Assignment 1 - Linear Systems

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1. Glossary

Linear Functions - graphically these functions are straight lines. This means that all the variables are in the power of 1. As in other functions, we find values of y depending on x (or more predictors).

Example:

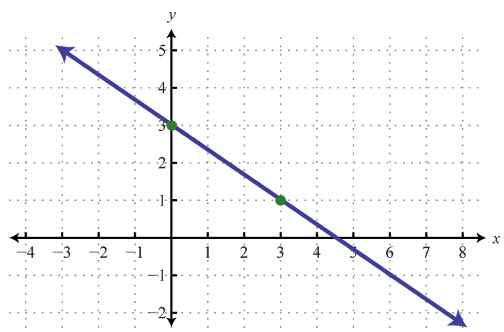

Figure 1. An example of a Linear Function.

Linear Systems - systems of equations that contain the same variables in the power of 1 or 0. They can have a unique solution, infinitely many solutions or no solution based on whether they are consistent and parametric.

Example:

$$x + y + z = 25$$

$$5x + 3y + 2z = 0$$

$$y - z = 6$$

This system has a unique solution which is :

$$x = \frac{-131}{5} \quad y = \frac{143}{5} \qquad z = \frac{113}{5}$$

Consistent Systems - Linear Systems of Equations which contain no inconsistencies, or equations that are known mathematically not to make sense.

Example:

0x= 5 *does not make from what we know in mathematics, as there is no* *such number when multiplied by 0 will give us 5. This is an example of an inconsistent system. When we make sure we have no inconsistency in the system, we have a consistent system.*

Augmented Matrices - represents important information such as the right-hand side of the equations or an identity matrix for the purpose of proving invertibility. We denote is as **[A|b].**

Example:

$$\begin{pmatrix} 1 & 1 & | & 6 \\ 2 & 2 & | & 12 \end{pmatrix}$$

The 6 and 12 represent the augmented part of the matrix (the solutions of the system), while the whole matrix represents the equation:

$x + y = 6$

$2x + 2y = 12$

Row Operations - Transforming the matrix into another matrix of the same size requires Row Operations. We can swap rows, multiply by a scalar and add one row to the other (or the scalar of the other row).

Example:

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \rightarrow \textbf{R1+R2} \rightarrow \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$$

Elementary Matrices - Elementary Matrices are matrices we use to solve Linear Systems of Equations, which are similar to identity matrices but change by a row operation that helps us reduce the original matrix to RREF.

Example:

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ → This is an identity matrix and,

$\begin{pmatrix} -1/2 & 0 \\ 0 & 1 \end{pmatrix}$ → This is an elementary matrix.

Invertible Matrix - is a special type of matrix, which is also known as non-singular. Its determinant is different from 0. Not all matrices are invertible, but those which are, when multiplied with the original matrix should produce an identity matrix (regardless of the side of multiplication). We can find the inverse of the matrix by augmenting it with an identity matrix and reducing it to RREF.

Example:

$$\begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 3 & 4 & | & 0 & 1 \end{pmatrix} \rightarrow \textbf{RREF} \rightarrow \begin{pmatrix} 1 & 0 & | & -2 & 1 \\ 0 & 1 & | & -3/2 & 1/2 \end{pmatrix}$$

The augmented part of the matrix in RREF represents the inverse of the original matrix.

RREF Matrix - (Reduced Row-Echelon Form) When the matrix is in this stage, we can see pivots of 1 and under and above pivots, 0s. We need those for solving linear systems when the matrix is augmented. The pivots represent the variables in a diagonal and the right-hand side represents the values of the variables for the given system. There are some criteria for a matrix to be in an RREF (more thoroughly explained, here). We can infer how many solutions the system has after turning the matrix into RREF.

Example:

$$\begin{pmatrix} 1 & 0 & 0 & | & 5 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & -9 \end{pmatrix}$$

All 1-s *are pivots, and* 5,2,-9 *are the values of* x,y,z *respectively.*

Homogeneous Systems - systems where all the solutions of the system (constants or right-hand side) are 0.

Example:

$$x + y + z = 0$$

$$3x + y + 4z = 0$$

$$x + 7y + 2z = 0$$

Determinants - are property of matrices, more specifically a calculation of the elements of matrices based on certain rules of linearity. They are useful to make inferences about the matrices, for instance, if the determinant of the matrix is 1, we can say the system has one unique solution or if the determinant of the matrix is 0, the matrix is not invertible. We denote the determinant with **det(A),** where A represents the matrix.

