Skill Builder: Single Variable Integration

Add it up:

a) $\sum_{i=1}^{2n}(3+(\frac{i}{n})^2)\frac{1}{n}$

Estimate the sum for $n=10$, $n=100$, $n=1000.$

From creating a function in Python, we can estimate the sum for the following values of n:

```
n=10 # the number of bins
the_sum=0 # the initial sum before each step

def summation (x, the_sum):
  for i in range(1, x, 1):
    the_sum+= (3+((i/n)**2))*(1/n)
  return the_sum

summation(2*n, the_sum) # n=10

>>> 8.17
```

$n=10$

$\sum_{i=1}^{20}(3+(\frac{i}{20})^2)\frac{1}{20}=8.17$

```
n=100
summation(2*n,the_sum) # n=100

>>> 8.616700000000003
```

$n=100$

$\sum_{i=1}^{200}(3+(\frac{i}{20})^2)\frac{1}{20}=8.61670$

```
n=1000
summation(2*n,the_sum) # n=100

>>> 8.661666999999982
```

$n = 1000$

$\sum_{i=1}^{2000}(3 + (\frac{i}{20})^2)\frac{1}{20} = 8.66166$

Evaluate (or estimate) the value of the sum as $n \to \infty$.

To **evaluate it**, we would use $\lim_{n\to\infty} \sum_{i=1}^{2n}(3 + (\frac{i}{n})^2)\frac{1}{n}$, and solve the rest of the summation with the limit.

Use the constant rule of summation:

$\lim_{n\to\infty} \sum_{i=1}^{2n}(3 + (\frac{i}{n})^2)\frac{1}{n} = \lim_{n\to\infty}(\frac{1}{n}\sum_{i=1}^{2n}(3 + (\frac{i}{n})^2)) =$

Use the summation partitioning of the summation rule:

$= \lim_{n\to\infty}(\frac{1}{n}\sum_{i=1}^{2n}(3 + \frac{i^2}{n^2})) = \lim_{n\to\infty}(\frac{1}{n}(\sum_{i=1}^{2n} 3 + \sum_{i=1}^{2n} \frac{i^2}{n^2})) =$

Use the summation of a constant rule, and the summation of i^2:

$= \lim_{n\to\infty}(\frac{1}{n}(6n + \frac{(2n+1)(4n+1)}{3n}) = \lim_{n\to\infty} \frac{18n^2+(2n+1)(4n+1)}{3n^2} =$

Divide and simplify the fraction and calculate its limit:

$= \lim_{n\to\infty} \frac{18n^2+8n^2+6n+1}{3n^2} = \lim_{n\to\infty} \frac{26n^2+6n+1}{3n^2} =$

$= \lim_{n\to\infty} \frac{26}{3} + \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} \frac{1}{3n^2}) = \frac{26}{3} + 0 + 0 = 8.66666$

However, **to estimate it**, we can iterate over the summation formula for a really big number (in our case, 10,000,000) and see the value that we get.

```
n=10000000
summation(2*n,the_sum) # n=10000000

>>> 8.666666166666289
```

That seems to converge around 8.66666, which is identical to our analytical results.

Rewrite the sum as a definite integral and compute it:

We substitute $2n$ with a new variable z, because we only know how to make an integral from a Riemann's sum whose sum iterates over n.

$2n = z$

$\sum_{i=1}^{z}(3 + (\frac{i}{\frac{z}{2}})^2) \cdot \frac{1}{\frac{z}{2}} = \sum_{i=1}^{z}(3 + \frac{4i^2}{z^2}) \cdot \frac{2}{z}$

Then we find the change in x, Δx, which in our case is given as the second term in the sum:

$\Delta x = \frac{b-a}{2n} = \frac{2}{z} = \frac{2}{2n} = \frac{1}{n}$

From the given fraction, we can deduce that:

$b - a = 2$

Then we need to find x_i, which is given by:

(Assumption: $a = 0$).

