CS111A Fall 2020

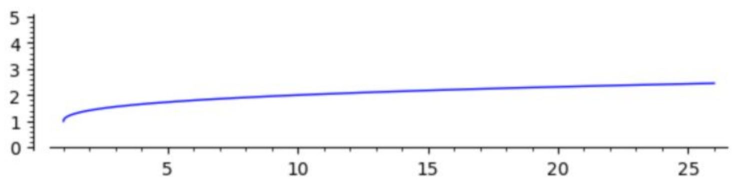
Final Deep Dive

Problem Statements

1 *High Dimensional Chains* (**#differentiation**, **#computationaltools**)

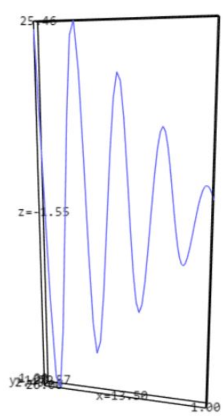
(a) Suppose that a particle is moving along the curve $x(t) = t^2 + 1$, $y(t) = \sqrt{t+1}$ in the plane. What do the variables x, y, and t represent? Make a parametric plot of the curve in $\mathbb{R}^2$ for $0 < t < 5$.

Answer These variables represent:
$x \rightarrow$ moving left and right
$y \rightarrow$ moving up and down
$t \rightarrow$ time
$x(t) = t^2 + 1$; $y(t) = \sqrt{t+1}$



(b) Suppose that the temperature of the particle at the point (x, y) is given by $T(x, y) = (x + y^2)\sin(x)$. Make a parametric plot of the temperature of the particle as it follows its path in $\mathbb{R}^2$.

Answer $T(x, y) = (x + y)^2 \cdot sin(x)$



(c) Compute $\dfrac{dT}{dt}$ at $t = 3$. Interpret what this quantity means in the context of this problem.

Answer $\frac{dT}{dt}$ at $t = 3$
To compute $\frac{dT}{dt}$, we would firstly want to substitute x and y from (a) to get the expression only with the variable t. We get the following expression:

$$T = ((t^2 + 1) + \sqrt{(t + 1)^2}) \cdot sin(t^2 + 1)$$

After simplification we get:

$$T = (t^2 + t + 2) \cdot sin(t^2 + 1)$$

Now we can find the derivative:

$$\frac{dT}{dt}(t^2 + t + 2) \cdot sin(t^2 + 1)$$

By applying the *Chain Rule* we get:

$$\frac{d}{dt}(t^2 + t + 2) \cdot sin(t^2 + 1) + \frac{d}{dt}sin(t^2 + 1) \cdot (t^2 + t + 2)$$

Now we can find the individual derivatives by the *Sum Rule* and *Chain Rule*:

$$\frac{d}{dt}(t^2 + t + 2) = \frac{d}{dt}t^2 + \frac{d}{dt}t + \frac{d}{dt}2 = 2t + 1 + 0 = 2t + 1$$

$$\frac{d}{dt}sin(t^2 + 1) = cos(t^2 + 1)\frac{d}{dt}(t^2 + 1) = cos(t^2 + 1) \cdot 2t$$

Plugging in the derivatives we found, we get:

$$(2t + 1) \cdot sin(t^2 + 1) + cos(t^2 + 1) \cdot 2t(t^2 + t + 2)$$

Now, to find the temperature at time 3, we plug-in 3 for each t in our expression:

$$(2\cdot 3+1)\cdot sin(3^2+1)+[2\cdot 3(3^2+3+2)]\cdot cos(3^2+1) = 7sin(10)+84cos(10) \approx -74.29015$$

In our context, this means that at time 3, the temperature is changing at approximately -74.29015 units, or in other words, it is decreasing.

(d) When you have a differentiable function $f(x, y)$ where the variables x and y are themselves differentiable functions of a single variable t,

$$\frac{d}{dt}\left(f(x(t), y(t))\right) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Show that this chain rule holds for $T(t)$ at $t = 1$.

Answer To show that the *Chain Rule* holds in the given context, we have to modify the following:

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The x sides becomes: [1]

$$\frac{\partial t}{\partial x}((x + y^2) \cdot sin(x)) \cdot \frac{dx}{dt}(t^2 + 1) = ((x + y^2) \cdot cos(x) + sin(x)) \cdot 2t)$$

The y side becomes:[2]

$$\frac{\partial t}{\partial y}((x + y^2) \cdot sin(x)) \cdot \frac{dy}{dt}(\sqrt{t + 1}) = 2y \cdot sin(x) \cdot \frac{1}{2\sqrt{t + 1}}$$

Thus, we get:

$$\frac{d}{dt}(f(x(t), y(t))) = ((x + y^2) \cdot cos(x) + sin(x)) \cdot 2t + 2y \cdot sin(x) \cdot \frac{1}{2\sqrt{t + 1}}$$

We simplify to get rid of xs and ys by plugging in their values:

$$((t^2 + 1 + (\sqrt{(t + 1)^2} \cdot cos(t^2 + 1) + sin(t^2 + 1)) \cdot 2t + 2\sqrt{t + 1} \cdot sin(t^2 + 1) \cdot \frac{1}{2\sqrt{t + 1}}$$