Example:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

2. Skill-Builder

1. 1 For each of the following linear systems:

I. **Define the associated augmented matrix.**
II. **Use row operations to reduce the matrix to RREF. Check your work using your favorite computational tool.**
III. **Determine whether or not the system is consistent.**
IV. **If the system is consistent, find all the solutions.**

a)

$$x + y + z = 3$$

$$x + 3y - z = 1$$

$$3x + 5y + 4z = 10$$

b)

$$2x - 4y + 8z = 4$$

$$x - y + z = 4$$

$$3x - 4y + 6z = 7$$

c)

$$2x + 2y + 6z = 2$$

$$6x + 9y - 3z = 9$$

$$10x + 14y + 2z = 14$$

ANSWERS:

a)

I. We write all the constants in the matrix. The left side represents the constant in front of x, y, z for all three equations respectively, while the right side represents the solutions of the given equations.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 3 & -1 & 1 \\ 3 & 5 & 4 & 10 \end{array}\right)$$

II. We begin with the original matrix and try to reduce the 0s below and above pivots:

1. In the first step, we add the $-1 * Row1$ to $Row2$ and we add $-3 * Row1$ to $Row3$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 3 & -1 & 1 \\ 3 & 5 & 4 & 10 \end{array}\right) \rightarrow R_2 - R_1 \ \& \ R_3 - 3R_1 \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & -2 & -2 \\ 0 & 2 & 1 & 1 \end{array}\right)$$

2. Now we have 0s under the first pivot, so we move to the second pivot. We add $-1 * Row2$ to $Row1$ multiplied by the scalar 2 and we add $-1 * Row2$ to $Row3$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & -2 & -2 \\ 0 & 2 & 1 & 1 \end{array}\right) \rightarrow 2R_1 - R_2 \ \& \ R_3 - R_2 \rightarrow \left(\begin{array}{ccc|c} 2 & 0 & 4 & 8 \\ 0 & 2 & -2 & -2 \\ 0 & 0 & 3 & 3 \end{array}\right)$$

3. After reducing the numbers above and below the second pivot to 0, we move to the third pivot. Here, we add the $-4 * Row3$ to $3 * Row1$, and we add the $2 * Row3$ to $3 * Row2$.

$$\begin{pmatrix} 2 & 0 & 4 & | & 8 \\ 0 & 2 & -2 & | & -2 \\ 0 & 0 & 3 & | & 3 \end{pmatrix} \rightarrow 3R_1 - 4R_3 \ \& \ 3R_2 - 2R_3 \rightarrow \begin{pmatrix} 6 & 0 & 0 & | & 12 \\ 0 & 6 & 0 & | & 0 \\ 0 & 0 & 3 & | & 3 \end{pmatrix}$$

4. We rescale all rows to get the pivots to 1. $Row1$ and $Row2$ are rescaled by $\frac{1}{6}$ and $Row3$ by $\frac{1}{3}$.

$$\begin{pmatrix} 6 & 0 & 0 & | & 12 \\ 0 & 6 & 0 & | & 0 \\ 0 & 0 & 3 & | & 3 \end{pmatrix} \rightarrow \frac{1}{6} * R_1 \ \& \ \frac{1}{6} * R_2 \ \& \ \frac{1}{3} * R_3 \rightarrow \begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 1 \end{pmatrix}$$

5. This is the RREF matrix obtained from row operations and the right-hand side represents the solutions for each variable.[1]

III. The system is consistent because each row has a variable and a solution which do not contradict each other or do not violate the mathematical rules.

IV. Based on the final form of the matrix (RREF) in step 5, we see that the solutions of the system are $x = 2$, $y = 0$, $z = 1$.

