2 ONE-FACTOR COMPLETELY RANDOMIZED DESIGN (CRD)

An experiment is run to study the effects of one factor on a response. The levels of the factor can be

- **quantitative (numerical) or qualitative (categorical)**
- with levels set by the experimenter or with randomly chosen levels.

When random selection, random assignment, and a (when possible) randomized run order of experimentation can be applied to the levels of the factor then the experimental design is called a **completely randomized design (CRD)**.

2.1 Notation

Assume that the factor of interest has $a \geq 3$ levels with n_i observations taken at level i of the factor. Let N be the total number of design observations.

The General Sample Size Case

Treatments	1	2	3	$\cdots$	a
	y_{11}	y_{21}	y_{31}	$\cdots$	y_{a1}
	y_{12}	y_{22}	y_{32}	$\cdots$	y_{a2}
	y_{13}	y_{23}	y_{33}	$\cdots$	y_{a3}
	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
	y_{1n_1}	y_{2n_2}	y_{3n_3}	$\cdots$	y_{an_a}
treatment totals	$y_{1\cdot}$	$y_{2\cdot}$	$y_{3\cdot}$	$\cdots$	$y_{a\cdot}$
treatment means	$\overline{y}_{1\cdot}$	$\overline{y}_{2\cdot}$	$\overline{y}_{3\cdot}$	$\cdots$	$\overline{y}_{a\cdot}$

$$\text{Grand total } y_{\cdot\cdot} = \sum_{i=1}^{a}\sum_{j=1}^{n_i} y_{ij}$$

$$\text{Grand mean } \overline{y}_{\cdot\cdot} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{n_i} y_{ij}}{\sum_{n_i}^{a} n_i} = \frac{y_{\cdot\cdot}}{N}$$

$$\text{Treatment total } y_{i\cdot} = \sum_{j=1}^{n_i} y_{ij}$$

$$\text{Treatment mean } \overline{y}_{i\cdot} = \frac{\sum_{j=1}^{n_i} y_{ij}}{n_i} = \frac{y_{i\cdot}}{n_i}$$

The Equal Sample Size Case ($n_i = n$ for $i = 1, 2, \ldots, a$)

Treatments	1	2	3	$\cdots$	a
	y_{11}	y_{21}	y_{31}	$\cdots$	y_{a1}
	y_{12}	y_{22}	y_{32}	$\cdots$	y_{a2}
	y_{13}	y_{23}	y_{33}	$\cdots$	y_{a3}
	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
	y_{1n}	y_{2n}	y_{3n}	$\cdots$	y_{an}
treatment totals	$y_{1\cdot}$	$y_{2\cdot}$	$y_{3\cdot}$	$\cdots$	$y_{a\cdot}$
treatment means	$\overline{y}_{1\cdot}$	$\overline{y}_{2\cdot}$	$\overline{y}_{3\cdot}$	$\cdots$	$\overline{y}_{a\cdot}$

$$\text{Grand total } y_{\cdot\cdot} = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}$$

$$\text{Grand mean } \overline{y}_{\cdot\cdot} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}}{an} = \frac{y_{\cdot\cdot}}{an}$$

$$\text{Treatment total } y_{i\cdot} = \sum_{j=1}^{n} y_{ij}$$

$$\text{Treatment mean } \overline{y}_{i\cdot} = \frac{\sum_{j=1}^{n} y_{ij}}{n} = \frac{y_{i\cdot}}{n}$$

Notation related to variability:

- SS_T = the total (corrected) sum of squares = $\sum_{i=1}^{a}\sum_{j=1}^{n_i}(y_{ij} - \overline{y}_{\cdot\cdot})^2$ = where s^2 is the sample variance of the $N = \sum_{i=1}^{a}\sum_{j=1}^{n_i} n_{ij}$ observations
- $N - 1$ = the degrees of freedom for total

Notation for variability treatments: ("E" stands for "Error")

- SS_E = the error sum of squares = $\sum_{i=1}^{a}\sum_{j=1}^{n_i}(y_{ij} - \overline{y}_{i\cdot})^2$ =
- $N - a$ = the error degrees of freedom
- MS_E = the mean square error = $\dfrac{SS_E}{N-a}$

Notation for variability treatments:

- SS_{Trt} = the treatment sum of squares = $\sum_{i=1}^{a}\sum_{j=1}^{n_i}(\overline{y}_{i\cdot}-\overline{y}_{\cdot\cdot})^2 = \sum_{i=1}^{a} n_i(\overline{y}_{i\cdot}-\overline{y}_{\cdot\cdot})^2$

  If all sample sizes are equal ($n_{ij}=n$), then $SS_{trt} = n\sum_{i=1}^{a}(\overline{y}_{i\cdot}-\overline{y}_{\cdot\cdot})^2$

- $a-1$ = the treatment degrees of freedom
- MS_{Trt} = the treatment mean square = $\dfrac{SS_{Trt}}{a-1}$

Alternate Formulas $SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n_i} y_{ij}^2 - \dfrac{y_{\cdot\cdot}^2}{N}$ $SS_{Trt} = \sum_{i=1}^{a}\dfrac{y_{i\cdot}^2}{n_i} - \dfrac{y_{\cdot\cdot}^2}{N}$ $SS_E = SS_T - SS_{Trt}$

Analysis of Variance (ANOVA) Table

Source of Variation	Sum of Squares	d.f.	Mean Square	F Ratio
Treatment	SS_{Trt}	$a-1$	MS_{Trt}	$F_0 = MS_{Trt}/MS_E$
Error	SS_E	$N-a$	MS_E	——
Total	SS_T	$N-1$	——	——

Sleep Deprivation Example (n_i are equal)

A study was conducted to determine the effects of sleep deprivation on hand-steadiness. The four levels of sleep deprivation of interest are 12, 18, 24, and 30 hours. 32 subjects were randomly selected and assigned to the four levels of sleep deprivation such that 8 subjects were randomly assigned to each level. The response is the reaction time to the onset of a light cue. The results (in hundredths of a second) are contained in the following table.

This is a one-factor CRD has $a = 4$ treatments (4 factor levels) and $n = 8$ replicates per treatment ($N = 4 \times 8 = 32$).

Treatment (in hours)			
12	18	24	30
20	21	25	26
20	20	23	27
17	21	22	24
19	22	23	27
20	20	21	25
19	20	22	28
21	23	22	26
19	19	23	27

Note: subscripts 1, 2, 3, 4 correspond to the 12, 18, 24, and 30 hour sleep deprivation treatments.

$\overline{y}_{1\cdot} = 19.375$ $\overline{y}_{2\cdot} = 20.75$ $\overline{y}_{3\cdot} = 22.625$ $\overline{y}_{4\cdot} = 26.25$ $\overline{y}_{\cdot\cdot} = 22.25$

Degrees of freedom $df_T = N-1 = 31$ $df_{trt} = a-1 = 3$ $df_E = N-a = 28$

Analysis of Variance (ANOVA) Table

	Source of Variation	Sum of Squares	d.f.	Mean Square	F Ratio	p-value
(Treatment)	Hours	213.25	3	$71.08\overline{3}$	$F_0 = 46.56$	$< .0001$
	Error	42.75	28	1.527		
	Total	256.00	31			

2.2 Linear Model Forms for Fixed Effects

- Assume the a levels of the factor are fixed by the experimenter. This implies the levels are specifically chosen by the experimenter.
- For any observation y_{ij} we can write: $y_{ij} = \overline{y}_{i\cdot} + (y_{ij} - \overline{y}_{i\cdot})$. Thus, an observation from treatment i equals the observed treatment mean $\overline{y}_{i\cdot}$ plus a deviation from that observed mean $(y_{ij} - \overline{y}_{i\cdot})$.
- This deviation is called the **residual** for response y_{ij}, and it is denoted: $e_{ij} = y_{ij} - \overline{y}_{i\cdot}$.

The linear **effects model** is $y_{ij} =$ where

- μ is the baseline mean and τ_i is the i^{th} treatment effect $(i = 1, \ldots, a)$ relative to μ.
- $\epsilon_{ij} \sim IIDN(0, \sigma^2)$. The random errors are **i**ndependent, **i**dentically **d**istributed following a **n**ormal distribution with mean 0 and variance σ^2.

The linear **means model** is $y_{ij} =$ where $\mu_i = \mu + \tau_i$ is the mean associated with the i^{th} treatment and $\epsilon_{ij} \sim IIDN(0, \sigma^2)$.

- The goal is to determine if there exist any differences in the set of a treatment means (or effects) in a CRD. We want to check the null hypothesis that $\mu_1, \mu_2, \ldots, \mu_a$, are all equal against the alternative that they are not all equal,

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_a \qquad H_1 : \mu_i \neq \mu_j \text{ for some } i \neq j.$$

  or, equivalently, that there are no significant treatment effects,

$$H_0 : \tau_1 = \tau_2 = \cdots = \tau_a \qquad H_1 : \tau_i \neq \tau_j \text{ for some } i \neq j.$$

- To answer this question, we determine statistically whether any differences among the treatment means could reasonably have occurred based on the variation that occurs BETWEEN treatment (MS_{Trt}) and WITHIN each of the treatments (MS_E).
- Our best estimate of the within treatment variability is the weighted average of the within treatment variances $(s_i^2, i = 1, 2, \ldots, a)$. The weights are the degrees of freedom $(n_i - 1)$ associated with each treatment:

$$\frac{\sum_{i=1}^{a}(n_i - 1)s_i^2}{\sum_{i=1}^{a}(n_i - 1)} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{n_a}(y_{ij} - \overline{y}_{i\cdot})^2}{N - a} =$$

- If $\epsilon_{ij} \sim N(0, \sigma^2)$, then the MSE is an unbiased estimate of σ^2. That is, $E(MS_E) = \sigma^2$.
- If the null hypothesis ($H_0 : \mu_1 = \mu_2 = \cdots = \mu_a$) is true then the MS_{trt} is also an unbiased estimate of σ^2. That is, ($E(MS_{trt}) = \sigma^2$ **assuming all the means are equal**. This implies the ratio:

$$F_0 =$$

  should be close to 1 because MS_{Trt} and MS_E are both unbiased estimates of σ^2 when H_0 is true.

- If F_0 is too large, we will reject H_0 in favor of the alternative hypothesis H_1.
- When H_0 is true and the linear model assumptions are met, the test statistic F_0 follows an F distribution with $(a-1, N-a)$ degrees of freedom ($F_0 \sim F(a-1, N-a)$).
- The formal statistical test is an **Analysis of Variance (ANOVA)** for a completely randomized design with one factor.

Sleep Deprivation Example: ANOVA Hypothesis Test Results

Recall: subscripts 1, 2, 3, 4 correspond to the 12, 18, 24, and 30 hour sleep deprivation treatments.

Hypotheses for Testing Equality of Means

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4 \qquad H_1 : \mu_i \neq \mu_j \text{ for } \quad \text{some } i \neq j.$$

Hypothesis for Testing Equality of Effects

$$H_0 : \tau_1 = \tau_2 = \tau_3 = \tau_4 \qquad H_1 : \tau_i \neq \tau_j \text{ for } \quad \text{some } i \neq j.$$

The Steps of the Hypothesis Test

- The **test statistic** is $F_0 = 46.56$.
- The **reference distribution** is the $F(3, 28)$ distribution.
- The $\alpha = .05$ **critical value** from the $F(3, 28)$ distribution is $F_{.05}(3, 28) \approx 2.95$.
- The **decision rule** is to reject H_0 if $F_0 \geq F_{.05}(3, 28)$ or fail to reject H_0 if $F_0 < F_{.05}(3, 28)$

  Or, reject if p-value $< \alpha$.
- The **conclusion** is to reject H_0 because $F_0 \geq F_{.05}(3, 28) \quad (46.56 > 2.95) \quad$ (and the p-value $< .0001$).

2.3 Expected Mean Squares

- Consider the following three possible constraints:

$$\text{(i)} \ \sum_{i=1}^{a} n_i \tau_i = 0 \qquad \text{(ii)} \ \ \tau_a = 0 \qquad \text{(iii)} \ \ \tau_1 = 0$$

- Constraint (i) is used in the Montgomery textbook, (ii) is the default in SAS, and (iii) is the default in R.

  Note: If all n_i are all equal ($n_1 = \cdots = n_a = n$), then constraint $\sum_{i=1}^{a} n_i \tau_i = 0$ reduces to $\sum_{i=1}^{a} \tau_i = 0$.
- For constraints (i), (ii), (iii), H_0 and H_1 for the equality of treatment effects can be written as:

$$H_0 : \tau_1 = \tau_2 = \cdots = \tau_a = 0 \qquad H_1 : \tau_i \neq \tau_j \ \text{ for some } i \neq j$$

- The expected values of the mean squares are

$$E(MS_{Trt}) = E\left[\frac{\sum_{i=1}^{a} n_i(\overline{y}_{i\cdot} - \overline{y}_{\cdot\cdot})^2}{a-1}\right] = \quad \sigma^2 +$$

$$E(MS_E) \ = E\left[\frac{\sum_{i=1}^{a}\sum_{j=1}^{n_i}(y_{ij} - \overline{y}_{i\cdot})^2}{N-a}\right] = \sigma^2$$

- If H_0 is <u>true</u> then $\tau_i = 0$ for $i = 1, 2, \ldots, a$. This implies

$$E(MS_{Trt}) = \sigma^2 + \frac{\sum_{i=1}^{a} n_i \cdot 0}{a-1} = \sigma^2 + 0 = \sigma^2.$$

- If H_0 is <u>not true</u> then $\tau_i \neq 0$ for at least one i. This implies

$$E(MS_{Trt}) = \sigma^2 + (\textit{positive quantity}) \quad \Longrightarrow \quad E(MS_{Trt}) > \sigma^2.$$

- As $|\tau_i|$ increases, the $E(MS_{Trt})$ also increases. This implies the $F-$ratio of the expected mean squares

$$F = \frac{E(MS_{Trt})}{E(MS_E)} = \frac{\sigma^2 + \sum_{i=1}^{a} n_i \tau_i^2/(a-1)}{\sigma^2}$$

increases.

- This summarizes part of the statistical theory behind using $F_0 = \dfrac{MS_{Trt}}{MS_E}$ to estimate $F = \dfrac{E(MS_{Trt})}{E(MS_E}$ and reject H_0 for large values of F_0.

2.4 Estimation of Model Parameters

- For the effects model, μ and $\tau_1, \ldots, \tau_a$ cannot be uniquely estimated without imposing a constraint on the model effects.

