4 FACTORIAL DESIGNS

4.1 Two Factor Factorial Designs

- A **two-factor factorial** design is an experimental design in which data is collected for all possible combinations of the levels of the two factors of interest.
- If equal sample sizes are taken for each of the possible factor combinations then the design is a **two-factor factorial** design.
- A balanced $a \times b$ factorial design is a factorial design for which there are a levels of factor A, b levels of factor B, and n independent replications taken at each of the $a \times b$ treatment combinations. The design size is $N = abn$.
- The effect of a factor is defined to be the average change in the response associated with a change in the level of the factor. This is usually called a .
- If the average change in response across the levels of one factor are not the same at all levels of the other factor, then we say there is an **interaction** between the factors.

TYPE	TOTALS	MEANS	(if $n_{ij} = n$)
Cell(i,j)	$y_{ij\cdot} = \sum_{k=1}^{n_{ij}} y_{ijk}$	$\overline{y}_{ij\cdot} = y_{ij\cdot}/n_{ij}$	$= y_{ij\cdot}/n$
i^{th} level of A	$y_{i\cdot\cdot} = \sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} y_{ijk}$	$\overline{y}_{i\cdot\cdot} = y_{i\cdot\cdot}/\sum_{j=1}^{b} n_{ij}$	$= y_{i\cdot\cdot}/bn$
j^{th} level of B	$y_{\cdot j\cdot} = \sum_{i=1}^{a}\sum_{k=1}^{n_{ij}} y_{ijk}$	$\overline{y}_{\cdot j\cdot} = y_{\cdot j\cdot}/\sum_{i=1}^{a} n_{ij}$	$= y_{\cdot j\cdot}/an$
Overall	$y_{\cdots} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} y_{ijk}$	$\overline{y}_{\cdots} = y_{\cdots}/\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}$	$= y_{\cdots}/abn$

where n_{ij} is the number of observations in cell (i,j).

EXAMPLE (A 2×2 balanced design): A virologist is interested in studying the effects of $a = 2$ different culture media (M) and $b = 2$ different times (T) on the growth of a particular virus. She performs a balanced design with $n = 6$ replicates for each of the 4 $M * T$ treatment combinations. The $N = 24$ measurements were taken in a completely randomized order. The results:

THE DATA

		M: Medium 1	M: Medium 2
T	12 hours	21 23 20 22 28 26	25 24 29 26 25 27
	18 hours	37 38 35 39 38 36	31 29 30 34 33 35

i = Level of T $\quad$ j = Level of M
k = Observation number
y_{ijk} = k^{th} observation from the i^{th} level of T and j^{th} level of M

TOTALS

	$T = 1$	$T = 2$	
$T = 12$	$y_{11\cdot} = 140$	$y_{12\cdot} = 156$	$y_{1\cdot\cdot} = 296$
$T = 18$	$y_{21\cdot} = 223$	$y_{22\cdot} = 192$	$y_{2\cdot\cdot} = 415$
	$y_{\cdot 1\cdot} = 363$	$y_{\cdot 2\cdot} = 348$	$y_{\cdots} = 711$

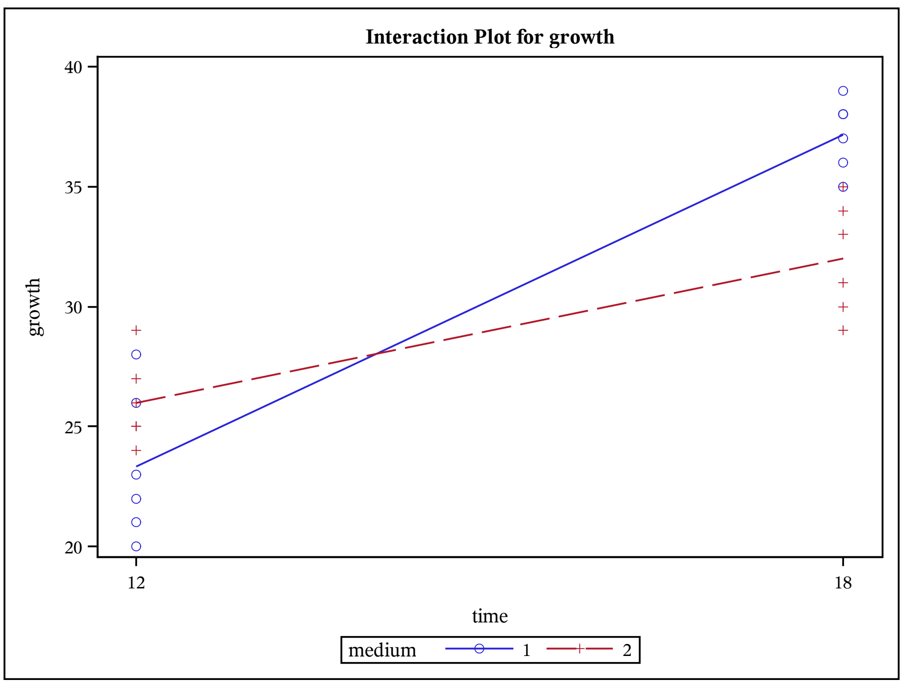
MEANS

	$M = 1$	$M = 2$	
$T = 12$	$\overline{y}_{11\cdot} = 23.\overline{3}$	$\overline{y}_{12\cdot} = 26$	$\overline{y}_{1\cdot\cdot} = 24.\overline{6}$
$T = 18$	$\overline{y}_{21\cdot} = 37.1\overline{6}$	$\overline{y}_{22\cdot} = 32$	$\overline{y}_{2\cdot\cdot} = 34.58\overline{3}$
	$\overline{y}_{\cdot 1\cdot} = 30.25$	$\overline{y}_{\cdot 2\cdot} = 29.00$	$\overline{y}_{\cdots} = 29.625$

- The effect of changing T from 12 to 18 hours on the response depends on the level of M.
  - For medium 1, the T effect = $37.1\overline{6} - 23.\overline{3}$ =
  - For medium 2, the T effect = $32 - 26$ =
- The effect on the response of changing M from medium 1 to 2 depends on the level of T.
  - For $T = 12$ hours, the M effect = $26 - 23.\overline{3}$ =
  - For $T = 18$ hours, the M effect = $32 - 37.1\overline{6}$ =

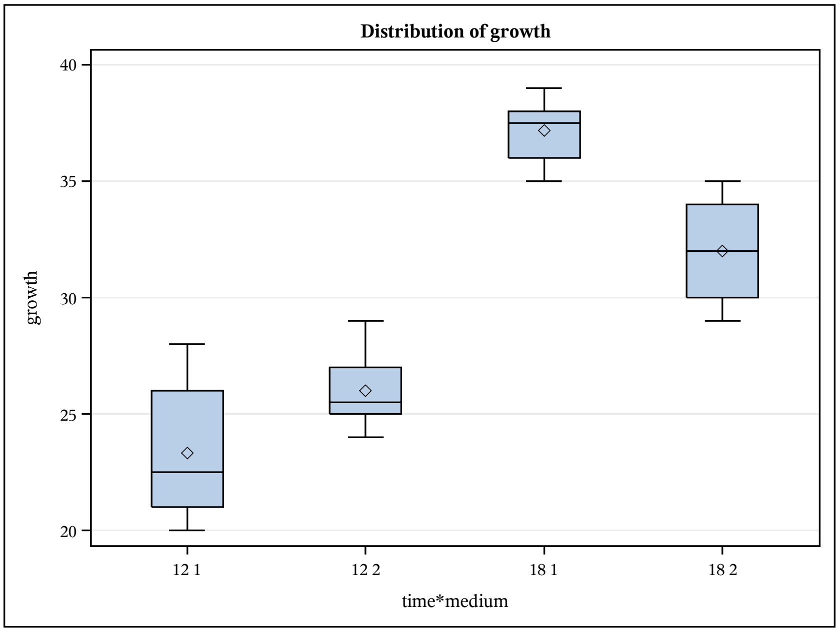
- If either of these pairs of estimated effects are significantly different then we say there exists a between factors M and T. For the 2×2 design example:
  - If $13.8\overline{3}$ is significantly different than 6 for the M effects, then we have a significant $M * T$ interaction.

    Or,
  - If $2.\overline{6}$ is significantly different than $-5.1\overline{6}$ for the T effects, then we have a significant $M * T$ interaction.
- There are two ways of defining an interaction between two factors A and B:
  - If the average change in response between the levels of factor A is not the same at all levels of factor B, then an **interaction** exists between factors A and B.
  - The lack of additivity of factors A and B, or the of the mean profiles of A and B, is called the **interaction** of A and B.
- When we assume there is no interaction between A and B, we say the effects are .
- An or is a graphical tool for checking for potential interactions between two factors. To make an interaction plot,
  1. Calculate the cell means for all $a \cdot b$ combinations of the levels of A and B.
  2. Plot the cell means against the levels of factor A.
  3. Connect and label means the same levels of factor B.



- The roles of A and B can be reversed to make a second interaction plot.
- Interpretation of the interaction plot:
  - Parallel lines indicate no significant interaction.
  - Severe lack of parallelism indicates a significant interaction.
  - Moderate lack of parallelism suggests a significant interaction may exist.

- Interactions can also be assessed in side-by-side boxplots.



- Statistical significance of an interaction effect depends on the magnitude of the ________ (or, within treatment variability) because it is the denominator of the F-test

  For small values of the MS_E, even small interaction effects (less nonparallelism) may be significant.

- When an $A * B$ interaction is large, the corresponding main effects A and B may have little practical meaning. Knowledge of the $A * B$ interaction is often more useful than knowledge of the main effect.

- We usually say that a significant interaction can mask the interpretation of significant main effects. That is, the experimenter must examine the levels of one factor, say A, at fixed levels of the other factor to draw conclusions about the main effect of A.