[1] #algorithms: Appendix

b)

I. We use the same method as in the previous question to populate the matrix in the augmented side.

$$\begin{pmatrix} 2 & -4 & 8 & | & 4 \\ 1 & -1 & 1 & | & 4 \\ 3 & -4 & 6 & | & 7 \end{pmatrix}$$

II. We begin with the original matrix and try to reduce the 0s below and above pivots:

1. First, we rescale the first pivot by the scalar $\frac{1}{2} * Row1$ in order to bring it to 1.

$$\begin{pmatrix} 2 & -4 & 8 & | & 4 \\ 1 & -1 & 1 & | & 4 \\ 3 & -4 & 6 & | & 7 \end{pmatrix} \to \frac{1}{2} * R_1 \to \begin{pmatrix} 1 & -2 & 4 & | & 2 \\ 1 & -1 & 1 & | & 4 \\ 3 & -4 & 6 & | & 7 \end{pmatrix}$$

2. Then we add $-1 * Row1$ to $Row2$ and $-3 * Row1$ to $Row3$ to bring them to 0.

$$\begin{pmatrix} 1 & -2 & 4 & | & 2 \\ 1 & -1 & 1 & | & 4 \\ 3 & -4 & 6 & | & 7 \end{pmatrix} \to R_2 - R_1 \text{ and } R_3 - 3R_1 \to \begin{pmatrix} 1 & -2 & 4 & | & 2 \\ 0 & 1 & -3 & | & 2 \\ 0 & 2 & -6 & | & 1 \end{pmatrix}$$

3. We reduce the number below the second pivot by adding $-1 * Row2$ to $Row3$.

$$\begin{pmatrix} 1 & -2 & 4 & | & 2 \\ 0 & 1 & -3 & | & 2 \\ 0 & 2 & -6 & | & 1 \end{pmatrix} \to R_3 - R_2 \to \begin{pmatrix} 1 & -2 & 4 & | & 2 \\ 0 & 1 & -3 & | & 2 \\ 0 & 0 & 0 & | & -3 \end{pmatrix}$$

4. We rescale the $Row3$ by $\frac{-1}{3}$.

$$\begin{pmatrix} 1 & -2 & 4 & \mid & 2 \\ 0 & 1 & -3 & \mid & 2 \\ 0 & 0 & 0 & \mid & -3 \end{pmatrix} \rightarrow \frac{-1}{3} * R_3 \rightarrow \begin{pmatrix} 1 & -2 & 4 & \mid & 2 \\ 0 & 1 & -3 & \mid & 2 \\ 0 & 0 & 0 & \mid & 1 \end{pmatrix}$$

5. Now, we try to turn all the numbers above the third pivot to 0, although the third pivot itself is 0. In that case, we add $-2 * Row3$ to both $Row2$ and $Row1$.

$$\begin{pmatrix} 1 & -2 & 4 & \mid & 2 \\ 0 & 1 & -3 & \mid & 2 \\ 0 & 0 & 0 & \mid & 1 \end{pmatrix} \rightarrow R_1 - 2R_3 \ \& \ R_2 - 2R_3 \rightarrow \begin{pmatrix} 1 & -2 & 4 & \mid & 0 \\ 0 & 1 & -3 & \mid & 0 \\ 0 & 0 & 0 & \mid & 1 \end{pmatrix}$$

6. We remove the number above pivot 2, by adding $2 * Row2$ to $Row1$.

$$\begin{pmatrix} 1 & -2 & 4 & \mid & 0 \\ 0 & 1 & -3 & \mid & 0 \\ 0 & 0 & 0 & \mid & 1 \end{pmatrix} \rightarrow R_1 + 2R_2 \rightarrow \begin{pmatrix} 1 & 0 & -2 & \mid & 0 \\ 0 & 1 & -3 & \mid & 0 \\ 0 & 0 & 0 & \mid & 1 \end{pmatrix}$$

7. This is the final step to reducing the matrix to RREF because under and above each pivot we have 0s.

III. This system is inconsistent because in the last row we get the equation $0x + 0y + 0z = 1$, which is absurd because it means $0 = 1$.

IV. There are no solutions to the system.

c)

I. We augment the matrix in the same way as the previous two steps.

$$\begin{pmatrix} 2 & 2 & 6 & | & 2 \\ 6 & 9 & -3 & | & 9 \\ 10 & 14 & 2 & | & 14 \end{pmatrix}$$

II. We follow these steps to reduce the matrix to RREF:

1. We rescale the $Row1$ by $\frac{1}{2}$.

$$\begin{pmatrix} 2 & 2 & 6 & | & 2 \\ 6 & 9 & -3 & | & 9 \\ 10 & 14 & 2 & | & 14 \end{pmatrix} \rightarrow \frac{1}{2} * R_1 \rightarrow \begin{pmatrix} 1 & 1 & 3 & | & 1 \\ 6 & 9 & -3 & | & 9 \\ 10 & 14 & 2 & | & 14 \end{pmatrix}$$

