Fall 2020 CSCI358 Midterm 2

Instructions:

- There are five questions in this paper. Please use the space provided (next to the questions) to write the answers, in the same way that you do for homework. You will submit your exam on Gradescope.
- Budget your time to answer various questions and avoid spending too much time on a particular question.
- The due date for this exam is 8:00am Nov 13. You have 72 hours to work on it.
- Since this is an exam, you can only ask clarification questions and are not supposed to discuss solutions on Piazza.

Problem 1. (8)

1. Let

$$A = \{x | x \in \mathbb{N} \text{ and } 1 < x < 50\}$$
$$B = \{x | x \in \mathbb{R} \text{ and } 1 < x < 50\}$$
$$C = \{x | x \in \mathbb{Z} \text{ and } |x| \geq 25\}$$

Which of the following are true? (You can put check mark "✓" after the true ones and cross "✗" after the false ones.)

a. $A \subseteq B$ ✓

b. $17 \in A$ ✓

c. $A \subseteq C$ ✗

d. $-40 \in C$ ✓

e. $\emptyset \in B$ ✗

f. $\{0, 1, 2\} \subseteq A$ ✗

g. $\sqrt{3} \in B$ ✓

h. $\{x | x \in \mathbb{Z} \text{ and } x^2 \geq 625\} \subseteq C$ ✓

i. $(A - B) \cup (B - C) = A - C$ ✓

j. $A \times B = B \times A$ ✗

Problem 2. (10 + 10 + 10)

1. Prove that

$$A \cap (B \cup A') = B \cap A$$

where A and B are arbitrary sets.

$$\begin{aligned} A \cap (B \cup A') &= B \cap A \\ (A \cap B) \cup (A \cap A') &= \\ (A \cap B) \cup \varnothing &= \\ A \cap B &= B \cap A \end{aligned}$$

Therefore, for any arbitrary sets A and B, $A \cap (B \cup A') = B \cap A$

2. Prove that $(A - B) \cap (B - A) = \varnothing$, where A and B are arbitrary sets.

   Proof by contradiction

   (a) $x \in (A - B) \rightarrow (x \in A \wedge x \notin B)$

   (b) $(x \in A \wedge x \notin B) \rightarrow (x \notin B)$

   (c) $(x \notin B) \rightarrow (x \notin (B - A))$

   Therefore, the claim must be true because the intersection between disjoint sets is $\varnothing$.

3. Prove that $2^A \cup 2^B \subseteq 2^{A \cup B}$, where A and B are arbitrary sets.

   Assume $x \in 2^A \cap 2^B$.

   $(x \in 2^A \cap 2^B) \to (x \in 2^A \wedge x \in 2^B)$
   $(x \in 2^A \wedge x \in 2^B) \to (x \subseteq A \wedge x \subseteq B)$
   $(x \subseteq A \wedge x \subseteq B) \to (x \subseteq (A \cap B))$
   $(x \subseteq (A \cap B)) \to (x \in 2^{A \cap B})$

   Therefore, $2^A \cup 2^B \subseteq 2^{A \cup B}$ is true for any arbitrary sets.

Problem 3. (10 + 10 + 10) Prove the following properties for Fibonacci sequence:

1. $F_{n+1}^2 = F_n^2 + F_{n-1}F_{n+2}$ for $n \geq 2$

   Basis Step ($n = 2$):

   $F_3^2 = F_2^2 + F_1F_4$

   $4 = 1^2 + (1)(3) = 4$

   Inductive Step:

$$F_{n+2} = F_{n+1}^2 + F_nF_{n+3}$$

$$\begin{aligned}
F_{n+2}^2 &= F_{n+2}F_{n+2} \\
&= F_{n+2} \cdot (F_n + F_{n+1}) \\
&= F_{n+2}F_n + F_{n+2}F_{n+1} \\
&= (F_{n+3} - F_{n+1}) \cdot F_n + (F_{n+1} + F_n) \cdot F_{n+1} \\
&= F_{n+3}F_n - F_{n+1}F_n + F_{n+1}^2 + F_{n+1}F_n \\
&= F_{n+1}^2 + F_nF_{n+3}
\end{aligned}$$

   Therefore, we have proven the statement to be true.

2. $F_1 + 2F_2 + 3F_3 + \cdots + nF_n = nF_{n+2} - F_{n+3} + 2$ for $n \geq 1$

   Basis Step ($n = 1$):

   $F_1 = 1$

   $F_1 = (1)F_3 - F_4 + 2$

   $F_1 = 2 - 3 + 2 = 1$

   Inductive Step:

$$\begin{aligned}
(n+1)F_{n+3} - F_{n+4} + 2 &= F_1 + 2F_2 + 3F_3 + \cdots + nF_n + (n+1)F_{n+1} \\
&= nF_{n+2} - F_{n+3} + 2 + (n+1)F_{n+1} \\
&= nF_{n+2} + F_{n+2} - F_{n+2} - F_{n+3} + (n+1)F_{n+1} + 2 \\
&= (n+1) \cdot F_{n+2} - (F_{n+2} + F_{n+3}) + (n+1)F_{n+1} + 2 \\
&= (n+1) \cdot F_{n+2} - F_{n+4} + (n+1)F_{n+1} + 2 \\
&= (n+1) \cdot (F_{n+1} + F_{n+2}) - F_{n+4} + 2 \\
&= (n+1) \cdot F_{n+3} - F_{n+4} + 2
\end{aligned}$$

   Therefore, we have proven the statement to be true.

3. $F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$ for $n \geq 1$

   Basis Step ($n = 1$):

   $F_2 = F_3 - 1 = 1$

   Inductive Step:

   $F_2 + F_4 + \cdots + F_{2n} + F_{2(n+1)} = F_{2(n+1)+1} - 1$

   $F_{2n+1} - 1 + F_{2(n+1)} = F_{2(n+1)+1} - 1$

   $F_{2n+1} + F_{2n+2} - 1 = F_{2n+3} - 1$

   $F_{2n+1} + F_{2n+2} = F_{2n+3}$

   Therefore, we have proven the statement to be true.

Problem 4. (8 + 8) Give a recursive definition for each of the following set:

1. The set of all binary strings starting with 1.

    - Basis Step: $1 \in S$
    - Recursive Step: $(x \in S) \to (x0 \in S \wedge x1 \in S)$

2. The set of all binary strings containing no more than a single 1.

   - Basis Step: $0 \in S, 1 \in S$, “” $\in S$
   - Recursive Step: $(x \in S) \rightarrow (x0 \in S \wedge 0x \in S)$

Problem 5. (8 + 8) Find the closed-form solution to each of the following recursively defined sequences:

1. $T_1 = 2$
   $T_n = T_{n-1} + n$ for $n \geq 2$.
   (Hint: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$)
   $T_n = \frac{n(n+1)}{2} + 1$

2. $S_1 = 2$
   $S_n = 2S_{n-1} + n2^n$ for $n \geq 2$.
   (Hint: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$)
   $S_n = 2^{n-1}(n^2 + n)$