Skill Builder: Linear Approximation and Related Rates

1. Linear Approximation

The algorithm:

> 💡 Points of tangency

> 💡 Set up the line

> 💡 Evaluate the approximation

> 💡 Relative error

a)

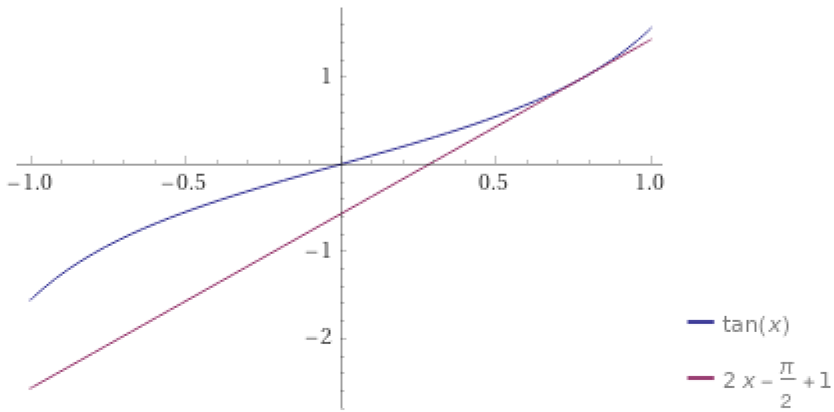

Figure 1. This graph represents the tangent line and its linear approximation in the given range in the problem (Wolfram Alpha).

If we take *a* to be π/4 then we can approximate for π/4. The rationale behind taking π/4 is because this point is close enough to the π/6, and we do know its value of tangent, thus we chose it.

💡 Points of tangency

$x_0 = \frac{\pi}{4}$

$y_0 = tan(\frac{\pi}{4}) = 1$

💡 Set up the line

$m = tan'(\frac{\pi}{4}) = \frac{1}{cos^2\frac{\pi}{4}} = \frac{1}{\frac{1}{2}} = 2$ → lope

$y = m(x - x_0) + y_0$ → line formula.

$y = 2(x - \frac{\pi}{4}) + 1$ → plug-in values.

$L(x) = y = 2x - \frac{\pi}{2} + 1$ → this is the linearization formula for this problem.

💡 Evaluate the approximation

$L(\frac{\pi}{6}) = 2 * \frac{\pi}{6} - \frac{\pi}{2} + 1$ → plug-in values.

$L(\frac{\pi}{6}) = 2 * 0.5233 - 1.57 + 1$

$L(\frac{\pi}{6}) = 1.0466 - 1.57 + 1 = 0.4766$ → the result.

> Relative error

$tan(\frac{\pi}{6}) = 0.5777$ → the actual value calculated by calculator.

$\eta = |\frac{\nu - \nu_{approx.}}{\nu}| = \frac{0.5777 - 0.4766}{0.5777} = 0.175$ → the error.

b)

Now we take *a* to be 0.

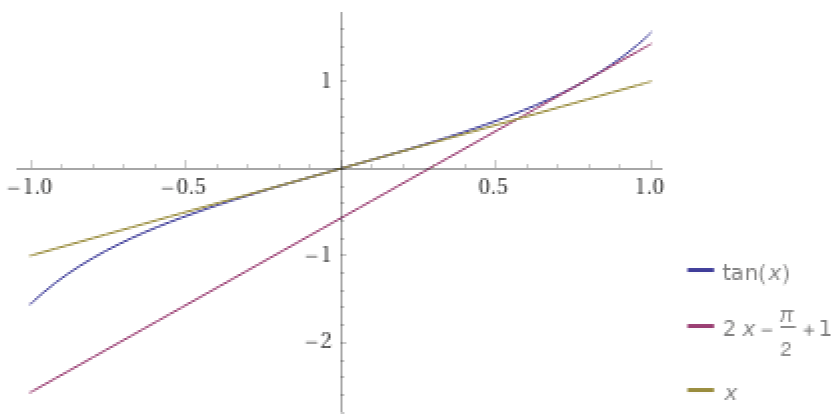

Figure 2. This graph represents the tangent line and its linear and quadratic approximation in the given range in the problem (Wolfram Alpha).