2. We reduce the elements below the pivot 1 to 0. Thus, we add $-6 * Row1$ to $Row2$ and $-10 * Row1$ to $Row3$.

$$\begin{pmatrix} 1 & 1 & 3 & | & 1 \\ 6 & 9 & -3 & | & 9 \\ 10 & 14 & 2 & | & 14 \end{pmatrix} \rightarrow R_2 - 6R_1 \; \& \; R_3 - 10R_1 \rightarrow \begin{pmatrix} 1 & 1 & 3 & | & 1 \\ 0 & 3 & -21 & | & 3 \\ 0 & 4 & -28 & | & 4 \end{pmatrix}$$

3. Rescale the $Row2$ by $\frac{1}{3}$

$$\begin{pmatrix} 1 & 1 & 3 & | & 1 \\ 0 & 3 & -21 & | & 3 \\ 0 & 4 & -28 & | & 4 \end{pmatrix} \rightarrow \frac{1}{3} * R_2 \rightarrow \begin{pmatrix} 1 & 1 & 3 & | & 1 \\ 0 & 1 & -7 & | & 1 \\ 0 & 4 & -28 & | & 4 \end{pmatrix}$$

4. Reduce the number below pivot 2 by adding $-1 * Row2$ to $Row1$ and $-4 * Row2$ to $Row3$.

$$\begin{pmatrix} 1 & 1 & 3 & | & 1 \\ 0 & 1 & -7 & | & 1 \\ 0 & 4 & -28 & | & 4 \end{pmatrix} \rightarrow R_1 - R_2 \ \& \ R_3 - 4R_4 \rightarrow \begin{pmatrix} 1 & 0 & 10 & | & 0 \\ 0 & 1 & -7 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

5. This is the final step to transforming the matrix to RREF, as the last row consists of 0s.

III. The system is consistent because all the equations make sense mathematically. The last row is basically saying that $0 = 0$, which remains true.

IV. We have 1 parametric and 2 pivot columns, therefore the system has infinitely many solutions. We represent this equation as

$x + 10b = 0 \rightarrow x = -10b$

$y - 7b = 1 \rightarrow y = 1 + 7b$,

where ***b*** represents any value.

1.2 Find numbers a, b, c and d so that the augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & a \\ 0 & 4 & 5 & b \\ 0 & 0 & d & c \end{bmatrix}$$

has:

a) No solutions

$$\begin{pmatrix} 1 & 0 & 0 & | & (a-b)/2 - c/2 \\ 0 & 1 & 0 & | & b/4 - 5/4c \\ 0 & 0 & d & | & c \end{pmatrix}$$

After we turn it into RREF, we see that we have no solutions only if d is equal to 0 and c is different from 0. That way, we create an inconsistency in the system.

b) Infinitely many solutions

$$\begin{pmatrix} 1 & 0 & 0 & | & (a-b)/2 - c/2 \\ 0 & 1 & 0 & | & b/4 - 5/4c \\ 0 & 0 & d & | & c \end{pmatrix}$$

We get infinitely many if we keep the system consistent and have more pivot columns that parametric. In that case, we use d to make the third column parametric and we turn it into 0. However, to remain consistent, we need to turn c into 0 as well. Thus, we have to turn c and d into 0 to have infinitely many solutions. An issue would arise with the fraction in the second row whose denominator is c, because if c becomes 0, then that fraction would not make sense.

- **Which numbers have no effect on the solvability?**

From the answers above, we can say we can alter the number of solutions regardless of a and b.

1.3 True or False: All homogeneous systems are consistent. Make sure you justify your answer.

True. A homogeneous system is one whose right-hand side is 0. If no other set of solutions satisfy the criteria, at least 0 will always be a solution. That is because any number multiplied by 0 will equal to 0.

2.1 For each of the problems in 1.1:

I. **Repeat the process of putting the augmented matrix into RREF using multiplication by elementary matrices.**

II. **Determine whether the matrix A is invertible. Give a one-sentence justification for your conclusion.**

III. **If the matrix A is invertible, find A^{-1}. How does it relate to the elementary matrices you found in part (I)?**

a)

I. In this step, we translate the row operations into elementary matrices and then we multiply it with the original matrix. If we do this correctly, this should lead us to the identity matrix.