- It is possible to have a significant interaction between two factors, while the main effects for both factors are not significant. This would happen when the interaction plot shows interactions in different directions that balance out over one or both factors (such as an X pattern). This type of interaction, however, is uncommon.

4.2 The Interaction Model

- The **interaction model** for a two-factor completely randomized design is:

$$y_{ijk} \quad = \qquad\qquad\qquad\qquad\qquad\qquad \tag{24}$$

where μ is the baseline mean, α_i is the i^{th} factor A effect, β_j is the j^{th} factor B effect, $(\alpha\beta)_{ij}$ is the $(i,j)^{th}$ $A * B$ interaction effect, ϵ_{ijk} is the random error of the k^{th} observation from the $(i,j)^{th}$ cell.

We assume $\epsilon_{ijk} \sim IID\ N(0, \sigma^2)$. For now, we will also assume all effects are fixed.

- If $(\alpha\beta)_{ij}$ is removed from (24), we would have the ________ :

$$y_{ijk} \quad = \quad \mu \quad + \quad \alpha_i \quad + \quad \beta_j \quad + \epsilon_{ijk} \tag{25}$$

- If we impose the constraints

$$\sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{b} \beta_j = 0 \qquad \sum_{i=1}^{a} (\alpha\beta)_{ij} = 0 \text{ for all } j \quad \text{and} \quad \sum_{j=1}^{b} (\alpha\beta)_{ij} = 0 \text{ for all } i, \tag{26}$$

then the least squares estimates of the model parameters are

$\widehat{\mu} \ =$ $\qquad$ $\widehat{\alpha}_i \ =$ $\qquad$ $\widehat{\beta}_j \ =$ $\qquad$ $\widehat{\alpha\beta}_{ij} \ =$

- If we substitute these estimates into (24) we get

$$y_{ijk} \ = \ \widehat{\mu} \ + \ \widehat{\alpha}_i \ + \ \widehat{\beta}_j \ + \ \widehat{\alpha\beta}_{ij} \ + \ e_{ijk}$$

$$= \ \overline{y}_{\cdot\cdot\cdot} \ + \ (\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot}) \ + \ (\overline{y}_{\cdot j\cdot} - \overline{y}_{\cdot\cdot\cdot}) \ + \ (\overline{y}_{ij\cdot} - \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot j\cdot} + \overline{y}_{\cdot\cdot\cdot}) \ + \ e_{ijk}$$

where e_{ijk} is the k^{th} residual from the treatment $(i,j)^{th}$ cell, and $e_{ijk} \ =$

Estimation of Parameters and the Normal Equations (equal n_{ij})

Goal: Find $\widehat{\mu}, \widehat{\alpha}_i, \widehat{\beta}_j$ and $\widehat{\alpha\beta}_{ij}$ for $i = 1, \ldots, a$ and $j = 1, \ldots, b$ that minimize

$$L \ = \ \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(y_{ijk} \ - \ \widehat{\mu} \ - \ \widehat{\alpha}_i \ - \ \widehat{\beta}_j \ - \ \widehat{\alpha\beta}_{ij}\right)^2 \ =$$

$$\begin{array}{llcl}
(1) & (3) & & (2) \\
(1)\ \widehat{\mu} & abn\widehat{\mu} + bn\cancel{\sum_i \widehat{\alpha}_i} + an\cancel{\sum_j \widehat{\beta}_j} + n\sum_i\cancel{\sum_j (\widehat{\alpha\beta})_{ij}} & = & y_{\cdot\cdot\cdot} \\
(2)\ \widehat{\alpha}_i & bn\widehat{\mu} + bn\widehat{\alpha}_i + n\cancel{\sum_j \widehat{\beta}_j} + n\cancel{\sum_j (\widehat{\alpha\beta})_{ij}} & = & y_{i\cdot\cdot} \quad i=1,\ldots,a \\
(3)\ \widehat{\beta}_j & an\widehat{\mu} + n\cancel{\sum_i \widehat{\alpha}_i} + an\widehat{\beta}_j + n\sum_i (\widehat{\alpha\beta})_{ij} & = & y_{\cdot j\cdot} \quad j=1,\ldots,b \\
(4)\ (\widehat{\alpha\beta})_{ij} & n\widehat{\mu} + n\widehat{\alpha}_i + n\widehat{\beta}_j + n\widehat{\alpha\beta}_{ij} & = & y_{ij\cdot}
\end{array}$$

CONSTRAINTS: $\sum_i \widehat{\alpha}_i = 0 \qquad \sum_j \widehat{\beta}_j = 0 \qquad \sum_i (\widehat{\alpha\beta})_{ij} = 0 \qquad \sum_j (\widehat{\alpha\beta})_{ij} = 0$

FROM (1) $\quad abn\widehat{\mu} = y_{\cdot\cdot\cdot} \quad \rightarrow \quad \boxed{\widehat{\mu} = \overline{y}_{\cdot\cdot\cdot}}$

SUBSTITUTE $\widehat{\mu}$ INTO (2) AND (3)

$\rightarrow$ (2) $\quad bn\overline{y}_{\cdot\cdot\cdot} + bn\widehat{\alpha}_i = y_{i\cdot\cdot} \quad \rightarrow \quad \overline{y}_{\cdot\cdot\cdot} + \widehat{\alpha}_i = \overline{y}_{i\cdot\cdot} \quad \rightarrow \quad \boxed{\widehat{\alpha}_i = \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot}}$

$\rightarrow$ (3) $\quad an\overline{y}_{\cdot\cdot\cdot} + an\widehat{\beta}_j = y_{\cdot j\cdot} \quad \rightarrow \quad \overline{y}_{\cdot\cdot\cdot} + \widehat{\beta}_j = \overline{y}_{\cdot j\cdot} \quad \rightarrow \quad \boxed{\widehat{\beta}_j = \overline{y}_{\cdot j\cdot} - \overline{y}_{\cdot\cdot\cdot}}$

SUBSTITUTE $\widehat{\mu}, \widehat{\alpha}_i, \widehat{\beta}_j$ INTO (4)

$\rightarrow$ (4) $\quad n\overline{y}_{\cdot\cdot\cdot} + n(\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot}) + n(\overline{y}_{\cdot j\cdot} - \overline{y}_{\cdot\cdot\cdot}) + n(\widehat{\alpha\beta})_{ij} = y_{ij\cdot}$

$$\cancel{\overline{y}_{\cdot\cdot\cdot}} + \overline{y}_{i\cdot\cdot} - \cancel{\overline{y}_{\cdot\cdot\cdot}} + \overline{y}_{\cdot j\cdot} - \overline{y}_{\cdot\cdot\cdot} + (\widehat{\alpha\beta})_{ij} = \overline{y}_{ij\cdot}$$

$$\boxed{(\widehat{\alpha\beta})_{ij} = \overline{y}_{ij\cdot} - \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot j\cdot} + \overline{y}_{\cdot\cdot\cdot}}$$

- For the 2×2 design,

$$\overline{y}_{\cdot\cdot\cdot} = 29.625 \qquad \overline{y}_{1\cdot} = 24.\overline{6} \qquad \overline{y}_{2\cdot} = 34.58\overline{6} \qquad \overline{y}_{\cdot 1} = 30.25 \qquad \overline{y}_{\cdot 2} = 29.00$$

- Assuming the constraints in (26),

$$\begin{aligned}
\widehat{\alpha}_1 &= 24.\overline{6} - 29.625 \;= \\
\widehat{\alpha}_2 &= 34.58\overline{3} - 29.625 \;= \\
\widehat{\beta}_1 &= 30.25\overline{6} - 29.625 \;= \\
\widehat{\beta}_2 &= 29.00\overline{6} - 29.625 \;= \\
\widehat{\alpha\beta}_{11} &= 23.\overline{3} - 24.\overline{6} - 30.25 + 29.625 \;= \\
\widehat{\alpha\beta}_{12} &= 26 - 24.\overline{6} - 29.00 + 29.625 \;= \\
\widehat{\alpha\beta}_{21} &= 37.1\overline{6} - 34.58\overline{3} - 30.25 + 29.625 \;= \\
\widehat{\alpha\beta}_{22} &= 32 - 34.58\overline{3} - 29.00 + 29.625 \;=
\end{aligned}$$

4.3 Notation for a Two-Way ANOVA

- n_{ij} = the sample size for the $(i,j)^{th}$ treatment cell, $\qquad N = \displaystyle\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}$

  $n_{i\cdot} = \sum_{j=1}^{b} n_{ij}$ = sample size for i^{th} level of A.

  $n_{\cdot j} = \sum_{i=1}^{a} n_{ij}$ = sample size for j^{th} level of B.