To solve this, we plug-in $t = 3$ in the expression:

$$((3^2 + 1 + (\sqrt{(3 + 1)^2} \cdot cos(3^2 + 1) + sin(3^2 + 1)) \cdot 2 \cdot 3 + 2\sqrt{3 + 1} \cdot sin(3^2 + 1) \cdot \frac{1}{2\sqrt{3 + 1}}$$

Now we can calculate this using Sage (See Appendix), where we get:

$$((10 + 4) \cdot cos(10) + sin(10)) \cdot 6 + 2 \cdot \sqrt{4} \cdot sin(10) \cdot (0.25) \approx -74.29015$$

We have gotten the exact same answer, as in the calculation in part (c), thus we have proven that the *Chain Rule* holds for $T(t)$ at $t = 3$.

[1] Apply Power Rule

[2] Apply Power Rule

(e) Now suppose that $z = f(x, y)$ is a differentiable function of x and y, and both $x = g(u, v)$ and $y = h(u, v)$ are differentiable functions of u and v. Generalize the rule from part (d) to find expressions for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Answer

$$\frac{dz}{du}(f(x(u,v),y(u,v))$$

$$\frac{dz}{du} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du}$$

$$\frac{dz}{dv} = \frac{\partial z}{\partial x}\frac{dx}{dv} + \frac{\partial z}{\partial y}\frac{dy}{dv}$$

Therefore:

$$\frac{dz}{du}(f(x(u,v),y(u,v)) = (\frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du}) + (\frac{\partial z}{\partial x}\frac{dx}{dv} + \frac{\partial z}{\partial y}\frac{dy}{dv})$$

(f) Suppose that f is a differentiable function of a single variable. Define $z = g(x, y) = xyf\left(\frac{x+y}{xy}\right)$. Show that z satisfies the *partial differential equation* of the form

$$x^2\frac{\partial z}{\partial x} - y^2\frac{\partial z}{\partial y} = G(x, y)z$$

and find the function $G(x, y)$.

Answer First of all we need to find the partial derivatives of the function z with respect to x and y:

$$\frac{\partial z}{\partial x}(x \cdot y \cdot f(\frac{x+y}{x \cdot y})) = y \cdot (f(\frac{x+y}{x \cdot y}) - (\frac{f'\frac{x+y}{x \cdot y}}{x})$$

$$\frac{\partial z}{\partial y}(x \cdot y \cdot f(\frac{x+y}{x \cdot y})) = x \cdot (f(\frac{x+y}{x \cdot y}) - (\frac{f'\frac{x+y}{x \cdot y}}{y})$$

Now, we can rewrite the given equation, using partial derivatives we found:

$$x^2(y(f(\frac{x+y}{x \cdot y}) - (\frac{f'\frac{x+y}{x \cdot y}}{x}))) - y^2(x(f(\frac{x+y}{x \cdot y}) - (\frac{f'\frac{x+y}{x \cdot y}}{y}))) = G(x,y)z$$

We replace z with the equation provided in the problem description:

$$x^2(y(f(\frac{x+y}{x \cdot y}) - (\frac{f'\frac{x+y}{x \cdot y}}{x}))) - y^2(x(f(\frac{x+y}{x \cdot y}) - (\frac{f'\frac{x+y}{x \cdot y}}{y}))) = G(x,y) \cdot x \cdot y \cdot f(\frac{x+y}{x \cdot y})$$

We divide both sides with $x \cdot y \cdot f(\frac{x+y}{x \cdot y})$ to get $G(x,y)$:

$$G(x,y) = \frac{x^2(y(f(\frac{x+y}{x \cdot y}) - (\frac{f' \frac{x+y}{x \cdot y}}{x}))) - y^2(x(f(\frac{x+y}{x \cdot y}) - (\frac{f' \frac{x+y}{x \cdot y}}{y})))}{x \cdot y \cdot f(\frac{x+y}{x \cdot y})}$$

If we use the *Chain Rule* for $\frac{\partial f}{\partial x}$, we get:

$$f'(\frac{x+y}{x \cdot y})\frac{\partial}{\partial x}(\frac{x+y}{x \cdot y}) = -f'(\frac{x+y}{x \cdot y})\frac{1}{x^2}$$

For $\frac{\partial f}{\partial y}$ we get:

$$f'(\frac{x+y}{x \cdot y})\frac{\partial}{\partial y}(\frac{x+y}{x \cdot y}) = -f'(\frac{x+y}{x \cdot y})\frac{1}{y^2}$$

Now, we can plug these values back in our patial derivative expressions:

$$\frac{\partial z}{\partial x}(x \cdot y \cdot f(\frac{x+y}{x \cdot y})) = y \cdot f(\frac{x+y}{x \cdot y}) - f'(\frac{x+y}{x \cdot y})\frac{y}{x}$$

$$\frac{\partial z}{\partial y}(x \cdot y \cdot f(\frac{x+y}{x \cdot y})) = x \cdot f(\frac{x+y}{x \cdot y}) - f'(\frac{x+y}{x \cdot y})\frac{x}{y}$$