> Point of tangency:

$x_0 = 0$

$y_0 = tan(0) = 0$

> Set up the line:

$Q(x) = tan(0) + tan'(0) * (x - 0) + tan''(0) * \frac{(x-a)^2}{2}$

Find the second derivative:

$tan"(0) = 2sec^2(0)tan(0) = 0$

$Q(x) = 0 + \frac{1}{cos^2(0)} * (x) + 0 * \frac{x^2}{2}$

$Q(x) = x$

> 💡 Evaluate the approximation:

$Q(\frac{\pi}{6}) = \frac{\pi}{6} = 0.5233$

> 💡 Relative Error:

$tan(\frac{\pi}{6}) = 0.5777$

$\eta = |\frac{\nu - \nu_{approx.}}{\nu}| = \frac{0.5777 - 0.5233}{0.5777} = 0.0941$

c)

The cubic equation form:

$g(x_n) = ax_n^3 + bx_n^2 + cx_n + d$

$g'(x_n) = 3ax_n^2 + 2bx_n + c$

$g''(x_n) = 6ax_n + 2b$

$g'''(x_n) = 6a$

From the above equations we find:

$a = \frac{g'''(x_n)}{6}$

$b = \frac{g''(x_n) - g'''(x_n)x_n}{2}$

$c = g'(x_n) - g''(x_n)x_n + \frac{g'''(x_n)}{2}x_n^2$

$d = g(x_n) - g'(x_n)x_n + \frac{g"(x_n)x_n^2}{2} - \frac{g'''(x_n)x_n^3}{6}$

Now we plug them in the original cubic equation form and we get:

$$C(x) = g(x) + g'(x_n)(x - x_n) + \frac{g''(x_n)(x-x_n)^2}{2} + \frac{g'''(x_n)(x-x_n)^3}{6}$$

Another way:

For quadratic approximation, we add a new term for the added accuracy. The term consists of the **second** derivative multiplied by the fraction between **squared *(x-a)*** and ***2!*** . Let's assume we are adding the same term once more. To make it cubic approximation, the following modifications should be made in the newest term:

1. switch from second→ third derivative,
2. switch from **squared *(x-a)*** → **cubed*(x-a)***
3. switch from **2!** → **3!** (I got to know about the factorial from the Taylor series, otherwise changing from second derivative to the third and from *squared (x-a)* to *cubed (x-a)* seemed rather intuitive).

Therefore the formula for cubic approximation is as follows:

$$C(x) = Q(x) + f'''(a) * \frac{(x-a)^3}{6}$$

$$C(x) = f(a) + f'(a) * (x - a) + f''(a) * \frac{(x-a)^2}{2} + f'''(a) * \frac{(x-a)^3}{6}$$

For *a=0,* in *g(x)*:

$$C(x) = g(0) + g'(0) * (x - 0) + g''(0) * \frac{(x-0)^2}{2} + g'''(0) * \frac{(x-0)^3}{6}$$

d)

> 💡 Plug-in values in the formula:

$$C(\frac{\pi}{6}) = tan(0) + tan'(0) * (\frac{\pi}{6} - 0) + f''(0) * \frac{(\frac{\pi}{6}-0)^2}{2} + f'''(0) * \frac{(\frac{\pi}{6}-0)^3}{6}$$

$$C(\frac{\pi}{6}) = Q(x) + f'''(0) * \frac{(\frac{\pi}{6}-0)^3}{6}$$

Find the third derivative:

$$tan'''(0) = 4sec^2(0)tan^2(0) + 2sec^4(0) = 4 * 0 + 2 = 2$$

$$C(\frac{\pi}{6}) = \frac{\pi}{6} + 2 * \frac{(\frac{\pi}{6})^3}{6} = 0.5233 + 0.0477 = 0.571$$

> 💡 Relative error:

$tan(\frac{\pi}{6}) = 0.5777$

$\eta = |\frac{\nu - \nu_{approx.}}{\nu}| = \frac{0.5777 - 0.571}{0.5777} = 0.011$

2. Related Rates:

The algorithm:

> 💡 Set up an equation that uses the given variables.

> 💡 Differentiate the equation in both sides (usually with respect to time, t).

> 💡 Substitute the given values in the derived formula at step 2.

a)

> 💡 Set up an equation that uses the given variables:

Need to find: How fast is the ice decreasing when it is 5 cm thick or $V_i(t)$ at x=5cm?