- $SS_T = \displaystyle\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} (y_{ijk} - \overline{y}_{\cdot\cdot\cdot})^2$ = the total sum of squares ($df = N - 1$)

- $SS_A = \displaystyle\sum_{i=1}^{a} n_{i\cdot}(\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot})^2$ = the sum of squares for factor A ($df = a - 1$)

  $MS_A = SS_A/(a-1)$ = the mean square for factor A

- $SS_B = \displaystyle\sum_{j=1}^{b} n_{\cdot j}(\overline{y}_{\cdot j\cdot} - \overline{y}_{\cdot\cdot\cdot})^2$ = the sum of squares for factor B ($df = b - 1$)

  $MS_B = SS_B/(b-1)$ = the mean square for factor B

- $SS_{AB} = \displaystyle\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}(\overline{y}_{ij\cdot} - \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot j\cdot} + \overline{y}_{\cdot\cdot\cdot})^2$

  = the $A * B$ interaction sum of squares ($df = (a-1)(b-1)$)

  $MS_{AB} = SS_{AB}/(a-1)(b-1)$= the mean square for the $A * B$ interaction

- $SS_E = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} \left(y_{ijk} - \overline{y}_{ij\cdot}\right)^2$ = the error sum of squares ($df = N - ab$)

  $MS_E = SS_E/(N - ab)$= the mean square error

For the equal sample size case ($n_{ij} = n$ for all i, j), the previous formulas reduce to:

- $SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \overline{y}_{...})^2, \quad df = abn - 1$
- $SS_A = nb\sum_{i=1}^{a}(\overline{y}_{i..} - \overline{y}_{...})^2, \quad df = a - 1, \quad MS_A = SS_A/(a-1)$
- $SS_B = na\sum_{j=1}^{b}(\overline{y}_{.j.} - \overline{y}_{...})^2, \quad df = b - 1, \quad MS_B = SS_B/(b-1)$
- $SS_{AB} = n\sum_{i=1}^{a}\sum_{j=1}^{b}(\overline{y}_{ij.} - \overline{y}_{i..} - \overline{y}_{.j.} + \overline{y}_{...})^2, \quad df = (a-1)(b-1), \quad MS_{AB} = SS_{AB}/(a-1)(b-1)$
- $SS_E = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(y_{ijk} - \overline{y}_{ij.}\right)^2, \quad df = ab(n-1), \quad MS_E = SS_E/ab(n-1)$
- **For both the equal and unequal sample size cases, the total sum of squares is partitioned into components corresponding to the terms in the model:**

$$SS_T \ = \ SS_A \ + \ SS_B \ + \ SS_{AB} \ + \ SS_E$$

Balanced Two-Factor Factorial ANOVA Table

Source of Variation	Sum of Squares	d.f.	Mean Square	F Ratio
A	SS_A	$a-1$	$MS_A = SS_A/(a-1)$	$F_A = MS_A/MS_E$
B	SS_B	$b-1$	$MS_B = SS_B/(b-1)$	$F_B = MS_B/MS_E$
$A*B$	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = SS_{AB}/(a-1)(b-1)$	$F_{A*B} = MS_{AB}/MS_E$
Error	SS_E	$ab(n-1)$	$MS_E = SS_E/(ab(n-1))$	——
Total	SS_{total}	$abn-1$	——	——

For the unbalanced case, replace $ab(n-1)$ with $N - ab$ for the d.f. for SS_E and replace $abn - 1$ with $N - 1$ for the d.f. for SS_{total} where $N = \sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}$.

4.4 Comments on Interpreting the ANOVA

- If we assume the constraints in (26), then the hypotheses can be rewritten as:

$$\text{H}_0: \ (\alpha\beta)_{11} = (\alpha\beta)_{12} = \cdots = (\alpha\beta)_{ab} = 0 \quad \text{vs.} \quad \text{H}_1: \text{ at least one } (\alpha\beta)_{ij} \neq 0$$
$$\text{H}_0: \ \alpha_1 = \alpha_2 = \cdots = \alpha_a = 0 \quad \text{vs.} \quad \text{H}_1: \text{ at least one } \alpha_i \neq 0$$
$$\text{H}_0: \ \beta_1 = \beta_2 = \cdots = \beta_b = 0 \quad \text{vs.} \quad \text{H}_1: \text{ at least one } \beta_j \neq 0$$

- If this test indicates that there is a significant interaction, then the interpretation of significant main effects hypotheses can be complicated. You will need to look at the interaction plots.
- To draw conclusions about a main effect, we will fix the levels of one factor and vary the levels of the other. Using this approach (combined with interaction plots) we may be able to provide an interpretation of main effects.

4.5 ANOVA for a 2 × 2 Factorial Design Example

We will now use SAS to analyze the 2 × 2 factorial design data discussed earlier.

		M	
		Medium 1	Medium 2
T	12 hours	21 23 20 22 28 26	25 24 29 26 25 27
	18 hours	37 38 35 39 38 36	31 29 30 34 33 35

The GLM Procedure

Dependent Variable: growth

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	691.4583333	230.4861111	45.12	<.0001
Error	20	102.1666667	5.1083333		
Corrected Total	23	793.6250000			

R-Square	Coeff Var	Root MSE	growth Mean	
0.871266	7.629240	2.260162	29.62500	$\bar{y}_{\cdots}$

Source	DF	Type III SS	Mean Square	F Value	Pr > F
time	1	590.0416667	590.0416667	115.51	<.0001
medium	1	9.3750000	9.3750000	1.84	0.1906
time*medium	1	92.0416667	92.0416667	18.02	0.0004

LOOK AT THIS FIRST. THERE IS STRONG EVIDENCE AGAINST $H_0: \alpha\beta_{11}=\alpha\beta_{12}=\alpha\beta_{21}=\alpha\beta_{22}=0$

Parameter	Estimate		Standard Error	t Value	Pr > |t|	
Intercept	32.00000000	B	0.92270737	34.68	<.0001	$\hat{\mu}$
time 12	-6.00000000	B	1.30490528	-4.60	0.0002	$\hat{\alpha}_1$
time 18	0.00000000	B	.	.	.	$\hat{\alpha}_2$
medium 1	5.16666667	B	1.30490528	3.96	0.0008	$\hat{\beta}_1$
medium 2	0.00000000	B	.	.	.	$\hat{\beta}_2$
time*medium 12 1	-7.83333333	B	1.84541474	-4.24	0.0004	$\widehat{\alpha\beta}_{11}$
time*medium 12 2	0.00000000	B	.	.	.	$\widehat{\alpha\beta}_{12}$
time*medium 18 1	0.00000000	B	.	.	.	$\widehat{\alpha\beta}_{21}$
time*medium 18 2	0.00000000	B	.	.	.	$\widehat{\alpha\beta}_{22}$

$\hat{\mu} = \bar{y}_{22\cdot}$ BASELINE MEAN

DEFAULT SAS CONSTRAINTS

$$\begin{array}{cc|c} \alpha\beta_{11} & \alpha\beta_{12} & \alpha_1 \\ \alpha\beta_{21} & \alpha\beta_{22} & \alpha_2 \\ \hline \beta_1 & \beta_2 & \end{array} \qquad \begin{array}{cc|c} \alpha\beta_{11} & 0 & \alpha_1 \\ 0 & 0 & 0 \\ \hline \beta_1 & 0 & \end{array}$$

USING SUMMATION CONSTRAINTS (SEE BOTTOM OF PAGE 109)

Parameter	Estimate	Standard Error	t Value	Pr > |t|	
mu	29.6250000	0.46135368	64.21	<.0001	$\hat{\mu}$
time=12	-4.9583333	0.46135368	-10.75	<.0001	$\hat{\alpha}_1$
time=18	4.9583333	0.46135368	10.75	<.0001	$\hat{\alpha}_2$
medium=1	0.6250000	0.46135368	1.35	0.1906	$\hat{\beta}_1$
medium=2	-0.6250000	0.46135368	-1.35	0.1906	$\hat{\beta}_2$
time=12 medium=1	-1.9583333	0.46135368	-4.24	0.0004	$\widehat{\alpha\beta}_{11}$
time=12 medium=2	1.9583333	0.46135368	4.24	0.0004	$\widehat{\alpha\beta}_{12}$
time=18 medium=1	1.9583333	0.46135368	4.24	0.0004	$\widehat{\alpha\beta}_{21}$
time=18 medium=2	-1.9583333	0.46135368	-4.24	0.0004	$\widehat{\alpha\beta}_{22}$

CONSTRAINTS

$\sum \alpha_i = 0$

$\sum \beta_j = 0$

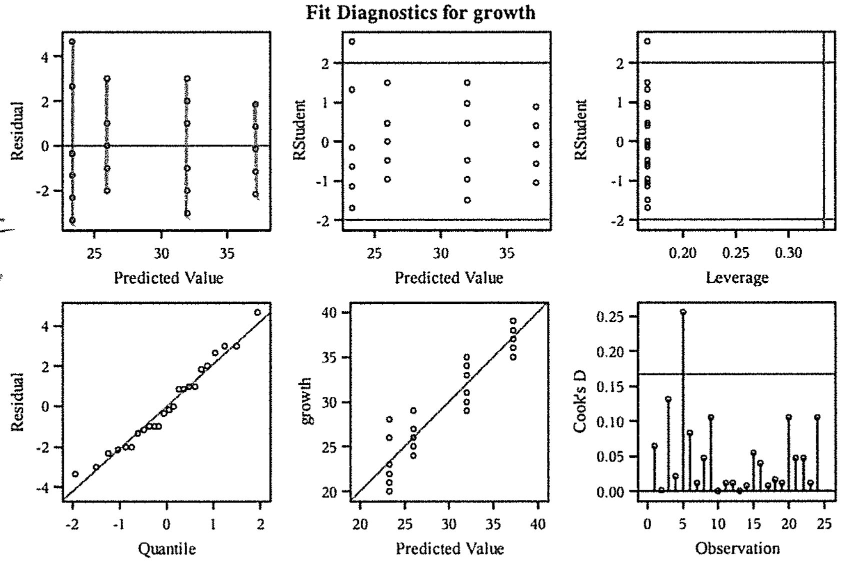
$\sum \alpha\beta_{1j} = 0$ (ROW 1)

$\sum \alpha\beta_{2j} = 0$ (ROW 2)

$\sum \alpha\beta_{i1} = 0$ (COLUMN 1)

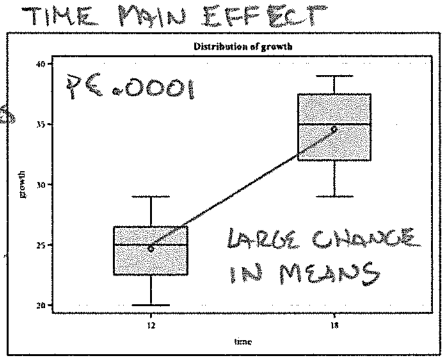
$\sum \alpha\beta_{i2} = 0$ (COLUMN 2)

NO SERIOUS VIOLATIONS OF ASSUMPTIONS

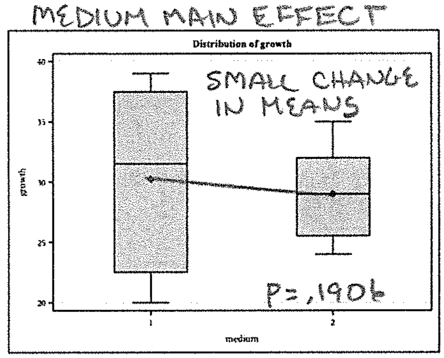


VERY STRONG EVIDENCE TO REJECT $H_0: \alpha_1 = \alpha_2 = 0$ → CONCLUDE A TIME MAIN EFFECT EXISTS.

