Assignment III

Linear Transformations

Minerva Schools at KGI
CS111B
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Table of Content:

Glossary

- **Linear Functions** - graphically these functions are straight lines. This means that all the variables are in the power of 1. As in other functions, we find values of y depending on x (or more predictors).

- **Linear Systems** - systems of equations that contain the same variables in the power of 1 or 0. They can have a unique solution, infinitely many solutions or no solution based on whether they are consistent and parametric.

- **Consistent Systems** - Linear Systems of Equations which contain no inconsistencies, or equations that are known mathematically not to make sense.

- **Augmented Matrices** - represents important information such as the right-hand side of the equations or an identity matrix for the purpose of proving invertibility. We denote is as $[A|b]$.

- **Row Operations** - Transforming the matrix into another matrix of the same size requires Row Operations. We can swap rows, multiply by a scalar and add one row to the other (or the scalar of the other row).

- **Elementary Matrices** - Elementary Matrices are matrices we use to solve Linear Systems of Equations, which are similar to identity matrices but change by a row operation that helps us reduce the original matrix to RREF.

- **Invertible Matrix** - is a special type of matrix, which is also known as non-singular. Its determinant is different from 0. Not all matrices are invertible, but those which are, when multiplied with the original matrix should produce an identity matrix (regardless of the side of multiplication). We can find the inverse of the matrix by augmenting it with an identity matrix and reducing it to RREF.

- **RREF Matrix** - (Reduced Row-Echelon Form) When the matrix is in this stage, we can see pivots of 1 and under and above pivots, 0s. We need those for solving linear systems when the matrix is augmented. The pivots represent the variables in a diagonal and the right-hand side represents the values of the variables for the given system. There are some criteria for a matrix to be in an RREF (more thoroughly explained, here). We can infer how many solutions the system has after turning the matrix into RREF.

- **Homogeneous Systems** - systems where all the solutions of the system (constants or right-hand side) are 0.

- **Determinants** - are property of matrices, more specifically a calculation of the elements of matrices based on certain rules of linearity. They are useful to make inferences about the matrices, for instance, if the determinant of the matrix is 1, we can say the system has one unique solution or if the determinant of the matrix is 0, the matrix is not invertible. We denote the determinant with det(A), where A represents the matrix.

- **Geometric Vector -** a vector with length and direction in space. Vector length: the magnitude of the vector. The magnitude of the vector is denoted as|| ||. We find it by summing the squares of all its elements under the square root.

- **Vector direction** - is the angle that the line makes with the horizontal Segment.

- **Dot Product** - the sum of respective components of two or more vectors (or one vector with itself). The result is a scalar. It requires two vectors of same length.

- **Standard Unit Vector** - a vector with magnitude 1. We use them to determine direction. Each vector can be represented in terms of standard unit vectors.

- **Projection of Vectors** - presenting one of the vectors in a straight line parallel to the other vector. We find it by the dot product of the two vectors and the unit vector defining the direction of the second Vector.

- **Vector space** - a set/space of vectors that share a set of properties such as:
  1. Closed under addition
  2. Closed under scalar multiplication
  3. There is a zero vector
  4. There are additive inverses
  6. The addition is commutative
  7. Addition and scalar multiplication are associative
  8. Addition and scalar multiplication have distributive properties

- **Vector subspace** - a vector space has a subspace when the subspace meets the three first properties of vector spaces.

- **Vector span** - is a set of all possible combinations of the vectors in S, if S is a subspace of vector space V.

- **Linear Independence** - is a linear combination of the vector that will result in 0, thus none of the vectors can be represented as a linear combination of others.

- **Basis of vectors** - is a set of vectors that span a vector space and are linearly independent simultaneously.

- **Dimensions of vectors** - the number of bases in a vector space. Orthonormal Vectors - A set of vectors is said to be orthogonal if each of the vectors has a length of 1 and their dot product is zero, thus they form a 90-degree angle with each other.

- **Orthonormal Basis** - is a basis for a vectors space whose vectors are orthonormal vectors.

- **Gram-Schmidt Algorithm** - is an algorithm for orthonormalising a set of vectors.
- **Orthogonal Matrices** - is a square matrix whose columns are orthogonal unit vectors (orthonormal vectors).

- **Change of Basis** - representing the vectors of one basis in terms of the vectors in the other basis.