Then, we can use these expressions for $G(x,y)$ to check if z satisfies the partial differential equation:

$$x^2\frac{\partial z}{\partial x} - y^2\frac{\partial z}{\partial y} = x^2 \cdot y \cdot f(\frac{x+y}{x \cdot y}) - x \cdot y \cdot f'(\frac{x+y}{x \cdot y}) - x \cdot y^2 \cdot f(\frac{x+y}{x \cdot y}) + x \cdot y \cdot f'(\frac{x+y}{x \cdot y}) = x \cdot y \cdot f(\frac{x+y}{x \cdot y}) \cdot (x-y)$$

And since, the answer we've gotten is equal to the value of z, provided in the prompt, we know that z satisfies the differential equation.

2 *Mixing Partials* (**#limitscontinuity**, **#differentiation**)

[Adapted from Marsden et.al. *Basic Multivariable Calculus*]

In many cases you can take the mixed second partial derivative in either order and get the same function. This is a result of Clairaut's Theorem (aka. the Equality of Mixed Partials): If $f(x,y)$ has continuous second partial derivatives, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

(a) Show that Clairaut's Theorem holds for the following examples:

(i) $C(x, y) = \cos(x^2y^2)$

Answer
First, take the derivative with respect to y, second with respect to x:

$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

First, take the derivative with respect to x, second with respect to y:

$$\frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

First mixed partial f_{xy}:

$$\frac{\partial^2 f}{\partial x \partial y}(cos(x^2y^2)) = \frac{\partial cos(u)}{\partial u}\frac{\partial u}{\partial x}$$

[3] We can use the $u-$substitution here as well to make the calculation easier:

$$u = x^2y^2$$

$$\frac{\partial}{\partial y}(cos(u)) = -sin(u)$$

When expanded, we get:

$$-sin(x^2y^2)(\frac{\partial}{\partial y}(x^2y^2)$$

Then, we find the partial derivative and treat x as a constant:

$$\frac{\partial}{\partial y}(x^2y^2) = x^2\frac{\partial}{\partial y}y^2 = 2x^2y$$

[4] Bringing everything together we get:

$$-sin(x^2y^2) \cdot 2x^2y$$

Now, we proceed to calculate the partial derivative with respect to x:

$$\frac{\partial}{\partial x}(-sin(x^2y^2) \cdot 2x^2y)$$

[3]Apply Chain Rule
[4]We firstly took the x out as a constant, and then applied the Power Rule and simplified.

Since y is a constant, we can take it out:

$$-2y \cdot \frac{\partial}{\partial x}(sin(x^2y^2)x^2)$$

Then we can apply the *Product Rule* [5] where $f = sin(x^2y^2)$ and $g = x^2$:

$$-2y(\frac{\partial}{\partial x}(sin(x^2y^2))x^2 + sin(x^2y^2)\frac{\partial}{\partial x}(x^2))$$

Separate partial derivatives are:

$$\frac{\partial}{\partial x}(sin(x^2y^2)) = cos(x^2y^2) \cdot 2y^2x$$

$$\frac{\partial}{\partial x}x^2 = 2x$$

Then the expression becomes:

$$-2y \cdot (cos(x^2y^2) \cdot 2y^2xx^2 + 2x \cdot sin(x^2y^2)$$

Simplify to get:

$$-2y \cdot (2x^3y^2 \cdot cos(x^2y^2) + 2x \cdot sin(x^2y^2))$$

Second mixed partial f_{yx}:
Firstly we take the partial derivative with respect to x:

$$\frac{\partial}{\partial x}(cos(x^2y^2))$$

We apply *Chain Rule*:

$$-sin(x^2y^2)\frac{\partial}{\partial x}(x^2y^2)$$

$$\frac{\partial}{\partial x}(x^2y^2) = 2y^2x$$

$$= -sin(x^2y^2) \cdot 2y^2x$$

Then, we take the partial with respect to y:

$$\frac{\partial}{\partial y}(-sin(x^2y^2) \cdot 2y^2)x)$$

We treat x as a constant and take it out:

$$-2x\frac{\partial}{\partial y}(sin(x^2y^2)y^2)$$

[5] $(\mathrm{f}{\cdot}g)' = f' \cdot g + f \cdot g'$

Applying the *Product Rule*:

$$-2x(\frac{\partial}{\partial y}(sin(x^2y^2)y^2) + \frac{\partial}{\partial y}(y^2)(sin(x^2y^2))$$

$$\frac{\partial}{\partial y}(sin(x^2y^2)) = cos(x^2y^2)2x^2y$$

$$-2x(2x^2y^3cos(x^2y^2) + 2y \cdot sin(x^2y^2))$$

Now, we can compare $\frac{\partial^2 C}{\partial x \partial y}$ and $\frac{\partial^2 C}{\partial y \partial x}$:

$$-2y \cdot (2x^3y^2 \cdot cos(x^2y^2) + 2x \cdot sin(x^2y^2)) = -2x(2x^2y^3cos(x^2y^2) + 2y \cdot sin(x^2y^2))$$

We see that the expressions are equal, thus we can conclude that Clairaut's Theorem holds.