TIME MAIN EFFECT



MEDIUM MAIN EFFECT

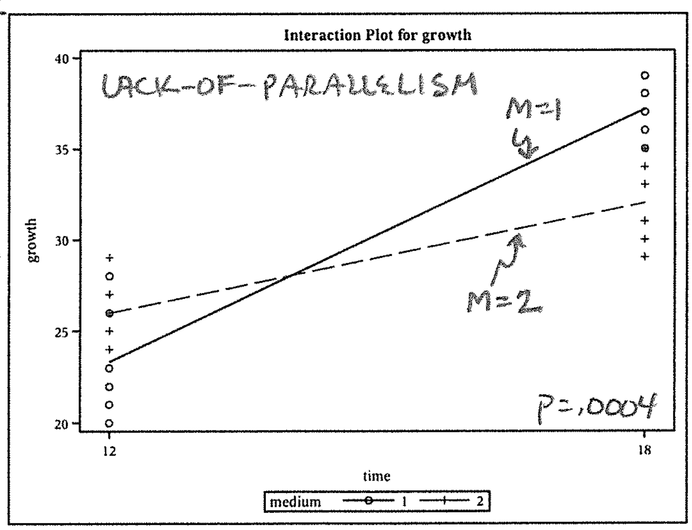


NO EVIDENCE TO REJECT $H_0: \beta_1 = \beta_2 = 0$ (FAIL TO REJECT) NO EVIDENCE OF A MEDIUM MAIN EFFECT

Level of time	N	growth Mean	growth Std Dev
12	12	24.6666667	2.77434131
18	12	34.5833333	3.28794861

Level of medium	N	growth Mean	growth Std Dev
1	12	30.2500000	7.58137670
2	12	29.0000000	3.71728151

THE PLOTS INDICATE THE MEAN RESPONSE INCREASES WHEN CHANGING TIME FROM 12 TO 18 HOURS, BUT AT DIFFERENT RATES. THE INCREASE IS LARGER WHEN M=1 THAN WHEN M=2.



VERY STRONG EVIDENCE TO REJECT $H_0: \alpha\beta_{11} = \alpha\beta_{12} = \alpha\beta_{21} = \alpha\beta_{22} = 0$

Tests for Normality				
Test	Statistic		p Value	
Shapiro-Wilk	W	0.966156	Pr < W	0.5737
Kolmogorov-Smirnov	D	0.140751	Pr > D	>0.1500
Cramer-von Mises	W-Sq	0.049547	Pr > W-Sq	>0.2500
Anderson-Darling	A-Sq	0.303234	Pr > A-Sq	>0.2500

LARGE P-VALUES INDICATE THERE IS NO EVIDENCE TO REJECT THE NORMALITY ASSUMPTION

4.5.1 SAS Code for 2 x 2 Factorial Design

TWOWAY1.SAS

H_0: The residuals are normally distributed.

H_1: The residuals are not normally distributed.

```
DM 'LOG; CLEAR; OUT; CLEAR;';

ODS GRAPHICS ON;
ODS PRINTER PDF file='C:\COURSES\ST541\TWOWAY1.PDF';
OPTIONS NODATE NONUMBER;

************************************************;
*** EXAMPLE: 2-FACTOR FACTORIAL (2x2) DESIGN ***;
************************************************;
DATA in;
   DO time = 12 to 18 by 6;
   DO medium = 1 to 2;
   DO rep = 1 to 6;
      INPUT growth @@; OUTPUT;
   END; END; END;
CARDS;
 21 23 20 22 28 26    25 24 29 26 25 27
 37 38 35 39 38 36    31 29 30 34 33 35
;
PROC GLM DATA=in PLOTS=(ALL);
     CLASS time medium;
     MODEL growth = time|medium / SS3 SOLUTION;
     MEANS time|medium;

*** Estimate mu ***;
     ESTIMATE 'mu' intercept 1;

*** Estimate the main effects for factor time';
     ESTIMATE 'time=12' time 1 -1 / divisor = 2 ;
     ESTIMATE 'time=18' time -1 1 / divisor = 2 ;

*** Estimate the main effects for factor medium';
     ESTIMATE 'medium=1' medium 1 -1 / divisor = 2 ;
     ESTIMATE 'medium=2' medium -1 1 / divisor = 2 ;

*** Estimate the interaction effects';
      *** Take the product of the tau_i and beta_j coefficients;
      *** from the main effects ESTIMATE statement. Divisor = a*b;

      *** To estimate taubeta i,j
      *** (1 -1) x (1 -1) = (1 -1 -1 1)  for i,j = 12,1;
      *** (1 -1) x (-1 1) = (-1 1 1 -1)  for i,j = 12,2;
      *** (-1 1) x (1 -1) = (-1 1 1 -1)  for i,j = 18,1;
      *** (-1 1) x (-1 1) = (1 -1 -1 1)  for i,j = 18,2;

     ESTIMATE 'time=12 medium=1' time*medium 1 -1 -1 1 / divisor = 4;
     ESTIMATE 'time=12 medium=2' time*medium -1 1 1 -1 / divisor = 4;
     ESTIMATE 'time=18 medium=1' time*medium -1 1 1 -1 / divisor = 4;
     ESTIMATE 'time=18 medium=2' time*medium 1 -1 -1 1 / divisor = 4;

     OUTPUT OUT=diag P=pred R=resid;

TITLE 'ANOVA and Estimation of Effects for a 2x2 Design';

PROC UNIVARIATE DATA=diag NORMAL;
     VAR resid;
RUN;
```

$\hat{\mu}$; $\hat{\alpha}_1$, $\hat{\alpha}_2$; $\hat{\beta}_1$, $\hat{\beta}_2$; $\widehat{\alpha\beta}_{11}$, $\widehat{\alpha\beta}_{12}$, $\widehat{\alpha\beta}_{21}$, $\widehat{\alpha\beta}_{22}$.

i
1
2

$a=2$ $a-1=1$ ON DIAGONAL DIVISOR $= a = 2$

j
1
2

$b=2$ $b-1=1$ ON DIAGONAL DIVISOR $= b = 2$

i	j
1	1
1	2
2	1
2	2

DIVISOR $= a \cdot b = 4$

→ NEEDED FOR NORMALITY TESTS

→ PERFORM TESTS FOR NORMALITY

4.6 Tests of Normality (Supplemental)

- For an ANOVA, we assume the errors are normally distributed with mean 0 and constant variance σ^2. That is, we assume the random error $\epsilon \sim N(0, \sigma^2)$.
- The Kolmogorov-Smirnov Goodness-of-Fit Test, the Cramer-Von Mises Goodness-of-Fit Test, and the Anderson-Darling Goodness-of-Fit Test can be applied to any distribution $F(x)$.
- Although the following notes use the general form $F(x)$, we will be assuming $F(x)$ represents a normal distribution with mean 0 and constant variance.
- We are also assuming that the random sample referred to in each test is the set of residuals from the ANOVA.
- Thus, in each each test we are checking the normality assumption in the ANOVA. In this case, we want to see a large p-value because we do not want to reject the null hypothesis that the errors are normally distributed.

4.6.1 Kolmogorov-Smirnov Goodness-of-Fit Test

Assumptions: Given a random sample of n independent observations

- The measurement scale is at least ordinal.
- The observations are sampled from a continuous distribution $F(x)$.

Hypotheses: For a hypothesized distribution $F^*(x)$

(i) Two-sided: $H_0 : F(x) = F^*(x)$ for all x vs. $H_1 : F(x) \neq F^*(x)$ for some x

(ii) One-sided: $H_0 : F(x) \geq F^*(x)$ for all x vs. $H_1 : F(x) < F^*(x)$ for some x

(iii) One-sided: $H_0 : F(x) \leq F^*(x)$ for all x vs. $H_1 : F(x) > F^*(x)$ for some x

Method: For a given α

- Define the empirical distribution function $S_n(x) = \dfrac{\text{Number of observations } \leq x}{n}$

(i) Two-sided test statistic: $T = \sup_x |F^*(x) - S_n(x)|$

- When plotted, T is the greatest vertical difference between the empirical and the hypothesized distribution.

(ii) One-sided test statistic: $T^+ = \sup_x (F^*(x) - S_n(x))$

(iii) One-sided test statistic: $T^- = \sup_x (S_n(x) - F^*(x))$

Decision Rule

- Critical values for T, T^+ and T^- are found in nonparametrics textbooks. For larger samples sizes, an asymptotic critical value can be used.
- We will just rely on p-values to make a decision.

4.6.2 Cramer-Von Mises Goodness-of-Fit Test

Assumptions: Same as the Kolmogorov-Smirnov test

Hypotheses: For a hypothesized distribution $F^*(x)$

$$H_0: \; F(x) = F^*(x) \;\text{ for all } x \quad \underline{\text{vs.}} \quad H_1: \; F(x) \neq F^*(x) \;\text{ for some } x$$

Method: For a given α

- Define the empirical distribution function $S_n(x) = \dfrac{\text{Number of observations} \quad \leq x}{n}$
- The Cramer-von Mises test statistic W^2 is defined to be
$$W^2 \;=\; n\int_{-\infty}^{\infty} [F^*(x) - S_n(x)]^2 dF^*(x).$$
- This form can reduces to $W^2 \;=\; \dfrac{1}{12n} + \sum_{i=1}^{n}\left(F^*(x_{(i)}) - \dfrac{2i-1}{2n}\right)^2$
  where $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ represents the ordered sample in ascending order.