- If we assume a linear constraint (i) $\sum_{i=1}^{a} n_i \tau_i = 0$, (ii) $\tau_a = 0$ (SAS default), or (iii) $\tau_1 = 0$ (R default), then $\mu, \tau_1, \ldots, \tau_a$ can be uniquely estimated from the grand mean $\overline{y}_{..}$ and the treatment means $\overline{y}_{1.}, \ldots, \overline{y}_{a.}$. The

  <u>assuming</u> (i) $\sum_{i=1}^{a} n_i \tau_i = 0$: $\widehat{\mu} = \overline{y}_{..}$ and $\widehat{\tau}_i = \overline{y}_{i.} - \overline{y}_{..}$ for $i = 1, 2, \ldots, a$

  <u>assuming</u> (ii) $\tau_a = 0$: $\widehat{\mu} = \overline{y}_{a}$ and $\widehat{\tau}_i = \overline{y}_{i.} - \overline{y}_{a}$ for $i = 1, 2, \ldots, a$

  <u>assuming</u> (iii) $\tau_1 = 0$: $\widehat{\mu} = \overline{y}_{1}$ and $\widehat{\tau}_i = \overline{y}_{i.} - \overline{y}_{1}$ for $i = 1, 2, \ldots, a$

- The criterion for estimating effects is the . The goal is to find parameter estimates that minimize SSE.

- Mathematically, find $\widehat{\mu}, \widehat{\tau}_1, \ldots, \widehat{\tau}_a$ that minimize $L = \sum_{i=1}^{a} \sum_{j=1}^{n_i} (y_{ij} - \widehat{\mu} - \widehat{\tau}_i)^2$.

- A solution can be found by solving the . That is, equate the partial derivatives of L to 0, and then solve:

$$\frac{\partial L}{\partial \widehat{\tau}_i} = -2 \sum_{j=1}^{n_i} (y_{ij} - \widehat{\mu} - \widehat{\tau}_i) = 0 \qquad \text{for } i = 1, \ldots, a$$

$$\longrightarrow \quad \sum_{j=1}^{n_i} y_{ij} = \sum_{j=1}^{n_i} \widehat{\mu} + \sum_{j=1}^{n_i} \widehat{\tau}_i \quad \longrightarrow \quad y_{i\cdot} = n_i \widehat{\mu} + n_i \widehat{\tau}_i \tag{1}$$

$$\frac{\partial L}{\partial \widehat{\mu}} = -2 \sum_{i=1}^{a} \sum_{j=1}^{n_i} (y_{ij} - \widehat{\mu} - \widehat{\tau}_i) = 0$$

$$\longrightarrow \quad \sum_{i=1}^{a} \sum_{j=1}^{n_i} y_{ij} = \sum_{i=1}^{a} \sum_{j=1}^{n_i} \widehat{\mu} + \sum_{i=1}^{a} \sum_{j=1}^{n_i} \widehat{\tau}_i \quad \longrightarrow \quad y_{\cdot\cdot} = N \widehat{\mu} + \sum_{i=1}^{a} n_i \widehat{\tau}_i \tag{2}$$

- Note that there are $a+1$ normal equations, but they are not linearly independent:

$$\begin{array}{rcll} y_{1\cdot} & = & n_1\widehat{\mu} \ + \ n_1\widehat{\tau}_1 & \text{for } i=1 \text{ in (1)} \\ y_{2\cdot} & = & n_2\widehat{\mu} \ + \ n_2\widehat{\tau}_2 & \text{for } i=2 \text{ in (1)} \\ \vdots & & \vdots & \vdots \\ y_{a\cdot} & = & n_a\widehat{\mu} \ + \ n_a\widehat{\tau}_a & \text{for } i=a \text{ in (1)} \\ \hline y_{\cdot\cdot} & = & N\widehat{\mu} \ + \ \sum_{i=1}^{a} n_i\widehat{\tau}_i & \text{The sum yields (2)} \end{array}$$

  Thus, there are an infinite number of solutions to the normal equations.

- To get a unique solution for $\widehat{\mu}$ and each $\widehat{\tau}_i$, we must impose a constraint.

- **Constraint I**: The constraint used in many textbooks is

$$\sum_{i=1}^{a} n_i\tau_i \ = \ 0 \quad \text{for unequal } n_i$$

$$\text{or, } \sum_{i=1}^{a} \tau_i \ = \ 0 \quad \text{for equal } n_i$$

- Substituting this constraint into $y_{\cdot\cdot} \ = \ N\widehat{\mu} \ + \ \sum_{i=1}^{a} n_i\widehat{\tau}_i$ in (2) and using estimates yields:

$$y_{\cdot\cdot} \ = \ N\widehat{\mu} \quad \longrightarrow \quad \widehat{\mu} \ =$$

- Substituting $\widehat{\mu} \ = \ \overline{y}_{\cdot\cdot}$ into (1) yields:

$$y_{i\cdot} \ = \ n_i\overline{y}_{\cdot\cdot} \ + \ n_i\widehat{\tau}_i \quad \longrightarrow \quad \widehat{\tau}_i \ =$$

- **Constraint II**: This constraint ($\tau_a = 0$) is the default used in Proc GLM in SAS:

- Substituting this constraint into the last normal equation $y_{a\cdot} \ = \ n_a\widehat{\mu} \ + \ n_a\widehat{\tau}_a$ in (1) and using estimates yields:

$$y_{a\cdot} \ = \ n_a\widehat{\mu} + n_a\widehat{\tau}_a \ = \ n_a\widehat{\mu}$$

  Thus,

$$\widehat{\mu} \ =$$

- Substitution of $\widehat{\mu} \ = \ \overline{y}_{a\cdot}$ into $y_{i\cdot} \ = \ n_i\widehat{\mu} \ + \ n_i\widehat{\tau}_i$ in (1) for $i = 1, 2, \ldots, a$ yields

$$\widehat{\tau}_i \ =$$

2.5 Sleep Deprivation Example (n_i are equal)

The four levels of sleep deprivation of interest are 12, 18, 24, and 30 hours. The response is the reaction time to the onset of a light cue (in hundredths of a second):

Treatment (in hours)				
12	18	24	30	
20	21	25	26	
20	20	23	27	
17	21	22	24	
19	22	23	27	
20	20	21	25	
19	20	22	28	
21	23	22	26	
19	19	23	27	
$\overline{y}_{1\cdot} = 19.375$	$\overline{y}_{2\cdot} = 20.75$	$\overline{y}_{3\cdot} = 22.625$	$\overline{y}_{4\cdot} = 26.25$	$\overline{y}_{\cdot\cdot} = 22.25$

Note: subscripts 1, 2, 3, 4 correspond to the 12, 18, 24, and 30 hour sleep deprivation treatments.

- Suppose we assume Constraint I: $\sum_{i=1}^{4} \widehat{\tau}_i = 0$. This is equivalent to the constraint $\sum_{i=1}^{4} n_i \widehat{\tau}_i = 0$ because all $n_i = 8$. The model parameter estimates are:

  $\widehat{\mu} = \overline{y}_{\cdot\cdot} = 22.25$

  $\widehat{\tau}_1 = \overline{y}_{1\cdot} - \overline{y}_{\cdot\cdot} = 19.375 - 22.25 = -2.875$ $\quad$ $\widehat{\tau}_2 = \overline{y}_{2\cdot} - \overline{y}_{\cdot\cdot} = 20.75 - 22.25 = -1.5$

  $\widehat{\tau}_3 = \overline{y}_{3\cdot} - \overline{y}_{\cdot\cdot} = 22.625 - 22.25 = 0.375$ $\quad$ $\widehat{\tau}_4 = \overline{y}_{4\cdot} - \overline{y}_{\cdot\cdot} = 26.25 - 22.25 = 4.0$

- Thus, our estimates $\widehat{\mu}_1$, $\widehat{\mu}_2$, $\widehat{\mu}_3$, and $\widehat{\mu}_4$ under Constraint I are:

  $\widehat{\mu}_1 = \widehat{\mu} + \widehat{\tau}_1 = 22.25 - 2.875 = 19.375 = \overline{y}_{1\cdot}$ $\quad$ $\widehat{\mu}_2 = \widehat{\mu} + \widehat{\tau}_2 = 22.25 - 1.5 = 20.75 = \overline{y}_{2\cdot}$

  $\widehat{\mu}_3 = \widehat{\mu} + \widehat{\tau}_3 = 22.25 + 0.375 = 22.625 = \overline{y}_{3\cdot}$ $\quad$ $\widehat{\mu}_4 = \widehat{\mu} + \widehat{\tau}_4 = 22.25 + 4.0 = 26.25 = \overline{y}_{4\cdot}$

- Suppose we assume (SAS) Constraint II: $\tau_a = 0$ where $a = 4$. The model parameter estimates are:

  $\widehat{\mu} = \overline{y}_{4\cdot} = 26.25$

  $\widehat{\tau}_1 = \overline{y}_{1\cdot} - \overline{y}_{4\cdot} = 19.375 - 26.25 = -6.875$ $\quad$ $\widehat{\tau}_2 = \overline{y}_{2\cdot} - \overline{y}_{4\cdot} = 20.75 - 26.25 = -5.5$

  $\widehat{\tau}_3 = \overline{y}_{3\cdot} - \overline{y}_{4\cdot} = 22.625 - 26.25 = -3.625$ $\quad$ $\widehat{\tau}_4 = \overline{y}_{4\cdot} - \overline{y}_{4\cdot} = 26.25 - 26.25 = 0$

- Thus, our estimates $\widehat{\mu}_1$, $\widehat{\mu}_2$, $\widehat{\mu}_3$, and $\widehat{\mu}_4$ under Constraint II are:

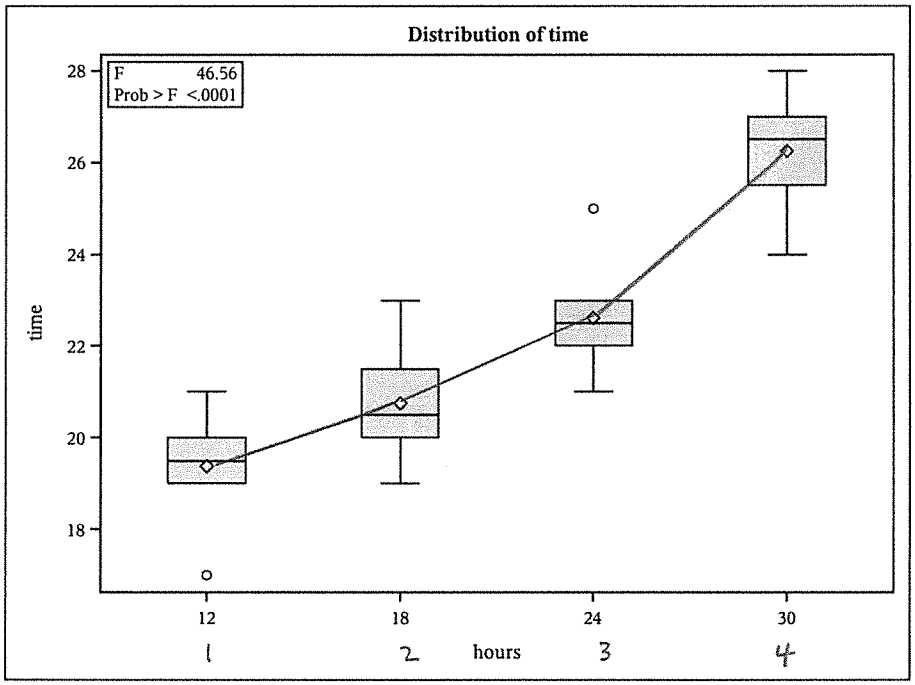
  $\widehat{\mu}_1 = \widehat{\mu} + \widehat{\tau}_1 = 26.25 - 6.875 = 19.375 = \overline{y}_{1\cdot}$ $\quad$ $\widehat{\mu}_2 = \widehat{\mu} + \widehat{\tau}_2 = 26.25 - 5.50 = 20.75 = \overline{y}_{2\cdot}$

  $\widehat{\mu}_3 = \widehat{\mu} + \widehat{\tau}_3 = 26.25 - 3.625 = 22.625 = \overline{y}_{3\cdot}$ $\quad$ $\widehat{\mu}_4 = \widehat{\mu} + \widehat{\tau}_4 = 26.25 - 0 = 26.25 = \overline{y}_{4\cdot}$

- Constraints I and II yield the same $\widehat{\mu}_i$ estimates although the $\widehat{\mu}$ and $\widehat{\tau}_i$ differ between constraints.
- A function that is uniquely estimated regardless of which constraint is used is said to be **estimable**.
- For a oneway ANOVA, $\mu + \tau_i$ for $i = 1, 2, \ldots, a$ are estimable functions, while individually μ, τ_1, τ_2 $\ldots$, τ_a are not estimable.
- Note: Any linear combination $\sum_{i=1}^{a} c_i(\mu + \tau_i)$ is estimable.

We will now analyze the data from the Sleep Deprivation example using SAS. The analysis will include

- Side-by-side boxplots of the time response across sleep deprivation treatments.
- ANOVA results.
- A table of treatment means and standard deviations.
- Parameter estimates assuming the constraint $\tau_4 = 0$. This is the default in SAS.
- Parameter estimates assuming the constraint $\sum_{i=1}^{4} \tau_i = 0$. These are calculated using ESTIMATE statements in SAS.



The GLM Procedure

Dependent Variable: time

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	213.2500000	71.0833333	46.56	<.0001
Error	28	42.7500000	1.5267857		
Corrected Total	31	256.0000000			

SSE

SS_{TOTAL}

(*)

R-Square	Coeff Var	Root MSE	time Mean
0.833008	5.553401	1.235632	22.25000

CV

$\bar{y}_{..}$

83.3% OF THE VARIABILITY IN THE RESPONSE (TIME) IS EXPLAINED BY THE EFFECTS MODEL

THERE IS VERY STRONG EVIDENCE TO REJECT $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$ OR $H_0: \tau_1 = \tau_2 = \tau_3 = \tau_4$

Source	DF	Type III SS	Mean Square	F Value	Pr > F
hours	3	213.2500000	71.0833333	46.56	<.0001

SS_{TRT} F_0 P-VALUE

(*) $CV = 100 \dfrac{\hat{\sigma}}{\hat{\mu}}$

$= 100 \dfrac{\sqrt{MS_E}}{\bar{y}_{..}} = 5.55$

SAS DEFAULT ESTIMATE (ASSUMES $\tau_4 = 0$)

$\hat{\mu} = \bar{y}_{4.}$

Parameter	Estimate		Standard Error	t Value	Pr > |t|	95% Confidence Limits	
Intercept	26.25000000	B	0.43686178	60.09	<.0001	25.35512921	27.14487079
hours 12	-6.87500000	B	0.61781585	-11.13	<.0001	-8.14053841	-5.60946159
hours 18	-5.50000000	B	0.61781585	-8.90	<.0001	-6.76553841	-4.23446159
hours 24	-3.62500000	B	0.61781585	-5.87	<.0001	-4.89053841	-2.35946159
hours 30	0.00000000	B	.	.	.	.	.

