Homework 6

Please note that you have to typeset your assignment (using LaTeX, Word, etc). Handwritten assignments will not be graded. You should submit a pdf version to Gradescope by 23:59 on November 5th. Be careful not to move questions or swap their numbers when using this template.

1. (3×5) Write the first five values in the sequence:

   (a) $C_1 = 5$
   $C_n = 6C_{n-1} + 3$ for $n \geq 2$

      i. $C_1 = 5$
      ii. $C_2 = 33$
      iii. $C_3 = 201$
      iv. $C_4 = 1,209$
      v. $C_5 = 7,257$

   (b) $S_1 = 2$
   $S_2 = 2$
   $S_n = 3S_{n-1} + 2S_{n-2}$ for $n \geq 3$

      i. $S_1 = 2$
      ii. $S_2 = 2$
      iii. $S_3 = 10$
      iv. $S_4 = 34$
      v. $S_5 = 122$

   (c) $S_1 = 10$
   $S_n = 4S_{n-1} + 10$ for $n \geq 2$

      i. $S_1 = 10$
      ii. $S_2 = 50$
      iii. $S_3 = 210$
      iv. $S_4 = 850$
      v. $S_5 = 3410$

(d) $S_1 = 1$

$S_n = 2S_{n-1} + \frac{1}{n}$ for $n \geq 2$

i. $S_1 = 1$
ii. $S_2 = \frac{5}{2}$
iii. $S_3 = \frac{16}{3}$
iv. $S_4 = \frac{131}{12}$
v. $S_5 = \frac{661}{30}$

(e) $A_1 = 2$

$A_2 = 3$

$A_n = 3A_{n-1}A_{n-2}$ for $n \geq 3$

i. $A_1 = 2$
ii. $A_2 = 3$
iii. $A_3 = 18$
iv. $A_4 = 162$
v. $A_5 = 8,748$

2. (12×2) Prove the following property of the Fibonacci sequence:

   (a) $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$

   Basis Step $(n = 1, n = 2)$:

   $F_1 = 1,\ F_2 = 1$

   $F_1 = F_2$

   $1 = 1$

   $F_1 + F_3 = F_4$

   $3 = 3$

   Inductive Step $(n > 2)$:

   Assume $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$

   $F_1 + F_3 + F_5 + \cdots + F_{2n-1} + F_{2n+1} = F_{2(n+1)}$

   $F_{2n} + F_{2n+1} = F_{2n+2}$

   $F_{2n+2} = F_{2n+2}$

(b) $F_n = 3F_{n-3} + 2F_{n-4}$ for $n \geq 5$

$F_n = F_{n-1} + F_{n-2} = 3F_{n-3} + 2F_{n-4}$

$F_n = F_{n-2} + F_{n-3} + F_{n-3} + F_{n-4} = 3F_{n-3} + 2F_{n-4}$

$F_n = F_{n-3} + F_{n-4} + F_{n-3} + F_{n-3} + F_{n-4} = 3F_{n-3} + 2F_{n-4}$

$F_n = 3F_{n-3} + 2F_{n-4} = 3F_{n-3} + 2F_{n-4}$

3. (16) Use induction to prove the following property of the Fibonacci sequence:

   $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$

   Basis Step ($n = 1, n = 2$):

   $F_1 = F_3 - 1 = 1$

   $F_1 + F_2 = F_4 - 1 = 2$

   Inductive Step ($n > 2$):

   Assume that $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$.

   Prove that $F_1 + F_2 + F_3 + \cdots + F_n + F_{n+1} = F_{n+3} - 1$

   $F_{n+2} - 1 + F_{n+1} = F_{n+3} - 1$

   $F_{n+2} + F_{n+1} = F_{n+3}$

   $F_{n+3} = F_{n+3}$

4. (10×3) Give a recursive definition for each of the following sets and operation:

   (a) The set of all binary strings containing an odd number of 0s.

   - Basis Step: $0 \in S$
   - Recursive Step: $x \in S \to 1x \in S \wedge x1 \in S \wedge 00x \in S \wedge x00 \in S \wedge 0x0 \in S$

   (b) The set of all binary strings ending with 0.

   - Basis Step: $0 \in S$
   - Recursive Step: $x \in S \to 0x \in S \wedge 1x \in S \wedge x0 \in S$

   (c) The set of all binary strings with an equal number of 0s and 1s.

   - Basis Step: $01 \in S, 10 \in S$
   - Recursive Step: $x \in S \to 01x \in S \wedge 10x \in S \wedge 0x1 \in S \wedge 1x0 \in S \wedge x01 \in S \wedge x10 \in S$

5. (5×3) Find the closed-form solution to each of the following recursively defined sequences:

(a) $S_1 = 5$

$S_n = S_{n-1} + 5$ for $n \geq 2$

$S_n = 5n$

(b) $A_1 = 1$

$A_n = 2A_{n-1} + 1$ for $n \geq 2$.

(Hint: $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$)

(c) $B_1 = 1$

$B_n = B_{n-1} + (2n - 1)$ for $n \geq 2$.

(Hint: $1 + 3 + 5 + \cdots + (2n - 1) = n^2$)