Deep Dive 1

1. **Deep Trouble**

a)

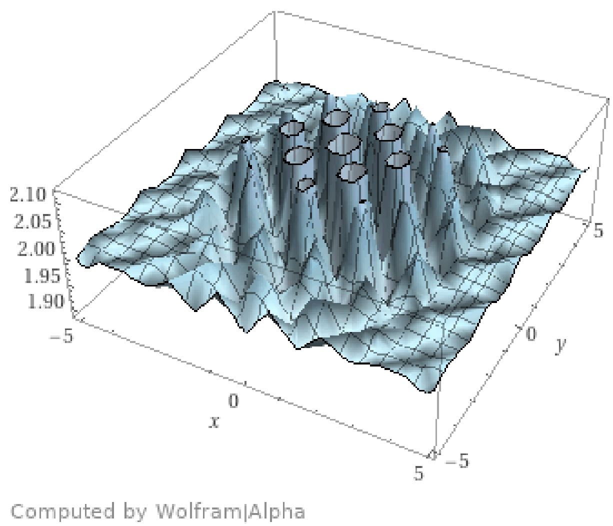

Figure 1. The plot of ocean depth from Deep Trouble Problem (Wolfram Alpha, n.d.)

b)

$d(x, y) = 2 + \frac{cos(4x)(cos3y)}{1+x^2+y^2}$

To analyze the asymptotic behavior of this function in all the directions, we need to plug in very large or very small values for x and y and see how the function behaves. Since in the denominator we use the square of x and y, we are allowed to take any negative value as well. In addition $cos(x) = cos(-x)$, so nothing about changing the sign will affect our equation.

The denominator grows indefinitely big for either x or y tending to $+\infty$ or $-\infty$, or when they both tend to ∞, while the numerator can take values between $[-1, 1]$. That makes the whole fraction $\frac{1}{\infty}$, whose limit is 0. Let's observe some cases of asymptotic behavior:

$\lim_{(x,y)\to(+\infty,+\infty)}(d(x, y)) = 2 + 0 = 2$

$\lim_{(x,y)\to(+\infty,-\infty)}(d(x,y)) = 2 + 0 = 2$
$\lim_{(x,y)\to(-\infty,+\infty)}(d(x,y)) = 2 + 0 = 2$
$\lim_{(x,y)\to(-\infty,-\infty)}(d(x,y)) = 2 + 0 = 2$
$\lim_{(x,y)\to(+\infty,y)}(d(x,y)) = 2 + 0 = 2$
$\lim_{(x,y)\to(-\infty,y)}(d(x,y)) = 2 + 0 = 2$
$\lim_{(x,y)\to(x,+\infty)}(d(x,y)) = 2 + 0 = 2$
$\lim_{(x,y)\to(x,-\infty)}(d(x,y)) = 2 + 0 = 2$

From these examples, we see that no matter which way we go indefinitely far, we will have the depth of 2.

We can generalize this into:

$\lim_{(x,y)\to(\infty,\infty)} = 2$

c)

1. $\lim_{t\to 0^-} \eta(x,y,t) = 0$

   This problem requires us to analyze the wave surface height before the rock has been thrown in the water. In that case, there is no wave surface height, simply because the cause is not there (the rock), thus the effect is not there (wave surface height), so we write the limit of it as 0.

2. $\lim_{t\to\infty} \eta(x,y,t) = 0$

   Do we see the wave surface height of a rock that we've thrown when we were 6 at the pond near our house? No. This is exactly what this equation means. If an infinite amount of time has passed since threw the rock, we do not observe any wave surface height.

3. $\lim_{x\to\infty} \eta(x,y,t) = 0$

   This one may depend more on the weight of the rock. Let's say an asteroid falls infinitely far away from the x-axis, the height of the surface wave may be observed in the current point of reference. However, if any small-sized rock has been thrown infinitely far away from x-axis, we expect to observe no wave surface height. Therefore, the limit is 0.