$\hat{\tau}_1$ $\hat{\tau}_2$ $\hat{\tau}_3$ $\hat{\tau}_4$

Note: The X'X matrix has been found to be singular, and a generalized inverse was used to solve the normal equations. Terms whose estimates are followed by the letter 'B' are not uniquely estimable.

THERE ARE INFINITELY MANY SOLUTIONS TO THE NORMAL EQUATIONS.

Level of hours	N	time Mean	time Std Dev
12	8	19.3750000	1.18773494
18	8	20.7500000	1.28173989
24	8	22.6250000	1.18773494
30	8	26.2500000	1.28173989

ASSUMING CONSTRAINT $\sum \tau_i = 0$ $\quad \bar{y}_{4.} \rightarrow$

Parameter	Estimate	Standard Error	t Value	Pr > |t|	95% Confidence Limits	
Baseline mu	22.2500000	0.21843089	101.86	<.0001	21.8025646	22.6974354
12 hour effect	-2.8750000	0.37833340	-7.60	<.0001	-3.6499808	-2.1000192
18 hour effect	-1.5000000	0.37833340	-3.96	0.0005	-2.2749808	-0.7250192
24 hour effect	0.3750000	0.37833340	0.99	0.3301	-0.3999808	1.1499808
30 hour effect	4.0000000	0.37833340	10.57	<.0001	3.2250192	4.7749808
12 hour mean	19.3750000	0.43686178	44.35	<.0001	18.4801292	20.2698708
18 hour mean	20.7500000	0.43686178	47.50	<.0001	19.8551292	21.6448708
24 hour mean	22.6250000	0.43686178	51.79	<.0001	21.7301292	23.5198708
30 hour mean	26.2500000	0.43686178	60.09	<.0001	25.3551292	27.1448708
12 vs 18 hrs	1.3750000	0.61781585	2.23	0.0343	0.1094616	2.6405384
12 vs 30 hrs	6.8750000	0.61781585	11.13	<.0001	5.6094616	8.1405384
18 vs 24 hrs	1.8750000	0.61781585	3.03	0.0052	0.6094616	3.1405384
Linear Trend	22.5000000	1.95370527	11.52	<.0001	18.4980162	26.5019838
Quadratic Trend	2.2500000	0.87372356	2.58	0.0156	0.4602584	4.0397416
Cubic Trend	1.2500000	1.95370527	0.64	0.5275	-2.7519838	5.2519838

$\hat{\mu} = \bar{y}_{..}$ $\hat{\tau}_1$ $\hat{\tau}_2$ $\hat{\tau}_3$ $\hat{\tau}_4$ $\quad \sum \hat{\tau}_i = 0$

$\hat{\mu}_1$ $\hat{\mu}_2$ $\hat{\mu}_3$ $\hat{\mu}_4$ $\quad \hat{\mu}_i = \bar{y}_{i.}$

$\hat{\mu}_2 - \hat{\mu}_1$ $\hat{\mu}_4 - \hat{\mu}_1$ $\hat{\mu}_2 - \hat{\mu}_2$ $\quad \hat{\mu}_i - \hat{\mu}_j = \bar{y}_{i.} - \bar{y}_{j.}$

(A) (B) (C)

(A) $\hat{\Gamma}_L = -3\hat{\mu}_1 - \hat{\mu}_2 + \hat{\mu}_3 + 3\hat{\mu}_4$

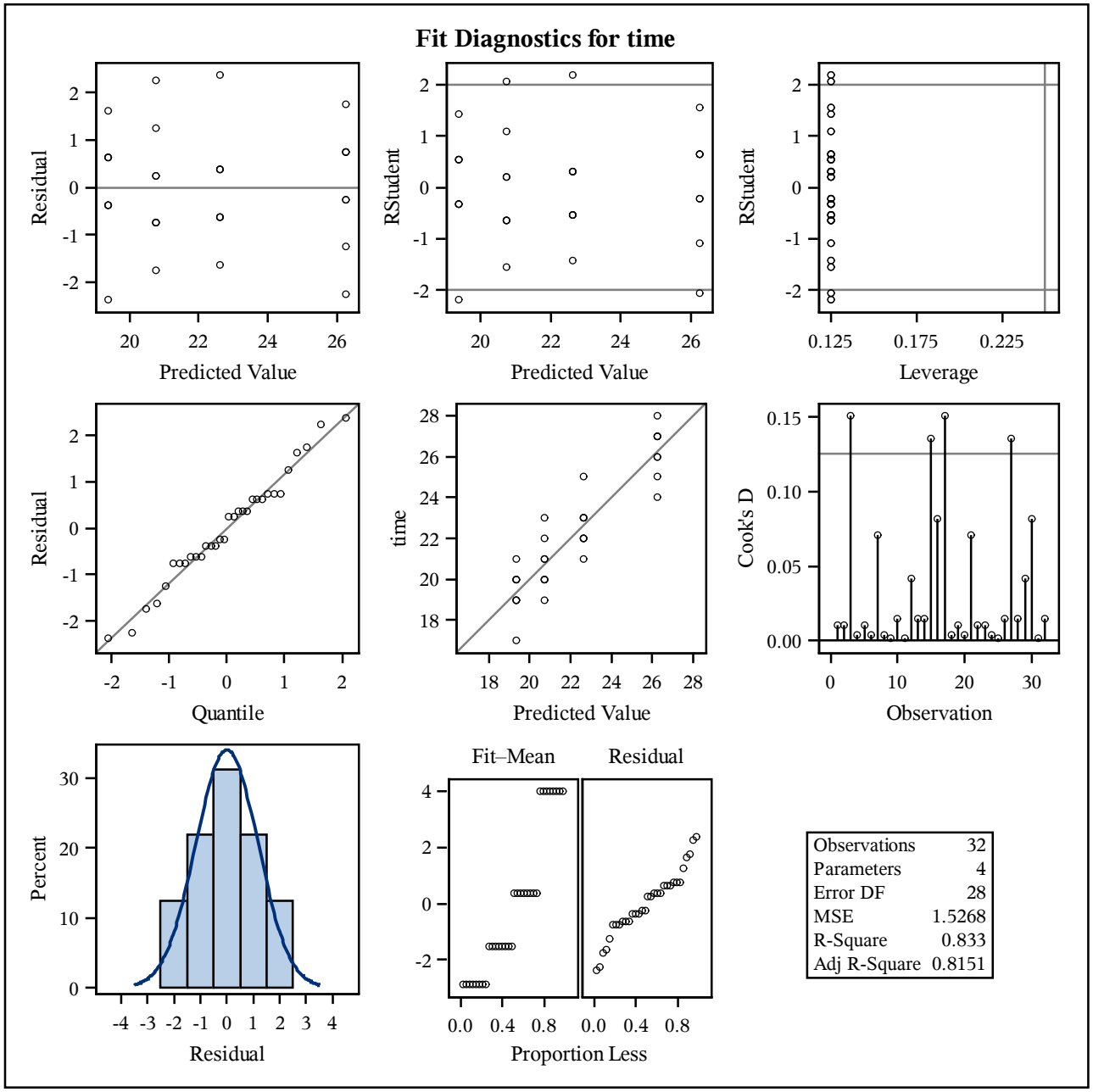
(B) $\hat{\Gamma}_Q = \hat{\mu}_1 - \hat{\mu}_2 - \hat{\mu}_3 + \hat{\mu}_4$

(C) $\hat{\Gamma}_C = -\hat{\mu}_1 + 3\hat{\mu}_2 - 3\hat{\mu}_3 + \hat{\mu}_4$

$$\text{S.E.} = \sqrt{MS_E \sum_{i=1}^{a} \frac{c_i^2}{n_i}}$$

$$t = \frac{\text{ESTIMATE}}{\text{S.E. (ESTIMATE)}}$$

- Diagnostic plots of the residuals to assess if any model assumptions are seriously violated. These include:
  - A **normal probability plot (NPP)** and a **histogram** of the residuals. These plots assess the assumption that the errors are normally distributed. The pattern in NP plot should be close to linear when the residuals are approximately normally distributed while the histogram should be bell-shaped (assuming there are a reasonable number of residuals. Any serious deviations from linearity suggests the normality assumption has been violated.
  - **Residual versus predicted (fitted) value plot**. This plot assesses the homogeneity of variance (HOV) assumption that the errors have the same variance for each treatment. The residuals should be centered about 0 and the spread of the residuals should be similar for each treatment. A pattern (such as funneling) indicates a potential violation of the HOV assumption.



2.5.1 SAS Code for Sleep Deprivation Example

```
DM 'LOG; CLEAR; OUT; CLEAR;';

ODS GRAPHICS ON;
ODS PRINTER PDF file='C:\COURSES\ST541\SLEEP.PDF';
OPTIONS NODATE NONUMBER;

*********************************;
*** Sleep deprivation example ***;
*********************************;

DATA in;
     DO hours = 12 to 30 by 6;
     DO rep = 1 to 8;
        INPUT time @@; OUTPUT;
     END; END;
CARDS;
20 20 17 19 20 19 21 19    21 20 21 22 20 20 23 19
25 23 22 23 21 22 22 23    26 27 24 27 25 28 26 27
;
PROC GLM DATA=in PLOTS = (ALL);
     CLASS hours;
     MODEL time = hours / SS3 SOLUTION CLPARM ALPHA=.05;
     MEANS hours ;
     ESTIMATE 'Baseline mu    ' INTERCEPT 1;

     ESTIMATE '12 hour effect' hours  3 -1 -1 -1 / DIVISOR=4;
     ESTIMATE '18 hour effect' hours -1  3 -1 -1 / DIVISOR=4;
     ESTIMATE '24 hour effect' hours -1 -1  3 -1 / DIVISOR=4;
     ESTIMATE '30 hour effect' hours -1 -1 -1  3 / DIVISOR=4;

     ESTIMATE '12 hour mean' INTERCEPT 1 hours 1 0 0 0;
     ESTIMATE '18 hour mean' INTERCEPT 1 hours 0 1 0 0;
     ESTIMATE '24 hour mean' INTERCEPT 1 hours 0 0 1 0;
     ESTIMATE '30 hour mean' INTERCEPT 1 hours 0 0 0 1;

     ESTIMATE '12 vs 18 hrs' hours -1  1  0  0;
     ESTIMATE '12 vs 30 hrs' hours -1  0  0  1;
     ESTIMATE '18 vs 24 hrs' hours  0 -1  1  0;

     ESTIMATE 'Linear Trend'    hours -3 -1  1 3;
     ESTIMATE 'Quadratic Trend' hours  1 -1 -1 1;
     ESTIMATE 'Cubic Trend'     hours -1  3 -3 1;

     CONTRAST 'Linear Trend'    hours -3 -1  1 3;
     CONTRAST 'Quadratic Trend' hours  1 -1 -1 1;
     CONTRAST 'Cubic Trend'     hours -1  3 -3 1;
TITLE 'SLEEP DEPRIVATION EXAMPLE';
TITLE2 'CONTRASTS AND MULTIPLE COMPARISONS';
RUN;
```

ORDER OF DATA ENTRY

HOUR	REP	HOUR	REP	HOUR	REP	HOUR	REP
12	1	18	1	24	1	30	1
12	2	18	2	24	2	30	2
⋮		⋮	⋮	⋮	⋮	⋮	⋮
12	8	18	8	24	8	30	8

OR, YOU CAN USE LINES;

OUTPUT RESIDUAL DIAGNOSTIC PLOTS (3×3)

OUTPUT ESTIMATES ASSUMING $\tau_a = 0$

95%

GENERATES CONFIDENCE INTERVALS

RESPONSE $\bar{y}_{i\cdot}$

TREATMENT

TYPE III SUMS OF SQUARES

LABELS

$\mu_1\ \mu_2\ \mu_3\ \mu_4$

OUTPUTS ESTIMATES ASSUMING $\sum_{i=1}^{a} \tau_i = 0$ (SEE PAGE 17)

$\mu\ \ \tau_1\ \tau_2\ \tau_3\ \tau_4$

$\mu+\tau_1$, $\mu+\tau_2$, $\mu+\tau_3$, $\mu+\tau_4$ — ESTIMATE μ_i

$\mu_1\ \mu_2\ \mu_3\ \mu_4$

$\mu_2-\mu_1$, $\mu_4-\mu_3$, $\mu_3-\mu_2$ — ESTIMATE $\mu_i-\mu_j$ OR $\tau_i-\tau_j$

ORTHOGONAL CONTRASTS

SEE (A), (B), (C) ON PAGE 14

- I will now show why the SAS ESTIMATE statements have the form

```
ESTIMATE '12 hour effect' hours  3 -1 -1 -1 / DIVISOR=4;
ESTIMATE '18 hour effect' hours -1  3 -1 -1 / DIVISOR=4;
ESTIMATE '24 hour effect' hours -1 -1  3 -1 / DIVISOR=4;
ESTIMATE '30 hour effect' hours -1 -1 -1  3 / DIVISOR=4;
```

- We will use the relationship between the means model and the effects model to derive this pattern.
- Assume the constraint $\sum_{i=1}^{4} \tau_i = 0$ with all n_i equal. Then $\mu =$.