- **Rank** - The number of linearly independent columns or the number of pivot columns. Rank(A)=dim(Acol)=dim(Arow).

- **Dimension** - Represents the number of vectors in the basis of the subspace.

- **Column space** - Represents the vector space given by the span of its columns.

- **Row space** - Represents the vector space given by the span of its rows.

- **Nullspace** - Represents the subspace that contains all the vectors which result in the homogeneous system of M (Mx=0).

- **Left nullspace** - Represents the vectors of A transpose that equal to the 0 vector as a solution.

- **Incidence matrix** - Matrices that are used to represent graphs. Rows represent edges, and columns denote nodes. Exit nodes are denoted as -1, if the edge leaves from the node to another node, which is called an entry node and it is denoted in the graph as +1.

- **Linear map** - Maps are functions that takes a vector from one vector space to the second vector space. A linear map, is a map for which vector addition and multiplication by a scalar still hold.

- **Eigenvector** - An eigenvector of a transformation is a special vector which can only change in magnitude (but not direction) after the transformation.

- **Eigenvalue** - A scalar value used to rescale eigenvectors.

Skill Builder

➔ Session 8.2

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1. **Find a basis and dimension for both of them:**

In order to find the basis we need to turn the matrix in Row Reduced Echelon Form and see the linearly independent columns:

$$RREF\ (A) = \begin{bmatrix} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \begin{matrix}-2\\1\\0\end{matrix} & \begin{matrix}1\\0\\0\end{matrix} \end{bmatrix}$$ (Produced by Sage)

We see that the matrix has two independent columns which serve as the basis for the matrix. Those two vectors span the space and can be used to represent the other vectors in this vector space. The dimension of the basis is 2 since there are two vectors in the basis.

Basis (A) = {[1, 0, 1], [2, 1, 2]}
Dim (A) = 2

With the given information we can find the basis of all the fundamental subspaces. Since we know that the dimension of the basis of column space of A is equal to the dimension of the basis of row space of A, and both of them equal to Rank (A), then:

1. Dim (Basis (A)) is the dimension of the basis of column space Col (A), thus Dim (Col (A)) = 2

2. Dim (Col (A)) = Dim (Row (A)) = Rank (A) = 2

3. Dim (Col (A)) + Dim (N^T (A)) = 3 … because the dimensions of A are 3x4.

a. Thus, Dim (N^T (A)) = 1

4. Dim (Row (A)) + Dim (N (A)) = 4 … because the dimensions of A are 3x4.

    a. Thus, Dim (N (A)) = 2

In conclusion:

Dim (Col (A)) = 2
Dim (Row (A)) = 2
Dim (N (A)) = 2
Dim (N^T (A)) = 1

We see that the Reduced Row Echelon Form of A is equivalent to U. Thus, U is already in RREF. From this, we can see that linearly independent columns are the same as A's, respectively first and second. The dimension is 2, too.

Basis (U) = {[1, 0, 0], [0, 1, 0]}
Dim (U) = 2

We follow a similar deductive procedure to find the dimensions of other subspaces:

Dim (Col (U)) = 2
Dim (Row (U)) = 2
Dim (N (U)) = 2
Dim (N^T (U)) = 1

2. **Compare the results. How are these matrices related? How are their subspaces related?**

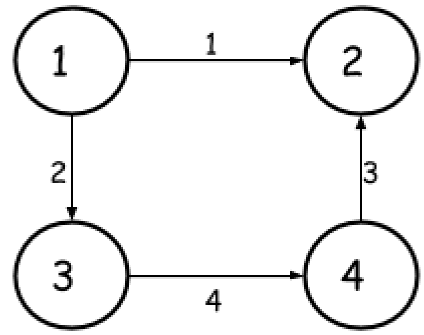
Since U is the RREF of A, we can say that those matrices are row-equivalent. If elementary matrices are used we can transform U to A and A to U.

Their subspaces have identical dimensions, as proven earlier, and their row space and left-null spaces are identical, but the vectors in the column space and nullspace are different (Sage).

➔ <u>Session 9.1 and 9.2</u>

1. Draw a graph with incidence matrix:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$



a. **Describe which edges form loops? How do they relate to the left-nullspace of A?**

From the graph, we see that all the nodes are involved in a big loop. If we consider that the loop starts at node 1 goes to 3, then 4, and finally 2, we can get back to node 1 by reversing the edge 1 (that connects node 2 and 1).