Decision Rule

- Tables of critical values exist for the exact distribution of W^2 when H_0 is true. Computers generate critical values for the asymptotic ($n \to \infty$) distribution of W^2.
- If W^2 becomes too large (or p-value $< \alpha$), then we will **Reject** H_0.

4.6.3 Anderson-Darling Goodness-of-Fit Test

Assumptions: Same as the Kolmogorov-Smirnov and Cramer-von Mises tests

Hypotheses: Same as the Cramer-von Mises test.

Method: For a given α

- Define the empirical distribution function $S_n(x) = \dfrac{\text{Number of observations} \quad \leq x}{n}$
- The Anderson-Darling test statistic A^2 is defined to be
$$A^2 \;=\; \int_{-\infty}^{\infty} \frac{1}{F^*(x)(1-F^*(x))}[F^*(x) - S_n(x)]^2 dx.$$
- This form can reduces to $A^2 \;=\; -\dfrac{1}{n}\left\{(2i-1)\left(lnF^*(x_{(i)}) + ln(1-F^*(x_{(n+1-i)}))\right)\right\} - n$ where $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ represents the ordered sample in ascending order.

Decision Rule

- Computers generate critical values for the asymptotic ($n \to \infty$) distribution of A^2.
- If A^2 becomes too large (or p-value $< \alpha$), then we will **Reject** H_0.

4.7 Matrix Forms for the Twoway ANOVA

Example: Consider a completely randomized 2×3 factorial design with $n = 2$ replications for each of the six combinations of the two factors (A and B). The following table summarizes the results:

Factor A	Factor B Levels		
Levels	1	2	3
1	1 , 2	4 , 6	5 , 6
2	3 , 5	5 , 7	4 , 6

- Model: $y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$ for $i = 1, 2$ $j = 1, 2, 3$ $k = 1, 2$ and $\epsilon_{ijk} \sim N(0, \sigma^2)$
- Assume (i) $\sum_{i=1}^{2} \alpha_i = 0$ (ii) $\sum_{j=1}^{3} \beta_j = 0$

  (iii) $\sum_{j=1}^{3} (\alpha\beta)_{ij} = 0$ for $i = 1, 2$ (iv) $\sum_{i=1}^{2} (\alpha\beta)_{ij} = 0$ for $j = 1, 2, 3$
- Thus, for the main effect constraints, we have $\alpha_2 = -\alpha_1$ and $\beta_3 = -\beta_1 - \beta_2$.
- The interaction effect constraints can be written in terms of just $\alpha\beta_{11}$ and $\alpha\beta_{12}$:

  $\alpha\beta_{12} =$ $\alpha\beta_{22} =$ $\alpha\beta_{13} =$ $\alpha\beta_{23} =$
- Thus, the reduced form of model matrix X requires only 6 columns: $\mu, \alpha_1, \beta_1, \beta_2, \alpha\beta_{11}$ and $\alpha\beta_{12}$.

$$
X = \begin{array}{c} \begin{array}{cccccc} \mu & \alpha_1 & \beta_1 & \beta_2 & \alpha\beta_{11} & \alpha\beta_{12} \end{array} \\ \left[\begin{array}{c|c|cc|cc}
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & 0 & 1 & 0 & -1 \\
1 & -1 & 0 & 1 & 0 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1
\end{array}\right] \end{array}
\qquad
y = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 6 \\ 5 \\ 6 \\ 3 \\ 5 \\ 5 \\ 7 \\ 4 \\ 6 \end{bmatrix}
$$

$$
X'X = \begin{bmatrix}
12 & 0 & 0 & 0 & 0 & 0 \\
0 & 12 & 0 & 0 & 0 & 0 \\
0 & 0 & 8 & 4 & 0 & 0 \\
0 & 0 & 4 & 8 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & 4 \\
0 & 0 & 0 & 0 & 4 & 8
\end{bmatrix}
\qquad
X'y = \begin{bmatrix} 54 \\ -6 \\ -10 \\ 1 \\ -6 \\ -3 \end{bmatrix}
$$

$$
(X'X)^{-1} = \frac{1}{12} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix}
\qquad
(X'X)^{-1}X'y = \begin{bmatrix} 4.5 \\ -0.5 \\ -1.75 \\ 1 \\ -0.75 \\ 0 \end{bmatrix}
\quad
\begin{bmatrix} \widehat{\mu} \\ \widehat{\alpha}_1 \\ \widehat{\beta}_1 \\ \widehat{\beta}_2 \\ \widehat{\alpha\beta}_{11} \\ \widehat{\alpha\beta}_{12} \end{bmatrix}
$$

Thus, $\widehat{\alpha}_2 = -\widehat{\alpha}_1 = 0.5$ $\quad \widehat{\beta}_3 = -\widehat{\beta}_1 - \widehat{\beta}_2 = 0.75$ $\quad \widehat{\alpha\beta}_{21} = -\widehat{\alpha\beta}_{11} = 0.75$

$\widehat{\alpha\beta}_{22} = -\widehat{\alpha\beta}_{12} = 0$ $\quad \widehat{\alpha\beta}_{13} = -\widehat{\alpha\beta}_{11} - \widehat{\alpha\beta}_{12} = 0.75$ $\quad \widehat{\alpha\beta}_{23} = \widehat{\alpha\beta}_{11} + \widehat{\alpha\beta}_{12} = -0.75$

Alternate Approach: Keeping a Column for Each Model Parameter

- In total, there are $2+a+b$ constraints, and there will be $1+a+b+(a*b)$ columns in X.

$$
X \;=\; \begin{array}{c}
\begin{array}{cccccccccccc} \mu & \alpha_1 & \alpha_2 & \beta_1 & \beta_2 & \beta_3 & \alpha\beta_{11} & \alpha\beta_{12} & \alpha\beta_{13} & \alpha\beta_{21} & \alpha\beta_{22} & \alpha\beta_{23} \end{array} \\
\left[\begin{array}{c|cc|ccc|cccccc}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}\right]
\end{array}
\qquad
y \;=\; \left[\begin{array}{c} 1 \\ 2 \\ 4 \\ 6 \\ 5 \\ 6 \\ 3 \\ 5 \\ 5 \\ 7 \\ 4 \\ 6 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]
$$

$$
X'X \;=\; \left[\begin{array}{c|cc|ccc|cccccc}
12 & 6 & 6 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 2 & 2 \\
\hline
6 & 7 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 \\
6 & 1 & 7 & 2 & 2 & 2 & 0 & 0 & 0 & 2 & 2 & 2 \\
\hline
4 & 2 & 2 & 5 & 1 & 1 & 2 & 0 & 0 & 2 & 0 & 0 \\
4 & 2 & 2 & 1 & 5 & 1 & 0 & 2 & 0 & 0 & 2 & 0 \\
4 & 2 & 2 & 1 & 1 & 5 & 0 & 0 & 2 & 0 & 0 & 2 \\
\hline
2 & 2 & 0 & 2 & 0 & 0 & 4 & 1 & 1 & 1 & 0 & 0 \\
2 & 2 & 0 & 0 & 2 & 0 & 1 & 4 & 1 & 0 & 1 & 0 \\
2 & 2 & 0 & 0 & 0 & 2 & 1 & 1 & 4 & 0 & 0 & 1 \\
2 & 0 & 2 & 2 & 0 & 0 & 1 & 0 & 0 & 4 & 1 & 1 \\
2 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 4 & 1 \\
2 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 1 & 4
\end{array}\right]
\qquad
X'y \;=\; \left[\begin{array}{c} 54 \\ \hline 24 \\ 30 \\ \hline 11 \\ 22 \\ 21 \\ \hline 3 \\ 10 \\ 11 \\ 8 \\ 12 \\ 10 \end{array}\right]
$$

$$
(X'X)^{-1} \;=\; \frac{1}{180}\left[\begin{array}{r|rr|rrr|rrrrrr}
86 & -45 & -45 & -20 & -20 & -20 & -6 & -6 & -6 & -6 & -6 & -6 \\
\hline
-45 & 70 & 20 & 0 & 0 & -0 & -10 & -10 & -10 & 10 & 10 & 10 \\
-45 & 20 & 70 & -0 & 0 & 0 & 10 & 10 & 10 & -10 & -10 & -10 \\
\hline
-20 & 0 & -0 & 80 & -10 & -10 & -30 & 15 & 15 & -30 & 15 & 15 \\
-20 & 0 & 0 & -10 & 80 & -10 & 15 & -30 & 15 & 15 & -30 & 15 \\
-20 & -0 & 0 & -10 & -10 & 80 & 15 & 15 & -30 & 15 & 15 & -30 \\
\hline
-6 & -10 & 10 & -30 & 15 & 15 & 76 & -14 & -14 & -4 & -4 & -4 \\
-6 & -10 & 10 & 15 & -30 & 15 & -14 & 76 & -14 & -4 & -4 & -4 \\
-6 & -10 & 10 & 15 & 15 & -30 & -14 & -14 & 76 & -4 & -4 & -4 \\
-6 & 10 & -10 & -30 & 15 & 15 & -4 & -4 & -4 & 76 & -14 & -14 \\
-6 & 10 & -10 & 15 & -30 & 15 & -4 & -4 & -4 & -14 & 76 & -14 \\
-6 & 10 & -10 & 15 & 15 & -30 & -4 & -4 & -4 & -14 & -14 & 76
\end{array}\right]
\qquad
(X'X)^{-1}X'y \;=\; \left[\begin{array}{r} 4.5 \\ \hline -.5 \\ .5 \\ \hline -1.75 \\ 1 \\ .75 \\ \hline -.75 \\ 0 \\ .75 \\ .75 \\ 0 \\ -.75 \end{array}\right]
\;=\; \left[\begin{array}{r} \widehat{\mu} \\ \hline \widehat{\alpha}_1 \\ \widehat{\alpha}_2 \\ \hline \widehat{\beta}_1 \\ \widehat{\beta}_2 \\ \widehat{\beta}_3 \\ \hline \widehat{\alpha\beta}_{11} \\ \widehat{\alpha\beta}_{12} \\ \widehat{\alpha\beta}_{13} \\ \widehat{\alpha\beta}_{21} \\ \widehat{\alpha\beta}_{22} \\ \widehat{\alpha\beta}_{23} \end{array}\right]
$$

4.8 Alternate Analysis as a Oneway ANOVA

- Suppose we have data from a two-factor factorial design. The following method can be used to perform a multiple comparison test to compare treatment means as well as Levene's Test to check the homogeneity of variance assumption.
- The main idea is to having $a \times b$ levels and analyze the data as if you had a oneway ANOVA with $a \times b$ treatments.