(ii) $r(\theta, t) = \dfrac{1}{\sin^2(\theta) + e^{-t}}$

Answer[6] For this problem, we repeat the same steps:
$\frac{\partial^2 r}{\partial \theta \partial t}$ First we take the derivative with respect to t and second with respect to θ
$\frac{\partial^2 r}{\partial t \partial \theta}$ First we take the derivative with respect to θ and second with respect to t

$$\frac{\partial^2 r}{\partial \theta \partial t}\frac{1}{\sin^2(\theta) + e^{-t}} = \frac{\partial^2 r}{\partial t \partial \theta}\frac{1}{\sin^2(\theta) + e^{-t}} = \frac{4cos(\theta)e^{-t}sin(\theta)}{(sin(\theta)^2 + e^{-t})^3}$$

(b) Let

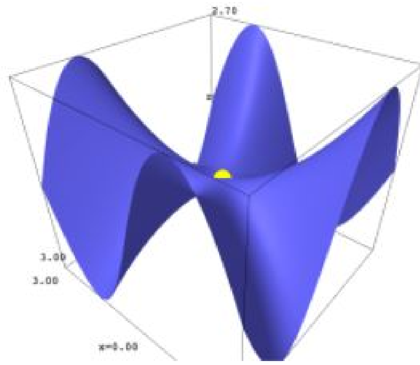
$$f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

(i) Plot $f(x, y)$. Where is f continuous? Where is f differentiable? Explain your reasoning.

Answer The function is continuous for all values of x and y. It is continuous for as long as the denominator is not 0. Since $f(x, y)$ is continuous along

[6] Appedix 2a(ii)

all values of x and y, it means that it is also differentiable along all those values.



(ii) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $(x, y) \neq (0, 0)$.

Answer For this we are treating y as a constant and taking it out:

$$y\frac{\partial}{\partial x}(\frac{x(x^2-y^2)}{x^2+y^2})$$

Applying the *Quotient Rule*:

$$y\frac{\frac{\partial}{\partial x}(x(x^2-y^2))(x^2+y^2)-\frac{\partial}{\partial x}(x^2+y^2)x(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x}(x(x^2-y^2)) = 3x^2-y^2$$

$$\frac{\partial}{\partial x}(x^2+y^2) = 2x$$

$$y\frac{(3x^2-y^2)(x^2+y^2)-2xx(x^2-y^2)}{(x^2+y^2)^2}$$

Simplify to get:

$$\frac{y(x^4+4x^2y^2-y^4}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(\frac{xy(x^2-y^2}{x^2+y^2})$$

Treat x as a constant and take it out:

$$x\frac{\partial}{\partial y}(\frac{y(x^2-y^2)}{x^2+y^2})$$

Apply *Quotient Rule*:

$$x\frac{\frac{\partial}{\partial y}(y(x^2-y^2))(x^2-y^2)-\frac{\partial}{\partial y}(x^2+y^2)y(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}(y(x^2-y^2))=x^2-3y^2$$

$$\frac{\partial}{\partial y}(x^2+y^2)=2y$$

$$x\frac{(x^2-3y^2)(x^2+y^2)-2yy(x^2-y^2)}{(x^2+y^2)^2}$$

Simplifying:

$$\frac{x(-y^4-4x^2y^2+x^4)}{(x^2+y^2)^2}$$

(iii) Show that $\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial y}(0,0)=0$. *(Hint: Use the limit definition of a partial derivative.)*

Answer Since $f(0,0)=0$, we can use this to get the h value:

$$f(h,0)=\frac{h\cdot 0(h^2-0^2)}{h^2+0}=\frac{0}{h^2}$$

Then, if we use h to calculate the partial derivative with respect to x we get:[7]

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{0}{h^2}-0}{h}=0$$

Now, we can repeat the same steps with y:

$$f(0,h)=\frac{0\cdot h(0^2-h^2)}{0^2+h^2}=\frac{0}{h^2}=0$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{0}{h^2}-0}{h}=0$$

Therefore [8],

$$\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial y}(0,0)=0$$

[7]The value we receive is 0and not undefined due to h being a very small number that's close to 0, but is never reaching 0.

[8]Appendix 2b(iii)

(iv) Using the last two parts, write expressions for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ as piecewise functions.

Answer

$$\frac{\partial f}{\partial x}\{\frac{y(x^4+4x^2y^2-y^4}{(x^2+y^2)^2}, (x,y) \neq (0,0); 0(x,y) = (0,0)\}$$

$$\frac{\partial f}{\partial y}\{\frac{x(-y^4-4x^2y^2+x^4}{(x^2+y^2)^2}, (x,y) \neq (0,0); 0(x,y) = (0,0)\}$$

(v) Compute the two mixed partials $\frac{\partial^2 f}{\partial x \partial y}(0,0)$ and $\frac{\partial^2 f}{\partial y \partial x}(0,0)$. Are the mixed partials equal at the point $(0,0)$? If not, why not?