1. $ElementaryMatrix1 \star OriginalMatrix$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 3 & 5 & 4 \end{pmatrix}$$

2. $ElementaryMatrix2 \star OriginalMatrix$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 3 & 5 & 4 \end{pmatrix}$$

3. $ElementaryMatrix3 \star OriginalMatrix$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 2 & 1 \end{pmatrix}$$

4. $ElementaryMatrix4 \star OriginalMatrix$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

5. $ElementaryMatrix5 * OriginalMatrix$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} * \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

6. $ElementaryMatrix6 * OriginalMatrix$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

7. $ElementaryMatrix7 * OriginalMatrix$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

8. $ElementaryMatrix8 * OriginalMatrix$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9. We have successfully turned the matrix into RREF.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

II. We know that this system has one unique solution so the matrix should be invertible. We can prove this by augmenting the original matrix with an identity matrix and performing row operations until we get to RREF on the left side.

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 1 & 3 & -1 & | & 0 & 1 & 0 \\ 3 & 5 & 4 & | & 0 & 0 & 1 \end{pmatrix} \rightarrow \text{RREF} \rightarrow \begin{pmatrix} 1 & 0 & 0 & | & 17/6 & 1/6 & -2/3 \\ 0 & 1 & 0 & | & -7/6 & 1/6 & 1/3 \\ 0 & 0 & 1 & | & -2/3 & -1/3 & 1/3 \end{pmatrix}$$

This means that the matrix is invertible and its inverse is the augmented part.

III. We have found that the inverse of the matrix is:

$$\begin{pmatrix} 17/6 & 1/6 & -2/3 \\ -7/6 & 1/6 & 1/3 \\ -2/3 & -1/3 & 1/3 \end{pmatrix}$$

After left-multiplying the elementary matrices from the last one to the first (*Elementary Matrix* 8 * *Elementary Matrix* 7...**Elementary Matrix* 1), we get the same matrix as the inverse of the original matrix (See Sage Solutions). Therefore the product of left-multiplication of elementary matrices is the inverse matrix of the original matrix.

b)

I. In this step, we translate the row operations into elementary matrices and then we multiply it with the original matrix. If we do this correctly, this should lead us to the identity matrix.

1. $ElementaryMatrix1 * OriginalMatrix$

$$\color{red}{\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} * \begin{pmatrix} 2 & -4 & 8 \\ 1 & -1 & 1 \\ 3 & -4 & 6 \end{pmatrix}$$

2. $ElementaryMatrix2 * OriginalMatrix$

$$\color{red}{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} * \begin{pmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 3 & -4 & 6 \end{pmatrix}$$

3. $ElementaryMatrix3 * OriginalMatrix$

$$\color{red}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}} * \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 3 & -4 & 6 \end{pmatrix}$$

4. $ElementaryMatrix4 * OriginalMatrix$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} * \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 2 & -6 \end{pmatrix}$$

5. $ElementaryMatrix5 * OriginalMatrix$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

6. We are unable to obtain an identity matrix due to the row of zeros.

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

II. We already proved that this matrix cannot be turned into RREF and create an identity matrix, therefore, we will not be able to obtain its inverse. This aligns with our previous solution that there are no solutions to this system as it is inconsistent.

III. Not applicable.

c)

I.

1. $ElementaryMatrix1 * OriginalMatrix$

$$\begin{pmatrix} 1 & 0 & 0 \\ -6 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 2 & 2 & 6 \\ 6 & 9 & -3 \\ 10 & 14 & 2 \end{pmatrix}$$

2. $ElementaryMatrix2 * OriginalMatrix$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 10 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 & 3 \\ 0 & 3 & -21 \\ 10 & 14 & 2 \end{pmatrix}$$

3. $ElementaryMatrix3 * OriginalMatrix$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 & 3 \\ 0 & 3 & -21 \\ 0 & 4 & 28 \end{pmatrix}$$

4. $ElementaryMatrix4 * OriginalMatrix$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -7 \\ 0 & 4 & 28 \end{pmatrix}$$

5. $ElementaryMatrix5 * OriginalMatrix$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

6. We are unable to obtain an identity matrix due to the row of zeros.

$$\begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

II. For the same reasons as in the previous problem, we see that we are unable to obtain an identity matrix, therefore we cannot get the inverse of the original matrix. Thus the matrix does not have a unique solution like we previously found out.

III. Not applicable.

2.2 Find examples of 2x2 matrices that exhibit the following behavior.

2.2 For this problem, we need to apply the rules of the determinant and invertibility. If the determinant is 0, the matrix is not invertible. So anytime we need a non-invertible matrix, we could change its entries to result in 0 at the end.