Multiple Comparison Procedures

Suppose the researcher is interested in comparing the cell means from a two factor factorial design. The following method can be used to perform a multiple comparison procedure (MCP):

1. Create a single factor having $a \times b$ levels. For the 2×2 design example, create a single factor having four levels from the two levels of time (T =12,18) and the two levels of medium (M =1,2). In the SAS code, I called these levels 12_1, 12_2, 18_1, and 18_2.
2. *Run Bonferroni's MCP (or any other MCP) on this single factor.* For the 2×2 design, we reject all $\mathrm{H}_0 : \mu_{ij} = \mu_{i'j'}$ except for $\mu_{12\_2} = \mu_{12\_1}$.

Levene's Test of the HOV Assumption

- In a twoway ANOVA, the HOV assumption implies that all $a \times b$ variances are equal. That is, we assume is the variance associated with the errors for treatment combination (i, j) based on the two design factors.
- Suppose the researcher is interested in testing the HOV assumption that all of the $a \times b$ variances are equal in a two factor factorial design.
- The following method can be used to perform Levene's HOV Test:
  1. *Create a single factor having $a \times b$ treatment levels.* For the 2×2 design example, create a single factor having four levels from the two levels of time (12,18) and the two levels of medium (1,2). In the SAS code, I called these levels 12_1, 12_2, 18_1, and 18_2.
  2. *Run Levene's HOV Test on this single factor.* For the 2×2 design example, we fail to reject the HOV assumption (p-value = .1793).

Dependent Variable: growth

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	691.4583333	230.4861111	45.12	<.0001
Error	20	102.1666667	5.1083333		
Corrected Total	23	793.6250000			

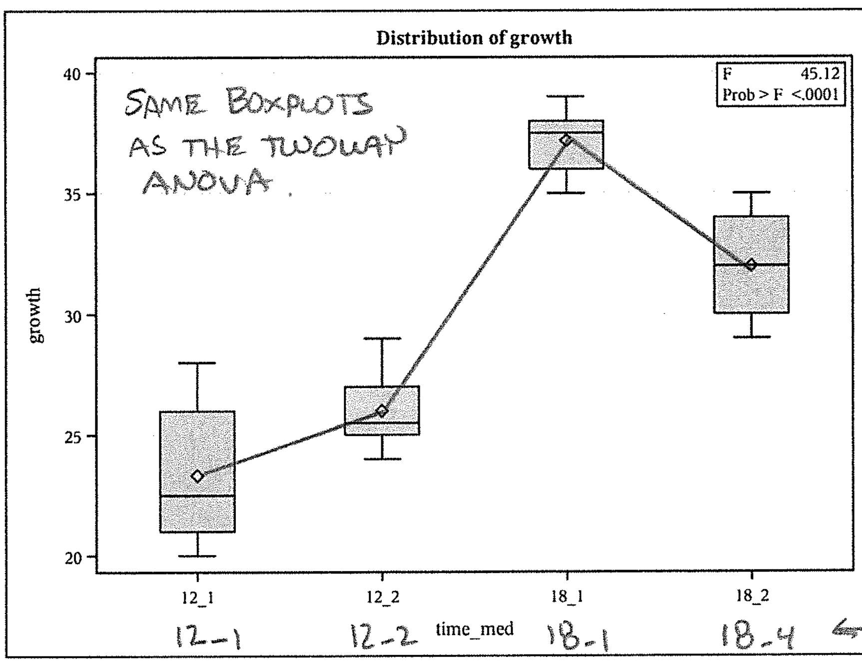
R-Square	Coeff Var	Root MSE	growth Mean
0.871266	7.629240	2.260162	29.62500

FROM THE TWOWAY ANOVA (PAGE 113)

SOURCE	DF	SS
A	1	$590.041\overline{6}$
B	1	9.375
A·B	1	$92.041\overline{6}$
		$691.458\overline{3}$

Source	DF	Type III SS	Mean Square	F Value	Pr > F
time_med	3	691.4583333	230.4861111	45.12	<.0001

$SS_A + SS_B + SS_{AB}$ FROM TWOWAY ANOVA



P<.0001

ONEWAY ANOVA WITH FOUR TREATMENTS

Levene's Test for Homogeneity of growth Variance ANOVA of Absolute Deviations from Group Means					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
time_med	3	6.3472	2.1157	1.80	0.1793
Error	20	23.4815	1.1741		

LEVENE'S TEST

FAIL TO REJECT $H_0: \sigma_{11}^2 = \sigma_{12}^2 = \sigma_{21}^2 = \sigma_{22}^2$

THE HOV ASSUMPTION IS NOT REJECTED.

Alpha	0.05
Error Degrees of Freedom	20
Error Mean Square	5.108333
Critical Value of t	2.92712
Minimum Significant Difference	3.8196

Means with the same letter are not significantly different.			
Bon Grouping	**Mean**	**N**	**time_med**
A	37.167	6	18_1
B	32.000	6	18_2
C	26.000	6	12_2
C			
C	23.333	6	12_1

```
DM 'LOG; CLEAR; OUT; CLEAR;';
ODS GRAPHICS ON;
ODS PRINTER PDF file='C:\COURSES\ST541\TWOALT.PDF';
OPTIONS NODATE NONUMBER;

******************************************************;
*** ALTERNATIVE ANALYSIS FOR A TWOWAY ANOVA PLUS   ***;
*** TESTING THE HOMOGENEITY OF VARIANCE ASSUMPTION ***;
*** AND MULTIPLE COMPARISON PROCEDURES             ***;
******************************************************;

DATA in;
   DO time_med = '12_1' , '12_2' , '18_1' , '18_2';
   DO rep = 1 to 6;
      INPUT growth @@; OUTPUT;
   END; END;
CARDS;
 21 23 20 22 28 26    25 24 29 26 25 27
 37 38 35 39 38 36    31 29 30 34 33 35
 ;
PROC GLM DATA=in;
     CLASS time_med;
     MODEL growth = time_med / SS3;
     MEANS time_med / BON HOVTEST=LEVENE(TYPE=ABS);
TITLE 'ALTERNATE ANOVA AND HOV TEST';
RUN;
```

4.9 Other Multiple Comparison Procedures

- You can also perform a MCP using the LSMEANS statement in Proc GLM in SAS. E.g., to perform a Bonferroni MCP:

  1. Include a **LSMEANS A*B** statement with option **/ ADJUST=BON** for factors A and B.
  2. Reject $H_0 : \mu_{ij} - \mu_{i'j'}$ if the adjusted p-value in the matrix of p-values is $\leq \alpha$. This is equivalent to taking the p-value $\leq \alpha^*$ where $\alpha^* = \alpha/C$ and C is the number of comparisons made. In essence, we are just multiplying the individual test p-values by the number of comparisons, and then checking if the adjusted p-value is $< \alpha$.

- For a different MCP, just change BON to TUKEY, SIDAK, or SCHEFFE. If you do not include the ADJUST option, you will get Fisher's LSD test by default.

4.10 ANOVA for a 2 $\times$ 3 Factorial Design Example

- An experiment was run to investigate how the type of glass and the type of phosphorescent coating affects the brightness of a light bulb. The response variable is the current (in microamps) to obtain a specified brightness. The data are

Glass Type		Phosphor Type A	Phosphor Type B	Phosphor Type C
	1	278	297	273
	1	291	304	284
	1	285	296	288
	2	229	259	228
	2	235	249	225
	2	241	241	235

- Look at the difference in glass means across the levels of phosphor:

$$\overline{y}_{2,A} - \overline{y}_{1,A} = -49.7 \qquad \overline{y}_{2,B} - \overline{y}_{1,B} = -49.3 \qquad \overline{y}_{2,C} - \overline{y}_{1,C} = -52.3$$

  The variability in these three differences ($MS_{glass*phosphor} = 4.0\overline{5}$) is small relative to the $MS_E = 44.\overline{2}$, so we fail to reject the null hypothesis that the interaction effects are equal.

- The Glass*Phosphor interaction is not significant (p-value = .9130). This is also obvious from the strong parallelism in the interaction plots.