We can find the left-nullspace of A by putting the transpose of A in RREF with the solutions as 0 vector.

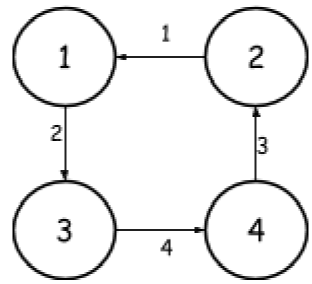
$$\text{RREF}(A^T) = \left[\begin{array}{cccc|c} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \rightarrow \begin{array}{l} -X_1 - X_2 = 0 \\ X_1 + X_3 = 0 \\ X_2 - X_4 = 0 \\ -X_3 + X_4 = 0 \end{array} \rightarrow \begin{array}{l} -X_1 = X_2 \\ X_1 = -X_3 \\ X_2 = X_4 \\ X_3 = X_4 \end{array}$$

From deduction, we find that $-X_1 = X_2 = X_3 = X_4$, which also could be represented as <-1,1,1,1>. We see that this vector also represents the directions of the edges. While edge 2, 3, and 4 have the same direction, edge 1 seems to go in the opposite. Another observation is that there is one vector on the basis of left-nullspace and there is one independent loop in the graph.

If we reach the stage that left-nullspace has an empty basis, then we can say there are no longer loops in the graph. That's because one full loop equals 0 when it returns to the initial position. By this inference, if we have two vectors on the basis of nullspace, then there are two independent loops that start and can get back to the same point resulting in zero (n- vectors on the basis of left-nullspace, n independent loops). But if the left-nullspace is empty, then we cannot seem to find edges that return to the initial point (whose sum would be 0), therefore there are no independent loops.

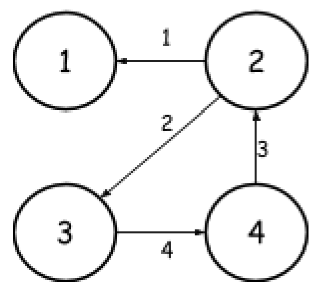
b. **Put A in RREF by hand. Draw the graph in each step.**

1. We multiply Row 1 by -1, to turn the pivot into a positive:

→ 

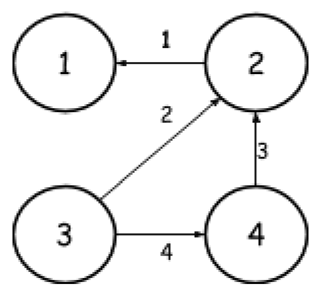
2. Add Row 1 to Row 2.

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \rightarrow$$



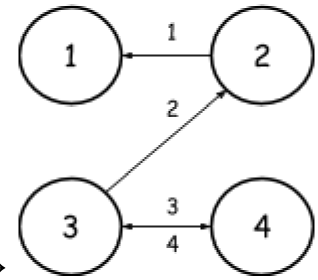
3. Multiply Row 2 by -1.

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \cdot (-1)} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \rightarrow$$



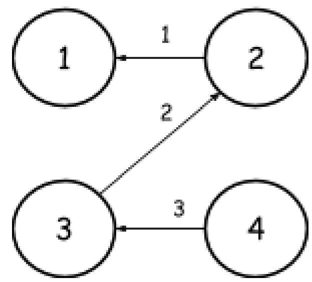
4. Subtract Row 2 from Row 3.

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \rightarrow$$



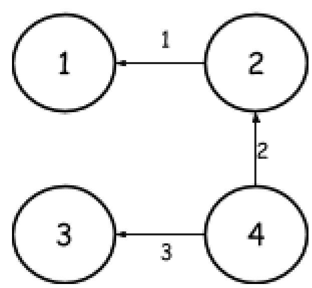
5. Subtract Row 4 from Row 3.

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_4 - R_3} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$



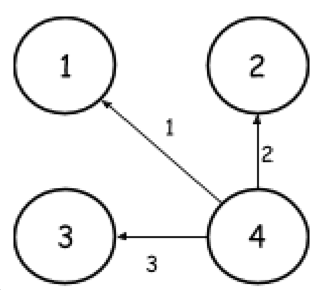
6. Add Row 3 to Row 2.

