Cross-covariance:

- Covariance between two series

$$\gamma_{xy}(s,t) = E[(x_s - \mu_{xs})(y_t - \mu_{yt})]$$

Cross-correlation function (CCF):

$$\rho_{xy}(s,t) = \frac{\gamma_{xy}(s,t)}{\sqrt{\gamma_x(s,s)\gamma_y(t,t)}}$$

White noise (in CC 2.3):

- uncorrelated random variables collected over time
- w_t~wn(0, σ_w^2) or e_t~wn(0, σ_e^2)
- all oscillations of all periods present in equal strength

Noise also comes in other colors, pink, red brown, black, blue, etc.

Brownian motion will also be of interest (random walk discussed more below)

- Go to http://en.wikipedia.org/wiki/Colors_of_noise for more details on other colors of noise (and to listen to them).

White independent noise:

w_t~iid(0, σ_w^2)

White independent, Gaussian noise:

w_t~iid N(0, σ_w^2)

Example: y

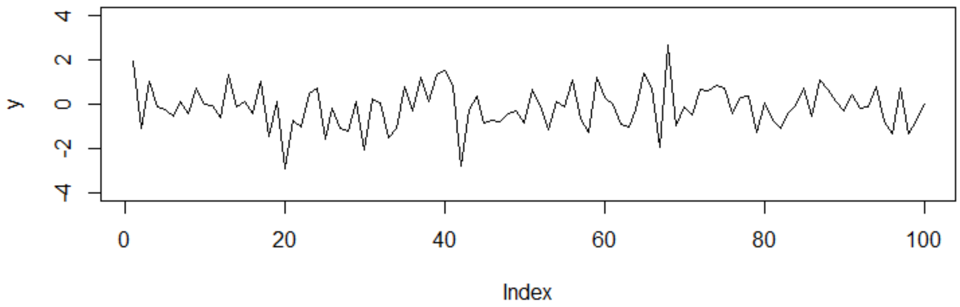
White independent, t noise:

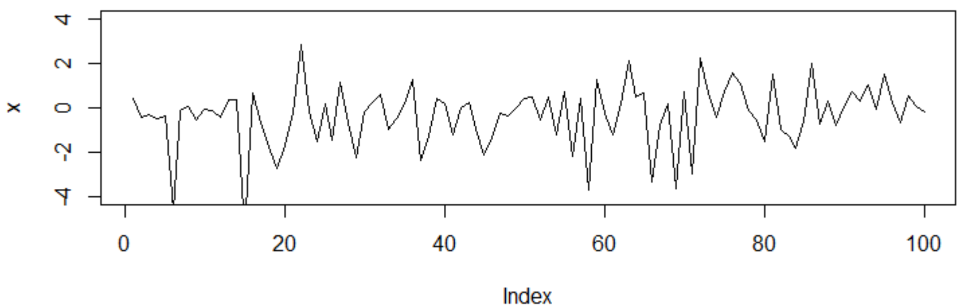
w_t~iid t_3

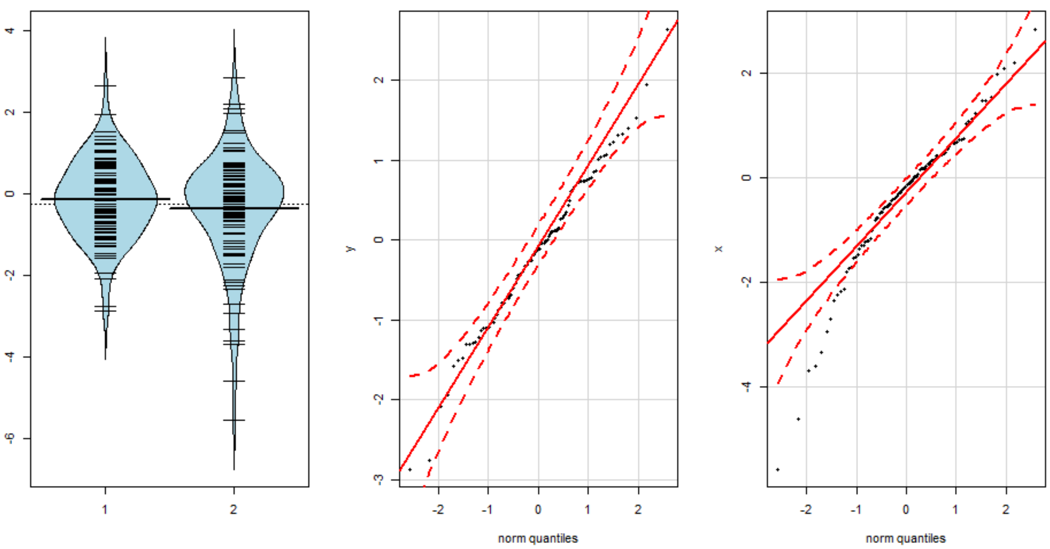
Example: x

```
> set.seed(2123)
> par(mfrow=c(2,1))
> y<-rnorm(100)
> plot(y,type="l",ylim=c(-4,4))
> x<-rt(100,df=3)
> plot(x,type="l",ylim=c(-4,4))
> par(mfrow=c(1,1))
> par(mfrow=c(1,3))
> require(beanplot)
> beanplot(y,x,col="lightblue",method="jitter")
> require(car)
```

```
> qqPlot(y,pch=16)
> qqPlot(x,pch=16)
```