- The Bonferroni MCT results are summarized below:

Glass/Phosphor	2 C	2 A	2 B	1 C	1 A	1 B
Mean	229.3	235.0	249.7	281.7	284.7	299.0

SAS Code and Output

```
DATA in; INPUT glass phosphor $ light @@;
LINES;
1 A 278  1 A 291  1 A 285  1 B 297  1 B 304  1 B 296
1 C 273  1 C 284  1 C 288  2 A 229  2 A 235  2 A 241
2 B 259  2 B 249  2 B 241  2 C 225  2 C 228  2 C 235
;
PROC GLM DATA=in ;
     CLASS glass phosphor;
     MODEL light = glass|phosphor / SS3 SOLUTION;
     MEANS glass|phosphor / BON;
     LSMEANS glass*phosphor / ADJUST=BON;
```

```
    ESTIMATE 'mu' intercept 1;
    ESTIMATE 'glass=1' glass 1 -1 / divisor = 2 ;
    ESTIMATE 'glass=2' glass -1 1 / divisor = 2 ;

    ESTIMATE 'phosphor=A' phosphor 2 -1 -1 / divisor = 3 ;
    ESTIMATE 'phosphor=B' phosphor -1 2 -1 / divisor = 3 ;
    ESTIMATE 'phosphor=C' phosphor -1 -1 2 / divisor = 3 ;

    ESTIMATE 'glass=1 phos=A' glass*phosphor 2 -1 -1 -2 1 1 / divisor = 6 ;
    ESTIMATE 'glass=1 phos=B' glass*phosphor -1 2 -1 1 -2 1 / divisor = 6 ;
    ESTIMATE 'glass=1 phos=C' glass*phosphor -1 -1 2 1 1 -2 / divisor = 6 ;
    ESTIMATE 'glass=2 phos=A' glass*phosphor -2 1 1 2 -1 -1 / divisor = 6 ;
    ESTIMATE 'glass=2 phos=B' glass*phosphor 1 -2 1 -1 2 -1 / divisor = 6 ;
    ESTIMATE 'glass=2 phos=C' glass*phosphor 1 1 -2 -1 -1 2 / divisor = 6 ;

TITLE '(2 x 3) TWO FACTOR ANALYSIS OF VARIANCE';
RUN;
```

$\hat{\mu}$, $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\beta}_3$

i	j
1	1
1	2
1	3
2	1
2	2
2	3

KRONECKER PRODUCT

$$\underset{\text{GLASS}}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \otimes \underset{\text{PHOSPHOR}}{\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}}$$

DIVISOR $= a \cdot b = 6$

(2 x 3) TWO FACTOR ANALYSIS OF VARIANCE

The GLM Procedure

Dependent Variable: light

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	5	12626.44444	2525.28889	57.10	<.0001
Error	12	530.66667	44.22222		
Corrected Total	17	13157.11111			

R-Square	Coeff Var	Root MSE	light Mean
0.959667	2.526375	6.649979	263.2222

Source	DF	Type III SS	Mean Square	F Value	Pr > F
glass	1	11450.88889	11450.88889	258.94	<.0001
phosphor	2	1167.44444	583.72222	13.20	0.0009
glass*phosphor	2	8.11111	4.05556	0.09	0.9130

— NO EVIDENCE AGAINST $H_0: \alpha\beta_{11} = \cdots = \alpha\beta_{23} = 0$

↓

NO EVIDENCE IN FAVOR OF AN INTERACTION

↓

NO EVIDENCE TO REJECT H_0.

Parameter	Estimate	Standard Error	t Value	Pr > |t|
mu	263.222222	1.56741511	167.93	<.0001
glass=1	25.222222	1.56741511	16.09	<.0001
glass=2	-25.222222	1.56741511	-16.09	<.0001
phosphor=A	-3.388889	2.21665970	-1.53	0.1522
phosphor=B	11.111111	2.21665970	5.01	0.0003
phosphor=C	-7.722222	2.21665970	-3.48	0.0045
glass=1 phos=A	-0.388889	2.21665970	-0.18	0.8637
glass=1 phos=B	-0.555556	2.21665970	-0.25	0.8063
glass=1 phos=C	0.944444	2.21665970	0.43	0.6776
glass=2 phos=A	0.388889	2.21665970	0.18	0.8637
glass=2 phos=B	0.555556	2.21665970	0.25	0.8063
glass=2 phos=C	-0.944444	2.21665970	-0.43	0.6776

$\hat{\mu}$

$\sum \hat{\alpha}_i = 0$

$\sum \hat{\beta}_j = 0$

① $\sum \alpha\beta_{iA} = 0$

② $\sum \alpha\beta_{iB} = 0$

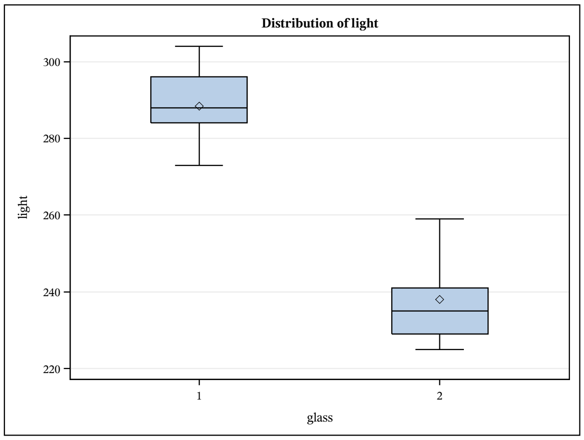
③ $\sum \alpha\beta_{iC} = 0$

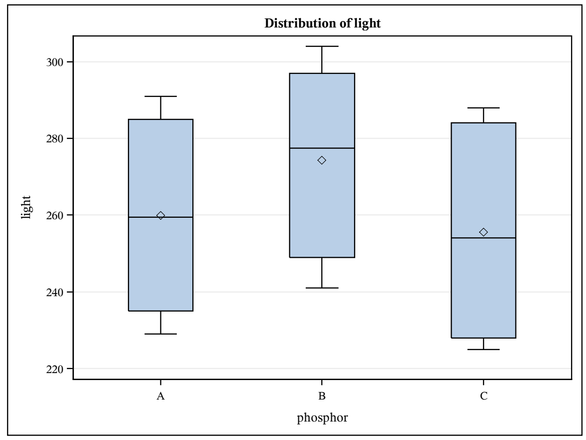
$\sum \alpha\beta_{1j} = 0$

$\sum \alpha\beta_{2j} = 0$

$\widehat{\alpha\beta}_{ij}$ VALUES

	A	B	C	Σ
1	$-.3\bar{8}$	$-.\bar{5}$	$.9\bar{4}$	0
2	$.3\bar{8}$	$.\bar{5}$	$-.9\bar{4}$	0
Σ	0	0	0	





Alpha	0.05
Error Degrees of Freedom	12
Error Mean Square	44.22222
Critical Value of t	2.17881
Minimum Significant Difference	6.8302

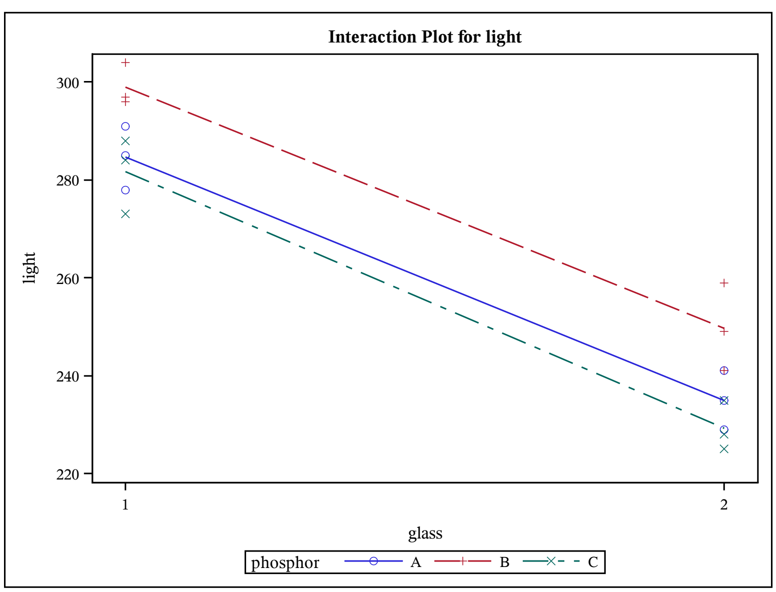
Means with the same letter are not significantly different.

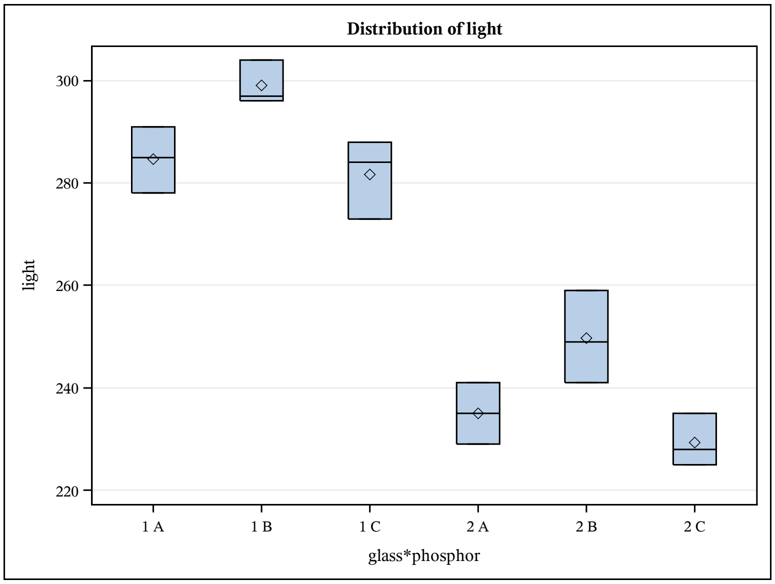
Bon Grouping	Mean	N	glass
A	288.444	9	1
B	238.000	9	2

Alpha	0.05
Error Degrees of Freedom	12
Error Mean Square	44.22222
Critical Value of t	2.77947
Minimum Significant Difference	10.671

Means with the same letter are not significantly different.