Answer We can use the limit definition of the derivative to calculate the mixed partial using Sage[9].
The Clairaut's Theorem fails in this case because partials are not continuous. The output for f_{xy} is -1, and for f_{yx} is 1, which means that both mixed partials are defined, but they are not equal since the mixed partials are discontinuous at their points of origin $(0,0)$.

3 *Taking it up a notch* (**#integration**, **#computationaltools**)

Just like we went from a single to a double integral, we can add another dimension to create a *triple integral.* One (hard-to-visualize) interpretation of a triple integral is the hyper-volume of a 4-dimensional region. We can interpret triple integrals in other ways depending on the context of the problem.

(a) Give a qualitative interpretation of $\iiint_E f dV$ for each of the following descriptions of f.

(i) $f(x,y,z) = 1$ for all points in a 3-dimensional region E. (Hint: Recall the meaning of the double integral of $f(x,y) = 1$).

Answer The triple integral of the 1 produces the hypervolume of the 3-dimensional region, E, which is defined by $f(x,y,z) = 1$.

(ii) $f(x,y,z)$ is the mass density at point (x,y,z) of a solid object occupying a region E.

[9] Appendix b(v)

Answer The integration of mass density of an object produces the mass of the object. The triple integral of the density of the object occupying the region E, is the mass of the same object.

(iii) $f(x, y, z)$ is the joint probability density function for three random variables.

Answer The integration of a joint probability density function gives a joint cumulative density function, which finds the intersecting hypervolume between probabilities of x, y and z after/before a certain value in the range (i.e. $P(X > x); P(Y > y); P(Z > z)$).

(b) Consider three (non-negative) random variables, X, Y, and Z. Let $f(x, y, z) = k$ inside the following region E, and 0 elsewhere. The region E is defined as the region under the plane $z = 4 - 2x - y$ and in the first octant ($x \geq 0, y \geq 0$, and $z \geq 0$). What value must k take such that $f(x, y, z)$ satisfies all of the conditions to be a probability density function?

Answer The conditions of being a density function is to have its triple integral equal to 1, because the triple integral is going to represent the cumulative density function, which has a volume of 1. To construct the integrals we use the constraints as boundaries. From the plane $z = 4 - 2x - y$, we could substitute the values of z in the main function:
For z, we already know from the plane formula.

$$f(x, y, 4 - 2x - y) = k$$

Each value x, y, z has to be greater than 0, which also represents the lower boundary of the integrals. The upper boundary is defined by the plane:

$$z \to 4 - 2x - y$$

For y, we set the z to 0, to see where it meets the axes:

$$4 - 2x - y = 0$$

$$y = 4 - 2x$$

$$y \to 4 - 2x$$

For x, we set y and z to 0.

$$4 - 2x - 0 = 0$$

$$4 = 2x$$

$$x = 2$$

$$x \to 2$$

which constructs the integral

$$\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} k \cdot dzdydx$$

Using Sage[10] we get the integral to be:

$$\frac{16}{3}k$$

We need the integral to be 1, so:

$$\frac{16}{3}k = 1$$

$$\frac{1}{\frac{16}{3}} = k$$

$$k = \frac{3}{16}$$

(c) Compute the probability $P(Y > X)$.

Answer Given that we want to assess the probability of Y being bigger than X, we need to set the lower boundary at X, so the integrated area will be from x to $4-2x$. This guarantees that Y will be greater than X. From this point, we integrate the interval[11]:

$$\int_0^2 \int_x^{4-2x} \int_0^{4-2x-y} \frac{3}{16} \cdot dzdydx$$

We get a finite value, which is $\frac{3}{4}$, which is equal to 75%. Thus, we conclude that the probability that Y is great than X is 75%.

(d) Find the expected values of X, Y, and Z, denoting them as $\bar{X}, \bar{Y}, \bar{Z}$ respectively. Plot the point $(\bar{X}, \bar{Y}, \bar{Z})$ in your figure and give a geometric interpretation of its location. What would happen if $f(x, y, z)$ was not a constant?

Answer The general formula to find the expected value, is summing the product of the set of values with their corresponding probabilities. As such, we incorporate the value of interest (x, y, z) inside the integral and multiply it with its corresponding