$x_i = a + \Delta x + i = 0 + \frac{1}{n} \cdot i = \frac{i}{n}$

$b - 0 = 2$

$b = 2$

Finally, we need to find our function, which is given by our first term of summation:

$f(x_i) = 3 + x_i^2 = 3 + \frac{i^2}{n^2}$

Now we have all the components to create the integral:

$\int_0^2 f(x_i)dx = \int_0^2 3 + x_i^2 dx$

To compute this integral, we need to know its antiderivative and use the Fundamental Theorem of Calculus:

With the reverse power-rule:

$\int_0^2 3 + x_i^2 dx = 3x + \frac{x^3}{3}|_0^2 = 3 \cdot 2 + \frac{2^3}{3} - 0 = 6 + \frac{8}{3} = 6 + 2.66666 = 8.66666$

The integral gives us the exact same value as the sum approach, and the computational tools.

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b) $\sum_{i=1}^{n} \sin(\frac{i\pi}{n})\frac{\pi}{n}$

Estimate the sum for $n = 10$, $n = 100$, $n = 1000$.

Reusing the same function, we get the estimated values:

```
n=10 # the number of bins
the_sum=0 # the initial sum before each step

def summation (x, the_sum):
  for i in range(1, x, 1):
    the_sum+= (np.sin((i*np.pi)/n)*(np.pi/n))
  return the_sum

summation(n, the_sum) # n=10

>>> 1.9835235375094544
```

$n = 10$:

$\sum_{i=1}^{10} \sin(\frac{i\pi}{10})\frac{\pi}{10} = 1.98352$

```
n=1000
summation(n, the_sum) # n=1000
>>>
1.9999983550656637
```

$n = 100$:

$\sum_{i=1}^{100} \sin(\frac{i\pi}{100})\frac{\pi}{100} = 1.99999$

```
n=10000000
summation(n, the_sum) # n=10000000

>>>
1.9999999999999394
```

$n = 1000$

$\sum_{i=1}^{100} \sin(\frac{i\pi}{100})\frac{\pi}{100} = 1.99999$

Evaluate (or estimate) the value of the sum as $n \to \infty$.

Estimation through Python, by setting $n = 10000000$.

```
n=10000000
summation(n, the_sum) # n=10000000
>>> 1.9999999999999394
```

This seems to converge at around 2.

Rewrite the sum as a definite integral and compute it:

We use similar strategy as in the first problem:

$\Delta x = \frac{\pi}{n}$

$b - a = \pi$

$a = 0; b = \pi$

$x_i = 0 + \frac{\pi}{n} \cdot i = \frac{i\pi}{n}$

$f(x_i) = \sin(x_i)$

$\int_0^\pi f(x_i)dx = \int_0^\pi \sin(x)dx$

To compute the integral we use the antiderivative of $\sin$ function:

$-\cos(x_i)|_0^\pi = -\cos(\pi) - (\cos(0)) = 1 + 1 = 2$

Our integral shows a very similar result to the estimation in the first part.

Integral Part:

a) Indefinite integrals

$$\int \frac{ln(x)}{x} dx$$

We can use the substitution method to compute this integral.

$u = ln(x);$

then we differentiate u over x, to get the value of dx in terms of u:

$\frac{du}{dx} = \frac{1}{x}$

$dx = x \cdot du$

And now we can set up a new integral:

$\int \frac{u}{x} \cdot du \cdot x$

$\int = udu$

We can apply the power rule, if we consider the u^1.

$\frac{u^{1+1}}{1+1} = \frac{u^2}{2}$

Now we use the real values of the function, and since it is an indefinite integral, we add a constant:

$\frac{(ln(x))^2}{2} + C = \frac{1}{2}(ln(x))^2 + C$

$$\int e^x \cos(e^x) dx$$

We use substitution method again:

$u = e^x,$

then we differentiate u over x, to express the value of dx in terms of u:

$du = e^x dx$

And we can rewrite the integral based on the substitutions we made:

$\int \cos(u) du$

The antiderivative of $\cos(x)$ is $\sin(x)$, thus:

$\int \cos(u) du$ = $\sin(u)$

Then we substitute u with the real values of the function, and add a constant because the integral is indefinite:

$\sin(e^x) + C$

$$\int \frac{e^x}{1 + e^{(2x)}} dx$$

We can first restructure the integral to make it easier to substitute:

$\int \frac{1}{1+(e^x)^2} \cdot e^x dx$

And then we are able to use the substitution method to compute this integral:

$u = e^x$

$du = e^x dx$

We rewrite the integral, with the new substitution:

$\int \frac{1}{1+u^2} du$

The antiderivative of this function is $\arctan(u)$.