Moving Average:

- Symmetric local average of observations (2-sided)
  - $$v_t = \sum_{j \in [t-k, t+k]} y_j / (2k+1)$$
- One-sided moving average (observation and its prior neighbor)

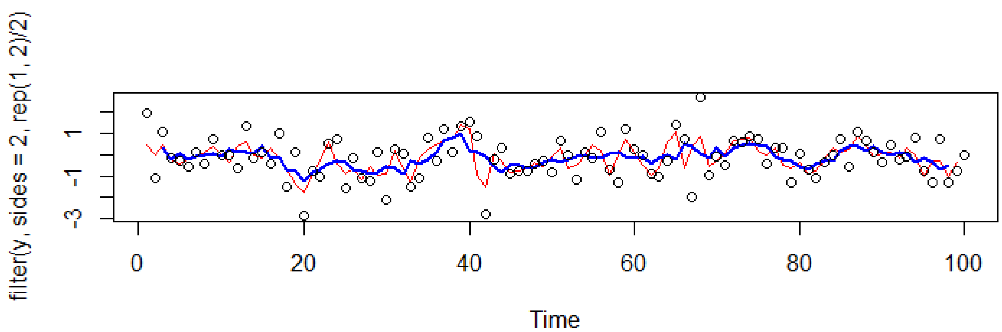
- $v_t = \sum_{j \in [t-1,t]} y_j / 2$

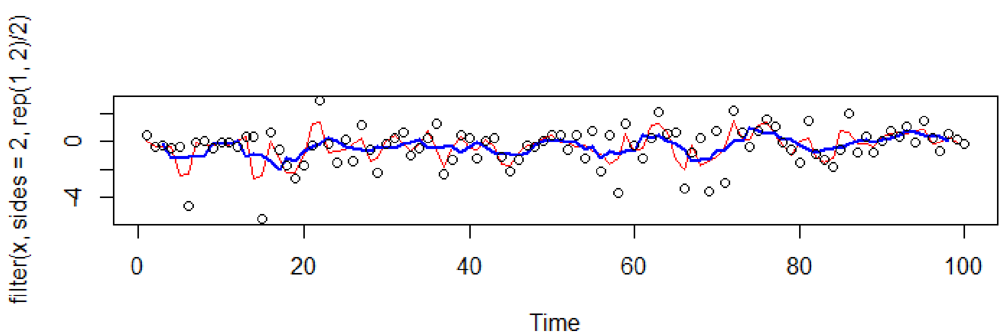
- Linear Filter
  - applied through time series

```
plot.ts(filter(wnex,sides=2,rep(1,5)/5))
```

- Filters, like all time series models that incorporate lagged dependencies, have to be modified at the edge(s) in some fashion.

```
> par(mfrow=c(2,1))
> plot.ts(filter(y,sides=2,rep(1,2)/2),col="red",ylim=range(y))
> lines(filter(y,sides=2,rep(1,5)/5),col="blue",lwd=2)
> points(y)
> plot.ts(filter(x,sides=2,rep(1,2)/2),col="red",ylim=range(x))
> lines(filter(x,sides=2,rep(1,5)/5),col="blue",lwd=2)
> points(x)
```