4. $\lim_{y\to\infty} \eta(x,y,t) = 0$

With the same rationale, as in c.3, the limit remains 0.

d)

$Z(x, y, t) = d(x, y) + \eta(x, y, t)$

1. $\lim_{t\to\infty} Z(x, y, t) = \lim_{t\to\infty} d(x, y) + \lim_{t\to\infty} \eta(x, y, t) = d(x, y) + 0 = d(x, y)$

   We take the limit as time approaches ∞ for both functions that make up $Z(x, y, t)$. Since the rock has been thrown a long time ago, that does not affect the depth of the ocean now. The depth will remain the same $2 + \frac{cos(4x)(cos3y)}{1+x^2+y^2}$, as t is not a relevant variable to $d(x, y)$; in other words, time approaching infinity does not affect the general formula for the depth.

2. $\lim_{x\to\infty} Z(x, y, t) = \lim_{x\to\infty} d(x, y) + \lim_{x\to\infty} \eta(x, y, t) = 2 + 0 = 2$

   In this case, we throw the rock infinitely far away on the left or in the right. We follow the same approach of taking the limit of both functions and we already know from part b and c that as x approaches infinity $d(x, y)$ becomes 2 and $\eta(x, y, z)$ is 0. The sum of the limits for both functions is 2, thus we say that the precise depth is also 2.

3. $\lim_{y\to\infty} Z(x, y, t) = \lim_{y\to\infty} d(x, y) + \lim_{y\to\infty} \eta(x, y, t) = 2 + 0 = 2$

   Similarly, as in the second part of this question, we follow the strategy of calculating the sum of the limits of both functions which is equal to 2. If we throw the rock infinitely far on the North or South, then we observe the precise depth to be 2.

2. **Time Lord**

The wheel takes 7 seconds to complete a full revolution.

The wheel has a radius of 1.

$x(t) = cos(\frac{2\pi}{7}t)$

$y(t) = sin(\frac{2\pi}{7}t)$

a)

The average rate of change of each coordinate is found by $\frac{f(x_1)-f(x_0)}{x_1-x_0}$ or $\frac{f(y_1)-f(y_0)}{y_1-y_0}$.
We need to know the coordinates of x and y, in order to calculate rate of change.

1. First photo

$$t_0 = 0$$

$$x_0(0) = cos(\frac{2\pi}{7}0) = cos(0) = 1$$

$$y_0(0) = sin(\frac{2\pi}{7}0) = sin(0) = 0$$

2. Second photo

$$t_1 = 7$$

$$x_1(7) = cos(\frac{2\pi}{7}7) = cos(2\pi) = 1$$

$$y_1(7) = sin(\frac{2\pi}{7}7) = sin(2\pi) = 0$$

3. Third photo

$$t_2 = 14$$

$$x_2(14) = cos(\frac{2\pi}{7}14) = cos(4\pi) = 1$$

$$y_2(14) = sin(\frac{2\pi}{7}14) = sin(4\pi) = 0$$

4. Fourth photo

$$t_3 = 21$$

$$x_3(21) = cos(\frac{2\pi}{7}21) = cos(6\pi) = 1$$

$$y_3(21) = sin(\frac{2\pi}{7}21) = sin(6\pi) = 0$$

Average rates of change:

$$\frac{x_1-x_0}{t_1-t_0} = \frac{1-1}{7-0} = \frac{0}{7} = 0$$

$$\frac{x_2-x_1}{t_1-t_2} = \frac{1-1}{14-7} = \frac{0}{7} = 0$$

$$\frac{x_3-x_2}{t_3-t_2} = \frac{1-1}{21-14} = \frac{0}{7} = 0$$

$$\frac{y_1-y_0}{t_1-t_0} = \frac{0-0}{7-0} = \frac{0}{7} = 0$$

$$\frac{y_2-y_1}{t_2-t_1} = \frac{0-0}{14-7} = \frac{0}{7} = 0$$

$\frac{y_3 - y_2}{t_3 - t_2} = \frac{0-0}{21-14} = \frac{0}{7} = 0$

The average rate of change for each coordinate is 0. The reason why these results make sense is, because the duration between taking the first and second photos gives the wheel enough time to make a full revolution and be back at exactly the same point. The units of our rate of change are $\frac{output(units)}{input(units)}$ which in our case is $\frac{distance}{time} = \frac{meter}{second}$.

b)