A. **A pair of matrices B and C such that $BC = -CB$ (excluding the case $BC = 0$).**

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

B. **A non-zero matrix M so that $M^2 = 0$.**

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

C. **Invertible matrices A and B so that $A + B$ is not invertible.**

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow$ Invertible.

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow$ Invertible.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \text{Non-invertible.}$$

D. **Non-invertible matrices C and D so that $C + D$ is invertible.**

$$\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \rightarrow \text{Non-invertible.}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \text{Non-invertible.}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \text{Invertible.}$$

3.1 For each of the following statements, prove that the statement is true or give a counterexample demonstrating that it is false.

A. **If $det(A) = 0$ then at least one of the cofactors must be 0.**

False.

Counterexample:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow Det(A) = 0$$

The cofactor of the first pivot is 1, while the cofactor of the second pivot is 1 as well, but the determinant is 0.

B. **A matrix whose entries are 0s and 1s has determinant 1, 0 or -1.**

False.

Counterexample:

For the matrix to be different from -1, 0, 1, we could write in a general form.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ h & i & j \end{vmatrix} = a\begin{vmatrix} e & f \\ i & j \end{vmatrix} - b\begin{vmatrix} d & f \\ h & j \end{vmatrix} + c\begin{vmatrix} d & e \\ h & i \end{vmatrix}$$

From this, we know that a and c should be 1, while b should be either -1 or 0, for positive results.

Furthermore, for the a and c to remain positive we need to know that their cofactors produce a positive result, thus,

- $e * j = 1$, which means that both e and j should be 1.
- $d * i = 1$, which means that both d and i should be 1
- $f * i = 0$, which means $f = 0$ because we already know the $i = 1$
- $e * h = 0$, which means $h = 0$, because must be 1.
- Now, we've got all the values. We just replace them.[2]

[2] #heuristics: Appendix

C. **If matrices and are identical except that ,
then .**

False.

Counterexample:

We see that the -2 at is a product of , yet is not equal to .

D. **The determinant of a matrix is the product of its pivots.**

False.

Counterexample:

From 3.1B we see that the product of the pivots is 1, but the determinant is 2.

3.2 A matrix is singular if . Prove the following statements are true for a 2x2 matrix.

A. **If is invertible and is singular, then is singular.**

True.

$$|AB| = |A|*|B|=|A|*0=0$$

This means that if either A or B is 0, their determinant will be 0, which means it is non-invertible.

B. If is singular, then is singular.

True.

Both sides are identical, so if one of them is 0, then the other must also be 0.

3. Deep Dive

1. **Two matrices are *row equivalent* if one matrix can be changed into another matrix by a series of elementary row operations.**
   A. **Are the following matrices row-equivalent?**

      ,

      Yes, because their RREFs are identical. Once we convert M1 to RREF, we can multiply by the elementary matrices of M2 and convert M1 to M2.

   B. **Removing the third column of each matrix gives**

      ,

      Use Sage (or another computational tool) to show whether or not these matrices are row equivalent. Discuss what happens if you remove the fourth column instead: would the resulting matrices be row-equivalent?

C. **Could removing the same column from a pair of equivalent matrices affect row equivalence? If so, give an example. If not, give a short justification.**

   No, because the row operations are not affected by the columns. Even if we remove any same column from both matrices, we could still turn the matrices into the same RREF. An important note is that both matrices should have the same dimensions.

D. **Show that the system of equations corresponding to the augmented matrix**

   has no solutions.

   We see that the last row contains an inconsistency, therefore this system has no solutions.

E. **What happens to the solution set of linear systems if you remove a row from the augmented matrix. What does this tell you about the effect of removing a row in row-equivalence?**

   The system of equations solutions change once a row is moved or modified.

   This means that the row-equivalence will change if a row is removed because the RREF of the particular matrix is going to change. As stated before, RREFs between row-equivalent matrices need to be identical in order for them to be row-equivalent. If one's RREF changes, then they are no longer row-equivalent.