Random Walk (from CC 2.2):

$e_t \sim wn(0, \sigma_e^2)$ from t=1,…

$Y_1 = e_1$
$Y_2 = e_1 + e_2$
…

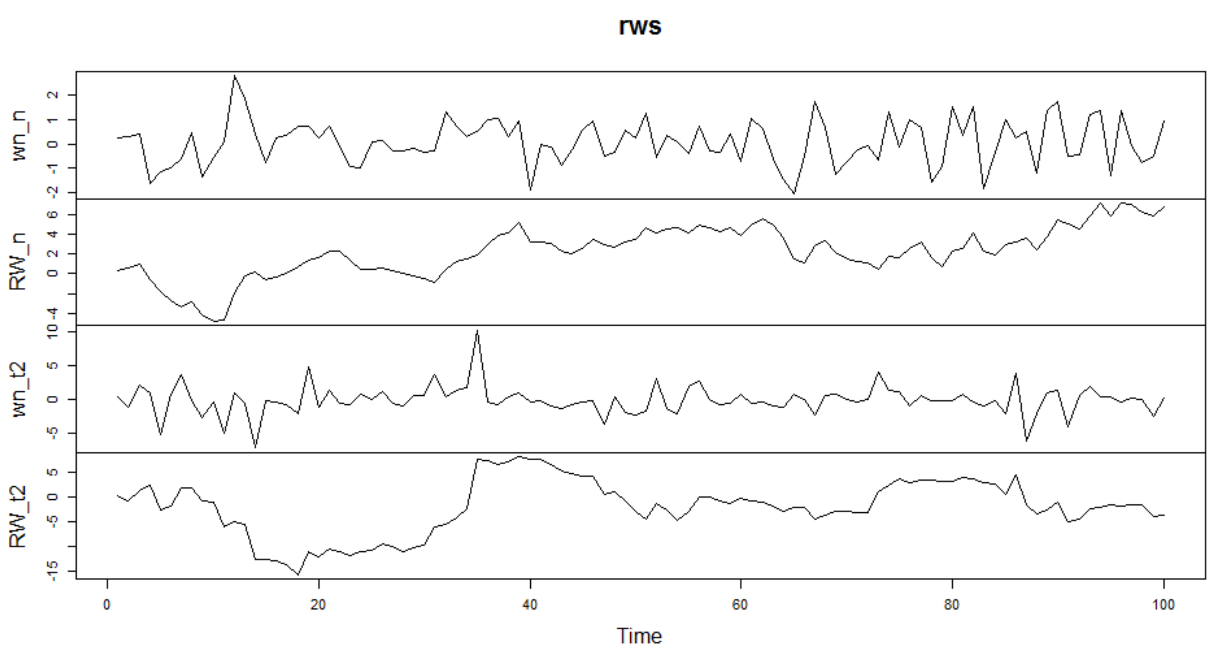
Generally, $Y_t =$

e_t's are sizes of "steps" and Y_t is the position of the (drunk) walker

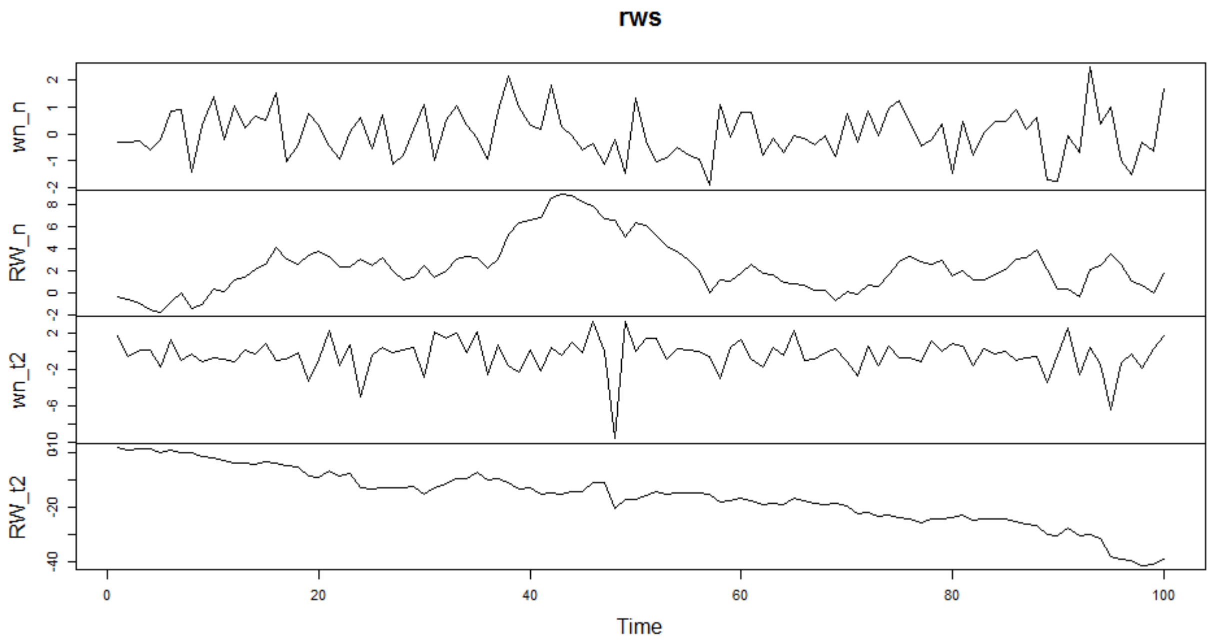
- Also called a drunkard's walk
- Each realization is a random fluctuation from the current starting point
- "the haphazard step of drunken sailor, bereft of his bearings, zapped with random shocks"
- Best guess of next location is the current location $Y_t = Y_{t-1}$
- Generated by adding up white noise process over time (using cumsum)

```
> set.seed(62345)
> wn_n<-rnorm(100)
> wn_t2<-rt(100,df=2)
> rws<-ts(data.frame(wn_n,RW_n=cumsum(wn_n),wn_t2,RW_t2=cumsum(wn_t2)))
> plot.ts(rws)
> set.seed(72345)
> wn_n<-rnorm(100)
> wn_t2<-rt(100,df=2)
> rws<-ts(data.frame(wn_n,RW_n=cumsum(wn_n),wn_t2,RW_t2=cumsum(wn_t2)))
> plot.ts(rws)
```

White noise from N(0,1) and t(2) used in random walk processes:



And a second realization of this:



Random walks have an equal chance of wandering to negative values as positive values…

- Assume that we have a white noise process used to generate a random walk:

$\mu_t = E(Y_t) =$

$Var(Y_t) =$

$Cov(Y_t, Y_s) = \gamma_{t,s} =$

$$\rho(t,s) = \frac{\gamma(t,s)}{\sqrt{\gamma(t,t)\gamma(s,s)}} =$$

Some examples (also think about $n \to \infty$)

Revisiting our **Moving Average** example: $v_t = \sum_{j \in [t-k, t+k]} y_j / (2k+1)$ with k=2

where y_j ~independent with $(0, \sigma_y^2)$

- Find the mean, variance, covariance, and autocorrelation functions:

2.3: Stationarity:

- Conventional time series methods often start with a quest for stationarity

- **Strictly stationary**: probabilistic behavior does not change over time (statistical equilibrium)

  - Joint distribution of $Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}$ is the same as $Y_{t_1-k}, Y_{t_2-k}, \ldots, Y_{t_n-k}$ for all k

  - Suppose that n=1:

  - Suppose that n=2:

- Properties of autocovariance and autocorrelation for a strictly stationary process:

- **Weakly stationary (second order stationary)**:
    - The mean function is constant over time and
    - $\gamma_{t,t-k} = \gamma_{0,k}$
    - This relates only to the first 2 moments (mean, variance) and not the full joint distribution
    - However, if the observations are multivariate normally distributed and weakly stationary, then strict stationarity holds
    - If we say "stationary" we probably mean weakly stationary
    - It turns out that making the mean stationary is often not too difficult, so stationarity in the variance/covariances is what concerns us most.