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 + R_3} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$



7. Add Row 2 to Row 1.

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$



c. **Conjecture the relationship between loops in the graph and rows in the incidence matrix.**

We see that the linearly independent rows of the incidence matrix are linearly independent resulting in no loops in the graph, which means that no row/edge can be created by summing the other rows/edges. This makes sense from Kirchoff's law perspective. The row space of A equated to 0 is equivalent to finding the left-nullspace of the matrix, and equivalent to finding out if rows are linearly independent. As long as the rows remain linearly dependent, then we will observe loops in the graph (Illinois Faculty, 2018).

d. **Find a basis for** Row (A). **What is the relationship between** Row (A) **and the spanning tree?**

By transposing A and putting it into RREF we get:

$$\text{Basis}(A) = \{[-1,1,0,0]^T, [-1,0,1,0]^T, [0,1,0,-1]^T\}.$$

This is identical to the incidence matrix of the spanning tree edges which we get in the last step of Part b. Back to the definition of basis, this means that all of the edges of the original matrix can be represented by the linear combination of these three vectors. We can say that the edges of the spanning tree can be represented by the vectors in the basis of Row space of A.

➔ Session 10.1 and 10.2

1. Hip to be square

a. Suppose all vectors v in the unit square [0, 1] × [0, 1] are transformed to Av where A is a 2 × 2 matrix. What is the shape of the transformed region when

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix}$$ **?**

The unit square coordinates are [0,0], [1,0], [0,1] and [1,1]. Multiplying each of these vectors with A will give us the coordinates of the transformation:

$$A*[0,0] = [0,0]^T;$$
$$A*[1,0] = [1,3]^T;$$
$$A*[0,1] = [3,7]^T;$$
$$A*[1,1] = [4,10]^T$$

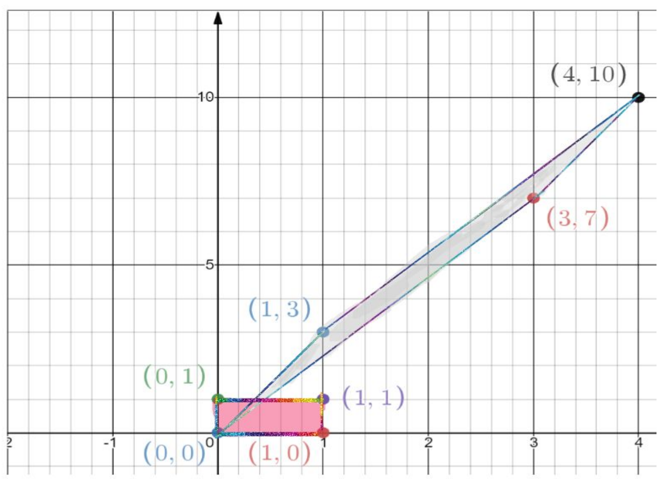


Figure 1. The graph formed from the coordinates of the transformed region (Desmos, n.d.).[1]

The role of A is to transform the non-zero vectors. Specifically, it adds $3*y$ to x and $3*x$ to $7*y$. We see from the picture that the transformed region shares the origin with the non-transformed shape. This is because the column vector was a zero vector, thus it did not shift in the graph.

b. In general, what is the shape of the transformed region?

The shape of a transformed region is a parallelogram.
For any given 2x2 transformation matrix A, we will find that the coordinates would always be [0,0], $[x_{11}, x_{21}]$, $[x_{12}, x_{22}]$, $[x_{11}+x_{12}, x_{21}+x_{22}]$. If we look closely, we see that the slope between the opposite sides is the same in the figure. In the same way, if we carry further calculations, we find that this is true for the general case as well, thus we can say that the shape of such transformation will always be a parallelogram (Princeton, n.d.).

[1] Self Reflective Comment: I drew the lines in the graph and the figure by hand, and I find it very useful how well I was able to understand it once I used my favorite colors to depict the transformation.

c. For which matrices A is that region a square?

The adjacents need to be normal on each other (form 90 degree angle), hence be parallel, and have the same size for the region to be a square.

1. Adjacents are equal to each other when $x_{11} = x_{22}$. This is because the role of these two entries is to rescale the adjacents. If the rescaling is unequal to at least one of the adjacents, then we violate the first condition of creating a square. By having the equal entries we make sure that the rescaling will always be identical.