  We also know that $\mu_i = \mu + \tau_i$. Therefore, $\tau_i = \mu_i - \mu$ for $i = 1, 2, 3, 4$.
- Consider the case with $i = 1$. Then

$$\begin{aligned} \tau_1 = \mu_1 - \mu &= \mu_1 - \frac{\mu_1 + \mu_2 + \mu_3 + \mu_4}{4} \\ &= \frac{3}{4}\mu_1 - \frac{1}{4}\mu_2 - \frac{1}{4}\mu_3 - \frac{1}{4}\mu_4 \\ &= \frac{1}{4}(3\mu_1 - \mu_2 - \mu_3 - \mu_4) \end{aligned}$$

- Note that τ_1 can be expressed as a contrast in the 4 means. Thus, $\widehat{\tau}_1$ is estimable because it is a linear combination of the $\widehat{\mu}_i$'s, and each $\widehat{\mu}_i = \overline{y}_{i\cdot}$ is estimable. Similarly,

$$\begin{aligned} \tau_2 &= \frac{1}{4}(-\mu_1 + 3\mu_2 - \mu_3 - \mu_4) \\ \tau_3 &= \frac{1}{4}(-\mu_1 - \mu_2 + 3\mu_3 - \mu_4) \\ \tau_4 &= \frac{1}{4}(-\mu_1 - \mu_2 - \mu_3 + 3\mu_4) \end{aligned}$$

  Thus, the coefficients in the ESTIMATE statements for $\tau_1, \tau_2, \tau_3, \tau_4$ are, respectively,
  $\left(\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$ $\left(-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$ $\left(-\frac{1}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}\right)$ $\left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{3}{4}\right)$
- In general, the coefficients in the ESTIMATE statements have the form

$$\begin{array}{lccccccc} \text{For } \tau_1\text{:} & a-1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ \text{For } \tau_2\text{:} & -1 & a-1 & -1 & -1 & \cdots & -1 & -1 \\ \text{For } \tau_3\text{:} & -1 & -1 & a-1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \text{For } \tau_{a-1}\text{:} & -1 & -1 & -1 & -1 & \cdots & a-1 & -1 \\ \text{For } \tau_a\text{:} & -1 & -1 & -1 & -1 & \cdots & -1 & a-1 \end{array}$$

  with a DIVISOR $= a$.
- Note that for τ_i, the i^{th} coefficient will be $(a-1)/a$ and all other coefficients will be $-1/a$.
- If all n_i are not equal, then we assume constraint $\sum_{i=1}^{a} n_i \tau_i = 0$. This will then yield $\mu = \frac{\sum_{i=1}^{a} n_i \mu_i}{N}$.
- You can then find the coefficients in the ESTIMATE statement by rewriting

$$\begin{aligned} \tau_i = \mu_i - \mu &= \mu_i - \frac{\sum_{i=1}^{a} n_i \mu_i}{N} \\ &= \frac{N\mu_i - \sum_{i=1}^{a} n_i \mu_i}{N} \end{aligned}$$

```
DM 'LOG; CLEAR; OUT; CLEAR;';

ODS GRAPHICS ON;
* ODS PRINTER PDF file='C:\COURSES\ST541\montprob\p3_20mod.PDF';
OPTIONS NODATE NONUMBER;

*****************************;
*** Problem 3-20 page 117 ***;
*****************************;

DATA in;
     DO circuit = 1 to 3;
     DO rep = 1 to 5;
        INPUT time @@; OUTPUT;
     END; END;
CARDS;
9 12 10 8 15  20 21 23 17 30  6 5 8 16 .
;

PROC GLM DATA=in ; * PLOTS = (ALL);
     CLASS circuit;
     MODEL time = circuit / SS3 SOLUTION ; * CLPARM ALPHA=.05;
   * MEANS circuit ;

     ESTIMATE 'Circuit 1' circuit  9 -5 -4  / DIVISOR=14;
     ESTIMATE 'Circuit 2' circuit -5  9 -4  / DIVISOR=14;
     ESTIMATE 'Circuit 3' circuit -5 -5  10 / DIVISOR=14;

TITLE 'Problem 3.20, Page 117 with missing observation';
RUN;
```

$\hat{\tau}_i$ ESTIMATES WITH UNEQUAL SAMPLE SIZES ASSUMING $\sum_{i=1}^{a} n_i \tau_i = 0$.

NOTE: $\sum n_i \mu_i = \sum n_i (\mu + \tau_i)$

$= \sum n_i \mu + \sum n_i \tau_i$ ($\sum n_i \tau_i = 0$)

$= \sum n_i \mu \Rightarrow \mu = \frac{\sum n_i \mu_i}{\sum n_i}$

$$\hat{\mu} = \frac{n_1\hat{\mu}_1 + n_2\hat{\mu}_2 + n_3\hat{\mu}_3}{\sum n_i} = \frac{5\hat{\mu}_1 + 5\hat{\mu}_2 + 4\hat{\mu}_3}{14} \quad (\bar{y}_{..})$$

$$\Rightarrow \hat{\tau}_i = \hat{\mu}_i - \hat{\mu} = \hat{\mu}_i - \frac{\sum n_i \hat{\mu}_i}{\sum n_i} = \hat{\mu}_i - \frac{5\hat{\mu}_1 + 5\hat{\mu}_2 + 4\hat{\mu}_3}{14}$$

$$\Rightarrow \hat{\tau}_1 = \frac{1}{14}(9\hat{\mu}_1 - 5\hat{\mu}_2 - 4\hat{\mu}_3) \qquad \hat{\tau}_2 = \frac{1}{14}(-5\hat{\mu}_1 + 9\hat{\mu}_2 - 4\hat{\mu}_3)$$

$$\hat{\tau}_3 = \frac{1}{14}(-5\hat{\mu}_1 - 5\hat{\mu}_2 + 10\hat{\mu}_3)$$

COEFFICIENTS FOR THE ESTIMATE STATEMENTS

$$\begin{bmatrix} \frac{9}{14} & \frac{-5}{14} & \frac{-4}{14} \\ \frac{-5}{14} & \frac{9}{14} & \frac{-4}{14} \\ \frac{-5}{14} & \frac{-5}{14} & \frac{10}{14} \end{bmatrix}$$

Problem 3.20, Page 117 with missing observation

The GLM Procedure

Dependent Variable: time

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	496.5071429	248.2535714	13.63	0.0011
Error	11	200.3500000	18.2136364		
Corrected Total	13	696.8571429			

R-Square	Coeff Var	Root MSE	time Mean
0.712495	29.87421	4.267744	14.28571

Source	DF	Type III SS	Mean Square	F Value	Pr > F
circuit	2	496.5071429	248.2535714	13.63	0.0011

$\sum n_i \hat{\tau}_i = 0$

$\hat{\tau}_1$
$\hat{\tau}_2$
$\hat{\tau}_3$

Parameter	Estimate	Standard Error	t Value	Pr > |t|
Circuit 1	-3.48571429	1.53027881	-2.28	0.0437
Circuit 2	7.91428571	1.53027881	5.17	0.0003
Circuit 3	-5.53571429	1.80345088	-3.07	0.0107

Parameter	Estimate		Standard Error	t Value	Pr > |t|
Intercept	8.75000000	B	2.13387185	4.10	0.0018
circuit 1	2.05000000	B	2.86288951	0.72	0.4889
circuit 2	13.45000000	B	2.86288951	4.70	0.0007
circuit 3	0.00000000	B	.	.	.

$$\sum_{i=1}^{3} n_i \hat{\tau}_i = (5)(-3.4857\ldots) + (5)(7.9142\ldots) + (4)(-5.5337\ldots)$$

$$= -17.428571\ldots + 39.57142\ldots - 22.14285\ldots$$

$$= 0$$

2.5.2 R Analysis for Sleep Deprivation Example

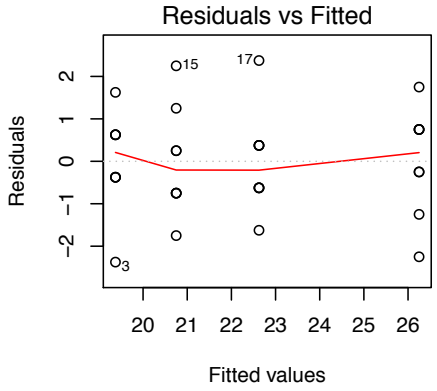
R Output for Sleep Deprivation Example

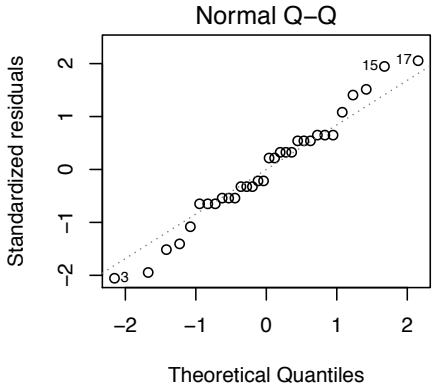
```
> #---------- Treatment means and std dev ----------------
>
> tapply(time,hours,mean)

    12     18     24     30
19.375 20.750 22.625 26.250

> tapply(time,hours,sd)

      12       18       24       30
1.187735 1.281740 1.187735 1.281740


> #----------- Generate ANOVA results ----------------
> summary (f1)
             Df Sum Sq Mean Sq F value   Pr(>F)
factor(hours)  3 213.25   71.08   46.56 5.22e-11 ***
Residuals     28  42.75    1.53
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

> summary(f2)
Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)      19.3750     0.4369  44.350  < 2e-16 ***
factor(hours)18   1.3750     0.6178   2.226   0.0343 *
factor(hours)24   3.2500     0.6178   5.260 1.36e-05 ***
factor(hours)30   6.8750     0.6178  11.128 8.64e-12 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.236 on 28 degrees of freedom
Multiple R-squared: 0.833,      Adjusted R-squared: 0.8151
F-statistic: 46.56 on 3 and 28 DF,  p-value: 5.222e-11
```

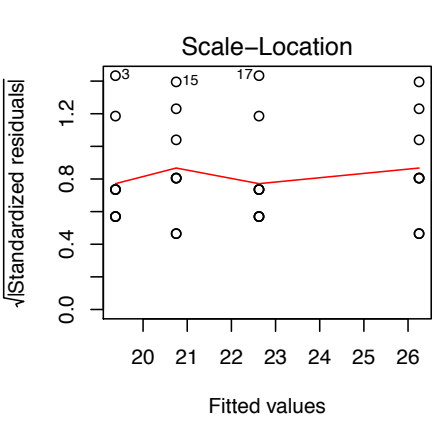
R Code for Sleep Deprivation Example

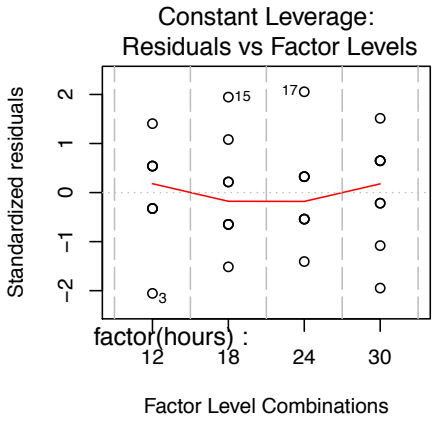
```
hours <- c(rep(12,8),rep(18,8),rep(24,8),rep(30,8))

time <- c(20,20,17,19,20,19,21,19,21,20,21,22,20,20,23,19,
25,23,22,23,21,22,22,23,26,27,24,27,25,28,26,27)

#---------- Treatment means and std dev ----------------
tapply(time,hours,mean)
tapply(time,hours,sd)

#----------- Generate ANOVA results ----------------
f1 <- aov(time~factor(hours))
summary (f1)
f2 <- lm(time~factor(hours))
summary(f2)

#----------- Generate diagnostic plots ----------------
windows()
par(mfrow=c(2,2))
plot(f1)
windows()
par(mfrow=c(2,2))
stripchart(time~hours,vertical=TRUE,main="Response Time vs Treatment")
plot(fitted(f1),resid(f1),main="Residuals vs Predicted Values")
qqnorm(resid(f1),main="Normal Probability Plot")
hist(resid(f1),nclass=8,main="Histogram of Residuals")
```

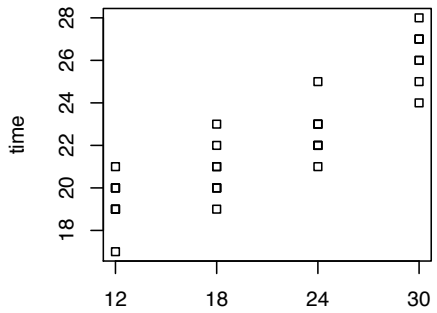
R Diagnostic Plots

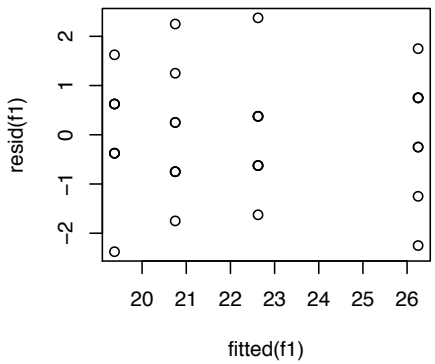


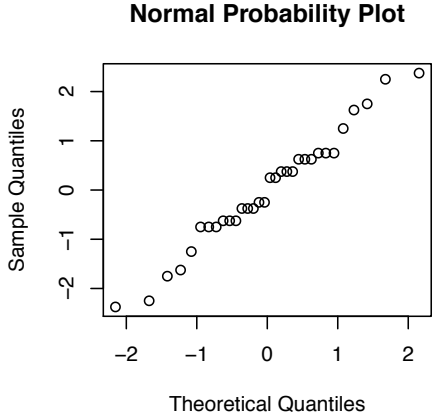


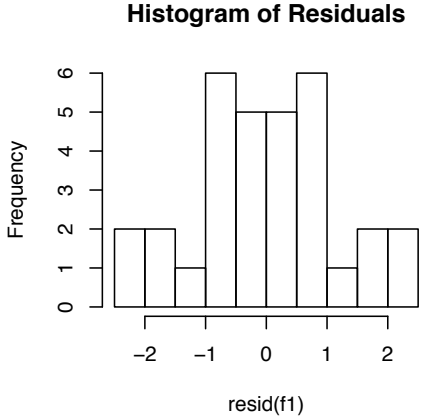












2.6 CRD Matrix Form Example

Suppose there are $a = 3$ treatments and $n = 3$ observations per treatment. The data were:

Treatment		
1	2	3
4	10	7
5	12	8
6	14	12

Summary Statistics			
$y_{1\cdot} = 15$	$y_{2\cdot} = 36$	$y_{3\cdot} = 27$	$y_{\cdot\cdot} = 78$
$\overline{y}_{1\cdot} = 5$	$\overline{y}_{2\cdot} = 12$	$\overline{y}_{3\cdot} = 9$	$\overline{y}_{\cdot\cdot} = 78/9 = 26/3$

CONSTRAINT I: $\sum_{i=1}^{a} \tau_i = 0$ (equal n_i case)

- Model: $y_{ij} = \mu + \tau_i + \epsilon_{ij}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$
- In full column rank matrix form $y = X\theta + \epsilon$ where $\theta' = [\mu, \tau_1, \tau_2]$.
- Goal: Find $\widehat{\theta}' = [\widehat{\mu}, \widehat{\tau}_1, \widehat{\tau}_2, \widehat{\tau}_3]$ that minimizes the residual sum of squares (SS_E).

$$X = \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix} \begin{matrix} \mu \\ 1 \\ 1 \\ 1 \\ \hline 1 \\ 1 \\ 1 \\ \hline 1 \\ 1 \\ 1 \end{matrix} \begin{matrix} \tau_1 \\ 1 \\ 1 \\ 1 \\ \hline 0 \\ 0 \\ 0 \\ \hline -1 \\ -1 \\ -1 \end{matrix} \begin{matrix} \tau_2 \\ 0 \\ 0 \\ 0 \\ \hline 1 \\ 1 \\ 1 \\ \hline -1 \\ -1 \\ -1 \end{matrix} \qquad y = \begin{bmatrix} 4 \\ 5 \\ 6 \\ \hline 10 \\ 12 \\ 14 \\ \hline 7 \\ 8 \\ 12 \end{bmatrix} \qquad \epsilon = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \hline \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_{23} \\ \hline \epsilon_{31} \\ \epsilon_{32} \\ \epsilon_{33} \end{bmatrix} \qquad \theta = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \end{bmatrix}$$