F. **Determine whether the following statements are true or false. If the statement is true, give a 2 to 3 sentence proof. If the statement is false, give a counterexample.**

   a. **Row-equivalent augmented matrices have the same solution set.**

      True. Since the RREF of the matrices needs to be identical, their right-hand side should also be identical for them to be row-equivalent. This means that the matrices with the same solution set, when multiplied by the elementary matrices of each other, produce the other matrix.

b. **Augmented matrices with the same solution set are row-equivalent.**

True. We can see the solution set from the RREF. This sentence is equivalent to saying that *if the RREFs of two matrices are the same, then they are row-equivalent*. We already discussed that once two matrices are turned into RREF, then they can be multiplied with the other's elementary matrix to turn into another matrix. There is one case when this does not hold, which is when both the solution sets are zeros (empty). Then, we cannot get the original matrix back.

2. **In Session 1.2 you explored a simple Closed Leontief Exchange Model with five interdependent industries. In the closed exchange model, no goods or services enter or leave a national economy. However, most nations import and export goods. For this problem, you will develop an open exchange model for the newly formed nation of Minervalia, then generalize the model to an economy with n sectors.**

A. **Suppose Minervalia has 5 sectors: agriculture (A), energy (N), real estate (R), education (E), and entertainment(T). The following table represents the number of units of goods from other sectors used to produce one unit with a given sector. The table also lists the demand for each good from Minervalia's trading partners.**

Let , , , and denote the number of units produced by the agriculture ( ), energy ( ), real estate ( ), education ( ), and entertainment ( ) sectors respectively. Write a system of equations that

defines the number of units each sector should produce to satisfy both internal and external demand.

1. We produce the equations based on the table.

2. We put all the unknown variables on the left side.

B. Rewrite your systems of equations in the form where is a matrix of coefficients. What do and represent in this model?

→ the matrix of coefficients

→ the variables, the unknowns

→ the solutions of the system

C. **Does the system have a solution? In other words, is there a set of production levels that would satisfy both internal and external demand. Is this solution unique?**

Yes. After augmenting with , we reduce it to RREF, then we find an identity matrix on the left side, and the solutions of the system on the right side (we rounded the number to the first three decimal digits).

We have found the solutions of the system for which, if we replace the entries in with the right-hand side of this matrix, both the internal and external needs are met. From this matrix, we can infer that the system requires most production of the units of Energy and the least of Entertainment.

D. **Adjust the parameters in the Demand table to find levels that lead to**

(i) infinitely many solutions

If we change the external demand column to , then we would get infinitely many solutions for the system. In Minervalia context, the system would be able to sustain itself with any combinations of production, as long as they are a closed system and only aspire to fulfill internal needs (Example in [Sage Solutions]).

(ii) no solutions.

If we turn any industries productions to , then we get no solutions for the system. This makes sense in the Minervalia since the other industries are dependent on this industry whose production was modified to . Let's say, if Energy shut down, then Agriculture demands for energy would not be fulfilled. The system would collapse (Example in Sage Solutions).

E. **Now suppose Minervalia has sectors. Let represent the number of units produced by the –th sector. Assume that Sector uses units from Sector**

   I. **Write a system of equations that models internal and external demand in this sector system. Rewrite your systems of equations in the form , where is a matrix of coefficients.**

   II. **Let be the matrix whose entries are the internal production requirements. Express in terms of**

      A. The equations, where , and express external demand:

...

...+

$$\begin{matrix} \vdots & \vdots & \vdots & \vdots \end{matrix}$$

...+

      B. The matrices:

III. Find a condition you can use to quickly determine whether the model has a unique solution.

There are two cases when the system does not have a unique solution:

1. when the external demand for each sector is (each element of is),
2. when the production of one industry is (one row of is a row of zeros).

In any other cases, the system should have a unique solution.

IV. Derive a formula for the vector in terms of and the demand vector that holds when the system has a unique solution.

Since we are dealing with matrices, we cannot factorize for by 1, therefore we use an Identity Matrix, .

We solve this equation for .

4. Sage Solutions

[Here.]

5. Appendix

#algorithms: Developed a step-by-step procedure to explain how to reduce the matrix to RREF, while illustrating it with visuals and explanatory name-coding (Row1, Row2, Row3). I made sure that if those steps are followed, we will always get the same result, by providing the same solution with Sage.

#heuristics: Identified a scenario where we could use previous knowledge of determinants to build new knowledge about matrices. In that case, I used means-ends analysis, by examining what the final goal is arithmetically, and then I went one step backwards from the final goal until I reached the initial stage of finding the matrix needed. This was a good opportunity to apply heuristics as we are facing a new problem and we can make use of what we know to solve it.