[10] Appendix 3b

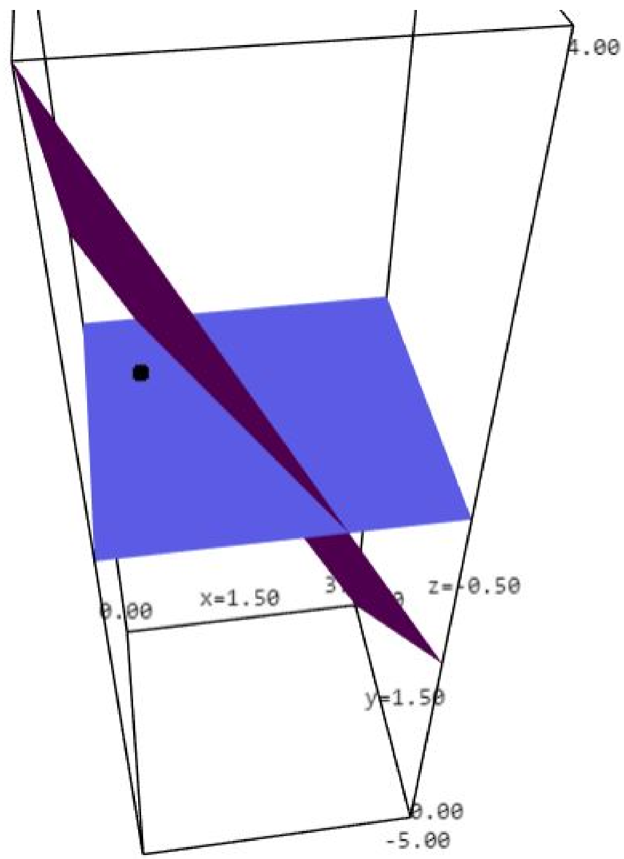
[11] Appendix 3c

probability given by $f(x, y, z)$.[12] If k was not a constant, then we would still follow the same procedure, however, our solutions may not produce an exact scalar.

$$\bar{X} = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} x \cdot \frac{3}{16} \cdot dzdydx = \frac{1}{2}$$

$$\bar{Y} = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} y \cdot \frac{3}{16} \cdot dzdydx = 1$$

$$\bar{Z} = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} z \cdot \frac{3}{16} \cdot dzdydx = 1$$



[12] Appendix 3d

4 *Euler's solver* (**#differentiation**, **#integration**)

Consider a differential equation of the type

$$\frac{dy}{dt} = f(t, y), \tag{1}$$

where f is a function that may depend on both t and y. We want to find the solution to this differential equation, $y(t)$, with initial conditions prescribed by $y(t_0) = y_0$. Our goal in this problem is to derive a computational scheme known as Euler's method, and see how it can be useful to approximate solutions to first-order differential equations.

(a) From above, we are given an initial point in the solution ($y(t_0) = y_0$), and an equation that describes the slope of the solution everywhere, Eq. 1. Use this information to find the **linear approximation** to the solution at the initial point (t_0, y_0).

Answer Let's start from the general line equation, which is:

$$y - y_0 = m(x - x_0)$$

where x_0 is the variable for which we know the value of $f(x_0) = y_0$. The variables x, is the what we are approximating for, while y is $f(x)$. Lastly, m represents the slope and is found from differentiating the function $f'(x_0)$. By placing y_0 on the right side of the equation we get:

$$y = m(x - x_0) + y_0$$

or put differently,

$$f(x) = f'(x_0)(x - x_0) + f(x_0)$$

From this setup, we can find the general linear approximation formula:

$$L(x) = y = f(x_0) + f'(x_0)(x - x_0)$$

Given this information, we substitute in the general formula with the information we have, $(x_0 = t_0; f(x_0) = y_0; x = t; m = f(t_0, y_0))$, which takes the shape of

$$L(t) = y(t_0) + y'(t_0)(t - t_0)$$

which is equal to

$$L(t) = y_0 + f(t_0, y_0)(t - t_0)$$

.

(b) Use the linear approximation to estimate the value of the solution at a nearby point, $t_1 = t_0 + h$. We will denote your approximation by y_1. Provided that h is sufficiently small, the line tangent and the solution will be close, and y_1 will be a reasonable approximation of the actual value of the solution, $y(t_1)$.

Answer From the part (a) equation $L(t) = y(t_0) + y'(t_0)(t - t_0)$, we could substitute for $t_1 = t_0 + h$ instead of t:

$$y_1 = y(t_0) + f(t_0, y_0)(t_1 - t_0)$$

From further simplification (by representing t_1 as $t_0 + h$), we get:

$$y_1 = y_0 + f(t_0, y_0)(t_0 + h - t_0)$$

which is equal to

$$y_1 = y_0 + f(t_0, y_0)(h)$$

(c) Now, find the equation of the line that goes through the point (t_1, y_1) and has the same slope the solution would have at that point, as prescribed by the differential equation.

Answer We reuse the general line equation from part (a):

$$y - y_0 = m(x - x_0)$$

In our context, instead of x_0 and y_0, we use t_1 and y_1:

$$y - y_1 = m(t - t_1)$$

Given that our slope m has the shape of $f(t_1, y_1)$, we substitute that in the equation:

$$y - y_1 = f(t_1, y_1)(t - t_1)$$

We could further extend this to:

$$y - y_1 = f(t_1, y_1)t - f(t_1, y_1)t_1$$

(d) We may once again use this line to estimate the value of the solution at a nearby point. Consider $t_2 = t_1 + h$. Estimate the value of the solution at the point t_2, which we will denote by y_2.