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix} \qquad (X'X)^{-1} = \frac{1}{9} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$X'y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \\ \hline 10 \\ 12 \\ 14 \\ \hline 7 \\ 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 78 \\ 15\text{-}27 \\ 36 - 27 \end{bmatrix} = \begin{bmatrix} 78 \\ -12 \\ 9 \end{bmatrix} = \begin{bmatrix} y_{\cdot\cdot} \\ y_{1\cdot} - y_{3\cdot} \\ y_{2\cdot} - y_{3\cdot} \end{bmatrix}$$

$$\widehat{\theta} = (X'X)^{-1}X'y \;=\; \frac{1}{9}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 78 \\ -12 \\ 9 \end{bmatrix} = \begin{bmatrix} 78/9 \\ (-24-9)/9 \\ (12+18)/9 \end{bmatrix} = \begin{bmatrix} 26/3 \\ -11/3 \\ 10/3 \end{bmatrix} = \begin{bmatrix} \widehat{\mu} \\ \widehat{\tau}_1 \\ \widehat{\tau}_2 \end{bmatrix}$$

Then $\widehat{\tau}_3 = -\widehat{\tau}_1 - \widehat{\tau}_2 = \dfrac{1}{3}$. Because $\widehat{\mu}_i = \widehat{\mu} + \widehat{\tau}_i$, we get $\widehat{\mu}_1 = 5$, $\widehat{\mu}_2 = 12$, and $\widehat{\mu}_3 = 9$,

<u>CONSTRAINT II</u>: $\tau_3 = 0$ (equal n_i case) Goal: Find $\widehat{\theta'} = [\widehat{\mu}, \widehat{\tau}_1, \widehat{\tau}_2]$ with $\widehat{\tau}_3 = 0$ because of the constraint.

$$\begin{array}{cc} & \begin{matrix} \mu & \tau_1 & \tau_2 \end{matrix} \\ X \;=\; & \left[\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \end{array} \qquad y \;=\; \left[\begin{array}{c} 4 \\ 5 \\ 6 \\ \hline 10 \\ 12 \\ 14 \\ \hline 7 \\ 8 \\ 12 \end{array}\right] \qquad \epsilon \;=\; \left[\begin{array}{c} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \hline \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_{23} \\ \hline \epsilon_{31} \\ \epsilon_{32} \\ \epsilon_{33} \end{array}\right] \qquad \theta \;=\; \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \end{bmatrix}$$

$$X'X \;=\; \begin{bmatrix} 9 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix} \qquad (X'X)^{-1} \;=\; \frac{1}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$X'y \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} y \;=\; \begin{bmatrix} y_{\cdot\cdot} \\ y_{1\cdot} \\ y_{2\cdot} \end{bmatrix}$$

$$\widehat{\theta} = (X'X)^{-1}X'y \;=\; \begin{bmatrix} (1/3)(y_{\cdot\cdot} - y_{1\cdot} - y_{2\cdot}) \\ (1/3)(-y_{\cdot\cdot} + 2y_{1\cdot} + y_{2\cdot}) \\ (1/3)(-y_{\cdot\cdot} + y_{1\cdot} + 2y_{2\cdot}) \end{bmatrix} \quad \text{Note : } y_{\cdot\cdot} = y_{1\cdot} + y_{2\cdot} + y_{3\cdot}$$

$$= \begin{bmatrix} (1/3)y_{3\cdot} \\ (1/3)(y_{1\cdot} - y_{3\cdot}) \\ (1/3)(y_{2\cdot} - y_{3\cdot}) \end{bmatrix} = \begin{bmatrix} \overline{y}_{3\cdot} \\ \overline{y}_{1\cdot} - \overline{y}_{3\cdot} \\ \overline{y}_{2\cdot} - \overline{y}_{3\cdot} \end{bmatrix} = \begin{bmatrix} 9 \\ 5-9 \\ 12-9 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \\ 3 \end{bmatrix}$$

Then $\widehat{\mu} = 9$, $\widehat{\tau}_1 = -4, \widehat{\tau}_2 = 3$, and $\widehat{\tau}_3 = 0$

- The estimates of the 3 means are
$$\begin{array}{ll} \widehat{\mu}_1 = \widehat{\mu} + \widehat{\tau}_1 = 9 - 4 & = 5 \\ \widehat{\mu}_2 = \widehat{\mu} + \widehat{\tau}_2 = 9 + 3 & = 12 \\ \widehat{\mu}_3 = \widehat{\mu} + \widehat{\tau}_3 & = 9 \end{array}$$
which are the same as those using Constraint I.

Alternate Matrix Form Solutions

- Suppose we want a full column rank matrix X having $a+1$ parameter columns. To do this, we can add a row to matrix X and vector y which includes the constraint. Thus, X and y will have $N+1$ rows, and $\theta' = [\mu, \tau_1, \tau_2, \tau_3]$.
- That is, to find the least squares solutions for $\mu,\ \tau_1, \ldots, \tau_a$ we append a row to matrix X and a value c to vector y based on the based on the linear constraint.

$$X = \begin{array}{c} \begin{matrix} \mu & \tau_1 & \tau_2 & \tau_3 \end{matrix} \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 \end{array}\right] \end{array} \qquad y = \left[\begin{array}{c} 4 \\ 5 \\ 6 \\ \hline 10 \\ 12 \\ 14 \\ \hline 7 \\ 8 \\ 12 \\ \hline 0 \end{array}\right] \qquad \epsilon = \left[\begin{array}{c} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \hline \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_{23} \\ \hline \epsilon_{31} \\ \epsilon_{32} \\ \epsilon_{33} \end{array}\right] \qquad \theta = \left[\begin{array}{c} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{array}\right]$$

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 3 & 3 \\ 3 & 4 & 1 & 1 \\ 3 & 1 & 4 & 1 \\ 3 & 1 & 1 & 4 \end{bmatrix} \qquad (X'X)^{-1} = \frac{1}{9} \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 3 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ -1 & 0 & 0 & 3 \end{bmatrix}$$

$$X'y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \\ 10 \\ 12 \\ 14 \\ 7 \\ 8 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 78 \\ 15 \\ 36 \\ 27 \end{bmatrix} = \begin{bmatrix} y_{\cdot\cdot} \\ y_{1\cdot} \\ y_{2\cdot} \\ y_{3\cdot} \end{bmatrix}$$

$$\widehat{\theta} = (X'X)^{-1}X'y = \frac{1}{9} \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 3 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ -1 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 78 \\ 15 \\ 36 \\ 27 \end{bmatrix} = \begin{bmatrix} 26/3 \\ -11/3 \\ 10/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \widehat{\mu} \\ \widehat{\tau}_1 \\ \widehat{\tau}_2 \\ \widehat{\tau}_3 \end{bmatrix}$$

- The estimates of the 3 means are
$$\begin{array}{lllll} \widehat{\mu}_1 = \widehat{\mu} + \widehat{\tau}_1 & = & (26/3) + (-11/3) & = 5 \\ \widehat{\mu}_2 = \widehat{\mu} + \widehat{\tau}_2 & = & (26/3) + (10/3) & = 12 \\ \widehat{\mu}_3 = \widehat{\mu} + \widehat{\tau}_3 & = & (26/3) + (1/3) & = 9 \end{array}$$

CONSTRAINT II: $\tau_3 = 0$ (equal n_i case)

$$X = \begin{array}{c} \begin{matrix} \mu & \tau_1 & \tau_2 & \tau_3 \end{matrix} \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array} \qquad y = \left[\begin{array}{c} 4 \\ 5 \\ 6 \\ \hline 10 \\ 12 \\ 14 \\ \hline 7 \\ 8 \\ 12 \\ \hline 0 \end{array}\right] \qquad \epsilon = \left[\begin{array}{c} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \hline \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_{23} \\ \hline \epsilon_{31} \\ \epsilon_{32} \\ \epsilon_{33} \end{array}\right] \qquad \theta = \left[\begin{array}{c} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{array}\right]$$

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 3 & 3 \\ 3 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 3 & 0 & 0 & 4 \end{bmatrix} \qquad (X'X)^{-1} = \frac{1}{3} \begin{bmatrix} 4 & -4 & -4 & -3 \\ -4 & 5 & 4 & 3 \\ -4 & 4 & 5 & 3 \\ -3 & 3 & 3 & 3 \end{bmatrix}$$

$$X'y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \\ 10 \\ 12 \\ 14 \\ 7 \\ 8 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 78 \\ 15 \\ 36 \\ 27 \end{bmatrix} = \begin{bmatrix} y_{\cdot\cdot} \\ y_{1\cdot} \\ y_{2\cdot} \\ y_{3\cdot} \end{bmatrix}$$

$$\widehat{\theta} = (X'X)^{-1}X'y = \frac{1}{3} \begin{bmatrix} 4 & -4 & -4 & -3 \\ -4 & 5 & 4 & 3 \\ -4 & 4 & 5 & 3 \\ -3 & 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 78 \\ 15 \\ 36 \\ 27 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \\ 13 \\ 0 \end{bmatrix} = \begin{bmatrix} \widehat{\mu} \\ \widehat{\tau}_1 \\ \widehat{\tau}_2 \\ \widehat{\tau}_3 \end{bmatrix}$$

- The estimates of the 3 means are
$$\begin{array}{lllll} \widehat{\mu}_1 & = & \widehat{\mu} + \widehat{\tau}_1 & = 9 - 4 & = 5 \\ \widehat{\mu}_2 & = & \widehat{\mu} + \widehat{\tau}_2 & = 9 + 3 & = 12 \\ \widehat{\mu}_3 & = & \widehat{\mu} + \widehat{\tau}_3 & & = 9 \end{array}$$

  which are the same as those using Constraint I.

2.7 Inferences for Linear Combinations and Contrasts of Treatment Means

- Let $\sum_{i=1}^{a} c_i\mu_i$ be a **linear combination** of the treatment means (i.e., at least one $c_i \neq 0$).
- A **contrast** Γ is a comparison involving a linear combination of two or more treatment means subject to the restriction that the sum of the coefficients = 0. That is,

$$\Gamma = \sum_{i=1}^{a} c_i\mu_i \qquad \text{where} \qquad \sum_{i=1}^{a} c_i = 0$$

- Any pairwise comparison of means $\Gamma = \mu_i - \mu_j$ is an example of a contrast.

Estimation

- $\widehat{\Gamma} = \sum_{i=1}^{a} c_i\overline{y}_{i\cdot}$ is an unbiased estimate of the linear combination $\Gamma = \sum_{i=1}^{a} c_i\mu_i$.
- The expected value of $\widehat{\Gamma}$: $E(\widehat{\Gamma}) = \sum_{i=1}^{a} c_i\mu_i = \Gamma$. Thus, $\widehat{\Gamma}$ is an unbiased estimator of Γ.
- Recall that the variance of a sum of independent statistics equals the sum of the variances of the statistics. Therefore, the variance of $\widehat{\Gamma}$ is

$$\sigma^2(\widehat{\Gamma}) = \sum_{i=1}^{a} \text{Var}(c_i\overline{y}_{i\cdot}) = \sum_{i=1}^{a} c_i^2\text{Var}(\overline{y}_{i\cdot}) =$$

- The standard deviation of $\widehat{\Gamma}$: $\sigma(\widehat{\Gamma}) = \sigma\sqrt{\sum_{i=1}^{a} \frac{c_i^2}{n_i}}$.
- Using MS_E to estimate σ^2, the standard error of $\widehat{\Gamma}$ is $se(\widehat{\Gamma}) =$.
- These estimates will be useful when we are interested in generating confidence intervals and testing hypotheses about $\Gamma = \sum_{i=1}^{a} c_i\mu_i$.

Confidence Intervals and Hypothesis Tests

- If the assumptions for the single factor ANOVA are valid, then

$$\frac{\widehat{\Gamma} - \Gamma}{se(\widehat{\Gamma})} = \frac{\widehat{\Gamma} - \Gamma}{\sqrt{MS_E\sum(c_i^2/n_i)}}$$

  follows a t-distribution with $N - a$ degrees of freedom (= MS_E degrees of freedom).

- Thus, if you want to test $\text{H}_0 : \Gamma = 0$ against $\text{H}_1 : \Gamma \neq 0$
  1. Calculate the test statistic $t = \dfrac{\widehat{\Gamma}}{se(\widehat{\Gamma})}$.
  2. For a specified α, determine $t^* = t(1 - \alpha/2, N - a)$, the critical t-value from the t-distribution with $N - a$ degrees of freedom (or find the p-value.)
  3. Reject H_0 if $t \geq t^*$ or if $t \leq -t^*$ (or compare α to the p-value.)
- A confidence interval for Γ is given by $\widehat{\Gamma} \pm t^* se(\widehat{\Gamma})$

- On computer output, you may get the results for a $(1, N-a)$ degrees of freedom F-test. However, the sum of squares for Γ is $SS_\Gamma = MS_\Gamma$. That is,

$$SS_\Gamma \;=\; MS_\Gamma \;=\; \frac{\widehat{\Gamma}^2}{\sum_{i=1}^{a} c_i^2/n_i}.$$

Thus, $F \;=\; \dfrac{MS_{trt}}{MS_E} \;=\; \dfrac{\widehat{\Gamma}^2}{MS_E \sum_{i=1}^{a} c_i^2/n_i} \;=\; \dfrac{\widehat{\Gamma}^2}{se(\widehat{\Gamma})} \;=\; t^2.$

2.7.1 Confidence Intervals for a Treatment Mean

- To calculate a confidence interval for μ_i, calculate $\quad \overline{y}_{i\cdot} \pm t^* \sqrt{MS_E/n_i}$ where $t^* = t(1-\alpha/2, N-a)$ is the critical t-value from the t-distribution with $N-a$ degrees of freedom
- Note: This is a special case of a linear combination where $c_i = 1$ and $c_j = 0$ for $j \neq i$.
- If you want to test $\quad \mathrm{H}_0 : \mu_i = c \quad$ against $\quad \mathrm{H}_1 : \mu_i \neq c \quad$ where c is a hypothesized value for μ_i, reject H_0 if c is not in the confidence interval.
- Rejecting H_0 is equivalent to having a p-value $< \alpha$.