Bon Grouping	Mean	N	phosphor
A	274.333	6	B
B	259.833	6	A
B			
B	255.500	6	C





Level of glass	Level of phosphor	N	light Mean	light Std Dev
1	A	3	284.666667	6.50640710
1	B	3	299.000000	4.35889894
1	C	3	281.666667	7.76745347
2	A	3	235.000000	6.00000000
2	B	3	249.666667	9.01849951
2	C	3	229.333333	5.13160144

The GLM Procedure
Least Squares Means
Adjustment for Multiple Comparisons: Bonferroni

glass	phosphor	light LSMEAN	LSMEAN Number
1	A	284.666667	1
1	B	299.000000	2
1	C	281.666667	3
2	A	235.000000	4
2	B	249.666667	5
2	C	229.333333	6

Least Squares Means for effect glass*phosphor
Pr > |t| for H0: LSMean(i)=LSMean(j)

Dependent Variable: light

i/j	1	2	3	4	5	6
1		0.3237	1.0000	<.0001	0.0005	<.0001
2	0.3237		0.1161	<.0001	<.0001	<.0001
3	1.0000	0.1161		<.0001	0.0011	<.0001
4	<.0001	<.0001	<.0001		0.2890	1.0000
5	0.0005	<.0001	0.0011	0.2890		0.0420
6	<.0001	<.0001	<.0001	1.0000	0.0420	

4.11 Example: Sample size determination and power estimation

Determine N given a nominal power level (Case 1) and determine power given N (Case 2) for a specified pattern of means or effects

- Suppose there are 6 treatments resulting from a 2×3 factorial design having n replicates, and based on a prior study we have estimates of the treatment means: $\mu_{11} = 35.1$, $\mu_{12} = 33.7$, $\mu_{13} = 30.2$, $\mu_{21} = 23.0$, $\mu_{22} = 25.9$, $\mu_{23} = 30.4$.
- Our prior estimate of σ is 1.4, and the significance level is set to $\alpha = .05$ for tests.
- For **Case 1**, determine the total sample size $N = 6n$ setting the power for the ANOVA F-tests for main effects and the interaction at levels $1 - \beta = .50,\ .80,\ .90,$ and $.95$.
- For **Case 2**, determine the power $1-\beta$ for the ANOVA F-tests for the main effects and the interaction when the total sample size $N = 18,\ 24,\ 30,$ and 36.

SAS code for Case 1 and Case 2

```
data twoway;
     input levelA $ levelB $ meanest;
datalines;
A1 B1 35.1  A1 B2 33.7  A1 B3 30.2  A2 B1 23.0  A2 B2 25.9  A2 B3 30.4
;
proc glmpower data=twoway;
  class levelA levelB;
  model meanest = levelA|levelB;
  power
     stddev = 1.4
     alpha  = 0.05
     ntotal = .
     power  = .5 .8 .9 .95 ;
title 'Determining design size for given power and mean estimates';
title2 'for a twoway (2 x 3) ANOVA';



proc glmpower data=twoway;
  class levelA levelB;
  model meanest = levelA|levelB;
  power
     stddev = 1.4
     alpha  = 0.05
     ntotal = 18 24 30 36
     power  = . ;
title 'Determining power for a given design size and mean estimates';
title2 'for a twoway (2 x 3) ANOVA';
run;
```

SAS output for Case 1: Determine N for a nominal power level

The GLMPOWER Procedure

Dependent Variable	meanest
Alpha	0.05
Error Standard Deviation	1.4

μ_{11} 35.3 Δ_1	μ_{12} 33.7 Δ_2	μ_{13} 30.2 Δ_3	$\bar{\mu}_{1\cdot}=33.0$
μ_{21} 23.0	μ_{22} 25.9	μ_{23} 30.4	$\bar{\mu}_{2\cdot}=26.4\bar{3}$
$\bar{\mu}_{\cdot 1}=29.05$	$\bar{\mu}_{\cdot 2}=29.8$	$\bar{\mu}_{\cdot 3}=30.3$	

Computed N Total

Index	Type	Source	Nominal Power	Test DF	Error DF	Actual Power	N Total	n	COMPARE
1	Effect	levelA	0.50	1	6	>.999	12	2	
2	Effect	levelA	0.80	1	6	>.999	12	2	
3	Effect	levelA	0.90	1	6	>.999	12	2	$\bar{\mu}_{1\cdot}, \bar{\mu}_{2\cdot}$
4	Effect	levelA	0.95	1	6	>.999	12	2	
5	Effect	levelB	0.50	2	36	0.520	42	7	
6	Effect	levelB	0.80	2	72	0.818	78	13	$\bar{\mu}_{\cdot 1}, \bar{\mu}_{\cdot 2}, \bar{\mu}_{\cdot 3}$
7	Effect	levelB	0.90	2	96	0.915	102	17	
8	Effect	levelB	0.95	2	114	0.954	120	20	
9	Effect	levelA*levelB	0.50	2	6	0.992	12		
10	Effect	levelA*levelB	0.80	2	6	0.992	12		$\Delta_1, \Delta_2, \Delta_3$
11	Effect	levelA*levelB	0.90	2	6	0.992	12		
12	Effect	levelA*levelB	0.95	2	6	0.992	12		

FOR POWER = .90, TAKE LARGEST n (2, 17, 12)
$\Rightarrow n = 17$

SAS output for Case 2: Determine power for a given N

The GLMPOWER Procedure

Dependent Variable	meanest
Alpha	0.05
Error Standard Deviation	1.4

Computed Power

Index	Type	Source	N Total	Test DF	Error DF	Power
1	Effect	levelA	18	1	12	>.999
2	Effect	levelA	24	1	18	>.999
3	Effect	levelA	30	1	24	>.999
4	Effect	levelA	36	1	30	>.999
5	Effect	levelB	18	2	12	0.215
6	Effect	levelB	24	2	18	0.296
7	Effect	levelB	30	2	24	0.375
8	Effect	levelB	36	2	30	0.450
9	Effect	levelA*levelB	18	2	12	>.999
10	Effect	levelA*levelB	24	2	18	>.999
11	Effect	levelA*levelB	30	2	24	>.999
12	Effect	levelA*levelB	36	2	30	>.999

WHY IS THE POWER SO LOW?

4.12 Tukey's Test for Nonadditivity

- Consider a two-factor $a \times b$ factorial design that has only $n = 1$ replicate for each of the ab treatment combinations. We sometimes refer to this as an **unreplicated** experiment or an experiment with a **one observation per cell design**.
- Recall that the MSE df $= ab(n-1)$ for the two-factor interaction model. Thus, there will be 0 df for the MSE when $n = 1$. That is, the df column in the ANOVA table looks like:

Source of Variation	df
A	$a-1$
B	$b-1$
AB	$(a-1)(b-1)$
Error	0
Total	$ab-1$

- If no interaction exists (i.e., we assume all $(\alpha\beta)_{ij}$ effects $= 0$), then the additive model $y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}$ is assumed to be adequate.
- Then the $(a-1)(b-1)$ df, SS, and MS for the interaction effects will now be the SS, df, and MS for the Error. This will allow us to perform tests on the main effects for A and B.
- If an interaction does exist, then what can we do? If we can assume the interaction effects possess a certain structure, then tests may exist.
- One such test is **Tukey's Test for Additivity** which tests if the interaction terms take on the form $(\alpha\beta)_{ij} = \gamma\tau_i\beta_j$. This says the $(ij)^{th}$ interaction effect is proportional to the product of the main effects.
- If this interaction structure is true, then the model will be:

$$y_{ij} =$$

- Note, that we have added only 1 parameter (γ) to the additive model leaving us $(a-1)(b-1)-1 = ab-a-b$ df for the new MSE.
- The following SAS code tricks PROC GLM into fitting this model and performing Tukey's Test for Nonadditivity.

SAS Code for Tukey's Test for Additivity Example

```
*******************************************************;
*** TUKEY'S NONADDITIVITY TEST FOR A TWO-FACTOR DESIGN ***;
***           WITH ONE OBSERVATION PER CELL            ***;
*******************************************************;

DATA in; INPUT temp pressure impurity @@; CARDS;
100 25 5   100 30 4   100 35 6   100 40 3   100 45 5
125 25 3   125 30 1   125 35 4   125 40 2   125 45 3
150 25 1   150 30 1   150 35 3   150 40 1   150 45 2
 ;
PROC GLM DATA=in NOPRINT;
     CLASS temp pressure;
     MODEL impurity = temp pressure ;
     OUTPUT OUT=diag PREDICTED=yhat;

DATA diag; SET diag;
     nonadd = yhat**2;

TITLE 'TUKEY TEST FOR NONADDITIVITY -- 1 OBS PER CELL';

PROC GLM DATA=diag;
     CLASS temp pressure;
     MODEL impurity = temp pressure nonadd / SS1 SOLUTION;
RUN;
```

TUKEY TEST FOR NONADDITIVITY -- 1 OBS PER CELL

The GLM Procedure

Variable: IMPURITY

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	7	35.03185550	5.00455079	18.42	0.0005
Error	7	1.90147783	0.27163969		
Corrected Total	14	36.93333333			

R-Square	Coeff Var	Root MSE	IMPURITY Mean
0.948516	17.76786	0.521191	2.933333

Source	DF	Type I SS	Mean Square	F Value	Pr > F
TEMP	2	23.33333333	11.66666667	42.95	0.0001
PRESSURE	4	11.60000000	2.90000000	10.68	0.0042
NONADD	1	0.09852217	0.09852217	0.36	0.5660

Parameter	Estimate		Standard Error	t Value	Pr > |t|
Intercept	1.812807882	B	0.47262874	3.84	0.0064
TEMP 100	2.312807882	B	1.18771619	1.95	0.0925
TEMP 125	0.844827586	B	0.41838222	2.02	0.0832
TEMP 150	0.000000000	B	.	.	.
PRESSURE 25	-0.255336617	B	0.44482151	-0.57	0.5839
PRESSURE 30	-1.070607553	B	0.60942962	-1.76	0.1224
PRESSURE 35	0.716748768	B	0.63427295	1.13	0.2957
PRESSURE 40	-1.070607553	B	0.60942962	-1.76	0.1224
PRESSURE 45	0.000000000	B	.	.	.
NONADD	0.036945813		0.06134722	0.60	0.5660