1. First photo

   $t_0 = 0$

   $x_0(0) = cos(\frac{2\pi}{7}0) = cos(0) = 1$

   $y_0(0) = sin(\frac{2\pi}{7}0) = sin(0) = 0$

2. Second photo

   $t_1 = 6$

   $x_1(6) = cos(\frac{2\pi}{7}6) = cos(\frac{12\pi}{7}) = 0.623$

   $y_1(6) = sin(\frac{2\pi}{6}7) = sin(\frac{12\pi}{6}) = -0.781$

3. Third photo

   $t_2 = 12$

   $x_2(12) = cos(\frac{2\pi}{7}12) = cos(\frac{24\pi}{7}) = -0.222$

   $y_2(12) = sin(\frac{2\pi}{7}12) = sin(\frac{24\pi}{7}) = -0.974$

4. Fourth photo

   $t_3 = 18$

   $x_3(18) = cos(\frac{2\pi}{7}18) = cos(\frac{36\pi}{7}) = -0.9$

   $y_3(18) = sin(\frac{2\pi}{7}18) = sin(\frac{36\pi}{7}) = -0.433$

Average rates of change:

$\frac{x_1 - x_0}{t_1 - t_0} = \frac{0.623-1}{6-0} = \frac{-0.377}{6} = -0.062$

$\frac{x_2-x_1}{t_1-t_2} = \frac{-0.222-0.6233}{12-6} = \frac{-0.845}{6} = -0.14$

$\frac{x_3-x_2}{t_3-t_2} = \frac{-0.9+0.222}{18-12} = \frac{-0.678}{6} = -0.113$

$\frac{y_1-y_0}{t_1-t_0} = \frac{-0.781-0}{6-0} = \frac{-0.781}{6} = -0.13$

$\frac{y_2-y_1}{t_2-t_1} = \frac{-0.974+0.781}{12-6} = \frac{-0.193}{6} = -0.032$

$\frac{y_3-y_2}{t_3-t_2} = \frac{-0.433+0.974}{18-12} = \frac{0.541}{6} = 0.09$

So, the average rate of change for y is also 0.

c)

x at $t_0 = 0$ → Instantaneous rate of change:

We need to take the derivative of the formula of x, to find the instantaneous rate of change.

$\frac{d}{dt}x = \frac{d}{dt}(cos(\frac{2\pi}{7}t))$ →

2. apply the Chain Rule → $\frac{d}{dt}cos(\frac{2\pi}{7}t) * \frac{d}{dt}(\frac{2\pi}{7}t)$ →

3. use $\frac{2\pi}{7}$ as a constant in the second term →

$-sin(\frac{2\pi}{7}t) * (\frac{2\pi}{7} * \frac{d}{dt}(t)) = -sin(\frac{2\pi}{7}t) * (\frac{2\pi}{7} * 1) = -sin(\frac{2\pi}{7}t) * \frac{2\pi}{7}$ →

4. substitute for $t = 0$ → $-sin(\frac{2\pi}{7}0) * \frac{2\pi}{7} = -sin(0) * \frac{2\pi}{7} = 0$

In part b, at the change between t_1 and t_0 the average rate of change is -0.062, whereas here we get the instantenous rate of change to be 0, at exactly t_0.

y at $t_0 = 0$ → Instantaneous rate of change:

$\frac{d}{dt}y = \frac{d}{dt}(sin(\frac{2\pi}{7}t))$ →

2. apply the Chain Rule → $\frac{d}{dt}sin(\frac{2\pi}{7}t) * \frac{d}{dt}(\frac{2\pi}{7}t)$ →

3. use $\frac{2\pi}{7}$ as a constant in the second term →

$cos(\frac{2\pi}{7}t) * (\frac{2\pi}{7} * \frac{d}{dt}(t)) = cos(\frac{2\pi}{7}t) * (\frac{2\pi}{7} * 1) = cos(\frac{2\pi}{7}t) * \frac{2\pi}{7}$ →

4. substitute for $t = 0$ → $cos(\frac{2\pi}{7}0) * \frac{2\pi}{7} = cos(0) * \frac{2\pi}{7} = 1 * \frac{2\pi}{7} = \frac{2\pi}{7} = 0.897$

In part b, at the change between t_1 and t_0 we have the average rate of change -0.13, whereas the instantaneous rate of change at exactly $t = 0$ is 0.897.