2.7.2 Inferences for Differences Between Two Treatment Means

- Consider the difference $D_{ij} = \mu_i - \mu_j$ between two of the treatment means μ_i and μ_j. This is referred to as a **pairwise comparison**.
- This is a a special case of a contrast where $c_i = 1$, $c_j = -1$, and $c_k = 0$ for $k \neq i, j$.
- To estimate the difference D_{ij}, we substitute the sample treatment means for μ_i and μ_j and get

$$\widehat{D}_{ij} =$$

- The expected value of $\widehat{D}_{ij}$: $\quad E(\widehat{D}_{ij}) =$
- The variance of $\widehat{D}_{ij}$: $\quad \sigma^2(\widehat{D}_{ij}) = \mathrm{Var}(\overline{y}_{i\cdot}) + \mathrm{Var}(\overline{y}_{j\cdot}) = \dfrac{\sigma^2}{n_i} + \dfrac{\sigma^2}{n_j} = \sigma^2 \left(\dfrac{1}{n_i} + \dfrac{1}{n_j} \right)$
- Thus, the standard deviation of $\widehat{D}_{ij}$: $\quad \sigma(\widehat{D}_{ij}) = \sigma \sqrt{\dfrac{1}{n_i} + \dfrac{1}{n_j}}$
- To estimate $\sigma(\widehat{D}_{ij})$, we replace σ with $\sqrt{MS_E}$. Thus, the standard error of $\widehat{D}_{ij}$ is

$$se(\widehat{D}_{ij}) = \sqrt{MS_E \left(\frac{1}{n_i} + \frac{1}{n_j} \right)}$$

- These estimates will be used when generating a confidence intervals for $\mu_i - \mu_j$.
- To calculate a confidence interval for $\mu_i - \mu_j$, calculate $\quad \widehat{D}_{ij} \pm t^* se(\widehat{D}_{ij})$

  where $t^* = t(1-\alpha/2, N-a)$ is the critical t-value from the t-distribution with $N-a$ degrees of freedom.
- If you want to test $\quad \mathrm{H}_0 : \mu_i = \mu_j \quad$ against $\quad \mathrm{H}_1 : \mu_i \neq \mu_j$ reject H_0 if the confidence interval does not contain 0.
- Rejecting H_0 is equivalent to having a p-value $< \alpha$.

Sleep Deprivation Example: A study was conducted to determine the effects of sleep deprivation on hand-steadiness. The four levels of sleep deprivation of interest are 12, 18, 24, and 30 hours. 32 subjects were randomly selected and assigned to the four levels of sleep deprivation such that 8 subjects were assigned to each level.

Note: subscripts 1, 2, 3, 4 correspond to the 12, 18, 24, and 30 hour sleep deprivation treatments.

$\overline{y}_{1\cdot} = 19.375$ $\overline{y}_{2\cdot} = 20.75$
$\overline{y}_{3\cdot} = 22.625$ $\overline{y}_{4\cdot} = 26.25$ $MS_E = 1.5267857$

95% Confidence Intervals for μ_i

Note: $se(\overline{y}_{i\cdot}) = \sqrt{MS_E/n_i} = \sqrt{1.526786/8} = .43686$ and $t^* = t_{.025,28} = 2.048$ for $i = 1, 2, 3, 4$.

$$\begin{aligned} \overline{y}_{i\cdot} \pm t^*\sqrt{MS_E/n_i} &= \overline{y}_{i\cdot} \pm (2.048)(.43686) \\ &= \overline{y}_{i\cdot} \pm .895 \end{aligned}$$

Hours	Mean	95% Confidence Interval	
12	μ_1	$19.375 \pm .895$	$(18.480, 20.270)$
18	μ_2	$20.750 \pm .895$	$(19.855, 21.645)$
24	μ_3	$22.625 \pm .895$	$(21.730, 23.520)$
30	μ_4	$26.250 \pm .895$	$(25.355, 27.145)$

- If you want to test $\mathrm{H}_0 : \mu_i = 22$ against $\mathrm{H}_1 : \mu_i \neq 22$, we would

  Reject H_0 for 12 hours ($i = 1$) because 22 is not in the confidence interval for μ_1.

  Reject H_0 for 18 hours ($i = 2$) because 22 is not in the confidence interval for μ_2.

  Fail to reject H_0 for 24 hours ($i = 3$) because 22 is in the confidence interval for μ_3.

  Reject H_0 for 30 hours ($i = 14$) because 22 is not in the confidence interval for μ_4.

95% Confidence Intervals for $\mu_i - \mu_j$

Note: $se(\widehat{D}_{ij}) = \sqrt{MS_E\left(\frac{1}{n_i} + \frac{1}{n_j}\right)} = \sqrt{1.526786\left(\frac{1}{8} + \frac{1}{8}\right)} = .61782$ and $t^* = t_{.025,28} = 2.048$ for $i, j = 1, 2, 3, 4$ $(i \neq j)$.

$$\begin{aligned} \widehat{D}_{ij} \pm t^* se(\widehat{D}_{ij}) &= \widehat{D}_{ij} \pm (2.048)(.61782) \\ &= \widehat{D}_{ij} \pm 1.265 \end{aligned}$$

Comparison	Difference	95% Confidence Interval	
12 vs 18	$\mu_2 - \mu_1$	1.375 ± 1.265	(0.109,2.641)
12 vs 30	$\mu_4 - \mu_1$	6.875 ± 1.265	(5.609,8.141)
18 vs 24	$\mu_3 - \mu_2$	1.875 ± 1.265	(0.609,3.141)

- If you want to test $\mathrm{H}_0 : \mu_2 - \mu_1 = 0$ against $\mathrm{H}_1 : \mu_2 - \mu_1 \neq 0$, we would

  Reject H_0 because 0 is not in the confidence interval for $\mu_2 - \mu_1$.

  We would also reject $\mathrm{H}_0 : \mu_4 - \mu_1 = 0$ and $\mathrm{H}_0 : \mu_3 - \mu_2 = 0$ because 0 is not in the confidence interval for $\mu_4 - \mu_1$ or the confidence interval for $\mu_3 - \mu_2$.

Parameter	Estimate	Standard Error	t Value	Pr > |t|	95% Confidence Limits	
Baseline mu	22.2500000	0.21843089	101.86	<.0001	21.8025646	22.6974354
12 hour effect	-2.8750000	0.37833340	-7.60	<.0001	-3.6499808	-2.1000192
18 hour effect	-1.5000000	0.37833340	-3.96	0.0005	-2.2749808	-0.7250192
24 hour effect	0.3750000	0.37833340	0.99	0.3301	-0.3999808	1.1499808
30 hour effect	4.0000000	0.37833340	10.57	<.0001	3.2250192	4.7749808
12 hour mean	19.3750000	0.43686178	44.35	<.0001	18.4801292	20.2698708
18 hour mean	20.7500000	0.43686178	47.50	<.0001	19.8551292	21.6448708
24 hour mean	22.6250000	0.43686178	51.79	<.0001	21.7301292	23.5198708
30 hour mean	26.2500000	0.43686178	60.09	<.0001	25.3551292	27.1448708
12 vs 18 hrs	1.3750000	0.61781585	2.23	0.0343	0.1094616	2.6405384
12 vs 30 hrs	6.8750000	0.61781585	11.13	<.0001	5.6094616	8.1405384
18 vs 24 hrs	1.8750000	0.61781585	3.03	0.0052	0.6094616	3.1405384
Linear Trend	22.5000000	1.95370527	11.52	<.0001	18.4980162	26.5019838
Quadratic Trend	2.2500000	0.87372356	2.58	0.0156	0.4602584	4.0397416
Cubic Trend	1.2500000	1.95370527	0.64	0.5275	-2.7519838	5.2519838

(A) (B) (C)

VERY STRONG LINEAR TREND

MODERATE EVIDENCE OF A QUADRATIC TREND

NO EVIDENCE OF A CUBIC TREND

$\hat{\Gamma}_L, \hat{\Gamma}_Q, \hat{\Gamma}_C$

$F = t^2$

(D)

Contrast	DF	Contrast SS	Mean Square	F Value	Pr > F
Linear Trend	1	202.5000000	202.5000000	132.63	<.0001
Quadratic Trend	1	10.1250000	10.1250000	6.63	0.0156
Cubic Trend	1	0.6250000	0.6250000	0.41	0.5275

$213.25 = SS_{TRT}$

The SAS output above was generated by the following SAS code which appears after the MODEL statement:

```
ESTIMATE '12 hour mean' INTERCEPT 1 hours 1 0 0 0;
ESTIMATE '18 hour mean' INTERCEPT 1 hours 0 1 0 0;
ESTIMATE '24 hour mean' INTERCEPT 1 hours 0 0 1 0;
ESTIMATE '30 hour mean' INTERCEPT 1 hours 0 0 0 1;

ESTIMATE '12 vs 18 hrs' hours -1  1  0  0;
ESTIMATE '12 vs 30 hrs' hours -1  0  0  1;
ESTIMATE '18 vs 24 hrs' hours  0 -1  1  0;

ESTIMATE 'Linear Trend'    hours -3 -1  1 3;
ESTIMATE 'Quadratic Trend' hours  1 -1 -1 1;
ESTIMATE 'Cubic Trend'     hours -1  3 -3 1;

CONTRAST 'Linear Trend'    hours -3 -1  1 3;
CONTRAST 'Quadratic Trend' hours  1 -1 -1 1;
CONTRAST 'Cubic Trend'     hours -1  3 -3 1;
```

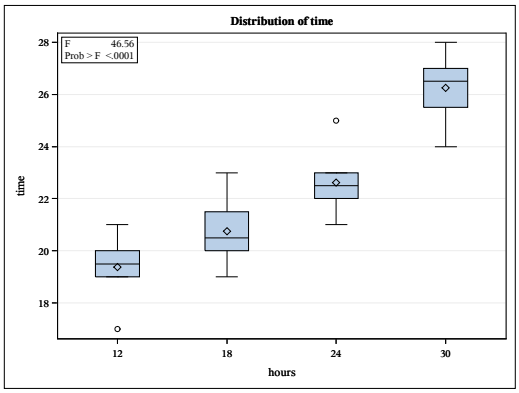
(A) (B) (C) (D)

SEE TABLE IX ON PAGE 31 ($a=4$) FOR ORTHOGONAL CONTRAST COEFFICIENTS

2.7.3 Orthogonal Contrasts

- Let $c_1, c_2, \ldots, c_a$ and $d_1, d_2, \ldots, d_a$ be the coefficients of two contrasts Γ_1 and Γ_2.
- If $\sum_{i=1}^{a} c_i d_i = 0$ then Γ_1 and Γ_2 are **orthogonal contrasts**.
- In a oneway ANOVA with a factor levels, any set of $a-1$ mutually orthogonal contrasts partition the SS_{trt} into $a-1$ single degree of freedom components. Therefore, all tests of orthogonal contrasts are independent.
- If the a levels are equally spaced on a numerical scale, then there exists a set of $a-1$ orthogonal contrasts that test for polynomial trends (linear, quadratic, cubic, etc.) up to order $a-1$.
- The contrast coefficients can be found in Table IX on the next page. In this table n represents the number of factor levels ($= a$ in our notes). The P_i columns ($i = 1, 2, \ldots, a-1$) are the coefficients for linear ($i = 1$), quadratic ($i = 2$), cubic ($i = 3$), ... orthogonal polynomial contrasts.
- Note that any P_i and P_j columns are orthogonal for each $n = a$. Thus, the $a-1$ columns $P_1, P_2 \ldots, P_{a-1}$ form a set of $a-1$ mutually orthogonal contrasts.

Orthogonal Contrasts for the Sleep Deprivation Example



- There are $a = 4$ factor levels. Therefore, we can test for linear ($i = 1$), quadratic ($i = 2$), and cubic ($i = 3$) orthogonal polynomial contrasts.
- We use the values in the P_1, P_2, and P_3 columns for $n = a = 4$ in Table IX. These can be seen in the SAS ESTIMATE and CONTRAST statements.
- Thus, we are testing for linear, quadratic, and cubic trends in across the factor levels 12, 18, 24, and 30 hours of sleep deprivation.

  Linear Contrast: $\Gamma_L = -3\mu_1 - \mu_2 + \mu_3 + 3\mu_4$

  Quadratic Contrast: $\Gamma_Q = \mu_1 - \mu_2 - \mu_3 + \mu_4$

  Cubic Contrast: $\Gamma_C = -\mu_1 + 3\,\mu_2 - 3\mu_3 + \mu_4$

- We now replace each μ_i with $\widehat{\mu}_i = \overline{y}_{i\cdot}$ to generate estimates $\widehat{\Gamma}_L, \widehat{\Gamma}_Q$, and $\widehat{\Gamma}_C$:

  Estimated Linear Contrast: $\widehat{\Gamma}_L = -3(19.375) - 20.75 + 22,625 + 3(26.25) = 22.5$

  Estimated Quadratic Contrast: $\widehat{\Gamma}_Q = 19.375 - 20.75 - 22,625 + 26.25 = 2.25$

  Estimated Cubic Contrast: $\widehat{\Gamma}_C = -19.375 + 3(20.75) - 3(22,625) + 26.25 = 1.25$

- Next, we will find the standard error for $\widehat{\Gamma}_L, \widehat{\Gamma}_Q$, and $\widehat{\Gamma}_C$. Recall $se(\widehat{\Gamma}) = \sqrt{MS_E \sum_{i=1}^{a} \frac{c_i^2}{n_i}}$.

- From the ANOVA, $MSE = 1.5267857$. For $\widehat{\Gamma}_L$ and $\widehat{\Gamma}_C$, $\sum_{i=1}^{4} \frac{c_i^2}{n_i} =$

  For $\widehat{\Gamma}_Q$, we have $\sum_{i=1}^{4} \frac{c_i^2}{n_i} =$

- And, $se(\widehat{\Gamma}_L) = se(\widehat{\Gamma}_C) = \sqrt{(1.5267857)(2.5)} =$

  $se(\widehat{\Gamma}_Q) = \sqrt{(1.5267857)(0.5)} =$

- To find the sum of squares for Γ_L, Γ_Q, and Γ_C, we use $SS_\Gamma = MS_\Gamma = \frac{\widehat{\Gamma}^2}{\sum_{i=1}^{a} c_i^2/n_i}$.

$$SS_{\Gamma_L} = \frac{22.5^2}{2.5} =$$
$$SS_{\Gamma_Q} = \frac{2.25^2}{0.5} =$$
$$SS_{\Gamma_C} = \frac{1.25^2}{2.5} =$$

- All of these values can be seen on the SAS output.