- Note that strict stationarity implies the following:
    - $P[Y_{t1} \leq x_1, \ldots Y_{tk} \leq x_k] = P[Y_{t1+h} \leq x_1, \ldots Y_{tk+h} \leq x_k]$
      for all k, t, x, and h

- Back to properties of a weakly stationary process
  If the mean function exists, $\Rightarrow \mu_s = \mu_t$
    - The mean must be constant

  If the variance function exists,
  $\Rightarrow \gamma(s,t) = \gamma(s\text{-}k,t\text{-}k)$

- The autocovariance function of the process depends only on the difference between s and t

Checking stationarity of a white noise process:

$e_t \sim wn(0, \sigma_e^2)$ from t=1,…

$E(e_t) = 0$

$$\gamma_k = \begin{cases} \sigma_e^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

- So a white noise process is at least weakly stationary since the mean and autocovariance do not depend on *t* based on these results.
  - Due to independence, it is actually strictly stationary but we only really need the weak result
  - Note that any distribution could be used here if the mean and variance exist

Jointly stationary:

- 2 series are jointly stationary if they are each stationary and cross-covariance function is a function of k alone
  - $\gamma_{xy}(k) = E[(x_{t+k} - \mu_x)(y_t - \mu_y)]$
- Cross-correlation of jointly stationary series
  - $\rho(k) = \gamma_{xy}(k) / \text{sqrt}[\gamma_x(0)\, \gamma_y(0)]$

<u>Chapter 3: TRENDS:</u>

3.1: Deterministic vs Stochastic Trends

- Deterministic: trend driven by a non-random process
  - trend is not a function of the previous observation(s)?
- Stochastic: trend driven by a random process
  - trend is a function of the previous observation(s)?
  - Model changes over time