Answer Starting from this line equation:

$$y_2 - y_1 = f(t_1, y_1)(t_2 - t_1)$$

Moving y_1 to the right-side, gives us:

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$$

Since we know that $t_2 = t_1 + h$, the equation takes the shape of:

$$y_2 = y_1 + f(t_1, y_1)(t_1 + h - t_1)$$

which is finally equal to:

$$y_2 = y_1 + f(t_1, y_1)(h)$$

(e) Generalize: come up with an algorithm to compute y_n, the estimated value of the solution at the point $t_n = t_0 + nh$.

Answer

```
1. Find f(t,y)
2. Define the initial points t_0, y_0
3. Select a step-size, h
4. Select an iteration number, n
5. Iterate over n
5.1 t = t+h
5.2 y = y+ h*f(t,y)
5.3 Output y.
```

(f) We can model the rate of change of money in a checking account by the differential equation

$$\frac{dy}{dt} = ry + q(t), \tag{2}$$

where r is the interest rate (compounded continuously), $q(t)$ is your rate of saving/spending, with t measured in months. Consider a checking account with an annual interest rate of 3% that is initially empty, and a constant savings rate of $q(t) = 100$ dollars/month. Use the algorithm you constructed in part (e) to estimate the amount of money you will have after a year. Try at least three different values of the **step-size** h. Which one provides the most accurate estimate.

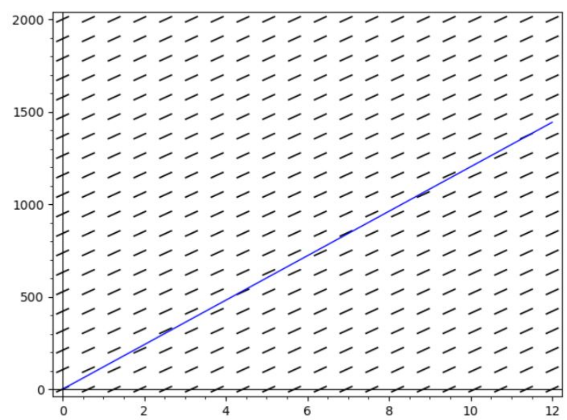
Answer[13]
With $t_0 = 0$, $y_0 = 0$, $h = 1$, $t = 12$, the answer outputted is 1419.2029561539296.
With $t_0 = 0$, $y_0 = 0$, $h = 0.5$, $t = 12$, the answer outputted is 1431.6760397634166.
With $t_0 = 0$, $y_0 = 0$, $h = 0.05$, $t = 12$, the answer outputted is 1443.1428477278268.

[13] Appendix 4e

The most accurate estimate is provided by the last case, where the step-size is the smallest. That is because the value of y is being evaluated more frequently, which is a more accurate depiction of the final stage. For instance, in real life, if you evaluated your expenses every week, you would have a more accurate description of the expenses after a year, rather than just calculating it every 1 month.

(g) Plot the slope field for the differential equation from Eq. 2. On the same axis, plot the points (t_i, y_i) you obtained from your algorithm, and finally, compare your approximations with the actual solution $(y(t))$.

Answer The slope field plotted from Sage with the line that our algorithm outputted is:



The actual solution can be found with Sage[14] is 1444.43138186780. The relative error for

i. step-size 1 is 1.747%
ii. step-size 0.5 it's 0.883%
iii. step-size 0.05, the relative error is 0.0893% [15]

We see that the smaller the step-size, the smaller the error. For $h = 0.05$, the relative error is almost insignificant.

[14] Appendix 4g

[15] Calculated by relative error = —absolute error / actual value— = —(actual value - measured value) / actual value—

Reflection Questions

1. Give two examples of applying HCs to solve a specific problem. How did you apply the HC? Was the application successful? If yes, why? If not, how could you improve your application?

   #heuristics: Throughout the assignment, we applied multiple heuristics, out of which analogous thinking and tallying were the most useful. When approaching a task, we referenced class material to find similarities and differences to get started with the problem's conceptual understanding. We used the study guide materials, pre-class videos, and breakout notebooks to refresh our understanding of the work's computational components. The analogous thinking was plausible for the given case since we covered similar problem sets during our class sessions and used these as references to build upon. Tallying is a useful heuristic for evaluating the results of your calculations. For instance, you get an answer you were not expecting in the final question. Thus, you can create a checklist to see your calculations' backbone (e.g., what assumption each step presents for the given problem). Then, once you see some mismatches between calculations and the implications of these calculations, you can make a plausible conclusion of why you arrived at the specific answer.

   #algorithms: In this assignment, we have consistently gone through a step-by-step analysis to solve the problem at hand. In question 4 particularly, we have created a general algorithm, which finds the numerical solution to differential equations. In part (e), we present a pseudocode, and then in Appendix 4e, we show its implementation on Python. The outputs different values for alterations of step-size, which shows that it is working. Moreover, we test it against the theoretical solution, and its empirical approximation has a small relative error. To improve this application, it would be useful to consider more edge cases (for instance, by setting every parameter of the *eulermethod* function to 0, we should expect the output to be 0). However, this was beyond the prompt requests.

2. Give two examples of applying HCs to complete this assignment as a group. How did you apply the HC? Was the application successful? If yes, why? If not, how could you improve your application?