$\int \frac{1}{1+u^2} du = \arctan(u)$

And then we use back the initial variables, and add a constant because it is an indefinite integral:

$\arctan(u) = \arctan(e^x) + C$

b) Definite integrals

$$\int_1^e \frac{ln(x)}{x} dx$$

We can use the final function from the first indefinite integral, without the C (because now the integral is defined), and we evaluate it for the given points with the Fundamental Theorem of Calculus.

$\frac{1}{2}(ln(x))^2|_1^e = \frac{1}{2}[(ln(e))^2 - (ln(1))^2]$ = $\frac{1}{2}(1-0) = \frac{1}{2}$

$$\int_0^{ln(\pi)} e^x \cos(e^x) dx$$

Similarly, we can evaluate the Fundamental Theorem of Calculus formula at the given endpoints in the final function from the indefinite integrals, without the constant (as now the integral is definite).

$\sin(e^x)|_0^{ln(\pi)} = \sin(e^{ln(\pi)}) - \sin(e^0) = 0 - \sin(1) = -0.84147$

$$\int_0^1 \frac{ln(x)}{x} dx$$

Since this is the same integral as in earlier in this assignment, we use the same strategy for the given endpoints:

$\frac{1}{2}(ln(x))^2|_1^e = \frac{1}{2}[(ln(1))^2 - (ln(0))^2]$→ DNE.

The second term suggests that we should calculate the natural logarithm of 0, which is undefined. Thus, we cannot compute the finite value of this integral, and for this given boundary it could be considered improper.

$$\int_0^{\infty} e^x \cos(e^x) dx$$

This is an improper integral, which is infinitely stretched over the x-axis, but that does not mean that the area does not converge to a finite value. To explore whether there is a finite value, we will use a trick; instead of using ∞, we will replace it with B, which represents a really large number, and our integral will take the form of:

$\lim_{B\to\infty} \int_0^B e^x \cos(e^x)dx$

We know the antiderivative of the second part of this integral, and it is:

$\sin(e^x)$

We can use the First Theorem of Calculus again to evaluate at endpoints, but while taking the limit of B to infinity:

$\lim_{B\to\infty} \sin(e^x)|_0^B = \lim_{B\to\infty}(\sin(e^B) - \sin(e^0))$ → DNE.

$e^B \to \infty$

$e^0 \to 1$

While sin(1) is well-defined, the sin of infinity is not. Thus, we cannot find a finite value of this integral. We say this integral is divergent, just our previous one.

Reflection Questions:

1. Well, the area between the curves, is just another area. So we would still use the integral. However, since this area is defined as a gap between f(x) and g(x), then we can take their subtracted value, just like how we would for other gaps between areas of well known shapes. Since this a curve, with a given range, we would use a definite integral $\int_a^b f(x) - g(x)dx$. What is more important here, is that we always need to take the first function to be the upper function (on the positive quadrants), to ensure we get a positive value at the end, because unlike the area under curve, the area between curves cannot be a negative value.

   Sometimes, we are not even evaluating the gaps, but the intersection area between the curves. We would still use the same strategy. An appropriate context where we would like to know about the area between the curves is for example, when one is buying two carpets, and wants to minimize the overlap between carpets, where both cover the floor.

2. An obvious application is #estimation. I attempted to create a reusable function in Python, which I used to estimate both sums in problem 1 for different values of n. The estimation turned out pretty correctly when compared with the analytical and integral solutions.

   A rather weaker use of HC is #constraints. Although in the improper integrals problems, I try to explain why the integrals do not converge, I could have better explained how these constraints inform our knowledge of integrals; an example of how constraints can help us solve a problem.

#cs111a-computationaltools: Here I attach the full code for the first part of the assignment:[Google Colaboratory](https://colab.research.google.com/drive/1EpHl0IzeM5Rr0ZI4uB9tUttMRqN4PrcL?usp=sharing)

Sources of information beyond class that informed this assignment:

For the optional question:

https://socratic.org/questions/how-do-you-find-the-antiderivative-of-e-x-1-e-2x

The Area Between the Curves: https://tutorial.math.lamar.edu/ProblemsNS/CalcI/AreaBetweenCurves.aspx

Summation Rules: http://ms.uky.edu/~123/lecturenotes/Chapter9_answers.pdf

Riemann's Sum and Integrals:

https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-3/a/definite-integral-as-the-limit-of-a-riemann-sum