We know:

diameter = 20 cm → radius = 10 cm

Volume of the sphere: $V = \frac{4}{3} * \pi * r^3$ = $\frac{4}{3} * 3.14 * 10^3 = 4186.66$

Change of ice: $\frac{dV_i}{dt} = -150 \frac{cm^3}{min}$

Volume of the layer:

Here we devise a new formula.

The layer + the sphere should have a radius of (10+x)cm.

Only the volume of the ice should be the volume of the sphere with the ice subtracted by the volume of the sphere without the ice.

Thus, $V_i = \frac{4}{3} * \pi * (r + 10cm)^3 - \frac{4}{3} * \pi * r^3$.

💡 Differentiate the equation in the both sides:

$V_i(t) = (\frac{4}{3} * \pi * (r+10)^3 - \frac{4}{3} * \pi * r^3)(t)$

$\frac{dV_i}{dt} = \frac{d\frac{4}{3}*\pi*(r+10cm)^3}{dt} - \frac{d\frac{4}{3}*\pi*r^3}{dt} = 4\pi(x+10cm)^2\frac{dx}{dt} - 0$

💡 Substitute the given values in the derived formula at step 2.

$-150\frac{cm^3}{min} = 4\pi(5cm + 10cm)^2\frac{dx}{dt}$ //: $4\pi(15cm)^2$

$\frac{-150\frac{cm^3}{min}}{4\pi(15cm)^2} = \frac{dx}{dt}$

$\frac{dx}{dt} = -0.053\frac{cm}{min}$

b)

💡 Set up an equation that uses the given variables.

Need to find:

$\frac{da}{dt}, \frac{db}{dt}, \frac{dc}{dt}$

Given:

$\frac{dh}{dt} = -1\frac{cm}{min}$

$\frac{dP}{dt} = 4\frac{cm}{min}$

$a = b = 5cm$

$c = 8cm$

$P = 2a + c$

$\frac{c^2}{4} + h^2 = a^2$ → Pythagorean Theorem. → h=3cm.

> 💡 Differentiate the equation in both sides (usually with respect to time, t).

$P(t) = (2a + c)(t)$

$\frac{dP}{dt} = 2\frac{da}{dt} + \frac{dc}{dt}$

$c^2(t) = 4(a^2 - h^2)(t)$

$c\frac{dc}{dt} = 4a\frac{da}{dt} - 4h\frac{dh}{dt}$

> 💡 Substitute the given values in the derived formula at step 2.

$8\frac{dc}{dt} = 20\frac{da}{dt} + 12 \ \ // : 8$

$\frac{dc}{dt} = 2.5\frac{da}{dt} + 1.5$

$\frac{dP}{dt} = 2\frac{da}{dt} + 2.5\frac{da}{dt} + 1.5$

$4\frac{cm}{min} - 1.5 = 4.5\frac{da}{dt}$

$2.5\frac{cm}{min} = 4.5\frac{da}{dt}$ //: 4.5

$\frac{da}{dt} = \frac{db}{dt} = 0.5555\frac{cm}{min}$

$\frac{dc}{dt} = 2.5\frac{da}{dt} + 1.5 = 2.5 * 0.555 + 1.5 = 2.8875\frac{cm}{min}$

Reflection:

1. From the previous feedback I have learnt to pay more attention to detail. I have made minute mistakes that created bigger errors at the end. The way I tried to integrate the feedback was by paying more attention this time to the graph plotting as well as arithmetical work, before I continue to the next step. I would like to get more feedback in the form of :

1. Things you need to revisit: x, y, z;
2. Things you know well: a, b, c.

2. #algorithms: I have used or devised algorithms for each of the problems in this assignment. I follow all the steps and made sure to highlight them throughout my problem-solving process. The reason I chose to use this HC is because I want to realize how separate problems are connected to one another by the shape of the problem or the way they are solved.

   #estimation: Throughout the assignment we have used the Linear Approximation and its extensions to estimate about points that we do not know the tangent of. The estimation process is facilitated by differentiation. With the help of relative error we have also estimated the error which gave us possibility to compare the different approximations and their accuracy.

   #levelsofanalysis: In the last problem with the related rates, we analyze the problem from various levels, considering the geometric one, where we derive height from Pythagorean Theorem, as well as the calculus perspective to find the change per time. With the help of multiple levels of analysis we were able to come to derive knowledge from the known variables.