If we look at the photos, we would feel like the wheel is moving much slower, and in the opposite direction. That is because we get to see the wheel at only some points in time, but its motion is continuous. As we saw from the algebra, the instantaneous rates are different from the average rates of change, and what we are observing in our photos is a collection of instantaneous shots that are divided by a period of time (6 or 7 seconds).

d)

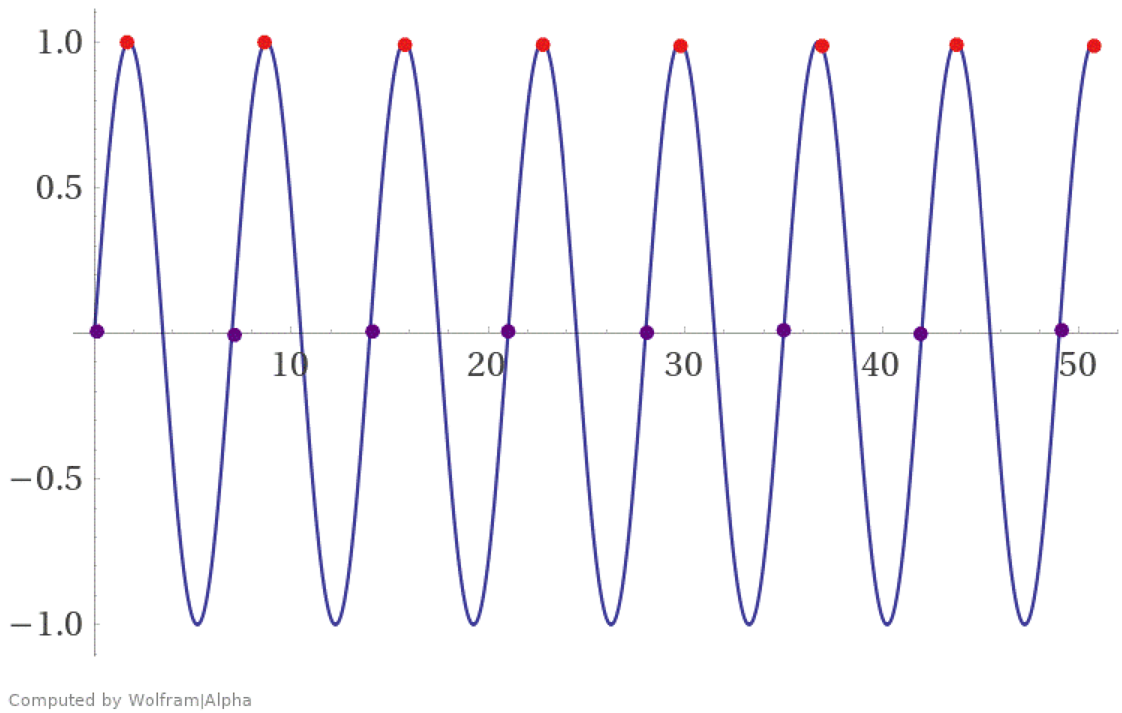

Figure 2. This graph represents the x values with the red dots and the y values with the purple dots, over the range of 50 seconds (Wolfram Alpha, n.d.).

If you connect the red dots in this graph you see that the function from that line would be constant, and it would be exactly 1 distance unit away from the x-axis. While if you connect the purple dots of y coordinate, we will again observe another constant function with 0 distance unit from the y-axis. Those two functions relate to part a, since we found the values of x for four shots to be 1, while the y value was 0. We are obtaining exactly the same results from the graph.

Reflection questions:

1. Good use:

   #context: Although the work was math-heavy, I tried to interpret and situate the solutions in the context given for the problem. This makes the math seem more applicable in the real life situations.

   Bad use:

   #dataviz: My visualizations lack legends and axes labels, although they are supplied with elaborate descriptions. I could improve this application by adding the mentioned features.

2. I have used spaced practice, as one of the components of science of learning. Instead of cramming all the material at once, I have done exercises in Deep Dive one per day, with two days interval in between. This has helped me come back with fresh eyes and spot mistakes that I hadn't notice in the first time.

3. I tend to forget the derivatives of specific values. I believe that with more practice, it will become easier to know the derivative without checking my bank of common derivatives solutions. However, I feel like having notes of the most common derivatives has proven to be helpful in terms of memorizing some of them. The good thing is that I can always use limits to compute the derivative, in case if my notes are not present.