2. The angle between the adjacents is 90 degrees when $x_{12} = - x_{21}$. These two entries are responsible for the angle between adjacents. When we make them negative to each other, we make sure that they will keep axes perpendicular to each other (Khan Academy, n.d.).

d. For which A is that region a line segment?

For the region to be a line segment, all the slopes should be equal and they should span the space. In order to achieve that, we need the angle between adjacents to be 0. As we said earlier, the entries responsible for rotation are x_{12} and x_{21}. So, solutions are always going to mean that one of these entries is 0.
A solution is making sure we eliminate x or y-axis. This can happen if, from this vector $[x_{11}+x_{12}, x_{21}+x_{22}]$, either x_{11} and x_{12} are 0, or x_{21} and x_{22} are 0 (Khan Academy, n.d.).

2. *All about that basis*[2]

Let $T: R_3 \rightarrow R_3$ **be the map given by** $T(x, y, z) = (x + 2y + 3z, 3x - 2y + z, 2x - 4y - 2z)$**.**

a. Show that T **is a linear transformation.**

There are two conditions to be met:

1. Closed under addition:

$T(a, b, c) + T(x, y, z) = T(a+x, b+y, c+z)$

$T(a, b, c) + T(x, y, z) =$
$= T(a + 2b + 3c, 3a - 2b + c, 2a - 4b - 2c) + T(x + 2y + 3z, 3x - 2y + z, 2x - 4y, -2z) =$

[2] Math Joke: What is the difference between a psychotic, a neurotic and a mathematician? A psychotic believes that 2+2=5. A neurotic knows that 2+2=4, but it kills him. A mathematician simply changes the base.

$= T(a + 2b + 3c + x + 2y + 3z, 3a - 2b + c + 3x - 2y + z, 2a - 4b - 2c + 2x - 4y, -2z)$

$T(a+x, b+y, c+z) = (a + x + 2b + 2y + 3c + 3z, 3a + 3x - 2b - 2y + c + z, 2a + 2x - 4b - 4y - 2c - 2z)$

2. Closed under multiplication

$kT(x,y,z) = T(kx, ky, kz)$

$kT(x,y,z) = (k(x + 2y + 3z), k(3x - 2y + z), k(2x - 4y - 2z) =$
$= (kx + 2ky + 3kz, (3kx - 2ky + kz, 2kx - 4ky - 2kz)$

$T(kx, ky, kz) = (kx + 2ky + 3kz, (3kx - 2ky + kz, 2kx - 4ky - 2kz)$

Thus, both conditions are met and the transformation is a linear map.

b. Find the matrix A of T with respect to the standard unit basis E of R_3.

The three unit basis vectors of R_3 are [1, 0, 0], [0, 1, 0], [0, 0, 1].

We need to transform each of them to find the matrix.

$T([1, 0, 0]) = (1 + 0 + 0, 3 - 0 + 0, 2 - 0 - 0) = [1, 3, 2]$
$T([0, 1, 0]) = (0 + 2 + 0, 0 - 2 + 0, 0 - 4 - 0) = [2, -2, -4]$
$T([0, 0, 1]) = (0 + 0 + 3, 0 - 0 + 1, 0 - 0 - 2) = [3, 1, -2]$

Each transformation serves as column vector, so we get the 3x3 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 2 & -4 & -2 \end{bmatrix}$$

c. Let $S = \{v1, v2, v3\} = \{<1, 0, -1>, <0, 2, 3>, <-1, 3, 0>\}$.

i. Show that S is a basis for R_3.

To prove that S is a basis for R_3, we need to check if there are three vectors and if they are linearly independent. As we see, that there are three vectors, we check if they are independent by putting the matrix in the RREF.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 3 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since we obtained the identity matrix, we know that the system has only the trivial solution, which makes the three columns vector linearly independent. Hence, S is a basis for R_3.

ii. Find the matrix B of T with respect to S.
iii. Find the change-of-basis matrix P from S to E.