IX Coefficients of Orthogonal Polynomials[a]

		$n=3$		$n=4$			$n=5$				$n=6$					$n=7$					
	X_j	P_1	P_2	P_1	P_2	P_3	P_1	P_2	P_3	P_4	P_1	P_2	P_3	P_4	P_5	P_1	P_2	P_3	P_4	P_5	P_6
	1	−1	1	−3	1	−1	−2	2	−1	1	−5	5	−5	1	−1	−3	5	−1	3	−1	1
	2	0	−2	−1	−1	3	−1	−1	2	−4	−3	−1	7	−3	5	−2	0	1	−7	4	−6
	3	1	1	1	−1	−3	0	−2	0	6	−1	−4	4	2	−10	−1	−3	1	1	−5	15
	4			3	1	1	1	−1	−2	−4	1	−4	−4	2	10	0	−4	0	6	0	−20
	5						2	2	1	1	3	−1	−7	−3	−5	1	−3	−1	1	5	15
	6										5	5	5	1	1	2	0	−1	−7	−4	−6
	7															3	5	1	3	1	1
$\sum_{j=1}^{n} [P_i(X_j)]^2$		2	6	20	4	20	10	14	10	70	70	84	180	28	252	28	84	6	154	84	924
	λ	1	3	2	1	$\frac{10}{3}$	1	1	$\frac{5}{6}$	$\frac{35}{12}$	2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{7}{12}$	$\frac{21}{10}$	1	1	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{7}{20}$	$\frac{77}{60}$

		$n=8$						$n=9$						$n=10$					
	X_j	P_1	P_2	P_3	P_4	P_5	P_6	P_1	P_2	P_3	P_4	P_5	P_6	P_1	P_2	P_3	P_4	P_5	P_6
	1	−7	7	−7	7	−7	1	−4	28	−14	14	−4	4	−9	6	−42	18	−6	3
	2	−5	1	5	−13	23	−5	−3	7	7	−21	11	−17	−7	2	14	−22	14	−11
	3	−3	−3	7	−3	−17	9	−2	−8	13	−11	−4	22	−5	−1	35	−17	−1	10
	4	−1	−5	3	9	−15	−5	−1	−17	9	9	−9	1	−3	−3	31	3	−11	6
	5	1	−5	−3	9	15	−5	0	−20	0	18	0	−20	−1	−4	12	18	−6	−8
	6	3	−3	−7	−3	17	9	1	−17	−9	9	9	1	1	−4	−12	18	6	−8
	7	5	1	−5	−13	−23	−5	2	−8	−13	−11	4	22	3	−3	−31	3	11	6
	8	7	7	7	7	7	1	3	7	−7	−21	−11	−17	5	−1	−35	−17	1	10
	9							4	28	14	14	4	4	7	2	−14	−22	−14	−11
	10													9	6	42	18	6	3
$\sum_{j=1}^{n} [P_i(X_j)]^2$		168	168	264	616	2184	264	60	2772	990	2002	468	1980	330	132	8580	2860	780	660
	λ	2	1	$\frac{2}{3}$	$\frac{7}{12}$	$\frac{7}{10}$	$\frac{11}{60}$	1	3	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{3}{20}$	$\frac{11}{60}$	2	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{5}{12}$	$\frac{1}{10}$	$\frac{11}{240}$

[a]Adapted with permission from *Biometrika Tables for Statisticians*, Vol. 1, 3rd edition by E. S. Pearson and H. O. Hartley, Cambridge University Press, Cambridge, 1966.

2.8 Multiple Comparison Procedures

This section is a compilation of material from your text, the *SAS/STAT User's Guide* (1992), Hochberg and Tamhane (1987), Toothaker (1991), Miller (1981), and Milliken and Johnson (1992).

- When performing hypothesis tests on a single mean, on differences between means, or on contrasts, we must realize the following limitations:
  - Individual tests using a specified α level are only appropriate if the tests were not suggested by the data.
  - If we run a series of hypothesis tests each at a specified α level, then the α level applies to the individual tests and not for a set of tests.
- A procedure that enables us to make more than one comparison among two or more means is called a **multiple comparison procedure (MCP)**.
- The most common situation occurs when the researcher suspects that there may be differences among the a means and it is important to determine which means can be considered significantly different from each other. The typical approach is to look at all $a(a-1)/2$ pairwise comparisons of the form $\mu_i - \mu_j$ and test for significant differences.
- Failure to reject $H_0 : \mu_i = \mu_j$ in a MCP should not lead you to conclude that H_0 is true. Failure to reject H_0 implies only that the difference between population means, if any, is not large enough to be detected for the given n_i sample sizes.
- Warnings: A MCP can lead to counter-intuitive results when the sample sizes are unequal and there are large disparities among the n_i sample sizes. For example, suppose there are 4 treatments (A, B, C, D) such that $\overline{y}_{A\cdot} > \overline{y}_{B\cdot} > \overline{y}_{C\cdot} > \overline{y}_{D\cdot}$. Suppose $n_A = n_D = 3$ while $n_B = n_C = 100$. Then the difference between μ_B and μ_C may be significant while the difference between μ_A and μ_D is not.
- Generating a set of confidence intervals may be more informative than the results of the set of significance tests in a MCP. Confidence intervals (i) show the degree of uncertainty in each comparison, (ii) indicate the statistical significance and (iii) may prevent nonstatisticians from falsely concluding that means that are not significantly different implies equality.
- Details will be presented for the following multiple MCPs: Fisher's Least Significant Difference (LSD), Bonferroni, Sidak, Tukey, and Dunnett.

Two Types of Hypotheses

The following two types of hypotheses describe two possible relationships that may exist among the population means. These will be related to different types of MCP error rates.

- A **complete null hypothesis** exists if all of the a means are equal. This corresponds to H_0 for the F-test in the ANOVA. This is also referred to as the 'full' or 'overall' null hypothesis.
- A **partial null hypothesis** exists when the complete null hypothesis is false, but at least one pair of means are equal.
  - For example, suppose there are 4 treatments (A, B, C, D) and that $\mu_A = \mu_B = \mu_C$ but that $\mu_i \neq \mu_D$ for $i = A, B, C$. Then, there is a partial null hypothesis in the three equal means.
- For Fisher's Least Significant Difference (LSD), Bonferroni, Sidak, and Tukey MCPs, we only need to consider the complete null hypothesis.

Types of Errors

- A **Type I error** is a rejection of the null hypothesis H_0 when it is true. It is typical to let α be the probability of a Type I error.
- A **Type II error** is a failure to reject the null hypothesis H_0 when it is false. For a pairwise comparison $\mu_i - \mu_j$, the probability of a Type II error, denoted β, depends on the value of $\mu_i - \mu_j$.
- The **power of a test** $= 1 - \beta =$ the probability of rejecting H_0 when it is false.

Error Rates

- Situation: Suppose that a MCP requiring a total of C pairwise comparisons or contrasts is to be performed.
- When choosing a MCP, the researcher (whether he or she realizes it) is selecting a method of controlling the error rate. That is, for each MCP there is a value of α^* assigned to each of the C comparisons. This choice also affects the power of the chosen testing procedure.
- The **comparisonwise error rate (CER)** is the Type I error rate for each comparison. We will use α^* to denote the CER.
- As an alternative to CER, we can control the **experimentwise error rate for the complete null hypothesis (EERC)** . That is, we may want to control the probability of making at least one Type I error among the C comparisons (and not per comparison) to be at most some prespecified value α when the complete null hypothesis $H_0 : \mu_1 = \mu_2 = \cdots = \mu_a$ is true. In this case the CER $\alpha^* < \alpha$.
- In addition to the CER and the EERC, there is one more error rate that will be useful in evaluating the performance of the various MCPs:
  - The **maximum experimentwise error rate (MEER)** under any complete or partial null hypothesis is the probability of making at least Type I error if any complete or partial null hypothesis is true (at least one pair and possibly all pairs of means are equal).
- Suppose C pairwise comparisons or contrasts are being performed with CER $= \alpha^*$. Then

$$\begin{aligned} \Pr(\text{of at least one Type I error}) &= 1 - \Pr(\text{of no Type I errors}) \\ &\leq \qquad (3) \\ &< \qquad (4) \end{aligned}$$

- Equality in (3) will occur if the C comparisons consist of orthogonal contrasts. Empirical studies have shown that the probability of at least one Type I error is fairly close to the bound $1-(1-\alpha^*)^C$ even when the contrasts are not orthogonal.
- Fixing the CER α^* can increase the power but at the expense of a larger EERC. Specifically, assuming α^* is fixed, the EERC among the C comparisons increases as C increases.
- By fixing the EERC at some value α (e.g., $\alpha = .05$), the experimenter wants to prevent declaring too many comparisons significant by chance alone. Whenever the experimenter is trying to answer many questions using MCPs in a single experiment, it is often recommended that the experimenter control the EERC.
- It is possible to control the MEER (and hence the EERC) by setting the CER α^* at a value smaller than the desired overall α level. Two ways this can be accomplished are by reconsidering the inequalities in (3) and (4).

Method 1: Set the CER $\alpha^* = \alpha$. Then from (3)

$$\Pr(\text{of at least one Type I error}) \leq 1-(1-\alpha)^C .$$

This method does not control the MEER to be $\leq \alpha$. This corresponds to the **Fisher's LSD (Least Significant Difference)** MCP.

Method 2: Set the CER $\alpha^* = \alpha/C$. Then from (4)

$$\Pr(\text{of at least one Type I error}) < C\alpha^* = \alpha.$$

This method assures that the MEER $< \alpha$. This corresponds to the **Bonferroni** MCP.

Method 3: Using (3), find α^* such that $\alpha = 1-(1-\alpha^*)^C$. This yields CER $\alpha^* = 1-(1-\alpha)^{1/C}$. Thus,

$$\Pr(\text{of at least one Type I error}) \leq 1-(1-\alpha^*)^C = \alpha.$$

This method assures that the MEER $\leq \alpha$. These corresponds to the **Sidak** MCP.

- **Method 4**: If all of the sample sizes are equal, the EERC can also be controlled by using the distribution of a **studentized range** (which is the range of a independent standard normal random variables (e.g., $(\overline{y}_i - \mu_i)/(\sigma/\sqrt{n})$) divided by the square root of an independent χ^2/ν variable with ν d.f. (e.g., $\sqrt{MS_E/\sigma^2}$ with $\nu = N-a$ d.f.). This approach corresponds to **Tukey's HSD (Honestly Significant Difference** MCP.

Simultaneous Testing Using Confidence Intervals

- Suppose we are interested in a family of C pairwise comparisons $D_{ij} = \mu_i - \mu_j$. The confidence limits for each D_{ij} are:

$$\widehat{D}_{ij} \pm t^* \, se(\widehat{D}_{ij})$$

where

$$\widehat{D}_{ij} = \overline{y}_{i\cdot} - \overline{y}_{j\cdot} \qquad se(\widehat{D}_{ij}) = \sqrt{MS_E(1/n_i + 1/n_j)}$$

  - For Fisher's LSD Method, $t^* = t(\alpha/2, N-a)$ is the critical value associated with the t-distribution with $N-a$ degrees of freedom.
  - For the Bonferroni Method, $t^* = t(\alpha/2C, N-a)$ is the critical value associated with the t-distribution with $N-a$ degrees of freedom.
  - For the Sidak Method, $t^* = t(\alpha^*/2, N-a)$ is the critical value associated with the t-distribution with $N-a$ degrees of freedom and $\alpha^* = 1-(1-\alpha)^{1/C}$
  - For the Tukey Method, $t^* = (q_\alpha(a, N-a))/\sqrt{2}$ where $q_\alpha(a, N-a)$ is the critical value associated with the studentized range distribution. A tables of $q_\alpha(a, N-a)$ values are in the Montgomery text.

- If a confidence interval for D_{ij} does not contain 0 then we **reject** $H_0 : \mu_i = \mu_j$ in favor of $H_1 : \mu_i \neq \mu_j$. If a confidence interval for D_{ij} contains 0 then we **fail to reject** $H_0 : \mu_i = \mu_j$.

Multiple Comparison Testing Using a Test Statistic

- The following procedure can be applied to the Fisher's LSD, Bonferroni, Sidak, and Tukey Procedures.

  1. For each $D_{ij} = \mu_i - \mu_j$, calculate $\widehat{D}_{ij} = \overline{y}_{i\cdot} - \overline{y}_{j\cdot}$ and $se(\widehat{D}_{ij})$.
  2. Calculate $b_d = t^* \, se(\widehat{D}_{ij})$.
  3. Decision rule: **Reject** $H_0 : \mu_i = \mu_j$ if $|\widehat{D}_{ij}| \geq b_d$. **Fail to reject** $H_0 : \mu_i = \mu_j$ if $|\widehat{D}_{ij}| < b_d$.

Comments on Fisher's LSD Procedure

- It is up to the user to first check the ANOVA F-test result to decide whether or not to proceed to look at the pairwise comparisons.
- Using the results of the preliminary ANOVA F-test will control the EERC. That is, the probability of rejecting $H_0 : \mu_1 = \mu_2 = \cdots = \mu_a$ when it is true, is α.
- Once the researcher proceeds to the set of pairwise comparisons given rejection from the ANOVA F-test, the MEER is no longer controlled. That is, as the number of treatments a gets larger, the MEER may be considerably larger than α.
- This test is also called the *protected t-test* because the individual t-tests are computed only if the ANOVA F-test indicates significance (i.e., the ANOVA F is seen as 'protecting' the usual t- tests). 'LSD' corresponds to the fact that α-level critical t-value is the smallest critical value that the t-statistic must exceed to be significant when considering only a single comparison.
- Because of the poor control of the MEER, the LSD procedure is generally not recommended. However, because the CER is controlled at α for each comparison. Thus, this procedure has the highest power among the four MCPs.
- If, however, the experiment was exploratory in nature, then the results for the LSD MCP could suggest which levels to consider for future experimentation.
- Fisher's LSD procedure does not require equal sample sizes.

Comments on the Bonferroni and Sidak Procedures

- Because the t^* value in the Bonferroni and Sidak Procedures depends on the actual number of comparisons C, the Bonferroni and Sidak procedures will have high power for small sets of planned comparisons, but low power for large sets of planned comparisons.
- The MEER $< \alpha$ for the Bonferroni MCP. the MEER $\leq \alpha$ for the Sidak MCP.
- The Bonferroni and Sidak MCPs use the actual number of comparisons C in the selection of critical t-values. Because of the reliance on C, these two procedures will have relatively good power for small sets of planned comparisons and relatively low power for large sets of planned comparisons.
- Both procedures are applicable whether or not the sample sizes are equal.
- Although the critical t-values for the Bonferroni and Sidak tests are close, the Sidak values will always be smaller than the Bonferroni values. Thus, the Sidak procedure will have both a slightly higher MEER and power than the Bonferroni procedure.

Comments on the Tukey Procedure

- The Tukey procedure is more powerful than the Bonferroni and Sidak procedures when considering <u>all</u> pairwise comparisons ($C = a(a-1)/2$) and all n_i are equal.
- The calculations for a Tukey procedure do not depend on the number of comparisons C. The procedure depends on the number of treatments a. This implies that when a subset of pairwise comparisons are of interest ($C < a(a-1)/2$), the Tukey procedure will have lower power than the Bonferroni and Sidak procedures.
- When the treatment sample sizes are equal, then the EERC $= \alpha$ and the MEER $< \alpha$.
- When the treatment sample sizes are not equal, then the EERC is less than α. Thus, the Tukey method is Type I error conservative when the sample sizes are unequal.

