hand-written assignments will not be graded. You should submit a pdf version to Gradescope by 23:59 on November 5th. Be careful not to move questions or swap their numbers when using this template.

1.  $(3 \times 5)$  Write the first five values in the sequence:

(a)  $C_1 = 5$

$$C_n = 6C_{n-1} + 3 \text{ for } n \geq 2$$

i.  $C_1 = 5$

ii.  $C_2 = 33$

iii.  $C_3 = 201$

iv.  $C_4 = 1,209$

v.  $C_5 = 7,257$

(b)  $S_1 = 2$

$$S_2 = 2$$

$$S_n = 3S_{n-1} + 2S_{n-2} \text{ for } n \geq 3$$

i.  $S_1 = 2$

ii.  $S_2 = 2$

iii.  $S_3 = 10$

iv.  $S_4 = 34$

v.  $S_5 = 122$

(c)  $S_1 = 10$

$$S_n = 4S_{n-1} + 10 \text{ for } n \geq 2$$

i.  $S_1 = 10$

ii.  $S_2 = 50$

iii.  $S_3 = 210$

iv.  $S_4 = 850$

v.  $S_5 = 3410$

(d)  $S_1 = 1$

$$S_n = 2S_{n-1} + \frac{1}{n} \text{ for } n \geq 2$$

i.  $S_1 = 1$

ii.  $S_2 = \frac{5}{2}$

iii.  $S_3 = \frac{16}{3}$

iv.  $S_4 = \frac{131}{12}$

v.  $S_5 = \frac{661}{30}$

(e)  $A_1 = 2$

$$A_2 = 3$$

$$A_n = 3A_{n-1}A_{n-2} \text{ for } n \geq 3$$

i.  $A_1 = 2$

ii.  $A_2 = 3$

iii.  $A_3 = 18$

iv.  $A_4 = 162$

v.  $A_5 = 8,748$

2.  $(12 \times 2)$  Prove the following property of the Fibonacci sequence:

(a)  $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$

Basis Step ( $n = 1, n = 2$ ):

$$F_1 = 1, F_2 = 1$$

$$F_1 = F_2$$

$$1 = 1$$

$$F_1 + F_3 = F_4$$

$$3 = 3$$

Inductive Step ( $n > 2$ ):

$$\text{Assume } F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$$

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} + F_{2n+1} = F_{2(n+1)}$$

$$F_{2n} + F_{2n+1} = F_{2n+2}$$

$$F_{2n+2} = F_{2n+2}$$

$$(b) \ F_n = 3F_{n-3} + 2F_{n-4} \text{ for } n \geq 5$$

$$F_n = F_{n-1} + F_{n-2} = 3F_{n-3} + 2F_{n-4}$$

$$F_n = F_{n-2} + F_{n-3} + F_{n-3} + F_{n-4} = 3F_{n-3} + 2F_{n-4}$$

$$F_n = F_{n-3} + F_{n-4} + F_{n-3} + F_{n-3} + F_{n-4} = 3F_{n-3} + 2F_{n-4}$$

$$F_n = 3F_{n-3} + 2F_{n-4} = 3F_{n-3} + 2F_{n-4}$$

3. (16) Use induction to prove the following property of the Fibonacci sequence:

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$$

Basis Step ( $n = 1, n = 2$ ):

$$F_1 = F_3 - 1 = 1$$

$$F_1 + F_2 = F_4 - 1 = 2$$

Inductive Step ( $n > 2$ ):

Assume that  $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$ .

Prove that  $F_1 + F_2 + F_3 + \cdots + F_n + F_{n+1} = F_{n+3} - 1$

$$F_{n+2} - 1 + F_{n+1} = F_{n+3} - 1$$

$$F_{n+2} + F_{n+1} = F_{n+3}$$

$$F_{n+3} = F_{n+3}$$

4. ( $10 \times 3$ ) Give a recursive definition for each of the following sets and operation:

(a) The set of all binary strings containing an odd number of 0s.

- Basis Step:  $0 \in S$
- Recursive Step:  $x \in S \rightarrow 1x \in S \wedge x1 \in S \wedge 00x \in S \wedge x00 \in S \wedge 0x0 \in S$

(b) The set of all binary strings ending with 0.

- Basis Step:  $0 \in S$
- Recursive Step:  $x \in S \rightarrow 0x \in S \wedge 1x \in S \wedge x0 \in S$

(c) The set of all binary strings with an equal number of 0s and 1s.

- Basis Step:  $01 \in S, 10 \in S$
- Recursive Step:  $x \in S \rightarrow 01x \in S \wedge 10x \in S \wedge 0x1 \in S \wedge 1x0 \in S \wedge x01 \in S \wedge x10 \in S$

5. ( $5 \times 3$ ) Find the closed-form solution to each of the following recursively defined sequences:

(a)  $S_1 = 5$

$$S_n = S_{n-1} + 5 \text{ for } n \geq 2$$

$$S_n = 5n$$

(b)  $A_1 = 1$

$$A_n = 2A_{n-1} + 1 \text{ for } n \geq 2.$$

$$(\text{Hint: } 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1)$$

$$A_n = 2^n - 1$$

(c)  $B_1 = 1$

$$B_n = B_{n-1} + (2n - 1) \text{ for } n \geq 2.$$

$$(\text{Hint: } 1 + 3 + 5 + \cdots + (2n - 1) = n^2)$$

$$B_n = n^2$$