Change of basis is representing the vectors of S in terms of the vectors of E. $P_{\square \to e}$ will remain the same as S, since E holds only the unit basis vectors. Thus:

$$P_{\square \to e} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ -1 & 3 & 0 \end{pmatrix}$$

$$P_{e \to \square} = \begin{pmatrix} \frac{9}{11} & \frac{3}{11} & -\frac{2}{11} \\ \frac{3}{11} & \frac{1}{11} & \frac{3}{11} \\ -\frac{2}{11} & \frac{3}{11} & -\frac{2}{11} \end{pmatrix}$$

Then we transform the vectors of S:

$$\begin{pmatrix} -2 & 13 & 5 \\ 2 & -1 & -9 \\ 4 & -14 & -14 \end{pmatrix}$$

which we multiply with the matrix $P_{e \to \square}$, to convert them to basis S . Finally, we get B, which is:[3]

$$\begin{pmatrix} -\frac{20}{11} & \frac{142}{11} & \frac{46}{11} \\ \frac{8}{11} & -\frac{4}{11} & -\frac{36}{11} \\ \frac{2}{11} & -\frac{1}{11} & -\frac{9}{11} \end{pmatrix}$$

[3] All those calculations are further elaborated in the Sage Notebook.

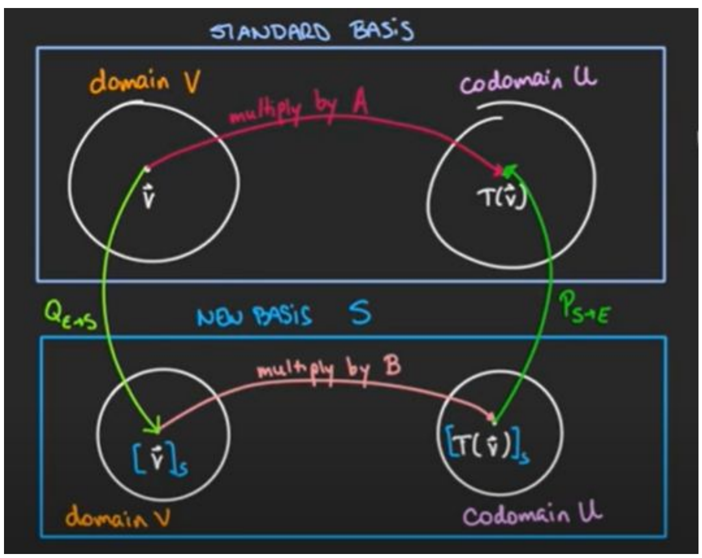
iv. Show that $B = P^{-1}AP$. Explain the mechanics behind this relationship.

When we multiply the inverse of P, the original matrix and P, we get the matrix that represents the vectors of S in terms of the vectors in E.

The parameters in this equation represent:

a) conversion from the vector basis to the standard basis (P).
b) transformation of the coordinates in the standard basis (A).
c) conversion to the original basis S (inverse of P).

The model from class depicts the relationship between the domains/codomains in different basis and the matrices that take you from one to the other:



➔ <u>Session 11.1</u>

Consider $P(3)$, the set of all the polynomials of degree 3 or less. Let

$T(ax^3 + bx^2 + cx + d) = (a + b)x^3 + (c-d)x$:

1. **Find a matrix A of T with respect to the standard unit vectors of $P(3)$ (i.e [1 0 0 0] for x^3, [0 1 0 0] for x^2, etc.).**

We begin by transforming the vectors in the standard basis of P(3). We know that the basis for P(3) is consistent of x^3, x^2, x and 1, which we will represent by the column vectors [1,0,0,0], [0,1,0,0], [0,0,1,0] and [0,0,0,1].
Then we apply transformation T to each of the vectors:

T ([1,0,0,0]) = ([1,0,0,0])
T ([0,1,0,0]) = ([1,0,0,0])
T ([0,0,1,0]) = ([0,0,1,0])
T ([0,0,0,1]) = ([0,0,-1,0])

$$\text{Matrix } A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. **What is the dimension of Image of T? How does this relate to the dimensions of the subspaces of A?**

After turning the matrix to RREF, we notice that there are two independent columns which imply that the rank is 2, therefore the dimension of Im(T) is also 2.
From this we already know that the dimension of column space is 2, thus the dimension of row space must be 2. Since the left-nullspace and the column space lie on Rn (which is 4 in this case) and the null space with row space lie in Rm (which is also 4), we know that dimensions of left-nullspace and null space must also be 2.

3. **Which polynomials would be mapped to the zero vector? How does this relate to subspaces of the matrix A?**

From the transformation function, if we equal it to zero, then we notice that a = -b and c=d. Therefore for any of these polynomials that satisfy those conditions, when transformed the solution is the zero vector.
The relation it has with the subspaces is that such combination indeed represents the nullspace of A.

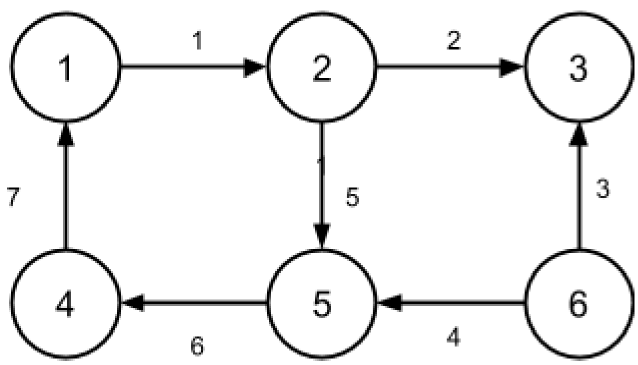
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Deep Dive

➔ *Avoiding Carmageddon*

a) The potentials of each node are the indicators of the flow of traffic. If we want to answer questions of the type, how many cars are entering and exiting the roundabout, we could use the information of the potentials of each node.
The potentials through the edges indicate the flow between two nodes; in other words, going from a roundabout to the other. The higher the potential difference, the bigger the traffic.

b) The graph looks like:



$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The incidence matrix represents the edges in and the nodes. To find its fundamental subspaces, we turn the matrix into RREF and check for linear independence.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Column space: we find that the first five column vectors are linearly independent. These vectors are the basis for column space. In this context, these vectors indicate all the combinations of the traffic flow in the nodes (enter/exit node).

Row space: From the RREF of the transpose of the matrix we see that the first 4 column vectors are linearly independent, together with the 6th vector. Thus, the basis for row space are the above mentioned vectors. In this context, those vectors indicate all combinations of the flows along the edges.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Null space: From Sage we find the two dimensional null space. In our context, this shows the potential of nodes for the traffic to be 0, or without any flow.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}$$ [In transposed form].

Left-nullspace: Similarly we find the one dimensional left-nullspace. In our context, the left null space informs us about having the same flow-in and out in the nodes.

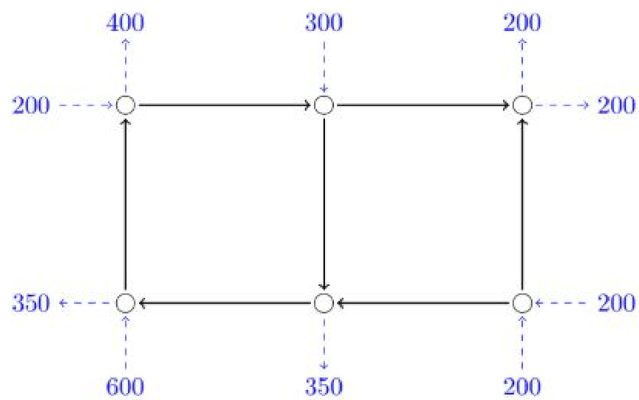
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$ [In transposed form].

c) Assumptions:

1. The vehicles must have access at all nodes. This means no matter where in the graph we start from, and what the destination is, we should be able to reach there. This means that all the nodes are somehow connected in the system, otherwise there would be nodes left out of the graph and without reachability.
2. There will be symmetrical information to the drivers who need to enter and exit particular nodes, if they need to go to a specific destination. For instance, drivers would at all cost avoid going to node 3, as you can only enter the node but not exit it.
3. The roads are one-directional.
4. The system is closed.

5. Edges should facilitate movement (each node must have enter and exit), so that there isn't overflow in one node. Again, if we look at node three, there is potential for overflow as each vehicle that enters it, stays there forever. We need the flow out of node 3 to be also open.

d) In this case, we should make sure that in the intersection inflow will equal flow out. From the given figure we find the potential differences, and make them equal to zero.



- Edge 1 + Edge 7 = 200
Edge 1 - Edge 2 - Edge 5 = 300
Edge 2 + Edge 3 = -400
- Edge 4 - Edge 3 = 400
Edge 4 + Edge 5 - Edge 6 = - 350
Edge 6 - Edge 7 = 250

Edge 7 = 200 + Edge 1
Edge 1 = 300 + Edge 2 + Edge 5
- Edge 3 = - Edge 2 + 400
- Edge 3 = Edge 4 + 400

When solving for the edges we should keep in mind that all edges should be positive at the end because the roads are one-directional.

After solving, we get a parametric solution with two free variables:

$$f\begin{bmatrix}1\\0\\0\\0\\1\\1\\1\end{bmatrix}+g\begin{bmatrix}0\\1\\-1\\1\\-1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\400\\0\\300\\-50\\200\end{bmatrix}=\begin{bmatrix}edge1\\edge2\\edge3\\edge4\\edge5\\edge6\\edge7\end{bmatrix}$$

Since the solutions should be positive, g cannot be more than 400, because the -1 in edge 3 will result in a negative for a bigger value of g. Seeing the negative at edge 5 as well, g has to be at least 300, if f=0. From edge 6, we see that f has to be bigger than 50.

There are many solutions to the system as long as those conditions are met. The extreme cases that would work would be f=50, g =400. In that case,

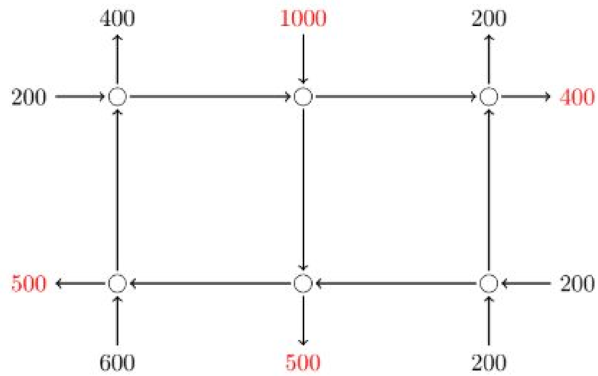
Edge 1 = 50
Edge 2 = 400
Edge 3 = 0
Edge 4 = 400
Edge 5 = 50
Edge 6 = 0
Edge 7 = 250

Realistically, we should be able to optimize this, by increasing f, so that edge 6 and 3 are able to accomodate more cars and make it easier for edge 4 and 2. Here, we have about 1150 cars. This solution would not be practical, as we would rather make use of all the segments of the road. With proper signage, we could regulate the distribution of cars more evenly.

e)

From the solutions above, we see that Edge 3 and Edge 6 could be completely removed and the traffic would still flow. However this would hinder the accessibility of all nodes. Generally, we can remove any of the edges that form independent loops in the graph.

f) We follow the same procedure of examining flow in and flow out:



- Edge 1 + Edge 7 = 200
Edge 1 - Edge 2 - Edge 5 = - 1000
Edge 2 + Edge 3 = 600
- Edge 3 - Edge 4 = - 400
Edge 4 + Edge 5 - Edge 6 = 500
Edge 6 - Edge 7 = - 100

As we try to solve this problem, we realize that there are no solutions because there are inconsistencies in the system. We realize that somewhere in the system, it is impossible for all the vehicles to be able to leave the node (inflow is not equal to outflow).

The inflow and outflow are specifically not equal in node 5, where the gridlock would happen. The gridlock would generally not happen if the inflow and outflow of node 5 would be equal, which is what we should aim for primarily.
To avoid the gridlock we can utilize the other edges by controlling the numbers of vehicles that enter node 4. Another possible solution would be changing the direction of edge 3, so that node 3 has an exit edge as well.

Appendix

- ➔ **Sage Solutions**

  Google Colab Notebook

- ➔ **HC Applications**

  #breakitdown: Used several techniques to divide problems in coherent and tractable chunks, which altogether lead to the solution. Effectively used steps in problems as a way of problem analysis/solving.

  #composition: Explained mathematical concepts in appropriate jargon for the audience, using clear, precise style. Added supplementary visual aids to enhance understanding.

References

Khan Academy:

1. Image of a subset under a transformation (video)
2. Rotating shapes about the origin by multiples of 90° (article)

Princeton:

1. Linear Transformations on the Plane

Desmos:

1. Plot Points

Illinois Faculty:

1. Applied Linear Algebra

I collaborated in solving problems with Sohit Miglani, Valdrin Jonuzi and Rhythm Mehta.