2.8.1 Dunnett's MCP for Comparisons to a Control

- Situation: The experimenter is comparing $a-1$ experimental treatments to a control.
- In this situation, the desired inference may be directional or one-sided. The question is often 'Is one of the treatments better than the control?'. When this is the case, one-tailed critical values are used.
- In other cases, the researcher just wants to detect if the treatment means are different than the control mean. For such a nondirectional (or two-sided) hypothesis, two-tailed critical values are used.
- Then for each experimental treatment i, we
  1. Calculate the $a-1$ differences $D_i = \overline{y}_{i\cdot} - \overline{y}_{control}$.
  2. For a one-sided alternative, calculate

$$D_\alpha = d_\alpha(a-1, N-a)\sqrt{MS_E\left(\frac{1}{n_i} + \frac{1}{n_{control}}\right)}.$$

Decision Rule for $H_1 : \mu_i > \mu_{control}$, reject $H_0 : \mu_i = \mu_{control}$ if $D_i > D_\alpha$.
Decision Rule for $H_1 : \mu_i < \mu_{control}$, reject $H_0 : \mu_i = \mu_{control}$ if $D_i < -D_\alpha$.

  3. For a two-sided alternative, calculate

$$D_\alpha = d_\alpha(a-1, N-a)\sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_{control}}\right)}.$$

Decision Rule for $H_1 : \mu_i \neq \mu_{control}$, reject $H_0 : \mu_i = \mu_{control}$ if $|D_i| > D_\alpha$.

- Tables of one and two-sided $d_\alpha(a-1, N-a)$ values are in the Appendix of the Montgomery text.

2.8.2 Multiple Comparison Procedure Example

Example: A single-factor CRBD was run with $a = 5$ treatments and $n = 5$ replications per treatment. The data are given below:

Treatment				
A	B	C	D	E
19.08	22.04	18.68	16.99	15.34
17.07	21.44	19.86	13.18	13.52
18.91	18.82	19.68	16.97	15.23
15.09	20.49	17.78	12.90	15.63
17.00	19.34	17.86	15.00	13.21

SAS Code for MCP Example

```
DATA in;
     DO trt = 'A', 'B', 'C', 'D', 'E';
     DO rep = 1 to 5;
        INPUT y @@; OUTPUT;
     END; END;
LINES;
19.08 17.07 18.91 15.09 17.00     22.04 21.44 18.82 20.49 19.34
18.68 19.86 19.68 17.78 17.86     16.99 13.18 16.97 12.90 15.00
15.34 13.52 15.23 15.63 13.21
;
PROC GLM DATA=in ; * PLOTS = (ALL);
     CLASS trt;
     MODEL y = trt / SS3 SOLUTION;
     MEANS trt / LSD TUKEY SIDAK BON ALPHA=.05 CLDIFF LINES;
     MEANS trt / DUNNETT('C') DUNNETTL('C') DUNNETTU('C');
TITLE 'Multiple Comparison Test Example';
RUN;
```

Multiple Comparison Test Example

The GLM Procedure

Dependent Variable: y

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	4	122.7958160	30.6989540	14.41	<.0001
Error	20	42.5974000	2.1298700		
Corrected Total	24	165.3932160			

R-Square	Coeff Var	Root MSE	y Mean
0.742448	8.463080	1.459407	17.24440

Source	DF	Type III SS	Mean Square	F Value	Pr > F
trt	4	122.7958160	30.6989540	14.41	<.0001

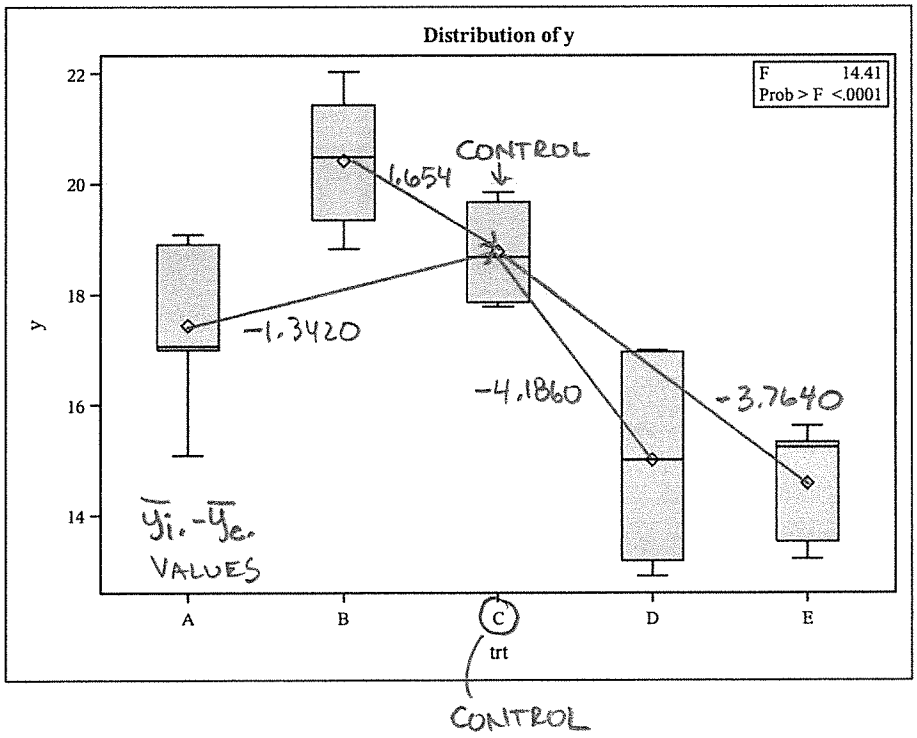
VERY STRONG EVIDENCE TO REJECT $H_0: \mu_A = \mu_B = \mu_C = \mu_D = \mu_E$

CONSTRAINT I

$\hat{\mu} = \bar{y}_{..}$

$\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..}$

Parameter		Estimate		Standard Error	t Value	Pr > |t|
Intercept		14.58600000	B	0.65266684	22.35	<.0001
trt	A	2.84400000	B	0.92301029	3.08	0.0059
trt	B	5.84000000	B	0.92301029	6.33	<.0001
trt	C	4.18600000	B	0.92301029	4.54	0.0002
trt	D	0.42200000	B	0.92301029	0.46	0.6525
trt	E	0.00000000	B	.	.	.



$\bar{y}_{i.} - \bar{y}_{c.}$ VALUES

FOR DUNNETT'S TEST, COMPARE TREATMENTS A, B, D, E TO THE CONTROL C

(a-1 = 4 COMPARISONS)

SEE PAGE 40

Fisher's LSD MCP

Alpha	0.05
Error Degrees of Freedom	20
Error Mean Square	2.12987
Critical Value of t	2.08596
Least Significant Difference	1.9254

Comparisons significant at the 0.05 level are indicated by ***.				
trt Comparison	**Difference Between Means**	**95% Confidence Limits**		
B - C	1.6540	-0.2714	3.5794	
B - A	2.9960	1.0706	4.9214	***
B - D	5.4180	3.4926	7.3434	***
B - E	5.8400	3.9146	7.7654	***
C - B	-1.6540	-3.5794	0.2714	
C - A	1.3420	-0.5834	3.2674	
C - D	3.7640	1.8386	5.6894	***
C - E	4.1860	2.2606	6.1114	***
A - B	-2.9960	-4.9214	-1.0706	***
A - C	-1.3420	-3.2674	0.5834	
A - D	2.4220	0.4966	4.3474	***
A - E	2.8440	0.9186	4.7694	***
D - B	-5.4180	-7.3434	-3.4926	***
D - C	-3.7640	-5.6894	-1.8386	***
D - A	-2.4220	-4.3474	-0.4966	***
D - E	0.4220	-1.5034	2.3474	
E - B	-5.8400	-7.7654	-3.9146	***
E - C	-4.1860	-6.1114	-2.2606	***
E - A	-2.8440	-4.7694	-0.9186	***
E - D	-0.4220	-2.3474	1.5034	

Tukey's HSD MCP

Alpha	0.05
Error Degrees of Freedom	20
Error Mean Square	2.12987
Critical Value of Studentized Range	4.23186
Minimum Significant Difference	2.762

Comparisons significant at the 0.05 level are indicated by ***.				
trt Comparison	**Difference Between Means**	**Simultaneous 95% Confidence Limits**		
B - C	1.6540	-1.1080	4.4160	
B - A	2.9960	0.2340	5.7580	***
B - D	5.4180	2.6560	8.1800	***
B - E	5.8400	3.0780	8.6020	***
C - B	-1.6540	-4.4160	1.1080	
C - A	1.3420	-1.4200	4.1040	
C - D	3.7640	1.0020	6.5260	***
C - E	4.1860	1.4240	6.9480	***
A - B	-2.9960	-5.7580	-0.2340	***
A - C	-1.3420	-4.1040	1.4200	
A - D	2.4220	-0.3400	5.1840	
A - E	2.8440	0.0820	5.6060	***
D - B	-5.4180	-8.1800	-2.6560	***
D - C	-3.7640	-6.5260	-1.0020	***
D - A	-2.4220	-5.1840	0.3400	
D - E	0.4220	-2.3400	3.1840	
E - B	-5.8400	-8.6020	-3.0780	***
E - C	-4.1860	-6.9480	-1.4240	***
E - A	-2.8440	-5.6060	-0.0820	***
E - D	-0.4220	-3.1840	2.3400	

Fisher's LSD MCP

Means with the same letter are not significantly different.				
t Grouping		**Mean**	**N**	**trt**
	A	20.4260	5	B
	A			
B	A	18.7720	5	C
B				
B		17.4300	5	A
	C	15.0080	5	D
	C			
	C	14.5860	5	E

Tukey's HSD MCP

Means with the same letter are not significantly different.				
Tukey Grouping		**Mean**	**N**	**trt**
	A	20.4260	5	B
	A			
B	A	18.7720	5	C
B				
B	C	17.4300	5	A
	C			
D	C	15.0080	5	D
D				
D		14.5860	5	E

Bonferroni's MCP

Alpha	0.05
Error Degrees of Freedom	20
Error Mean Square	2.12987
Critical Value of t	3.15340
Minimum Significant Difference	2.9106

Comparisons significant at the 0.05 level are indicated by ***.				
trt Comparison	**Difference Between Means**	**Simultaneous 95% Confidence Limits**		
B - C	1.6540	-1.2566	4.5646	
B - A	2.9960	0.0854	5.9066	***
B - D	5.4180	2.5074	8.3286	***
B - E	5.8400	2.9294	8.7506	***
C - B	-1.6540	-4.5646	1.2566	
C - A	1.3420	-1.5686	4.2526	
C - D	3.7640	0.8534	6.6746	***
C - E	4.1860	1.2754	7.0966	***
A - B	-2.9960	-5.9066	-0.0854	***
A - C	-1.3420	-4.2526	1.5686	
A - D	2.4220	-0.4886	5.3326	
A - E	2.8440	-0.0666	5.7546	
D - B	-5.4180	-8.3286	-2.5074	***
D - C	-3.7640	-6.6746	-0.8534	***
D - A	-2.4220	-5.3326	0.4886	
D - E	0.4220	-2.4886	3.3326	
E - B	-5.8400	-8.7506	-2.9294	***
E - C	-4.1860	-7.0966	-1.2754	***
E - A	-2.8440	-5.7546	0.0666	
E - D	-0.4220	-3.3326	2.4886	

Bonferroni's MCP

Means with the same letter are not significantly different.				
Bon Grouping		**Mean**	**N**	**trt**
	A	20.4260	5	B
	A			
B	A	18.7720	5	C
B				
B	C	17.4300	5	A
	C			
	C	15.0080	5	D
	C			
	C	14.5860	5	E

Sidak's MCP

Alpha	0.05
Error Degrees of Freedom	20
Error Mean Square	2.12987
Critical Value of t	3.14330
Minimum Significant Difference	2.9013

Comparisons significant at the 0.05 level are indicated by ***.				
trt Comparison	**Difference Between Means**	**Simultaneous 95% Confidence Limits**		
B - C	1.6540	-1.2473	4.5553	
B - A	2.9960	0.0947	5.8973	***
B - D	5.4180	2.5167	8.3193	***
B - E	5.8400	2.9387	8.7413	***
C - B	-1.6540	-4.5553	1.2473	
C - A	1.3420	-1.5593	4.2433	
C - D	3.7640	0.8627	6.6653	***
C - E	4.1860	1.2847	7.0873	***
A - B	-2.9960	-5.8973	-0.0947	***
A - C	-1.3420	-4.2433	1.5593	
A - D	2.4220	-0.4793	5.3233	
A - E	2.8440	-0.0573	5.7453	
D - B	-5.4180	-8.3193	-2.5167	***
D - C	-3.7640	-6.6653	-0.8627	***
D - A	-2.4220	-5.3233	0.4793	
D - E	0.4220	-2.4793	3.3233	
E - B	-5.8400	-8.7413	-2.9387	***
E - C	-4.1860	-7.0873	-1.2847	***
E - A	-2.8440	-5.7453	0.0573	
E - D	-0.4220	-3.3233	2.4793	

Sidak's MCP

Means with the same letter are not significantly different.				
Sidak Grouping		**Mean**	**N**	**trt**
	A	20.4260	5	B
	A			
B	A	18.7720	5	C
B				
B	C	17.4300	5	A
	C			
	C	15.0080	5	D
	C			
	C	14.5860	5	E

Dunnett's MCP (Two-Sided)

Alpha	0.05
Error Degrees of Freedom	20
Error Mean Square	2.12987
Critical Value of Dunnett's t	2.65103
Minimum Significant Difference	2.4469

Comparisons significant at the 0.05 level are indicated by ***.				
trt Comparison	**Difference Between Means**	**Simultaneous 95% Confidence Limits**		
B - C	1.6540	-0.7929	4.1009	
A - C	-1.3420	-3.7889	1.1049	
D - C	-3.7640	-6.2109	-1.3171	***
E - C	-4.1860	-6.6329	-1.7391	***

Dunnett's MCP (Lower One-Sided)

Alpha	0.05
Error Degrees of Freedom	20
Error Mean Square	2.12987
Critical Value of Dunnett's t	2.30443
Minimum Significant Difference	2.127

Comparisons significant at the 0.05 level are indicated by ***.				
trt Comparison	**Difference Between Means**	**Simultaneous 95% Confidence Limits**		
B - C	1.6540	-0.4730	Infinity	
A - C	-1.3420	-3.4690	Infinity	
D - C	-3.7640	-5.8910	Infinity	
E - C	-4.1860	-6.3130	Infinity	

Dunnett's MCP (Upper One-Sided)

Alpha	0.05
Error Degrees of Freedom	20
Error Mean Square	2.12987
Critical Value of Dunnett's t	2.30443
Minimum Significant Difference	2.127

Comparisons significant at the 0.05 level are indicated by ***.				
trt Comparison	**Difference Between Means**	**Simultaneous 95% Confidence Limits**		
B - C	1.6540	-Infinity	3.7810	
A - C	-1.3420	-Infinity	0.7850	
D - C	-3.7640	-Infinity	-1.6370	***
E - C	-4.1860	-Infinity	-2.0590	***

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