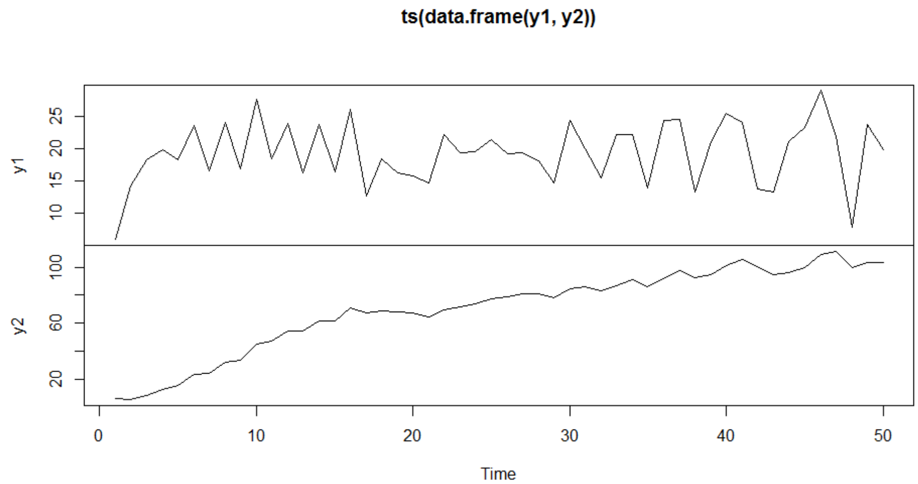
- Two examples

$$Y_t = \alpha + \beta t + e_t, e \sim N(0, \sigma_e^2)$$

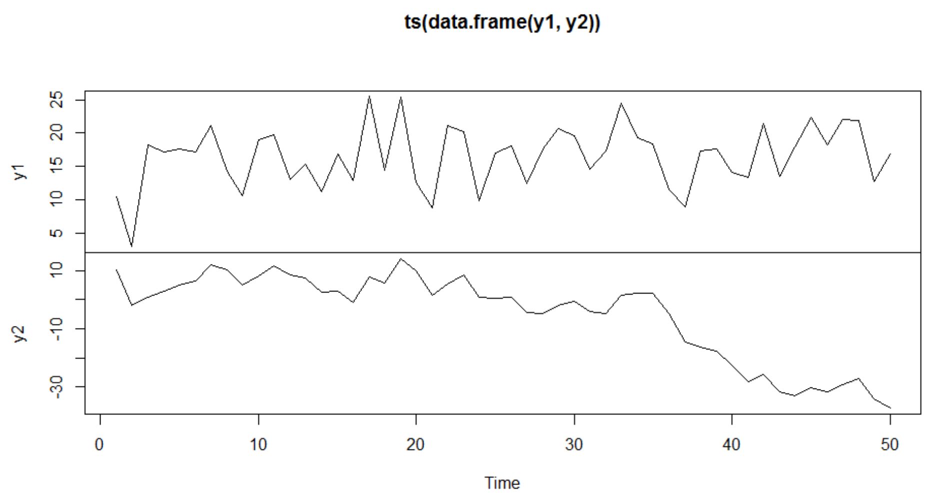
$$Y_t = Y_{t-1} + v_t, v \sim N(0, \sigma_v^2)$$

```
> set.seed(22345)
> e<-rnorm(50,0,5)
>  time<-1:50
>  beta0<-15
>  beta1<-.1
>  y1<-beta0+beta1*time+e
>  y2<-15+cumsum(e)
>  plot.ts(ts(data.frame(y1,y2)))
```

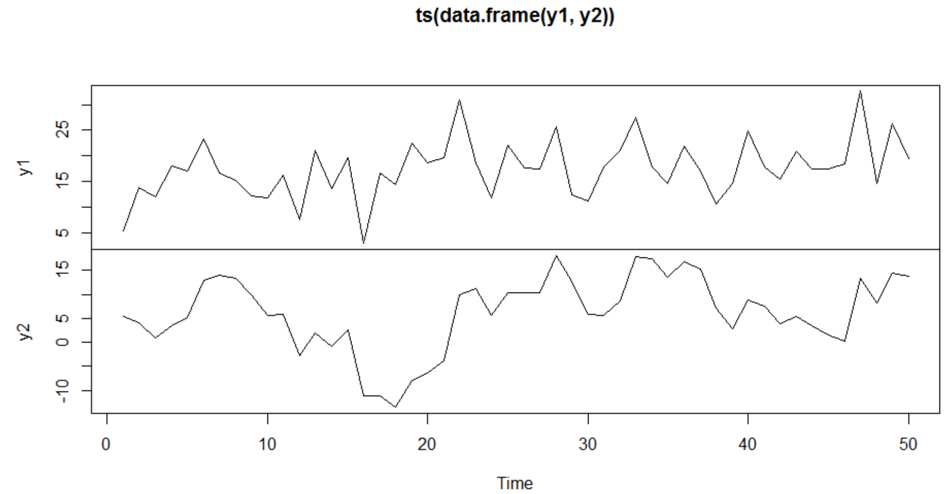
Repeat the simulation a few times…




```
> set.seed(5545)
```




```
> set.seed(7745)
```



It is interesting to see how this can change if you try modifying the sd of the innovations into the RW and/or the length of time that you observe the series.

- But the definitions and models sometimes land in a grey area regarding stochastic and deterministic models…

$$Y_t = \alpha + \beta_t time + e_t, e \sim N(0, \sigma_e^2)$$

$$Y_t = \alpha + s(x) time + e_t, e \sim N(0, \sigma_e^2)$$

$$Y_t = \alpha + Y_{t-1} + v_t, v \sim N(0, \sigma_v^2)$$

Back to CC 3.1:

- $Y_t = \mu_t + X_t$
    - Suppose that $\mu_t = \mu_{t-12}$ which would be deterministic
    - But $Y_t = Y_{t-12}$ is stochastic
        - and even though both suggest a 12 unit lagged dependence, they are different models
    - Note that deterministic trends imply that the relationship holds forever and stochastic trends can change over time
    - It can be difficult (impossible?) over shorter time frames to distinguish between stochastic and deterministic behavior

- As in CC, a random walk can locally have a positive or negative trend

- Generally using regular regression techniques in the presence of nonstationary explanatory or response variables is dangerous (See Ch. 11 on Spurious Correlations)
    - Is "time" a stationary process?

CC 3.2: Estimation of a Constant Mean:

- Assume $Y_t = \mu + X_t$
    - where $E(X_t)=0$

- Typical estimator of μ is

    - It is easily shown to be an unbiased estimator

    - But what about its variance (CC 3.2.3 using Exercise 2.17)?

    - And connections to having a simple random sample of observations: