Deep Dive - Integration (Location-Based Assignment)

Deep Dives

1. *'Round and 'round we go* (**#integration**)

   Our goal in this part is to develop a framework for computing the volume of a specific class of solids, specifically those that are symmetric and can be constructed as a solid of revolution. We begin by considering a function $f(x)$ in the interval $[a, b]$. As our first example, let's take the function $f(x) = 4 - x^2$, in the interval $[0, 2]$, as shown below in Figure 1. The solid of revolution obtained by revolving $f(x)$ around the x-axis is shown in Figure 2.

   (a) What shape is obtained if we take a vertical slice from the solid of revolution, say at $x = 1$? What is the area of such slice? How do the shape and its area change if we look at a vertical slice through $x = 0$?

   Solution

   Taking the vertical slice of the solid revolution at x=1 would give us the shape of a circle. The area of that circle would be computed by the general circle area formula.

   $$A = \pi \cdot r^2$$

   However, we need to find the radius r, before we proceed. From the given function $f(x) = 4-x^2$, we could substitute for $x = 1$. Let's assume the unit we are working with is meters.

   $$f(1) = 4 - 1^2 = 4 - 1 = 3m$$

   Now, we substitute back to the original area formula. Ultimately, that would be:

   $$A = \pi \cdot 3^2 m^2 = 9\pi m^2$$

   When we look at the vertical slice at $x = 0$, then we still have the shape of a circle, thus the formula does not change, only the radius does.

   $$f(0) = 4 - 0^2 = 4m$$

$$A = \pi \cdot 4^2 m^2 = 16\pi m^2$$

(b) Find a general formula for the area of a vertical slice at any value x. Explain all the terms in your formula.

SOLUTION

As we noticed earlier, the shapes we get from vertical slices in the given interval are circles, thus the general formula contains the circle area formula.

$$A = \pi r^2 = \pi \cdot (4 - x^2)^2$$

where A represents the area, π is a constant, $(4 - x^2)$ represents the function $f(x)$, which also represents the radius r.

(c) We can think of the volume of the solid of revolution as a "stack" of all these slices. We should thus be able to compute its total volume by summing up the volume of each of these slices. In order to do that, we must first assign some thickness to each slice, which we will denote Δx. Write an expression for the volume of:

(i) an individual slice at any value x

(ii) the sum of all slices if we have n total slices.

SOLUTION

i) Given that we need to compute a volume, now we need not only the radius but also the height (thickness) of the radius. That will be denoted by Δx.

$$V_s = A \cdot \Delta x = \pi(4 - x^2)^2 \cdot \Delta x$$

ii) Now that we have the volume for one slice, we can find the volume for the whole solid revolution as that would be the sum of each slice volume, while the slice thickness becomes very small.

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} V_s = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi(4 - x_i^2)^2 \cdot \Delta x$$

This could be written as:

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi \cdot f(x_i)^2 \cdot \Delta x$$

(d) What limit expression would give us the exact volume of the solid?

SOLUTION

The general limit formula is:

$$\lim_{n\to} \sum_{i=1}^{n} (f(x_i^*)\Delta x)$$

We get Δx by the given formula, $\Delta x = \frac{b-a}{n}$, where b and a, are the intervals of the function, and n represents the number of slices.
In our case, $b = 2$ and $a = 0$.

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

For x_i^*, we have another formula,

$$x_i^* = a + i \cdot \Delta x = 0 + i \cdot \frac{2}{n} = \frac{2i}{n}$$

And now we are considering our area function:

$$f(x_i^*) = \pi(4 - x_i^{*2})^2$$

$$f(\frac{2i}{n}) = \pi(4 - (\frac{2i}{n})^2)^2 = \pi(4 - \frac{4i^2}{n^2})^2$$

Now we have every piece to add to the general formula:

$$\lim_{n\to} \sum_{i=1}^{n} \pi \cdot (4 - \frac{4i^2}{n^2})^2 \cdot \frac{2i}{n}$$

(e) Find the integral expression that is equivalent to the limit expression you found above. Justify your reasoning and explain the values of the bounds on the integral.

SOLUTION

The general formula of an signed area integral is:

$$\int_a^b f(x)dx = \lim_{n\to} \sum_{i=1}^{n} (f(x_i^*)\Delta x)$$

From (d), we can see that we have already written the right-side of the equation. To write that into a integral, we should use the same bounds and function:

$$\int_0^2 \pi(4-x^2)^2 dx = \lim_{n\to} \sum_{i=1}^{n} \pi \cdot (4 - \frac{4i^2}{n^2})^2 \cdot \frac{2i}{n}$$

(f) Use this expression to compute the volume of the solid of revolution shown in Figure 2.

SOLUTION

$$\int_0^2 \pi(4-x^2)^2 dx$$

We can take π out of the integral as it is a constant:

$$\pi \int_0^2 (16 + x^4 - 8x^2) dx$$

We find the antiderivative of $(16 + x^4 - 8x^2)$, by applying the reverse power rule (at x^4 and x^2):

$$\pi \int_0^2 (16 + x^4 - 8x^2) dx = \pi \int_0^2 16x + \frac{x^5}{5} - \frac{8x^3}{3}$$

Then, we evaluate the limit at the boundaries, as Fundamental Theorem of Calculus suggests:

$$\pi(16x + \frac{x^5}{5} - \frac{8x^3}{3})|_0^2 = \pi(32 + \frac{32}{5} - \frac{64}{3}) - 0 = \pi \cdot (32 + 6.4 - 21.33) = 17.07 \cdot \pi = 53.59$$

$$V = 53.59m^3$$

(g) Would the formula change if we had revolved around the y-axis instead? Do you expect the volume of the solid of revolution obtained by revolving $f(x)$ around the y-axis to be larger, smaller, or the same as that from part (f)?

SOLUTION

Yes, as our main function $f(x)$, from where we get the radius changes.

$$f(y) = \sqrt{4-y} \to A = \pi(\sqrt{4-y})^2 = \pi(4-y)$$

The bounds of the integral would also be different, in our case $[0, 4]$, as well as the antiderivative would be taken with respect to y.

$$V = \int_0^4 \pi(4 - y)dy$$

By finding the antiderivative (applying reverse power rule) and evaluating it at its bounds (FTC), we get:

$$V = \pi(4y - \frac{y^2}{2})|_0^4 = \pi(16 - \frac{16}{2}) - 0 = 8\pi = 25.12m^3$$

The volume is smaller, when revolved around the y-axis.

(h) **[The Method of Disks]** Generalize: given any function $g(x)$, find the formula of the volume of the solid of revolution obtained by revolving $g(x)$ around the (i) x-axis; (ii) y-axis.

SOLUTION

i) x-axis:

$$V = \int_a^b A(x)dx$$

ii) y-axis:

$$V = \int_t^s A(y)dy$$

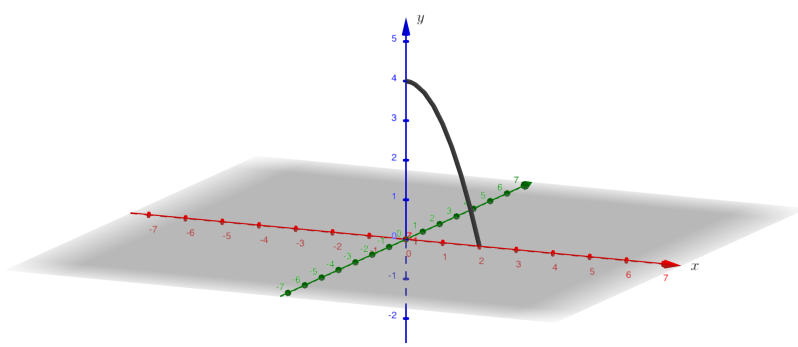

Figure 1: Graph of the function $f(x) = 4 - x^2$ in the interval $[0, 2]$.

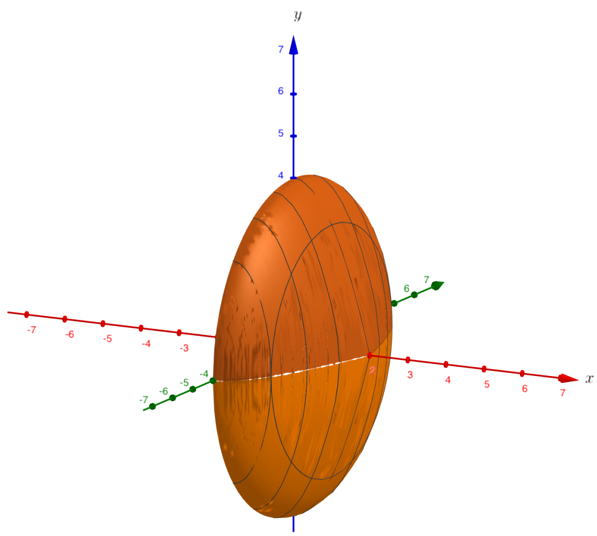

Figure 2: Graph of the solid of revolution obtained by rotating the function $f(x) = 4 - x^2$ in the interval $[0, 2]$ around the x-axis.

2. *Cookie-cutting* (**#integration**)

   In the previous part, you developed the **Method of Disks** for computing the volume of solids of revolution. We will now find an alternative method, the **Method of Shells**. We will consider the same function $f(x) = 4 - x^2$, in the interval $[0, 2]$, as shown in Figure 1, but now, we will first consider the solid of revolution obtained by revolving $f(x)$ around the y-axis is shown in Figure 3.

   (a) Say we take two circular cookie cutter, one of radius $r_1 = 1$ and the other with radius $r_2 = 1 + \Delta r$, and use them to cut out a slice of our solid of revolution. What is the 3-dimensional shape of the slice? What is the surface area of this shape?

   SOLUTION

   The area of this shape would look like cylinder.

The general formula of a cylinder area is:

$$A = 2 \cdot \pi \cdot r \cdot h + 2 \cdot \pi \cdot r^2$$

In our case, $r_1 = 1$ and $r_2 = 1 + \Delta r$:

$$A = 2\pi(1 + 1 + \Delta r)(1 + \Delta r - 1 + h) = 2\pi((2 + \Delta r)(\Delta r + h))$$

(b) Find a general formula for the surface area of the slice obtained by cutting the solid with cookie cutters of radii $r_1 = x$ and $r_2 = x + \Delta x$. Explain all the terms in your formula.

SOLUTION

The inner area would be computed by only taking into consideration r_1:

$$A_{in} = 2\pi x f(x)$$

[1] The outer area would consider r_2:

$$A_o = 2\pi(x + \Delta x) f(x + \Delta x)$$

[2] When $\Delta x \to 0$, the second formula takes the shape of:

$$A_o = 2\pi x f(x)$$

which is identical to the inner area.

(c) We can think of the volume of the solid of revolution as a collection of slices of cookie cutters of many radii. We should thus be able to compute its total volume by summing up the volume of each of these slices. In order to do that, we must first assign some thickness to each slice, which we will denote Δx. Write an expression for the volume of:

(i) an individual slice of radius x;

(ii) the sum of all slices if we have n total slices.

SOLUTION

i) Similarly to Question 1 (c), we will multiply the area and the new variable of thickness, to produce the volume:

$$V = A_{in} \cdot \Delta x = 2\pi f(x) \Delta x$$

[1] A_{in}- Inner Area; x - r_1; $f(x)$ - h

[2] A_i- Outer Area; $x + \Delta x$ - r_2; $f(x + \Delta x)$ - h

ii) Now we calculate the whole volume, again with the same strategy as in Question 1 (c):

$$V = \lim_{n \to \inf} \sum_{i=1}^{n} 2\pi f(x_i) \Delta x$$

(d) What limit expression would give us the exact volume of the solid?

SOLUTION

We could find the integral given our summation:

$$V = \lim_{n \to \inf} \sum_{i=1}^{n} 2\pi f(x_i^*) \Delta x$$

Our boundaries have not changed, thus Δx is equal to $\frac{2-0}{n} = \frac{2}{n}$.
Now, we no longer have x_i, instead we use, $x_i^* = i \cdot \Delta x = \frac{2i}{n}$.

(e) Find the integral expression that is equivalent to the limit expression you found above. Justify your reasoning. Use this expression to compute the volume of the solid of revolution shown in Figure 3. How does your answer compare with problem 1(g)?

SOLUTION

Given part (d), our integral takes the shape of

$$\int_0^2 A_{in} dx$$

. To solve this integral, we begin with taking the constants out:

$$\int_0^2 A_{in} dx = \int_0^2 2\pi x f(x) dx = 2\pi \int_0^2 x \cdot (4 - x^2) dx$$

Then we calculate the antiderivative of $x \cdot (4 - x^2)$:

$$\int_0^2 x \cdot (4 - x^2) dx$$

$$u = (4 - x^2); \frac{du}{dx} = -2x; dx = -\frac{1}{2x} du$$

From this substitution, we get $-\frac{(x^2-4)^2}{4}$, where we apply the FTC:

$$2\pi \cdot (-\frac{(x^2 - 4)^2}{4})|_0^2 = 2\pi \cdot (-\frac{(2^2 - 4)^2}{4} - (-\frac{(0^2 - 4)^2}{4})) = 2\pi \cdot 4 = 8\pi = 25.12m^3$$

This is the exact same answer that we've gotten from Question 1 (g), when we revolved around y-axis.

(f) Would the formula change if we had revolved around the x-axis instead?

Solution

Yes, the formula would change, because we'd take the terms of the function with respect to y. Particularly:

$$\int_s^t A_{in}(y)dy$$

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(g) **[The Method of Shells]** Generalize: given any function $g(x)$, use the Method of Shells to find the formula of the volume of the solid of revolution obtained by revolving $g(x)$ around the (i) y-axis; (ii) x-axis.

Solution

i)

$$V = 2\pi \int_a^b xg(x)dx$$

ii)

$$V = 2\pi \int_s^t yg(y)dy$$

(h) From your experience in these first two problems, which method yielded a simpler calculation? What do you think are the best uses of each method?

Solution

In my opinion, the method of discs yielded a simpler solution, although I see quite a lot of similarities in both. A counterintuitive thing at Shells' Method is taking the functions in terms of x, when we revolve around y, whereas in Discs' method, we would take the functions in terms of x, when we revolve around x. Depending whether the slices of your function make more sense to be circles, or cylinders, one should choose the respective method.

(i) Discuss specific cases of solids of revolution in real life whose volumes could be computed by the methods described here.

Solution

If we sliced a fruit, and we only had that one slice to measure the whole volume of a fruit, we could approximate with both methods, since we can measure the thickness of the slice and its radius (if we have a horizontal slice).

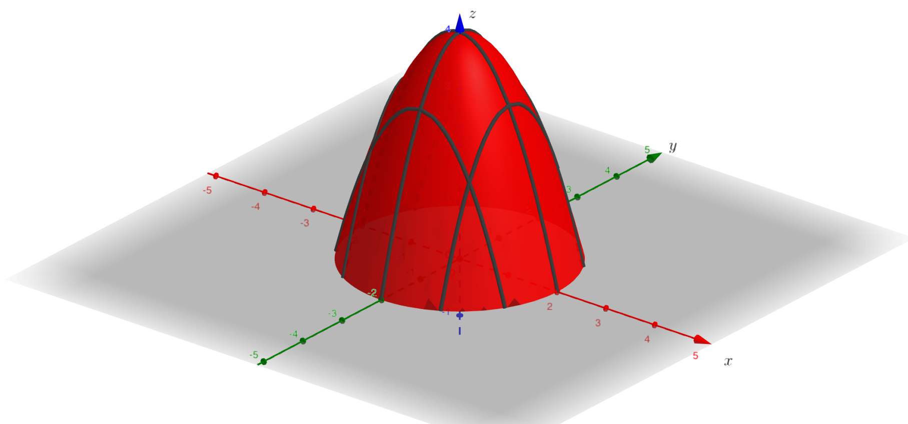


Figure 3: Graph of the solid of revolution obtained by rotating the function $f(x) = 4 - x^2$ in the interval $[0, 2]$ around the y-axis.

3. *Math is sweet!/ Math is beautiful!/ Math is useful! [LBA]* (**#integration**, **#computationaltools**)

   (a) Find an object that represents your current surroundings and can be modeled as a solid of revolution. It could be a pastry (sweet), a piece of art (beautiful), or a tool(useful). Explain how your choice is representative of your location.

   Solution

   "It's Tea Time in London. Bring your teacups and milk-tea." It doesn't happen rarely that people get judged based on their teacup choice in London. Since you're supposed to be drinking tea throughout the day (High Tea, Afternoon Tea, Elevenses), you better have a good teacup. I chose this object, as I associate a lot of my London experience with a teacup, and it is a common thing to speak about in London. I thought, after all, finding volume of a teacup might be "my cup of tea".

(b) Create a single variable function that, when revolved, generates your chosen item. For consistency, we will take the y-axis as the axis of revolution, thus your function should depend only on x, $y = f(x)$. *(Note: both x and y values have dimensions. Make sure to explicitly state their dimensions and check that they effectively model your chosen object.)*

Solution

$$y = f(x) = 2x^2; 0 \leq y \leq 5$$

The upper bound of y is $5cm$, as that's how tall the teacup is.

(c) Plot your function and attach a photo of your object.

Solution

Figure 4: Holding a teacup.

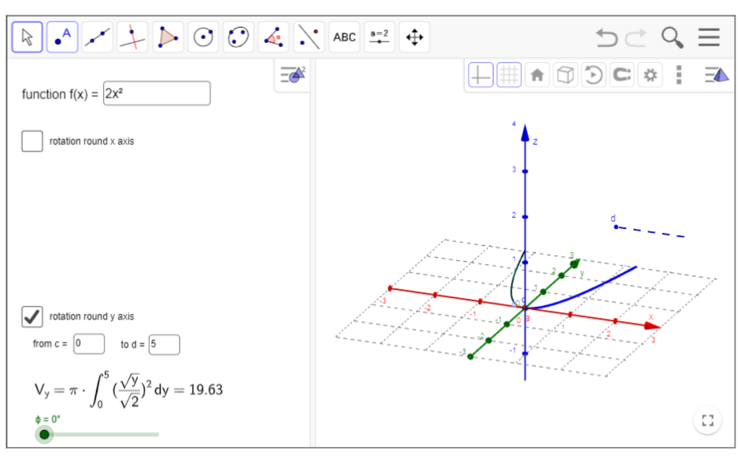


Figure 5: 2D representation of a teacup (Geogebra).

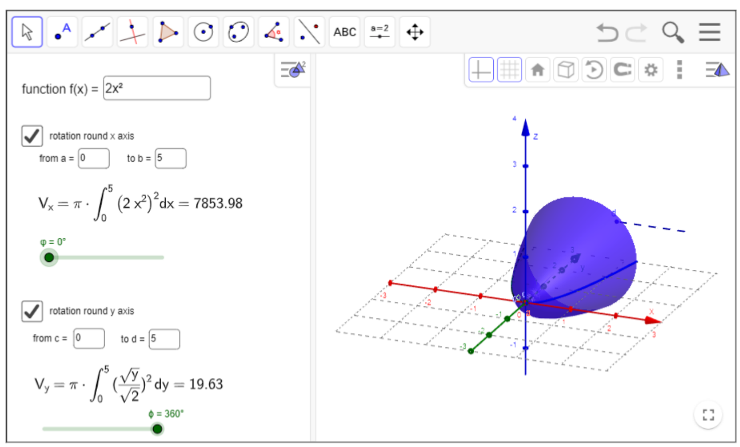


Figure 6: 3D representation of a teacup (Geogebra).

(d) Use either of the frameworks developed above to find an estimate for the volume of your object. Discuss the method that you used, justifying your choice.

Solution

$$f(x) = 2x^2; 0 \leq y \leq 5$$
$$f^{-1}(x) = f(y) = \frac{\sqrt{y}}{\sqrt{2}}$$

Now we can use the method of discs:

$$V = \pi \int_t^s f(y)^2 dy = \int_0^5 (\frac{\sqrt{y}}{\sqrt{2}})^2 dy$$

$$V = \pi \int_0^5 \frac{y}{2} dy$$

We need to find the antiderivative of $\frac{y}{2}$, so we take $\frac{1}{2}$ out of the integrand as a constant:

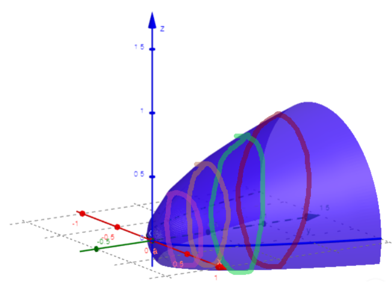
$$V = \pi \cdot \frac{1}{2} \int_0^5 y dy$$

Then we apply the power rule, by considering y^1, and we evaluate the result in the boundaries.

$$V = (\frac{\pi}{2}) \cdot (\frac{y^2}{2}|_0^5) = (\frac{\pi}{2}) \cdot (\frac{25}{2} - 0) = \frac{\pi}{2} \cdot \frac{25}{2} = 19.625 cm^3$$

This means I can drink about $20ml$ tea from my teacup.

(e) Create a visualization of your chosen method.



(f) Estimate the total volume of your object and compare your measurement with your integral calculations. Discuss any discrepancies between the two.

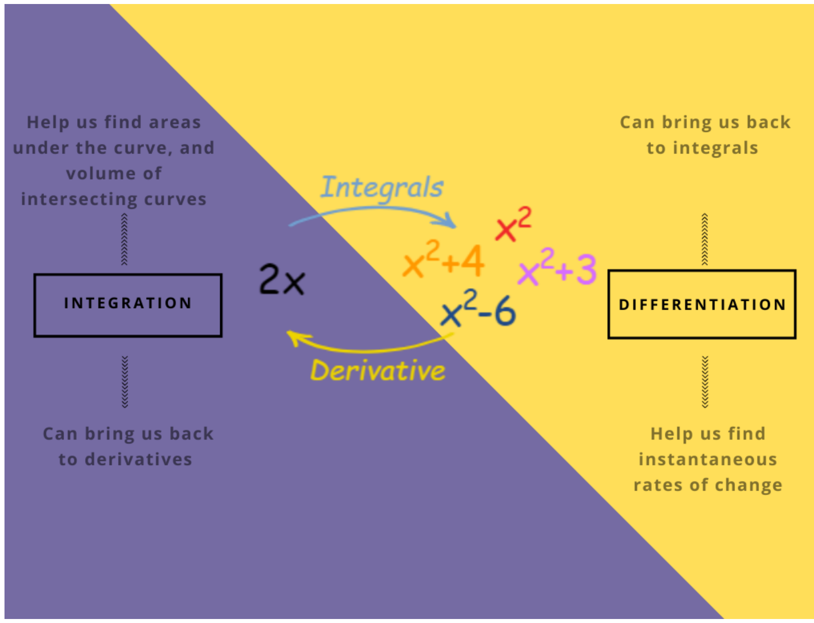
SOLUTION

$$\int_0^5 \int_0^2 2x^2 dx dy = \int_0^5 \frac{16}{3} dy = \frac{80}{5} = 26.666 cm^3$$

In order to compute this volume with double-derivatives, I need to define the area as a function. The function right now gives us two times the area of a square, which is a far estimation of the shape of the cup. However, the volumes are not very far off from one another.

Reflection Questions

1. Create a concept map that relates differentiation and integration. Include at least two applications of each technique on your map.



2. Give two examples of applying HCs to solve a problem. How did you apply the HC? Was the application successful? If yes, why? If not, how could you improve your application
#variables: This HC application could be improved. I already applied it while using multiple variables in appropriate contexts, but I did not categorize them. I think it would have made my explanations richer if I explained the dependent variables, and the nature of the dependence.
#heuristics: I have used means-ends analysis to come up with the teacup function. I had no idea what function would give us a teacup (for instance designers of teacups make these functions first, and then they develop the product); but I was in front of the finished product, and I had to map it to its initial state (the function). I conceptualized the edges, and tried to remember functions that look similar to those. However, I ended up creating a table with the slope at the end, and that successively step-by-step brought me to the more applicable function.

3. How have you applied the science of learning to improve your understanding of this material? Reflect on your application. How could you improve your approach?

> To improve my understanding I have assessed how I learn best, and how I am used to learning in general; and then I have used the generation effect. I have consistently solved problems by hand and in a physical notebook. This has proven particularly effective because one remembers more when they generate notes from new concepts in the way they are used to grasping new concepts. In addition, this is the technique of studying maths I am most familiar with, and sometimes I used my handwritten notes to explain concepts to my friends (elaborative interrogation), which helped me strengthen my understanding.

I have collaborated with Albin Siriniqi during this assignment. We have checked our final results and methods of solving problems.