   #differences: We effectively applied this HC from the beginning of our cooperation on the Final Deep Dive assignment. During Session 14.1, we discussed the strategy to tackle the assignment and divided the problems according to our strengths and preferences. This approach was successful because it allowed for efficient teamwork, where each member could contribute to the best of their abilities and share their knowledge with the other person and discuss the challenges they are facing. Such a practical assessment of our skills created a friendly and approachable environment that prompted cooperation. Apart from the academic strategy, we also worked on the communications strategy, selecting the means of communication that were the most suitable for both of us, along with making decisions on the format of our assignment and the technical tools we plan to use. Altogether, this framework, based on our differences, worked well given the assignment's type and the time constraints.

#responsibility We approached the division of responsibilities based on individual preferences and skills. For instance, Alma writes her assignments in LaTeX, and she suggested writing up the Deep Dive using LaTeX. Other organizational parts included reflecting on our experience and describing HC applications, which became Liuda's subtask. Along with this, we worked separately on our problems(Alma on problems 3 and 4, Liuda on problems 1 and 2). We set up regular daily check-ins with each other to see what progress we are making. Besides this, we set a deadline for submitting our parts and another deadline for the Deep Dive submission, which kept us accountable.

Appendix

2a(ii)

```
var("th,t")
f=((sin(th)*sin(th)+e**(-t))**(-1))
show(f.hessian())
```

$$\begin{pmatrix} -\frac{e^{(-t)}}{\left(\sin(th)^2+e^{(-t)}\right)^2}+\frac{2\,e^{(-2\,t)}}{\left(\sin(th)^2+e^{(-t)}\right)^3} & -\frac{4\,\cos(th)e^{(-t)}\sin(th)}{\left(\sin(th)^2+e^{(-t)}\right)^3} \\ -\frac{4\,\cos(th)e^{(-t)}\sin(th)}{\left(\sin(th)^2+e^{(-t)}\right)^3} & \frac{8\,\cos(th)^2\sin(th)^2}{\left(\sin(th)^2+e^{(-t)}\right)^3}-\frac{2\,\cos(th)^2}{\left(\sin(th)^2+e^{(-t)}\right)^2}+\frac{2\,\sin(th)^2}{\left(\sin(th)^2+e^{(-t)}\right)^2} \end{pmatrix}$$

2b(iii)

```
var('delta_x,delta_y')
print(limit(f(x+delta_x,0)-0/delta_x,delta_x=0))
print(limit(f(0,y+delta_y)-0/delta_y,delta_y=0))
```

```
0
0
```

2b(v)

```
limit((partial_x(0,0+delta_y)-0)/delta_y,delta_y=0)
```

```
-1
```

```
limit((partial_y(0+delta_x,0)-0)/delta_x,delta_x=0)
```

```
1
```

3b

```
var('x' ,'y', 'z' ,'k')
f(x,y,z)=k
z_plane = 4- 2*x - y
integrate(integrate(integrate(f,(z, 0, 4- 2*x - y)), (y, 0, 4-2*x)), (x,0,2))
```

Evaluate

Language: Sage

Share

```
16/3*k
```

3c

```
var('x' ,'y', 'z' ,'k')
f(x,y,z)=3/16
z_plane = 4- 2*x - y
integrate(integrate(integrate(f,(z, 0, 4- 2*x - y)), (y, x, 4-2*x)), (x,0,2))
```

Evaluate

Language: Sage

Share

```
3/4
```

3d

```
var('x' ,'y', 'z' ,'k')
f(x,y,z)=3/16
z_plane = 4- 2*x - y
expected_x = integrate(integrate(integrate(f*x,(z, 0, 4-2*x - y)), (y, 0, 4-2*x)), (x,0,2))
expected_y = integrate(integrate(integrate(f*y,(z, 0, 4-2*x - y)), (y, 0, 4-2*x)), (x,0,2))
expected_z = integrate(integrate(integrate(f*z,(z, 0, 4-2*x - y)), (y, 0, 4-2*x)), (x,0,2))
print(expected_x, expected_y, expected_z)
```

Evaluate

Language: Sage

Share

```
1/2 1 1
```

4e

```
# Differential equation
# diff = y'= ry+100
def slope(q,y):
  return ((0.03*y)+q)

def euler_method(t_0, y_0, h, t):
  while t_0<t:
    y_0= y_0+slope(100, y_0)*h
    t_0=t_0+h
  return y_0
```

```
euler_method(0, 0, 1, 12) # step-size 1
```

```
1419.2029561539296
```

```
euler_method(0,0, 0.5, 12) # step-size 0.5
```

```
1431.6760397634166
```

```
euler_method(0, 0, 0.05, 12) # step-size 0.05
```

```
1443.1428477278268
```

4g

```
var ('t')
y=function('y')(t)
dt=diff(y,t)-0.03*y-100
solution=desolve(dt, y, ics=[0,0])

print(N(solution(12)))
```

Evaluate

Language: Sage

Share

```
1444.43138186780
```